EXISTENCE h–PRINCIPLE FOR ENGEL STRUCTURES

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Abstract. In this article we prove that the inclusion of the space of Engel structures of a smooth 4–manifold into the space of full flags of its tangent bundle induces surjections in all homotopy groups. In particular, we construct Engel structures representing any given full flag.

1. Introduction

The main result of the article is an existence h–principle for Engel structures. The geometry of distributions has been essential in the development of differential geometry, and the topology of the spaces of distributions on a smooth manifold represents a central question in the field.

It is said that a class of distributions satisfies an h–principle if the homotopy groups of the space of such distributions can be related to the homotopy groups of an understood space of algebraic geometric nature. This is the case of foliations due to the seminal work of W. Thurston [16, 17], and of contact and even–contact structures after the work of M. Gromov, Y. Eliashberg and co–authors in [2, 6, 11].

Engel structures constitute the least understood class of topologically stable distributions. The first hint towards a possible h–principle in Engel geometry was T. Vogel's construction of Engel structures in any smooth parallelizable 4–manifold [19]. The present article provides a proof of a general existence h–principle for Engel distributions.

1.1. Engel geometry.

Let M be a smooth n–dimensional manifold. An m–distribution on M is a smooth correspondence assigning an m–dimensional subspace in T_pM to each point $p \in M$ – equivalently, it is a smooth section of the grassmanian bundle $\mathrm{Gr}_m(TM) \longrightarrow M$. The group of diffeomorphisms of the manifold acts in the space of distributions by push–forward. A dense, open subclass of distributions for which this action is locally transitive at the level of germs is called topologically stable. E. Cartan proved [3, 14] that topologically stable distributions necessarily conform to the inequality

$$m(n-m) \leq n.$$

E. Cartan's theorem states that there are only four such classes. These are line fields, contact structures, even–contact structures and Engel structures; confer [18] for a more detailed account. The common feature of these distributions is that they do not possess local invariants. The existence or non–existence of geometric global invariants is therefore a crucial factor for them to be mathematically relevant. This is the reason why the global theory of line fields – Dynamical Systems – and the study of contact structures – Contact Topology – are so rich. In contrast, even–contact structures satisfy a complete h–principle (see [8, Theorem 14.2.3],[9, 13]) and hence their global structure is determined by their formal algebraic topology invariants, making them uninteresting from a Differential Topology perspective.

Engel structures still await for an answer: it is not known whether they can be classified only in terms of the underlying algebraic topology. The present article focuses on the existence problem, providing a complete answer. As for classification, the situation is much harder than in the contact case. In the contact setting, Gray's global stability theorem reduces the problem to the study of connected components of the moduli space of contact structures; see [7] for a historical perspective. The notion of overtwisted contact structure [6, 2] singles out special connected components whose homotopy type

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can be fully understood. The methods of this article clearly suggest possible definitions of "overtwisted Engel structure" but, due to the lack of global stability, it is hard to provide one that is invariant under homotopies. Therefore, the notion of overtwistedness applies not to whole connected components but to open sets inside the space of Engel structures. As such, any working definition is not canonical: different choices of homotopically equivalent open sets do the job. More importantly, there are still no invariants to tell whether some connected component of Engel structures contains an overtwisted representative or not.

For these reasons we have chosen not to give a tentative definition of overtwistedness. Let us state now our results.

Let M be a smooth 4–manifold. An **Engel structure** $\mathcal{D}$ is a maximally non–integrable 2–distribution, i.e. a 2–distribution $\mathcal{D} \subseteq TM$ conforming to the following two properties:

(1) $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ is a 3–distribution,
(2) $[\mathcal{E}, \mathcal{E}] = TM$.

By definition, the second condition is equivalent to the 3–distribution $\mathcal{E}$ being even–contact. The kernel of the bilinear map

$$[\quad,\quad] : \mathcal{E} \times \mathcal{E} \to TM/\mathcal{E}$$

is a line field $\mathcal{W} \subseteq \mathcal{E}$ which also satisfies $\mathcal{W} \subseteq \mathcal{D}$. Indeed, otherwise we would have a point $p \in M$ where $\mathcal{W}_p \oplus \mathcal{D}_p = \mathcal{E}_p$ and therefore $[\mathcal{D}_p, \mathcal{D}_p] = [\mathcal{E}_p, \mathcal{E}_p] = T_pM$, a contradiction.

Therefore, any Engel structure canonically induces a full flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$. Moreover, the Lie bracket induces two bilinear antisymmetric bundle maps:

$$\mathcal{D} \times \mathcal{D} \to \mathcal{E}/\mathcal{D}$$

$$\mathcal{E}/\mathcal{W} \times \mathcal{E}/\mathcal{W} \to TM/\mathcal{E}.$$

By taking the determinant in both cases, we obtain the following bundle isomorphisms (see [15]):

$$\det(\mathcal{D}) \simeq \mathcal{E}/\mathcal{D} \tag{C1}$$

$$\det(\mathcal{E}/\mathcal{W}) \simeq TM/\mathcal{E}. \tag{C2}$$

Then we define a **formal Engel structure** to be a full flag $\mathcal{W}^1 \subset \mathcal{D}^2 \subset \mathcal{E}^3 \subset TM^4$ in the tangent space satisfying Conditions (C1) and (C2). Such a flag is denoted by $(\mathcal{W}, \mathcal{D}, \mathcal{E})$ and the isomorphisms are implied. Fixing the isomorphism in Condition (C1.) is equivalent to fixing an orientation in $\mathcal{E}$.

A consequence of Conditions (C1) and (C2) is the existence of a natural isomorphism $\mathcal{W} \simeq TM/\mathcal{E}$ which implies that TM is orientable if and only if $\mathcal{W}$ is. In particular, if the plane field $\mathcal{D}$ is an oriented distribution in an oriented 4–manifold M, then M is parallelizable and the formal Engel flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$ is oriented.

In the literature, orientability is usually assumed when defining (formal) Engel structures. It was proven by T. Vogel [19] that the necessary condition of M being parallelizable is also sufficient for the existence of an Engel structure. His proof deeply uses the interaction between Engel structures and contact structures. In brief, a theorem of D. Asimov [1] states that a closed n–manifold, $n \neq 3$, with vanishing Euler characteristic admits a round handle decomposition, and T. Vogels argument uses this result and the assumption on parallelizability to reduce the existence of an Engel structure to the construction of appropriate Engel structures with contact boundary on every round handle.

Vogel's work suggested that the question posed in [8, Intrigue F2]: *do Engel structures in closed parallelizable* 4*–manifolds satisfy an h–principle?* could have a positive answer, but his techniques provided limited control of the homotopy type of the resulting Engel flag. The present article improves the result in this direction by providing an existence h–principle for Engel structures.

1.2. **Statement of the results.** Let M be a smooth 4–manifold, and consider the space of formal Engel structures $\mathfrak{F}(M)$ and the space of Engel structures $\mathfrak{E}(M)$, which are defined as

$$\mathfrak{F}(M) = \{(\mathcal{W}^1, \mathcal{D}^2, \mathcal{E}^3) \mid \mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM \text{ is a full flag satisfying (C1) and (C2)}\}$$

$$\mathfrak{E}(M) = \{(\mathcal{W}^1, \mathcal{D}^2, \mathcal{E}^3) \mid \mathcal{D} \text{ is Engel with } \mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \text{ its associated full flag }\}$$

and endowed with the compact–open topology; observe that there is an inclusion $i : \mathfrak{E}(M) \longrightarrow \mathfrak{F}(M)$ provided by the forgetful map. The main result of this article concerns the behaviour of the inclusion in all homotopy groups:

Theorem 1. *The map $\pi_k(i) : \pi_k(\mathfrak{E}(M)) \longrightarrow \pi_k(\mathfrak{F}(M))$ is surjective for every $k \geq 0$. In particular, every formal Engel structure is homotopic to the flag of a genuine Engel structure.*

In these terms, Vogel's construction [19] states that the set $\pi_0(\mathfrak{E}(M))$ is not empty as soon as M admits a completely oriented full flag. The methods used in the proof of Theorem 1 imply the following

Corollary 2. *Let $(M, \partial M)$ be a 4–manifold with boundary and $(\mathcal{W}, \mathcal{D}, \mathcal{E})$ a formal Engel structure such that the line field $\mathcal{W}$ is transverse to the boundary ∂M and $(\partial M, T(\partial M) \cap \mathcal{E})$ is a contact structure. Then there is a deformation of the formal Engel structure into a genuine Engel structure inducing the same contact structure on the boundary ∂M.*

In particular, this implies that the notion of Engel cobordism or Engel fillability is not particularly relevant in order to distinguish contact manifolds. Corollary 2 implies that given any smooth 3–manifold V and two homotopic contact structures ξ_0 and ξ_1 with vanishing first Chern class, there exists an Engel structure on the trivial cobordism $V \times [0, 1]$ inducing the contact manifolds $(V \times \{0\}, \xi_0)$ and $(V \times \{1\}, \xi_1)$ on the boundary components.

In view of Theorem 1, a reasonable question is whether the map i is also injective on π_k, possibly after restricting to some subclass within $\mathfrak{E}(M)$. The proof of Theorem 1 suggests possible definitions for such a class. This will be the content of future work.

Let us consider a second corollary from our main theorem. Consider a 4–dimensional foliation $\mathcal{F}$ in a smooth n–dimensional manifold M. Then a flag $\mathcal{W}^1 \subseteq \mathcal{D}^2 \subseteq \mathcal{E}^3 \subseteq T\mathcal{F}^4$ satisfying Condition (C1.) and

$$\det(\mathcal{E}/\mathcal{W}) = T\mathcal{F}/\mathcal{E} \tag{C2'}$$

is said to be a formal foliated Engel structure for the foliation $\mathcal{F}$; a 2–distribution $\mathcal{D} \subseteq T\mathcal{F}$ is called a foliated Engel structure if it is an Engel structure when restricted to each leaf of the foliation $\mathcal{F}$. Denote the spaces of formal foliated Engel structures and foliated Engel structures by $\mathfrak{F}(\mathcal{F})$ and $\mathfrak{E}(\mathcal{F})$, respectively. The parametric nature of Theorem 1 implies the following result.

Theorem 3. *The inclusion map $\pi_k(i) : \pi_k(\mathfrak{E}(\mathcal{F})) \longrightarrow \pi_k(\mathfrak{F}(\mathcal{F}))$ is surjective for every $k \geq 0$. Hence, formal foliated Engel structures are homotopic to flags of genuine foliated Engel structures.*

1.3. **Structure of the paper.** The article is organized as follows. In Section 2 we define all the objects involved and we discuss some known results. Subsection 2.2 is classical to an extent, since it can be mainly found in the works of E. Cartan [4], though it has been overlooked for many years. It can be condensed into Proposition 7, which is a fundamental ingredient in the proof of Theorem 1.

The article primarily focuses in the proof of the π_0–statement of Theorem 1; the argument in this case is central for the remaining results, and once described in detail it can be readily adapted to the parametric case in order to prove the π_k–statements. The proof of the π_0–statement of Theorem 1 consists of three parts.

First, given some formal Engel structure, we turn it into a flag whose 3–distribution is an even contact structure and whose 1–distribution is its kernel. This is achieved using the h–principle for even–contact structures [13].

Second, in Section 4, we triangulate M in a manner adapted to the kernel of the formal Engel structure, and subsequently deform the formal Engel structure to a genuine Engel structure in a neighborhood of the 3–skeleton. This reduction process provides a collection of 4–cells endowed with formal Engel structures that are genuine Engel structures in the boundary. We also prove in this section that such formal Engel structures on the 4–cells can be assumed to be of a particular form, which we call the 6π–**radial shells**.

Third, in Section 3, we construct an object called the **four–leaf clover** which allows us to deform any 6π–radial shell into a genuine Engel structure, thus concluding the proof of the π_0–statement in Theorem 1. Note that the argument is not presented in a linear fashion, and the chosen order serves to better motivate the constructions.

Finally, Section 5 discusses the parametric nature of the construction, and concludes Theorem 1, Theorem 3 and Corollary 2.

