THE FOLIATED LEFSCHETZ HYPERPLANE THEOREM

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ABSTRACT. A foliation $(M, \mathcal{F})$ is said to be 2–calibrated if it admits a closed 2-form ω making each leaf symplectic. By using approximately holomorphic techniques, a sequence W_k of 2–calibrated submanifolds of codimension–2 can be found for $(M, \mathcal{F}, \omega)$. Our main result says that the Lefschetz hyperplane theorem holds for the pairs $(F, F \cap W_k)$, with F any leaf of $\mathcal{F}$. This is applied to draw important consequences on the transverse geometry of such foliations.

1. Introduction and statement of the main result

A foliation $\mathcal{F}$ by surfaces on a 3–dimensional closed manifold M is called taut if for every leaf there exists a loop through it that is everywhere transverse to $\mathcal{F}$. This topological definition is equivalent to the following differential geometric one: there exists a closed 2-form inducing an area form on each leaf [11]. Tautness implies strong topological restrictions on the pair $(M, \mathcal{F})$: by the work of Novikov we know that the fundamental group of any leaf injects into the fundamental group of the ambient manifold, and that every loop $C \pitchfork \mathcal{F}$ must be non–trivial in homotopy.

The straightforward generalization of tautness to arbitrary dimension requires the existence, through any leaf, of a loop everywhere transverse to the foliation. However, in deep contrast to the 3–dimensional case, these objects are quite flexible, as shown by the h–principle proved by Meigniez [8]. In [6], the first author proposed the following alternative generalization of taut foliations to higher dimensions:

Definition 1. *A codimension–1 foliation $\mathcal{F}^{2n}$ of M^{2n+1} is said to admit a 2-calibration if there exists a closed 2-form ω such that the restriction of ω^n to the leaves of $\mathcal{F}$ is nowhere vanishing. A triple $(M, \mathcal{F}, \omega)$, where ω is a 2-calibration for $\mathcal{F}$, is referred to as a 2-calibrated foliation.*

Definition 2. *A submanifold $W \hookrightarrow (M, \mathcal{F}, \omega)$ is a* 2–calibrated submanifold *if it is everywhere transverse to $\mathcal{F}$ and it intersects each leaf of $\mathcal{F}$ in a symplectic submanifold w.r.t. ω.*

As in the symplectic and contact settings, Donaldson's approximately holomorphic techniques [2] can be applied to study 2-calibrated foliations. In particular, they can be used for the construction of *2–calibrated divisors*:

Proposition 1. [6, Corollary 1.2] *Let $(M^{2n+1}, \mathcal{F}; \omega)$ be a 2–calibrated foliation on a closed manifold with ω of integral class. Then, for any integer k large enough, there are 2–calibrated submanifolds W_k^{2n-1} representing the Poincaré dual of $[k\omega]$.*

Additionally, the maps

$$i_* : \pi_j(W_k) \to \pi_j(M)$$
$$i_* : H_j(W_k, \mathbb{Z}) \to H_j(M, \mathbb{Z})$$

are isomorphisms for $j < n - 1$ and surjections for $j = n - 1$.

The submanifolds W_k in the proposition will be called *Donaldson divisors*. The second part of the statement is the *2–calibrated Lefschetz hyperplane theorem*: much like in the projective and the symplectic cases, the divisors recover some of the topology of the ambient. The purpose of this note is to prove the following analogous result:

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Theorem 1. *Let $(M^{2n+1}, \mathcal{F}, \omega)$ be a 2–calibrated foliation on a closed manifold. Let W be a Donaldson divisor of dimension $2n-1$. Then, for every leaf F of $\mathcal{F}$ it holds that*[1]

$$\pi_k(F, F \cap W) = \{1\},\ 0 \leq k \leq n-1.$$

This result says that, despite the fact that a leaf F of $\mathcal{F}$ might be non–compact, the symplectic Lefschetz hyperplane theorem holds for the pair $(F, F \cap W_k)$. It should be remarked that the case of π_0 was already proved in [6, 7], where the question was tackled constructing 2–calibrated Lefschetz pencils. In fact, it is clear that the relative π_1 can be computed out of [7]. The method of proof in this note is new, not an adaptation of the Lefschetz pencil techniques, and yields a much shorter and simpler proof for the π_0 and the π_1 case as well.

