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Abelian gauge symmetries and proton decay in global F-theory GUTs

Thomas W. Grimm[1,3,*] and Timo Weigand[2,3,†]

[1]Bethe Center for Theoretical Physics, Nussallee 12, 53115 Bonn, Germany
[2]Institute for Theoretical Physics, University of Heidelberg, Philosophenweg 19, 69120 Heidelberg, Germany
[3]Kavli Institute for Theoretical Physics, Santa Barbara, California 93106, USA

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The existence of Abelian gauge symmetries in four-dimensional F-theory compactifications depends on the global geometry of the internal Calabi-Yau four-fold and has important phenomenological consequences. We study conceptual and phenomenological aspects of such $U(1)$ symmetries along the Coulomb and the Higgs branch. As one application we examine Abelian gauge factors arising after a certain global restriction of the Tate model that goes beyond a local spectral cover analysis. In $SU(5)$ grand unified theory (GUT) models this mechanism enforces a global $U(1)_X$ symmetry that prevents dimension-4 proton decay and allows for an identification of candidate right-handed neutrinos. We invoke a detailed account of the singularities of Calabi-Yau four-folds and their mirror duals starting from an underlying E_8 and $E_7 \times U(1)$ enhanced Tate model. The global resolutions and deformations of these singularities can be used as the appropriate framework to analyze F-theory GUT models.

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I. INTRODUCTION

The prospects of F-theory for the construction of realistic grand unified theory (GUT) models [1–4] have recently revived interest also in more formal aspects of F-theory compactifications on Calabi-Yau four-folds. While many phenomenological challenges of GUT model building can be and have been addressed already at the level of local models,[1] important issues remain which defy a treatment without reference to the global properties of the compactification. Among these are most notably almost all questions pertinent to Abelian gauge symmetries. Already the GUT breaking mechanism with the help of hypercharge flux, which is one of the characteristics of the F-theory GUT models advanced in [1–4], is sensitive to global compactification data because such flux can only be turned on along two-cycles that are trivial in the homology of the full Calabi-Yau four-fold. More fundamentally, the very definition of Abelian gauge symmetries hinges upon global information. This phenomenon is well-known already in the context of perturbative type II or heterotic string vacua, where Stückelberg couplings to axionic fields can degrade gauge symmetries to merely global selection rules below the mass scale of the gauge boson.

The global data of an elliptic Calabi-Yau four-fold are in general encoded in the Weierstrass model. In this paper we focus on elliptic fibrations that can be written globally in Tate form. The localized gauge degrees of freedom can be read off from the singularities of the elliptic fiber as reviewed in Sec. II A. In constructing global examples it is important to have a method to explicitly resolve the singularities. This is crucial not only to control the topology of the compactification and to reliably compute the Euler characteristic, but also to determine the physical spectrum such as the number of $U(1)$ bosons below the Kaluza-Klein scale. Such an explicit construction of a class of singular Calabi-Yau four-folds and their resolution has been achieved in Refs. [18,19] in terms of toric geometry, extending the program of global toric type IIB orientifold GUT model building advanced in [20]. See [21–23] for F-theory models on a different singular Calabi-Yau four-fold. Very recently, a similar approach as in [18,19] to the explicit construction and resolution of Calabi-Yau four-folds has been taken in [24]. The manifolds of [18,21] appear also in the models [25].

What has made the construction of three-generation $SU(5)$ GUT [18,19,21–23] and flipped $SU(5)$ GUT [24,25] models possible is, in addition, the description of gauge flux with the help of the spectral cover construction [1,4,7,26]. The latter can be thought of as the restriction of the Tate model to the vicinity of the $SU(5)$ GUT brane and as such necessarily discards some of the global information of the model. To decide in concrete global examples whether the spectral cover nonetheless captures the main aspects of the geometry correctly requires further tests. For the examples of [18,19] a spectral cover based formula for the Euler characteristic has been compared to the value computed independently via toric geometry and a match was found. This was taken as an indication that the spectral cover methods are applicable in these cases. Let us stress, however, that the spectral cover formula of [18] was not meant as an unambiguous method to compute the Euler characteristic in examples where independent computations are not available, but rather as a check on the applicability of the spectral cover methods to these Calabi-Yau manifolds. An even stronger indicator will be given in

*grimm@th.physik.uni-bonn.de
†t.weigand@thphys.uni-heidelberg.de

[1]For an incomplete list of references see e.g. [5–17].

Sec. IV of the present article via mirror symmetry, which in suitable examples exchanges the gauge group and spectral cover group.

As a special class of constructions, the so-called split spectral cover was discussed in [21,22] (and applied therein and in [18,19]) as a method to implement Abelian selection rules in the gauge theory of the GUT, e.g. $U(1)_X$ in the decomposition $SO(10) \to SU(5) \times U(1)_X$. These can forbid unwelcome couplings such as the dangerous dimension-4 proton decay operators. Given the global nature of Abelian gauge symmetries, however, it has continued to be an open question whether these selection rules are unbroken also in the full global model. Besides, the nature of $SU(5)$ gauge singlets playing the role of right-handed neutrinos has remained elusive in the spectral cover picture because these states are not localized on the GUT brane and are thus beyond the actual scope of the spectral cover.

In fact, the result of the analysis put forward in the present paper is that questions such as the existence of (massive) $U(1)$ gauge symmetries cannot be answered in a satisfactory manner from a spectral cover perspective. Rather, we find a method to ensure the presence of Abelian symmetries and thus to remove dangerous dimension-4 proton decay operators by considering special limits of the Tate model of the compact Calabi-Yau four-fold itself. We call the resulting construction $U(1)$-*restricted Tate models*. Note that quite recently, it has been argued in [27] from the perspective of the monodromies of the F-theory model that $U(1)_X$ is generically broken in split spectral cover models. Our approach has been independent of these findings and offers a complimentary view on the breaking of $U(1)_X$. We also describe a way to ensure that $U(1)_X$ is preserved as a—possibly massive—symmetry below the Kaluza-Klein scale in global models. Implementing this construction in concrete examples is of phenomenological importance not only in order to guarantee the existence of Abelian selection rules advertised in the split spectral cover context; rather, the very computation of individual chiral matter indices in split spectral cover constructions can be invalidated if the $U(1)$ symmetries are Higgsed at a global level. Determining whether this is the case requires a global analysis of the type put forward in a sequel paper.

In Sec. II we start from a Tate model with gauge group enhanced to E_8 along the divisor S. The final model with gauge group $G \subset E_8$ appears as a deformation thereof. In Sec. II B we study the Cartan $U(1)$s within E_8 by moving to the Coulomb branch via direct resolution and identifying the extended Dynkin diagram within the resolved fiber. We argue that the deformation of the Tate model to gauge group G generically Higgses all $U(1)$ symmetries within $H = E_8/G$. In nongeneric cases, however, some $U(1)$ symmetries remain. The gauge flux associated with the Cartan generators of H are most conveniently studied along the Coulomb branch. After the deformation they describe data of a truly non-Abelian H-bundle. A special role is played by the extended node in the extended Dynkin diagram. In Sec. II C we argue that this node carries information about massive $U(1)$ gauge symmetries that have acquired a Kaluza-Klein scale mass via the F-theoretic analogue of the type II Stückelberg mechanism. This clarifies the F-theory fate of the ubiquitous type II $U(1)$s.

We then specialize to the $U(1)_X$ symmetry in $SU(5)$ GUT models. In Sec. III A we recall its local description in terms of an $S[U(4) \times U(1)_X]$ spectral cover. We argue that generically this symmetry is Higgsed by VEVs of $U(1)_X$ charged $SU(5)$ GUT singlets localized away from the GUT brane and invisible to the spectral cover analysis. In this case effective dimension-4 proton violating couplings may ruin the model in a way inaccessible from the spectral cover perspective. We then resolve the problem of dimension-4 proton decay in Sec. III B by promoting the split of the spectral cover to a global restriction of the sections appearing in the Tate model.[2] This gives an unambiguous way to determine the presence of $U(1)_X$ by detection of a curve of $SU(2)$ enhancement on the I_1 divisor of the discriminant. We identify this curve as the proper localization curve of the states charged only under $U(1)_X$ which have the correct quantum numbers to play the role of right-handed neutrinos. This resolves also the puzzle of the localization of neutrinos in the spectral cover context [18,22]. The appearance of an extra Abelian gauge symmetry is further confirmed in Sec. III C, where we resolve the singular curve. In the spirit of M/F-duality the resulting increase in $h^{1,1}$ indicates a $U(1)$ boson which is massless in the absence of gauge flux. The latter can render the Abelian symmetry massive, which then survives only as a global symmetry valid below the Kaluza-Klein scale. For applications such as the engineering of $U(1)_X$ this is exactly what one is interested in for model building. We furthermore show that the underlying gauge symmetry in the Tate model is $E_7 \times U(1)$, as opposed to the previously described E_8 for the generic Tate model. A drawback of the $U(1)$ restriction, however, is seen to be a significant decrease in the Euler characteristic of the four-fold. This challenges the construction of models satisfying the $D3$-tadpole constraint. Finally, in Sec. III D we describe how the appearance of $U(1)_X$ can be further understood in analogy with type IIB orientifolds, where the restricted Tate model describes brane-image brane splitting.

In Sec. IV we study in greater detail the deformation of the underlying E_8 Tate model to a compactification with $G \subset E_8$, thereby putting the logic of the local spectral cover approach into perspective with global models.

[2]Note that six-dimensional F-theory compactifications with a restricted Tate model and additional $U(1)$ factors have been studied from the perspective of heterotic/F-theory duality in Refs. [28–30].

This section fills in the technical details for some of the claims in Sec. II and stresses the use of mirror symmetry to study the deformations. In Sec. IVA we work out the natural appearance of the underlying E_8 structure for a certain class of F-theory models based on a $\mathbb{P}_{1,2,3}[6]$ Tate model. In fact the breaking of E_8 to G along a divisor via a bundle with structure group $H = E_8/G$ is recovered entirely in terms of the Tate model. These observations are independent of a local gauge theory perspective or a heterotic dual. This picture is corroborated further in Sec. IV B with the help of an analysis of the mirror dual Calabi-Yau four-folds and their gauge enhancements. It is found that in the mirror dual precisely those gauge groups appear which are responsible for the unfolding of E_8 to the various codimension loci of singularity enhancement. Finally in Sec. IV C this logic is applied to the mirror dual of the restricted Tate model corresponding to a split spectral cover based on an underlying $E_7 \times U(1)$.

II. COMPACT CALABI-YAU FOUR-FOLDS AND ABELIAN GAUGE SYMMETRIES

A. Complete-intersecting four-folds and the Tate form

To set the stage we introduce the Calabi-Yau four-folds on which we compactify F-theory. The class of four-folds we consider is rather general so as to cover the geometries which have been recently used in the study of F-theory GUT models in Refs. [18,19,24]. We explicitly realize the Calabi-Yau four-fold Y via *two* hypersurface constraints

$$P_{\text{base}}(y_i) = 0, \qquad P_{\text{T}}(x, y, z; y_i) = 0 \tag{2.1}$$

in a six-dimensional projective or toric ambient space. Here P_{base} is the constraint of the base B which is independent of the coordinates (x, y, z) of the elliptic fiber. This more general setting also includes hypersurfaces encoded by a single constraint $P_{\text{T}} = 0$ if P_{base} is chosen to be trivial. $P_{\text{T}} = 0$ is the constraint that describes the structure of the elliptic fibration. We consider fibrations that can be given in Tate form,

$$P_{\text{T}} = x^3 - y^2 + xyza_1 + x^2z^2a_2 + yz^3a_3 + xz^4a_4 + z^6a_6 = 0, \tag{2.2}$$

where (x, y, z) are coordinates of the torus fiber. In a sequel we will often be working with the inhomogeneous Tate form by setting $z = 1$. The $a_n(y_i)$ depend on the complex coordinates y_i of the base B and have to transform as sections of K_B^{-n}, with K_B being the canonical bundle of the base B. Setting all $a_n = 1$ one finds that (2.2) reduces to the elliptic fiber $\mathbb{P}_{1,2,3}[6]$. A fibration based on a representation of the elliptic curve as $\mathbb{P}_{1,2,3}[6]$ is called E_8 fibration for reasons that will be become clearer in Sec. IVA.

