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Mathematical selves in the shaping of mathematical modernism: The circulation and disputation of Chasles' formula (1864-1893)

Abstract

For more than three decades, fierce debates raged both in private letters and across public spaces over a formula expressed in 1864 by French geometer Michel Chasles. Proofs and refutations thereof abounded, to no avail: the formula was too useful to be abandoned by its defenders, too elusive to be made rigorous for its detractors. The disputes over Chasles' formula would not be solved by a definitive proof or rebuttal; rather, the core epistemic issues at stake shifted from truth to geometrical significance. This paper tracks the main lines of circulation of Chasles' formula, and shows how the disputes to which it gave rise embody conflicting *mathematical selves* – that is to say, different normative accounts of what being a mathematician entails. This perspective allows for a renewed understanding of what historians have described as the conflicted rise of modernism in mathematics, and a firmer rooting of it within broader late 19th-century cultural trends.

Short biography

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Introduction: A formula in flux

It was June 1890, and German mathematician Felix Klein was growing ever more dissatisfied with the state of enumerative geometry, a newly-emerged branch of mathematics. Klein was an influential professor and powerful organizer of mathematical research based at the University of Göttingen.[1] For some time already, he had been puzzled by the undecidedness surrounding a geometrical formula first expressed and justified on inductive grounds some 26 years prior by French geometer Michel Chasles.[2] A centerpiece of Chasles' much-celebrated theory of characteristics, this formula purportedly solved a difficult mathematical problem whose general solution had escaped geometers for a long time, namely the enumeration of curves satisfying given conditions (such as passing through a given point, or touching a given line). This problem had stumped renowned geometers such as Jakob Steiner, who had previously conjectured that there were 7776 conics touching five other given conics, only for Chasles to show that the correct number was in fact 3264.[3] The theory of characteristics itself had quickly garnered praise across Europe, with translations and summaries being published in Italian, Danish, English, and French, and the London Royal Society's awarding Chasles with the Copley medal in 1865 – a coveted honor then rarely bestowed upon a mathematician.[4]

In 1873, some seven years after Chasles' initial publication, independent and quasi-simultaneous proofs of this formula were given by two mathematicians: the Göttingen-based Alfred Clebsch, in a posthumous paper; and Georges-Henri Halphen, a French artillery officer freshly graduated from the Ecole Polytechnique.[5] Three years later, yet another proof was given by a student of Clebsch's while editing the latter's *Vorlesungen über Geometrie* under Klein's supervision.[6] Shortly thereafter, however, these efforts would be deemed profoundly misguided by Halphen himself. Near the end of the year 1876, he changed his mind regarding his own proof, presented a counter-example to Chasles' formula to the Académie des Sciences in Paris, and announced the forthcoming publication of an alternative theory.[7] For Halphen, undue reliance on intuition and vague notions had led geometers astray, and thorough analysis of the problem revealed that Chasles' methods counted objects which were no satisfactory solutions to the problem at hand, but mere computational artefacts.

Halphen's refutation, however, was not accepted by all. Among the dissenters was German mathematician Hermann Schubert. From 1874 onwards, Schubert had been devising a fruitful symbolical calculus, building on formal regularities he had observed in the results of his colleagues, and, crucially, on Chasles' formula. Schubert's calculus, and in particular his 1879 book *Kalkül der abzählenden Geometrie* impressed many, if only by the sheer number of new and difficult results Schubert had been able to obtain with his idiosyncratic methods.[8] To accept Halphen's sharp arguments against what he perceived to be the lack of rigor and analytical precision of his predecessors, for geometers at large, meant to agree that the numerous proofs of Chasles' formula produced by esteemed mathematicians were flawed, and to renounce the embarrassment of riches provided by Schubert's methods.

Klein, who by the late 1870s was actively corresponding with Schubert, could not fail to notice the problematic state of Chasles' theory of characteristics.[9] In 1884, while still a professor in Leipzig, he assigned the problem of assessing the validity of Chasles' formula to Eduard Study, a promising young student. Despite his initial reluctance to work on this problem, by 1885 Study had obtained a new proof for Chasles' formula, and attempted to fully respond to Halphen's criticism.[10] To that end, Study put forth a new kind of argument: Halphen's counter-example did not refute Chasles' formula *per se*, but only one interpretation thereof – and not necessarily the most appropriate one. Halphen's untimely death in 1889 came before his potential responses to Study's work could be published – and the few interactions the two mathematicians had had were largely unproductive.

Shortly after the publication of Study's dissertation, Klein received a rather bitter letter by Danish geometer Hieronymous Zeuthen, who lamented Study's reluctance to discuss the matter with him, and rejected the claim that the problem had been solved once and for all. Having studied under Chasles in Paris in 1865, written a dissertation in Copenhagen on the theory of characteristics, and corresponded with both Halphen and Schubert throughout the 1870s on these matters, Zeuthen was a renowned expert of enumerative geometry; and firmly on Halphen's side in his disagreement with Schubert and Study. Despite Klein's insistence, however, Study refused to engage in the discussions Zeuthen had called for.[11]

In 1890, therefore, with no end to these disputes in sight, Klein asked Zeuthen to write a public and official response to Study, to be published in the pages of the *Mathematische Annalen* which Klein then edited. Zeuthen obliged, and in so doing reiterated his opinion that Study's work was based on a misunderstanding of the very problem Halphen had set out to solve, and that the latter's results still held. Study replied in the same journal in 1892, and so did Zeuthen in 1893, but at no point did their respective positions evolve: despite Klein's intervention, the indeterminacy over the validity of Chasles' formula subsisted after all these years.[12]

There was a profound mathematical reason for the persistence of these disputes; and geometers nowadays all acknowledge that the validity of Chasles' formula crucially hinges upon the formalism adopted to translate the terms at the heart of the theory of characteristics. By the 1930s already, mathematicians had largely eschewed these debates, with many viewing them as nothing more than a matter of "*honor*."[13] And yet, the story of Chasles' formula cannot be read as that of the vain disputes of mathematicians insufficiently equipped to realize the ambiguity of their problem. Indeed, the final words on this topic would not come in the way of a definitive proof or refutation; but only at the close of a reinvention of the cultural and scientific identity of mathematics itself.

Modernism, truth, and language in *fin de siècle* mathematics

In his landmark 1990 study *Moderne Sprache Mathematik*, Herbert Mehrtens described the transformation of mathematics at the turn of the 20th-century as "*a shockwave blasting through the concepts of truth, meaning, object, and existence.*"[14] Throughout this transformation, he argued, criteria for truth and validity, modes of objectivity, and textual practices had to be reinvented and renegotiated amongst mathematicians. This was no peaceful transformation, as bespeaks Mehrtens' distinction between the "moderns" (such as Georg Cantor and David Hilbert) and the "counter-moderns" (such as Henri Poincaré or Klein himself) as two contrasting forms of self-understanding and style in mathematics, which clashed with zeal and fervor. In a thorough assessment of Mehrtens' thesis, Jeremy Gray has characterized mathematics under the modernist conception as "*as an autonomous body of ideas, having little or no outward reference, [...] maintaining a complicated—indeed, anxious—rather than a naïve relationship with the day-to-day world*."[15] The modernist mathematician, in Mehrtens' view, is a "*free creator*" whose proofs and propositions derive meaning from no external system of references, be they physical objects, a model of some phenomenal field, or even abstract objects conceived prior to the utterance of mathematical speech. The counter-moderns, in turn, pushed back against the perceived dangers of leaving the creation of mathematical concepts up to a matter of arbitrary will, and sought to quell related epistemic anxieties by grounding mathematical truths into either appeals to intuition or some transcendent order.[16]

Mehrtens' categories have been criticized as insufficient for informing a precise understanding of the multi-national, cross-cultural, decade-spanning modernist transformation of mathematics. Gray himself noted that they suffered from too exclusive a focus on Germany over the whole of Europe, and on programmatic or philosophical texts over actual samples of mathematical practice.[17] Debates over Chasles' formula span the decades and the countries most crucial to the conflicted emergence of modern mathematics, and were never understood by their protagonists as discussions of foundational or philosophical issue, but rather of technical, albeit important results.[18] Thus, this historical episode is an ideal candidate to test Mehrtens' sweeping narrative and categories, and to confront them to actual mathematical practice, on a multi-national scale. Another line of critique of this narrative, forcefully argued by Leo Corry, lies in the notion that the worth of the concept of modernism in the historiography of mathematics hinges upon its ability to incorporate contemporary, broader cultural changes.[19] The very term "modernism" suggests comparisons to transformations in the arts, many of which have been tentatively put forth.[20] Drawing from the history of scientific objectivity, as proposed by Lorraine Daston and Peter Galison in their much-discussed book *Objectivity*, this paper sets out to show another way to tie the modernist transformation of mathematics to several late 19th-century cultural trends and ruptures, by framing it as the confrontation of successive *mathematical selves*.

The history of objectivity, as envisioned by Daston and Galison, is not that of the conceptualizations and philosophical accounts of objectivity, but rather of the various epistemic virtues that regulated and enabled said objectivity. Such virtues were not only preached, but also practiced and embodied by various practices such as note-taking, self-erasing, attentive observation etc. Together, these virtues and practices constituted scientific personas, such as that of the sage or the expert, which projected different kinds of ontologies onto the same phenomenon, from the shape of snowflakes to the anatomy of insects. In so doing, Daston and Galison's ambition was to put forth a mesoscopic, *longue-durée* history of scientific objectivity across disciplinary borders, framed as the history of "the manifestations and mutations of the *scientific* self."[21]

This paper makes uses of this analytical framework, albeit on a resolutely micro-historical and local scale. In what follows, we will contrast the epistemic ideals which can be found in the highly-normative descriptions of proper mathematical practice produced by four of the key actors of the historical episode previously sketched; namely Chasles, Halphen, Schubert, and Study. To each of these accounts are associated different epistemic virtues, and which in turn give rise to different textual practices, ontologies and regimes of truth. Our main contention is that these virtues and practices were constitutive of different *mathematical selves*, whose incompatibility accounts for the inconclusiveness of the disputes over Chasles' formula. These *selves* are then all situated differently on the quadrants drawn by the two axes along which the modernist transformation of mathematics has been described, namely the absence of outward reference for mathematical discourse, and the growing anxiety amongst practitioners after the emergence of new standards of rigor. Furthermore, they will be shown to have been shaped against the decisive backdrop of various cultural trends and intellectual debates beyond mathematics.[22] Thus, the emergence of mathematical modernism is here depicted as a composite phenomenon, inseparable from cultural history at large.

