FLEXIBILITY FOR TANGENT AND TRANSVERSE IMMERSIONS IN ENGEL MANIFOLDS

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Abstract. We study the space of immersions of $\mathbb{S}^1$ that are tangent to an Engel structure $\mathcal{D}$. We show that the full h–principle holds as soon as one excludes the closed orbits of $\mathcal{W}$, the characteristic foliation of $\mathcal{D}$. This is sharp: we elaborate on work of Bryant and Hsu to show that curves tangent to $\mathcal{W}$ sometimes form additional isolated components that cannot be detected at a formal level. We then show that this is an exceptional phenomenon: if $\mathcal{D}$ is C^∞–generic, curves tangent to $\mathcal{W}$ are not isolated anymore.

These results, in conjunction with an argument due to M. Gromov, prove that a full h–principle holds for immersions transverse to the Engel structure.

1. History of the problem and outline of the article

Let ξ be a bracket–generating distribution. In [8, p. 84], M. Gromov posed the following two exercises for the reader:

a. The sheaf of *tangent* immersions $\mathbb{R} \to (M, \xi)$ is microflexible.
b. For any manifold N, the sheaf of *transverse* maps $N \to (M, \xi)$ is flexible.

R. Bryant and L. Hsu disproved Statement (a.) in [2]: They were able to show that rigid tangent segments – segments that, relative to their ends, are isolated in the C^1 topology – exist in maximally non–integrable 2–distributions in dimension 4 and onwards.

Bryant and Hsu's result heavily suggests that a complete h–principle cannot possibly hold for immersions tangent to an Engel structure $\mathcal{D}$. In Subsection 2.6 we elaborate slightly on their work to show that indeed this is the case. Then, we prove Theorem 19: the C^0–dense, parametric, and relative h–principle can be salvaged by restricting to those immersions that are somewhere not tangent to the characteristic foliation $\mathcal{W}$. A key ingredient in the proof is Theorem 17, where we show that generic families of tangent curves not everywhere tangent to $\mathcal{W}$ are in general position with respect to $\mathcal{W}$. In Section 4 we discuss deformations of curves tangent to $\mathcal{W}$ and we show that, if $\mathcal{D}$ is C^∞–generic, Theorem 19 can be strengthened to a full h–principle at the π_0 level, see Theorem 27.

One of the corollaries of Theorem 19 is that horizontal embeddings that are not orbits of $\mathcal{W}$ are purely classified by the rotation number and the homotopy class they represent as smooth maps. This was already known for the case of standard Engel $\mathbb{R}^4$ by work of Adachi [1] and Geiges [7]. Recently, elaborating on some of the techniques presented on this article, the first author and R. Casals have proven a full h–principle for horizontal embeddings that are somewhere not tangent to $\mathcal{W}$ [3].

In Section 5, we show that transverse maps and immersions satisfy the h–principle in the Engel case, as in Statement (b.), by following the argument outlined by Gromov in [8, p. 84]. Indeed, his argument is correct once one uses our Theorem 19 as a replacement for Statement (a.) above.

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2. Preliminaries

Throughout the article we will use Gromov's notation $\mathcal{O}p(A)$ to denote a neighbourhood of the set A of unspecified but sufficiently small size for our arguments to go through.

2.1. Elementary facts about Engel structures.

Definition 1. *Let M be a smooth closed 4–manifold and let $\mathcal{D} \subset TM$ be a smooth 2–distribution. $\mathcal{D}$ is said to be* Engel *if it is maximally non–integrable, that is, $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ is a 3–distribution and $[\mathcal{E}, \mathcal{E}] = TM$. The pair $(M, \mathcal{D})$ is said to be an* Engel manifold*.*

$\mathcal{E}$ is called the associated even–contact structure*. The line field $\mathcal{W} \subset \mathcal{D}$ uniquely defined by $[\mathcal{W}, \mathcal{E}] \subset \mathcal{E}$ is called the* kernel *or* characteristic foliation *of the Engel structure. The flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$ is called the* Engel flag *associated to $\mathcal{D}$.*

$\mathcal{E}$ is orientable and comes with a canonical orientation due to the identification $\mathcal{E} = \langle X, Y, [X, Y]\rangle$, where $\{X, Y\}$ is a local framing for $\mathcal{D}$; this does not depend on the choice of framing. Similarly, the non–integrability of $\mathcal{E}$ implies that there is a bundle isomorphism $\det(\mathcal{E}/\mathcal{W}) \cong TM/\mathcal{E}$ given by Lie bracket. Using this isomorphism, one can then readily show that orienting the line field $\mathcal{W}$ amounts to orienting the manifold M:

$$\det(TM) = \det(\mathcal{E}) \otimes TM/\mathcal{E} = \mathcal{W} \otimes \det(\mathcal{E}/\mathcal{W}) \otimes TM/\mathcal{E} = \mathcal{W}.$$

As such, if $\mathcal{W}$ and $\mathcal{D}$ are orientable, we may choose orientations and regard the Engel flag as a parallelisation of M up to homotopy.

A relevant feature of Engel structures is that they possess a local model [6]:

Proposition 2 (F. Engel)**.** *Let $(M_i, \mathcal{D}_i)$, $i = 1, 2$, be two Engel manifolds. Let $p_i \in M_i$, $i = 1, 2$. Then there are neighbourhoods $U_i \ni p_i$ and a diffeomorphism $\phi : U_1 \to U_2$ such that $\phi_*\mathcal{D}_1 = \mathcal{D}_2$.*

Using the Proposition, we deduce that the following two local models can be found around any given point in an Engel manifold:

Example 3. Consider $M = \mathbb{R}^4$ with coordinates (x, y, z, t). The 2–distribution $\mathcal{D}_{\mathrm{std}} = \ker(dy - zdx) \cap \ker(dz - tdx)$ is sometimes called the *standard Engel structure* or the *Darboux model*. Regard x as a variable, y as a function on x, and z and t as its first and second derivatives, respectively. The structure equations of $\mathcal{D}_{\mathrm{std}}$ precisely encode this differential relation between y, z, and t and therefore $\mathcal{D}_{\mathrm{std}}$ is tangent to any curve of the form $(x, y(x), y'(x), y''(x))$. Because of this, we say that $\mathcal{D}_{\mathrm{std}}$ is the *tautological distribution* in the jet space $J^2(\mathbb{R}, \mathbb{R}) \cong \mathbb{R}^4$.

Note that $\mathcal{W}$ is spanned by ∂_t. The 3-distribution $\mathcal{E} = [\mathcal{D}_{\mathrm{std}}, \mathcal{D}_{\mathrm{std}}]$ induces on every slice $\mathbb{R}^3 \times \{t_0\}$ the contact structure $\ker(dy - zdx)$. □

Example 4. Take $M = \mathbb{R}^4$ with coordinates (x, y, z, t) again. The 2–distribution

$$\mathcal{D}_{\mathrm{lorentz}} = \ker(dy - tdx) \cap \ker(dz - t^2dx) = \langle \partial_t, \partial_x + t\partial_y + t^2\partial_z\rangle$$

is an Engel structure, which we call the *Lorentzian model*. $\mathcal{W}$ is spanned by $\partial_x + t\partial_y + t^2\partial_z$. □

We denote by $\mathfrak{Engel}(M)$ the space of Engel structures in M endowed with the C^2–topology. The following construction first appeared in the work of E. Cartan; it provided the first examples of Engel structures in closed manifolds:

Example 5. Fix a contact manifold (M^3, ξ). Denote the projectivisation of the bundle ξ as $\pi : \mathbb{P}(\xi) \to M$. Define the *Cartan prolongation* (of one projective turn) as the manifold $\mathbb{P}(\xi)$ equipped with the 2–distribution

$$\mathcal{D}(\xi)(p, L) = (d\pi_{(p,L)})^{-1}(L),$$

where p is a point in M, L is a line in ξ_p, and thus (p, L) is a point in $\mathbb{P}(\xi)$. $\mathcal{D}(\xi)$ is an Engel structure with kernel $\mathcal{W}(\xi) = \ker(d\pi)$. The even–contact structure associated to it is $(d\pi)^{-1}(\xi)$. Observe that the Darboux model from Example 3 can be understood as the Cartan prolongation of standard contact $\mathbb{R}^3$ where $\mathbb{R}^3 \times \{\infty\}$ has been removed. $\square$

More recently, T. Vogel [12] showed that $\mathfrak{Engel}(M)$ is non–empty as soon as M is parallelisable. Further, the existence h–principle in [4] states that any given homotopy class of complete flags can be realised by an Engel structure.

2.2. **Horizontal immersions.** Recall that, due to the maximal non–integrability, all submanifolds tangent to an Engel structure have dimension at most 1. This motivates our interest in the following definition:

Definition 6. *Let $(M, \mathcal{D})$ be an Engel manifold. An immersion $\gamma : \mathbb{S}^1 \to M$ is said to be* horizontal *if $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in \mathbb{S}^1$.*

When γ is an embedding, we say that it is an Engel knot *or a* horizontal knot.

Following the h–principle philosophy, one can understand an immersion as two separate maps, the map itself and its derivative, that are coupled together. Decoupling this relation leads to the definition of *formal immersion*: a pair (γ, F) with a smooth map $\gamma : \mathbb{S}^1 \to M$ and $F : T\mathbb{S}^1 \to \gamma^*TM$ a monomorphism covering it. We can proceed analogously in the horizontal setting:

Definition 7. *Let $(M, \mathcal{D})$ be an Engel manifold. A* formal horizontal immersion *is a pair (γ, F) satisfying:*

1. *$\gamma : \mathbb{S}^1 \to M$ is a smooth map,*
2. *$F : \mathbb{S}^1 \to \mathcal{D}|_\gamma$ is a non–vanishing section satisfying $F(s) \in \mathcal{D}_{\gamma(s)} \subset T_{\gamma(s)}M$.*

We write $\mathcal{I}$ for the space of immersions of $\mathbb{S}^1$ into M. Its formal counterpart is $\mathcal{FI}$, the space of formal immersions. There is a natural inclusion $\mathcal{I} \to \mathcal{FI}$. Under this map the C^0-topology in $\mathcal{FI}$ induces the C^1–topology in $\mathcal{I}$, which is the natural one because the immersion condition is a differential condition of order 1.

We denote by $\mathcal{HI}(\mathcal{D}) \subset \mathcal{I}$ and $\mathcal{FHI}(\mathcal{D}) \subset \mathcal{FI}$ the subspaces of horizontal and formal horizontal immersions, respectively. As before, there is a continuous inclusion $\mathcal{HI}(\mathcal{D}) \to \mathcal{FHI}(\mathcal{D})$. Since these definitions make sense as well for immersions of the interval, we define $\mathcal{I}([0,1])$, $\mathcal{FI}([0,1])$, $\mathcal{HI}([0,1], \mathcal{D})$, and $\mathcal{FHI}([0,1], \mathcal{D})$ analogously.