To put the ansatz (2.2) into perspective we recall that most generally every elliptic four-fold with a section admits a description as a Weierstrass model

$$P_{\text{W}} = x^3 - y^2 + fxz^4 + gz^6 = 0. \tag{2.3}$$

Clearly, every Tate model (2.2) can be brought into this form via the relation

$$f = -\frac{1}{48}(\beta_2^2 - 24\beta_4), \qquad g = -\frac{1}{864}(-\beta_2^3 + 36\beta_2\beta_4 - 216\beta_6), \tag{2.4}$$

where

$$\beta_2 = a_1^2 + 4a_2, \qquad \beta_4 = a_1a_3 + 2a_4, \qquad \beta_6 = a_3^2 + 4a_6. \tag{2.5}$$

In turn the Tate model is a specialization of the Weierstrass model (2.3), which emerges naturally in toric elliptic fibrations. We will focus on this class of elliptic fibrations.

The sections a_n encode the discriminant Δ of the elliptic fibration given by

$$\Delta = -\frac{1}{4}\beta_2^2(\beta_2\beta_6 - \beta_4^2) - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_4\beta_6. \tag{2.6}$$

The discriminant locus Δ may factorize with each factor describing the location of a 7-brane on a divisor D_k in B. The precise gauge group along D_k is encoded in the vanishing degree $\delta(D_k)$ of Δ and the vanishing degrees $\kappa_n(D_k)$ of the a_n,

$$a_1 = \mathfrak{b}_5 w^{\kappa_1}, \quad a_2 = \mathfrak{b}_4 w^{\kappa_2}, \quad a_3 = \mathfrak{b}_3 w^{\kappa_3}, \quad a_4 = \mathfrak{b}_2 w^{\kappa_4}, \quad a_6 = \mathfrak{b}_0 w^{\kappa_6}, \quad \Delta = \Delta' w^{\delta}, \tag{2.7}$$

as classified in Table 2 of Ref. [31]. For example, for an $SU(5)$ gauge group along the divisor S: $w = 0$, where w is one of the base coordinates y_i, the κ_n are given by

$$(\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_6) = (0, 1, 2, 3, 5). \tag{2.8}$$

The sections $\mathfrak{b}_n$ generically depend on all coordinates (y_i, w) of the base B but do not contain an overall factor of w. Note that in an $SU(5)$ GUT model the $\mathfrak{b}_n$ identify the so-called matter curves along which zero modes charged under the $SU(5)$ gauge group are localized. The matter curves for the **10** and **5** representations,

$$P_{\mathbf{10}}: z = w = 0 \cap \mathfrak{b}_5 = 0, \qquad P_{\mathbf{5}}: z = w = 0 \cap \mathfrak{b}_3^2\mathfrak{b}_4 - \mathfrak{b}_2\mathfrak{b}_3\mathfrak{b}_5 + \mathfrak{b}_0\mathfrak{b}_5^2 = 0, \tag{2.9}$$

are the curves of singularity enhancement to $SO(10)$ and, respectively, $SU(6)$. Note that for a generic choice of sections $\mathfrak{b}_i$ the $P_{\mathbf{5}}$-curve does not factorize so that all matter in the fundamental representation, both $\bar{\mathbf{5}}_M$ and the Higgs $\mathbf{5}_H + \bar{\mathbf{5}}_H$ is localized on the same curve. It will be a crucial task of Sec. III to identify a class of compact Calabi-Yau four-folds for which the $P_{\mathbf{5}}$-curve splits into two curves hosting $\bar{\mathbf{5}}_M$ and $\mathbf{5}_H + \bar{\mathbf{5}}_H$, respectively.

It is important to stress that in the case of such a higher degeneration, not only the elliptic fibration will be singular, but rather the Calabi-Yau four-fold itself. We will call this singular four-fold Y_G. The singularities of gauge group G can be resolved into a nonsingular Calabi-Yau four-fold $\bar{Y}_G$. This has been demonstrated explicitly for the $SU(5)$ GUT examples in Refs. [18,19]. The existence of such a resolved space is crucial to determine the topological data, as e.g. required for tadpole cancellation. $\bar{Y}_G$ is also plays a vital role in the study of Cartan symmetries and hence will be discussed more thoroughly in the next section.

B. Cartan $U(1)$s and underlying E_8 structure

In this section we study the connection between the Cartan $U(1)$ potentials and fluxes for the gauge theory on the brane S and the geometry of Y_G. Note that the globally varying dilaton profile makes it hard to obtain the spectrum and effective action of an F-theory compactification on a singular Y_G. What rescues us is the duality of F-theory to an M-theory reduction, which leads to an identification of $U(1)$ symmetries in geometric terms. Our strategy is to use the fact that in the M-theory reduction one can access the Coulomb branch of the gauge theory in which G is broken to $U(1)^{\mathrm{rk}(G)}$, where rk is the rank of G. In fact this branch is attained upon resolving Y_G into $\bar{Y}_G$ as follows.

The singular fiber of Y_G over the discriminant Δ contains a tree of zero-volume $\mathbb{P}^1$s, which intersect as the nodes of the *extended* Dynkin diagram of G in the fiber of the four-fold. Resolution corresponds to blowing-up the $\mathbb{P}^1$s to nonzero volume. More precisely, one can resolve the generic G singularity over S by introducing rk(G) blow-up divisors D_i, $i = 1, \ldots, \mathrm{rk}(G)$, which are $\mathbb{P}^1$ bundles over S. The extended node of the affine Dynkin diagram is obtained as the linear combination

$$D_0 = \hat{S} - \sum_i a_i D_i, \qquad (2.10)$$

where the divisor $\hat{S}$ in $\bar{Y}_G$ is the elliptic fibration over S, and a_i are the Dynkin numbers of the Dynkin node associated with D_i (see e.g. [32]). Explicit constructions of these D_i are known for various compact Calabi-Yau manifolds [31,33,34], including Calabi-Yau four-folds relevant for GUT model building [18,19]. Let us denote by ω_i, $i = 0, \ldots, \mathrm{rk}(G)$ the elements of $H^2(\bar{Y}_G, \mathbb{Z})$ which are Poincaré dual to D_0 and the blow-up divisors D_i. The intersections of the fiber $\mathbb{P}^1$s are captured by the identify

$$\int_{\bar{Y}_G} \omega_i \wedge \omega_j \wedge \tilde{\omega} = -C_{ij} \int_S \tilde{\omega}, \qquad (2.11)$$

where $\tilde{\omega}$ is a four-form on the base B and C_{ij} is the extended Cartan matrix of G. The appearance of C_{ij} links the group theory of G with the intersection theory on the resolved four-fold $\bar{Y}_G$. The two-forms ω_i are thus in direct relation with the simple roots of the gauge group G. Note that the integral (2.11) localizes onto S in agreement with the assertion that all non-Abelian gauge dynamics localizes onto S.

The connection between the gauge theory and geometry can be exploited further by noting that for a Tate model of the form (2.2) the gauge enhancements follow a structure inherited from an underlying E_8 symmetry into which G can be embedded. This E_8 structure emerges naturally in the context of the globally realized Tate models in projective or toric ambient spaces in which it is possible to enhance the gauge symmetry to E_8 along the divisor S by a suitable choice of sections a_i. This leads to a singular four-fold Y_{E_8}. For the examples relevant to this work one can show explicitly that Y_{E_8} can be resolved into a Calabi-Yau manifold $\bar{Y}_{E_8}$. The original four-fold Y_G is a deformation of Y_{E_8} such that along S the underlying E_8 is broken to $G = E_8/H$, where G is the commutant of E_8 with the group H. Physically this corresponds to a recombination of the 7-branes. The details of this E_8 enhancement and the deformations to $G \subset E_8$ are described in Sec. IVA. We hasten to add, though, that more general examples of Weierstrass models with higher rank gauge groups that cannot be embedded into E_8 are known (see e.g. the recent [35,36]). It would be important to understand the generalization of the E_8 construction to these more general situations.

In order to not directly work with singular geometries, one can also attempt to interpret the transition between the two resolved Calabi-Yau four-folds $\bar{Y}_{E_8}$ and $\bar{Y}_G$. As for Y_G one moves to the Coulomb branch of E_8 via resolution of Y_{E_8} into $\bar{Y}_{E_8}$ by introducing the resolution divisors $D_i^{E_8}$, $i = 0, \ldots, 8$. On $\bar{Y}_{E_8}$ the group theory of E_8 is realized via global four-fold intersections as in (2.11), now for the dual two-forms $\omega_i^{E_8}$. One can divide the resolution divisors $D_i^{E_8}$ into two sets D_i^G and D_i^H such that the dual two-forms intersect as the respective Dynkin diagrams of the two commuting E_8 subgroups G and H in $E_8 \to G \times H$. This makes the Cartan generators of the commutant H of $G \subset E_8$ visible in terms of the two-forms ω_i^H with dual divisors D_i^H.

One may think of the transition from $\bar{Y}_{E_8}$ to $\bar{Y}_G$ as follows: The deformation introduces monodromies[3] for the individual $\mathbb{P}^1$ fibers of the D_i^H. Generically this Higgses H along S completely. The $\mathbb{P}^1$ fibers of D_i^H have been pulled off S so as to lie in the fiber over the remaining I_1 locus which intersects S along the curves. Because of the monodromies these $\mathbb{P}^1$s are no longer independent, corresponding to the Higgsing of the Cartan group H. The change from $\bar{Y}_{E_8}$ to $\bar{Y}_G$ is thus captured by a geometric transition, in which the resolving Kähler volumes are replaced by complex structure deformations. Schematically one can summarize this as

[3]The role of monodromies in local F-theory models has been discussed in [7,8,22,27].

$$\begin{array}{ccc} Y_{E_8} & \xrightarrow{\text{brane recomb.}} & Y_G \\ \downarrow & & \downarrow \\ \bar{Y}_{E_8} & \xrightarrow{\text{geom. transition}} & \bar{Y}_G \end{array} \tag{2.12}$$

In Sec. IV B we will show that starting from a reference geometry $\bar{Y}_{E_8}$ the dual group H and its splittings can be studied using mirror symmetry for $\bar{Y}_G$.

Having collected some geometric properties of $\bar{Y}_G$ and $\bar{Y}_{E_8}$ we now turn to the discussion of Cartan $U(1)$s and gauge fluxes. In M-theory the gauge fields A^i in the Cartan algebra of G arise by reduction of C_3 along the two-forms ω_i^G, $i = 0, \ldots, \mathrm{rk}(G)$. Note that only $\mathrm{rk}(G)$ of these two-forms are cohomologically independent so that the expansion reads

$$C_3 = \sum_{i=1}^{\mathrm{rk}(G)} A^i \wedge \omega_i^G + \cdots. \tag{2.13}$$

In order to find the actual Cartan $U(1)$s one first has to introduce the linear combinations of ω_i^G representing the Cartan generators of G. Recall that the ω_i^G correspond to the simple roots and intersect as in (2.11).