Chasles: Geometry as exploration of natural order

On February 15th 1864, when French geometer Michel Chasles took the podium during one of the weekly public meetings of the Paris Académie des Sciences, he was a familiar sight. There, the aging mathematician was a well-respected figure, famous for his tireless promotion of geometry, as well as his historical erudition regarding all things mathematical and historical. For almost twenty years, he had been holding the chair of Higher Geometry at the Faculté de

Paris, where he taught and developed his own geometrical methods in front of advanced students.[23] His lectures were informed by his contemporary research, the content of which he thus polished and prepared for communication to his fellow *académiciens*. The theory of characteristics, which Chasles began presenting in February 1864, far exceeded his previous results both in impact on fellow European mathematicians and in sheer volume. Between 1864 and 1867, Chasles took the podium over twenty times to address the Académie on this topic, either to add new results or to share those his colleagues outside the Académie had sent him. The center of this attention was Chasles' uniform method for the determination of the number of conic sections satisfying any five conditions, requiring no computation other than elementary sums and products of a few integers.[24]

This theory constituted for Chasles the culmination of a life-long endeavour to renew the language and methods of pure geometry. Chasles defined pure geometry as geometrical theories and methods resting on "*pure reasoning alone*", in contrast with analytical geometry, which relied on the use of coordinate systems and algebraic computations.[25] His first book, a historical survey on the development of geometrical methods, drew conclusions from historical studies on how to elevate pure geometry to the same level of generality as its analytical counterpart.[26] In so doing, Chasles saw himself as part of a tradition which started at the Ecole Polytechnique through the influential teaching of one its founders, Gaspard Monge.[27] He viewed his task as one of synthesis and systematisation of the disorganised yet powerful new methods and concepts that Monge and his spiritual students and followers, such as Lazare Carnot, Charles Dupin, and Jean-Victor Poncelet, had brought forth in the early 19th century.[28]

These authors all had slightly different understandings of why analytical methods had risen to such lofty standards of generality and efficiency throughout the 18th century, and consequently they held different views on how to remediate to the inferiority of pure methods. However, few of them doubted the epistemic certainty or rigor of the mathematical knowledge obtained through analytical means. Rather, they were unhappy with the epistemic quality of said truths, which they reckoned insufficiently illuminated the mind. Chasles' conception of mathematical knowledge rested on one central creed, namely that "*all mathematical truths can become simple and intuitive, once the narrow path [to said truths] that is natural and characteristic has been found*."[29] This postulate led Chasles to draw a stark contrast between the knowledge provided by ingenious, human-made analytical machineries, and that of natural, effortless geometrical studies: "*Analysis"*, claimed Chasles, suffers from the same weaknesses as "*all human conceptions: its swift and penetrating march does not always sufficiently enlighten the mind*."[30] This assessment stood in complete opposition to Chasles' description of the ideal, pure geometer, once they are equipped with the modern methods and theories first discovered by Monge. This modern geometer, Chasles claimed, simply had to "*pick any arbitrary known truth, and submit it to the various general principles of transformation ; they [would] derive from it other truths, different or more general."* What's more, added Chasles, anyone could now become a geometer: "*genius is no longer required*."[31] At the end of Chasles' historical narrative stood the figure of the geometer as a student of (spatial) extension, able to effortlessly and systematically combine truths within naturally-grounded theories. Just like the technical drawing devised and promoted by Monge and Dupin years before, Chasles' teaching was meant to unfetter its users by "*bringing their practices in lines with the dictate of nature and reason*."[32]

This description of the ideal mathematician was also something Chasles transmitted through his teaching and promotion of his own geometrical practice. The naval officer Ernest de Fauque de Jonquières was a student and friend of Chasles' until a bitter priority dispute tore them apart, and worked for some time on Chasles' theory of characteristics. In a review of Chasles' 1860 book on Euclid's lost Porisms, he warned prospective geometers against the alluring promises of the "l*ively gait*" of analytical geometry, which he thought of as appealing to

> this frenzy, this frantic need to reach any arbitrary goal, which is one of the dominant characters of our times. But it is good, for the sake of science itself, to temper this character. For, even if we left the laurels of celerity in research to analytical methods, science could not be exclusively served in this way. To use a vulgar comparison, one may get the lay of a land fairly quickly by travelling along the major railways that criss-cross it; but to know in-depth the details, the productions, the resources of this land, one must step off the locomotive, and set out to explore by foot its ancient roads and unbeaten paths. In so doing, one acquires habits of patience, observation, and criticism, which might well disappear, if we couldn't bear to go back to this primitive mode of travel.[33]

At a time of fast industrialisation [34], and as calculating machines ceased to be mere abstractions[35], Chasles and his students promoted a counter-figure of the ideal mathematician, one that would go against the analytical trends, and aim for a slow-moving, but steady and methodical knowledge.[36] They embodied a mathematical self that bore crucially on the choices in conceptual tools and textual practices that these actors elected to use when solving geometrical problems.

In his courses, Chasles would craft new notations and linguistic devices in order to structure the whole of Geometry around a few central concepts, and to systematize the writing of geometrical proofs and propositions. One such line of enquiry pursued by Chasles from the early 1850s would be of particular importance for the theory of characteristics: the shaping of a new mode of description of curves through what he sometimes called "*geometrical equations*." In particular, Chasles studied procedures to construct certain curves determined by some of their points that were perfectly general, that is to say that the instructions involved in it would be applicable to any possible configuration of given points. Such procedures, Chasles claimed, had to rest on properties of these curves that are so absolutely fundamental as to completely characterize them. In turn, such properties would act as the "*true equations*"[37] of these curves. Unlike traditional Cartesian equations, however, they involved no algebraic symbols or variables. In his 1865 *Traité des Sections Coniques* (the content of which had been written and taught much earlier), Chasles built on this framework to form two central propositions, namely Pascal's and Brianchon's theorems, wherefrom the entire theory of these curves he thought to derive. Of these theorems Chasles would then say that they were the "*punctual and tangential equations of conics*."[38] In both instances, Chasles would emphasize the search for a fundamental and characteristic property of a class of curves as the geometer's main task. Such properties then act as geometrical equations, which involve no extrinsic elements (unlike Cartesian equations, with their artificial coordinate systems), while displaying the same level of generality as analytical equations. From these geometrical equations, the pure geometer could effortlessly, and with no use of shrewd computations, derive an infinity of higher-level truths.[39]

Chasles' presentation of the theory of characteristics made explicit the connection with his past research on geometrical equations.[40] He first explained why analytical methods cannot solve the general problem of enumerating conics: the computations required by the procedure of elimination that this would entail are simply intractable. They require the combination of five algebraic equations (of potentially high degrees) in six unknowns, which is, in general, more than the human mind can handle. His own theory had no such problem. Chasles considered what he called systems of conics, that is to say infinite collections of conics satisfying four conditions (see fig. 1 below). In a given system, for any condition Z, there is a finite number of conics satisfying Z. He then defined, for each of these systems, two numbers μ and ν, which he named the "characteristics" of these systems[41]. This terminology was not arbitrary: Chasles' central observation was that all properties of systems of conics can be expressed through a number obtained by adding these two numbers a certain number of times, that is to say through a number of the form $\alpha\mu + \beta\nu$ (where α and β depend solely on Z). In other words, Chasles had found a systematic method which, to every geometrical condition, was able to associate two coefficients α and β, so that in any system of conics of characteristics (μ,ν), the number of conics satisfying this condition Z was $\alpha\mu + \beta\nu$; this latter expression being called the 'module' of the condition. To showcase this remarkable regularity, Chasles then drew up long, monotonous lists of geometrical properties, each of which matched a geometrical condition. To enumerate conics satisfying five conditions, one only had to refer to these lists, find the properties corresponding to these five conditions, and carry out a series of simple additions and multiplications of these α's and β's. For instance, knowing that in a system of characteristics (μ, ν), there are $2\mu + 2\nu$ conics touching one given conic (that is to say that, for this condition, α and β are equal to 2), a series of elementary arithmetical operations given by the general procedure allowed Chasles to enumerate the 3264 conics in a plane which touch five given conics, a result for which he is still remembered.[42] Thus, for Chasles, characteristics allow for a natural representation of the properties of systems of curves, which in turn necessarily lead to effortless enumerations where algebraic representations, based upon artificial coordinate systems and variables yielded but inextricable computations.

Figure 1 inserted here (see captions at the end of manuscript)

The claim that all properties of systems of conics can be expressed in such a compact form, is what would be identified as Chasles' theorem. However, for his method to function and for it to rest on solid epistemic ground, Chasles did not need to prove this general formula: all he needed was to establish these lists of propositions covering any conceivable condition, and to do so in a systematic manner. And this is exactly what he did: in the archives preserved at the Paris Académie des Sciences can be found thousands of leaflets upon which Chasles sketched proofs of such propositions in a highly systematic and condensed form (see fig. 2 below), undated but most likely produced between 1864 and 1876. That a mathematician might consider this a worthwhile use of their time shows how active a role Chasles' normative ideal of the mathematician played in his scientific practice, up to the very identification of what should constitute valuable output, and what proves the value of a theory.

Insert figure 2 here.

The $\alpha\mu + \beta\nu$ formula for Chasles was less of a theorem than the sign that, through the notion of characteristics, he had captured an essential, natural and fundamental property of conic sections. For this very reason, he would liken the writing of characteristics (μ,ν) to that of the geometrical equation of a system of conics. It is not surprising, therefore, that Chasles later showed skepticism or even disinterest when he was confronted with the new generation's attempts at proving this formula.[43] This endeavour, in Chasles' view, amounted to proving something as fundamental as the adequacy of the concept of a degree of an equation, and was not part of what he as a mathematician and a geometer identified as a valuable or meaningful epistemic task. The theory of characteristics, in Chasles' understanding, was the crowning achievement that displayed in full the worth of his way of acting *qua* geometer: a geometer who refuses the help of artificial and computational machineries, in favor of slower, pedestrian, but deeper and richer surveys of the fundamental properties of a theory, and was thus rewarded with a method that would solve any problem related to conics. Of this, the facility and systematicity of his thousands of leaflets was a surer sign to Chasles than the complicated proofs later published would ever be.

Halphen: The Fall from geometrical Grace

As Chasles' theory circulated across national, mathematical, and cultural boundaries, much was lost in translation; even at the Paris-based Société Mathématique de France (SMF). Chasles himself was the first president of the SMF, which had been formed in November 1872 partly after his lament, expressed in his 1870 *Rapport*, that French mathematics was doomed to lag behind their German, English, or Italian counterparts lest such a society be created immediately.[44] A mathematical journal, the *Bulletin de la SMF* (BSMF), was immediately associated to this Society. A mere glance at the papers published therein between 1873 and 1876 shows the dominating influence of Chasles, as the theory of characteristics and related geometrical problems form a much larger proportion of the publications than in any other major mathematical journal in Europe, aside from the *Comptes-Rendus de l'Académie des Sciences* (CRAS), where Chasles' influence was equally strong.