Remark 8. Observe that there is a forgetful map $\mathcal{FHI}(\mathcal{D}) \to \mathfrak{Maps}(\mathbb{S}^1, M)$. The fibre over $\gamma \in \mathfrak{Maps}(\mathbb{S}^1, M)$ is $\mathrm{Mon}(T\mathbb{S}^1, \mathcal{D}|_\gamma)$, where Mon denotes bundle monomorphisms. In particular, depending on whether $\mathcal{D}$ is orientable or not over γ, this corresponds to $\mathrm{Mon}_{\mathbb{S}^1}(\mathbb{R}, \mathbb{R}^2)$ or $\mathrm{Mon}_{\mathbb{S}^1}(\mathbb{R}, \mathbb{R} \oplus L)$, with L the non orientable line bundle over the circle. In any case, the map is indeed a locally trivial fibration.

If $\mathcal{D}$ is orientable and a global framing has been chosen, it follows that:

$$\pi_0(\mathcal{FHI}(\mathcal{D})) \cong \pi_0(\mathfrak{Maps}(\mathbb{S}^1, M)) \times \mathbb{Z}$$

where the integer in $\mathbb{Z} \cong \pi_0(\mathfrak{Maps}(\mathbb{S}^1, \mathbb{S}(\mathcal{D}) \cong \mathbb{S}^1))$ represents the turning of F with respect to the framing. This integer is usually called the *rotation number* of the formal immersion.

Similarly, if we fix some $(\gamma, F) \in \mathcal{FHI}(\mathcal{D})$:

$$\pi_1(\mathcal{FHI}(\mathcal{D}), (\gamma, F)) \cong \pi_1(\mathfrak{Maps}(\mathbb{S}^1, M), \gamma) \times \mathbb{Z}.$$

The integer on the right hand side is computed as follows: given any loop $(\gamma_\delta, F_\delta)_{\delta \in \mathbb{S}^1}$ in $\mathcal{FHI}(\mathcal{D})$, it measures the turning of $\delta \to F_\delta(0)$ with respect to the framing of $\mathcal{D}$. This realises the isomorphism $\mathbb{Z} \cong \pi_1(\mathfrak{Maps}(\mathbb{S}^1, \mathbb{S}(\mathcal{D}) \cong \mathbb{S}^1), F)$.

Finally, reasoning similarly and using the fact that $\pi_j(\mathfrak{Maps}(\mathbb{S}^1, \mathbb{S}(\mathcal{D}) \cong \mathbb{S}^1), F) = 0$ for $j > 1$:

$$\pi_j(\mathcal{FHI}(\mathcal{D}), (\gamma, F)) \cong \pi_j(\mathfrak{Maps}(\mathbb{S}^1, M), \gamma) \text{ for } j > 1.$$

□

The aim of this article is to understand the nature of the inclusion $\mathcal{HI}(\mathcal{D}) \to \mathcal{FHI}(\mathcal{D})$. If it is a weak homotopy equivalence, we say the the *h-principle* holds for horizontal immersions. In Subsection 2.6 we will strengthen a result of Bryant and Hsu that will play a central role in proving that this cannot possibly hold. In Section 3 we will restrict our attention to a subset of $\mathcal{HI}(\mathcal{D})$ for which the inclusion into $\mathcal{FHI}(\mathcal{D})$ is indeed a weak homotopy equivalence.

2.3. **The development map.** Let $(M, \mathcal{D})$ be an Engel manifold. Assume that the Engel flag is oriented and fix a parallelisation $\{W \in \mathcal{W}, X \in \mathcal{D}, Y \in \mathcal{E}, Z\}$. We now introduce the notion of development map [10], which measures the "turning" of $\mathcal{D}$ with respect to W as we follow a W–orbit.

Fix a point $p \in M$. Let ϕ be an embedding of $\mathbb{D}^3$ transverse to $\mathcal{W}$ and satisfying $\phi(0) = p$. Transversality with respect to $\mathcal{W}$ implies that $\xi_0 = \phi^*\mathcal{E}$ is a contact structure containing the line field $\phi^*\mathcal{D}$. The parallelisation identifies $\xi_0(0)$ with $\mathbb{R}^2$ and $(\phi^*\mathcal{D})(0)$ with the oriented line $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$. We extend the map ϕ to an immersion

$$\Psi_p : \mathbb{D}^3 \times \mathbb{R} \to M$$

$$\Psi_p(x, y, z, t) = \varphi_{\phi(x,y,z)}(t),$$

where $\varphi_q(t)$ is the time–t flow of W starting at the point $q \in M$. Since φ_q preserves $\mathcal{E}$, the contact structure induced by $\Psi_p^*\mathcal{E}$ on every slice $\mathbb{D}^3 \times \{t\}$ is precisely ξ_0. This implies that $(\Psi_p^*\mathcal{D})|_{\mathbb{D}^3 \times \{t\}}$ is a line field contained in ξ_0.

Definition 9. *The map*

$$\begin{array}{rcl} \hat{p} : \mathbb{R} & \to & \mathbb{RP}^1 \cong \mathbb{P}(\xi_0(0)) \\ t & \to & \mathbb{P}((\Psi_p^*\mathcal{D})|_{\mathbb{D}^3 \times \{t\}}(0)) \end{array}$$

is called the development map *at p.*

The development map only depends on the parallelisation we chose. Thus, any parametrised curve $\gamma : [0, 1] \to M$ tangent to $\mathcal{W}$ has a well–defined development map:

$$\hat{\gamma} : [0, 1] \to \mathbb{RP}^1$$

$$\hat{\gamma}(s) = \widehat{\gamma(0)}(\varphi^{-1}_{\gamma(0)}(\gamma(s))).$$

In particular, we will say that two points $\gamma(t_0)$ and $\gamma(t_1)$ differ by an integer number of (projective) turns if $\hat{\gamma}(t_0) = \hat{\gamma}(t_1)$. Observe that these are only half–turns if we analogously consider the development map into the *oriented sphere bundle* of the contact structure ξ_0.

2.4. **The Geiges projection.** Let us focus on horizontal immersions and embeddings into the standard Engel structure $(\mathbb{R}^4, \mathcal{D}_{\text{std}})$. The following result was proven by Adachi [1] and Geiges [7]:

Proposition 10. *Horizontal knots in $(\mathbb{R}^4, \mathcal{D}_{\text{std}})$ are classified, up to homotopy as embedded horizontal curves, by their rotation number with respect to $\mathcal{W}$.*

First of all, observe that in dimension 4 there is no smooth knotting of $\mathbb{S}^1$ and, mimicking the discussion in Remark 8, the only formal invariant is precisely the rotation number.

Let us outline Geiges' approach; note that our naming convention for the variables is different from his. Take coordinates (x, y, z, t) in $\mathbb{R}^4$, so $\mathcal{D}_{\text{std}}$ is given as the kernel of the 1–forms $\alpha = dy - zdx$ and $\beta = dz - tdx$. We say that the map $\pi_{\text{Geiges}} : (x, y, z, t) \to (x, z, t)$ is the *Geiges projection*. Horizontal immersions are transverse to the projection direction, so π_{Geiges} maps horizontal immersions to Legendrian immersions in $(\mathbb{R}^3(x, z, t), \ker(\beta))$ additionally satisfying the area constraint $\int_\gamma zdx = 0$. The rotation number of γ agrees precisely with the rotation of its Geiges projection.

The key observation is that, given two horizontal immersions, we can use the h–principle for Legendrian immersions to find a 1-parametric family $(\nu_t)_{t\in[0,1]}$ of Legendrian immersions interpolating between their Geiges projections ν_0 and ν_1. The curves ν_t do not bound area zero and therefore do not lift to closed horizontal curves. However, we can add a bump to the front projection of ν_t, parametrically in t, to achieve the area constraint. This gives the result for immersions.

Self-intersections of the lift of ν_t correspond to self-intersections of ν_t in which both branches of the front projection of ν_t bound area zero. It is easy to see that this is codimension 2 phenomenon, thus it does not show up for generic 1-parametric families, proving Proposition 10. □

2.5. **Local models and deformations of tangent curves.** Let $(M, \mathcal{D})$ be an Engel manifold and let $\gamma : [-1, 1] \to M$ be a horizontal immersion. Consider the following problem: to describe the deformations of γ, relative to its ends, as a horizontal curve.

Lemma 11. *Let γ be horizontal and everywhere transverse to $\mathcal{W}$. Then, there is a constant $\varepsilon > 0$, a tubular neighbourhood $\nu(\gamma)$, and a submersion:*

$$\phi : ([-1, 1] \times \mathbb{D}^2_\varepsilon \times [-\varepsilon, \varepsilon] \subset \mathbb{R}^4, \mathcal{D}_{\text{std}}) \to (\nu(\gamma) \subset M, \mathcal{D})$$

satisfying $\phi^*\mathcal{D} = \mathcal{D}_{\text{std}}$ and $\phi(x, 0, 0, 0) = \gamma(x)$.

In particular, the C^1–perturbations of γ relative to the ends are, in the model, of the form

$$x \to (x, y(x), y'(x), y''(x)),$$

with $y : [-1, 1] \to \mathbb{R}$ vanishing to order two at $x = \pm 1$.

Proof. Take two linearly independent vector fields Y, Z along γ such that the 3–distribution $\langle \gamma', Y, Z\rangle$ is a complement for $\mathcal{W}$. Use the exponential map on $\langle Y, Z\rangle$ along γ to find an immersion $\psi : [-1, 1] \times \mathbb{D}^2_\varepsilon \to M$. Since $\psi^*\mathcal{E}$ is a contact structure on this slice, with $\psi^{-1} \circ \gamma$ a Legendrian curve, the Legendrian neighbourhood theorem applies: there exists a small tubular neighbourhood of γ within the slice that is contactomorphic to $([-1, 1] \times \mathbb{D}^2_\varepsilon, \ker(dy - zdx))$ with γ corresponding to $y = z = 0$. We fix such an identification.

Fix a line field W spanning $\mathcal{W}$ and use it to flow the slice, yielding a submersion $\phi : [-1, 1] \times \mathbb{D}^2_\varepsilon \times [-\varepsilon, \varepsilon] \to M$ extending ψ; write t for the coordinate in $[-\varepsilon, \varepsilon]$. By construction, the Legendrian line field determined by $\psi^*\mathcal{D}$ on the slice $\{t = t_0\}$ is spanned by $\partial_x + f_{t_0}(x, y, z)\partial_y$, where f_t is a $[-\varepsilon, \varepsilon]$-family of functions vanishing at $t_0 = y = z = 0$. Due to the Engel condition, they satisfy $\partial_t f_t > 0$. Then, the implicit function theorem implies that we can reparametrise ϕ along the t-direction in a manner depending on (x, y, z) to ensure $\phi^*\mathcal{D} = \mathcal{D}_{\text{std}}$.