The following theorem states an important consequence of Theorem 1.

Theorem 2. *Let $(M^{2n+1}, \mathcal{F}, \omega)$ be a 2–calibrated foliation on a closed manifold. Then there exists a closed 3-dimensional 2–calibrated submanifold $(W, \mathcal{F}_W = W \cap \mathcal{F}, \omega|_W) \hookrightarrow M$ satisfying the following equivalent properties:*

(i) *the map between holonomy groupoids induced by the inclusion*

$$\iota\colon \mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F})$$

is an essential equivalence;

(ii) *any total transversal T for $(W, \mathcal{F}_W)$ is also a total transversal for $(M, \mathcal{F})$, and the holonomy pseudogroups $\mathcal{H}(\mathcal{F}, T)$ and $\mathcal{H}(\mathcal{F}_W, T)$, induced on T by $\mathcal{F}$ and $\mathcal{F}_W$, respectively, coincide.*

Proof. This follows by first observing that, if W is a submanifold transverse to $\mathcal{F}$, then, for either condition (i) or (ii) to hold (see [10] for background material on essential equivalences and holonomy groupoids), it suffices that for each leaf $F \in \mathcal{F}$

$$\pi_0(F, F \cap W) = \pi_1(F, F \cap W) = \{1\}.$$

Theorem 1 can be applied as long as $n > 1$. Doing so iteratively yields a descending chain of Donaldson divisors $M^{2n+1} = W_0 \supset W_1^{2n-1} \supset \cdots \supset W_{n-1}^3$ satisfying

$$\pi_0(F \cap W_k^{2(n-k)+1}, F \cap W_{k+1}^{2(n-k)-1}) = \{1\},$$

and

$$\pi_1(F \cap W_k^{2(n-k)+1}, F \cap W_{k+1}^{2(n-k)-1}) = \{1\}.$$

This proves the claim. □

Note that a 3-dimensional 2-calibrated submanifold as in Theorem 2 is a classical 3-dimensional taut foliation. The map $\iota\colon \mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F})$ being an essential equivalence implies not just that the map induced on leaf spaces $W/\mathcal{F}_W \to M/\mathcal{F}$ is a homeomorphism (c.f. [7]), but that both foliations have the same *transverse geometry*.

To spell this out more precisely, this implies in particular that:

- The homeomorphism on leaf spaces preserves the growth type of the leaves [4].
- There is a bijection between the *transverse geometric structures* on $(W, \mathcal{F}_W)$ and those on $(M, \mathcal{F})$. These are, for instance: holonomy invariant transverse (Radon) measures, Riemannian metrics – in general, structures defined by (invariant) sheaves over the Haefliger groupoid Γ^1_∞ –, and real analytic structures (i.e. reductions to Γ^1_ω).
- There is an isomorphism between the periodic, Hochschild and periodic cyclic homologies of the convolution algebra of the holonomy groupoids [1].

[1] For $A \subset B$, we have $\pi_0(B, A) = \pi_0(B)/\pi_0(A)$. From the definition, this extends the long exact sequence for the pair to the π_0-level, see [5] pag. 476.

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2. Ingredients of the proof

Our proof of Theorem 1 follows Donaldson's proof of the Lefschetz hyperplane theorem for approximately holomorphic divisors. His proof followed the Andreotti-Frankel proof in the affine/projective case: in the complement of a divisor, the modulus of its defining approximately holomorphic section can be regarded, after a small perturbation, as a Morse function with critical points of index at least n, which implies that the ambient manifold is obtained from the divisor by attaching handles of index at least n. It readily follows that the relative homology and homotopy groups of degree less than n vanish.

In the foliated case however, the critical points of this function come in $\mathbb{S}^1$ families and a non–compact leaf will, in general, have infinitely many critical points. Hence, the relative homotopy type of a leaf with respect to the divisor is not readily understood.

We shall first review the essentials of the approximately holomorphic machinery that we will need for the proof of Theorem 1. Then we will discuss some conditions for Morse functions in open manifolds that will guarantee a nice behavior for their gradient flow.