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2. Preliminaries

In this section we first recall the properties of even–contact structures that are relevant for the proof of Theorem 1. Then we characterize certain Engel structures on the 4–cell $\mathbb{D}^3 \times [0,1]$ in terms of curves in the 2–sphere $\mathbb{S}^2$, and provide two valuable examples of Engel structures due to E. Cartan [4].

Given topological spaces $A \subset B$, we will denote by $\mathcal{O}p(A)$ any sufficiently small open neighborhood of A inside B.

2.1. **Even–contact structures.**

In the previous section we briefly introduced even–contact structures. Let us describe this notion in detail.

Definition 4. *Let M be a smooth $(2n+2)$–dimensional manifold. A $(2n+1)$–distribution $\mathcal{E}^{2n+1} \subseteq TM$ is said to be an even–contact structure if it is locally described as the kernel of a 1–form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$. $(M, \mathcal{E})$ is called an even–contact manifold, and $\mathcal{E}$ is said to be coorientable if α can be defined globally.*

Even–contact structures can be regarded as transverse contact structures associated to line fields: the even–contact condition amounts to the 2–form $d\alpha$ being of maximal rank in $\ker(\alpha)$, and the kernel $\mathcal{W}$ of the 2–form $d\alpha|_{\ker(\alpha)}$ is a real line field. Furthermore:

Lemma 5. *Let $(M^{2n+2}, \mathcal{E})$ be an even–contact manifold and $N^{2n+1} \subseteq M$ a $(2n+1)$–dimensional submanifold transverse to the kernel $\mathcal{W}$ of $\mathcal{E}$. Then $(N, \mathcal{E} \cap TN)$ is a contact manifold.*

Proof. Given a locally defining 1–form α for the even–contact $\mathcal{E}$, the $(2n+1)$–form $\alpha \wedge (d\alpha)^n|_N$ is a volume form on N since the kernel $\mathcal{W}$ of the 2–form $d\alpha$ is transverse to N. □

Since even–contact structures $(M^4, \mathcal{E}^3)$ induce contact structures in 3–manifolds N transverse to their kernel $\mathcal{W}$, so do Engel structures $(M^4, \mathcal{D}^2)$. In this case, the line field $TN \cap \mathcal{D} \subseteq TN \cap \mathcal{E}$ is a distinguished Legendrian vector field of the contact manifold $(N, \mathcal{E} \cap TN)$.

The contact Darboux theorem states that any contact structure is locally isomorphic to $(\mathbb{R}^{2n+1}, \ker(dz - \sum_{i=1}^n x_i dy_i))$, where the coordinates in $\mathbb{R}^{2n+1}$ are $(x_1, y_1, \dots, x_n, y_n, z)$. Using the Lemma, we deduce that even–contact structures also possess a local normal form given by $(\mathbb{R}^{2n+2}, \ker(dz - \sum_{i=1}^n x_i dy_i))$, with $(x_1, y_1, \dots, x_n, y_n, z, t)$ the coordinates in $\mathbb{R}^{2n+2}$.

There are two significant differences between contact and even–contact structures. A first invariant associated to an even–contact structure is its kernel: this line field often has complicated dynamics, and these are unstable under smooth perturbations of the even–contact structure. Therefore, Gray's stability cannot hold in full generality, but it does hold if the line field $\mathcal{W}$ is fixed [10].

The second difference is the existence of global geometric invariants. D. McDuff showed, using convex integration, that the classification of even–contact structures satisfies a complete h–principle. A pair $\mathcal{W}^1 \subset \mathcal{E}^3 \subset TM^4$ satisfying Condition (C2.), as in the introduction, is called an almost even–contact structure. Then, McDuff's result in dimension 4 states the following:

Theorem 6. ([8, Section 20.6],[13]) *For any given smooth manifold M^4, the canonical map from the space of even contact structures into the space of formal even contact structures is a weak homotopy equivalence.*

Theorem 6 is used in the proof of Theorem 28, the main reduction result used in Theorem 1. This reduces the construction of an Engel structure from a formal Engel structure to the construction of an Engel structure from an even–contact structure.

Recall that M. Gromov's h–principle applies to Engel structures in *open* manifolds. In particular, for an open 4–manifold, the space of Engel structures is weakly homotopy equivalent to the space of formal Engel structures [8, Theorem 7.2.3]. The core contribution of Theorem 1 is the surjection h–principle for Engel structures on *closed* smooth 4–manifolds.

2.2. Engel structures in $\mathbb{D}^3 \times [0,1]$.

In this subsection we discuss the relation between Engel structures and families of convex curves in the 2–sphere $\mathbb{S}^2$. This will be used in Section 3 in order to construct a genuine Engel structure in a 4–cell with appropriate boundary conditions.

2.2.1. *Engel structures as curves.*

Consider the 4–cell $\mathbb{D}^3 \times [0,1]$ with coordinates (x, y, z, t). Let $\mathcal{D} = \langle \partial_t, X \rangle$ be a 2–plane distribution where the non–vanishing vector field X is tangent to the foliation by level sets $\mathbb{D}^3 \times \{t_0\}$, $t_0 \in [0,1]$. Let us write $\dot{X} = [\partial_t, X]$ and $\ddot{X} = [\partial_t, \dot{X}]$, which are two vector fields also tangent to these level sets.

The three vector fields X, $\dot{X}$ and $\ddot{X}$ on $\mathbb{D}^3 \times [0,1]$ can be regarded as 1–parametric families of vector fields in $\mathbb{D}^3$, with parameter $t \in [0,1]$; these families are denoted by X_t, $\dot{X}_t$ and $\ddot{X}_t$.

Trivialize $T\mathbb{D}^3$ with the coordinate frame $\langle \partial_x, \partial_y, \partial_z \rangle$ to identify all the fibres of the 2–sphere bundle $\mathbb{S}(T\mathbb{D}^3)$ with a fixed 2–sphere $\mathbb{S}^2$. For $p \in \mathbb{D}^3$ fixed, define the curve:

$$\gamma_p : [0,1] \longrightarrow \mathbb{S}^2$$

$$\gamma_p(t) = \frac{X(p,t)}{||X(p,t)||}.$$

The 2–plane distribution $\mathcal{D} = \langle \partial_t, X \rangle$ can then be given by a $\mathbb{D}^3$–family of such curves. In order to characterize the Engel condition from this viewpoint, we briefly discuss convex curves.

2.2.2. *Convex curves in* $\mathbb{S}^2$.

Consider a parametrized smooth curve $\gamma : [0,1] \longrightarrow \mathbb{S}^2$; its unit tangent vector field is given by $\mathfrak{t}(t) = \gamma'(t)/||\gamma'(t)||$. Define $\mathfrak{n}(t)$ to be the unique vector field such that $\{\mathfrak{t}(t), \mathfrak{n}(t)\}$ is an orthonormal oriented basis of the tangent space $T_{\gamma(t)}\mathbb{S}^2$, and we define the Frenet map of the curve γ by

$$\mathfrak{F}(\gamma) : [0,1] \longrightarrow SO(3), \quad \mathfrak{F}(\gamma)(t) = (\gamma(t), \mathfrak{t}(t), \mathfrak{n}(t)).$$

A point $\gamma(t)$ is said to be an **inflection point** of the curve γ if $\langle \mathfrak{t}'(t), \mathfrak{n}(t) \rangle = 0$, and the curve γ is said to be **convex** if it has no inflection points. In Section 3 we consider curves that are not convex but fail to be so in an explicit manner: These curves are C^∞–limits of convex curves that become increasingly tangent to the equator $\{z = 0\} \subseteq \mathbb{S}^2$.

2.2.3. *The Engel condition.* Following the description of Engel structures $\mathcal{D} = \langle \partial_t, X \rangle$ on the 4-cell $\mathbb{D}^3 \times [0,1]$ in terms of families of curves γ_p on the 2–sphere, we now provide a sufficient condition for these families to define Engel structures.

Proposition 7. *A 2–distribution $\mathcal{D} = \langle \partial_t, X \rangle$ is Engel in a neighborhood of the point $(p,t) \in \mathbb{D}^3 \times [0,1]$ if both $\gamma_p'(t) \neq 0$ and at least one of the following two conditions holds:*

1. *the curve $\gamma_p : [0,1] \longrightarrow \mathbb{S}^2$ has no inflection point at time t,*
2. *the 2–distribution $\langle X_t, \dot{X}_t \rangle$ is a contact structure on $\mathcal{O}p(p) \times \{t\} \subseteq \mathbb{D}^3 \times \{t\}$.*

Proof. First, the 2–distribution $\mathcal{D}$ being non–integrable translates to the condition $\dot{X}(p,t) \neq 0$. Indeed, since the vector field $[\partial_t, X]$ is tangent to the foliation by level sets, the condition for the vector $[\partial_t, X]_{(p,t)}$ not to be in $\mathcal{D}$ is equivalent to the vector field $\dot{X}_t$ not being colinear with X_t at the point p, so the curve γ_p has non–zero velocity at time t.

Set $\mathcal{E} = \langle \partial_t, X, \dot{X} \rangle$. For the 2–distribution $\mathcal{D}$ to be an Engel structure, the 3–distribution $\mathcal{E}$ must be non–integrable, so at least one of the two vectors $\ddot{X} = [\partial_t, \dot{X}]$ and $[X, \dot{X}]$ should not be contained in $\mathcal{E}$ at the point (p,t). If $t \in [0,1]$ is not an inflection point of the curve γ_p, the acceleration $\ddot{X}_t(p)$ is not contained in the space spanned by the position $X_t(p)$ and the speed $\dot{X}_t(p)$. If the 2–distribution $\langle X_t, \dot{X}_t \rangle$ is a contact structure on the level $\mathcal{O}p(\{p\}) \times \{t\}$, the Lie bracket satisfies $[X_t, \dot{X}_t] \notin \langle X_t, \dot{X}_t \rangle$ at the point p. □

Proposition 7 generalizes two classical constructions due to E. Cartan [4]:

Example 8 (contact prolongation)**.** *Let (N, ξ) be a contact 3–manifold and $\xi = \langle Y, Z \rangle$ a frame. The contact prolongation of (N, ξ) is the Engel structure $(N \times [0,1], \mathcal{D})$ defined by*

$$\mathcal{D}(p,t) = \langle \partial_t, X \rangle, \quad \text{where } X(p,t) = \cos(t)Y(p) + \sin(t)Z(p).$$

In this case, Condition (2.) in Proposition 7 is satisfied, which proves that $\mathcal{D}$ is an Engel structure.

Example 9 (lorentzian prolongation)**.** *Consider a lorentzian 3–manifold (N, g) with a type $(1,2)$ framing $\langle L^+, Y_-, Z_- \rangle$. The kernels of the lorentzian metric at each point define a family of cones on the tangent bundle TN which (after trivialization with the framing) provide a family of non–degenerate quadric curves C_p in the unit 2–sphere $\mathbb{S}^2$. Parametrize each curve $X_p : [0,1] \longrightarrow C \subseteq \mathbb{S}^2$ and define the lorentzian prolongation $(N \times [0,1], \mathcal{D})$ as the 2–distribution defined by*

$$\mathcal{D}(p,t) = \langle \partial_t, X_p(t) \rangle.$$

This is an Engel structure because Condition (1.) of Proposition 7 is satisfied.

Notice that the legendrian line field $\mathcal{W}$ is expanded by ∂_t in the case of the contact prolongation and it is transverse to the direction ∂_t in the lorentzian prolongation.

The combination of the two constructions in Examples 8 and 9 allows to create flexibility in the space of Engel structures. In particular, Proposition 7 and Example 9 show that the Engel condition is very closely related to the convexity of the corresponding $\mathbb{D}^3$–family of curves. This will be the crucial ingredient for the extension result stated in Theorem 18, which is one of the two steps in the argument of Theorem 1.

3. The Hole and Its Filling

In this section we address the problem of extending a particular germ of Engel structure on $\mathcal{O}p(\partial \mathbb{D}^4)$ to an Engel structure in the interior of $\mathbb{D}^4$. The reduction process explained in Section 4, subsumed in Theorem 28, implies that such an extension suffices in order to prove Theorem 1.