Any C^1-small deformation of $\phi^{-1} \circ \gamma(x) = (x, 0, 0, 0)$ is graphical over it and therefore of the form $x \to (x, y(x), z(x), w(x))$. In Example 3, we remarked that $\mathcal{D}_{\text{std}}$ is precisely the tautological structure in the space of 2-jets $J^2(\mathbb{R},\mathbb{R})$, so horizontality of the deformation implies that it is given by $x \to (x, y(x), y'(x), y''(x))$ for some function $y : [-1,1] \to \mathbb{R}$. If the deformation is relative to the ends, we must impose $y(\pm 1) = y'(\pm 1) = y''(\pm 1) = 0$. $\square$

If instead of curves transverse to the characteristic foliation we consider curves tangent to it, we can instead observe rigid behaviour:

Lemma 12. *Let γ be an integral curve of $\mathcal{W}$ with development map less than a turn. Then, there is a tubular neighbourhood $\nu(\gamma)$, an embedding $x : [-1,1] \to \mathbb{R} \subset \mathbb{R}^4$, $x(0) = 0$, with tubular neighbourhood $\nu(x)$, and a submersion:*
$$\phi : (\nu(x) \subset \mathbb{R}^4, \mathcal{D}_{\mathrm{lorentz}}) \to (\nu(\gamma) \subset M, \mathcal{D})$$
satisfying $\phi^*\mathcal{D} = \mathcal{D}_{\mathrm{lorentz}}$ and $\gamma(s) = \phi(x(s), 0, 0, 0)$.

Any C^1–perturbation of γ is given as $(x(s), y(x(s)), z(x(s)), t(x(s)))$ with
$$y(x) = y(0) + \int_0^x t(s)ds$$
$$z(x) = z(0) + \int_0^x t^2(s)ds$$
In particular, the expression for z implies that there are no deformations relative to the ends.

Proof. Let us do a preliminary computation. By using the flow of $\partial_x + t\partial_y + t^2\partial_z$, we can obtain the following change of coordinates:
$$\Psi : \mathbb{R}^4 \to \mathbb{R}^4$$
$$\Psi(x, y, z, t) = (x, y + tx, z + t^2x, t)$$
It is easy to check that it satisfies:
$$\Psi^*(\mathcal{D}_{\mathrm{lorentz}} = \ker(dy - tdx) \cap \ker(dz - t^2dx)) = \mathcal{D}_0 = \ker(dz - 2tdy) \cap \ker(dy + xdt)$$
which can be thought as a Cartan prolongation of one projective turn of the contact structure $\ker(dz - 2tdy)$ with the slice at infinity removed.

Find some disk transverse to $\mathcal{W}$ passing through $\gamma(0)$ and parametrise it so that the contact structure induced by $\mathcal{E}$ is precisely $\ker(dz - 2tdy)$. Then, the flow of a vector field spanning $\mathcal{W}$ yields an Engel submersion into $\nu(\gamma)$ whose domain is a tubular neighbourhood of some segment contained in $\{0\} \times \mathbb{R} \subset (\mathbb{R}^4, \mathcal{D}_0)$. The claim follows by precomposing this immersion with Ψ^{-1}. The claim regarding deformations is immediate from the model. $\square$

Remark 13. It follows from their proofs that Lemmas 11 and 12 hold parametrically. It is also clear that, if the curve γ was embedded, the models can be taken to be diffeomorphisms.

2.6. **A result of Bryant and Hsu.** According to Lemma 12, sufficiently short integral curves of $\mathcal{W}$ are isolated in the C^1–topology. This phenomenon was fully characterised by Bryant and Hsu:

Proposition 14 (Proposition 3.1 in [2])**.** *Let $(M, \mathcal{D})$ be an Engel manifold. The following two conditions are equivalent for a horizontal immersion $\gamma : [0,1] \to M$:*

- *all the C^1-perturbations of γ relative to its endpoints are reparametrisations,*
- *γ is everywhere tangent to $\mathcal{W}$ and its associated development map is injective away from its endpoints.*

We can slightly extend Bryant and Hsu's result to show that the closed orbits of $\mathcal{W}$ (which are horizontal immersions of the circle and not the interval as before) sometimes have a small space of C^1-deformations:

Proposition 15. *Let (M,ξ) be a contact 3-manifold. In the Cartan prolongation $(\mathbb{P}(\xi),\mathcal{D}(\xi))$ of one projective turn, the unparametrised embedded curves tangent to $\mathcal{W}(\xi)$ come in two connected components, each of which is diffeomorphic to M. Each connected component conforms a connected component of $\mathcal{HI}(\mathcal{D}(\xi))$.*

Proof. Everything is reduced to showing that the C^1–perturbations of an embedded closed orbit γ of $\mathcal{W}(\xi)$ are exactly the nearby closed orbits. Take some curve η that is C^1–close to γ. If it is tangent to a $\mathcal{W}(\xi)$–orbit $\tilde\gamma$ in an open interval $I\subset\mathbb{S}^1$, then $\eta|_{\mathbb{S}^1\setminus I}$ is a compactly supported deformation of $\tilde\gamma|_{\mathbb{S}^1\setminus I}$, which makes less than one projective turn. Applying Lemma 12 implies that η is a reparametrisation of $\tilde\gamma$.

Otherwise, take some time t_0 where η is transverse to $\mathcal{W}(\xi)$. Lemma 11 yields a neighbourhood U of $\eta(t_0)$. The vector field $\eta'(t)|_{[t_0-\delta,t_0+\delta]}$, for δ small, can be extended in U to a vector field X whose flowlines are C^1–close to η (and thus graphical over γ). Now, in $\mathcal{O}p(\eta(t_0))\subset U$, we perturb X to make it tangent to $\mathcal{W}(\xi)$. A suitable choice yields flowlines that are C^1–close to η in $\mathcal{O}p(\partial U)$, that are still graphical over γ, and that are tangent to $\mathcal{W}$ in an interval. Lemma 11 implies that we can take one such flowline and interpolate back to η in $\mathcal{O}p(\partial U)$ to yield a curve $\tilde\eta$ that is a C^1–small deformation of γ. The proof concludes by applying to $\tilde\eta$ the reasoning in the first paragraph. □

Using Proposition 15 and our main Theorem 19 it follows that the h-principle does not hold for the inclusion $\mathcal{HI}(\mathcal{D})\to\mathcal{FHI}(\mathcal{D})$. This is explained in Remark 23 below.

In more general Engel manifolds it is not always true that sufficiently short curves tangent to $\mathcal{W}$ are isolated as tangent immersions. This is explored in Section 4, where we also provide a different, more geometric, proof of Proposition 15.

3. h–principle for horizontal immersions

Proposition 15 motivates us to focus our attention in the following two classes of curves:

Definition 16. *We denote by $\mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})\subset\mathcal{HI}(\mathcal{D})$ the open subspace of those horizontal curves that are not everywhere tangent to $\mathcal{W}$.*

Similarly, we write $\mathcal{HI}^{\text{gen}}(\mathcal{D})\subset\mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})$ for the collection of horizontal curves whose set of $\mathcal{W}$–tangencies has empty interior.

Observe that $\mathcal{HI}^{\text{gen}}(\mathcal{D})$ is a nicer space than $\mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})$ in the sense that the curves it contains are defined by a local condition.

Using Lemma 11 we deduce that the space of curves in $\mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})$ up to reparametrisation is infinite dimensional (in the sense that space of deformations of each curve is infinite dimensional). In contrast to this, the subspace of closed orbits of $\mathcal{W}$, the complement of $\mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})$ in $\mathcal{HI}(\mathcal{D})$ (up to reparametrisation), has finite Hausdorff dimension. This implies that, when we study $\mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})$ instead of $\mathcal{HI}(\mathcal{D})$, we are discarding a very small subspace of horizontal curves.

3.1. A genericity result.

Our first theorem states that, once one restricts to the subspace $\mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})$, there are enough deformations to guarantee a "*generic*" picture:

Theorem 17. *Let M be a 4–manifold. Let K be a closed manifold and fix a map $\mathcal{D}:K\to\mathfrak{Engel}(M)$. Denote by $\mathcal{W}(k)$ the characteristic foliation of $\mathcal{D}(k)$, $k\in K$.*

Consider a family $\gamma:K\to\mathcal{I}(M)$ satisfying $\gamma(k)\in\mathcal{HI}^{\text{n.e.t.}}(\mathcal{D}(k))$. Then, there is a C^∞–small perturbation $\tilde\gamma$ of γ so that the set

$$\{(k,s)\in K\times\mathbb{S}^1\ ;\ \tilde\gamma(k)'(s)\in\mathcal{W}(k)_{\tilde\gamma(k)(s)}\}$$

is a submanifold of codimension 1 *in generic position with respect to the foliation* $\coprod_{k\in K}\{k\}\times\mathbb{S}^1$. *In particular,* $\tilde\gamma(k)$ *lies in* $\mathcal{HI}^{\rm gen}(\mathcal{D}(k))$.

We shall dedicate the rest of this subsection to its proof; it relies on local C^∞–small deformations using Lemmas 11 and 12. Before we do so, let us make a small remark: The set of tangencies with $\mathcal{W}$ is, by definition, closed. However, it might be very badly behaved and have non-empty interior; such an example is shown in Figure 1 in the second row from the bottom. Theorem 17 shows that such behaviour is not C^∞-generic, which is not clear a priori due to the horizontality condition.

Setup. Let us construct an adequate cover of the space $K\times\mathbb{S}^1$. Locally, for every $(k,s)\in K\times\mathbb{S}^1$, we can find vector fields $W(k',s)$ spanning $\mathcal{W}(k')$. Denote by $\eta(k',s)$ the integral curve of $W(k',s)$ with domain $[s-\varepsilon,s+\varepsilon]$ and satisfying $\eta(k',s)(s)=\gamma(k')(s)$.

We can apply Lemma 12 parametrically to the curves $\eta(k',s)$. If $\gamma(k)'(s)\in\mathcal{W}(k)$, there is a product neighbourhood $\mathbb{D}_\varepsilon(k)\times[s-\varepsilon,s+\varepsilon]\ni(k,s)$ in which every curve $\gamma(k')$, $k'\in\mathbb{D}_\varepsilon(k)$, is graphical over $\eta(k',s)$ in the model. We say that this neighbourhood is of type I.

