Mathematical Selves and the Shaping of Mathematical Modernism: Conflicting Epistemic Ideals in the Emergence of Enumerative Geometry (1864–1893)

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Abstract: For more than three decades, fierce debates raged both in private letters and across public spaces over a formula expressed in 1864 by the French geometer Michel Chasles. Proofs and refutations thereof abounded, to no avail: the formula was too useful to be abandoned by its defenders, too elusive to be made rigorous for its detractors. The disputes over Chasles's formula would not be solved by a definitive proof or rebuttal; rather, the core epistemic issues at stake shifted from generality to rigor and from truth to geometrical significance. This essay tracks the main lines of circulation of Chasles's formula and shows how the disputes to which it gave rise embody conflicting *mathematical selves*—that is to say, different normative accounts of what being a mathematician entails. This perspective allows for a renewed understanding of what historians have described as the conflicted rise of modernism in mathematics and a firmer rooting of it within broader late nineteenth-century cultural trends.

INTRODUCTION: A FORMULA IN FLUX

It was June 1890, and the German mathematician Felix Klein was growing ever more dissatisfied with the state of enumerative geometry, a newly emerged branch of mathematics. Klein was an influential professor and powerful organizer of mathematical research based at the University of Göttingen.[1] For some time, he had been puzzled by the uncertainty surrounding

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[1] David Rowe, "Klein, Hilbert, and the Göttingen Mathematical Tradition," *Osiris*, 1989, N.S., 5:186–213, https://doi.org/10.1086/368687. While I shall not emphasize it here, the globalization of mathematics was an important feature of this historical episode. On global mathematics see Michael Barany, "Integration by Parts: Wordplay, Abuses of Language, and Modern Mathematical Theory on the Move," *Historical Studies in the Natural Sciences*, 2018, 48:259–299, https://doi.org/10.1525/hsns.2018.48.3.259.

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a geometrical formula first expressed and justified on inductive grounds some twenty-six years earlier by the French geometer Michel Chasles.[2] A centerpiece of Chasles's much-celebrated theory of characteristics, this formula purportedly solved a difficult mathematical problem whose general solution had escaped geometers for a long time—namely, the enumeration of curves satisfying given conditions (such as passing through a given point or touching a given line). This problem had stumped renowned geometers such as Jakob Steiner, who had previously conjectured that there were 7,776 conics touching five other given conics—only for Chasles to show that the correct number was in fact 3,264.[3] The theory of characteristics itself had quickly garnered praise across Europe, with translations and summaries being published in Italian, Danish, English, and French; moreover, the Royal Society in London awarded Chasles its Copley Medal in 1865—a coveted honor then rarely bestowed on pure mathematicians.[4]

In 1873, some seven years after Chasles's initial publication, independent and quasi-simultaneous proofs of this formula were given by two mathematicians: the Göttingen-based Alfred Clebsch, in a paper published posthumously; and Georges-Henri Halphen, a French artillery officer freshly graduated from the École Polytechnique. Three years later, yet another proof was given by a student of Clebsch's who was editing the latter's *Vorlesungen über Geometrie* under Klein's supervision. Shortly thereafter, however, these efforts would be deemed profoundly misguided by Halphen himself. Near the end of 1876, he changed his mind regarding his own proof, presented a counterexample to Chasles's formula to the Académie des Sciences in Paris, and announced the forthcoming publication of an alternative theory.[5] For Halphen, undue reliance on intuition and vague notions had led geometers astray, and thorough analysis of the problem revealed that Chasles's methods counted objects that were not satisfactory solutions to the problem at hand but mere computational artifacts.