Subsection 3.1 introduces in detail this extension problem and Subsection 3.3 relates different extension problems in order to obtain a simpler model. Subsection 3.4 provides a useful rephrasing in terms of curves. In Subsection 3.5 we explain the solution up to three technical lemmas, whose statement and proof we defer to Subsection 3.7. The influence of the articles [2, 5] is manifest in this section.

3.1. **Engel shells.** The following definition describes an Engel germ in the boundary $\partial(\mathbb{D}^3 \times [0,1])$ of the 4–cell $\mathbb{D}^3 \times [0,1]$ that extends to the interior as a formal Engel structure. Consider coordinates $(x, y, z; t)$ in the cartesian product $\mathbb{D}^3 \times [0,1]$.

Definition 10. *An* ***Engel shell*** *is a formal Engel structure $(\mathcal{W}, \mathcal{D}, \mathcal{E})$ on the 4–cell $\mathbb{D}^3 \times [0,1]$ conforming to the following properties:*

1. *$\mathcal{D} = \langle \partial_t, X \rangle$, where X is tangent to the level sets $\mathbb{D}^3 \times \{t\}$,*
2. *In a neighborhood $\mathcal{O}p(\partial(\mathbb{D}^3 \times [0,1]))$ of the boundary:*
   I. *The 2–distribution $\mathcal{D}$ is an Engel structure,*
   II. *$\mathcal{E} = \xi \oplus \partial_t$, with ξ a t–invariant contact structure on the level sets $\mathbb{D}^3 \times \{t\}$,*
   III. *$\mathcal{W} = \langle \partial_t \rangle$ and X is tangent to the 2–distribution ξ,*
   IV. *$\{\partial_t, X, [\partial_t, X]\}$ is a positive frame for $\mathcal{E}$ for the orientation induced by Condition (C1.).*

If $(\mathcal{W}, \mathcal{D}, \mathcal{E})$ defines an Engel structure on $\mathbb{D}^3 \times [0,1]$, the Engel shell is said to be solid.

Let us discuss the homotopic properties of those formal Engel structures that agree with a fixed Engel structure near the boundary. Extending $\mathcal{W}$ to the interior amounts to finding a non–vanishing section of $\mathcal{D}$ that agrees with a given section along the boundary of $\mathbb{D}^4$. Similarly, extending $\mathcal{E}$ amounts to extending a section of $T\mathbb{D}^4/\mathcal{D}$. Since the space of sections of a circle bundle over the pair $(\mathbb{D}^4, \partial\mathbb{D}^4)$ is a non–empty contractible space, the homotopy class of the formal Engel structure is determined by the homotopy class of the 2–distribution $\mathcal{D}$, which at the same time is determined by the homotopy class of the line field X. This justifies the notation $\mathcal{D}$ or also X for an Engel shell $(\mathbb{D}^3 \times [0,1], \mathcal{W}, \mathcal{D}, \mathcal{E})$.

3.2. **Angular shells.** The first step in the reduction process stated in Theorem 28 is deforming the given formal Engel structure to a formal Engel structure in which the 3–distribution $\mathcal{E}$ is even–contact and the line field $\mathcal{W}$ is its kernel. The Engel shells resulting from this procedure are of the form

$$\mathcal{W} = \langle \partial_t \rangle, \quad \mathcal{E} = \xi \oplus \mathcal{W}, \quad X \in \xi \times \{t\} \subset T(\mathbb{D}^3 \times \{t\})$$

not only on the boundary $\mathcal{O}p(\partial(\mathbb{D}^3 \times [0,1]))$, but on the interior $\mathbb{D}^3 \times [0,1]$. This particular type of Engel shell $\mathcal{D}$ is called an **angular shell**. The advantage of working with angular shells is that they can be described by $\mathbb{D}^3$–families of real–valued functions, as we now explain.

Consider the euclidean metric in $\mathbb{D}^3 \times [0,1]$. Let $\mathcal{D} = \langle \partial_t, X \rangle$ be an angular shell and assume that the vector field X is unitary. Fix also an orthonormal legendrian framing $\{Y, Z\}$ for the contact structure $(\mathbb{D}^3, \xi)$ so that $\{\partial_t, Y, Z\}$ is a positive frame for the orientation induced by Condition (C1.) in the 3–distribution $\mathcal{E}$. Then the following formula

$$X(p,t) = \cos(c(p,t))Y + \sin(c(p,t))Z, \tag{1}$$

assigns to each angular shell a real–valued function $c : \mathbb{D}^3 \times [0,1] \longrightarrow \mathbb{R}$ that is uniquely defined up to shifting by 2π.

Given an angular shell $\mathcal{D}$, the function $c = c(\mathcal{D})$ defined by Equation 1 is called its **angular function**. The discussion on Subsection 2.2 and the orientation conventions imply the following fact:

Lemma 11. *The angular shell $\mathcal{D}$ is an Engel structure at the point (p,t) if and only if $\partial_t c(\mathcal{D})(p,t) > 0$.*

In particular, the differential inequality $\partial_t c(\mathcal{D}) > 0$ always holds on a neighborhood $\mathcal{O}p(\partial(\mathbb{D}^3 \times [0,1]))$. Conversely, suppose that a function $c : \mathbb{D}^3 \times [0,1] \longrightarrow \mathbb{R}$ satisfies $\partial_t c(\mathcal{D}) > 0$ on a neighborhoods $\mathcal{O}p(\partial(\mathbb{D}^3 \times [0,1]))$. Then c is the angular function of some angular model $\mathcal{D}(c)$ which is uniquely defined. In consequence, there is a bijective correspondence between angular functions up to shifting by 2π and angular models; recall that this correspondence depends on the legendrian framing $\{Y, Z\}$ we choose for ξ. Contractibility of the space of real functions relative to the boundary implies that:

Lemma 12. *Let $\mathcal{D}(c_1)$ and $\mathcal{D}(c_2)$ be two angular shells with the same underlying framed contact structure. They are homotopic relative to the boundary as angular shells if and only if their angular functions $c_1, c_2 : \mathbb{D}^3 \times [0,1] \longrightarrow \mathbb{R}$ agree on $\mathcal{O}p(\partial(\mathbb{D}^3 \times [0,1]))$.*

The following example illustrates the simplest case in which an angular shell can be homotoped to a genuine Engel structure. Bolzano's theorem shows that, in general, the extension problem is obstructed if one tries to solve it within the space of angular shells.

Example 13. *Suppose that $c(p,1) > c(p,0)$ for all $p \in \mathbb{D}^3$. Lemmas 11 and 12 imply that the angular shell $\mathcal{D}(c)$ is homotopic relative to the boundary to a solid Engel shell on $\mathbb{D}^3 \times [0,1]$.*

3.3. **Domination and radial shells.** The extension problem for the germ of an Engel structure in the boundary of $\mathbb{D}^3 \times [0,1]$ to its interior introduces a partial order between angular shells.

Definition 14. *Let $\mathcal{D}(c_1)$ and $\mathcal{D}(c_2)$ be two angular shells, $\mathcal{D}(c_1)$ **dominates** $\mathcal{D}(c_2)$ if*

$$c_1(p,0) \leq c_2(p,0) \text{ and } c_2(p,1) \leq c_1(p,1).$$

The following proposition reduces the problem of filling angular shells to filling angular shells with simple angular functions presenting symmetry.

Proposition 15. *Let $\mathcal{D}(c_1)$ and $\mathcal{D}(c_2)$ be two angular shells such that $\mathcal{D}(c_1)$ dominates $\mathcal{D}(c_2)$. If $\mathcal{D}(c_2)$ admits a deformation to a solid Engel shell through Engel shells, then so does $\mathcal{D}(c_1)$.*

Proof. Using Lemma 11, we can deform c_1 to be strictly increasing in the intervals

$$\mathcal{O}p([0, h_0(p)]) \cup \mathcal{O}p([h_1(p), 1]),$$

where $h_1, h_2 : \mathbb{D}^3 \to [0,1]$ are smooth functions satisfying $c_1(p, h_i(p)) = c_2(p,i)$, $i = 0,1$. Now there is a unique embedding $\Phi : \mathcal{O}p(\partial\mathcal{D}(c_2)) \to \mathcal{D}(c_1)$ satisfying $\Phi^* c_1 = c_2$. Extending Φ to the interior of $\mathcal{D}(c_2)$ arbitrarily, $\mathcal{D}(c_1)$ can be homotoped within $\Phi(\mathcal{D}(c_2))$, relative to its boundary, to achieve $\Phi^* c_1 = c_2$. Perform the Engel deformation in $\Phi(\mathcal{D}(c_2))$ provided by assumption and the claim follows. □

Given an angular function $c : \mathbb{D}^3 \times [0,1] \longrightarrow \mathbb{R}$ and a point $p \in \mathbb{D}^3$, the difference $c(p,1) - c(p,0)$ measures the amount of rotation of the Legendrian vector field X. The extension problem that we solve in Theorem 18 concerns a particular class of angular shells in which X rotates *enough* along each vertical segment of the boundary $\partial\mathbb{D}^3 \times [0,1]$. In order to describe in precise terms this geometric intuition, we introduce the following definition. Hereafter, the symbol ρ denotes a fixed numeric real value such that $[\rho, 2\rho] \subset (0,1)$.

Definition 16. *Let $K \in \mathbb{R}^+$ be a constant. A function $c : \mathbb{D}^3 \times [0,1] \longrightarrow \mathbb{R}$ is said to be K–**radial** if it conforms to the following three properties:*

I. *$c(p,t)$ is increasing in $t \in [0, 2\rho]$ and it satisfies*
$$c(p,t) = c(p,\rho) + \frac{(t-\rho)}{\rho} \cdot K, \text{ for } t \in [\rho, 2\rho],$$
II. *$c(p,t) - c(p,\rho)$ is invariant under the action of $SO(3)$ on $\mathbb{D}^3$,*
III. *$c(p,t)$ is independent of p whenever $(p,t) \in \mathcal{O}p(\{0\} \times [0,1])$,*

*The angular shell $\mathcal{D}(c)$ associated to a K–radial angular function c is said to be a K–**radial shell**.*

In this notation, the most relevant use of Proposition 15 is the following corollary.

Corollary 17. *Let c be an angular function, and $K \in \mathbb{R}^+$ a constant satisfying:*

$$K < \min_{p \in \partial\mathbb{D}^3} (c(p,1) - c(p,0)).$$

Then there exists a K–radial function c' such that $\mathcal{D}(c)$ dominates $\mathcal{D}(c')$.

It is time to state Theorem 18, the main result of this section and, along with Theorem 28, one of the two key ingredients in the proof of the existence h–principle stated in Theorem 1. The rest of this section is dedicated to its proof, and its statement reads as follows:

Theorem 18. *A 6π–radial shell is homotopic through Engel shells to a solid Engel shell.*

Theorem 28 in the next section implies that, in order to prove Theorem 1, it suffices to deform angular shells with difference angle $c(p,1)-c(p,0)$ greater than 6π everywhere to solid Engel shells. Consequently, Corollary 17 and Theorem 18 indeed conclude the proof of Theorem 1.

The proof of Theorem 18 is essentially contained in the sequence drawn in Figure 4, and it features the four–leaf clover curve as a crucial ingredient. The fact that the contact 2–plane field $\xi \subseteq T\mathbb{D}^3 \times \{t\}$ in an angular shell cuts the unit sphere $\mathbb{S}^2 \cong T_p\mathbb{D}^3 \times \{t\}$ in an equator depending on the point p, and the essential role of the inflection points of the curves on this 2–sphere, require an additional technicality that we now address by defining Engel combs.