Otherwise, if (k,s) is such that $\gamma(k)'(s)$ is transverse to $\mathcal{W}(k)$, so are the nearby curves. We use Lemma 11 parametrically to yield a product neighbourhood of (k,s) in which the curves $\gamma(k')$ look like the zero section in $J^2(\mathbb{R},\mathbb{R})$. We call this a neighbourhood of type II.

Then, by compactness of $K\times\mathbb{S}^1$, we can find a finite cover $\{U_{i,j}\}$ comprised of neighbourhoods like the ones we just described. We assume that it is the product of a covering $\{W_i\}$ in K and a covering $\{V_j=\mathcal{O}p([\frac{j}{N},\frac{j+1}{N}])\}$, $j=0,..,N-1$, in $\mathbb{S}^1$. We order the neighbourhoods $\{U_{i,j}=W_i\times V_j\}$ as follows: for any fixed W_i, we find some $j_i\in\{0,..N-1\}$ such that $W_i\times V_{j_i}$ is of type II and we order the $W_i\times V_j$ cyclically increasing from $j=j_i+1$ to $j=j_i$. The order in which we consider each W_i is not important and hence we just proceed as we numbered them. See Figure 1.

The idea now is to modify γ over the neighbourhoods $U_{i,j}$ inductively using the order we just constructed. Over those of type I we will deform to achieve the desired transversality. Over those of type II we have more flexibility, so we shall use them to ensure that the deformation $\tilde\gamma$ does close up.

Take a neighbourhood $U_{i,j}$. Denote by $\tilde U_{i,j}$ the union of the neighbourhoods over which a C^∞–close deformation $\tilde\gamma$ of γ has been defined already.

Type I neighbourhoods. Assume that $U_{i,j}$ is of type I. Applying Lemma 12, we have a family of curves

$$\gamma(k):V_j\to(\mathbb{R}^4,\ker(dy-tdx)\cap\ker(dz-t^2dx)),\qquad k\in W_i$$

that are graphical over the x axis and thus given by functions:

$$\gamma(k)(s)=(x_k(s),y_k(x_k),z_k(x_k),t_k(x_k))$$

with $x_k(s)$ a diffeomorphism with its image, $t_k(x)$ some arbitrary function, and

$$\left\{\begin{aligned} y_k(x_k(s)) &= y_k(x_k(-1))+\int_{x_k(-1)}^{x_k(s)} t_k(x)dx\\ z_k(x_k(s)) &= z_k(x_k(-1))+\int_{x_k(-1)}^{x_k(s)} t_k^2(x)dx\end{aligned}\right. \tag{1}$$

where the dependence on k is smooth. Analogously, $\tilde\gamma$ is defined by functions $(\tilde x_k,\tilde y_k,\tilde z_k,\tilde t_k)$ which are only defined over $U_{i,j}\cap\tilde U_{i,j}$.

Tangencies with $\mathcal{W}$ are given by $t_k',\tilde t_k'=0$. We extend $\tilde t_k$ from $U_{i,j}\cap\tilde U_{i,j}$ to the whole of $U_{i,j}$ arbitrarily, ensuring that it remains C^∞–close to t_k and that it has generic critical points (for a family of dimension $\dim(K)$). We can extend $\tilde y_k$ and $\tilde z_k$ to the whole of $U_{i,j}$ using the integral expressions (1) with initial values those in $U_{i,j}\cap\tilde U_{i,j}$. The order that we chose for the induction means that $U_{i,j}\cap\tilde U_{i,j}\cap(\{k\}\times\mathbb{S}^1)$ is connected at every such step, so in particular we are defining $\tilde y_k$ and $\tilde z_k$ as integrals with boundary

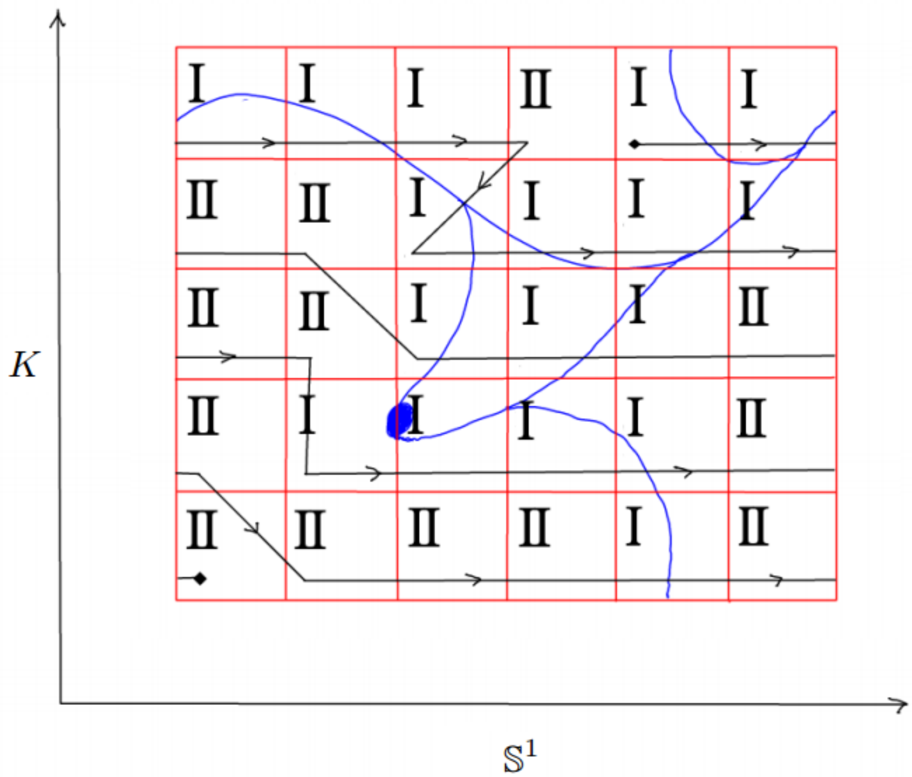

FIGURE 1. In red we depict the manifold $K \times \mathbb{S}^1$. The tangencies with $\mathcal{W}$ are the points in blue. The sets $U_{i,j}$ correspond to neighbourhoods of the smaller red squares; they have been numbered as I or II depending on whether they are of the first or second type. The black line with arrows indicates the order in which we proceed for the induction.

conditions given only at one end of the interval V_j. Note that this construction is indeed relative to $\tilde{U}_{i,j}$.

Type II neighbourhoods. Assume that $U_{i,j}$ is of type II. In its local model, given by Lemma 11, the perturbations $\tilde{\gamma}(k)$ (which are defined only over $\tilde{U}_{i,j}$) can be assumed to be graphical over $\gamma(k)$, which are themselves seen as subintervals of the zero section in $J^2(\mathbb{R},\mathbb{R})$. $\tilde{\gamma}$ is thus described by a family of functions $\tilde{y}_k$ and their first and second derivatives $\tilde{z}_k$ and $\tilde{t}_k$, respectively. Extend $\tilde{y}_k$ arbitrarily to $U_{i,j}$ while keeping it C^∞ close to y_k; take $\tilde{z}_k$ and $\tilde{t}_k$ to be the corresponding derivatives of $\tilde{y}_k$. This can be done regardless of the boundary conditions (this is the reason why the last step must be over a neighbourhood of type II). No additional tangencies with $\mathcal{W}$ are introduced doing this. $\square$

Remark 18. In type II neighbourhoods, after extending $\tilde{y}_k$, one could construct a bump function $\psi : U_{i,j} \to \mathbb{R}$ that is identically 1 near $\partial U_{i,j}$ and identically zero in a slightly smaller ball and then take $\psi\tilde{y}_k$ and its derivatives as the desired extensions to the whole of $U_{i,j}$. In this manner, by taking the cover to be fine enough, one can strengthen Theorem 17 saying that the deformation $\tilde{\gamma}$ can be taken to agree with γ in an arbitrarily large closed set disjoint from the tangencies.

3.2. **The main theorem.** Let us state our main result:

Theorem 19. *Let K be a manifold, possibly with boundary, and fix $\mathcal{D} : K \to \mathfrak{Engel}(M)$. Consider a family $\phi : K \to \mathcal{FI}(M)$ satisfying*

- *$\phi(k) \in \mathcal{HI}^{\text{n.e.t.}}(\mathcal{D}(k))$ for $k \in \partial K$,*

- *$\phi(k) \in \mathcal{FHI}(\mathcal{D}(k))$ for all k.*

Then, ϕ is homotopic, relative to ∂K, to a map $\tilde{\phi} : K \to \mathcal{I}(M)$ satisfying $\tilde{\phi}(k) \in \mathcal{HI}^{\text{n.e.t.}}(\mathcal{D}(k))$ for all k.

In particular, for a given Engel manifold $(M, \mathcal{D})$, the inclusions $\mathcal{HI}^{\text{gen}}(\mathcal{D}) \subset \mathcal{HI}^{\text{n.e.t.}}(\mathcal{D}) \subset \mathcal{FHI}(\mathcal{D})$ are weak homotopy equivalences.

Most of the work needed for the theorem is contained in the following proposition, which states that a parametric, relative in the domain (with respect to some subset B of the interval), relative in the parameter (with respect to some subset A of the parameter space K), C^0–close h–principle holds for horizontal immersions of the interval.

Proposition 20. *Let $(M = \mathbb{R}^3 \times (-\varepsilon, \varepsilon), \mathcal{D} = \mathcal{D}_{\text{std}})$. Let $A \subset \partial\mathbb{D}^m$ be a closed CW–complex. Let $B \subset [0,1]$ be either $\{0,1\}$, $\{0\}$ or the empty set. Fix a map $\psi_k \in \mathcal{FHI}([0,1], \mathcal{D})$, $k \in \mathbb{D}^m$, conforming to the following properties:*

- *$\psi_k \in \mathcal{HI}^{\text{n.e.t.}}([0,1], \mathcal{D})$ for $k \in \mathcal{O}p(A)$.*
- *ψ_k is horizontal with respect to $\mathcal{D}$ for $s \in \mathcal{O}p(B)$.*

Then, there is a homotopy $\psi_k^\delta \in \mathcal{FHI}([0,1], \mathcal{D})$, $\delta \in [0,1]$, starting at $\psi_k^0 = \psi_k$, such that:

- *$\psi_k^1 \in \mathcal{HI}^{\text{n.e.t.}}([0,1], \mathcal{D})$ for all $k \in \mathbb{D}^m$,*
- *$\psi_k^\delta = \psi_k$ for $k \in \mathcal{O}p(A)$ or $s \in \mathcal{O}p(B)$,*
- *γ_k^δ is C^0–close to γ_k^0, where $(\gamma_k^\delta, F_k^\delta)$ are the two components of ψ_k^δ.*

Let us explain how to deduce our main theorem using Proposition 20.