For the most generic deformation to $\bar{Y}_G$, the Cartan $U(1)$s of H are Higgsed completely in the course of the recombination process of (2.12). This corresponds to the maximal possible monodromy group acting on the $\mathbb{P}^1$ fibers of D_i^H. However, some of the Cartan elements of H may survive as massless $U(1)$ gauge symmetries in the full compactification if the Higgsing is incomplete. One of the results of our analysis is to identify in Sec. III B such a nongeneric deformation of Y_{E_8} to Y_G which does leave a certain $U(1)$ unHiggsed.

Since it will become relevant in that context, we now recall how to identify $U(1)$ factors in a full Weierstrass model. This gives an unambiguous way to determine the total rank of the brane gauge groups. It is sufficient to count the number of $U(1)$ factors along the Coulomb branch $\bar{Y}_G$. The existence of an Abelian gauge group below the Kaluza-Klein scale then hinges upon the availability of harmonic two-forms on the resolved Calabi-Yau four-fold $\bar{Y}$ along which the M-theory three-form can be reduced.[4] Not all these two-forms will correspond to four-dimensional $U(1)$ gauge bosons. First, there are $h^{1,1}(B)$ elements of $H^2(B, \mathbb{Z})$ which lead to chiral Kähler moduli of the F-theory compactification. Second, there is one element in $H^2(\bar{Y}_G, \mathbb{Z})$ which corresponds to the class of the elliptic fiber and captures the extra metric degrees of freedom in the lift of a three-dimensional M-theory compactification to a four-dimensional F-theory compactification.

[4]In addition, the reduction along harmonic (2, 1)-forms of the base B of Y leads to so-called bulk $U(1)$s which are the equivalent of the Ramond-Ramond (RR) $U(1)$s obtained in a type IIB reduction of C_4.

In summary, the number of Abelian brane gauge symmetries in the Coulomb branch is now given by

$$n_v = h^{1,1}(\bar{Y}_G) - h^{1,1}(B) - 1. \tag{2.14}$$

Note that if Y_G gives rise to the non-Abelian gauge symmetry G, then the number of extra $U(1)$ factors that are not Cartan elements of G is given by $\tilde{n}_v = n_v - rk(G)$.

To end this section, let us note that the construction of $\bar{Y}_{E_8}$ allows us to think about the description of gauge flux as follows: On $\bar{Y}_{E_8}$ a subclass of gauge flux derives from the Cartan generators of E_8. The gauge flux within that class leaving G intact is of the form

$$G_4 = F_2^{(i)} \wedge \omega_i^H + \cdots, \tag{2.15}$$

where ω_i^H are the two-forms dual to the divisors D_i^H introduced above. This can be made concrete when considering, for example, an $SU(5)$ GUT model on S. To define G_4 on Y_{E_8} one first introduces a basis of fundamental weights of $H = SU(5)_\perp$ as

$$\lambda_i = \sum_{k=1}^{i} \omega_{5-k}, \qquad i = 1, \ldots 5, \tag{2.16}$$

where the ω_i correspond to the simple roots of E_8 with intersection (2.11), including the extended node ω_0. For the definition of the flux G_4 it is convenient to also introduce a "dual" basis of two-forms λ_i^* corresponding to the dual weights as linear combinations of the ω_i. One demands that these satisfy

$$\int_{\bar{Y}_{E_8}} \lambda^j \wedge \lambda_i^* \wedge \tilde{\omega} = \delta_i^j \int_S \tilde{\omega}. \tag{2.17}$$

The four-form flux G_4 can then be defined as

$$G_4 = \sum_{i=1}^{5} F_2^{(i)} \wedge \lambda_i^*. \tag{2.18}$$

To determine the analogous flux for Y_G one therefore has to trace back (2.15) under the geometric transition from $\bar{Y}_{E_8}$ to $\bar{Y}_G$. For the most generic deformation to $\bar{Y}_G$ which Higgses H completely, the formerly Abelian fluxes become data of a genuinely non-Abelian H-bundle. A precise study of these transitions is beyond the scope of this work, and will be presented elsewhere [37]. After the deformation the G_4 flux is no longer of the simple form (2.15) because it will in general not be a sum of four-forms that can be written as a product of two two-forms. However, in the transition from $\bar{Y}_{E_8}$ to $\bar{Y}_G$ the number of four-forms in $H^{2,2}(\bar{Y}_G)$ increases significantly. These new four-forms indeed cannot be represented as a wedge of two nontrivial two-forms on $\bar{Y}_G$. Fluxes of this type are known to appear in the superpotential rather then the D term (see Ref. [38] for a recent discussion). This is in agreement with the fact that after the deformation the Cartan $U(1)$s of H are Higgsed so that no field-dependent Fayet-Iliopoulos term can arise from the fluxes. If on the other hand a certain $U(1) \subset H$ survives, there exists Abelian gauge flux associated with that $U(1)$ on Y_G, and we will argue in Sec. III C for the presence of an associated Fayet-Iliopoulos term.

C. Massive $U(1)$s and the extended node

In the previous section we discussed the appearance of the Cartan $U(1)$s in the breaking of G to $U(1)^{\text{rk}(G)}$. In the Kaluza-Klein expansion we have used rk(G) independent forms ω_i^G. However, the local geometry sees an additional two-form associated with the extended node of the Dynkin diagram (2.10). This extra form does not extend to an element of $H^2(\bar{Y}_G, \mathbb{Z})$. In the language of resolving $\mathbb{P}^1$ fibers one thus finds a homological relation among the nodes e_i of the extended Dynkin diagram and the elliptic fiber e. Geometrically this means that there exists a three-chain $\mathcal{C}$ with boundary $\partial\mathcal{C}$ given by

$$\partial\mathcal{C} = \hat{e}, \qquad \hat{e} = e - \sum_{i=0}^{\text{rk}(G)} a_i e_i, \tag{2.19}$$

where $a_0 = 1$ for the $\mathbb{P}^1$ e_0 associated with the extended node. One can now attempt to include an additional two-form $\hat{\omega}$ in the reduction of C_3 which is nonclosed and precisely captures the relation (2.19) as

$$d\hat{\omega} = \Psi, \qquad \int_{\mathcal{C}} \Psi = \int_{\hat{e}} \hat{\omega} = 1. \tag{2.20}$$

It is possible to systematically include such nonclosed and exact forms in the dimensional reduction.[5] This leads to compactification on non-Calabi-Yau manifolds. To understand this in our context we first consider the M-theory compactification on an elliptic fibration that features $\hat{\omega}$ and later take the F-theory limit of vanishing fiber. In particular we also include $\hat{\omega}$ in the expansion of a globally defined two-form $J = \hat{v}\,\hat{\omega} + \cdots$. This implies that dJ does not vanish and one would interpret this as an M-theory compactification on a non-Kähler manifold. Moreover, $\hat{\omega}$ and Ψ also appear in the expansion of C_3 as

$$C_3 = \hat{A} \wedge \hat{\omega} + c\Psi + \cdots, \tag{2.21}$$

where $\hat{A}$ is a vector and c is a scalar in the noncompact dimensions. The condition (2.20) directly leads to the appearance of the covariant derivative $\mathcal{D}c = dc + \hat{A}$ in the field strength F_4 of C_3 as

$$F_4 = \mathcal{D}c \wedge \Psi + \hat{F} \wedge \hat{\omega} + \cdots. \tag{2.22}$$

This is the M/F-theory analogue of a Stückelberg term known in weakly coupled type II theories and implies that the $U(1)$ field can be rendered massive by absorbing the scalar c with a gauge transformation. Now we take the limit of vanishing fiber in the lift from M-theory to F-theory. The non-Kähler-ness disappears in the limit $\hat{v} \to 0$ while the extra $U(1)$ fields remain as massive fields at the Kaluza-Klein scale. Hence, one expects that these massive $U(1)$ factors play a prominent role in an F-theory compactification.

A clean interpretation of the geometrically massive $U(1)$s in (2.22) is provided in F-theory compactifications with an $SU(N)$ gauge group which admit a well-defined orientifold limit. In the orientifold picture one actually starts from a $D7$-brane construction with a $U(N) = SU(N) \times U(1)$ gauge group. Such examples arise if a stack of N $D7$-branes is identified with an image stack in the Calabi-Yau three-fold double cover Z of B. If these two stacks wrap four-cycles in different homology classes, the $U(1)$ in the decomposition of $U(N)$ becomes massive since a scalar appearing in the expansion of the RR two-form C_2 is gauged [43]. This can be inferred from the Stückelberg term of the $D7$-brane Chern-Simons action

$$S_{\text{CS}} \supset \int \hat{F} \wedge C_6, \tag{2.23}$$

where C_6 is the RR six-form dual to C_2, and $\hat{F}$ is the field strength of the brane $U(1)$. The associated mass term is purely geometric and independent of any fluxes. Hence, the massive $U(1)$ is not directly visible in an F-theory compactification with only harmonic forms, since in the absence of fluxes all gaugings and D-terms disappear. However, the gauging induced by (2.23) precisely maps to the gauging (2.22) and hence identifies the correct massive $U(1)$ in F-theory. Clearly also the scalar c in (2.21) is identified with the scalar arising in the RR two-form C_2. Note that this mechanism clarifies why in a type IIB compactification a stack of N branes not invariant under the orientifold action gives rise to gauge group $U(N)$, while in F-theory one generically sees only the $SU(N)$: The perturbative Stückelberg mechanism due to (2.23) is built into the geometry automatically and thus does not allow us to disentangle the massive $U(1)$ boson from the other Kaluza-Klein states. However, linear combinations F of such $U(1)$s may remain massless as far as the *geometric* Stückelberg is concerned, both in IIB and in F-theory. In the presence of internal gauge flux $\langle F \rangle$ these remaining $U(1)$ potentials may still receive a mass term from the independent coupling

$$S_{\text{CS}} \supset \int F \wedge \langle F \rangle \wedge C_4 \tag{2.24}$$

in IIB language. For a clean distinction between the two Stückelberg terms (2.23) and (2.24) in the recent type IIB literature see [44]. The M/F-theory analogue is the Chern-Simons coupling

$$S = \int C_3 \wedge F_4 \wedge G_4, \tag{2.25}$$

[5] See Refs. [39–41] for initial discussions of $\mathcal{N} = 2$ and $\mathcal{N} = 1$ non-Calabi-Yau reductions. The $U(1)$ sector of type II compactifications on non-Kähler manifolds has also been studied in detail in Ref. [42].

where we distinguish the flux G_4 notationally from the field strength F_4 as defined in (2.22). Only the combinations F of gauge fields that lie also in the kernel of the mass matrix resulting from (2.24) remain as true gauge symmetries. Unlike the geometric mass terms due to couplings of the form (2.23) the flux-induced mass term turns out to lie below the Kaluza-Klein scale in F-theory. More details will be provided in [37].

III. $U(1)$-RESTRICTED TATE MODELS AND DIMENSION-4 PROTON DECAY

A. Split spectral covers and dimension-4 proton decay

The structure outlined in the previous section on the basis of the global Tate model is captured locally by the spectral cover construction. While eventually we aim at going beyond this local picture, we now recall its basic features. This makes contact with the F-theory GUT model building literature and allows us to analyze potential limitations of this technology.