3.4. **Engel combs.** Subsection 2.2 implies that Engel shells can be described in terms of $\mathbb{D}^3$–parametric families of parametrized curves in the 2–sphere $\mathbb{S}^2$. These curves are given by the unitary vector field X that determines the 2–distribution $\mathcal{D} = \langle \partial_t, X\rangle$. The vector field X is a section of the unit tangent bundle of the level sets $\mathbb{D}^3 \times \{t\}$ and thus (once this bundle is trivialized) can be considered as a map $X : \mathbb{D}^3 \times [0,1] \longrightarrow \mathbb{S}^2$.

Instead of the trivialization $\langle \partial_x, \partial_y, \partial_z\rangle$ provided by the coordinates, we trivialize the tangent bundles $T(\mathbb{D}^3 \times \{t\})$ of the level sets in a manner more suited to such families of curves. This is done as follows: for each point $(p,t) \in \mathbb{D}^3 \times [0,1]$, consider the t–invariant orientation–preserving linear isometry $\varphi_{(p,t)} : T_{(p,t)}(\mathbb{D}^3 \times \{t\}) \longrightarrow \mathbb{R}^3$ defined by the conditions

$$\varphi_{(p,t)}(Y(p,t)) = \partial_x, \quad \varphi_{(p,t)}(Z(p,t)) = \partial_y,$$

where $\{Y, Z\}$ is a frame for the contact structure $(\mathbb{D}^3, \xi)$. The isometries $\varphi_{(p,t)}$ identify the unit sphere of the contact plane $T_{(p,t)}(\mathbb{D}^3 \times \{t\}) \cap \mathcal{E}_{(p,t)}$ with the horizontal equator $\mathbb{S}^2 \cap \{z=0\}$.

Consider the following rotation of angle θ around the z–axis:

$$\mathrm{Rot}(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and write } e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then a given radial shell $\mathcal{D}(c)$ yields a 1–parametric family of curves

$$(2) \qquad \begin{cases} \gamma_r^c : [0,1] \longrightarrow \mathbb{S}^2, \quad r \in [0,1], \\ \gamma_r^c(t) = \mathrm{Rot}(c(p,t) - c(p,\rho))e_1 \quad \text{where } p \text{ is any point in } \mathbb{D}^3 \text{ with radius } r. \end{cases}$$

In the proof of Theorem 18 we work with Engel shells that are not necessarily angular. This leads to the following definition and subsequent lemma.

Definition 19. *Let $K \in \mathbb{R}^+$ be a positive constant. A K–**Engel comb** is a $[0,1]$–family of curves $\gamma_r : [0,1] \longrightarrow \mathbb{S}^2$, $r \in [0,1]$, such that:*

I. *there exists a K–radial function $c : \mathbb{D}^3 \times [0,1] \longrightarrow \mathbb{R}$ such that:*
$$\gamma_r(t) = \gamma_r^c(t), \text{ for } t \in \mathcal{O}p([0,2\rho] \cup \{1\}) \text{ and } r \in \mathcal{O}p(\{1\}),$$
II. *The curves γ_r are independent of r whenever $r \in \mathcal{O}p(\{0\}) \subseteq [0,1]$,*
III. *The curves γ_r are C^∞–tangent to the horizontal equator $\{z=0\}$ at their inflection points.*

An Engel comb is thus an interval family of curves, all of which agree with each other in the interval $[0,2\rho]$. This notation is motivated by the fact that the different curves resemble the teeth of a comb. An Engel comb describes an Engel shell with radial symmetry:

Lemma 20. *Consider a K–Engel comb γ_r, and a smooth function $d : \mathbb{D}^3 \to \mathbb{R}$. Then the 2–plane distribution*

$$\mathcal{D}(\gamma_r) = \langle \partial_t, X(p,t)\rangle = \langle \partial_t, \varphi^{-1}_{(p,t)} \mathrm{Rot}(d(p))\gamma_{|p|}(t)\rangle$$

defines an Engel shell.

Proof. Condition (II.) in Definition 19 implies that $\mathcal{D}(\gamma_r)$ is smooth near $\{0\} \times [0,1] \subseteq \mathbb{D}^3 \times [0,1]$, whereas Condition (I.) recovers the boundary conditions of the Engel shell. □

The function d in the Lemma corresponds to $d(p) = c(p, 0)$, where c is the corresponding K–radial function.

Engel combs form a strict subclass of Engel shells that is well suited for the extension problem. However, the resulting Engel shells are not necessarily solid due to the lack of control on either the velocities or the inflection points of the curves. The following definition includes an additional condition which guarantees that the Engel comb yields a solid Engel shell. Note that the function d plays no role in the fillability of the Engel shell.

Definition 21. *An Engel comb γ_r is said to be* **tame** *if it satisfies the following two properties:*

1. *γ_r' is non–vanishing,*
2. *Consider the set $\mathcal{I}_{\gamma_r} = \{(r,t) \in [0,1]^2 |\ t$ is an inflection point of $\gamma_r\}$. For every $(r,t) \in \mathcal{I}_{\gamma_r}$, $\exists a, b \in \mathbb{R}^+$, $a < b$, such that $(r,t) \in [a,b] \times \{t\} \subset \mathcal{I}_{\gamma_r}$.*

Indeed, these two conditions imply that tame Engel combs induce, through Lemma 20, solid Engel shells.

Proposition 22. *The Engel shell induced from a tame Engel comb is a solid Engel shell.*

Proof. By Condition (1.) in Definition 21, the 2–plane $\mathcal{D}$ defined by γ_r is non–integrable. Suppose that $(|p|, t)$ lies in the complement of $\mathcal{I}_{\gamma_r}$, then Proposition 7 shows that $\mathcal{D}$ is Engel at (p, t).

If, on the contrary, $(|p|, t) \in \mathcal{I}_{\gamma_r}$, consider the interval $[a, b]$ provided by Condition (2.) in Definition 21. The points (p', t) with $|p'| \in [a, b]$ conform a region $S = \mathbb{S}^2 \times [a, b] \subset \mathbb{D}^3$ with non–empty interior. By Condition (III.) in Definition 19, $\langle \gamma_r, \gamma_r' \rangle|_{[a,b]\times\{t\}} = \{z = 0\}$, which means that the corresponding Engel shell has $\langle X, \dot{X} \rangle|_{S\times\{t\}} = \xi$. Since the contact condition is open, we can apply Proposition 7 to conclude. □

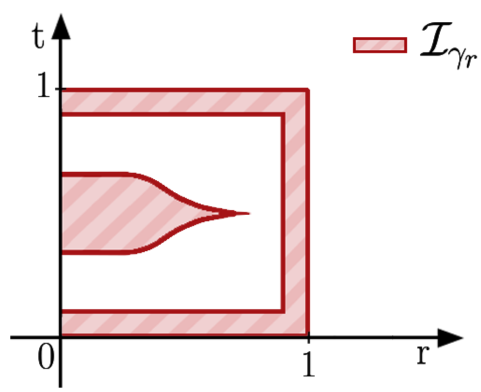

FIGURE 1. A possible configuration for the inflection points of a tame family similar to the one in the proof of Theorem 18.

In light of Proposition 22, we now focus on deforming any 6π–Engel comb into a tame Engel comb.

3.5. **Reducing to tame Engel combs.** In this subsection, we first describe two families of curves: the kink and the four–leaf clover. These are then used in the proof of Theorem 18 to deform a given K–Engel comb in the region $t \in [\rho, 2\rho]$.

3.5.1. *The kink curve.* By definition, a K–Engel comb γ_r is given in $[0,1]\times[\rho, 2\rho]$ by $\mathrm{Rot}(K(t-\rho)/\rho)e_1$, i.e. a rotation along the horizontal equator which does not depend on $p \in \mathbb{D}^3$. For instance, $K \geq 6\pi$ results in the curves γ_r turning more at least 3 times around the equator at constant speed as t goes from ρ to 2ρ. The kink curves serve to interpolate, relative to the boundary, between a segment encircling once the equator and a short convex segment strictly contained in the upper hemisphere, see Figure 2.

For each $\theta \in [0, \pi/2]$, consider the plane given by the equation $\{\sin(\theta)(x-1)+\cos(\theta)z = 0\}$. For $\theta = 0$ this describes the plane $\{z = 0\}$ and for $\theta = \pi/2$ the vertical plane $\{x = 1\}$. Considering $\theta \in [0, \pi/2)$, the intersection of these planes with the 2–sphere $\mathbb{S}^2$ yields the following parametrized curves

$$\beta_\theta(t) = (\sin^2(\theta) + \cos^2(\theta)\cos(t), \cos(\theta)\sin(t), \sin(\theta)\cos(\theta)(1 - \cos(t))), \quad t \in [0, 2\pi].$$

The curve β_0 parametrises the equator with constant angular speed, and $\beta_{\pi/2}$ is a constant map with image the point $(1, 0, 0)$. The remaining curves for $\theta \in (0, \pi/2)$ are convex since they present rotational symmetry with respect to the normal axis of the corresponding plane. Note also that the Frenet frame remains constant at the origin of these curves: $\mathfrak{F}(\beta_\theta)(0) = \mathrm{Id}$ for $\theta \in [0, \pi/2)$.

3.5.2. *The four-leaf clover.* The geometric reason for us to introduce the four–leaf clover is that it allows to arbitrarily decrease the value of the angular function in $t = 2\rho$ at the expense of deforming to convex curves in $t \in [\rho, 2\rho]$. Note then that once the angular function is small enough at the point $t = 2\rho$, the formal Engel structure $(\mathbb{D}^3 \times [2\rho, 1], \mathcal{D})$ will be homotopic to a solid Engel shell.

Although the four–leaf clover is a curve on the 2–sphere, it is simpler to describe it in an affine chart.

Lemma 23. *The affine chart* $\Pi : H^2 = \{(x, y, z) \in \mathbb{S}^2 : x > 0\} \longrightarrow \mathbb{R}^2$, $\Pi(x, y, z) = (y/x, z/x)$ *maps great circles to straight lines, and convex curves to convex curves.*

Proof. The correspondence between great circles and planes passing through the origin implies the first claim. Since convexity can be defined in terms of the order of contact with great circles or straight lines, respectively, the second claim follows. □

The parametrized plane curve $f(t) = (\cos(t)\sin(2t), \sin(t)\sin(2t))$, $t \in [0, 2\pi]$, which we call the four–leaf clover, is convex and, by Lemma 23, so is the curve $(\Pi^{-1} \circ f) \subseteq \mathbb{S}^2$. Figure 3 depicts the clover.

Let us reparametrize the resulting curve to $\kappa(t) = \Pi^{-1} \circ f(2\pi t)$, and set $\beta(t) = \beta_\theta(6\pi t)$ for an arbitrary but fixed $\theta \in (0, \pi/2)$. The following lemma is an immediate consequence of Lemma 23.

Lemma 24. *The curves* $\beta(t)$ *and* $\kappa(t)$ *are homotopic through a smooth family* $\tau_s(t)$ *of convex curves,* $s \in [0, 1]$, *with Frenet frames* $\mathfrak{F}(\tau_s)(0) = \mathrm{Id}, \forall s \in [0, 1]$.

Proof. Both curves β and κ lie in the same connected component of the space of convex curves [12, Theorem 5]: a family of convex curves τ_s joining $\tau_0 = \beta$ and $\tau_1 = \kappa$, and satisfying $\mathfrak{F}(\tau_s)(0) = \mathrm{Id}$, can be constructed by using an affine chart Π, reparametrising both curves by angle, and linearly interpolating between them. This is possible since their images by Π have Gauss maps with winding number 3. □

3.6. **The proof of Theorem 18.** The argument uses three technical lemmas whose statements and proofs are postponed to the following subsection; their geometric content is however intuitive. Lemmas 25 and 26 state that a family of curves tangent to the equator at a given point can be deformed to be C^∞–tangent to the equator at that point, and also ensuring that the curves that were convex remain convex away from that point. Lemma 27 states that a segment that is C^∞–tangent to the equator on one of its ends can be deformed so that it first parametrizes the original segment and then an arbitrarily long piece of equator.