Proof of Theorem 19. Fix an open subdomain $\tilde{K} \subset K$ such that, for any k in its complement, $\phi(k) \in \mathcal{HI}^{\text{n.e.t.}}(\mathcal{D}(k))$. After applying Theorem 17, we can assume that the curves $\phi(k)$, $k \in \mathcal{O}p(\partial\tilde{K})$, are in general position with respect to $\mathcal{W}(k)$. In [13, Section 5], W. Thurston devised a method, which he called *jiggling*, to perform barycentric subdivision and C^0-perturb a triangulation until every simplex is transverse to a given distribution. In this manner we find a triangulation $\mathcal{T}$ of $(\tilde{K} \times \mathbb{S}^1, \partial\tilde{K} \times \mathbb{S}^1)$ that is in general position with respect to the foliation $\mathcal{F} = \coprod\{k\} \times \mathbb{S}^1$. Write π_K and $\pi_{\mathbb{S}^1}$ for the projections of $K \times \mathbb{S}^1$ to its factors.

We will proceed inductively on the dimension of the simplices of $\mathcal{T}$, deforming ϕ to achieve horizontality in a neighbourhood of each simplex σ. Since the triangulation $\mathcal{T}$ is transverse to $\mathcal{F}$, we can choose a small neighbourhood $\mathcal{U}(\sigma)$ of σ and a parametrisation:

$$g(\sigma) : \mathbb{D}^m \times [0,1] \to \mathcal{U}(\sigma) \subset K \times \mathbb{S}^1$$
$$g(\sigma)^*\langle\partial_s\rangle = \langle\partial_t\rangle$$

where t is the coordinate in $[0,1]$, and s is the coordinate in $\mathbb{S}^1$. Since $\mathcal{T}$ is arbitrarily fine, it can be assumed that each segment $s \to \phi(k)(s)$, with $(k,s) \in g(\sigma)(\mathbb{D}^m \times [0,1])$, maps into a Darboux ball B_k for $\mathcal{D}(k)$. We can parametrically identify the balls B_k with the k–independent Darboux ball $(M = \mathbb{R}^3 \times (-\varepsilon, \varepsilon), \mathcal{D} = \mathcal{D}_{\text{std}})$.

Our induction hypothesis is that ϕ has been modified close to the subsimplices of σ to be horizontal. The transversality hypothesis for $\mathcal{T}$ implies that the set $\cup_{\tau \subsetneq \sigma} g(\sigma)^{-1}(\mathcal{O}p(\tau))$ can be taken to be as in the statement: of the form $\mathcal{O}p(A) \times [0,1]$ if σ is not a top-simplex, and of the form $\mathcal{O}p(A) \times [0,1] \cup \mathbb{D}^m \times \mathcal{O}p(\{0,1\})$ if it is. Now we apply Proposition 20 to the map $\psi_k(s) = \phi(\pi_K \circ g(k,s))(\pi_{\mathbb{S}^1} \circ g(k,s))$, homotoping ϕ to a horizontal family in $\mathcal{U}(\sigma)$ and concluding the inductive step.

Do note that we can apply Theorem 17 to obtain the other weak homotopy equivalence $\mathcal{HI}^{\text{gen}}(\mathcal{D}) \subset \mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})$. □

Now we break down the proof of Proposition 20 in several steps. The strategy is similar to Geiges' proof of Proposition 10; in particular, we will make use of the h–principle for Legendrian immersions. Let us recall its statement:

Proposition 21. *A C^0–dense, parametric, relative, and relative to the parameter h–principle holds for Legendrian immersions in contact manifolds.*

What this means precisely can be spelt out explicitly in the fashion of Proposition 20; refer to [5, 16.1.3]

Step I. The image of $M = \mathbb{R}^3 \times (-\varepsilon, \varepsilon)$ under the Geiges projection π_{Geiges} is $V = \mathbb{R}^2 \times (-\varepsilon, \varepsilon)$ with coordinates (x, z, t). Horizontal immersions descend to Legendrian immersions for the standard contact structure $\xi = \ker(dz - tdx)$. In particular, tangencies with $\mathcal{W}$ upstairs are in correspondence with tangencies downstairs with $\langle \partial_t \rangle$. From this, it follows that whenever ψ_k is horizontal and generic (in particular, whenever $k \in \mathcal{O}p(A)$ or $s \in \mathcal{O}p(B)$), $\pi_{\text{Geiges}} \circ \psi_k$ is in general position with respect to the Legendrian foliation given by $\langle \partial_t \rangle$, and thus the singularities of its front are generic. Do note that, since we work with higher dimensional families, singularities more complicated than cusps do appear.

Let us denote $\mathcal{L}eg(V, \xi)$ for the Legendrian immersions of the interval $[0, 1]$ into (V, ξ) and $\mathcal{FL}eg(V, \xi)$ for its formal counterpart. Much like in the case of horizontal immersions, a formal Legendrian immersion is a pair comprised of a map into V and a monomorphism into ξ (in this case, both with domain the interval).

Since $d\pi_{\text{Geiges}}$ maps $\mathcal{D}$ isomorphically onto ξ, the Geiges projection yields a family

$$\begin{aligned} \Psi_k^0 = \pi_{\text{Geiges}} \circ \psi_k \in \quad & \mathcal{FL}eg(V, \xi), & k \in \mathbb{D}^m, \\ \Psi_k^0 \in \quad & \mathcal{L}eg(V, \xi), & k \in \mathcal{O}p(A), \end{aligned}$$

which is already Legendrian for $s \in \mathcal{O}p(B)$. By Proposition 21, Ψ_k^0 is homotopic, relative to A and B, to a family $\Psi_k^{1/2} \in \mathcal{L}eg(V, \xi)$ for all k. We can further assume that the front of $\Psi_k^{1/2}$ has generic singularities as well. Denote by $\Psi_k^\delta = (\eta_k^\delta, G_k^\delta)$, $\delta \in [0, 1/2]$, the homotopy as formal Legendrians.

Step II. Let us construct a lift $\psi_k^\delta = (\gamma_k^\delta, F_k^\delta)$ of Ψ_k^δ. Since $\mathcal{D}$ projects to ξ under the Geiges projection, we define F_k^δ to be the unique lift of G_k^δ. For γ_k^δ, let us focus first on the case where B is empty or $\{0\}$. Define its y–coordinate $y_k^\delta(s)$ to be given by:

$$y_k^\delta(s) = y_k^0(0) + \int_{\eta_k^\delta|_{[0,s]}} z dx.$$

This construction guarantees $\gamma_k^\delta = \gamma_k^0$ for $k \in \mathcal{O}p(A)$.

Step III. If $B = \{0, 1\}$, defining y_k^δ by integration means that the y–coordinate of γ_k^δ will not necessarily agree with that of γ_k^0 at $s = 1$, as it should. The idea is to deform $\eta_k^{1/2}$ to yield a new Geiges projection η_k^1 having this integral adjusted. Note that we cannot do wild deformations: for a Legendrian not to escape the local model $V = \mathbb{R}^2 \times (-\varepsilon, \varepsilon)$, its front must have a slope bounded in terms of ε. Instead, we introduce type I Reidemeister moves to add or substract area.

Recall that the front of $\eta_k^{1/2}$ has generic singularities. In particular, given any point $k \in \mathbb{D}^m$, there is s_k such that the curve $\eta_k^{1/2}$ is not tangent to $\langle \partial_t \rangle$ at time s_k. It follows that we can find a small disk $\mathcal{U}_k \subset \mathbb{D}^m$ containing k and an interval $I_k \subset [0, 1]$ containing s_k such that the curves $s \to \psi_{k'}^{1/2}(s)$, $(k', s) \in \mathcal{U}_k \times I_k$, are transverse to $\langle \partial_t \rangle$ and therefore their front projection is an interval without cusps. By compactness, a finite number of open subsets $\mathcal{U}_k$ disjoint from A cover $\mathbb{D}^m \setminus \mathcal{O}p(A)$.

Given any even integer N, find an ordered sequence of times $s_k^1, .., s_k^N \in I_k$ and a width $\epsilon > 0$ such that the segments $[s_k^j - \epsilon, s_k^j + \epsilon] \subset I_k$ do not overlap. We construct η_k^1 as follows. Replace the front of the curves $\eta_k^{1/2}|_{[s_k^j-\epsilon, s_k^j+\epsilon]}$, for $k \in \mathcal{U}_k$, by adding a "Reidemeister I" loop such that the sign of the area it encloses is given by the parity of j. Modify the fronts of $\eta_k^{1/2}$, for $k \in \mathcal{O}p(\mathcal{U}_k) \setminus \mathcal{U}_k$, so that they transition, through Reidemeister I moves, from agreeing with those of $\eta_k^{1/2}$ in $\partial \mathcal{O}p(\mathcal{U}_k)$ to agreeing with those of η_k^1 in $\mathcal{U}_k$. Denote by η_k^δ, $\delta \in [1/2, 1]$, the corresponding Legendrian homotopy.

A remark is in order. The slopes of the fronts of η_k^δ, $\delta \in [1/2, 1]$, can be assumed to remain arbitrarily close to those of $\eta_k^{1/2}$; in particular, the deformation does not escape the Darboux ball M. In particular, we can find a bound, independent of N but depending on how much we want to C^0–approximate $\eta_k^{1/2}$, for how large the areas enclosed by the Reidemeister I loops can be. This implies that we can adjust N and the size of the loops to modify the area to be exactly the amount we require.

Since η_k^δ is Legendrian for $\delta \in [1/2, 1]$, its tangent map extends G_k^δ to the whole of $\delta \in [0, 1]$. G_k^δ lifts to F_k^δ as above. We define ψ_k^1 (or, rather, its y–coordinate) by integrating zdx over η_k^1. Since the $\mathcal{U}_k$ cover $\mathbb{D}^m$, we have that for all $k \in \mathbb{D}^m$ this integral can be adjusted to ensure $\psi_k^1(1) = \psi_k^0(1)$. We define the y–coordinate of ψ_k^δ, $\delta \in (0, 1)$, by lifting it arbitrarily relative to $s = 0, 1$ and $\delta = 0, 1$. □

An immediate consequence is an extension of the Adachi–Geiges result to any Engel manifold:

Corollary 22. *Let $(M, \mathcal{D})$ be an Engel manifold and let $\gamma_1, \gamma_2 \in \mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})$ be two horizontal loops. Then, they are isotopic as horizontal loops if and only if they are homotopic as maps and they have the same rotation number.*

Proof. Apply Theorem 19 to obtain a connecting family of immersions. One can then proceed in a cover by Darboux charts, much like in Theorem 17, in which intersection points, under the Geiges projection, appear as self–tangencies satisfying an area condition. Generically, curves with self–tangencies can be assumed to be isolated in a 1–parametric family. By adding or substracting area around said points, they can be assumed not to lift to intersections. □

Finally, we compare Theorem 19 with Proposition 15 to deduce that hte inclusion $\mathcal{HI}(\mathcal{D}) \to \mathcal{FHI}(\mathcal{D})$ is not a weak homotopy equivalence:

Remark 23. Gromov had conjectured that the sheaf of horizontal immersions is microflexible (Statement (a.) in the introduction). In Subsection 5.1 we will provide a definition, but for our purposes now it suffices to say that microflexibility means that many C^1-perturbations exist. Bryant and Hsu's result (Proposition 14) shows that this is not true.