And yet, many of the papers published there which seem to tackle the problems opened by Chasles preserved little of his notations, mathematical style, or of the central tasks he identified as motivating his work. Throughout the 1870s, Halphen was, by far, the most prolific author on enumerative questions both in the *CRAS* and the *BSMF*. His very first paper, published in 1869, consisted in a successful attempt to replicate Chasles' theory of characteristics for a different geometrical object, namely straight lines in space.[45] The first paragraph of the paper, however, spelled a crucial difference in their approaches. Halphen's strategy was to express and prove a general formula for the number of straight lines in space satisfying four conditions, which, in his notations, is written $\alpha M + \beta N$. While the form of the main result is similar to Chasles' formula, the theorem-oriented structure of Halphen's papers thus contrasts starkly with that of Chasles'. In lieu of lists of particular propositions exemplifying his $\alpha M + \beta N$ formula, Halphen's paper begins with the introduction of definitions and notations, moves on to a complete, algebraic proof of said formula, and concludes with an exploration of the theoretical consequences thereof – an inferential move seemingly of no interest to Chasles. Having identified different key epistemic goals for their work, Chasles and Halphen crafted different textual and literary resources to achieve them.

Unlike Chasles, Halphen maintained a constant engagement with German mathematics: not only did he read and communicate with German mathematicians, he also sent some of his work to German institutions and journals.[46] In 1882, Halphen shared an award given by the Berlin Academy with Max Noether for work on skew curves; and some of his work on the theory of characteristics would be republished in the *Mathematische Annalen* at Klein's express demand. By the time the SMF was created, Halphen was in possession of what he thought to be the first and definitive proof of Chasles' formula for conics. In 1873, he published it in the form of three short memoirs in the very first installment of the BSMF, only to discover he had been beaten to it by an earlier paper of Clebsch.

Clebsch, just like Halphen, had identified Chasles' formula as an important theorem that remained to be proven. Clebsch's intent, as outlined in the programmatic statement which opened his lectures which Lindemann edited in 1876, was the "*use of simple auxiliary means for the clothing by algebraic forms of geometrical problems*."[47] His 1872 posthumous paper, consequently, consists to a large extent in an attempt at extracting the algebraic content of Chasles' notions, so that the $\alpha\mu + \beta\nu$ formula be tractable and provable by the new theory of invariants. The two first sections aim to produce algebraic equations for systems of curves in less crude a way than mere Cartesian coordinates would allow for, while the third section is a discussion of the geometrical concept of the satisfaction of a condition, which Clebsch aims to show is equivalent to the vanishing of an invariant, a key concept in the new approaches to algebraic geometry.[48]

Halphen read Clebsch's memoir closely, and made his own the project of an investigation into the algebraic content of geometrical notions. However, he soon came to reckon this project had been insufficiently pursued by Clebsch, and that a more rigorous twist ought to be brought to it. In later recollections, written as part of an application for the Académie des Sciences, Halphen wrote:

> I immediately noticed that I still had to make precise a notion which had until then remained vague, namely that of the *independence* of, on the one hand, the system of conics, and on the other hand, the extra condition that is imposed on the conics of this system. Often M. Chasles had neglected to mention it, but everyone restored it effortlessly. In each example, indeed, nothing is simpler. In the general theory, however, it is not clear at first how to make this independence precise.[49]

Paradoxically, his reading of a paper that supposedly agreed with his own work stirred up doubt in Halphen's mind. According to his later retelling of the story, in investigating further into the analytical expression of this independence, Halphen discovered that there were not two, as Chasles, Clebsch, and many others previously thought, but three kinds of degenerate conics, and this discovery became the foundation upon which he constructed his counter-examples to Chasles' formula.[50]

Looking back at this turn of events, the French artillery officer immediately framed the irruption of this refutation via a military vocabulary:

> This theory, which led to so many controversies, seems today to be fixed. But, one must admit, what a strange fate it's had! Where to find the source of these vicissitudes? Too much imagination, perhaps, prematurely led geometers into an ill-prepared campaign. How much uncertainty, fumbling, how many mistakes even, soon to be corrected, were seen in this century's attempts at a general Geometry, which mingles with the *theory of algebraic functions*![51]

This juxtaposition of mathematical rigor and strategical preparation is not entirely unique, especially amongst a generation of Polytechniciens who graduated right before, or during, the bitter defeat against Prussia in 1870.[52] In a notice written after Halphen's early death in 1889, Henri Poincaré quoted the following assessment by Charles Hermite:

> Halphen, Faidherbe, after so many others, have been faithful to the double mission of the Ecole Polytechnique, and have continued its glorious traditions. Isn't there indeed, in the habits of intelligence, in this particular nature which the teaching of our great School creates, a normal link, a concordance with the soldier's qualities? A rigorous

> discipline of the mind prepares one for military duties, and doubtlessly mathematical studies contribute to form this faculty of abstraction which proves indispensable to the chief who needs to form an interior representation, an image of the action by which he leads himself, forgetting danger, into the tumult and obscurity of combat.[53]

Poincaré, himself another Polytechnique alumnus, concurred. Amongst the other notices written after Halphen's death, most of which were written by leading mathematicians from the same generation, there emerges a rather precise description of the kind of mathematician that Halphen supposedly was. Émile Picard, for instance, in his own obituary of Halphen, distinguished between two "*tendencies of mind*" one can find amongst mathematicians: there were those who "*busy themselves mainly with widening the perimeter of known notions*", and those who "*prefer to remain in the purview of more developed notions, to deal with them in depth*."[54] Halphen would then be characterized as an extreme example of this second tendency. His *modus operandi*, Picard tells us, was to leave no question incompletely solved, to never stop investigating a matter until absolute precision and rigor had been attained. Halphen's counter-examples, in this narrative, became emblematic of a certain mathematical frame of mind, of a certain way of acting *qua* mathematician. The very focus on counter-examples as a threat to generality shows well how this self translates into a specific mathematical practice. Of course, at a superficial level, no mathematician views their theorems as true in some cases only. However, betwixt the strict logical interpretation of this statement and actual mathematical practice, some leeway exists: unlike Halphen, some other contemporary algebraic geometers of a different milieu practiced a form of generic reasoning, wherein expressions which explicitly include every possible counter-example are not needed, nor perhaps even wanted.[55]

While the epistemic virtue of rigor has long been present in mathematicians' representations of their craft[56], its presence in the discourse of Halphen and his colleagues had a peculiar flavor. In a context of growing anxiety, after a military defeat which had been largely attributed to an imbalance in scientific advancements between France and Germany[57], as well as rising internal tensions within the body of mathematical knowledge itself[58], Halphen's work was viewed as the salvation brought by a new kind of mathematician: an analyst whose scientific ethos, mathematical methods, and epistemic ends were foreign to Chasles' pure geometer. Whereas the latter strived for the naturalization of theoretical settings, and set their mind to searching the simplest and most fundamental properties of geometrical figures, the analyst investigated with utmost precision the domain of validity of each theorem, discussed every possible counter-example, and used analytical means to expurge all possible vagueness from mathematical language. To the authority of the genial *académicien* that was Chasles, Halphen opposed that of the *'specialized disciplinary expertise'* of those with a hard-earned mastery of the modern techniques of algebraic analysis.[59] Between these two figures, a fall from geometrical grace happened: Halphen would not think of mathematics as a domain of knowledge in which Nature provides simple and general formulas to the acute observer, but rather as a set of hidden truths to be coldly besieged and eventually attacked. His alternative theory of characteristics leads to no neat and concise formula for the enumeration of conics; it even rejects the possibility for any finite number of terms to express a general solution to Chasles' problem. For someone who evaluates a theory on the ground of the ease and systematicity of its use, this would be a major setback. To Halphen, for whom such lofty hopes of simplicity were unfounded, this was nothing more than another sign of the deceiving character of naïve intuition.

Schubert: Human, all too human mathematics

Halphen's refutation built on intricate algebraic computations, which several of his colleagues admittedly struggled to understand. Even after the delayed publication of Halphen's memoirs in three of Europe's most famous mathematical journals, the explanations and arguments for his alternative formulae met a mixed reaction. While no one contested the mathematical skill displayed in these texts, several geometers elected to keep on using a formula which had been so fruitful and seemingly correct for years, sometimes merely adding a footnote or a passing remark mentioning Halphen's criticism to memoirs or papers which fully depended on Chasles' formula. This was the case of Hermann Schubert, a Gymnasium teacher in Hamburg, who had begun working on the theory of characteristics at around the same time as Halphen. In 1870, Schubert had written for a doctoral thesis a faithful and competent adaptation of Chasles' theory to second-order surfaces, with little in the way of notational or conceptual innovation.[60] Things would change drastically toward the end of the year 1873. In a seemingly anodyne paper, Halphen had made an observation to which he would never come back or attribute any particular importance. To Schubert, however, this observation would mark the birth of an entirely new way of writing, proving, and understanding enumerative properties of figures.

In 1864, Chasles had already obtained a complex expression for the number of conics satisfying five given conditions with associated coefficients (α_1, β_1), (α_2, β_2), .. , (α_5, β_5), by applying to them the $\alpha\mu + \beta\nu$ formula five times in a row. This expression was too unwieldy for theoretical use, but allowed for moderately faster computations when given five concrete geometrical conditions. Halphen, in 1873, at a time when he still believed in the validity of the $\alpha\mu + \beta\nu$ theorem, noticed that the final expression could be expressed through the much simpler formula
$(\alpha_1 p + \beta_1 d)\ (\alpha_2 p + \beta_2 d) \dots (\alpha_5 p + \beta_5 d)$
where the letters p and d are to be understood as variables, like the x's and y's of a polynomial equation.[61] Halphen showed that the expression above yielded the desired number of conics, provided that it be developed, and that each symbol $p^i d^{5-i}$ be respectively replaced by the numbers of conics passing through i points and touching 5-i lines. For Halphen, this was no more than a symbolic manipulation, helpful to make the general formula easier to handle. In particular, the above compact expression had no intrinsic meaning, and the letters p and d denoted empty variables, only to be instantiated at the end of a symbolic computation via concrete numbers.

Schubert, however, read something far more general and powerful in Halphen's paper, and immediately started publishing a series of articles making a very creative and fruitful use of this observation. In the first of these papers, he immediately reinstored the symbols for characteristics μ and ν in Halphen's general formula instead of the empty variables p and d.[62] He read this formula as expressing the fact that the product of five $(\alpha_1\mu + \beta_1\nu)$ modules of five given conditions does in fact represent the number of conics satisfying these five conditions, for the symbols $\mu^i\ \nu^{5-i}$ represent a composed condition of dimension 5, that is to say a condition which can only be satisfied by a finite number of conics. In Schubert's view, one could simply combine symbols for conditions, as if they were algebraic entities, and factors of maximal dimension would simply represent finite numbers of solutions.[63] For instance, if the symbols P and G denoted respectively the conditions 'passing through a given point' and 'touching a given line', then the symbol PG would denote the composed condition 'passing through a given point and touching a given line'. When turning to the enumerative geometry of a certain figure, Schubert would enumerate basic conditions which can be imposed to such curves, represent them with symbols, and proceed to give elementary algebraic formulas ruling the use of these symbols. The juxtaposition of two such symbols would stand for the conjunction of the two initial conditions, while additions of symbols would denote the disjunction of the corresponding conditions; and the symbolism of algebra would provide a way to compute with geometrical conditions.