Proof of Theorem 18. Consider $c : \mathbb{D}^3 \times [0, 1] \longrightarrow \mathbb{R}$ a 6π–radial function, the curves corresponding to its associated 6π–Engel comb γ^c_r are tangent to the horizontal equator of the 2–sphere. The only a priori information we have on γ^c_r is the existence of a time interval $t \in [\rho, 2\rho]$ during which the curves wind around the horizontal equator three times. We are going to deform these curves as the radius $|p|$ decreases; observe that if $c(p, 1) - c(p, 2\rho)$ were positive on $\mathbb{D}^3$, we could apply Lemma 11 to obtain a solid Engel shell.

The deformation provided by the kink curves β_θ modifies the three laps around the equator into a curve with three kinks. The curve with three kinks can be homotoped to the four–leaf clover curve, which now can be used to arbitrarily decrease the value of $c(p, 2\rho)$. The decreasing process consists

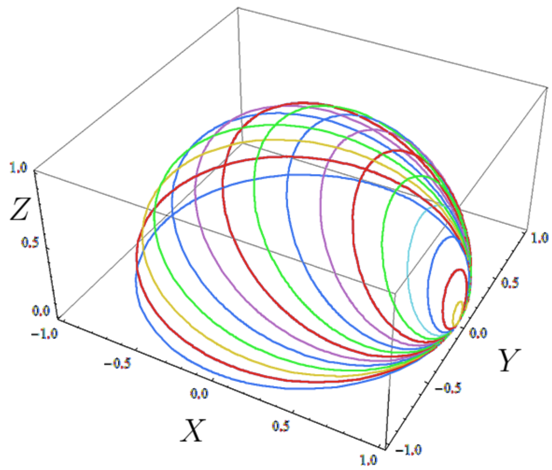

FIGURE 2. The curves β_θ for different values of the parameter θ.

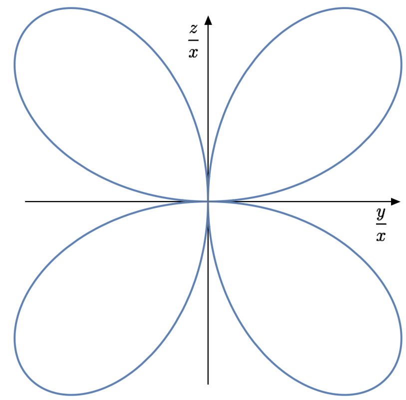

FIGURE 3. The curve $\Pi \circ \kappa$.

of clockwise pulling the two left–most leaves of the four–leaf clover around the equator as many times as needed; this process is illustrated in Figure 4.

This geometric explanation is now detailed with the corresponding analysis. First the curve

$$\beta(t) = \gamma_r^c(\rho(1+t)) = \mathrm{Rot}(6\pi t)e_1, \quad t \in [0,1],$$

is deformed to the four–leaf clover; this is achieved by applying Lemma 24 to obtain the family $\tau_s : \mathbb{S}^1 := [0,1]/\{0 \simeq 1\} \longrightarrow \mathbb{S}^2$, $s \in [0,1]$, which we understand as maps with domain the interval $[0,1]$. The family τ_s can be modified at its ends to glue smoothly with a curve tangent to the equator; this is done by applying Lemma 25 to τ_s at times $t \in \{0,1\}$. This yields a $[0,1]$–family of curves f_s satisfying that:

- $f_0(t) = \beta(t) = \mathrm{Rot}(6\pi t)e_1$, for $t \in [0,1]$,
- there exists a small $\varepsilon > 0$ such that $f_s(t) = \tau_s(t)$, for $t \in [\varepsilon, 1-\varepsilon]$ and $s \in [0,1]$,
- for $s \in (0,1]$, $f_s(t)$ is convex for $t \in (0,1)$ and it has an ∞–order of contact with the equator $\{z = 0\}$ at the endpoints $f_s(0)$ and $f_s(1)$,
- the Frenet frame in the midpoint of the four–leaf clover is

$$\mathfrak{F}(f_1(1/2)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is the deformation in the region $s \in [0,1]$, we now define a deformation for $s \in [1,2]$.

Note that $t = 1/2$ is the time in which the four-leaf clover f_1 (or κ) has turned and is pointing in the opposite direction. In order to clockwise pull the two left–most leaves of the four–leaf clover, we first need to flatten the point $t = 1/2$ so that it has an ∞–order tangency with the equator: this is done by applying Lemma 26 to the curve f_1 at $t = 1/2$. This provides a family of curves $f_s : [0,1] \longrightarrow \mathbb{S}^2$, $s \in [1,2]$ such that:

- there exists a small $\varepsilon > 0$ such that $f_s(t) = f_1(t)$, for $t \notin [1/2-\varepsilon, 1/2+\varepsilon]$,
- the curves $f_s(t)$, $s \in [1,2)$ are convex if and only if $t \in (0,1)$, and the curve $f_2(t)$ is convex if and only if $t \in (0,1) \setminus \{1/2\}$. The inflection points of f_s, $s \in [1,2]$, are ∞–order tangencies with the equator $\{z = 0\}$,
- the Frenet frame in the midpoint of these modified four–leaf clovers remains constant:

$$\mathfrak{F}(f_s(1/2)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The ∞–order tangency point that we have introduced at $t = 1/2$ allows us to stretch the point into an arbitrarily large interval, and hence clockwise pulling the two left–most leaves of the flattened four–leaf clover. This deformation will occur for those values of the parameter $s \in [2, 3]$. Consider an arbitrary constant $C < 0$ to be chosen later which captures the amount of stretching and clockwise pulling.

Consider a small $\varepsilon \in \mathbb{R}^+$ and define $\varepsilon_s = (s-2)\varepsilon$. By applying Lemma 27 to the flattened four–leaf clover $f_2 : [0, 1] \longrightarrow \mathbb{S}^2$ we obtain a family of curves $f_s : [0, 1] \longrightarrow \mathbb{S}^2$, $s \in [2, 3]$, satisfying:

- $f_s(t/(1 - 2\varepsilon_s)) = f_2(t)$, for $t \in [0, 1/2 - \varepsilon_s]$,
- $f_s(t) = \mathrm{Rot}(C(s - 2)) \cdot f_2((t - 2\varepsilon_s)/(1 - 2\varepsilon_s))$, for $t \in [1/2 + \varepsilon_s, 1]$,
- the curve $f_s(t)$ negatively winds around the horizontal equator $\{z = 0\}$ in the interval $t \in [1/2 - \varepsilon_s, 1/2 + \varepsilon_s]$ with non–vanishing speed.

The first two conditions just reparametrize curve $f_2(t)$ away from $\mathcal{O}p(\{t = 1/2\})$ to a curve $f_3(t)$ such that the first half remains the same and the second half is moved by a rotation of angle C. The third condition is the clockwise–pulling process along the horizontal equator; Figure 4 describes the family f_s, $s \in [0, 3]$.

Let us now insert the deformation given by the $[0, 3]$–family of curves f_s inside the initial 6π–Engel comb γ_r^c. The curves in the Engel comb γ_r^c have a specified behaviour on the interval $t \in [\rho, 2\rho]$ and this is the interval where the deformation provided by f_s is to be inserted; we now provide the analytical details for this.

Consider a small $\delta \in \mathbb{R}^+$ such that the 6π–radial function c is increasing for $|p| \in [1 - 3\delta, 1]$. Define a smooth decreasing cut–off function $\chi : [0, 1] \longrightarrow [0, 3]$ so that the family $f_{\chi(r)}$ is smooth in the parameter and
$$\chi(t) = 3 \text{ for } t \in [0, 1 - 3\delta], \quad \chi(t) = 0 \text{ for } t \in [1 - \delta/3, 1].$$

The point now is to replace the initial family of curves of the 6π–Engel comb γ_r^c by the family of curves $F_r = f_{\chi(r)}((t - \rho)/\rho)$ in the interval of time $t \in [\rho, 2\rho]$. Observe that they do not glue immediately, since they have differing values at the points $t = \{\rho, 2\rho\}$. The absolute value $|F_r(\rho) - \gamma_r^c(\rho)|$ can be made arbitrarily small according to Lemma 25 and, since γ_r^c describes a solid angular shell in $t \in [0, \rho]$, it can be perturbed slightly to allow for the smooth glueing of both families while still describing a solid angular shell for $t \in [0, \rho]$. The endpoint quantities $F_r(2\rho)$ and $\gamma_r^c(2\rho)$ differ by a rotation by a positive angle in the equator and hence we can stretch γ_r^c to glue both families at $t = 2\rho$. The resulting Engel comb $\widetilde{\gamma}_r$ is homotopic through Engel combs to γ_r^c and thus provides a deformation of the initial Engel shell.

The Engel shell associated to $\widetilde{\gamma}_r$ can be made solid. By construction, the resulting formal Engel structure is still an Engel structure in the region $[0, 2\rho]$ as an application of Proposition 22. In the region $t \in [2\rho, 1]$, $\widetilde{\gamma}_r$ admits an angular function $\widetilde{c}(p, t)$ which satisfies $\widetilde{c}(p, 2)\rho) = c(p, 2\rho) - C$ (as a consequence of the clockwise–pulling of the two left–most leaves) and $\widetilde{c}(p, 1) = c(p, 1)$. The constant $C \in (-\infty, 0)$ can then be chosen such that $\widetilde{c}(p, 2\rho) < \widetilde{c}(p, 1)$, and then Lemma 11 provides a deformation of $\widetilde{\gamma}_r$ in the region $\mathbb{D}^3 \times [2\rho, 1]$ to a solid Engel shell. This concludes the deformation of the initial 6π–Engel comb into a solid Engel shell and thus proves the statement of Theorem 18. □

3.7. **Technical lemmas.** In the proof of Theorem 18 we have used two geometric facts regarding deformations of curves in the 2–sphere: modification of a horizontal inflection point into an ∞–order point of contact with the horizontal equator and the stretching of an ∞–order point of contact into an arbitrarily large segment. For completeness, we now include the statements and part of the analytic details of their proofs.

3.7.1. *Two lemmas on achieving ∞–order of contact.* The following two lemmas are quite similar in nature, both concerning deformations of a family of curves near a point in order to create ∞–order of contact with a certain curve (and at the same time preserving any existing convexity).

Lemma 25. *Consider a smooth family of curves $\gamma_s : [0, 1] \longrightarrow \mathbb{S}^2$, $s \in K$, where K is a compact space. Suppose that the curves γ_s are either convex or reparametrizations of an equatorial arc, and that the initial Frenet frame is $\mathfrak{F}(\gamma_s)(0) = \mathrm{Id}$.*