Halphen's refutation, however, was not accepted by all. Among the dissenters was the German mathematician Hermann Schubert. From 1874 onward, Schubert had been devising a fruitful symbolic calculus, building on formal regularities he had observed in the results of his colleagues and, crucially, on Chasles's formula. Schubert's calculus, and in particular his 1879 book *Kalkül der abzählenden Geometrie*, impressed many, if only by the sheer number of new and difficult results he had been able to obtain with his idiosyncratic methods.[6] For geometers

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[2] Michel Chasles, "Formules générales comprenant la solution de toutes les questions relatives aux sections coniques," *Comptes Rendus de l'Académie des Sciences de Paris*, 1864, 59:209–218. Enumerative geometry has received scant historical attention thus far. Notable exceptions include Steven Kleiman, "Chasles' Enumerative Theory of Conics," in *Studies in Algebraic Geometry*, ed. Abraham Seidenberg (Washington, D.C.: Mathematical Association of America, 1980), pp. 117–138; and Yvonne Hartwich, "Eduard Study (1862–1930): Ein mathematischer Mephistopheles im geometrischen Gärtchen" (Ph.D. diss., Johannes Gutenberg Univ. Mainz, 2005).

[3] Michel Chasles, "Détermination du nombre de sections coniques qui doivent toucher cinq courbes données d'ordre quelconque etc.," *Compt. Rend. Acad. Sci. Paris*, 1864, 58:222–226. On Steiner's geometry see Jemma Lorenat, "Synthetic and Analytic Geometries in the Publications of Jakob Steiner and Julius Plücker (1827–1829)," *Archive for History of Exact Sciences*, 2016, *70*:413–462, https://doi.org/10.1007/s00407-015-0174-8.

[4] Sir Lieutenant-General Edward Sabine, "President's Address," *Proceedings of the Royal Society of London*, 1865, *14*:482–513. Created in 1731, and initially intended to celebrate the best experiment of the year, the Copley Medal soon became a coveted prize to recognize contributions to "natural knowledge"; see M. Yakup Bektas and Maurice Crosland, "The Copley Medal: The Establishment of a Reward System in the Royal Society, 1731–1839," *Notes and Records of the Royal Society of London*, 1992, *46*:43–76. The award of this medal to Chasles was mostly the result of intense lobbying by the English mathematician Thomas Archer Hirst.

[5] Alfred Clebsch, "Zur Theorie der Charakteristiken," *Mathematische Annalen*, 1873, 6:1–15; Georges-Henri Halphen, "Mémoire sur la détermination des coniques et des surfaces du second ordre," *Bulletin de la Société Mathématique de France*, 1872–1873, *1*:130–142; Ferdinand von Lindemann, *Vorlesungen über Geometrie von Alfred Clebsch* (Leipzig: Teubner, 1876), pp. 397–401; and Halphen, "Sur les caractéristiques des systèmes de coniques," *Compt. Rend. Acad. Sci. Paris*, 1876, 83:537–539.

[6] Hermann Schubert, *Kalkül der abzählenden Geometrie* (Leipzig: Teubner, 1879).

at large, to accept Halphen's sharp arguments against what he perceived to be the lack of rigor and analytical precision of his predecessors was to agree that the numerous proofs of Chasles's formula produced by esteemed mathematicians were flawed and to renounce the embarrassment of riches provided by Schubert's methods.

Klein, who by the late 1870s was actively corresponding with Schubert, could not fail to notice the problematic state of Chasles's theory of characteristics.[7] In 1884, while still a professor in Leipzig, he assigned the problem of assessing the validity of Chasles's formula to Eduard Study, a promising young student. Despite his initial reluctance to work on this problem, by 1885 Study had obtained a new proof for Chasles's formula and attempted to respond fully to Halphen's criticism.[8] To that end, Study put forth a new kind of argument: Halphen's counterexample did not refute Chasles's formula *per se* but only one interpretation thereof—and not necessarily the most appropriate one. Halphen's untimely death in 1889 came before any response he might have made to Study's work could be published, and the few interactions the two mathematicians did have were largely unproductive.