2.1. **The approximately holomorphic theory for 2-calibrated foliations.** Let M^{2n+1} be a closed manifold endowed with a 2-calibrated foliation $(\mathcal{F}, \omega)$. After a small perturbation, we may assume without loss of generality that $[\omega]$ is a rational class; by scaling the class, we may also assume that it is integral. We let $\mathcal{L} \to M$ be the pre–quantum line bundle associated to ω; this is a Hermitian line bundle with a compatible connection ∇ whose curvature is $-2\pi i\omega$.

We let $\nabla^{\mathcal{F}}$ denote the component of ∇ tangential to $\mathcal{F}$. After choosing an almost complex structure J compatible with ω, the tangential connection can be further decomposed into its complex linear and antilinear parts, yielding $\nabla^{\mathcal{F}} = \partial + \bar{\partial}$.

According to [6], Corollary 1.2, upon choosing the almost complex structure J, it is possible to construct a family $s_k : M \to \mathcal{L}^k$ of sections of the k-th tensor powers of $\mathcal{L}$, for k large enough, such that $W_k := s_k^{-1}(0)$ are closed, 2-calibrated submanifolds of codimension two.

To state the conditions that are required for the sequence s_k, we fix a metric g on M which over the leaves satisfies $g = \omega(\cdot, J\cdot)$. Further, we define a family of scaled metrics $g_k = kg$.

Definition 3.

(1) *A sequence of sections $s_k : M \to \mathcal{L}^k$ is said to be approximately holomorphic if there is a universal constant $C > 0$ such that:*
$$|s_k|_{g_k}, |\nabla s_k|_{g_k} < C; \qquad |\bar{\partial} s_k|_{g_k}, |\nabla\bar{\partial} s_k|_{g_k} < Ck^{-1/2},$$
for k large enough.

(2) *A sequence of sections $s_k : M \to \mathcal{L}^k$ is said to be ν-transverse to zero along the foliation $\mathcal{F}$ if at any point either $|s_k|_{g_k} \geq \nu$ or $|\nabla^{\mathcal{F}} s_k|_{g_k} \geq \nu$.*

To every such an approximately–holomorphic transverse to zero sequence s_k one associates a sequence of functions $f_k : M \setminus W_k \to \mathbb{R}$ by $f_k = \log |s_k|^2$. The Lefschetz hyperplane theorem for Donaldson-type submanifolds ([2, 6]) states:

Proposition 2. *Fixing a leaf F, the function $f_k : F \setminus (W_k \cap F) \to \mathbb{R}$, which might not be Morse, has only critical points of index at least n.*

This proposition, when applied to a closed leaf, implies Theorem 1 immediately –as seen in [2, 7]– for Donaldson divisors.

2.2. **Gradient flows and the topology of open manifolds.** The study of flows which behave well on open manifolds already appears in the literature on foliation theory [3]. For the sake of completeness, we review these facts tailored to the applications we have in mind.

Let f be a Morse function on a manifold M. For any $a \in \mathbb{R}$ set $M_a = \{x \in M \mid f(x) \leq a\}$, and denote by $\mathrm{Crit}_a(f)$ the subset of critical points of f lying in $M \backslash M_a$.

Let a be a regular value for f and let $b > a$. Assume for the moment that M is compact. It is customary to study the relative topology of the pair (M_b, M_a) using minus the gradient flow of f with respect to some fixed metric g. The key point is that the following dichotomy holds: for any $x \in M_b \backslash M_a$ the trajectory of $-\nabla_g f$ starting at x either enters M_a in finite time, or converges to one of the finitely many critical points in $\mathrm{Crit}_a(f)$.

If M is no longer compact but f is proper, then of course the study of the relative topology of the pair (M_b, M_a) goes exactly as in the compact case. There might be cases –as in our setting coming from approximately holomorphic geometry– that the natural Morse functions to be used are not proper, and one needs to impose an appropriate form of the above dichotomy for trajectories of $-\nabla_g f$:

Lemma 1. *Let f be a Morse function on a manifold M and let g be a metric on M so that $\nabla_g f$ is complete. Let a be a regular value, $b > a$, and assume that the following holds:*

(1) *For every compact subset $X \subset M_b$, there exist finitely many critical points $c_1, \dots, c_{i_X}$ in $\mathrm{Crit}_a(f)$ such that the following dichotomy holds: a trajectory of $-\nabla_g f$ starting at $x \in X$ either reaches M_a in finite time, or converges to a critical point in $\{c_1\} \cup \dots \cup \{c_{i_X}\}$.*
(2) *Every $c \in Crit_a(f)$ has index $\geq j$.*

Then we have that $\pi_k(M_b, M_a) = 0$, for $k = 0, \dots j-1$.