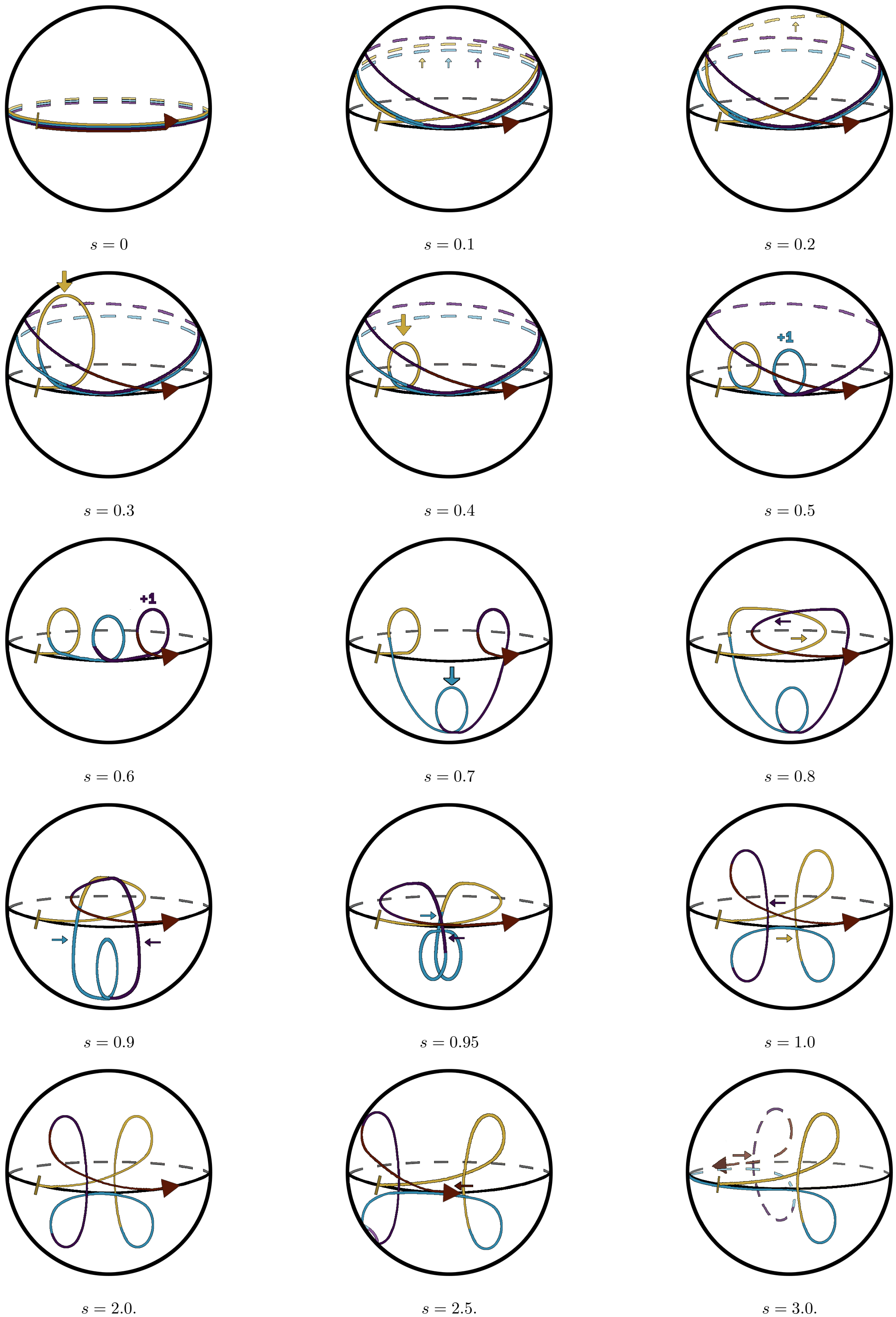

FIGURE 4. The family of curves f_s from the proof of Theorem 18.

Then, for any $\varepsilon \in \mathbb{R}^+$ small enough, there is a smooth family of curves $\eta_s : [0,1] \longrightarrow \mathbb{S}^2$, $s \in K$, satisfying:

I. $\|\eta_s - \gamma_s\|_{C^1} \leq \varepsilon$, $\eta_s|_{[\varepsilon,1]} = \gamma_s|_{[\varepsilon,1]}$, *and* $\mathfrak{F}(\eta_s)(0) = \mathrm{Rot}(-\varepsilon)$.

IIa. *If the curve γ_s is convex, the curve η_s is convex for $t \in (0,1]$ and $\eta_s(0)$ is an ∞–order tangency with the horizontal equator $\{z = 0\}$.*

IIb. *If the curve γ_s is a reparametrization of an equatorial arc, so is the curve η_s.*

Proof. Here we use Lemma 23 to translate this into a problem of real–valued functions; the affine chart for the 2–sphere is $\Pi : H^2 \longrightarrow \mathbb{R}^2$. Consider $\delta \in (0,\varepsilon)$ such that $\gamma_s|_{[0,\delta]} \subseteq H^2$ and $(\Pi \circ \gamma_s)|_{[0,\delta]}$ are graphical over the horizontal line $\Pi(\{z = 0\}) \subseteq \mathbb{R}^2$. The image of the family of curves γ_s can be expressed as a family of plane curves $(t, f_s(t)) \subseteq \mathbb{R}^2$, with $f_s : [0,\delta] \longrightarrow \mathbb{R}^+$ a family of smooth functions. It now suffices to stretch the domain of the functions.

Construct an increasing cut–off function $\chi_1 : [-\varepsilon,\delta] \longrightarrow [0,\delta]$ satisfying:

$$\chi^{(k)}(-\varepsilon) = 0 \text{ for } k \in \mathbb{N}, \quad \chi''|_{[-\varepsilon,\delta/2)} > 0, \text{ and } \chi(t)|_{[\delta/2,\delta]} = t.$$

Since the composition of increasing convex functions is also convex, the family of curves

$$\eta_s : [-\varepsilon,\delta] \longrightarrow H^2, \quad \eta_s(t) = \Pi^{-1} \circ (t, f_s \circ \chi(t))$$

preserves any existing convexity and it can be glued with the family of curves $\gamma_s|_{[\delta/2,1]}$. We can then reparametrize to obtain the required family of curves. □

In this same vein, we can smoothly flatten a given point in a convex curve to achieve an ∞–order of contact with respect to an equator, and also preserve the Frenet frame at that point. This is the content of the following lemma:

Lemma 26. *Consider a smooth convex curve $\gamma : [-1,1] \longrightarrow \mathbb{S}^2$ with $\mathfrak{F}(\gamma)(0) = \mathrm{Id}$ in its midpoint. For any $\varepsilon \in \mathbb{R}^+$ small enough, there exist smooth curves $\gamma_s : [-1,1] \longrightarrow \mathbb{S}^2$, $s \in [0,1]$, such that*

I. $\gamma = \gamma_0$.

II. $\|\gamma_s - \gamma\|_{C^1} \leq \varepsilon$, $\gamma_s|_{[-1,-\varepsilon]\cup[\varepsilon,1]} = \gamma|_{[-1,-\varepsilon]\cup[\varepsilon,1]}$, *and* $\mathfrak{F}(\gamma_s)(0) = \mathrm{Id}$.

III. *For $s \in [0,1)$, the curves γ_s are convex. γ_1 is convex in $[-1,0) \cup (0,1]$ and has an ∞–order of contact with the equator $\{z = 0\}$.*

Proof. Consider the affine chart Π from Lemma 23. Describe the curve $\Pi \circ \gamma$ near the midpoint $t = 0$ as the graph of a convex function $f : [-\delta,\delta] \longrightarrow \mathbb{R}$, with $\delta > 0$ sufficiently small. There exist constants $c_0, c_1 \in \mathbb{R}^+$ such that

$$0 < c_0 \leq f''(t), \quad 0 \leq \|f'(t)\| \leq c_1 \quad \forall t \in [-\delta,\delta].$$

Given a smooth function $g : [-\delta,\delta] \longrightarrow [-\delta,\delta]$, the condition for $F = f \circ g$ to be convex is the differential inequality

$$F'' = (f'' \circ g)(g')^2 + (f' \circ g)g'' > 0.$$

The bounds given by c_0, c_1 above imply that it is sufficient that g satisfies the inequality

$$c_0(g')^2 - c_1|g''| > 0. \tag{3}$$

Consider a function $h : [-\delta,\delta] \longrightarrow [0,1]$ such that:

a. $h(-t) = h(t)$, $h^{(k)}(0) = 0$ for $k \in \mathbb{N}$,

b. $\int_0^\delta h(t)dt = \delta$ and $h|_{[3\delta/4,\delta]} = 1$,

c. $h'|_{(0,\delta/4)} > 0$ and $h'|_{[\delta/4,\delta/2]} = 0$,

d. $c_0 > c_1|h'| \geq 0$ in $[\delta/2, 3\delta/4]$.

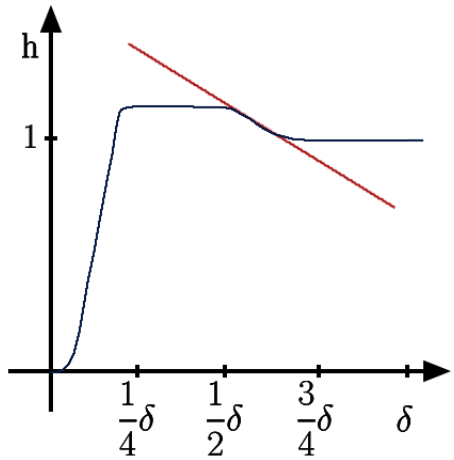

FIGURE 5. Graph of h.

See Figure 5 for a pictorial description of h. Now we construct a family g_s, $s \in [0,1]$, such that $f \circ g_s$ provides the desired flattening of f. Define $g_s(t) = \int_0^t [(1-s) + sh(t)]dt$, $s \in [0,1]$, $t \in (-\delta, \delta)$. Property (b.) allows us to define $f \circ g_s$ in $[-1,-\delta) \cup (\delta, 1]$ to be simply f. Property (c.) implies the convexity of $f \circ g_s$ in $(0, \delta/2)$. Property (e.) implies the convexity in $[\delta/2, 3\delta/4]$. Property (a.) ensures convexity in $[-1, 0)$ by symmetry and gives the ∞–order tangency at 0. The curve γ_s is uniquely defined by requiring $\Pi \circ \gamma_s$ to be the graph of $f \circ g_s$ in $t \in \mathcal{O}p(0)$. □

3.7.2. *The stretching lemma.* The following lemma concerns the stretching of a flattened point into a segment, the details of the proof are left to the reader.

Lemma 27. *Let $\gamma : [0,1] \longrightarrow \mathbb{S}^2$ be a smooth curve having an ∞–order of contact with the equator $\{z = 0\}$ at $\gamma(1)$. Further, assume that its Frenet frame at the endpoint $t = 1$ is*

$$\mathfrak{F}(\gamma)(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Given a smooth function $f : [0,1] \longrightarrow \mathbb{R}$ with $f(0) = 0$ and $f'(0) < 0$, the family of curves

$$\begin{cases} \gamma(t/(1-s/2)), & t \in [0, 1-s/2] \\ \mathrm{Rot}(f(s(2t-2+s)))\gamma(1), & t \in [1-s/2, 1] \end{cases}$$

can be reparametrized by a smooth family of smooth curves $\gamma_s : [0,1] \longrightarrow \mathbb{S}^2$, $s \in [0,1]$. □

4. Reducing to the angular model

The proof of Theorem 1 consists of a reduction process and an extension problem. In Section 3 we introduced a particular family of germs of Engel structures on the boundary of the 4–disk. Then, we showed how to extend them to the interior using Theorem 18. Hence, in order to conclude Theorem 1, it is sufficient to show that any formal Engel structure can be homotoped to be a genuine Engel structure except on finitely many 4–disks, each of them having an Engel germ on their boundaries fitting the hypothesis of Theorem 18. The main result of this section is this reduction process, which we now state:

Theorem 28. *Let $(\mathcal{W}_0, \mathcal{D}_0, \mathcal{E}_0)$ be a formal Engel structure on a closed 4–manifold M and $K \in \mathbb{R}^+$ a constant. Then there exists a homotopy of formal Engel structures $(\mathcal{W}_t, \mathcal{D}_t, \mathcal{E}_t)$, $t \in [0,1]$, and a collection of 4–disks $B_1, \dots, B_p \subseteq M$ such that:*

I. $(\mathcal{W}_1, \mathcal{D}_1, \mathcal{E}_1)$ *is a genuine Engel structure in the complement* $M \setminus \bigcup_{i=1}^p B_i$.

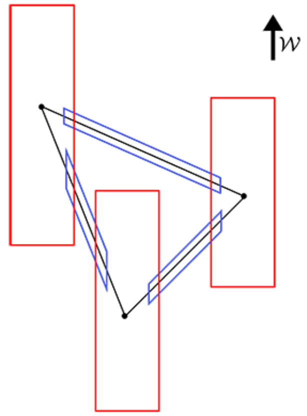

Figure 6. Case $n = 2$. Red: closed disks for the 0-simplices. Blue: closed disks for the 1-simplices.

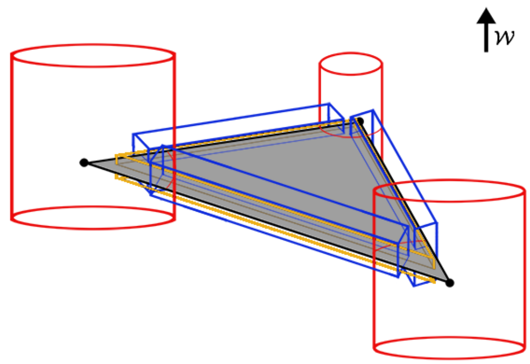

Figure 7. Case $n = 3$. Red: closed disks for the 0-simplices. Blue: closed disks for the 1-simplices. Yellow: closed disks for the 2-simplices.