Shortly after the publication of Study's *Habilitationsschrift*, Klein received a rather bitter letter from the Danish geometer Hieronymus Zeuthen, who lamented Study's reluctance to discuss the matter with him and rejected the claim that the problem had been solved once and for all. Having studied under Chasles in Paris in 1865, written a dissertation in Copenhagen on the theory of characteristics, and corresponded with both Halphen and Schubert throughout the 1870s on these matters, Zeuthen was a renowned expert on enumerative geometry—and firmly on Halphen's side in his disagreement with Schubert and Study. Despite Klein's insistence, however, Study refused to engage in the discussions Zeuthen had called for.[9]

In 1890, therefore, with no end to these disputes in sight, Klein asked Zeuthen to write a public and official response to Study, to be published in the pages of the *Mathematische Annalen*, which Klein then edited. Zeuthen obliged; in so doing he reiterated his opinion that Study's work was based on a misunderstanding of the very problem Halphen had set out to solve and that the latter's results still held. Study replied in the same journal in 1892, and Zeuthen weighed in again in 1893, but at no point did their respective positions evolve: despite Klein's intervention, the uncertainty as to the validity of Chasles's formula remained.[10]

There was a profound mathematical reason for the persistence of these disputes; and geometers nowadays all acknowledge that the validity of Chasles's formula crucially hinges on the formalism adopted to translate the terms at the heart of the theory of characteristics. By the 1930s mathematicians had largely eschewed these debates, with many viewing them as nothing more than a matter of "honor."[11] And yet, the story of Chasles's formula cannot be read as one of vain disputes among mathematicians insufficiently equipped to realize the ambiguity of their problem. Indeed, the final word on this topic would not come by way of a definitive proof or

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[7] Several letters from Schubert to Klein are preserved at the Göttingen Staatsbibliothek, Cod Ms Klein 11.

[8] Eduard Study, *Über die Geometrie der Kegelschnitte, insbesondere deren Charakteristikenproblem* (Habilitationsschrift, Univ. Leipzig) (Leipzig: Teubner, 1885). Thomas Hawkins contends that "Klein [did not exert] any significant influence upon [Study]": Thomas Hawkins, "The *Erlanger Programm* of Felix Klein: Reflections on Its Place in the History of Mathematics," *Historia Mathematica*, 1984, *11*:442–470, on pp. 449–450. A more nuanced account of their relationship can be found in Hartwich, "Eduard Study (1862–1930)" (cit. n. 2), pp. 50–64.

[9] Hartwich, "Eduard Study (1862–1930)," pp. 73–78.

[10] Hieronymus Zeuthen, "Sur la révision de la théorie des caractéristiques de M. Study," *Math. Ann.*, 1890, 37:461–464; Eduard Study, "Entgegnung," *ibid.*, 1892, *40*:559–562; and Zeuthen, "Exemples de la détermination des coniques dans un système donné qui satisfont à une condition donnée," *ibid.*, 1893, *41*:539–544.

[11] Baertel Van der Waerden, "Zur algebraischen Geometrie, XV: Lösung des Charakteristikenproblems für Kegelschnitte," *Math. Ann.*, 1938, *115*:645–655, esp. p. 646 n 3.

refutation, but only at the close of a reinvention of the cultural and scientific identity of mathematics itself.

MODERNISM, TRUTH, AND LANGUAGE IN *FIN-DE-SIÈCLE* MATHEMATICS

In his landmark 1990 study *Moderne Sprache Mathematik*, Herbert Mehrtens described the transformation of mathematics at the turn of the twentieth century as "a shockwave blasting through the concepts of truth, meaning, object, and existence." Throughout this transformation, he argued, criteria for truth and validity, modes of objectivity, and textual practices had to be reinvented and renegotiated among mathematicians. As Mehrtens's distinction between the "moderns" (such as Georg Cantor and David Hilbert) and the "counter-moderns" (such as Henri Poincaré and Klein himself) suggests, this was no peaceful transformation: he presents two camps, with contrasting forms of self-understanding and style in mathematics, that clashed with zeal and fervor. In a thorough assessment of Mehrtens's thesis, Jeremy Gray has characterized mathematics under the modernist conception as "as an autonomous body of ideas, having little or no outward reference, . . . maintaining a complicated—indeed, anxious—rather than a naïve relationship with the day-to-day world."[12] The modernist mathematician, in Mehrtens's view, is a "free creator" whose proofs and propositions do not derive meaning from any external system of reference—whether physical objects, a model of some phenomenal field, or even abstract objects conceived prior to the utterance of mathematical speech. The counter-moderns, in turn, pushed back against the perceived dangers of leaving the creation of mathematical concepts up to arbitrary will and sought to quell related epistemic anxieties by grounding mathematical truth in appeals either to intuition or to some transcendent order.[13]