Since microflexibility is one of the ingredients often used to prove h-principle statements, its failure suggests a failure of the h-principle for the inclusion $\mathcal{HI}(\mathcal{D}) \to \mathcal{FHI}(\mathcal{D})$. Indeed, we can show that this is the case when $\mathcal{D}$ is a Cartan prolongation: According to Theorem 19, there is a subspace $\mathcal{HI}^{\text{n.e.t.}}(\mathcal{D})$ which is weakly homotopy equivalent to $\mathcal{FHI}(\mathcal{D})$. By construction this subspace contains no $\mathcal{W}$-orbits. Proposition 15 states that there are two connected components of $\mathcal{HI}(\mathcal{D})$ that are comprised of $\mathcal{W}$-orbits. Putting these two facts together we deduce that $\mathcal{HI}(\mathcal{D}) \to \mathcal{FHI}(\mathcal{D})$ is not injective in π_0.

4. Curves tangent to $\mathcal{W}$ and their deformations

In Subsection 2.6 we showed that the h–principle for horizontal immersions does fail, in general, in the presence of closed orbits of the characteristic foliation. This motivated us to restrict our attention to the subclass of curves $\mathcal{HI}^{\text{n.e.t.}}$. Going in the opposite direction, in this section we explore the phenomenon of rigidity in more detail.

Let us explain our setup. Take the standard $(\mathbb{R}^3, \xi = \ker(dy - zdx))$ and let $\phi : (\mathbb{R}^3, \xi) \to (\mathbb{R}^3, \xi)$ be a contactomorphism that fixes the origin. Think about the mapping torus M_ϕ as the quotient $\mathbb{R}^3 \times [0, 1]/\phi$ with coordinates (x, y, z, t). M_ϕ can be endowed with a natural even–contact structure:

the pull–back of ξ, whose kernel is spanned by ∂_t. An Engel structure $\mathcal{D} = \langle \partial_t, L\rangle$ can be defined on M_ϕ, where $L \subset \xi$ is some t–dependent Legendrian vector field rotating positively in the t–direction and satisfying $\langle \phi^*(L(0))\rangle = \langle L(1)\rangle$.

Fix a framing $\langle X = \partial_x + z\partial_y, Z = \partial_z\rangle$ of ξ. Fix $L(0) = X$. We write $L(1)$ as $\cos(F(x, y, z))X + \sin(F(x, y, z))Z$, where $F : \mathbb{R}^3 \to \mathbb{R}$ is the smallest possible such function that is still positive. F can be extended to the whole mapping torus to define a possible L:

$$f : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}$$
$$f|_{\mathbb{R}^3\times\{1\}} = F, \qquad f|_{\mathbb{R}^3\times\{0\}} = 0, \qquad \partial_t f > 0,$$
$$L = \cos(f(x, y, z, t))X + \sin(f(x, y, z, t))Z.$$

Therefore, the structure equations for the Engel structure $\mathcal{D}$ read as:

$$\alpha = dy - zdx, \qquad \beta = \cos(f(x, y, z, t))dz - \sin(f(x, y, t, z))dx.$$

Consider the $\mathcal{W}$–integral curve $\gamma(\theta) = (0, 0, 0, \theta)$. Any C^1–small deformation of γ is of the form $\eta(\theta) = (x(\theta), y(\theta), z(\theta), \theta)$, and satisfies the equations:

$$\begin{aligned} \tan(f(x, y, z, t)) &= \frac{z'}{x'}, \\ y(t) - y(0) &= \int_0^t zx'ds, \\ \phi(\eta(1)) &= \eta(0). \end{aligned}$$

We say that the plane curve $\pi \circ \eta(\theta) = (x(\theta), z(\theta))$ is the front of η. These formulas in particular describe how to recover η from its front. Using this language, we can provide a more geometrical proof of Proposition 15.

Alternative proof of Proposition 15. Given a C^1–perturbation η of a $\mathcal{W}$–tangent curve γ, we want to show that η is tangent to $\mathcal{W}$ as well. Suppose otherwise; by Theorem 17, we can assume that η is in general position with respect to $\mathcal{W}$. We can find a neighbourhood of γ that is a mapping torus M_ϕ with ϕ the identity and $L(1) = -L(0)$; we are in the setup above, with $f(x, y, z, t) = \pi t$. The front $\pi \circ \eta$ is a closed plane curve with cusps. It must possess at least one cusp and, choosing our neighbourhood suitably, we assume that $\pi\circ\eta$ has, at $\pi\circ\eta(0) = 0$, a cusp pointing to the left. The first equation above states that the slope of η rotates clockwise π degrees, and thus the curve is piecewise convex. The second one says that the signed area bounded by η must be zero.

Observe that the number of cusps must be odd since the oriented slope approaching $t = \pi$ must be horizontal and pointing to the left and at every cusp the orientation changes sign. Denote the values of the parameter for which the curve has a cusp by $\{t_0 = 0 = \pi, t_1, \dots, t_{2n}\}$. Since the slope is only horizontal at the endpoints, the cusps are alternating; that is, at t_{2i-1} the curve leaves the horizontal line $\{z = z(t_{2i-1})\}$ going downwards and at t_{2i} it leaves it going upwards. In other words, the function $z(t)$ is strictly increasing in the intervals (t_{2i}, t_{2i+1}) and strictly decreasing otherwise. We now deform $\pi \circ \eta$ by *enlarging the cusps*: We add a straight segment to the end of each of the cusps, except for the one at the origin. After a C^∞-small perturbation in an arbitrarily small neighbourhood of each segment, we can still assume that the resulting curve bounds area zero and is everywhere convex; in particular, it is still the front projection $\pi \circ \eta$ of a C^1-perturbation η of γ. This procedure allows us to push upwards the odd cusps and push downwards the even ones. Therefore, for $i > 0$:

$$\begin{aligned} z(t_{2i-1}) &= z(t_1) > 0, \\ z(t_{2i}) &= z(t_2) < 0. \end{aligned}$$

Consider the segments $\pi\circ\eta|_{(t_{2i-1},t_{2i})}$ and $\pi\circ\eta|_{(t_{2i},t_{2i+1})}$, and reverse the parametrisation of the former. Then, both of them are segments starting from $\pi \circ \eta(t_{2i})$ and finishing in the same z–coordinate, but the latter has greater slope. This reasoning readily implies that:

$$x(t_1) > x(t_3) > \cdots > x(t_{2n-1}),$$

$$x(t_2) < x(t_4) < \cdots < x(t_{2n}).$$

In particular, the segments $\pi \circ \eta|_{(t_{2i-1},t_{2i})}$ and $\pi \circ \eta|_{(t_{2i+1},t_{2i+2})}$ intersect at a point s_i. This means that inbetween t_{2i-1} and t_{2i+2} a *Reidemeister I move* configuration appears, bounding some positive area. Refer to Figure 2.

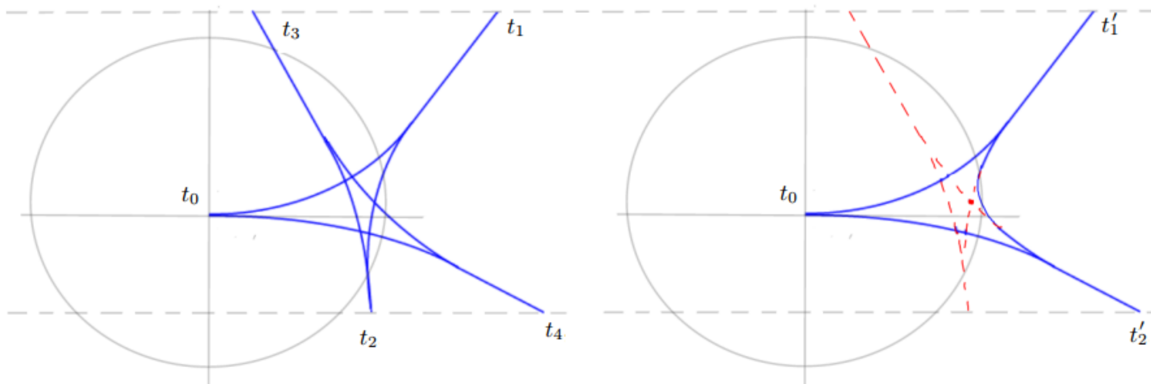

FIGURE 2. On the left hand side, a possible projection for a deformation η with five cusps. On the right hand side, we outline in red the area of the Reidemeister I loop, that has been removed, yielding a curve with only three cusps. Note that the cusps have been made longer so that they would reach the horizontal gray lines.

Now we conclude by induction on $2n+1$, the number of cusps. Our induction hypothesis is that a front conforming to the properties above must bound positive area. This is straightforward for $2n+1 = 3$. For the induction step, the reasoning on the previous paragraph shows that, for $2n+1 > 3$, a Reidemeister I move appears. By removing it (along with the points t_{2i} and t_{2i+1}) and smoothing the curve at s_i, the points t_{2i-1} to t_{2i+2} are now connected by a segment with no cusps. Since the area under this operation decreases and now the number of cusps is $2n-1$, the induction hypothesis concludes the proof. □

On the other hand, we now describe two examples of "short" $\mathcal{W}$–orbits that admit deformations not everywhere tangent to $\mathcal{W}$. These models can be inserted into a Cartan prolongation by deforming $\mathcal{W}$ using some contact vector field.

Example 24 (Curves making one projective turn.)**.** Take the mapping torus M_ϕ with $\phi(x, y, z) = (x, y/2, z/2)$. Fix $L(0) = -L(1) = \partial_x + z\partial_y$. Let η be the desired deformation of $(0, 0, 0, \theta)$, which we assume is in general position with respect to $\mathcal{W}$. Its front $\pi \circ \eta(\theta) = (x(\theta), z(\theta))$ satisfies $(x(0), z(0)) = (x(1), 2z(1)) = (0, 0)$ and encloses an area of $y(1)/2$. On the left hand side of Figure 3, such a curve is presented; it is clear that the area it bounds can be adjusted to be exactly $y(1)/2$. □

Example 25 (Curves having an arbitrarily short development map.)**.** Fix some angle $\alpha \in (0, \pi)$. The following contactomorphism is the lift of the turn of angle $-\alpha$ in the plane (x, z):

$$\psi(x, y, z) = (\cos(\alpha)x + \sin(\alpha)z, y - \sin^2(\alpha)zx + \frac{1}{2}\cos(\alpha)\sin(\alpha)(z^2 - x^2), \cos(\alpha)z - \sin(\alpha)x).$$

We consider the mapping torus of ψ.