The spectral cover approach to F-theory model building [1,4,7,26] can be applied in situations with non-Abelian gauge symmetry along just a single divisor S: $w = 0$. Its essence is to focus on the local neighborhood of S within Y by discarding all terms of higher power in the normal coordinate w that appear in the sections $\mathfrak{b}_n$. The restrictions of $\mathfrak{b}_n$ to the GUT divisor,

$$b_n = \mathfrak{b}_n|_{\omega=0}, \tag{3.1}$$

are therefore sections entirely on S. In this local picture, the GUT brane is described as the base of the bundle $K_S \to S$, given by $s = 0$. The neighborhood of S is then modeled by a spectral surface viewed as a divisor of the total space of K_S. In the sequel we will concentrate on the Tate model for an $SU(5)$ GUT symmetry along S with associated spectral surface

$$\mathcal{C}^{(5)}: b_0 s^5 + b_2 s^3 + b_3 s^2 + b_4 s + b_5 = 0. \tag{3.2}$$

One can think of $\mathcal{C}^{(5)}$ as encoding the information about the discriminant locus in the local vicinity of S. In particular, the intersections of $\mathcal{C}^{(5)}$ and S determine the matter curves (2.9) on S. It is also clear from the relation (3.1), though, that all the information in $\mathfrak{b}_n$ contained in terms higher in w is lost in the spectral cover approach.

In agreement with our previous remarks, the gauge group G along S arises by breaking an underlying E_8 gauge symmetry via a Higgs bundle of structure group $H = E_8/G$. For $G = SU(5)_{\rm GUT}$ this means that $H = SU(5)_\perp$. Note that this interpretation of the gauge group G was motivated in [1,26] by the unfolding of an E_8 symmetry that underlies the local gauge theory on S (see also [2,3]) and the spectral cover construction is a geometrization of this local gauge theory. As anticipated in Sec. II B we will argue in Sec. IV A that the interpretation of the gauge group G along S as the compliment of the spectral cover group H in E_8 arises naturally from the geometric Tate model. Our interpretation of the spectral cover makes no reference to a local gauge theory description let alone to a heterotic dual.

If $\mathcal{C}^{(5)}$ splits into two or more divisors, the structure group H factorizes accordingly. As a result, the gauge group $G = E_8/H$ is expected to increase. For example, let us specify to the situation of a factorized divisor [22]

$$\mathcal{C}^{(5)} = \mathcal{C}^{(4)} \times \mathcal{C}^{(1)}. \tag{3.3}$$

This split corresponds to the factorization of (3.2) into

$$(c_0 s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4)(d_0 s + d_1) = 0. \tag{3.4}$$

By comparison of (3.4) with (3.2) one can express the sections b_n as

$$\begin{gathered} b_5 = c_4, \qquad b_4 = c_3 + c_4 d_0, \qquad b_3 = c_2 + c_3 d_0, \\ b_2 = c_1 + c_2 d_0, \qquad b_0 = -c_1 d_0^2, \end{gathered} \tag{3.5}$$

where we have restricted to $c_0 = -c_1 d_0$ such that the term proportional to s^4 vanishes in (3.4). All b_n, c_n are appropriate sections on S such that the Tate sections a_n (2.7) are elements of $H^0(B, K_B^{-n})$, and d_1 is taken as a constant in order to avoid unwanted 10-matter curves; see [22] for details.

Since under this split $H = SU(4) \times U(1)_X$, the gauge group G is expected to enhance to $SU(5) \times U(1)_X$ and the massless $SU(5)_{\rm GUT}$ matter picks up the following $U(1)_X$ charges:

$$\mathbf{10}_1, \qquad \mathbf{10}_{-4}, \qquad (\bar{\mathbf{5}}_m)_{-3}, \qquad (\mathbf{5}_H)_{-2} + (\bar{\mathbf{5}}_H)_2, \tag{3.6}$$

where the exotic $\mathbf{10}_{-4}$ is absent for the choice (3.5). The different $U(1)_X$ charges of the $\mathbf{5}$ representations reflect the split of the $\mathbf{5}$-matter curves on S which is induced by the factorization of the spectral cover.

The charge assignments in the $S[U(4) \times U(1)_X]$ follow group theoretically from the decomposition of $SU(5)_\perp \to S[U(4)_\perp \times U(1)_X]$ by identifying the generator of $U(1)_X$ as the $SU(5)$ Cartan generator $T = \text{diag}(1, 1, 1, 1, -4)$. Correspondingly, the representations of $SU(5)_\perp$ in the decomposition of the $\mathbf{248}$ of E_8 into $SU(5) \times SU(5)_\perp$,

$$\mathbf{248} \mapsto (\mathbf{24}, 1) + (1, \mathbf{24}) + [(\mathbf{10}, \mathbf{5}) + (\bar{\mathbf{5}}, \mathbf{10}) + \text{H.c.}], \tag{3.7}$$

become

$$\begin{gathered} \mathbf{5} \to \mathbf{4}_1 + 1_{-4}, \qquad \mathbf{10} \to \mathbf{6}_2 + \mathbf{4}_{-3} \\ \mathbf{24}_0 \to \mathbf{15}_0 + 1 + \mathbf{4}_5 + \bar{\mathbf{4}}_{-5}. \end{gathered} \tag{3.8}$$

At the level of roots and weights this can be phrased as follows: The nodes ω_i of the extended Dynkin diagram of E_8 are partitioned into the simple roots ω_i^G and ω_i^H with $G = SU(5)_{\rm GUT}$ and $H = SU(5)_\perp$. The fundamental weights λ_i, $i = 1, \ldots, 5$ of $SU(5)_\perp$ have the well-known representation in terms of ω_i given in (2.16). The elements of the Cartan subalgebra of H are identified with the dual elements λ_i^*, introduced in (2.17). In particular, the generator T_X corresponds to the combination

$$\lambda_1^* + \lambda_2^* + \lambda_3^* + \lambda_4^* - 4\lambda_5^*. \tag{3.9}$$

This guarantees the correct $U(1)_X$ charges for the above states. E.g. the representation $(1, \mathbf{4}_5)$ under $SU(5)_{\rm GUT} \times S[U(4) \times U(1)_X]$ that descends from $(1, \mathbf{24})$ corresponds to the weight $\lambda_i - \lambda_5$, $i = 1, \ldots, 5$, and thus has $U(1)_X$ charge $1 - (-4) = 5$. Treating all λ_i independent as above is the local version of the Coulomb branch for $\bar{Y}_{E_8}$ advocated in Sec. II B. The spectral cover analogue of the deformation to $\bar{Y}_{SU(5)}$ corresponds to the introduction of monodromies acting on the roots within H [7,8,22,27]. In the split spectral cover of the type (3.4), the monodromies act not on the full $SU(5)_\perp$ by permutation of all λ_i, $i = 1, \ldots, 5$, but only on an $SU(4)$ subgroup of $SU(5)_\perp$ by identifying λ_i, $i = 1, \ldots, 4$. This implies that the $U(1)_X$ associated with the generator T_X might have a chance to survive the Higgsing/deformation.

If the $U(1)_X$ factor really survives globally it leads to phenomenologically appealing selection rules for the Yukawa couplings of the GUT. In particular split spectral covers are used in the compact models of [18,19,21,23] as a method to avoid dimension-4 proton decay operators because couplings of the type $\mathbf{10}\bar{\mathbf{5}}_m\bar{\mathbf{5}}_m$ are forbidden while the desirable Yukawa coupling $\mathbf{10}\bar{\mathbf{5}}_m\bar{\mathbf{5}}_H$ is allowed by $U(1)_X$.

Another virtue of the spectral cover construction is that it yields a description also of gauge flux [1,4,26]. For a split spectral cover of the type (3.4) one can express a certain class of gauge flux in terms of an $S[U(4) \times U(1)_X]$ bundle W [18] on S. This is a spectral cover bundle for which the spectral sheaf factorizes into $\mathcal{N}^{(4)}$ and $\mathcal{N}^{(1)}$ defined, respectively, on $\mathcal{C}^{(4)}$ and $\mathcal{C}^{(1)}$. Its description involves an element $\zeta \subset H^2(S, \mathbb{Z})$ such that $\pi_4^*\mathcal{N}^{(4)} = \zeta = -\pi_1^*\mathcal{N}^{(1)}$, where π_i is the projection $\mathcal{C}^{(i)} \to S$. Following the logic of heterotic spectral cover constructions with $S[U(N) \times U(1)]$ bundles [45,46] it is natural to assume a D-term potential for $U(1)_X$ of the standard form

$$\sum_i q_i|\phi_i|^2 + \xi = 0, \qquad \xi \propto \int_S J \wedge \zeta. \tag{3.10}$$

Here ϕ_i denotes charged matter under $U(1)_X$ and we have also displayed the Fayet-Iliopoulos D-term for the gauge bundle. Associated with this D-term is a Stückelberg-type mass term for the $U(1)_X$ boson induced by nonzero gauge flux. The fact that the $U(1)_X$ boson acquires a Stückelberg mass is well-known not to affect its relevance as a global symmetry constraining the Yukawas.

However, an important caveat concerning the existence of $U(1)_X$ as a gauge symmetry, albeit a massive one, is that the local split of the spectral cover is by construction insensitive to information away from the GUT brane S. In particular there can be matter states localized away from S which are uncharged under the non-Abelian part of the GUT group G, but charged under $U(1)_X$. In fact in $SU(5)_{\rm GUT}$ models based on split spectral covers of the type (3.4) the role of right-handed neutrinos is played by states

$$N_R^c\colon\ 1_5. \tag{3.11}$$

These are the states commented on after (3.9). They have the correct $U(1)_X$ quantum numbers to participate in the Dirac Yukawa coupling $\bar{\mathbf{5}}_m\mathbf{5}_H N_R^c$. Since these states arise away from the GUT brane S, their precise location is hard to determine in the spectral cover approach; see [18,23] for proposals.

Now, the problem is that the *local* split of the spectral cover does not guarantee that matter of the type (3.11) does not acquire a nonzero vacuum expectation value (VEV) such as to *Higgs* $U(1)_X$. In fact, from a field theory perspective any VEV of such recombination moduli in agreement with the D-term condition (3.10) is compatible with the local split (3.4), which only takes into account the vicinity of S. Turning tables around, the unambiguous appearance of *localized* massless matter away from S can be taken as an indication that $U(1)_X$ is un-Higgsed. To really determine the presence of such massless matter one has to go beyond the spectral cover approximation and consider the full Tate model. The matter in question must then be localized on a curve C on the I_1 part of the discriminant over which the singularity type of the fiber enhances from I_1 to I_2. This corresponds to an enhancement to $A_1 \simeq SU(2)$ and extra matter will appear from the decomposition of the adjoint $\mathbf{3}$ of $SU(2)$ under the branching $SU(2) \to U(1)$.

To appreciate the consequences of a Higgsing of $U(1)_X$ for proton decay let us assume that indeed a recombination modulus Φ localized away from S has acquired a VEV that would not be detected from the *local* split of the spectral cover. Such a field could have the quantum numbers of N_R^c with $U(1)_X$ charge $+5$ or of its conjugate with charge -5. In the first case Φ could participate in a dimension-5 coupling $W \supset \frac{1}{M}\mathbf{10}\bar{\mathbf{5}}_m\bar{\mathbf{5}}_m\mathbf{\Phi}$, where M is a mass scale. Clearly a VEV for Φ induces a dimension-4 proton decay operator

$$\frac{\langle\mathbf{\Phi}\rangle}{M}\mathbf{10}\bar{\mathbf{5}}_m\bar{\mathbf{5}}_m. \tag{3.12}$$

If, on the other hand, it is the field $\tilde{\mathbf{\Phi}}$ with charge -5 that acquires a VEV this way of generating dangerous $U(1)_X$ violating couplings does not occur. Which of the two scenarios arises depends on an interplay of the sign of the Fayet-Iliopoulos term and of further F-terms constraining the VEVs of $\mathbf{\Phi}$ and $\tilde{\mathbf{\Phi}}$. For a discussion of similar effects in the heterotic context, see [47–49].