Mehrtens's categories have been criticized as insufficient for informing a precise understanding of the multinational, cross-cultural, decade-spanning modernist transformation of mathematics. Gray himself noted that they suffered from too exclusive a focus on Germany rather than looking at the whole of Europe and on programmatic or philosophical texts rather than actual samples of mathematical practice.[14] The debates over Chasles's formula spanned the decades and the countries most crucial to the conflicted emergence of modern mathematics and were never understood by their protagonists as discussions of foundational or philosophical issues but, rather, of technical, albeit important, results.[15] Thus, this historical episode is an ideal candidate to test Mehrtens's sweeping narrative and categories and to confront them with actual mathematical practice on a multinational scale. Another line of critique of this narrative, forcefully argued by Leo Corry, lies in the notion that the worth of the concept of modernism in the historiography of mathematics hinges on its ability to incorporate broader contemporary cultural changes. The very term "modernism" suggests comparisons to transformations in the arts, many of which have been tentatively put forth.[16] Drawing on the history of scientific objectivity, as set

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[12] Herbert Mehrtens, *Moderne Sprache Mathematik: Eine Geschichte des Streits um die Grundlagen der Disziplin und des Subjekts formaler Systeme* (Frankfurt: Suhrkamp, 1990), p. 8 (here and throughout the essay, translations into English are mine unless otherwise specified); and Jeremy Gray, *Plato's Ghost: The Modernist Transformation of Mathematics* (Princeton, N.J.: Princeton Univ. Press, 2008), p. 1.

[13] Mehrtens, *Moderne Sprache Mathematik*, pp. 9–11, 394–404.

[14] Gray, *Plato's Ghost* (cit. n. 12), pp. 9–12.

[15] The novelty of the modernist approach to mathematics is usually measured against the backdrop of "a consensus in 1880"; see Gray, *Plato's Ghost*, p. 112.

[16] Leo Corry, "How Useful Is the Term 'Modernism' for Understanding the History of Early Twentieth-Century Mathematics?" in *Science as a Cultural Practice*, Vol. 2: *Modernism in Science, 1900–1940*, ed. Moritz Epple and Falk Müller (Berlin: Akademie, forthcoming); the essay can be found at https://www.leocorry.com/articles. For work suggesting comparisons to transformations in the arts see, e.g., William Everdell, *The First Moderns: Profiles in the Origins of Twentieth-Century Thought* (Chicago: Univ. Chicago Press, 1997); Daniel Albright, *Quantum Poetics: Yeats, Pounds, Eliot, and the Science of Modernism*

out by Lorraine Daston and Peter Galison in their much-discussed book *Objectivity*, this essay seeks to show another way to tie the modernist transformation of mathematics to several late nineteenth-century cultural trends and ruptures by framing it as the confrontation of successive *mathematical selves*.

The history of objectivity, as presented by Daston and Galison, is not one of conceptualizations and philosophical accounts of objectivity but, rather, one of various epistemic virtues that regulated and enabled said objectivity. Such virtues were not only preached but also practiced, and they were embodied in various practices such as note-taking, self-erasing, attentive observation, and the like. Together, these virtues and practices constituted scientific personas—such as that of the sage or the expert—that projected different kinds of ontologies onto the same phenomenon, from the shape of snowflakes to the anatomy of insects. Daston and Galison's ambition was to put forth a mesoscopic, *longue durée* history of scientific objectivity across disciplinary borders, framed as the history of "the manifestations and mutations of the *scientific* self."[17]