The wildly alogical nature of Schubert's mathematical act is striking. The validity of symbolic manipulations in Halphen's memoir rested on the truth of the $\alpha\mu + \beta\nu$ formula, and constituted little more than a rewriting of this formula, with algebra being used to prove, delineate and explore geometrical truths. Schubert, however, had turned this elementary observation into a full-fledged, autonomous symbolic apparatus, the formal justification of which would remain lacking in the eyes of the majority of his colleagues.[64] From a fruitful re-reading of a consequence of Chasles' formula for conics, Schubert developed a symbolism whose stated goal was to combine geometrical conditions of all kinds, and to deal with all sorts of geometrical figures, with algebra serving now as a model for a new geometrical language, rather than as a conceptual tool for stating and proving theorems.

The radical novelty of Schubert's approach cannot be understated. Not only did Schubert obtain a plethora of new results – some of which were deemed extremely impressive, such as the enumeration of 666,841,088 quadrics tangent to 9 others; but his was also an entirely new way of researching, writing and presenting the results of geometrical investigations. Unlike Halphen's theorem-oriented memoirs, and more like Chasles' texts, Schubert found value in the production of long lists of formulae. These lists, however, differed significantly from Chasles'. They were not the systematic and voluntarily monotone enumeration of properties of geometrical conditions, but rather tables of symbolic expressions to be used in the course of enumerative computations (see fig. 3 below).

Insert figure 3 here.

Schubert's shaping of the *Kalkül* was contemporary to, and partly shaped by an epistolary exchange with Halphen and Zeuthen which began in 1876, shortly after Schubert published his first complete presentation of enumerative geometry[65], and Halphen his first counter-examples to Chasles' formula. The tone of the discussion was at first cordial, as Halphen helped Schubert become a member of the SMF, and as the Royal Danish Academy, of which Zeuthen was a leading member, had just awarded him a gold medal for his work on cubics.[66] As Halphen published his counter-examples, however, the exchange turned polemical. Schubert continued to write papers deploying his *Kalkül* on various geometrical figures, systematically searching for analogs of Chasles' formula (of which he himself co-authored a proof with Adolf Hurwitz in the immediate wake of Halphen's refutation).[67] Halphen viewed this enterprise as doomed, since he had just shown that such a problem could not be solved even in the simple case of conics; but failed to convince Schubert to either renounce his project, or to take the necessary precautions to account for these new counter-examples. Schubert's letters, to which the answers are not extant, show the German mathematician maintaining a friendly and even disciple-like tone, even at some point acknowledging Halphen's merits (whose proofs he half-admitted he could not fully understand), while making no changes whatsoever in his own published work. In turn, contemporary letters from Zeuthen to Halphen (to which the replies are not extant either, save for a few which were transcribed in Halphen's *Oeuvres complètes*) reveal that Halphen was becoming progressively annoyed with his interlocutor's reluctance to take note of the newly-established falsity of Chasles' formula.

Toward the end of the year 1879, Halphen grew restless and demanded that he and Schubert make their disagreement public and discussed during one of the bi-weekly meetings of the SMF (which Schubert could not attend). Having revisited his previous work on conics, Halphen was now able to produce counter-examples for each general formula of Schubert's whatever the geometrical figure at hand, almost on command. Schubert would at first attempt to save his formulas, by arguing that they are simply meaningless in the problematic cases that Halphen was pointing to.[68] In other documents, he would propose reinterpretations of Halphen's counter-examples to turn them into "*interesting verifications"* of his own formulas, without ever engaging with the Frenchman's intricate analytical discussion of what it means to satisfy a geometrical condition.[69] Such defenses would be printed in December 1879, until eventually, in January 1880, Schubert gave up on defending his formulas.[70] While a retraction was soon published in the *BSMF,* Schubert continued with his mathematical practice unaltered whatsoever in articles published at the same time in German journals.[71] This concession made on French soil was largely borne out of a desire to maintain peaceful scientific communications with a notorious society, which was of paramount importance to a Gymnasiumlehrer in Hamburg, isolated from the main German mathematical communities, rather than a sincere renunciation.[72]

In letters he wrote to Zeuthen at the same time, Halphen explained his gripes with Schubert's persistence to look for formulas such as Chasles' $\alpha\mu + \beta\nu$:

> Of all the reasons one can enlist against the allegedly *general* theorems, the best is the following: the arguments with which they can be covered disappear when the two beings (C), (Σ) [*(C), (Σ) here refer respectively to a figure which solves a problem, and a system of such figures*] are each defined by more than one equation. In these circumstances, we must abandon *intuition* and come back to Analysis. By this term I mean true reasoning ; I demand no equation, of course. M.Schubert absolutely wants to alter nature to accommodate it to his formulas. We deal with a problem that admits one solution: *One! Are you joking? The formula yields two: therefore there are two!* Do you know how I replied? I took the question to be a particular case of another, wherein the formula yields 5, and then of another, wherein the same formula gives one.[73]

Halphen's criticism focused on Schubert's supposed belief that formulas can give rise to their own meaning. Schubert, according to Halphen, had inverted the proper epistemological order: he accepted the results produced by his symbols, and left no room for critical appraisal of these results. And yet, Halphen thought he had uncovered and revealed the insufficiently precise determination of what this unchecked symbolism represents: to one formula of Schubert's, Halphen had associated two possible, yet contradictory, analytical representations. For this reason, he disparaged Schubert's *Kalkül* as mere "*intuition*", despite there being little that one would spontaneously describe as intuitive in this highly formal and symbolic geometrical practice: intuition, for Halphen, served as a generic term to pejoratively refer to any mathematical practice not based on the careful analysis of the equations or definitions of its concepts. A few days prior to this letter, Halphen had asserted to Zeuthen that he "*knew the meaning of these formulas much better than Schubert, without a doubt*."[74] There again, he pitched his ability to gain insight into the correspondence between complex formulae and geometrical configurations through his mastery of analysis, against whatever loose, potentially fruitful, combination of symbols Schubert had devised. This line of defense could not have contrasted more strongly with Schubert's own depiction and understanding of mathematical practice.

Indeed, Schubert's enumerative geometry can be advantageously read against the backdrop of his own philological and philosophical interests, thus casting new light on his mathematical work and his refusal to really engage with Halphen's criticism. In the wake of his 1879 book and after a decade of intense work on enumerative geometry, Schubert began authoring articles and books on a wider range of subjects, and in a wider range of journals. Among these stand out works on the philology and ethnography of numbers, recreational mathematics, elementary textbooks on algebra, and popular essays on the nature of mathematical knowledge.

In a booklet published in 1885, Schubert expounded what he described as a "cultural-historical study" of the formation of numbers.[75] This text consisted in a sketch of the developmental stages through which the formations of number-words ("*Zahlwortbildung*") and number-signs ("*Zahlzeichenbildung*") allegedly ought to pass. For Schubert, "*the system of numbers that we take for self-explanatory in our childhood is not something that can be taken for self-explanatory, but rather the highest offshoot of a cultural-historical process that began when man became man, when he began to speak and write*."[76] Schubert went on to show how various peoples devised various ways to write numbers and to represent them on account of both their cultural and ecological landscape. Religions and mythologies, as well as surrounding seas or mountains, are possible factors in the development of said number-words and number-signs. In a particularly striking passage, Schubert wrote:

> There is [in the oldest literature of the Brahmans] talk of a king who advanced his wealth to a hundred thousand trillion jewels, of a Monkey Prince who could confront his enemies with 10,000 sextillion monkeys in battle. And in Buddhist times one read of 24,000 trillion deities and of the 600,000 million sons of Buddha. [..] The Greeks were too friendly to the natural and the true, to love such exaggerations. Homer lets a wounded Ares scream like 9- or 10,000 men in the fifth book of the Iliad. In India, a god of war who could only scream like 10,000 men, would be considered asthmatic.[77]

Schubert goes on to explain why certain peoples ("*Volk*") have a need and desire for large numbers, which in turn led them to devise ways of conveniently writing words for large numbers. To craft a word for the number ten thousand, the Greeks created the new word μύριοι (myriad), because they couldn't reasonably foresee a real need for many more such words. The Indians, on the contrary, yearned for ever larger numbers, and so devised a way to express them using number-words which already existed, not unlike contemporary English does with the juxtaposition of the words 'ten' and 'thousand'. Such systems of number-words are ultimately classified by Schubert on a scale which goes from "*Natürliche Zahlreichen*" (numbers being represented by collections of points or other tokens) to the "*Prinzip des Stellenwerthes*", which corresponds to our modern way of writing the so-called Arabic numerals.[78]

In this book and further publications, Schubert displayed an exhaustive knowledge of the contemporary philological and ethnographical ("*Forschungssreisende*") literature. He was in close contact with both explorers and philologists, publishing a summary of his views in the second edition of German explorer Georg von Neumayer's *Guides to scientific observations on travels*[79], and participating to the *Kongress deutscher Philologen und Schulmänner* in 1905. Schubert's *kulturgeschichtliche* project also bears the mark of a larger German tradition of cultural history of mathematics, to which was most famously associated Moritz Cantor, but which actually goes back to Arthur Arneth. A professor of mathematics at the Heidelberg

Lyceum, Arneth "*viewed the abstraction process leading to mathematical content as being conditioned by cultural factors*."[80] In tying these locally- and culturally-rooted mathematics together into an ultimately universal science, which surmounts national and regional characteristics, Schubert shifted this tradition closer to the cosmopolitan and Humboldtian historiography of Hermann Hankel, another German mathematician at the crossroads of philology, mathematics, and history of mathematics.[81]

Schubert's interest in philology and ethnography is crucial for understanding his epistemology of mathematics and the regulative ideal of mathematical activity which underlay his geometry.[82] Indeed, in a series of articles for the newly-created journal *The Monist*[83], as well as in the very first chapter of Klein's and Wilhelm Meyer's *Encyklopädie der mathematischen Wissenschaften*, Schubert built on this aforementioned study of the (cultural) history of numbers.[84] From his study of "primitive" systems of numerations, and his understanding of the developmental stages of the path to ideal number systems, he attempted to derive a philosophical account of what numbers are, as well as what strings of symbols of numbers and operations represent.[85] "*Counting a group of things*", Schubert proposed, "*is to regard the things as the same in kind and to associate ordinally, accurately, and singly with them other things. In writing, we associate with the things to be counted simple signs, like points, strokes, or circles*."[86] Philological and ethnographical studies paint before our eyes the original mathematician as a crafter of signs, words, and symbols, who progressively emancipates their science from the local cultural and ecological landscape it originated from. Once such emancipation has been achieved, the mathematician's numbers are pure cultural creations:

> Observation of the world of actual facts, as revealed to us by our senses, can naturally lead us only to positive whole numbers, such only, and no others, being results of actual counting. All other kinds of numbers are nothing but artificial inventions of mathematicians.[87]

How, then, are we to know how to operate on these unnatural numbers? Schubert's solution to this question, while not completely unoriginal, borrows extensively from Hankel's work on systems of numbers, and in particular on his *principle of permanence.* Schubert renamed it the "principle of no exception", and summarized it as follows:

> In the construction of arithmetic every combination of two previously defined numbers by a sign for a previously defined operation (plus, minus, times, etc.) shall be invested with meaning, even where the original definition of the operation used excludes such a combination ; and the meaning imparted is to be such that the combination considered shall obey the same formula of definition as a combination having from the outset a signification, so that the old laws of reckoning shall still hold good and may still be applied to.[88]

Crucial for both Schubert's and Hankel's understandings of what systems of numbers are, is the latter's proof of the theorem that there can't possibly be any extension of the system of complex numbers which preserves basic algebraic laws, such as commutativity (ab = ba). Hamilton's quaternions, for instance, are an extension of complex numbers in which the order of multiplication matters.[89] For Schubert, this shows that "*the building up of arithmetic is thus completed*", and that this science has reached absolute perfection because it derives from a single, '*monistic principle*'.[90]

The connection between these views and geometry would appear most clearly in Schubert's rebuttal of the spiritualist theses of German astrophysicist Johann Zöllner.[91] Toward the end of his life, Zöllner had argued that the mathematics of four-dimensional spaces and its physical interpretation form a rational and scientific basis for spiritualism, that is to say the study of the spirits of the dead. Rejecting any attempt to use pure mathematics to naturalize such phenomena, Schubert insisted on the purely artificial character of the numbers the mathematician freely constructs in the course of their work. Dimensions are but one example of such artificial numbers:

> Is it permissible to extend the notion of space by the introduction of point-aggregates of more than three dimensions? [..] In mathematics, in fact, the extension of any notion is admissible, provided such extension does not lead to contradictions with itself or with results which are well established. Whether such extensions are necessary, justifiable, or important for the advancement of science is a different question. It must be admitted, therefore, that the mathematician is justified in the extension of the notion of space as a point-aggregate of three dimensions, and in the introduction of space or point-aggregates of more than three dimensions, and in the employment of them as means of research.[92]

Schubert's views echoed once more those of Hankel's, who had famously claimed that "*number is no longer an object, a substance which exists outside the thinking subject and the objects giving rise to it, an independent principle, as it was for instance for the Pythagoreans. [..] Only that counts as impossible for the mathematician which is logically impossible, i.e. that which contradicts itself*."[93] For Schubert, the mathematician wields symbols and concepts with no intrinsic relation to natural objects whatsoever. The sole rules of such an activity are that it should preserve past discoveries, and introduce no new contradiction. This is not to say that anything goes: mathematics, for Schubert, is always located on a path of progression, of which the end goal is the "*[unification] under a high point of view of theories heretofore regarded as different*."[94]

To view Schubert as a philosopher of mathematics is bound to lead to disappointments: his writings do not have the finesse and argumentative solidity to withstand assaults from the likes of Frege, who harshly dismantled his views on numbers in an ironic review.[95] There is, however, much to gain from reading Schubert's texts as depicting a regulative ideal of mathematical activity, one that already ruled his geometrical research. Indeed, Schubert's enumerative geometry, as the name suggests, is a science of the numbers of geometry. In some instances, he even uses expressions such as "geometrical numbers" to refer to the symbols of his *Kalkül*.[96] Thus, the conclusions of his later philosophical papers are strongly tied to the way he envisioned and ruled his geometrical practice. We can now understand why Halphen's criticism failed to elicit a strong reaction from Schubert. Halphen accused Schubert of "altering Nature", but this accusation could not sway the German geometer, for whom mathematicians were free to craft symbols and numbers as they saw fit, as long as no contradictions were thus introduced, in the hope of finding a path to a unitary formulation of the solution to a geometrical problem. The rhetorical recourse to Nature, whether in the form of Chasles' account of geometrical practice as the search for fundamental properties from which theories can be effortlessly derived, or of Halphen's description of the Analyst using his expert training and tools to track the traps and counter-examples which lay in our imprecise intuition of Geometry, was ultimately meaningless for Schubert. Here again, a clash of geometers who shared little understanding of what mathematical activity consists in and what its goals are, made constructive dialogue nigh impossible.

Study: A matter of point of view?

A subtle way out of Schubert's and Halphen's impossible dialogue had been previously proposed by Hieronymus Zeuthen. In a long letter, written as a response to Halphen's announcement that he 'had no more doubts regarding the falsity of the $\alpha\mu + \beta\nu$ theorem',[97] Zeuthen suggested that it might be possible to "*preserve Chasles' theorem by adopting another point of view*."[98] Zeuthen then distinguished between three *points of view* on what conics are. The first one, which he attributed to de De Jonquières (Chasles' aforementioned student), consisted in "*defining conics exclusively by their punctual properties*"; the second one, which assumedly was Chasles', consisted in "*regarding in an equal manner punctual and tangential properties*." Halphen's point of view, which had lead him to reject the $\alpha\mu + \beta\nu$ formula, was not characterized by Zeuthen in a similar manner, but only said to be "*entirely clear and well-defined*." These points of view were not equal: while the first one is "*simple and very clear*", it suffers from being altered by the principle of duality, and leads to infinite numbers of solutions or other such meaningless results in some enumerative problems. As for the second one, claims Zeuthen, "*Chasles' theorem is such an intimate consequence [thereof], that its proof would present itself, were we only able to define it precisely*." Halphen's viewpoint was the only one deemed sufficiently clear and precise, while not presenting decisive geometrical flaws.

Halphen adopted this presentation in early publications about his counter-examples, but never fully committed to it.[99] His last letters to Schubert, such as the one quoted previously, completely breaks from it: in lieu of contrasting viewpoints, Halphen rooted the authority of his theory in the nature of geometrical objects and its examination through analysis. Zeuthen's solution, however, was not lost on everyone: Henri Poincaré, who acted as editor of Halphen's collected works after his death, had initially offered Zeuthen to publish their correspondence. While only a small fraction of Halphen's letters were eventually published, Poincaré was able to survey the exchanges with Schubert and Zeuthen while editing Halphen's complete works. As he wrote his obituary for Halphen, Poincaré would reformulate Zeuthen's presentation of said exchanges, with a twist of his own: "*points de vue*" had become "*conventions*", and Halphen was now credited with being the first to make explicit and perfectly precise the possible conventions one can adopt regarding the question of generality in enumerative geometry.[100] The notion that the validity of a theorem may depend on conventions or viewpoints, however, would be put to more critical use by Eduard Study, to whom Klein had advised in 1884 to write his *Habilitationsschrift* on the disputes plaguing enumerative geometry. Indeed, Study's dissertation would display yet another understanding of the meaning of Chasles' formula, as well as another way to understand the nature of the epistemic task at hand for the enumerative geometer.

Not unlike Halphen, Study initially pit the intuition of the geometers of the past against logical deductions and concepts, which alone could end the turmoil surrounding Chasles' formula:

> If, however, one wants to settle with complete generality a problem which in special cases is treated in an intuitive manner, then one must move from intuition (*Anschauung*) to concepts (*Begriffe*), and put logical deductions in place of appeals to appearances. Often, in individual cases, the latter are merely one's silent confession of the insufficient awareness of the true reasons behind a result.[101]

Study viewed Clebsch as having taken a decisive step forward in that direction, and, like Halphen, he aimed to clarify the content of the vague geometrical concepts that had led Chasles to his formula. However, Study was not ready to give up on Chasles' formula. To defend it, he expounded a new way of conceptualising generality in geometry:

> One has [..] to distinguish between the properties of the figures whose presence is regarded as the necessary and sufficient condition for the existence of those other properties which represent the geometric proposition, and the others which are regarded as consequences of said proposition, or only conditioned by the arbitrary manifestation of the general proposition. The former must be elevated to the rank of definitions and made the basis of proof. This operation is performed by anyone who makes a generalization, intentionally or not. Since it consists mainly in clarifying one's conception of what is essential for a proposition, it can be carried out without appearing to one's consciousness as a progress of thought.[102]

Study put the emphasis on the clear delineation of these fundamental properties of a figure, which characterize the permanence of the other properties of said figure. His strategy, going forward, would be to search for the property which, in Chasles' theory, had implicitly characterized conics in enumerative context. Study claimed that this was also Clebsch's strategy, but that the latter was misled in his search. Halphen's counter-examples, thus, only showed that Clebsch's algebraic characterization of conics (for enumerative purposes) was not adequate. However, Study added, the "*definition of solutions is arbitrary*"; and there was another interpretation of Chasles' formula which makes it absolutely precise and valid.[103] To expound it was the purpose of Study's dissertation.

Later on, as Study travelled to Paris with Hilbert, he met Halphen in person, but neither of them was capable of changing the other's mind.[104] From the few letters they also exchanged around this period, it appears that their divergences were more than simply mathematical. Throughout this exchange, Study insisted that "Chasles did not have a sufficiently clear idea of the nature of the solutions which were to be counted; so that [Halphen's] conception of the theorem and [Study's] both should be regarded as interpretations, and indeed as equally valuable interpretations of the original formulation to be determined."[105] Study wanted to frame the relation between their memoirs as that of two equally possible interpretations of Chasles' theory of characteristics, which lead to two different truth-values for the $\alpha\mu + \beta\nu$ formula, thereby concluding that the validity of this formula is indeed a matter of convention. Study then attempted to convince Halphen that the latter's interpretation is insufficiently faithful to what Chasles had in mind. But Halphen cared little for this new framing of his own work. In a brief reply, he "*[persisted] in finding nothing new or useful in [Study's] interpretation of Chasles' theory*."[106] The discussion between Study and Halphen was not just one between two mathematicians who disagree on a technical issue: it was, yet again, the confrontation of two different figures of the mathematician.

Despite his defense of Chasles' formula and his emphasis on the arbitrariness of mathematical definitions, Study was no ally of Schubert's, but rather one of his staunchest critics. In a later paper, he attacked one of the principles at the heart of Schubert's *Kalkül*, which he viewed as a symptom of another overarching problem plaguing contemporary geometry:

> In countless cases, the objects of geometrical investigations are so unclearly explained, that one has to guess the meaning (*Sinn*) of individual concepts (*Begriffe*) from the

> assertions made about them, whereby differences in opinion can naturally arise. [..] First of all, the concept of geometric figure, as defined by M.Schubert and explained by his examples, has such an unusual scope (*Umfang*), that it is most unlikely that anything universally valid could be said about it at this point. One is immediately forced to resort to an interpretation.[107]

Like Halphen, Study accused Schubert of inverting the proper epistemic order between definition and investigation, as the latter lets his symbols freely operate, and never searches for the concept behind them. A staunch realist, even in the face of Einstein's introduction of non-Euclidean geometries in physics, Study railed against axiomatizers who never inquired about what objects fell under their definitions, and those who, like Hilbert, equated the coherence of an axiomatics to the existence of its objects. Arbitrariness ran amok, Study thought, could just as well lead to a state of generalized incomprehension amongst a mathematicians, and the creation of mathematical concepts, while free, must always be "*motivated*", lest "*we let the creature* (das Geschöpf) *become the Creator* (zum Herren werden)."[108] Against Schubert's concept-free symbolism, or Chasles' intuitive geometry, but also against Halphen's restrictive reliance on a natural, yet possibly irregular, theory of conics, Study reconciled arbitrary definitions with an intransigeant emphasis on the importance of the mathematician's duty to precisely measure and delineate the extension of the concepts they freely produce.