Take a deformation η ending at $(x(1), y(1), z(1))$. The projection $\pi \circ \eta$ must bound a signed area of

$$y(1) - y(0) = \sin^2(\alpha)z(1)x(1) - \frac{1}{2}\cos(\alpha)\sin(\alpha)[z(1)^2 - x(1)^2].$$

The right hand side is precisely the integral of zdx over the curve β given by going from $(x(0), z(0))$ to the origin and then to $(x(1), z(1))$ following straight lines, as a computation shows.

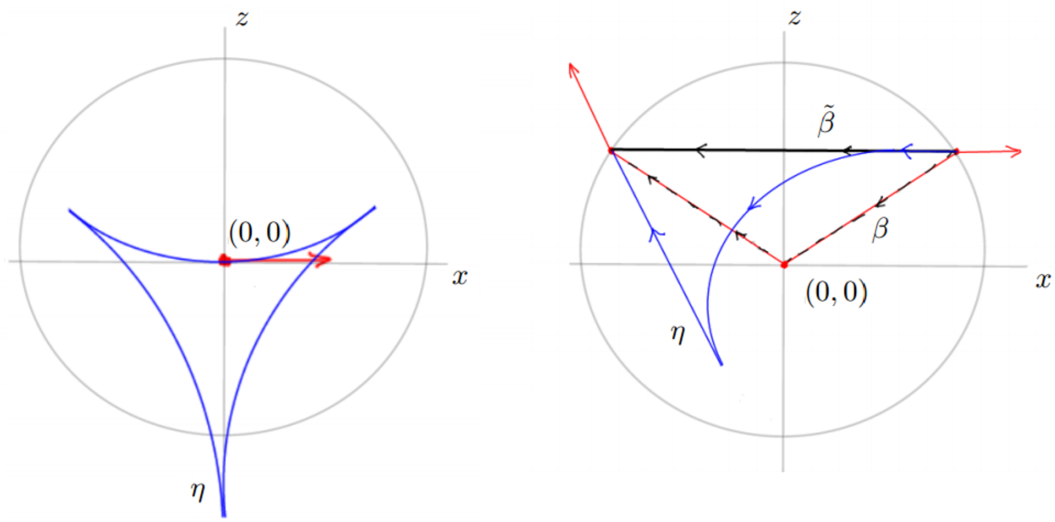

FIGURE 3. On the left hand side, a possible deformation for a curve making one projective turn. On the right, a deformation for a curve with short development map. The curves are depicted in blue. The tangent vectors at $t = 0, 1$ are shown in red.

Consider $(x(1), z(1))$ lying in the first quadrant and making an angle of $\alpha/2$ with the vertical axis. Let $\tilde{\beta}$ be the straight horizontal segment connecting $(x(0), z(0))$ and $(x(1), z(1))$. In particular, it lies above β and thus $\int_\beta z dx > \int_{\tilde{\beta}} z dx$. Now it is straightforward to create a curve η such that $\int_\eta z dx = \int_\beta z dx = y(1) - y(0)$ by adding some (positive) area to $\tilde{\beta}$ and adjusting it to ensure that it consistently turns clockwise. Refer to the right hand side of Figure 3. □

Slightly generalising the first example, it is not hard to show that:

Proposition 26. *Let ϕ be a contactomorphism of $(\mathbb{R}^3, \xi = \ker(dy - zdx))$ fixing the origin and with conformal factor different from 1. Let M_ϕ be the corresponding mapping torus with coordinates (x, y, z, t) and endowed with the Engel structure with smallest turning. Then, the $\mathcal{W}$–curve $\gamma(\theta) = (0, 0, 0, \theta)$ admits deformations somewhere not tangent to $\mathcal{W}$.*

Proof. Take $d_0\phi$, the linearisation at the origin. $d_0\phi|_\xi$ is a linear map in $\mathbb{R}^2$ that can be lifted to a contactomorphism $\tilde{\phi}$. By zooming in with the contactomorphism $(x, y, z) \to (\lambda x, \lambda^2 y, \lambda z)$, ϕ becomes C^∞ close to $\tilde{\phi}$, and therefore it is enough to prove the statement for ϕ linear.

If the conformal factor at the origin is different from 1, there is a dilation in the y–coordinate. Then we construct a deformation starting and finishing at the origin and bounding an area $y(1) - y(0) > 0$, which is possible if we select $y(0)$ small enough and with the adequate sign. □

Let us elaborate on an interesting consequence of Proposition 26. Observe that, at a linear level, contactomorphisms fixing the origin and having conformal factor different from 1 are generic. This readily implies a Kupka–Smale type of theorem for kernels of even–contact structures. Let us spell it out:

Theorem 27. *A C^∞–generic even–contact structure has isolated $\mathcal{W}$–orbits having Poincaré map not a strict contactomorphism.*

The same holds for a generic Engel structure. In particular, the inclusion $\pi_0(\mathcal{HI}(\mathcal{D})) \to \pi_0(\mathcal{FHI}(\mathcal{D}))$ is a bijection if $\mathcal{D}$ is C^∞–generic.

A key ingredient will be the following standard fact, whose proof we recall:

Lemma 28. *Let $(\mathbb{D}^3, \xi = \ker(\alpha))$ be the standard contact Darboux ball. Let $\phi_0 : \mathcal{O}p(\mathbb{S}^2) \subset \mathbb{D}^3 \to \mathbb{D}^3$ be a smooth contact map (i.e. $(\phi_0)_*\xi = \xi$ but ϕ_0 not necessarily bijective) with no fixed points.*

Consider the space of smooth contact maps $\phi : \mathbb{D}^3 \to \mathbb{D}^3$ agreeing with ϕ_0 in $\mathcal{O}p(\mathbb{S}^2)$. Then, the subset of those having non–degenerate fixed points is open and dense in the C^∞-topology.

Proof. Consider the manifold $V = \mathbb{D}^3 \times \mathbb{D}^3 \times \mathbb{R}$, and let π_1 and π_2 be the projections onto its first and second factors, respectively. V can be endowed with the contact structure $\ker(\lambda = \pi_1^*\alpha - e^t\pi_2^*\alpha)$; any contact map $\phi : \mathbb{D}^3 \to \mathbb{D}^3$ lifts to a Legendrian $\Gamma_\phi(x) = (x, \phi(x), \log[\alpha/(\phi^*\alpha)])$, where the last term accounts for the conformal factor of ϕ.

We need for Γ_ϕ to be transversal to $\Delta \times \mathbb{R}$, with $\Delta \subset \mathbb{D}^3 \times \mathbb{D}^3$ the diagonal. This claim follows from Thom's transversality (see, for instance, [5, p. 17, 2.3.2]): Indeed, let $p \in \Gamma_\phi \cap (\Delta \times \mathbb{R})$. Then, there is a neighbourhod $U \ni p$ contactomorphic to $J^1(\mathbb{R}^3, \mathbb{R})$ with $\Gamma_\phi \cap U$ being taken to the zero section. Then, Thom's transversality states that a generic C^∞–small deformation of $\Gamma_\phi \cap U$ (which is given as the graph of a function) is transversal to the submanifold $(\Delta \times \mathbb{R}) \cap U$. Proceeding chart by chart, capping the deformations off, and using progressively smaller deformations allows us to conclude. Since the deformations are C^∞-small, they are graphical over the first factor of V, and hence give rise to a contact map. □

Note that reasoning in this fashion yields the analogous result for contactomorphisms in compact contact manifolds of any dimension as well.

Proof of Theorem 27. Fix a metric on M. For simplicity, focus on even–contact structures having orientable and oriented kernel. Any such $\mathcal{E}$ has an associated unitary vector field W spanning $\mathcal{W}$ positively. Otherwise, observe that the argument that follows can be applied by taking a double cover and proceeding in a $\mathbb{Z}_2$–equivariant fashion.

Consider the subset of even–contact structures such that the W–orbits of length at most $T > 0$ are non degenerate. We claim that it is open and dense. We claim that it is still open and dense if we further require for the Poincaré return maps of the orbits to be non–strict contactomorphisms. Assuming these statements, the subset of even–contact structures such that this is true for orbits of all periods is a countable intersection of open and dense sets.

Our claims readily follow from arguments of Peixoto [11][p. 219-220], which we briefly sketch. Take $(M, \mathcal{E})$. Given any W–orbit γ of period $\tau < T$, Lemma 28 produces a C^∞–small deformation of W such that the Poincaré return has only isolated fixed points. However, this might produce new orbits of period $2\tau - \varepsilon \geq N\tau < T$ for some integer N. Therefore, one starts deforming orbits that are close to the minimal period and introduces progressively smaller deformations as the period goes up to T. If we additionally want the orbits not to have return map a strict contactomorphism, we take the isolated orbits we have produced and we replace their Poincaré return maps by their linearised version, which we then make generic.

This concludes the proof for the statement regarding even–contact structures. We then note that any C^∞–perturbation of $\mathcal{E} = [\mathcal{D}, \mathcal{D}]$ can be realised by a C^∞–perturbation of $\mathcal{D}$, so we conclude that the same holds for a C^∞–generic Engel structure. Having all $\mathcal{W}$–orbits isolated, Proposition 26 allows us to deduce that, at least at the π_0 level, there is a complete h–principle for the inclusion $\mathcal{HI}(\mathcal{D}) \to \mathcal{FHI}(\mathcal{D})$ if $\mathcal{D}$ is generic. □

An immediate consequence of Theorem 27 is:

Corollary 29. *Let $\mathcal{D}$ be C^∞–generic Engel structure. Then, horizontal embeddings are classified by the free homotopy class they represent and their rotation number.*

Remark 30. Theorem 27 should still be true for higher π_k. This would require carefully analysing families of curves and ensuring that the model from Figure 3 can be introduced parametrically.

5. Transverse maps and immersions

Having proven our results on horizontal immersions, we can study the other condition that is geometrically meaningful for a map to satisfy in the presence of a distribution: that of being transverse. We shall review Gromov's strategy for proving flexibility. This was already worked out in detail by Y. Eliashberg and N. Mishachev in [5][p. 136] for the contact case, and indeed the proof goes through without any major differences.