This essay makes use of this analytical framework, albeit on a resolutely microhistorical and local scale. In what follows, I will contrast the epistemic ideals that can be found in the highly normative descriptions of proper mathematical practice produced by four of the key actors in the historical episode previously sketched: Chasles, Halphen, Schubert, and Study. Each of these accounts is associated with different epistemic virtues, which in turn give rise to different textual practices, ontologies, and regimes of truth. My main contention is that these virtues and practices were constitutive of different *mathematical selves*, whose incompatibility largely accounts for the inconclusiveness of the disputes over Chasles's formula. These selves are thus all situated differently on the quadrants formed by the two axes along which the modernist transformation of mathematics has been described—namely, the absence of outward reference for mathematical discourse and the growing anxiety among practitioners after the emergence of new standards of rigor. Furthermore, they will be shown to have been shaped against the decisive backdrop of various cultural trends and intellectual debates beyond mathematics.[18] Thus, the emergence of mathematical modernism is here depicted as a composite phenomenon, inseparable from cultural history at large.

CHASLES: GEOMETRY AS EXPLORATION OF NATURAL ORDER

On 15 February 1864, when the French geometer Michel Chasles took the podium during one of the weekly public meetings of the Paris Académie des Sciences, he was a familiar sight. The aging mathematician was a well-respected figure, famous for his tireless promotion of geometry as well as his historical erudition regarding all things mathematical. For almost twenty years he had held the Chair of Higher Geometry at the Faculté de Paris, where he taught and developed his own geometrical methods for an audience of advanced students.[19] His lectures were informed

(Cambridge: Cambridge Univ. Press, 1997); and Nina Engelhardt, *Modernism, Fiction, and Mathematics* (Edinburgh: Edinburgh Univ. Press, 2018).

[17] Lorraine Daston and Peter Galison, *Objectivity* (New York: Zone, 2007), p. 198; for fuller discussions of the concept of the scientific self see pp. 35–50, 216–233, 367–371. For more recent scholarship on scientific selves see, e.g., Paul White, "Darwin's Emotions: The Scientific Self and the Sentiment of Objectivity," *Isis*, 2009, *100*:811–826, https://doi.org/10.1086/652021; and Ian Hesketh, "Technologies of the Scientific Self: John Tyndall and His Journal," *ibid.*, 2019, *110*:460–482, https://doi.org/10.1086/704672.

[18] These "selves" are not to be equated with moral biographies: behind each of them lie broader characters, which for the purpose of brevity I locate here only in certain individuals. For another approach that builds on Daston and Galison's work on a resolutely more local scale see Hermann Paul, "The Virtues and Vices of Albert Naudé: Toward a History of Scholarly Personae," *History of Humanities*, 2016, *1*:327–338, https://doi.org/10.1086/688036.

[19] This post was created specifically for Chasles, at the instigation of Louis Poinsot. Chasles officially occupied it until 1879, when he was eighty-six. In 1868 Pierre-Ossian Bonnet began regularly to assist Chasles in his teaching duties. Ten years later,

by his contemporary research, the content of which he polished and prepared for communication to his fellow *académiciens*. The theory of characteristics, which Chasles began presenting in February 1864, was one output of this work regimen—but one that far exceeded his previous results both in the praise it attracted from fellow European mathematicians and in sheer volume. Between 1864 and 1867 Chasles took the podium over twenty times to address the Académie on this topic, either to add new results or to share those his colleagues outside the Académie had sent him. The center of this attention was Chasles's uniform method for the determination of the number of conic sections satisfying any five conditions, which required no computations other than elementary sums and products of a few integers.[20]