Proof. Let us start by making the following observation: if X is as in assumption (1) and the collection $\{c_1\} \cup \dots \cup \{c_{i_X}\}$ is empty, then we claim that X is taken in finite time to M_a by the flow ϕ of $-\nabla_g f$. Indeed, for every $x \in X$ there exists a time $t_x > 0$ such that $f(\phi_{t_x}(x)) < a$; further, since for fixed t, ϕ_t is continuous, there is a small ball $B_g(x, \varepsilon_x)$ centered at x such that $\phi_{t_x}(B_g(x, \varepsilon_x)) \subset M_a$. Then, the result follows by compactness of X.

Now, let N be a compact manifold and $h \colon (N, \partial N) \to (M_b, M_a)$ be a smooth map. Let U be a relatively compact neighborhood of $h(N)$. Then assumption (1) implies that trajectories starting at points in $\bar{U}$ can only enter M_a in finite time or converge to one of the finitely many critical points $\{c_1, \dots, c_{i_{\bar{U}}}\}$.

Observe that there is a small relatively compact neighborhood V of $h(\partial N)$ such that the flow of $-\nabla_g f$ sends V into M_a: this follows if $V \subset U$ is selected so that $f(V)$ lies below the critical values $\{f(c_1), \dots, f(c_{i_{\bar{U}}})\}$.

We now construct h', an arbitrarily small perturbation of h relative to V. Proceeding inductively over the finite list $\{c_1, \dots, c_{i_{\bar{U}}}\}$, as in [9], we obtain h' that is transverse to the ascending disks of the critical points and that satisfies $h'(N) \subset U$.

If N has dimension at most $j-1$ then, by hypothesis (2), transversality to the ascending disks means empty intersection. The hypotheses of the claim at the start of the proof are satisfied and it follows that $\pi_k(M, M_a) = 0$, for $k = 0, \dots j-1$. □

The following result describes quantitative conditions on the gradient vector field granting the dichotomy in point (1) of Lemma 1.

Proposition 3. *Let f be a Morse function, g be a complete metric on M, and $a < b \in \mathbb{R}$. Assume that there exist real constants $D, E > 0$ and open subsets $\mathcal{C}_i \subset M_b$, $i \in I$, such that:*

(1) *For any pair $i, i' \in I$, $i \neq i'$, we have $d_g(\mathcal{C}_i, \mathcal{C}_{i'}) > D$.*
(2) *The diameter of the sets $\mathcal{C}_i$ is at most E.*
(3) *There exist real numbers $\delta_1, \delta_2 > 0$, such that*

$$\delta_2 \geq |\nabla_g(f)(p)| \geq \delta_1, \forall p \in M_b \setminus \left(\bigcup_{i \in I} \mathcal{C}_i\right).$$

Then $-\nabla_g f$ is complete and the dichotomy in point (1) of Lemma 1 for $-\nabla_g f$ holds.

Essentially, the proposition states that the critical points of f come in families, indexed by I and contained in the sets $\mathcal{C}_i$, that are far from each other. In order to prove Proposition 3, let us introduce some notation and prove an auxiliary lemma. Given any $x \in M$, we denote by γ_x the positive half of the flow line that contains x. Denote by ϕ_t the flow of f at time t. Let γ_x^t designate the segment of the curve γ_x between x and $\phi_t(x)$. Then:

Lemma 2. *Under the assumptions of Proposition 3, there is a constant R, independent of $t \in \mathbb{R}$ and $x \in M_b$, such that $d_g(\phi_t(x), x) > R$ implies $f(\phi_t(x)) < a$.*

Proof. For every curve γ we denote by $\widetilde{\gamma}$ the (possibly disconnected) curve:

$$\widetilde{\gamma} = \left\{ p \in \gamma : p \notin \bigcup_{i \in I} \mathcal{C}_i \right\},$$

that is, the union of segments of γ that are disjoint from the sets $\mathcal{C}_i$.