II. *For each $i \in \{1, \ldots, p\}$, the restriction of the formal Engel structure $(\mathcal{W}_1, \mathcal{D}_1, \mathcal{E}_1)$ to each 4–disk B_i is a K–radial shell.*

The argument for Theorem 28 uses an adequate triangulation Σ of the 4–manifold M, and then deforms the formal Engel structure along the 3–skeleton of Σ to conform to the two properties in the statement. These two steps are detailed in Subsections 4.1 and 4.2, respectively.

4.1. **An adequate triangulation.** Consider a 4–manifold M with a formal Engel structure $(\mathcal{W}, \mathcal{D}, \mathcal{E})$. We construct a triangulation of M adapted to the flowlines of the line field $\mathcal{W}$. To it, we associate a collection of flowboxes – closed 4–disks $\mathbb{D}^3 \times [0,1] \subseteq M$ with coordinates $(x, y, z; t)$ where the line field $\mathcal{W}$ has the linear description $\langle \partial_t \rangle$ – satisfying a certain nesting property. The specific dimension of the manifold is not important for this argument and hence we will keep it general for later use in the parametric case.

Proposition 29. *Let M be an n–dimensional compact manifold, $n \geq 2$, endowed with a line field $\mathcal{W}$. Then there exists a triangulation Σ of M, understood as a finite collection of simplices, and a finite collection $\{S(\sigma)\}_{\sigma \in \Sigma}$ of closed n–disks, such that*

I. *Each simplex σ is contained in the union $\cup_{\tau \subseteq \sigma} S(\tau)$.*
II. *For each pair of simplices σ, σ', neither of them containing the other, we have $S(\sigma) \cap S(\sigma') = \emptyset$.*
III. *For each simplex $\sigma \in \Sigma$, $\exists \phi(\sigma) : S(\sigma) \longrightarrow \mathbb{D}^{n-1} \times [0,1]$ such that $\phi(\sigma)_* \mathcal{W} = \langle \partial_t \rangle$.*
IV. *For each lower dimensional simplex $\sigma \in \Sigma^{(j)}$, $j < n$, any orbit of the line field $\mathcal{W}|_{S(\sigma)}$ is either disjoint or completely contained in the set $\cup_{\tau \subsetneq \sigma} S(\tau)$.*

Note that the first two properties are of a topological nature, whereas the remaining two requirements belong to a dynamical setting. See Figures 6 and 7 for 2 and 3 dimensional examples of the required triangulations.

Proof. Fix a Riemannian metric g on the n–manifold M and consider a cover of M by open disks such that each disk is a flowbox of the line field $\mathcal{W}$. In each flowbox, fix a trivialization of $\mathcal{W}$ by a unitary vector field; denote it by $\mathcal{W}$ too. Apply Thurston's Jiggling Lemma [17, Section 5] to the 1–distribution $\mathcal{W}$ to find a triangulation Σ. This implies that the line field $\mathcal{W}$ is transverse to each lower dimensional simplex, i.e. the angle they make is strictly positive. Additionally, we can assume that each simplex of Σ is contained in one of the open disks of the cover.

For each j–simplex $\sigma \in \Sigma^{(j)}$, $j < n$, we fix a triple of positive real numbers $(r_0, r_1, r_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ on which the set $S(\sigma) \subseteq M$ will depend. Consider a j–dimensional disk $\widetilde{\sigma} \subseteq \sigma$ such that the distance

$r_0 < d_g(\partial\widetilde{\sigma}, \partial\sigma) \leq 2r_0$. Use the time–$r_1$ exponential map on an orthonormal basis of $(T\sigma \oplus \mathcal{W})^\perp \subseteq TM$ and the time–r_2 flow of $\mathcal{W}$ to construct the set:

$$S(\sigma) \cong \widetilde{\sigma} \times \mathbb{D}^{n-1-j}(r_1) \times [-r_2, r_2]$$

The region of the boundary $\partial(\tilde{\sigma} \times \mathbb{D}^{n-1-j}(r_1)) \times [-r_2, r_2]$ will be called the lateral boundary of $S(\sigma)$. Let us prove that suitable choices of (r_0, r_1, r_2) create a collection $S(\sigma)$ satisfying all the properties required in the statement for $j < n$. The sets $S(\sigma)$ can be chosen to additionally satisfy the following property:

- Va. For any j–simplex σ, $j < n$, and any $\tau \supsetneq \sigma$, τ intersects the boundary of the set $S(\sigma)$ in its lateral region.
- Vb. For any j–simplex σ, $j < n$, and any $\tau \subsetneq \sigma$, $S(\sigma)$ intersects the boundary of $S(\tau)$ in its lateral region.

These properties imply Property (IV.). Let us prove this by induction on the dimension of the simplices.

For $j = 0$, the first radius r_0 is not defined; but observe that the five properties in the statement are satisfied by choosing $r_1, r_2 > 0$ small enough. Further, Property (Va.) can be satisfied by choosing $r_2 \to 0$ and $r_1/r_2 \to 0$, and Property (Vb.) is empty. Indeed, if for each sequence of pairs (r_1, r_2) satisfying $r_2 \to 0$ and $r_1/r_2 \to 0$, Property (Va.) does not hold, then the angle between some simplex τ containing the point σ and $\mathcal{W}$ would be zero, and this is impossible.

Let us explain the inductive step: we suppose that the six properties hold for the k–simplices, $k = 0, \ldots, j-1$, and we consider a j–dimensional simplex σ. Choose the first two radii (r_0, r_1) small enough such that

$$\partial(\tilde{\sigma} \times \mathbb{D}^{n-1-j}(r_1)) \subseteq \cup_{\tau \subsetneq \sigma} S(\tau),$$

and shrink (r_1, r_2) to guarantee Property (II.). Property (Va.) is achieved by choosing the quotient r_2/r_1 to be large enough and then Property (Vb.) is guaranteed if r_2 is chosen small enough.

It remains to consider the n–dimensional simplices $\sigma \in \Sigma^{(n)}$: for each such σ we consider the PL–smooth disk D_+ constructed as the union of the faces of σ where $\mathcal{W}$ is inward pointing. This yields a flowbox for $\mathcal{W}$ contained in σ by considering the forward flow (for differing times) of a disk contained in D_+; this flowbox can be smoothed and assumed to have boundary C^0–close to $\partial\sigma$. □

4.2. **Engel energy.** The starting point in this subsection is that of Theorem 28, a formal Engel structure $(\mathcal{W}^1, \mathcal{D}^2, \mathcal{E}^3)$ on a smooth 4–manifold M. By applying Theorem 6, we can suppose that the 3–distribution $\mathcal{E}$ is even contact with the line field $\mathcal{W}$ being its kernel. The deformations considered henceforth maintain both $\mathcal{E}$ and $\mathcal{W}$.

Let us introduce a measure of the Engelness of a formal Engel structure, which we refer to as the Engel energy; the argument for Theorem 28 is phrased in terms of the creation of such Engel energy. Fix an auxiliary Riemannian metric g in M. Given a point p, fix an oriented orthonormal frame $\{W \in \mathcal{W}, X \in \mathcal{D}, Y \in \mathcal{E}\}$ in $\mathcal{O}p(p)$ for the 3–distribution $\mathcal{E}$.

Definition 30. *The Engel energy of the 2–distribution $\mathcal{D}$ at the point $p \in M$ is*

$$\mathcal{H}(\mathcal{D})(p) = \langle \mathcal{L}_W X, Y \rangle.$$

The Engel energy is a globally defined quantity that, up to constants, captures the derivative of the angular function. In particular:

Lemma 31. *Let $(M; \mathcal{W}, \mathcal{D}, \mathcal{E})$ be a formal Engel structure with $(\mathcal{W}, \mathcal{E})$ even–contact. Then $\mathcal{H}(\mathcal{D})(p) > 0$ if and only if $\mathcal{D}$ is Engel at p and $[\mathcal{D}, \mathcal{D}] = \mathcal{E}$ as oriented distributions.*

For a closed domain $U \subseteq M^4$, a chart $\phi : U \longrightarrow \mathbb{D}^3 \times [0,1]$ is said to be adapted if $\phi_* \mathcal{W} = \langle \partial_t \rangle$; the charts associated to the triangulation provided by Proposition 29 are adapted. Then, the Engel energy can be described in terms of the local angular functions introduced in Section 3:

Lemma 32. *Fix the pair $(\mathcal{W},\mathcal{E})$. Let $\phi : U \subset M \longrightarrow \mathbb{D}^3 \times [0,1]$ be an adapted chart. Let ξ be the oriented contact structure in $\mathbb{D}^3$ induced by $\phi_*\mathcal{E}$. Fix a positively oriented framing for ξ.*

Then, there exists a positive constant C_ϕ such that

$$\frac{1}{C_\phi}\cdot \mathcal{H}(\mathcal{D})(\phi^{-1}(p,t)) < \partial_t c(\phi_*\mathcal{D})(p,t) < C_\phi \cdot \mathcal{H}(\mathcal{D})(\phi^{-1}(p,t))$$

for any 2–plane $\mathcal{D}$ satisfying $\mathcal{W}\subset\mathcal{D}\subset\mathcal{E}$. □

The constant C_ϕ depends on ϕ and on the framing chosen for ξ but not on $\mathcal{D}$. This concludes the discussion on Engel energy, which is used in the forthcoming proof of Theorem 28.

4.3. **Proof of Theorem 28.** Recall $(\mathcal{W}_0,\mathcal{D}_0,\mathcal{E}_0)$ and $K \in \mathbb{R}^+$ from the statement. Using Theorem 6 we homotope the pair $(\mathcal{W}_0,\mathcal{E}_0)$ into an even–contact structure; $\mathcal{D}_0$ can be taken along during the homotopy. Proposition 29 applied to $\mathcal{W}_0$ provides a triangulation Σ and a covering by flowboxes $\{S(\sigma)\}_{\sigma\in\Sigma}$

The next step is homotoping $(\mathcal{W}_0,\mathcal{D}_0,\mathcal{E}_0)$ into a formal Engel structure $(\mathcal{W}_1 = \mathcal{W}_0,\mathcal{D}_1,\mathcal{E}_1 = \mathcal{E}_0)$ with large Engel energy close to the 3–skeleton $\Sigma^{(3)}$; this geometrically translates into the Legendrian vector field rotating sufficiently fast. This is achieved by creating Engel energy inductively on the skeleta of the triangulation Σ.

Engel energy in the lower skeleta. Let $j = 0,1,2,3$ and denote:

$$S_j := \bigcup_{\sigma\in\Sigma^{(j)}} S(\sigma).$$

Assume that $\mathcal{D}_0$ has already been deformed on S_{j-1} to yield an Engel structure $\mathcal{D}$ satisfying $\mathcal{H}(\mathcal{D})|_{S_{j-1}} > K_0$. K_0 is a positive constant that can be assumed to be arbitrarily large. We will now show that a similar bound holds over S_j after further homotoping $\mathcal{D}$. In all these homotopies it holds that $\mathcal{W}_0\subset\mathcal{D}\subset\mathcal{E}_0$.

For each j–simplex σ, we thicken $S(\sigma)$ into a slightly bigger flowbox and we consider an adapted chart $\phi(\sigma)$ on this thickening, which identifies it with the 4–disk

$$\mathbb{D}^3_{1+\varepsilon}\times[-\varepsilon,1+\varepsilon],$$

for some small $\varepsilon\in\mathbb{R}^+$, and identifies $S(\sigma)$ with the 4–subdisk $\mathbb{D}^3\times[0,1]$.