This theory constituted for Chasles the culmination of a lifelong endeavor to renew the language and methods of pure geometry. Chasles defined pure geometry as geometrical theories and methods resting on "pure reasoning alone," in contrast with analytical geometry, which relied on the use of coordinate systems and algebraic computations.[21] His first book, a historical survey on the development of geometrical methods, drew conclusions from historical studies on how to elevate pure geometry to the same level of generality as its analytical counterpart.[22] In so doing, Chasles saw himself as part of a tradition that started at the École Polytechnique through the influential teaching of one its founders, Gaspard Monge. He viewed his task as one of synthesis and systematization of the disorganized yet powerful new methods and concepts that Monge and his spiritual students and followers, such as Lazare Carnot, Charles Dupin, and Jean-Victor Poncelet, had brought forth in the early nineteenth century.[23]

These authors all had slightly different understandings of why analytical methods had risen to such lofty standards of generality and efficiency throughout the eighteenth century, and consequently they held different views as to how to remediate the inferiority of purely geometrical methods. However, few of them doubted the epistemic certainty or rigor of the mathematical truths obtained through analytical means. Rather, they were unhappy with the epistemic quality of said truths, which they reckoned insufficiently illuminated the mind. Chasles's conception of mathematical knowledge rested on one central creed—namely, that "all mathematical truths can become simple and intuitive, once the narrow path [to said truths] that is natural and characteristic

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Bonnet obtained the Chair of Astronomy left vacant by Urbain Le Verrier's death, and Gaston Darboux, a rising star of French mathematics, replaced Chasles as professor of higher geometry.

[20] "Conic sections," here, refers to curves that can be obtained as the intersection of a cone and a plane; alternatively, they can be defined as the curves whose equations are polynomials of the second degree in two variables x and y, namely: $ax^2 + by^2 + cxy + dx + ey = 1$. A conic is uniquely defined by five coefficients (a, b, c, d, e); therefore, only a finite number of such curves simultaneously satisfy five independent conditions. For the technical details of this mathematical problem see Andrew Bashelor, Amy Ksir, and Will Traves, "Enumerative Algebraic Geometry of Conics," *American Mathematical Monthly*, 2008, *115*:701–728, https://doi.org/10.1080/00029890.2008.11920584.

[21] For an overview of the rivalry between pure and synthetic geometries in the nineteenth century see Jemma Lorenat, "'Die Freude an der Gestalt': Methods, Figures, and Practices in Early Nineteenth-Century Geometry" (Ph.D. diss., Univ. Pierre et Marie Curie, 2015), esp. Ch. 1; and Jeremy Gray, *Worlds out of Nothing: A Course in the History of Geometry in the Nineteenth Century* (London: Springer, 2007), Chs. 1–6.

[22] Michel Chasles, *Aperçu historique sur le développement des méthodes etc.* (Paris: Gauthier-Villars, 1837). See Karine Chemla, "The Value of Generality in Michel Chasles' Historiography of Geometry," in *The Oxford Handbook of Generality in Mathematics and the Sciences*, ed. Chemla, Renaud Chorlay, and David Rabouin (Oxford: Oxford Univ. Press, 2016), pp. 47–89; and Nicolas Michel, "The Values of Simplicity and Generality in Chasles's Geometrical Theory of Attraction," *Journal for General Philosophy of Science*, 2020, *51*:115–146, https://doi.org/10.1007/s10838-019-09451-z.

[23] Chasles, *Aperçu historique sur le développement des méthodes etc.*, pp. 253–254. Neither Poncelet nor Chasles was able to attend Monge's teaching in person, as the latter ceased to teach and exert any real institutional power after 1810, in part owing to illness. See Bruno Belhoste, *La formation d'une technocratie: L'École polytechnique et ses élèves de la Révolution au Second Empire* (Paris: Belin, 2003), pp. 195–212. On Monge's teaching of geometry see Belhoste and René Taton, "L'invention d'une langue des figures," in *L'École normale de l'an III*, ed. Jean Dhombres, 5 vols. (1992–2016), Vol. 1 (Paris: Presses de l'École Normale, 1992), pp. 271–400, https://doi.org/10.4000/books.editionsulm.442.