A related danger for models with Higgsed $U(1)_X$ concerns the precise counting of massless matter. As stressed in this context in [18] (see also [9]) a VEV of the recombination moduli is identical to a deformation of the

$S[U(4) \times U(1)_X]$ bundle W into a proper $SU(5)$ bundle. Loosely speaking this can be thought of as forming a nonsplit extension from a direct sum of bundles, even though in this context W is actually not a direct sum of two independently defined vector bundles. In any case, while in this process the total chirality of the model is unaffected, the chirality of individual matter species might change. Concretely if the recombination modulus $\mathbf{\Phi}$ with charge $+5$ couples to $\bar{\mathbf{5}}_m$ and $\mathbf{5}_H$ as $W \supset \mathbf{\Phi} \mathbf{5}_H \bar{\mathbf{5}}$ a VEV for $\mathbf{\Phi}$ produces also a mass term of the form $\langle \mathbf{\Phi} \rangle \mathbf{5}_H \bar{\mathbf{5}}_m$. Therefore in general the computation of the individual chiralities of models with factorized spectral cover is guaranteed to be valid only if $U(1)_X$ is un-Higgsed.

B. $U(1)$-restricted Tate models and un-Higgsed $U(1)_X$

In this section we will introduce the geometries which guarantee the existence of an un-Higgsed $U(1)_X$. In fact we will see that the existence of the desired $U(1)_X$ symmetry can be ensured provided one extends the factorization of the spectral cover to a global restriction of the full Tate model. In the following we will refer to these geometries as $U(1)$-*restricted Tate models*. Let us promote the split (3.4) to a *global* modification of the sections $\mathfrak{b}_n$ as

$$\mathfrak{b}_5 = \mathfrak{c}_4, \qquad \mathfrak{b}_4 = \mathfrak{c}_3 + \mathfrak{c}_4 d_0, \qquad \mathfrak{b}_3 = \mathfrak{c}_2 + \mathfrak{c}_3 d_0, \tag{3.13}$$

$$\mathfrak{b}_2 = \mathfrak{c}_1 + \mathfrak{c}_2 d_0, \qquad \mathfrak{b}_0 = -\mathfrak{c}_1 d_0^2, \tag{3.14}$$

where $\mathfrak{b}_n$, $\mathfrak{c}_n$ now depend on all coordinates of the base B. With this form of the sections inserted into the Tate form (2.2), the coordinate transformation

$$x \to \tilde{x} + w^2 d_0^2, \qquad y \to \tilde{y} - w^3 d_0^3 \tag{3.15}$$

brings the Tate model polynomial into the form

$$P_{\mathrm{T}} = \tilde{x}^3 - \tilde{y}^2 + \tilde{x}\tilde{y}\tilde{a}_1 + \tilde{x}^2\tilde{a}_2 + \tilde{y}\tilde{a}_3 + \tilde{x}\tilde{a}_4 = 0 \tag{3.16}$$

with

$$\begin{aligned} \tilde{a}_1 &= \tilde{\mathfrak{b}}_5 = \mathfrak{c}_4, \\ \tilde{a}_2 &= \tilde{\mathfrak{b}}_4 w = (\mathfrak{c}_3 + \mathfrak{c}_4 d_0 + 3 w d_0^2) w, \\ \tilde{a}_3 &= \tilde{\mathfrak{b}}_3 w^2 = (\mathfrak{c}_2 + \mathfrak{c}_3 d_0 + \mathfrak{c}_4 d_0^2 + 2 d_0^3 w) w^2, \\ \tilde{a}_4 &= \tilde{\mathfrak{b}}_2 w^3 = (\mathfrak{c}_1 + \mathfrak{c}_2 d_0 + 2\mathfrak{c}_3 d_0^2 + \mathfrak{c}_4 d_0^3 + 3 w d_0^4) w^3. \end{aligned} \tag{3.17}$$

Crucially one notes that $\tilde{a}_6 = \tilde{\mathfrak{b}}_0 = 0$ and that the coefficients $\tilde{\mathfrak{b}}_n$ are generic since the $\mathfrak{c}_n$ are generic. We denote the resulting singular four-fold by $X_{SU(5)}$ or, for more general gauge groups over S, by X_G. Note that on $X_{SU(5)}$ the $\mathbf{5}$ curve given in (2.9) now splits as

$$\tilde{P}_{\mathbf{5}} = \tilde{\mathfrak{b}}_3(\tilde{\mathfrak{b}}_4 - \tilde{\mathfrak{b}}_2\tilde{\mathfrak{b}}_5) = 0, \tag{3.18}$$

which allows for the localization $\mathbf{5}_m$ and $\mathbf{5}_H + \bar{\mathbf{5}}_H$ on different curves. Moreover, the discriminant of $X_{SU(5)}$ is now of the form

$$\begin{aligned} \Delta = w^5(&\tilde{\mathfrak{b}}_2\tilde{\mathfrak{b}}_3\tilde{\mathfrak{b}}_5[\tilde{\mathfrak{b}}_5^4 + 8\tilde{\mathfrak{b}}_4\tilde{\mathfrak{b}}_5^2 w + 16 w^2(\tilde{\mathfrak{b}}_4^2 - 6\tilde{\mathfrak{b}}_2 w)] \\ &+ \tilde{\mathfrak{b}}_3^3\tilde{\mathfrak{b}}_5 w(\tilde{\mathfrak{b}}_5^2 + 36\tilde{\mathfrak{b}}_4 w) - \tilde{\mathfrak{b}}_3^2[\tilde{\mathfrak{b}}_4\tilde{\mathfrak{b}}_5^4 + 8\tilde{\mathfrak{b}}_4^2\tilde{\mathfrak{b}}_5^2 w \\ &+ 2(8\tilde{\mathfrak{b}}_4^3 + 15\tilde{\mathfrak{b}}_2\tilde{\mathfrak{b}}_5^2)w^2 - 72\tilde{\mathfrak{b}}_2\tilde{\mathfrak{b}}_4 w^3] + \tilde{\mathfrak{b}}_2^2 w[\tilde{\mathfrak{b}}_5^4 \\ &+ 8\tilde{\mathfrak{b}}_4\tilde{\mathfrak{b}}_5^2 w + 16 w^2(\tilde{\mathfrak{b}}_4^2 - 4\tilde{\mathfrak{b}}_2 w)] - 27\tilde{\mathfrak{b}}_3^4 w^3). \end{aligned} \tag{3.19}$$

This implies that the elliptic fibration is singular over the curve

$$C\colon \tilde{\mathfrak{b}}_2 = 0, \qquad \tilde{\mathfrak{b}}_3 = 0 \tag{3.20}$$

in the base B. Application of Tate's algorithm confirms an $SU(2)$ enhancement with $\tilde{a}_n$ weights (0, 0, 1, 1, 2) and Δ-weight 2 along C. This singular curve describes the self-intersection locus of the I_1 part of the discriminant appearing in the brackets. Here extra massless degrees of freedom appear in the singular limit according to the decomposition of the adjoint of $SU(2)$. The $U(1)_X$ charge of the massless states along C derive from the specific embedding into E_8 and will be scrutinized further in the next section. We thus interpret the specialization to (3.17) as the un-Higgsing of the $U(1)_X$ gauge symmetry, signalled by the appearance of massless charged matter. These charged massless degrees of freedom play the role of the recombination moduli whose VEV in turn smoothens out the singularity away from the locus (3.17). This intuitive picture will be corroborated further below for a simple F-theory model with orientifold limit. Note that despite the appearance of a singular self-intersection curve C the full I_1 piece does not factorize. However, this is not required for the existence of the Abelian $U(1)_X$ gauge boson.

Note that this analysis generalizes to other gauge groups G localized on the divisor S, as long as $G \subset E_7$. Because of the additional $U(1)$ factor the maximal gauge group E_8 is no longer attainable along a divisor. This does not mean, though, that higher codimension E_8 enhancements along curves or points are forbidden. In fact, points of E_8 enhancement have been argued to lead to a favorable flavor structure in [50].

C. Abelian gauge bosons from resolution and connection to E_7 fibers

So far we have motivated the existence of an un-Higgsed $U(1)_X$ by slightly indirect methods. A direct argument is via the logic described in Sec. II B, where we described how in M-theory on a Calabi-Yau four-fold one can study $U(1)$ gauge factors by moving to the Coulomb branch of the gauge theory.

Applying the general relation (2.14) to the Tate model (3.16) one first encounters four $U(1)$ factors which correspond to the Cartan generators of $G = SU(5)_{\mathrm{GUT}}$ and which arise from blow-up divisors D_i^G with Poincaré dual two-forms ω_i^G. These have been explicitly constructed for GUT models in Refs. [18,19]. Crucially, here we will encounter an additional $U(1)_X$, since Y also contains the singular curve C, given in (3.20), away from the GUT brane S. The singular curve can be canonically resolved into a divisor $\hat{D}_C$ by introducing a new coordinate s and replacing $\tilde{y} \to \tilde{y}s$, $\tilde{x} \to \tilde{x}s$. $\hat{D}_C$ is then given by $s = 0$ and increases the number $h^{1,1}(\bar{Y})$ in (2.14) by one. We denote the Poincaré dual two-form by ω_C. One checks that $\bar{Y}$ is indeed nonsingular after appropriately introducing scaling relations for s. Because of the extra element in $H^{1,1}(\bar{Y}, \mathbb{Z})$ there must exist a harmonic two-form ω_X which is related to ω_C and which encodes the surviving $U(1)_X$ factor. Simple examples show that a candidate for ω_X is

$$\omega_X = \omega_C - \omega_B + \pi^* c_1(B), \tag{3.21}$$

where π is the map from X_G to its base B. Using this two-form the expansion for C_3 reads

$$C_3 = A_X \wedge \omega_X + \sum_i A^i \wedge \omega_i^G + \cdots, \tag{3.22}$$

where A_X is the $U(1)_X$ potential and the A^i correspond to the Cartan $U(1)$s in the non-Abelian gauge group. Note that one again has to define linear combinations of the A^i and the ω_i^G, associated with the simple roots, to obtain the Cartan $U(1)$s. Moving to the singular Y_G, the A^i will become part of the non-Abelian gauge group over S. However, because Y away from the GUT brane did not become singular over a divisor, but over a curve C, there is no non-Abelian gauge enhancement involving the A_X factor, and the $U(1)_X$ remains untouched. This establishes the appearance of the $U(1)_X$ gauge boson.