Given any curve $\gamma \subset B(x, R)$ starting at x and intersecting the boundary of $B(x, R)$ at y, we can associate to it another curve, which we denote by $\eta = \eta_\gamma$, using the following procedure:

(1) list, in order, all the sets $\mathcal{C}_i$ that γ intersects. Remove all the consecutive repetitions of the same $\mathcal{C}_i$, listing just the first one in each series of repetitions. Write $\{\mathcal{C}_{i_j}\}_{j \in [1,..k]}$ for this finite list,
(2) mark the entry and exit points e_j and f_j of γ into each $\mathcal{C}_{i_j}$. In the case of consecutive repetitions of the same $\mathcal{C}_i$, just mark the first entry point and the last exit point of the series. For simplicity, denote $f_0 = x$ and $e_{k+1} = y$,
(3) call η the piecewise smooth curve formed by connecting these marked points in the order they appear. From e_j to f_j, take the shortest geodesic between the two points. From f_j to e_{j+1}, take the shortest path not intersecting any C_i. Denote these paths by $l(e_j, f_j)$ and $l(f_j, e_{j+1})$ respectively.

Assume $R > E + D$. If $k = 0, 1$, it is immediate that

$$\frac{\text{length}(\widetilde{\eta})}{\text{length}(\eta)} \geq \frac{D}{E + D},$$

otherwise, the following estimate holds:

$$\frac{\text{length}(\widetilde{\eta})}{\text{length}(\eta)} = \frac{\sum_{j=0}^{k} \text{length}(l(f_j, e_{j+1}))}{\sum_{j=0}^{k} \text{length}(l(f_j, e_{j+1})) + \sum_{j=1}^{k} \text{length}(l(e_j, f_j))} \geq$$

$$\frac{\sum_{j=1}^{k-1} \text{length}(l(f_j, e_{j+1}))}{\sum_{j=1}^{k-1} \text{length}(l(f_j, e_{j+1})) + kE} \geq \frac{(k-1)D}{(k-1)D + kE} \geq \frac{D}{2(E + D)}.$$

For any radius $r > E + D$, denote by τ the time at which the curve γ_x first intersects $\partial B(x, r)$. Denote this intersection point by y. Consider the segment γ_x^τ and its associated curve $\eta = \eta_{\gamma_x^\tau}$. Use the fact that over $\widetilde{\gamma}_x^\tau$ we have a lower bound for the gradient $|\nabla_g f| > \delta_1 > 0$:

$$|f(y) - f(x)| \geq \delta_1 \, \text{length}(\widetilde{\gamma}_x^\tau) \geq \delta_1 \, \text{length}(\widetilde{\eta}) \geq \delta_1 \, \text{length}(\eta) \frac{D}{2(E + D)} \geq r \frac{\delta_1 D}{2(E + D)}$$

which implies that, if r is taken to be large enough, $|f(y) - f(x)| > b - a$, and hence $y \in M_a$. □

Proof of Proposition 3. Let $X \subset M_b$ be a compact set. Let R be the universal constant given by Lemma 2. Denote by $X(R)$ the R-neighborhood of X, which is a relatively compact set. Lemma 2 implies that any trajectory starting at X either reaches the interior of M_a – which is equivalent to saying that it reaches M_a in finite time – or it remains in $X(R)$ for all time.

It must be shown that if a trajectory γ_x remains within $X(R)$ for all times then it must converge to a critical point. Since $X(R)$ is relatively compact and f is a Morse function, there is a finite number k of critical points in its closure. Each of those critical points $\{c_i\}_{i=1}^k$ has an arbitrarily small neighborhood V_i which corresponds to a ball in the standard Morse model around c_i. In particular, a trajectory that intersects V_i must intersect just once, either converging to c_i or escaping from V_i eventually. From this it follows that there is a time $t_0 > 0$ such that $\gamma_x(t) \notin V_i$, for all $t > t_0$ and every i. Since the gradient $|\nabla_g f| > \delta > 0$ is bounded from below in $X(R) \setminus \cup_{i=1..k} V_i$, this shows that $f(\gamma_x(t)) < a$ for t large enough, which is a contradiction. □

3. Proof of Theorem 1.

Fix some leaf $F \in \mathcal{F}$. All we need to do now is to check that, for a suitable choice of Morse function and metric on F, the hypotheses of Proposition 3 are satisfied for F. Our candidate is the restriction to the leaf of the function $f_k = \log |s_k|^2$, and the restriction to the leaf of any Riemannian metric on M.