Theorem 31. *Let $(M, \mathcal{D})$ be an Engel manifold and V be any manifold. Maps $f : V \to M$ with $df : TV \to TM \to TM/\mathcal{D}$ surjective satisfy a C^0–close, parametric, relative, and relative to the parameter h–principle.*

Remark 32. If V is 2–dimensional, the statement amounts to asking for V to be an immersion transverse to $\mathcal{D}$. The analogous statement for V having subcritical dimension 1 is already proven in [5][Prop. 8.3.2].

Remark 33. Assume that the Engel flag $\mathcal{W} \subset \mathcal{D} \subset \mathcal{E} \subset TM$ is orientable. Then, if V is an immersed closed transverse 2–dimensional manifold, it must be a torus with trivial normal bundle. If we drop the orientability assumption, V can be a Klein bottle as well.

Remark 34. The formal data is a mapping f and a formal derivative $F : TV \to TM$ that is surjective onto the quotient $TM/\mathcal{D}$.

5.1. The h–principle for Diff–invariant, microflexible and locally integrable relations.

Let us explain the main ingredients needed to prove Theorem 31. The interested reader might want to refer to [5][Chap. 13].

Fix two manifolds: W, of dimension n, and M. Let $\pi : J^r(W, M) \to W$ be the bundle of r–jets of maps from W to M. The constant r can be a non-negative integer or take the values ∞ or g, by which we mean germs of maps. A subset $\mathcal{R} \subset J^r(W, M)$ is called a *differential relation*. Given a map $f : W \to M$, we can consider its r–order associated jet, which is a section $j^r f : W \to J^r(W, M)$. A section of $J^r(W, M)$ of this form is said to be *holonomic*. A map $f : W \to M$ satisfying $j^r f \in \mathcal{R}$ is said to be a *solution* of the differential relation.

Definition 35. *A differential relation $\mathcal{R}$ is* locally integrable *if, for any m, and for any two maps*

$$h : [0, 1]^m \to J^r(W, M)$$

$$g_p : \mathcal{O}p(\pi \circ h(p)) \to M,\ p \in \mathcal{O}p(\partial[0, 1]^m)$$

satisfying $j^r g_p(\pi \circ h(p)) = h(p)$, there is

$$f_p : \mathcal{O}p(\pi \circ h(p)) \to M,\ p \in [0, 1]^m$$

satisfying $j^r f_p(\pi \circ h(p)) = h(p)$ for all p, and $f_p = g_p$ for all $p \in \mathcal{O}p(\partial[0, 1]^m)$.

That is, $\mathcal{R}$ is locally integrable if any pointwise differential condition given by $\mathcal{R}$ can be locally extended to a solution. We introduce the parameter space $[0, 1]^m$ to state that this local solvability holds parametrically and relatively as you vary the pointwise condition.

Let us denote $\theta_l = (A = [-1, 1]^n, B = \partial([-1, 1]^n) \cup ([-1, 1]^l \times \{0\}))$.

Definition 36. *A relation $\mathcal{R}$ is* microflexible *if, for any small ball $U \subset W$, any m, any $l \in \{0, \cdots, n-1\}$, and any maps*

$$h_p : \theta_l \to U,\ p \in [0, 1]^m,\ \textit{embeddings},$$

$$F_p : \mathcal{O}p(h_p(A)) \to \mathcal{R}\ \textit{holonomic},$$

$\tilde{F}_p^t : \mathcal{O}p(h_p(B)) \to \mathcal{R}$, $t \in [0,1]$, *holonomic and satisfying* $\tilde{F}_p^t = F_p$ *for* $p \in \mathcal{O}p(\partial[0,1]^m)$ *or* $t = 0$, *there is, for small* t, *a holonomic family* $F_p^t : \mathcal{O}p(h_p(A)) \to \mathcal{R}$ *extending* $\tilde{F}_p^t$, *and satisfying* $F_p^t = F_p$ *if* $p \in \mathcal{O}p(\partial[0,1]^m)$ *or* $t = 0$. *If the extension exists for all* t, *we say that* $\mathcal{R}$ *is* flexible.

That is, being microflexible amounts to proving that semi-local deformations of a solution of the differential relation can be extended to global solutions, as least for small times. Relations that are open are immediately microflexible and locally integrable.

The following proposition [5, 13.5.3] holds:

Proposition 37 (Gromov)**.** *Let* $\mathcal{R} \subset J^r(V \times \mathbb{R}, M)$ *be a locally integrable and microflexible relation that is invariant with respect to diffeomorphisms that leafwise preserve the foliation* $\coprod\{v\} \times \mathbb{R}$. *Then, a* C^0*–close, parametric, relative, and relative to the parameter* h*–principle holds in* $\mathcal{O}p(V \times \{0\})$.

Saying that the h–principle holds means that the space of holonomic sections is weak homotopy equivalent, under the inclusion, to the space of all sections into $\mathcal{R}$. Note that by C^0–close it is meant that the zeroeth order components are C^0–close, not its derivatives.

5.2. **Proof of Theorem 31.** Let $(M, \mathcal{D})$ be an Engel manifold. We claim that the relation $\mathcal{R}_1$ in $\pi : J^1(\mathbb{R}, M) \to \mathbb{R}$ of being tangent to $\mathcal{D}$ but transverse to $\mathcal{W}$ is locally integrable. Suppose we are given maps

$$h : [0,1]^m \to (\mathcal{D} \setminus \mathcal{W}) \subset TM$$
$$g_p : \mathcal{O}p(0) \subset \mathbb{R} \to M,\ p \in \mathcal{O}p(\partial[0,1]^m)$$

where the g_p are horizontal curves transverse to $\mathcal{W}$ satisfying $dg_p(0) = h(p)$. For all $p \in [0,1]^m$ and depending smoothly on p, we can extend the vector $h(p)$ to a vector field H_p in $\mathcal{O}p(\pi \circ h(p))$. We can assume that the maps g_p are embeddings by shrinking the domain. Therefore, for those $p \in \mathcal{O}p(\partial[0,1]^m)$, H_p can be assumed to be an extension of the tangent vector g_p'. Following the flow of H_p for short times gives the desired local extension of g_p.

We claim that $\mathcal{R}_1$ is microflexible as well. Observe that we only have to consider the case θ_0, which can be phrased as follows. Let $F_p^0 : [0,1] \to \mathcal{R}_1$, $p \in [0,1]^m$, be a family of holonomic maps. Let $F_p^t : \mathcal{O}p(\{0,1\}) \to \mathcal{R}_1$, $t \in [0,1]$, be a family of deformations defined around the endpoints of the interval. Let $\psi : [0,1] \to \mathbb{R}$ be a bump function which is identically 1 around $\{0,1\}$ and zero in an arbitrarily large interval in the interior of $[0,1]$. According to Lemma 11, the curves F_p^0 possess a local model in which they correspond to the zero section in $J^2(\mathbb{R}, \mathbb{R})$; this implies that, for small t, F_p^t is graphical over F_p^0 and therefore given by a function y_p^0. The extension is given by ψy_p^0 and its derivatives.

Let V be some manifold. Let $\mathcal{R}_2 \subset J^1(V, M)$ be the open relation of having the formal derivative be surjective onto $TM/\mathcal{D}$. The relation $\mathcal{R}_3 \subset J^1(V \times \mathbb{R}, M)$ consists of those maps with formal derivative surjective onto $TM/\mathcal{D}$ that, along the fibres $\{v\} \times \mathbb{R}$, are tangent to $\mathcal{D}$ but transverse to $\mathcal{W}$. Local integrability for $\mathcal{R}_3$ follows by mimicking the argument for $\mathcal{R}_1$.

We claim that $\mathcal{R}_3$ is also microflexible. Take $\theta_j = (A, B)$. Suppose we are given a holonomic family F_p^0 on A and a corresponding deformation F_p^t over $\mathcal{O}p(B)$. Find neighbourhoods $\mathcal{O}p_1(B) \subset \mathcal{O}p_2(B) \subset \mathcal{O}p(B)$ and build a bump function ψ that is 1 in $\mathcal{O}p_1(B)$ and 0 outside of $\mathcal{O}p_2(B)$. Since F_p^t is fibrewise graphical over F_p^0 for small t, we use ψ to interpolate back to F_p^0, as above; this can be achieved even if B is embedded wildly with respect to the foliation $\coprod\{v\} \times \mathbb{R}$. For small times the resulting deformation is C^∞–close to F_p^0, so in particular it is still surjective onto $TM/\mathcal{D}$ in the transverse direction.

By construction, $\mathcal{R}_3$ is invariant under diffeomorphisms preserving the foliation $\coprod\{v\} \times \mathbb{R}$ leafwise. Then, Proposition 37 allows us to conclude that in $\mathcal{O}p(V \times \{0\})$ a complete h–principle holds, so in particular a complete h–principle holds in V for the relation $\mathcal{R}_2$. □

Remark 38. Observe that we did not need the h–principle for tangent immersions of Theorem 19, instead we just checked the much more simple properties of being microflexible and locally integrable for the relation $\mathcal{R}_1$.

5.3. **Immersed 3–dimensional submanifolds.** The reader might have noticed that the most interesting case for a transverse submanifold was left out: codimension 1. Inspecting the proof presented in the previous subsection, it is clear that it cannot possibly go through, since immersions $V^3 \times \mathbb{R} \to M$ cannot avoid the $\mathcal{W}$–direction, which was a key ingredient in the 2–dimensional case to obtain microflexibility. Still, the following result holds:

Proposition 39. *Let $(M, \mathcal{D})$ be an Engel manifold. Let V be an arbitrary 3–manifold. Then, immersions $V \to M$ that are transverse to $\mathcal{D}$ satisfy a C^0–close, relative, relative to the parameter h–principle.*

Proof. Let V be a 3–manifold. Define the following differential relation $\mathcal{R} \subset J^g(V \times \mathbb{R}, M)$: germs that are transverse to $\mathcal{D}$ along $V \times \{s\}$ and lie in $\mathcal{HI}^{\text{gen}}(\mathcal{D})$ along $\{v\} \times \mathbb{R}$. There exists an obvious projection $J^g(V \times \mathbb{R}, M) \to J^1(V \times \mathbb{R}, M)$ and the image of $\mathcal{R}$ is the relation $\mathcal{R}^1$: maps with formal differential transverse to $\mathcal{D}$ along $V \times \{s\}$ and tangent to $\mathcal{D}$ along $\{v\} \times \mathbb{R}$. $\mathcal{R} \to \mathcal{R}^1$ is a Serre fibration with contractible fibre.

The proof of Proposition 39 amounts to showing that $\mathcal{R}$ is microflexible and locally integrable and then applying Proposition 37. The full h–principle for $\mathcal{HI}^{\text{gen}}(\mathcal{D})$ and the openness of the transverse immersion condition in codimension 1 imply microflexibility and the local integrability is tautological. The claim follows. □

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