Let us now resolve the $SU(2)$ singularity along C explicitly. For simplicity we will only concentrate on C and refer for resolution of the $SU(5)$ singularity along the GUT brane S to Refs. [18,19]. The resolved Tate form is given by

$$P_{\mathrm{T}} = \tilde{x}^3 s^2 - \tilde{y}^2 s + \tilde{x}\,\tilde{y}\,\tilde{z}\,a_1 + s\tilde{x}^2\tilde{z}^2 a_2 + \tilde{y}\tilde{z}^3 a_3 + \tilde{x}\tilde{z}^4 a_4 = 0. \tag{3.23}$$

The new Calabi-Yau $\bar{X}_G$ manifold has two fibers: Setting $\tilde{a}_n = 1 = s$ we recover the original $\mathbb{P}_{1,2,3}[6]$ fiber, while for $\tilde{a}_n = 1 = \tilde{x}$ one finds a so-called E_7 fiber. The latter is given by the hypersurface $\mathbb{P}_{1,1,2}[4]$ in the ambient space $(\tilde{y}, \tilde{z}, s) \cong (\lambda y, \lambda z, \lambda^2 s)$. This fiber has two sections. Note that the section $z = 0$ is shared by the new $\mathbb{P}_{1,1,2}[4]$ fiber as well as the original $\mathbb{P}_{1,2,3}[6]$ fiber, and yields the base B. In six-dimensional F-theory compactifications such geometries have been studied in the context of heterotic/F-theory duality in Refs. [29,30]. Note that the resolution (3.23) can be performed for all gauge groups $G \subset E_7$ on S. It turns out that enhancement to a gauge group E_8 is no longer possible on the divisor S. Clearly, this fits with the group theory interpretation presented at the end of Sec. III B, since $E_7 \times U(1)$ is of maximal rank in E_8. In other words the diagram (2.12) gets now replaced by

$$\begin{array}{ccc} X_{E_7\times U(1)} & \xrightarrow{\text{brane recomb.}} & X_{G\times U(1)} \\ \downarrow & & \downarrow \\ \bar{X}_{E_7\times U(1)} & \xrightarrow{\text{geom. transition}} & \bar{X}_{G\times U(1)} \end{array} \tag{3.24}$$

where we have included the $U(1)$ factor to stress the difference with the geometries in (2.12). In Sec. IV C we will show that the split spectral cover group and its splittings can be studied by using mirror symmetry for $\bar{X}_G$ and a reference geometry $\bar{X}_{E_7}$.

Since the underlying gauge group for the $U(1)$-restricted Tate model is not E_8, but rather $E_7 \times U(1)$, it is natural to ask how the Abelian gauge group in the final model embeds into E_8. To demonstrate this let us specialize to $G = SU(5)$. In this case we expect the surviving Abelian group to be given by $U(1)_X$ from the breaking $E_8 \to SU(5) \times SU(4) \times U(1)_X$. That this is indeed the case can be seen as follows: The Abelian gauge group visible in the underlying X_{E_7} is the up-to-normalization unique Cartan $U(1)$ within E_8 responsible for the branching

$$E_8 \to E_7 \times U(1)_a,$$
$$\mathbf{248} \to \mathbf{133}_0 + \mathbf{56}_1 + \bar{\mathbf{56}}_{-1} + 1_2 + 1_{-2}.$$

The deformation to $X_{SU(5)}$ can be understood via the branching $E_7 \to SU(5) \times SU(3)_\perp \times U(1)_b$. The full branching rules to $SU(5) \times SU(3)_\perp \times U(1)_a \times U(1)_b$ are

$$\begin{aligned} \mathbf{133}_0 &\to (1, \mathbf{8})_{0,0} + (\mathbf{24}, 1)_{0,0} + (1, 1)_{0,0} + [(\mathbf{5}, 1)_{0,6} \\ &\quad + (\mathbf{5}, \mathbf{3})_{0,-4} + (\mathbf{10}, \mathbf{3})_{0,2} + \text{c.c.}], \\ \mathbf{56}_1 &\to (1, \mathbf{3})_{1,-5} + (\mathbf{5}, \mathbf{3})_{1,1} + (\mathbf{10}, 1)_{1,-3} + \text{c.c.}, \\ 1_2 &\to 1_{2,0}, \end{aligned} \tag{3.25}$$

with the subscripts denoting the charges (q_a, q_b). The deformation to the $U(1)$-restricted, but otherwise most generic Tate model (3.13) Higgses $SU(3)_\perp$ and one linear combination of $U(1)_a$ and $U(1)_b$. The remaining $U(1)$ symmetry can be determined by noting that the model contains only one matter curve on which an $SU(5)$ singlet localizes. Recall that this is precisely the curve C appearing in (3.20). Indeed the up-to-rescaling unique combination of $U(1)_a$ and $U(1)_b$ compatible with this is

$$U(1)_X = \frac{1}{2}(-5U(1)_a + U(1)_b), \tag{3.26}$$

which results in the anticipated $U(1)_X$ charges displayed in (3.6). Note that the surviving $U(1)_X$ generator embeds indirectly into E_8 via the branchings and Higgsing described above.

The $U(1)$ restriction of the Tate model has another important effect: The Euler characteristic of the resolved fourfold $\bar{X}_G$ decreases considerably compared to the original fibration. This can be traced back to the fact that the $U(1)$ restriction (3.16) and (3.23) forces us to fix many complex structure moduli to restrict the I_1-locus. The change in the

Euler characteristic can be given as a closed expression if one restricts to the case of a smooth Tate model Y with no non-Abelian enhancement. The Euler characteristic of the four-fold can be computed as [51,52]

$$\chi(Y) = 12\int_B c_1(B)c_2(B) + 360\int_B c_1^3(B). \tag{3.27}$$

After the $U(1)$ restriction and resolution the Euler characteristic is reduced as

$$\chi(\bar{X}) = \chi(Y) - 216\int_B c_1^3(B), \tag{3.28}$$

which corresponds to the value obtained for an E_7 fibration. This can be checked explicitly in examples with the help of toric geometry. Using the general formula $\chi = 6(8 + h^{1,1} + h^{3,1} - h^{2,1})$ one determines straightforwardly the number of complex structure moduli which have to be fixed in order to ensure the presence of the addition $U(1)$. For the phenomenologically interesting cases of $SU(5)$ GUT models the effect of the $U(1)$ restriction has to be computed by direct resolution. For instance, consider the 3-generation model in the main text of [19]: The Tate model corresponding to the *nonsplit* $SU(5)_\perp$ spectral cover (3.2) gives rise to $\chi = 5718$. Once we implement the global $U(1)$ restriction (3.13) we find instead $\chi = 2556$ after resolution. On the other hand, if one computes the value of χ based just on the split spectral cover (3.5) as opposed to the globally $U(1)$-restricted Tate model (3.13), one finds $\chi = 5424$ using the formula of [18]. This shows that promoting the split spectral cover to a global $U(1)$-restricted Tate model decreases the value of χ significantly. Note that this affects all previous models [18,19,21,22] based on split spectral covers in the literature and makes a reevaluation of the $D3$-tadpole condition necessary if one wants to promote the split globally to save the $U(1)$ selection rules. This demonstrates once more the entanglement of global properties of the model and the appearance of Abelian symmetry and selection rules.

We conclude this section with some remarks on the description of $U(1)_X$ gauge flux in $U(1)$-restricted Tate models of type (3.16). Recall from Sec. II B that for a generic deformation from $\bar{Y}_{E_8}$ to $\bar{Y}_G$ the Cartan flux associated with $H = E_8/G$ turns into data describing a non-Abelian H-bundle. Such flux is represented by elements of $H^{2,2}(\bar{Y}_G)$ which cannot be written as the wedge of two two-forms. This is in agreement with the absence of a Fayet-Iliopoulos D-term because for generic deformations all $U(1)$ symmetries are Higgsed. For the $U(1)$ restricted Tate model, by contrast, due to the appearance of an un-Higgsed $U(1)_X$ potential we do expect the presence of a Fayet-Iliopoulos term. There must therefore exist a special type of $U(1)_X$ flux which involves a truly Abelian component. This is the global analogue of the extra $U(1)_X$ flux described by the class $\zeta \in H^2(S, \mathbb{Z})$ within the split spectral cover approach, see the discussion around (3.10).

In M-theory the D-terms arise from the Chern-Simons coupling

$$S_{\rm CS} = -\frac{1}{12}\int C_3 \wedge G_4 \wedge G_4. \tag{3.29}$$

In terms of the element ω_X associated with the $U(1)_X$ generator the Fayet-Iliopoulos term is

$$\xi \propto \int \omega_X \wedge J \wedge G_4. \tag{3.30}$$

This is the global version of the expected $U(1)_X$ Fayet-Iliopoulos term in the split spectral cover picture. We leave it for future work to study the concrete description of this type of Abelian gauge flux [37].

D. Connection to brane recombination in orientifolds

We would like to end this discussion by giving yet another, complementary, argument for the appearance of an Abelian gauge factor in $U(1)$ restricted Tate models of the form (3.16). This argument makes contact with the weakly coupled type IIB orientifold picture as follows: For weakly coupled models, the restriction to $a_6 = 0$ describes the split of an orientifold invariant 7-brane into a brane-image brane pair. To see this we need to recall that for a Weierstrass model (2.3) the connection with the IIB picture arises by the well-known Sen limit [53]. One parametrizes without loss of generality

$$f = -3h^2 + \epsilon\eta, \qquad g = -2h^3 + \epsilon h\eta - \epsilon^2\chi/12. \tag{3.31}$$

For the Tate model, which is related to the Weierstrass model via (2.4) and (2.5), one identifies

$$\beta_2 = -12h, \qquad \beta_4 = 2\epsilon\eta, \qquad \beta_6 = -\frac{1}{3}\epsilon^2\chi. \tag{3.32}$$

The orientifold limit consists in taking $\epsilon \to 0$ such that the string coupling becomes weak away from $h = 0$. The leading order discriminant then reads

$$\Delta_\epsilon = -9\epsilon^2 h^2(\eta^2 - h\chi) + \mathcal{O}(\epsilon^3). \tag{3.33}$$

For generic η and χ, corresponding to a smooth Weierstrass model without non-Abelian gauge symmetries, one identifies one $D7$-brane and one $O7$-plane located at

$$O7\colon\ h = 0, \qquad D7\colon\ \eta^2 = h\chi. \tag{3.34}$$

On the Calabi-Yau three-fold double cover Z: $\xi^2 = h$ of the base space $B = Z/\mathbb{Z}_2$ one finds only one set of 7-branes [54,55] wrapping the invariant cycle Q: $\eta^2 - \xi^2\chi = 0$. Correspondingly the Abelian gauge boson is projected out by the orientifold action.

Let us now specialize the complex structure moduli such that the orientifold invariant brane along Q splits into a brane-image brane pair. As discussed in [55] this requires that

$$\chi = \psi^2, \tag{3.35}$$

which induces, again on the double cover Calabi-Yau Z, the split $Q \rightarrow Q_+ \cup Q_-$ with $Q_\pm$: $\eta \mp \xi\psi = 0$. The orientifold action $\xi \rightarrow -\xi$ exchanges Q_+ and Q_-.

In view of the identification (3.32) together with the relation $\beta_6 = a_3^2 + 4a_6$ given in (2.5) the factorization (3.35) means nothing other than $a_6 = 0$. Thus, the Tate model corresponding to the split brane-image brane pair is exactly of the $U(1)$ restricted form (3.16). This orientifold picture makes the appearance of an extra $U(1)$ gauge symmetry clear: Prior to orientifolding the brane-image brane pair $Q_+ \cup Q_-$ carries gauge group $U(1)_+ \times U(1)_-$. The orientifold action identifies the two factors such that a single $U(1)$ boson survives corresponding to $U(1)_+ - U(1)_-$. This is the $U(1)$ boson observed for restricted Tate models of the form (3.16). Note the crucial fact that for the simple model (3.35) the branes Q_+ and Q_- lie in the same homology class on Z. Therefore the geometric mass term discussed around (2.23) does not make the linear combination $U(1)_+ - U(1)_-$ massive in agreement with the appearance of a $U(1)$ boson. In the generic situation, however, the factorization $Q \rightarrow Q_+ \cup Q_-$ is lost because of a deformation of $\chi = \psi^2$ into a nonfactored form. This simply describes the Higgsing of the $U(1)$ symmetry, whereby the extra matter states localized on the curve C of A_1 singularities acquire a VEV. From a more technical perspective, our analysis illustrates the connection between brane-image brane pairs and the appearance of restricted fibers in the Tate model, here fibers of E_7-type,; see the discussion around (3.23).[6]

Note that away from the strict orientifold limit $\epsilon \rightarrow 0$ the terms in the discriminant of higher order in ϵ become important. Taking them into account, the discriminant no longer factorizes in the O-plane and brane-image brane, but becomes a single component. This process is to be interpreted as the nonperturbative recombination of the brane and the O-plane system. However, it does not affect the presence of the Abelian gauge boson. For recent advances in the context of weak-coupling type IIB vs F-theory models, see [20,57–63].