Conclusion: From Truth to Significance

As the disciplinary and cultural identity of the wielders of the formula differed, so did the definition of its terms and its status *qua* mathematical proposition.[109] Such were the shifts in epistemic ideals and norms to which the groups of mathematicians involved in this circulation were beholden, that even the truth-value of this formula fluctuated. By the time Chasles' formula finally reached the Dutch geometer Van der Waerden, who produced in 1938 the first widely-accepted proof thereof, mathematics had largely gone through the so-called modernist transformation, which had been merely nascent in the writings of Schubert, Halphen, and Study. For Van der Waerden, mathematics was solely about the derivation of "*flawless proofs*" from rigorously defined frameworks: in the case of Chasles' formula, this included redefining every single term of its statement, even including that of a "*number*" of solutions.[110] Not subject to Study's strict realist regimen, Van der Waerden did not expect concepts to capture a pre-theoretical (and, indeed, pre-axiomatic) meaning, and thus did not feel obliged to measure the extension of said concepts: squabbles over which framework best captures Chasles' original intuition were of no interest to him. With Van der Waerden's proof, Chasles' formula had not only gained entry into the commonly accepted body of mathematical knowledge: it had finally been absorbed into modern mathematical practice.

As Daston and Galison have pointed out in *Objectivity*, however, epistemic ideals never disappear; and the succession of portraits presented here is not a series of replacements, but of confrontations.[111] The tensions constitutive of the modernist transformation of mathematics can and indeed still do reappear, albeit with a decidedly novel ring to them. If concepts are freely postulated, how can one ensure that they indeed capture the original geometrical intuition, and that the results they encapsulate are indeed those that were being sought after? This classical philosophical critique to naïve mathematical formalism (or, in Lakatos' term, *Euclideanism*) largely motivated a somewhat recent attempt to reinvigorate Halphen's criticism of Chasles' formula. Indeed, while the formalism most commonly used in contemporary algebraic geometry is one in which Chasles' formula is true, some argue that frameworks in which the formula is false have a more profound "*enumerative significance*."[112] Beyond Chasles' formula, the interconnected values of naturalness and simplicity in mathematics feature prominently in the autobiography of French mathematician Alexandre Grothendieck[113], while a re-emerging anxiety of the uncertainty of a growing part of the body of mathematical knowledge has led many to call for new standards of proof and communication amongst practitionners.[114] To account for the return of these epistemic virtues, a new cultural history of the figure of the mathematician is needed.

Captions for figures :

Figure 1: A system of conics touching one line and passing through three points, in F. von Lindemann, *Vorlesungen über Geometrie* (Leipzig:Teubner, 1876), p.395.

Figure 2: Chasles' lists of propositions derived from the uniform application of the 'principle of correspondence', courtesy of the Académie des sciences de Paris, Archives et patrimoine historique (Dossier Chasles, 35J/4).

Figure 3: Another list-making practice displayed in Schubert's *Kalkül der abzählenden Geometrie* (1879), p.150.

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1 David Rowe, "Klein, Hilbert and the Göttingen Mathematical Tradition", *Osiris,* 1989, 5:186-213, doi.org/10.1086/368687. While we shall not emphasize it, the globalization of mathematics was an important feature of this historical episode. On global mathematics, see Michael Barany, "Integration by Parts: Wordplay, Abuses of Language, and Modern Mathematical Theory on the Move", *Historical Studies in the Natural Sciences*, 2018, 48-3:259-299, doi.org/10.1525/hsns.2018.48.3.259.

2 Michel Chasles, "Formules générales comprenant la solution de toutes les questions relatives aux sections coniques", *Comptes Rendus de l'Académie des Sciences de* Paris, 1864, 59:209-218. Enumerative geometry has received scant historical attention thus far; notable exceptions include Steven Kleiman, "Chasles' enumerative theory of conics", in Abraham Seidenberg (ed.), *Studies in Algebraic Geometry* (Washington:Mathematical Association of America, 1980), pp.117-138; Yvonne Hartwich, *Eduard Study (1862-1930) – ein mathematischer Mephistopheles im geometrischen Gärtchen*, PhD Thesis, Johannes Gutenberg Universität in Mainz, 2005.

3 Michel Chasles, "Détermination du nombre de sections coniques qui doivent toucher cinq courbes données d'ordre quelconque etc.", *Comptes-Rendus de l'Académie des Sciences*, 1864, 58:222-226. On Steiner's geometry, see Jemma Lorenat, "Synthetic and analytic geometries in the publications of Jakob Steiner and Julius Plücker (1827–1829)", *Archive for History of Exact Sciences*, 2016, 70-4: 413-462, doi.org/10.1007/s00407-015-0174-8.

4 Sir Lieutenant-General Edward Sabine, "President's address", *Proceedings of the Royal Society of London*, 1865, 14:482-513. Created in 1731, and first intended to celebrate the best experiment of the year, the Copley Medal soon became a coveted prize to celebrate contributions to "natural knowledge"; see Y. Bektas and M. Crosland, "The Copley Medal: The Establishment of a Reward System in the Royal Society, 1731-1839", *Notes and Recordings of the Royal Society of London*, 1992, 46:43-76. The attribution of this medal was mostly the result of intense lobbying by English mathematician Thomas Archer Hirst.

5 Alfred Clebsch, "Zur Theorie der Charakteristiken", *Mathematische Annalen,* 1873, 6:1-15. Georges-Henri Halphen, "Mémoire sur la détermination des coniques et des surfaces du second ordre", *Bulletin de la Société Mathématique de France*, 1872-3, 1:130-142.

6 Ferdinand von Lindemann, *Vorlesungen über Geometrie von Alfred Clebsch* (Leipzig:Teubner, 1876), pp.397-401.

7 Georges-Henri Halphen, "Sur les caractéristiques des systèmes de coniques", *Comptes Rendus de l'Académie des Sciences de Paris*, 1876, 83:537-539.

8 Hermann Schubert, *Kalkül der abzählenden Geometrie* (Leipzig:Teubner, 1879). In what follows, we shall refer to this book as *Kalkül*.

9 Several letters from Schubert to Klein are preserved at the Göttingen Staatsbibliothek, Cod Ms Klein 11.

10 Eduard Study, "Über die Geometrie der Kegelschnitte, insbesondere deren Charakteristikenproblem", *Habilitationsschrift, Universität Leipzig*, (Leipzig:Teubner, 1885). Thomas Hawkins, in "The *Erlanger Programm* of Felix Klein: Reflections on Its Place in the History of Mathematics", *Historia Mathematica*, 1984, 11:442–470, contends that 'Klein [did not exert] any significance upon [Study]' (pp.449-450). A more nuanced account of their relationship can be found in Yvonne Hartwich, op.cit., pp.50-64.

11 Yvonne Hartwich, op. cit., pp.73-78.

12 Hieronymous Zeuthen, "Sur la révision de la théorie des caractéristiques de M.Study", *Mathematische Annalen,* 1890, 37:461-464; Eduard Study, "Entgegnung", *Mathematische Annalen*, 1892, 40:559-562; Hieronymous Zeuthen, "Exemples de la détermination des coniques dans un système donné qui satisfont à une condition donnée", *Mathematische Annalen*, 1893, 41:539-544.

13 Baertel Van der Waerden, "Zur algebraischen Geometrie. XV. Lösung des Charakteristikenproblems für Kegelschnitte", *Mathematische Annalen*, 1938, 115:645-655, see footnote 3, p.646.

14 Herbert Mehrtens, *Moderne Sprache Mathematik: eine Geschichte des Streits um die Grundlagen der Disziplin und des Subjekts formaler Systeme* (Frankfurt:Suhrkamp, 1990), p.8.

15 Jeremy Gray, *Plato's Ghost: The Modernist Transformation of Mathematics* (Princeton:Princeton University Press, 2008), p.1.

16 Mehrtens, op. cit., pp.9-11; 394-404.

17 Gray, op. cit., pp.9-12.

18 Modernism in mathematics is usually measured against the backdrop of a consensus-etalon in 1880, see Gray, op. cit., p.112.

19 Leo Corry, "How Useful is the Term 'Modernism' for Understanding the History of Early Twentieth-Century Mathematics?", in Moritz Epple and Falk Müller (eds.), *Science as a Cultural Practice, Vol.II: Modernism in Science 1900-1940* (Berlin:Akademie Verlag, forthcoming).

20 For instance, William Everdell, *The First Moderns: Profiles in the Origins of Twentieth-Century Thought* (Chicago: The University of Chicago Press, 1997); Daniel Albright, *Quantum Poetics: Yeats, Pounds, Eliot, and the Science of Modernism* (Cambridge:Cambridge University Press, 1997); Nina Engelhardt, *Modernism, Fiction and Mathematics* (Edinburgh:Edinburgh University Press, 2018).

21 Lorraine Daston and Peter Galison, *Objectivity* (New-York:Zone Books, 2007), p.198. For ampler discussions of the concept of scientific self, see pp.35-50; 216-233; 367-371. For more recent scholarship on scientific selves, see for instance Paul White, "Darwin's Emotions. The Scientific Self and the Sentiment of Objectivity", *Isis*, 2009, 100-1:811-826, doi/10.1086/652021; and Ian Hesketh, "Technologies of the Scientific Self: John Tyndall and his Journal", *Isis*, 2019, 110-3:460-482, doi:10.1086/704672.

22 These selves are not to be equated with moral biographies: behind each of them lie broader characters, which for the purpose of brevity we locate only in certain individuals. For an approach that builds off Daston's and Galison's work on a resolutely more local scale, see Hermann Paul, "The Virtues and Vices of Albert Naudé: Toward a History of Scholarly Personae", *History of Humanities*, 2016; 1-(2):327-338, doi:10.1086/688036.

23 This post was created specifically for Chasles, under the impulsion of Louis Poinsot. Chasles officially occupied it until the year 1879, at the age of 86 years old. In 1868, Pierre-Ossian Bonnet began to regularly assist Chasles in his teaching duties. Ten years later, Bonnet obtained the chair of Astronomy left vacant by Le Verrier's death, and Gaston Darboux, a rising star of French mathematics, replaced Chasles as professor of high geometry.

24 Conic sections, here, refer to curves which can be obtained as the intersection of a cone and a plane; alternatively, they can be defined as the curves whose equations are polynomials of the second degree in two variables x and y, viz.: $ax^2 + by^2 + cxy + dx + ey = 1$. A conic is uniquely defined by five coefficients (a,b,c,d,e); therefore, only a finite number of such curves simultaneously satisfy five independent conditions. For the technical

details of this mathematical problem, see Andrew Bashelor, Amy Ksir, Will Traves, "Enumerative Algebraic Geometry of Conics", *American Mathematical Monthly*, 2008, 115:701-728, doi.org/10.1080/00029890.2008.11920584.