We shall prove a couple of preliminary lemmas, for which we need to recall some notation. Given a function f, defined on a manifold endowed with a codimension one foliation $(M, \mathcal{F})$, the tangential differential $d^{\mathcal{F}} f$ is the composition of the differential with the projection $T^*M \to (T\mathcal{F})^*$. The points in which $d^{\mathcal{F}} f$ vanishes are the *tangential critical points of* f, which we denote by $\Sigma^{\mathcal{F}}(f)$. Of course, $\Sigma^{\mathcal{F}}(f)$ are nothing but the critical points of the restriction of f to each leaf of $\mathcal{F}$.

Lemma 3. *For every k large enough, the 2–calibrated submanifold $W_k \subset M$ has a tubular neighborhood that contains a full regular level set of $f_k = \log |s_k|^2$ and which is also disjoint from $\Sigma^{\mathcal{F}}(f_k)$.*

Proof. It is enough to check that $h_k = ||s_k||^2$ satisfies the Lemma, since log is an increasing monotone function.

We claim that the neighborhood $U = \{x \in M \mid ||s_k(x)|| < \nu\}$ of the submanifold W_k does not intersect $\Sigma^{\mathcal{F}}(f_k)$. Assume that $p \in U$. By the ν–transversality along $\mathcal{F}$ of the section s_k, there is a unitary vector field $v \in T_p\mathcal{F}$ such that $||\nabla_v s_k(p)|| \geq \nu$. By asymptotic holomorphicity, for k large, we have that the unitary vector field $Jv \in T_p\mathcal{F}$ satisfies $||\nabla_{Jv} s_k(p) - i\nabla_v s_k(p)|| = O(k^{-1/2})$. Therefore, the map $\nabla^{\mathcal{F}} s_k(p)$ is surjective. We conclude that $p \notin \Sigma^{\mathcal{F}}(f_k)$. □

Lemma 4. *Let F, a leaf of $\mathcal{F}$, be fixed. After a perturbation of the sequence s_k, preserving transversality to zero and approximately holomorphicity, it can be assumed that:*

1. *the restrictions of the f_k to F are Morse functions.*
2. *$\Sigma^{\mathcal{F}}(f_k)$ is a finite union of disjoint circles in general position with respect to $\mathcal{F}$. Their tangency points are turning points, i.e., birth-death type singularities for the restriction of f_k to the corresponding leaf.*

Proof. According to [3], after an arbitrarily small C^r perturbation, $r \geq 2$, the set of tangential critical points $\Sigma^{\mathcal{F}}(f_k)$ can be assumed to fit into a 1-dimensional manifold that is transverse to $\mathcal{F}$ everywhere but at the finite collection of turning points $c_1, \dots, c_d$. Every other point is a non–degenerate critical point for the restriction of f_k to the corresponding leaf. The turning points satisfy the following relevant property: in a small foliated chart, a plaque not containing the turning point intersects $\Sigma^{\mathcal{F}}(f_k)$ either in the empty set or in two tangential critical points.

Assertion (1) in the Lemma follows by showing that none of the $c_1, \dots, c_d$ belong to the fixed leaf F: if any of them do, a C^r–small isotopy, transverse to $\mathcal{F}$ at the turning point, can be used to move it to a nearby leaf. This is described in detail in [3].