IV. GLOBAL SPECTRAL COVERS AND MIRROR SYMMETRY

A. Spectral cover constructions

In this section we revisit the general philosophy behind the spectral cover approach to F-theory models. In particular we will argue for the appearance of a spectral cover directly from the form of the (globally defined) Tate model.

As reviewed, the general idea of the spectral cover is to describe the gauge group G along a divisor S by unfolding an underlying E_8 symmetry. This picture arose in the description of local ALE fibrations over S [1,4,7,26]. Formally the same structure appears as in F-theory examples with a perturbative heterotic dual description [64]. In these cases the four-dimensional F-theory gauge group can be understood as the commutant of the structure group of a vector bundle embedded into the perturbative heterotic $E_8 \times E_8$. The heterotic vector bundle can be directly determined from the constraint of the F-theory manifold [65,66]. In the sequel we find more evidence for the relevance of an E_8 bundle V in the description of the four-dimensional gauge dynamics along S in a genuine F-theory compactification of the type described in Sec. II. Our considerations rely entirely on the structure of the Tate model without any reference to the local gauge dynamics on S or a heterotic dual. We therefore believe that this view sheds new light on the relevance of the spectral cover that helps understand also its role in compact models.

We consider F-theory compactifications on elliptically fibered Calabi-Yau four-folds as introduced in Sec. II A and with a gauge enhancement G over a *single* divisor S given by the constraint $w = 0$. In a Tate model with E_8 elliptic fiber the generic fiber is given by $\mathbb{P}_{1,2,3}[6]$. As we will see in this case it is natural to consider gauge groups G *contained in* E_8. Given a $\mathbb{P}_{1,2,3}[6]$ fibration there is a natural split of the Tate constraint (2.2) as

$$P_{\mathrm{T}} = P_0 + P_V = 0. \tag{4.1}$$

It turns out that P_V specifies a gauge bundle V with structure group H which breaks E_8 to its commutant $G = E_8/H$. The simplest case is an E_8 singularity over S corresponding to the Tate form

$$\begin{aligned} P_0 &= x^3 - y^2 + xyzwh_6 + x^2z^2w^2h_4 + yz^3w^3h_3 \\ &\quad + xz^4w^4h_2 + z^6w^6h_0, \\ P_V &= z^6w^5b_0. \end{aligned} \tag{4.2}$$

Note that b_0 can be chosen to be independent of w by absorbing all higher order dependence on w into h_0. Such a singular Y can be constructed by studying the *resolved* four-fold $\tilde{Y}_{E_8}$ in which a set of resolution $\mathbb{P}^1$s is fibered over S.

Exactly in the case of an E_8 gauge group the leading powers of w and z match in P_0. One thus can introduce local coordinates $\tilde{w} = wz$ and $v = z^5w^4$ and write (4.2) as

$$\begin{aligned} P_0 &= x^3 - y^2 + xy\tilde{w}h_6 + x^2\tilde{w}^2h_4 + yh_3 + x\tilde{w}^3h_2 + \tilde{w}^6h_0, \\ P_V &= v\tilde{w}b_0. \end{aligned} \tag{4.3}$$

The point is that P_V is the defining equation for an $SU(1)$ bundle in the sense of the spectral cover as introduced in [64]. Note that the coordinate redefinition will generally induce inverse powers of v in P_0, and hence is only valid in the patch of nonvanishing v. Before turning to a more global analysis, let us first focus on P_V. To reduce the gauge group from E_8 to a subgroup G one can

[6]This connection was observed by the authors of the present article during completion of [18]. For an independent analysis see [56].

TABLE I. Spectral covers P_V and their generalizations.

G	E_8/G	P_V
E_8	$SU(1)$	$v\tilde{w}b_0$
E_7	$SU(2)$	$v(\tilde{w}^2b_0 + xb_2)$
E_6	$SU(3)$	$v(\tilde{w}^2b_0 + x\tilde{w}b_2 + yb_3)$
$SO(10)$	$SU(4)$	$v(\tilde{w}^4b_0 + x\tilde{w}^2b_2 + y\tilde{w}b_3 + x^2b_4)$
$SU(5)$	$SU(5)$	$v(\tilde{w}^5b_0 + x\tilde{w}^3b_2 + y\tilde{w}^2b_3 + x^2\tilde{w}b_4 + xyb_6)$
$SU(4)$	$SO(10)$	$v(\tilde{w}^2yb_3 + \tilde{w}^5b_{0,2} + yxb_6 + \tilde{w}x^2b_4) + v^2(\tilde{w}^4b_{0,1} + \tilde{w}^2xb_2)$

systematically add new terms to P_V which lower the vanishing orders κ_n in (2.7). This implies the following interpretation of the Tate model: The groups G on S are obtained as the *deformation* of the original E_8 singularity in (4.2) by allowing for new monomials with lower powers in w. This introduces new complex structure deformations of the Calabi-Yau four-fold so as to change P_V while keeping P_0 unaltered. In absence of a topological obstruction enforcing a minimal gauge group, this process can be performed until all non-Abelian gauge symmetry has been Higgsed. For example, using the Tate formalism as in [31] one finds the P_V listed in Table I.

In Table I we have performed a coordinate redefinition to $\tilde{w} = zw$ and $v = z^{6-N}w^{5-N}$, $N = 1, \ldots, 5$ to bring P_V into the form of a spectral cover for $SU(N)$ bundles [64]. For $SO(10)$ bundles one redefines $v = z$, $\tilde{w} = zw$ and thus captures the terms $z^6(b_{0,1}w^4 + b_{0,2}w^5)$ in the Tate form (4.2) for an $SU(4)$ singularity. As the polynomial for the $SO(10)$ bundle has a term v^2, such a bundle cannot be constructed directly by a spectral cover [64]. However, the data of this bundle are encoded by the generalized spectral cover with higher powers of v. A more complete list including various other bundle groups can be found in Ref. [66].

Let us analyze the $SU(N)$ spectral covers in more detail by specifying the transformation of the coordinates $(v, \tilde{w}, x, y)$ and the b_n as sections of appropriate line bundles. Recall that (z, x, y) appearing in (2.2) are sections $z \in H^0(\mathcal{L})$, $x \in H^0(\mathcal{L}^2 \otimes K_B^{-2})$, $y \in H^0(\mathcal{L}^3 \otimes K_B^{-3})$, where $\mathcal{L}$ is the line bundle for the scaling of $\mathbb{P}_{1,2,3}$, the ambient space of the elliptic fiber. By definition w is a section of $N_{S/B}$, the normal bundle to the divisor S of B over which we engineer non-Abelian gauge enhancement. With the above definition $\tilde{w} = zw$ and $v = z^{6-N}w^{5-N}$ one arrives at $\tilde{w} \in H^0(\mathcal{L} \otimes N_{S/B})$, $v \in H^0(\mathcal{L}^{6-N} \otimes N_{S/B}^{5-N})$. Homogeneity of the polynomial $P_{\rm T}$ therefore uniquely determines the coefficients b_n as sections

$$b_n \in H^0(S, \eta - nc_1(S)), \qquad \eta = 6c_1(S) + c_1(N_{S/B}). \quad (4.4)$$

This uses the adjunction formula $K_B|_S = K_S \otimes N_{S/B}^{-1}$ as well as the fact that b_n are truly sections of S since all further dependence on w has been shifted to h_n. Recall that the construction of an $SU(N)$ bundle over S via spectral covers involves a spectral surface of class $N\sigma + \eta$, with σ the section corresponding to S. In the presented construction one recovers the spectral cover with η from the geometry of Y via P_V.

It is crucial to keep in mind that the split (4.1) was only possible because we assumed that the non-Abelian gauge symmetry G appears over the single divisor S and that $G \subset E_8$. Despite this restriction, the geometry can be general and no reference to the existence of a heterotic dual has to be made. The nontrivial global information about the Calabi-Yau four-fold is captured by the sections h_n in P_0, which are given already in the Calabi-Yau four-fold Y_{E_8} with E_8 singularity (4.2) with trivial P_V. One consequence of this construction appears to be the existence of a simple formula for the Euler characteristic of the resolved four-fold $\bar{Y}_G$ as [18]

$$\chi(\bar{Y}_G) = \chi(\bar{Y}_{E_8}) + \chi_V. \quad (4.5)$$

Here χ_V is determined in a trivial fashion from the second Chern class of V and can be computed for various bundles V as a function of η and the Chern classes of S using [64]. For example, for a vector bundle V with structure group $SU(N)$ one has

$$\chi_V^{SU(N)} = \int_S c_1(S)^2(N^3 - N) + 3N\eta(\eta - Nc_1(S)). \quad (4.6)$$

Note that there is no reason for (4.5) to be generally valid. Rather one should compare the Euler characteristic computed directly for an explicitly constructed and resolved Calabi-Yau four-fold $\bar{Y}_G$ with the value (4.5). A match indicates the global applicability of the spectral cover formalism. Such matches have been found explicitly for many examples [18,19].

B. Mirror symmetry and spectral covers

In the study of F-theory compactifications with non-Abelian gauge symmetry one can use two techniques to analyze the gauge sector. The first method is to study singularity enhancements over the divisor S. At each codimension the singularity can enhance further as

$$G \subset G_C \subset G_P, \quad (4.7)$$

where C is an intersection curve of S with the I_1 locus and P is a point of intersection of C with other I_1 curves in S.

At the enhancement loci new matter fields and couplings can localize. Much of the information about the singularity is encoded in the canonical resolution $\bar{Y}_G$ by gluing in resolving $\mathbb{P}^1$s into the singular fibers over S, curves and points. In particular, the group enhancements are captured by the fact that the $\mathbb{P}^1$s intersect as the Dynkin diagrams of G, G_C, G_P at the various locations in S (see e.g. [67]). The second method to describe non-Abelian enhancements is the generalization of the constructions of [64] as described in Sec. IVA. Here the situation is somewhat inverse to (4.7) since the bundles breaking the E_8 become more trivial over curves and points. Thus the structure group reduces as

$$H \supset H_C \supset H_P, \tag{4.8}$$

where $H = E_8/G$, $H_C = E_8/G_C$ and $H_P = E_8/G_P$ are the respective commutants. We have already stressed that this construction is much less general. In particular, only the analysis of F-theory compactifications with single groups G in E_8 has been carried out. It would be interesting to explore generalizations of this construction.

In order to get deeper insights into the global applicability of the spectral cover construction and its extensions described in Sec. IVA one can attempt to make H, H_C, H_P visible as a physical gauge group in a dual theory. In Ref. [66] it was suggested to use mirror symmetry for Calabi-Yau four-folds to study F-theory models with heterotic dual (see also [68]). In the following we will show how mirror symmetry can be applied to the geometries studied in this paper, which, however, do not admit a heterotic dual.