25 For an overview of the rivalry between pure and synthetic geometries in the 19th century, see Jemma Lorenat, *"'Die Freude an der Gestalt": methods, figures and practices in early nineteenth century geometry,* PhD thesis, Université Pierre et Marie Curie, 2015, especially ch.1, and Jeremy Gray, *Worlds Out of Nothing. A Course in the History of Geometry in the 19th Century* (London:Springer-Verlag, 2007), ch.1-6.

26 Michel Chasles, *Aperçu historique sur le développement des méthodes etc.* (Paris:Gauthier-Villars, 1837). See Karine Chemla, "The value of generality in Michel Chasles' historiography of geometry", in Karine Chemla, Renaud Chorlay, David Rabouin (eds.), *The Oxford Handbook of Generality in Mathematics and the Sciences* (Oxford:Oxford University Press, 2016), and Nicolas Michel, "The values of simplicity and generality in Chasles' geometrical theory of attraction", *Journal for General Philosophy of Science,* 2020, 51:115-146, doi.org/10.1007/s10838-019-09451-z.

27 On Monge's teaching of geometry, see Bruno Belhoste et René Taton, "L'invention d'une langue des figures", in Jean Dhombres (ed.), *L'école normale de l'an III*, 5 vols. (1992-2016), vol.1 (Paris:Presses de l'Ecole Normale, 1992), doi.org/10.4000/books.editionsulm.442.

28 Michel Chasles, *op.cit.*, pp.253-254. Neither Poncelet nor Chasles were able to attend Monge's teaching in person, as the latter ceased to teach and exert any real institutional power after 1810, in part due to illness. See Bruno Belhoste, *La formation d'une technocratie. L'Ecole Polytechnique et ses élèves de la Révolution au Second Empire* (Paris:Belin, 2003), pp.195-212.

29 Michel Chasles, *op. cit.*, pp.2-3.

30 *Ibid.*, p.114.

31 *Ibid.*, pp.268-269.

32 Ken Alder, *Engineering the Revolutions: Arms & Enlightenment in France, 1763-1815* (Chicago:University of Chicago Press, 1997), pp.316-317.

33 Ernest de Jonquières, review of Michel Chasles, *Les trois livres de Porismes d'Euclide* (Paris:Gauthier-Villars, 1860), in "Bulletin de bibliographie, d'histoire et de biographie mathématique", *Nouvelles Annales de Mathématiques*, 1861, 20:1-11. The quote is pp.8-9.

34 The Marxist historian of industrialization Tom Kemp notes that French industrial capitalism was stimulated with great effect by the development of railways in the wake of the 1848 Revolution, see T. Kemp, *Industrialization in Nineteenth-Century Europe* (London & New-York:Routledge, 2014, second edition), pp.49-77. In parallel, industrialization came to polarize French academic circles as well, as shows Le Verrier's ascension at the head of the Ecole Polytechnique in 1850 as part of a general push by industrialist lobbies for more practical teaching, or by the “bifurcation”, a reform of the Baccaulauréat separating the sciences and the humanities, passed in 1852 (see Belhoste, *op. cit.*, pp.97-102). Chasles was publicly opposed to both reforms.

35 This was the case in England, most famously in the work of Babbage, but also in France, where De Colmar had developed his own calculating machine. See Eduardo L. Ortiz, "Babbage and French *Idéologie*: Functional Equations, Language, and the Analytical Method", in J. Gray and K.H. Parshall (eds.), *Episodes in the History of Modern Algebra (1800-1950)* (US:American Mathematical Society, vol.32, 2011), pp.13-47, and Matthew Jones, *Reckoning with Matter: Calculating Machines, Innovation, and Thinking about Thinking from Pascal to Babbage* (Chicago:University of Chicago Press, 2016), ch.6.

36 For a suggestive analysis of the epistemology of early 19th-century French pure geometry in empiricist terms, see Lorraine Daston, "The Physicalist Tradition in Early Nineteenth Century French Geometry", *Studies in History and Philosophy of Science*, 1985, 17:269-295.

37 Michel Chasles, "Construction de la courbe du troisième ordre déterminée par neuf points", *Comptes-Rendus de l'Académie des Sciences*, 1853, 36:943-952.

[38] Michel Chasles, *Rapport sur les progrès de la géométrie* (Paris:Hachette, 1870), p.269. This means that one is the equation whose roots are the points of a conic, while the roots of the other yield the tangent lines to the conic. The duality intrinsic to the theory of conics means that these propositions cannot be subsumed under a single one.

[39] As argued in Karine Chemla, *op. cit.*, p.60, Chasles' ideal of geometrical practice is one '*with almost no proof*'.

[40] Michel Chasles, "Considérations générales sur la méthode générale etc.", *Comptes-Rendus de l'Académie des Sciences*, 1864, 58:1167-1175.

[41] μ is the number of conics in the system passing through a given point, while ν is the number of conics in the system touching a given straight line. See Steven Kleiman, *op. cit.*, for the mathematical details.

[42] For instance, a popular textbook in algebraic geometry was named after this result: David Eisenbud and Joe Harris, *3264 & All That: A Second Course in Algebraic Geometry* (Cambridge:Cambridge University Press, 2016).

[43] Georges-Henri Halphen, *Notice sur les travaux mathématiques de M.Georges-Henri Halphen* (Paris:Gauthier-Villars, 1885), p.7.

[44] Michel Chasles, Rapport sur les progrès de la géométrie, pp.378-379. On the early history of this society, see Hélène Gispert, *La France mathématique. La société mathématique de France, 1870-1914* (Paris:Cahiers d'histoire des sciences et de philosophie des sciences, n°34, 1991).

[45] Georges-Henri Halphen, "Sur le nombre des droites qui satisfont à quatre conditions données", *Comptes-Rendus de l'Académie des Sciences*, 1869, 68:142-145.

[46] Halphen was fully aware of the recent developments in algebraic geometry, and his work builds extensively on that of Charles Hermite and Arthur Cayley, but also of German mathematicians such as Max Noether and Clebsch, who had been colleagues in Giessen during the late 1860s. See Laurent Gruson, "Un aperçu des travaux mathématiques de G.-H. Halphen", in G.Ellingsrud, C.Peskine, G.Sacchiero, S.A. Stromme (eds.), *Complex Projective Geometry* (Cambridge:Cambridge University Press, 1992), pp.189-198.

[47] Ferdinand von Lindemann, *op. cit.*, p.1. For more on the nuanced interplay between algebra and geometry in Clebsch's work, see François Lê, "Alfred Clebsch's 'Geometrical Clothing' of the theory of the quintic equation", *Archive for the History of Exact Sciences*, 2017, 71:39-70, doi.org/10.1007/s00407-016-0180-5.

[48] See Karen Hunger Parshall, "Toward a History of Nineteenth-Century Invariant Theory", in David E. Rowe and John McClearly (eds.), *The History of Modern Mathematics*, *vol.1: Ideas and their Reception* (San Diego:Academic Press, 1989), pp.155-206.

[49] George-Henri Halphen, "Notices sur les travaux", pp.9-10.

[50] A degenerate conic is a curve that can be decomposed into lower-degree curves, such as for instance a pair of lines.

[51] George-Henri Halphen, *op. cit.*, p.14.

[52] On the aftermath of this defeat in French scientific communities, see Robert Fox, *The Savant and the State* (Baltimore:The John Hopkins University Press, 2012), pp.227-273.

[53] Henri Poincaré, "Notice sur Halphen", *Journal de l'Ecole Polytechnique*, 1890, 60:137-161. The quote is p.138.

[54] Emile Picard, "Notice sur la vie et les travaux de Georges-Henri Halphen", *Comptes-Rendus de l'Académie des Sciences*, 1890, 110:489-497. The quotes are p.489. This portrait of Halphen's mathematical mind was also partially the result of the latter self-styling, in particular in his Notices.

[55] Thomas Hawkins, "Hesse's Principle of Transfer and the Representation of Lie Algebras", *Archive for history of exact sciences*, 1988, 39:41-73, doi.org/10.1007/BF00329985. In another obituary for Halphen, written by Camille Jordan, comparisons were drawn between his mathematical style and that of Abel, another important figure in the emergence of the notion of mathematical counter-examples; see Henrik K. Sørensen, "Exceptions and counterexamples: Understanding Abel's comment on *Cauchy's Theorem*", *Historia Mathematica*, 2005, 32:453-480,

doi:10.1016/j.hm.2004.11.010. The production of counter-examples, and the shift in epistemic norms it betrays, can be compared to what Daston and Galison presented as the passage from "truth-to-nature objectivity" to "mechanical objectivity" in the Cajal-Golgi dispute, see *Objectivity*, pp.115-125.

56 Judith Grabiner, *The Origins of Cauchy's Rigorous Calculus* (Cambridge, MA:The MIT Press, 1981).

57 Robert Fox, op. cit., pp.259-273.

58 Jeremy Gray, "Anxiety and Abstraction in Nineteenth-Century Mathematics", *Science in Context*, 2004, 17:23-47, doi.org/10.1017/S0269889704000043.

59 Robert Fox, op. cit., pp.236-237.

60 Hermann Schubert, "Zur Theorie der Charakteristiken", *Journal für die reine und angewandte Mathematik*, 1870, 71:368-382. For biographical information on Schubert, see Werner Burau, "Der Hamburger Mathematiker Hermann Schubert", *Mitteilungen der Mathematischen Gesellschaft in Hamburg*, 1966, 9:10-19

61 Georges-Henri Halphen, "Sur les caractéristiques, dans la théorie des coniques, sur le plan et dans l'espace, et des surfaces du second ordre", *Comptes-Rendus de l'Académie des Sciences*, 1873, 76:1074-1077.

62 Hermann Schubert, "Die Characteristiken der ebenen Curven dritter Ordnung im Raume", *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen*, 1874, pp.267-283.

63 For a more detailed exposition of Schubert's calculus, see Steven Kleiman and Dan Laksov, "Schubert Calculus", *The American Mathematical Monthly*, 1972, 19:1061-1082.

64 The power of Schubert's methods, combined to their apparent lack of theoretical justification, led two prominent modern-day mathematicians to describe some of Schubert's results as "landing a jumbo jet blindfolded!", see Eisenbud and Harris, op. cit., p.2.

65 Hermann Schubert, "Beiträge zur abzählende Geometrie", *Mathematische Annalen*, 1876, 10:1-116. This text also marks the first occurence of the term 'enumerative geometry'.

66 See the letters from Schubert to Halphen dated August 12th, Nov 5th, Dec 4th 1876, Ms 5264 168-170. The prized paper was eventually published, see Hermann Schubert, "Die 13 Ausartungen und die Fundamentalzahlen der ebenen Curven dritter Ordnung mit Spitze", *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen*, 1875, pp.359-387.