These C^r perturbations of f_k can be taken to be the result of a C^r perturbation of s_k. Indeed, let ε_k be a C^r perturbation of f_k. The function ε_k can be assumed to be identically zero away from an arbitrary small neighborhood of $\Sigma^{\mathcal{F}}(f_k)$ so, by lemma 3, the following expression is well defined:

$$\widetilde{s}_k = s_k\sqrt{1+\varepsilon_k/f_k},$$

since f_k is bounded from below in the support of ε_k. It is clear that

$$||\widetilde{s}_k|| = f_k + \varepsilon_k.$$

The asymptotic holomorphicity of the sequence $\tilde{s}_k$ can be readily checked:

$$\nabla\widetilde{s}_k = \nabla s_k\sqrt{1+\varepsilon_k/f_k} + s_k\frac{f_k\nabla\varepsilon_k - \varepsilon_k\nabla f_k}{2f_k^2\sqrt{1+\varepsilon_k/f_k}},$$

where the second term is C^r-small and the first is C^r-close to ∇s_k. A similar computation for the higher order derivatives concludes the claim. □

We can finally address the proof of the theorem.

Proof of Theorem 1. Fix a leaf F and assume that we have all the data needed for developing approximately–holomorphic geometry in M^{2n+1}. The metrics g_k induce complete metrics in F. Given an approximately–holomorphic sequence s_k, with corresponding Donaldson-type submanifolds W_k, an application of Lemma 4 yields a new approximately–holomorphic sequence, still denoted by s_k, that induces Morse functions $(f_k)_{|F}$ in $F \setminus W_k$.

By Lemma 3, W_k has an ε-neighborhood containing a regular level a_k. Lemmata 3 and 4 together mean that $\Sigma^{\mathcal{F}}(f_k)$ has a small tubular neighborhood of positive radius not intersecting the level a_k.

By Lemma 4, the manifold $\Sigma^{\mathcal{F}}(f_k)$ is transverse to $\mathcal{F}$ except in a finite number of turning points $c_1, \ldots, c_d$. Fix a closed geodesic arc T_i through each c_i, transverse to the foliation. Let $B^{2n}(0,r) \subset \mathbb{R}^{2n}$ be the closed ball of radius r. For $r > 0$ sufficiently small, the exponential map for the leafwise metric $g_k^{\mathcal{F}}$ yields disjoint foliated charts $\phi_i : U_i \to [0,1] \times B^{2n}(0,r)$ satisfying $\phi_i(T_i) = [0,1] \times \{0\}$. Having fixed r, by taking the T_i sufficiently short – effectively shrinking U_i in the vertical direction – it can be assumed that:

$$\phi_i(\Sigma^{\mathcal{F}}(f_k) \cap U_i) \subset [0,1] \times B^{2n}(0, r/2)$$

Consider the family of open arcs $I_j \cong (0,1) \subset \Sigma^{\mathcal{F}}(f_k)$, $j \in [1,2,..,l]$, and circles $I_j \cong \mathbb{S}^1 \subset \Sigma^{\mathcal{F}}(f_k)$, $j \in [l+1,2,..,m]$, comprising $\Sigma^{\mathcal{F}}(f_k) \setminus (\cup_{i=1..d} U_i)$. For sufficiently small $0 < s < r$, the exponential map for the metric $g_k^{\mathcal{F}}$ defines disjoint charts $\psi_j : V_j \to I_j \times B^{2n}(0,s)$. The union of the U_i and the V_j covers $\Sigma^{\mathcal{F}}(f_k)$.

The subsets $\mathcal{C}_i$, as in Proposition 3, can be defined and they come in two families:

(1) $s/2$–neighborhoods, in the metric $g_k^{\mathcal{F}}$, of the points $x \in I_j \cap F$, for any j,
(2) $r/2$–neighborhoods, in the metric $g_k^{\mathcal{F}}$, of the points $x \in T_i \cap F$, for any i.

By construction, the $g_k^{\mathcal{F}}$–diameter of the $\mathcal{C}_i$ is bounded above by $r/2$. Further, the $g_k^{\mathcal{F}}$–distance between any two sets $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ is bounded below by s. Therefore conditions (1) and (2) in Proposition 3 hold. Condition (3) follows immediately from the fact that the union of the $\mathcal{C}_i$ is the intersection of a neighborhood of $\Sigma^{\mathcal{F}}(f_k)$ with the leaf F.

An application of Lemma 1 shows that the relative homotopy groups $\pi_j(F, F \cap W_k)$ vanish for $j < n$ and for k large enough, since we already did the index computation in Proposition 2. □

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