Consider the image through $\phi(\sigma)$ of the finite union $\cup_{\tau\subsetneq\sigma}S(\tau)$. Property (IV.) of the triangulation Σ implies that this closed set can be described as $A\times[-\varepsilon,1+\varepsilon]$, for some closed set A, if the thickening is small enough. The hypothesis $\mathcal{H}(\mathcal{D})|_{S_{j-1}} > K_0$ translates into the inequality

$$\partial_t c(\phi(\sigma)_*\mathcal{D})|_{A\times[-\varepsilon,1+\varepsilon]} > \frac{K_0}{C_{\phi(\sigma)}},$$

where $C_{\phi(\sigma)}$ is the constant provided by Lemma 32 when applied to the chart $\phi(\sigma)$.

Consider a function $h : \mathbb{D}^3_{1+\varepsilon}\times[-\varepsilon,1+\varepsilon]\longrightarrow\mathbb{R}$ such that

$$h|_{A\times[-\varepsilon,1+\varepsilon]} = \partial_t c(\phi(\sigma)_*\mathcal{D})|_{A\times[-\varepsilon,1+\varepsilon]}, \qquad \text{and } h > \frac{K_0}{C_{\phi(\sigma)}}.$$

The linear interpolation between $c(\phi(\sigma)_*\mathcal{D})$ and h serves now as the required deformation of $\mathcal{D}$. In detail, consider a cut–off function $\beta : \mathbb{D}^3_{1+\varepsilon}\times[-\varepsilon,1+\varepsilon]\to[0,1]$ such that

$$\beta|_{\mathbb{D}^3\times[0,1]}\equiv 1, \quad \beta|_{\mathcal{O}p(\partial(\mathbb{D}^3_{1+\varepsilon}\times[-\varepsilon,1+\varepsilon]))}\equiv 0,$$

and the angular function $d : \mathbb{D}^3_{1+\varepsilon}\times[-\varepsilon,1+\varepsilon]\to\mathbb{R}$ defined as the linear interpolation

$$d(p,t) = (1-\beta(p,t))c(\phi(\sigma)_*\mathcal{D})(p,t) + \beta(p,t)\left(c(\phi(\sigma)_*\mathcal{D})(p,0) + \int_0^t h(p,t)dt\right).$$

The two angular functions $c(\phi(\sigma)_*\mathcal{D})$ and d are isotopic relative to the boundary and $A\times[-\varepsilon,1+\varepsilon]$. Hence, d induces a deformation of the 2–distribution $\mathcal{D}$ through structures contained in $\mathcal{E}_0$ and transverse to $\mathcal{W}_0$, relative to S_{j-1}. By applying this deformation to each j–simplex $\sigma \in \Sigma^{(j)}$, $\mathcal{D}$ can be assumed to satisfy

$$\mathcal{H}(\mathcal{D})|_{S_j} > \frac{K_0}{C^2_{\phi(\sigma)}}.$$

Since K_0 was an arbitrary constant and there are a finite number of simplices, the inductive character of the argument implies that we can assume, after deforming $\mathcal{D}$, that $\mathcal{H}(\mathcal{D})|_{S_3}$ is arbitrarily large. □

This provides a deformation satisfying Property (I.) in the statement of Theorem 28. The second step in the proof of Theorem 28 is thus to translate the bound on the Engel energy on S_3 into a K–radial shell model for the 4–cells. Note that the constant $K \in \mathbb{R}^+$ is given in the statement, whereas the constant $K_0 \in \mathbb{R}^+$ in the previous argument can be chosen arbitrarily.

Engel energy in the 4-cells. Consider a 4–simplex $\sigma \in \Sigma^{(4)}$ and a constant $K_0 \in \mathbb{R}^+$. Let $(\mathcal{W}_0, \mathcal{E}_0)$ be an even–contact structure. Let $\mathcal{D}$ be a 2–plane field with $\mathcal{W}_0 \subset \mathcal{D} \subset \mathcal{E}_0$ such that $\mathcal{H}(\mathcal{D})|_{S_3} > K_0$. Such an $\mathcal{D}$ exists by the previous stages of the proof.

Property (I.) of the triangulation Σ ensures that $\partial\sigma \subseteq \cup_{\tau \subsetneq \sigma} S(\tau)$, which implies

$$\partial_t c(\phi(\sigma)_*\mathcal{D})|_{\partial\mathbb{D}^3} > \frac{K_0}{C_{\phi(\sigma)}}.$$

Choose the constant $K_0 \in \mathbb{R}^+$ satisfying

$$\frac{K_0}{\max_{\sigma \in \Sigma^{(4)}} C_{\phi(\sigma)}} > K.$$

Since the number of 4–cells is finite, such a constant K_0 exists. This implies the inequality $c(\phi(\sigma)_*\mathcal{D})|_{\partial\mathbb{D}^3} > K$ for the angular function and we can then apply Corollary 17 to obtain a deformation into a K–radial shell. This concludes the proof of Theorem 28. □

5. Proof of Theorem 1 and its corollaries

In this section we first detail the proofs of Theorems 1 and 3, and then deduce Corollary 2.

5.1. **Proof of Theorem 1 and Theorem 3.**

The π_0–statement of Theorem 1, that is, *every formal Engel structure can be deformed through formal Engel structures to an Engel structure*, is a consequence of the reduction result Theorem 28 and the extension result Theorem 18. Let us introduce the appropriate language for the parametric analogues of these results.

Consider a $\mathbb{S}^k$–family of formal foliated Engel structures $(\mathcal{W}_x, \mathcal{D}_x, \mathcal{E}_x)$, $x \in \mathbb{S}^k$, in a smooth foliated manifold $(M^{m+4}, \mathcal{F}^4)$. The Cartesian product manifold $W = M \times \mathbb{S}^k$ is endowed with the product foliation $\mathcal{F}_W = \coprod_{x\in\mathbb{S}^k} \mathcal{F} \times \{x\}$ and then the family $\{(\mathcal{W}_x, \mathcal{D}_x, \mathcal{E}_x)\}_{x\in\mathbb{S}^k}$ can be understood as a formal foliated Engel structure $(\mathcal{W}, \mathcal{D}, \mathcal{E})$ in the foliated manifold $(W^{4+m+k}, \mathcal{F}^4_W)$. Homotoping this formal Engel flag to a genuine Engel flag amounts to deforming the original family of formal foliated Engel structures to a family of genuine foliated Engel structures.

In consequence, the π_0–surjectivity of Theorem 3 applied to the formal foliated Engel structure $(W^{m+4+k}, \mathcal{F}^4_W, \mathcal{W}, \mathcal{D}, \mathcal{E})$ implies the higher π_k–surjectivity for the formal foliated Engel structure $(M^{m+4}, \mathcal{F}^4, \mathcal{W}, \mathcal{D}, \mathcal{E})$. Note that the statement of Theorem 3 in the case $m = 0$ implies Theorem 1, and thus it suffices to discuss the proof of Theorem 3.

The two central ingredients in the proof for the π_0–surjectivity in Theorem 3 are Theorem 28 and Theorem 18 (in order of application). Let us discuss their parametric counterparts; the definitions of Engel, angular, and K–radial shells can be generalized to the foliated case:

Definition 33. *A formal foliated Engel structure* $(\mathbb{D}^3 \times [0,1] \times \mathbb{D}^m, \coprod_{x\in\mathbb{D}^m} \mathbb{D}^3 \times [0,1] \times \{x\}; \mathcal{W}, \mathcal{D}, \mathcal{E})$ *is said to be a foliated Engel* (*angular or K–radial*) *shell if:*

I. $(\mathbb{D}^3 \times [0,1] \times \{x\}, \mathcal{W}, \mathcal{D}, \mathcal{E})$ *if an Engel* (*angular or K–radial*) *shell for all* $x \in \mathbb{D}^m$,
II. $(\mathbb{D}^3 \times [0,1] \times \{x\}, \mathcal{W}, \mathcal{D}, \mathcal{E})$ *is solid for* $x \in \mathcal{O}p(\partial\mathbb{D}^m)$.

A foliated Engel shell is solid if its formal foliated Engel structure is a foliated Engel structure.

Note that the parameter space in these foliated definitions is the m–disk $\mathbb{D}^m$. The parametric generalization of the reduction result Theorem 28 can be stated as follows:

Theorem 34. *Let $(W^{4+m}, \mathcal{F}^4; \mathcal{W}_0, \mathcal{D}_0, \mathcal{E}_0)$ be a formal foliated Engel structure and $K \in \mathbb{R}^+$ a constant. Then there exists a homotopy of formal foliated Engel structures $(\mathcal{W}_t, \mathcal{D}_t, \mathcal{E}_t)$, $t \in [0,1]$, and a collection of $(4+m)$–disks $B_1, \ldots, B_p \subseteq W$ such that:*

1. *$(\mathcal{W}_1, \mathcal{D}_1, \mathcal{E}_1)$ is a foliated Engel structure in the complement of $W \setminus \bigcup_{i=1}^p B_i$.*
2. *For each $i \in \{1, \ldots, p\}$, $(B_i, \mathcal{F}|_{B_i}; \mathcal{W}_1, \mathcal{D}_1, \mathcal{E}_1)$ is a foliated K–radial shell.*

Proof. Theorem 6 provides a deformation of the formal foliated Engel structure $(\mathcal{W}_0, \mathcal{D}_0, \mathcal{E}_0)$ into a formal foliated Engel structure such that $\mathcal{E}$ is a leafwise even–contact structure and $\mathcal{W}$ is its leafwise kernel. Proposition 29 applied to the pair $(W, \mathcal{W})$ provides a triangulation Σ and an associated cover by sets $\{S(\sigma)\}_{\sigma\in\Sigma}$ such that the closed neighbourhoods $S(\sigma)$ are of the form $\mathbb{D}^3 \times [0,1] \times \mathbb{D}^m$, and are at the same time flowboxes for the line field $\mathcal{W}$ and foliated charts for the foliation $\mathcal{F}$. This can be achieved by requiring in its proof that we first follow the exponential flow in the leaf and then in the ambient manifold.

The proof for the non–parametric case works verbatim by observing that in each closed neighborhood $S(\sigma)$, the angular functions of the leafwise Engel structures can be described by a smooth function

$$c(p,t,x) : \mathbb{D}^3 \times [0,1] \times \mathbb{D}^m \longrightarrow \mathbb{R}$$

to which the deformations in the non–parametric Theorem 28 can be applied. □

The foliated generalization of the extension result Theorem 18 reads as follows:

Theorem 35. *A foliated 6π–radial shell is homotopic through foliated Engel shells to a solid foliated Engel shell.*

Proof. Since all the 6π–radial shells have the same model in the interval $t \in [\rho, 2\rho]$, the construction in Theorem 18 can be applied without introducing additional parameters and we obtain Engel shells with four–leaf clover curves in the interval $t \in [\rho, 2\rho]$. □

Theorem 34 and Theorem 35 imply the π_0–statement of Theorem 3, which suffices to prove Theorem 1 and the remaining π_k–surjectivity in Theorem 3. □

5.2. **Proof of Corollary 2.** This cobordism statement requires a relative version of the reduction Theorem 28. Once a relative reduction is performed, Theorem 18 implies the statement. Let us explain the relative reduction.

Consider a collar neighborhood $\mathcal{O}p(\partial M) \cong \partial M \times [0,1)$ and thicken the filling M to

$$\overline{M} := M \cup_{\partial M \times \{0\}} \partial M \times [-\varepsilon, 0];$$

this allows us to modify the formal Engel structure in $\mathcal{O}p(\partial M \times \{0\})$ as an interior open set of the manifold $\overline{M}$. Triangulate ∂M and extend this triangulation to the interior of M. Proposition 29 also holds restricted to triangulations of this form, because the simplices contained in the boundary ∂M are already transverse to the field $\mathcal{W}$ and Thurston's Jiggling Lemma has a relative character. This provides suitable neighbourhoods $S(\sigma) \subseteq \overline{M}$ for each simplex σ of the triangulation. Then the rest of the proof of Theorem 28 goes through and provides an Engel structure in a neighborhood $\mathcal{O}p(M) \subseteq \overline{M}$. By construction, the Engel structure close to ∂M is still an angular model that induces the given contact structure. □

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