67 Adolf Hurwitz und Hermann Schubert, "Ueber den Chasles'schen Satz $\alpha\mu + \beta\nu$", *Nachrichten von der Königl. Gesellschaft und der Georg-Augusts-Universität zu Göttingen*, 1876, 503-517. Adolf Hurwitz, who would become a well-known mathematician in his own right, was then Schubert's pupil in a high-school in Hildesheim, see Nicola Oswald and Jörn Steuding, "Complex continued fractions: early work of the brothers Adolf and Julius Hurwitz", *Archive for the history of exact sciences*, 2014, 68:499-528, doi.org/10.1007/s00407-014-0135-7. In a letter from Schubert to Halphen dated May 21st 1876, the main idea of the proof is attributed to Hurwitz (Bibliothèque de l'Institut, Cod Ms 5264 166).

68 "Now, you said that these formulas are inaccurate (ungenau), because they do not work in those cases, but I said that the formulas are even meaningless (sinnlos) then, and that their non-applicability [in these cases] is self-evident from the context of my book", writes Schubert in a letter to Halphen dated December 1879 (Cod Ms 5264).

69 A note entitled "Sur le principe concernant la constance des nombres géométriques", intended for publication in the BSMF, appended to a letter from Schubert to Halphen dated November 31st 1879 (Cod Ms 5264).

70 For the full sequence, see Georges-Henri Halphen, "Observations sur la théorie des caractéristiques", *Bulletin de la Société Mathématique de France*, 1880, 8:31-34, and Hermann Schubert, "Réponse aux observations de M.Halphen sur la théorie des caractéristiques" and "Note sur l'évaluation du nombre des coniques faisant partie d'un système et satisfaisant à une condition simple", *Bulletin de la Société Mathématique de France*, 1880, 8:60-61.

71 For instance Hermann Schubert, "Anzahlgeometrische Behandlung des Dreiecks", *Mathematische Annalen*, 1880, 17:153-212. In a short passage (pp.157-159), Schubert simply mentions '*Halphen's degeneration*' as a

possible counter-example to his results, and excludes them without discussing how one is supposed to detect their presence. Schubert's strategy may be compared to what Lakatos called the '*monster-barring strategy*', in Imre Lakatos, *Proofs and Refutations*, Cambridge:Cambridge University Press, 1976, ch.1, section 4.b.

72 In his letters to both Halphen and Klein, Schubert complains frequently of the price and difficulty of having mathematical journals delivered to him, or to the relatively small Gesellschaft of Hamburg-based mathematicians.

73 Georges-Henri Halphen, letter to Zeuthen dated December 7th 1879, quoted in Camille Jordan, Henri Poincaré, Emile Picard (eds.), *Oeuvres de G.-H. Halphen*, vol.4 (Paris:Gauthier-Villars, 1924), p.637.

74 *Ibid.*, p.636.

75 Hermann Schubert, *Zählen und Zahl: Eine kulturgeschichtliche Studie* (Hamburg:Richter, 1887).

76 *Ibid.*, p.1.

77 *Ibid.*, pp.14-15. In other passages, ecological features such as the number of neighbouring lakes are similarly taken to crucially determine a people's system of numbers.

78 *Ibid.*, p.36.

79 Hermann Schubert, "Das Zählen", in Georg von Neumayer (eds.), *Anleitung zu wissenschaftlichen Beobachtung auf Reisen (second edition)*, vol.2 (Berlin:Oppenheim, 1888), pp.288-294. On these guides, see Peter Monteath, "German anthropology, nationalism and imperialism: Georg von Neumayer's *Anleitung zu wissenschaftlichen Beobachtungen auf Reisen*", *History and Anthropology* (2018), pp.1–22, doi.org/10.1080/02757206.2018.1524758.

80 Ivahn Smadja, "Sanskrit *versus* Greek 'Proofs': History of Mathematics at the Crossroads of Philology and Mathematics in Nineteenth-Century Germany", *Revue d'histoire des mathématiques* (2015) 21, pp.217-349, doi.org/10.24033/rhm.189, the quote can be found p.266.

81 Ivahn Smadja, *op.cit.*, pp.301-307.

82 The crucial role played by the rise of philology as a science even in the seemingly remote province of mathematics (in the German context) adduces evidence to what has recently been argued in Lorraine Daston and Glenn Most, "History of Science and History of Philologies", *Isis*, 2015, 106:378-390, doi.org/10.1086/681980.

83 On these popular essays and the creation of *The Monist*, see Jemma Lorenat, "A Okapi Hypothesis", forthcoming.

84 Hermann Schubert, "Grundlagen der Arithmetik", in Felix Klein and Wilhelm Meyer (eds)., *Encyklopädie der mathematischen Wissenschaften*, 6 vols. (1898-1933), vol.I A 1 (Leipzig:Teubner, 1898), pp.1-27.

85 Of course, this was a shared concern for many German mathematicians at the time. Famous examples include Richard Dedekind, Gottlob Frege, and Edmund Husserl. See Moritz Epple, "The End of the Science of Quantity: Foundations of Analysis, 1860-1910", in Hans Niels Jahnke (eds.), *A History of Analysis* (American Mathematical Society, vol.24, 2003), pp.291-323.

86 Hermann Schubert, "Notion and Definition of Number", *The Monist*, 1894, 4:396-402. The quote is p.397.

87 Ibid., p.402.

88 Hermann Schubert, "Monism in Arithmetic", *The Monist*, 1894, 4:561-579 (quote p.567). This principle closely resembles George Peacock's famous '*principle of equivalent forms*', whose shaping had also hinged upon his author's interest in natural history. The mathematical and social contexts of Peacock's and Hankel's work, however, are very different; see Kevin Lambert, "A Natural History of Mathematics. George Peacock and the Making of English Algebra", *Isis*, 2013, 104:278-302, doi.org/10.1086/670948, Joan Richards, "Augustus de Morgan, the History of Mathematics, and the Foundations of Algebra", *Isis*, 1987, 78:6-30, doi.org/10.1086/354328.

89 On the social history of Hamilton's quaternions, see Josipa Petrunic, *Quaternion engagements and terrains*

of knowledge (1858-1880), PhD thesis, University of Edinburgh, 2009.

90 Hermann Schubert, "Monism in Arithmetic", pp.578-579.

91 Diethard Sawicki, *Leben mit den Toten: Geisterglauben und die Entstehung des Spiritsmus in Deutschland 1770-1900*, 2nd edition, Paderborn:Schöningh, 2016, pp.299-310. Zöllner's theses were discussed by a range of philosophers and scientists, from Nietzsche to Helmholtz, and even caused a scandal in Leipzig in the early 1870s.

92 Hermann Schubert, "The Fourth Dimension. Mathematical and Spiritualistic", *The Monist*, 1893, 3:402-449. The quote is p.410.

93 Hermann Hankel, *Theorie der complexen Zahlensysteme* (Leipzig:Voss, 1867), pp.6-7.

94 Hermann Schubert, "On the Nature of Mathematical Knowledge", *The Monist*, 1896, 6:294-305. The quote is p.301.

95 Gottlob Frege, *Über die Zahlen des Herrn H. Schubert* (Jena:Pohle, 1889), translation by Hans Kaal, in Brian McGuiness (eds.), *Gottlob Frege. Collected Papers on Mathematics, Logic, and Philosophy* (Oxford:Blackwell, 1984), pp.249-272.

96 Hermann Schubert, "*Sur le principe concernant la constance des nombres géométriques"* (Cod Ms 5264).

97 Letter dated July 29th 1876, in *Oeuvres de G.-H. Halphen*, pp.629-635.

98 Letter dated August 11th 1876, Cod Ms 5624.

99 Georges-Henri Halphen, "Sur les caractéristiques des systèmes de coniques et de surfaces du second ordre", *Comptes Rendus de l'Académie des Sciences de Paris*, 1876, 83:886-888. This note is a summary of a memoir sent to the Académie by Halphen on November 13th 1876, written a few months after Zeuthen's letter.

100 Henri Poincaré, "Notice sur Halphen", p.152. On conventions in Poincaré's philosophy of mathematics, see Gerhard Heinzmann, "Hypotheses and Conventions in Poincaré", in Michael Heidelberger, Gregor Schiemann (eds.), *The Significance of the Hypothetical in the Natural Sciences* (De Gruyter, 2009), pp.169-192, doi.org/10.1515/9783110210620.

101 Eduard Study, "Ueber die Geometrie der Kegelschnitte, insbesondere deren Characteristikenproblem", *Mathematische Annalen*, 1886, 27:58-101. The quote is p.61.

102 Ibid., pp.61-62.

103 Ibid., p.64.

104 On Hilbert's and Study's visit to Paris, see Constance Reid, *Hilbert-Courant* (New-York:Springer-Verlag, 1986), pp.22-28. Two letters from Study to Halpen are kept in the *Bibliothèque de l'Institut*, Paris, Ms 5624, 186-187. Two brief letters from Halphen to Study are in the archives of the *Justus-Liebig Universität Giessen*, as part of the Nachlassverzeichnis Friedrich Engel, NE090416-17.

105 Eduard Study, letter to George Halphen, 21st April 1886, Cod Ms 5624 187.

106 George Halphen, letter to Eduard Study, 22nd April 1886, NE090416.

107 Eduard Study, "Über das Prinzip der Erhaltung der Anzahl", in *Verhandlungen des Dritten Internationalen Mathematiker-Kongresses in Heidelberg 1904* (Leipzig:Teubner, 1905), pp.388-395. The quote is p.388 and 391. This rebuttal is strikingly similar the criticism of Schubert's philosophy of numbers expressed in Frege, op.cit.: '*If we did not know the sense of these words ['equal', 'greater', 'smaller'..], we would not know what thoughts were contained in these propositions and could not therefore prove these thoughts to be true*'.

108 Eduard Study, "Ein neuer Zweig der Geometrie", *Jahresbericht der Deutschen Mathematiker-Vereignung*, 11:97-123. On Study's epistemology of mathematics, see Hartwich, op. cit., pp.130-159, and Gray, *Plato's Ghost*, pp.293-296.

109 For a case-study in the *longue-durée* history of the different readings one same theorem can elicit, see Catherine Goldstein, *Un théorème de Fermat et ses lecteurs* (Saint-Denis:Presses Universitaires de Vincennes, 1995).

[110] Van der Waerden, *op. cit.*, pp.1-2. On Van der Waerden's geometrical works, see Norbert Schappacher, "A Historical Sketch of B. L. Van der Waerden's Work on Algebraic Geometry, 1926-1946", in Jeremy Gray, Karen Hunger Parshall (eds.), *Episodes in the History of Modern Algebra (1800-1950)*, USA:American Mathematical Society, vol.32, 2011, pp.245-284.

[111] Daston and Galison, *op. cit.*, p.19.

[112] Eduardo Casas-Alvero and Sebastian Xambò-Descamps, *The Enumerative Theory of Conics after Halphen* (Berlin Heidelberg:Springer Verlag, 1986), p.iii. For Lakatos' critique of modern(ist) mathematics, see Lakatos, *op. cit.*, ch.2, section 1.

[113] Alexandre Grothendieck, *Récoltes et Semailles. Réflexions et témoignage sur un passé de mathématicien* (undated, available online), for instance pp.50-52.

[114] Alma Steingart, "A group theory of group theory: Collaborative mathematics and the 'uninvention' of a 1000-page proof", *Social Studies of Science* (2012), 42-2:185-213.