Let us start by considering the mirror four-fold $\bar{Y}^*_G$ to the resolved space $\bar{Y}_G$. Our aim is to determine the gauge group obtained by compactifying F-theory on $\bar{Y}^*_G$. Hence we have to impose that $\bar{Y}^*_G$ itself is elliptically fibered. In fact, this is the case for the explicit elliptically fibered Calabi-Yau four-folds used for GUT model-building given by two constraints [18,19], as well as the elliptically fibered Calabi-Yau hypersurfaces studied in [52]. This can be traced back to the fact that these Calabi-Yau spaces are realized in a toric ambient space. Thus their mirror [69–72] and fibration structure [73] can be analyzed in detail using toric techniques (see Refs. [19,38], which include a review of these techniques and references). We first analyze the mirror of the space $\bar{Y}_{E_8}$ with a resolved E_8 singularity over S. The gauge group $\mathcal{H}(\bar{Y}^*_{E_8})$ associated with the resolved elliptic fibration of $\bar{Y}^*_{E_8}$ can be explicitly determined for the examples considered in this work, and will be of rather high rank. We can proceed in the same way for $\bar{Y}_G$, i.e. the space in which the gauge group E_8 on S is unfolded to G. One then shows that the new mirror gauge group is

$$\mathcal{H}(\bar{Y}^*_G) = \mathcal{H}(\bar{Y}^*_{E_8}) \times \mathcal{H}. \tag{4.9}$$

Here the new factor $\mathcal{H}$ is composed out of the structure groups (4.8) of the bundle V appearing over S as well as the enhancement groups over curves and points,

$$\mathcal{H} = H^{k_S} \times H_C^{k_C} \times \cdots \times H_P^{k_P} \times \cdots. \tag{4.10}$$

The dots indicate that one has to consider the bundle groups over all possible enhancement curves and points in S. Note that this picture makes direct contact with the spectral cover description and its extensions of Sec. IVA. The precise form of $\mathcal{H}$ determines P_V and vice versa. The largest group H determines G and hence the form of P_V to be picked out of Table I. The exponents in (4.10) are best explained by considering a specific example for G, H, as we will do next.

Given the P_V in Table I encoding the bundles on the divisor S one can count the number of monomials in each of the defining b_n. Let us explain this for the example of $G = SU(5)$. Clearly, if $b_n = 0$ for all $n > 0$ one obtains an E_8 gauge group or $SU(1)$ bundle. Let k_n denote the number of possible nonzero monomials in b_n. Starting with an $SU(1)$ bundle, there are k_2 deformations to an $SU(2)$ bundle, k_3 deformations to an $SU(3)$ bundle and k_4 deformations to an $SU(4)$ bundle. Finally, one has k_6 possible $SU(5)$ bundles corresponding to the different monomials in b_6. One can then show that in the mirror four-fold $\bar{Y}^*_{SU(5)}$ one finds as gauge factor in (4.9) the group

$$\mathcal{H} = SU(5)^{k_6} \times SU(4)^{k_4} \times SU(3)^{k_3} \times SU(2)^{k_2} \times SU(1)^{k_0}. \tag{4.11}$$

This gauge group can be determined by the Tate algorithm implemented via toric methods [31,33,34]. Very basically, the weighted projective space $\mathbb{P}_{1,2,3}$ is encoded torically by the vertices

$$\nu_1 = (0, -1), \qquad \nu_2 = (-1, 0), \qquad \nu_3 = (3, 2), \tag{4.12}$$

which correspond to the x, y, z coordinates in the Tate equation (2.2). In the above construction the mirror manifold $\bar{Y}^*$ admits the dual two-torus as the generic elliptic fiber. It is given by the vertices

$$\nu_1^* = (1, -2), \qquad \nu_2^* = (-1, 1), \qquad \nu_3^* = (1, 1), \tag{4.13}$$

which correspond to the x, y, z coordinates in the *mirror* of the Tate model. It is this dual Tate model in which one reads off the gauge group $\mathcal{H}$.

Let us discuss this result in more general terms. First, the exchange of G and $\mathcal{H}$ under mirror symmetry arises as a simple combinatorial fact intrinsic to elliptic fibrations with generic fiber $\mathbb{P}_{1,2,3}[6]$ of E_8 type. In other words, it is possible to show that the identification (4.9) is rooted in the application of mirror symmetry for reflexive polyhedra and does not rely on the duality to a heterotic model. Hence, mirror symmetry will likely turn out to be a powerful tool to argue for the global validity of the spectral cover construction for Calabi-Yau examples in which all non-Abelian gauge dynamics localizes on S. In particular, the split (4.5) of the Euler characteristic appears to be in accord with the factorization of the dual gauge group (4.9).

C. Split spectal covers and mirror symmetry for $U(1)$-restricted Tate models

In Sec. II we discussed an interesting specialization of the Tate model by demanding that globally $a_6 = 0$. This led to a Calabi-Yau four-fold X which admits an additional singularity over a curve C. We have argued that then an Abelian factor $U(1)_X$ remains un-Higgsed and can forbid dangerous dimension-4 proton decay operators. We now seek to apply mirror symmetry to the resolved manifold $\bar{X}$ and generalize the discussion of Sec. IV B. In particular, we want to determine the dual gauge group $\mathcal{H}$ for a singular four-fold $X_{SU(5)}$ with $SU(5)$ singularity over S and $SU(2)$ singularity over C.

To begin with, we note that the restriction $a_6 = 0$ can be implemented at the level of the Tate equation (2.2) by introducing a new coordinate s with appropriate scaling relations to forbid a term $z^6 a_6$. This has been done in (3.23), where we also noted that the divisor $s = 0$ corresponds to the blow-up divisor $\hat{D}_C$ of the singular curve C. In contrast to a generic $\mathbb{P}_{1,2,3}[6]$ elliptic fiber, one now has an elliptic fiber encoded by the two-torus vertices

$$\{\nu_i\} = \{(0,-1), (-1,0), (3,2), (-1,-1)\}. \tag{4.14}$$

The new vertex $\nu_4 = (-1,-1)$ corresponds to the coordinate s and restricts the Tate form to be (3.23). Note that this four-fold X still admits the section $z = 0$ corresponding to the base B.

The analysis of the mirror $\bar{X}^*$ of the Calabi-Yau four-fold $\bar{X}$ proceeds as before. In particular, one can perform the restriction $a_6 = 0$ for the GUT examples of Refs. [18,19] and show that the mirror is again elliptically fibered. The generic two-torus fiber of $\bar{X}^*$ is the dual to the fiber of $\bar{X}$ and can hence be inferred from (4.14) to be

$$\{\nu_i^*\} = \{(1,-2), (-1,1), (1,0), (0,1)\}. \tag{4.15}$$

By comparison with the dual of the $\mathbb{P}_{1,2,3}[6]$ elliptic fiber (4.13) one notes that the vertex (1, 1) corresponding to the mirror z-coordinate has split into two vertices (1, 0), (0, 1). We denote the corresponding coordinates by z_1 and z_2. This implies that the corresponding Tate form is modified as

$$P_{\rm T}^* = P_1 + P_{U(1)} \tag{4.16}$$

$$P_1 = y^2 z_2 + x^3 z_1 + a_{1,1}^* xyz_1 z_2 + a_{2,1}^* x^2 z_1^2 z_2 + a_{2,2}^* yz_1^2 z_2^2 + a_{3,2}^* xz_1^3 z_2^2 + a_{4,3}^* z_1^4 z_2^3$$
$$P_{U(1)} = a_{0,0} x^2 y. \tag{4.17}$$

This form of $P_{\rm T}^*$ is inferred by using the scaling relations of the coordinates (x, y, z_1, z_2), which are the linear relations among the points (4.15). Note that in contrast to the standard Tate model (2.2) based on $\mathbb{P}_{1,2,3}[6]$ one finds a term proportional to $x^2 y$. This is precisely the deformation mirror dual to the blow-up $\hat{D}_C$. Hence, the extra term $P_{U(1)}$ signals the existence of the extra $U(1)$. In the restriction $a_6 = 0$ with no further modification $a_{0,0}$ is a single monomial and corresponds to a single $U(1)_X$ gauge boson.

Since the elliptically fibered four-fold $\bar{X}^*$ has the form (4.16) we have to reinvestigate the dual gauge group $\mathcal{H}$. Using the scaling relations to set $z_1 = 1 = z_2$, one finds that P_1 has the standard Tate form (2.2). Hence, we can study the gauge group by analyzing the vanishing of the polynomial over different divisors using the Tate algorithm. One shows that the dual gauge group naturally splits as

$$\mathcal{H}(\bar{X}_G^*) = \mathcal{H}(\bar{X}_{E_7}^*) \times \mathcal{H}_X. \tag{4.18}$$

This is the analogue of (4.9) for the Tate model without the additional $U(1)$ factor. Note that now the maximal gauge group attained on X_G is E_7 as stressed in Secs. III B and III C. This matches nicely with the split (4.18) and can be checked for numerous examples. In particular, for $\bar{X}_{SU(5)}^*$ one finds

$$\mathcal{H}_X = SU(4)^{k_6} \times SU(3)^{k_4} \times SU(2)^{k_3} \times SU(1)^{k_2}. \tag{4.19}$$

Here k_n are the number of monomials in the b_n of the split spectral cover, which is equal to the k_n in (4.11). Note that $k_0 = 0$ in agreement with the fact that one has set $a_6 = b_0 = 0$. In contrast to the dual gauge group in (4.11) one thus finds that each factor has been broken by a $U(1)$. This is the global analogue of the split spectral cover construction $S[U(4) \times U(1)]$.

V. CONCLUSIONS

In this article we have analyzed aspects of Abelian gauge symmetries in global F-theory models. We have proposed a mechanism to guarantee un-Higgsed $U(1)$ factors by a special restriction of the form of the Tate model. The presence of the $U(1)$ factor has been explained from different perspectives: The $U(1)$-restricted Tate model gives rise to localized massless states charged under the Abelian group which signal the un-Higgsing of the Abelian symmetry. The Abelian gauge boson can also be detected directly due an increase in $h^{1,1}$ of the Calabi-Yau four-fold after resolution. Finally in models with a IIB description the $U(1)$ can be traced back to the presence of a brane-image brane pair. We have been able to match one sector of the Stückelberg mechanism in type IIB orientifolds with the appearance of the extended node in the affine Dynkin diagram in the fibers, a picture which we will describe in greater detail in an upcoming paper [37].

As a phenomenologically relevant application this mechanism allows one to implement a global $U(1)_X$ symmetry in F-theory GUT models that forbids dimension-4 proton decay. Crucially, our analysis goes beyond the split spectral cover approach, which is insensitive to the global question of $U(1)$ symmetries. The price one has to pay for the presence of $U(1)_X$ is a decrease of the Euler characteristic of the four-fold, which directly enters the $D3$-tadpole cancellation condition.

An important open question concerns the precise definition of the gauge flux. Along the Coulomb phase of the underlying E_8 and, respectively, $E_7 \times U(1)$ Tate model one can easily study the Cartan fluxes, which are then transformed into non-Abelian flux data after deformation to the actual Tate model of interest. We have argued that for $U(1)$ restricted Tate models a D-term naturally appears due to extra available flux data that involve a four-form written as the wedge product of two two-forms. The most pressing remaining question in this context is to identify the four-form describing the $U(1)$ flux concretely in terms of the resolved geometry in order to reliably compute its $D3$-tadpole charge and D-term. Progress is underway [37].

In the last part of the paper we have worked out the connection between the spectral cover construction and the global Tate model of an elliptic four-fold. We have argued that the E_8 structure underlying a generic Tate model is responsible for the relevance of spectral covers, independently of a local gauge theory description or a heterotic dual. This picture has been corroborated by an analysis of the mirror dual four-folds, both for generic and for $U(1)$ restricted Tate models. Let us stress that mirror symmetry is used here as a calculational tool and appears not to correspond to a physical duality. Clearly, this is in sharp contrast to heterotic/F-theory duality for which one expects a map of all physical quantities. It will be of considerable interest to focus on examples which admit no heterotic dual and highlight their global properties.

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