has been found." This postulate led Chasles to draw a stark contrast between the knowledge provided by ingenious, human-made analytical machinations and that of natural, effortless geometrical studies: "Analysis," claimed Chasles, suffers from the same weaknesses as "all human conceptions: its swift and penetrating march does not always sufficiently enlighten the mind." This assessment stood in complete opposition to Chasles's description of the ideal pure geometer, once equipped with the modern methods and theories first discovered by Monge. These modern geometers, Chasles claimed, simply had to "pick any arbitrary known truth, and submit it to the various general principles of transformation; they [would] derive from it other truths, different or more general." What's more, he added, anyone could now become a geometer: "genius is no longer required."[24] At the end of Chasles's historical narrative stood the figure of the geometer as a student of (spatial) extension, able effortlessly and systematically to combine truths within naturally grounded theories. Just like the technical drawing devised and promoted by Monge and Dupin years before, Chasles's teaching was meant to unfetter its users by "bringing their practices in line with the dictates of nature and reason."[25]

This description of the ideal mathematician was also something Chasles transmitted through his teaching and promotion of his own geometrical practice. The naval officer Ernest de Fauque de Jonquières was a student and friend of Chasles until a bitter priority dispute tore them apart and worked for some time on Chasles's theory of characteristics. In a review of Chasles's 1860 book on Euclid's lost *Porisms*, he warned prospective geometers against the alluring promises of the "lively gait" of analytical geometry, which he thought of as appealing to

> this frenzy, this frantic need to reach any arbitrary goal, which is one of the dominant characters of our times. But it is good, for the sake of science itself, to temper this character. For, even if we left the laurels of celerity in research to analytical methods, science could not be exclusively served in this way. To use a vulgar comparison, one may get the lay of a land fairly quickly by travelling along the major railways that criss-cross it; but to know in-depth the details, the productions, the resources of this land, one must step off the locomotive, and set out to explore by foot its ancient roads and unbeaten paths. In so doing, one acquires habits of patience, observation, and criticism, which might well disappear, if we couldn't bear to go back to this primitive mode of travel.[26]

At a time of rapid industrialization,[27] and as calculating machines ceased to be mere abstractions,[28] Chasles and his students promoted a counter-figure of the ideal mathematician, one who

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[24] Chasles, *Aperçu historique sur le développement des méthodes etc.*, pp. 2–3, 114, 268–269.

[25] Ken Alder, *Engineering the Revolution: Arms and Enlightenment in France, 1763–1815* (Chicago: Univ. Chicago Press, 1997), pp. 316–317.

[26] Ernest de Jonquières, rev. of Michel Chasles, *Les trois livres de Porismes d'Euclide* (Paris: Gauthier-Villars, 1860), in "Bulletin de bibliographie, d'histoire et de biographie mathématique," *Nouvelles Annales de Mathématiques*, 1861, *20*:1–11, on pp. 8–9.

[27] The Marxist historian of industrialization Tom Kemp notes that French industrial capitalism was stimulated to great effect by the development of railways in the wake of the 1848 revolution; see Tom Kemp, *Industrialization in Nineteenth-Century Europe*, 2nd ed. (London: Routledge, 2014), pp. 49–77. In parallel, industrialization came to polarize French academic circles as well, as is shown by Le Verrier's ascension to head of the École Polytechnique in 1850 as part of a general push by industrialist lobbies for more practical teaching, or by the "bifurcation," a reform of the Baccalauréat separating the sciences and the humanities, passed in 1852 (see Belhoste, *La formation d'une technocratie* [cit. n. 23], pp. 97–102). Chasles was publicly opposed to both reforms.

[28] This was the case in England, most famously in the work of Charles Babbage, but also in France, where Thomas De Colmar had developed his own calculating machine. See Eduardo L. Ortiz, "Babbage and French *Idéologie*: Functional Equations, Language, and the Analytical Method," in *Episodes in the History of Modern Algebra (1800–1950)*, ed. Jeremy Gray and Karen Hunger Parshall (Providence, R.I.: American Mathematical Society, 2011), pp. 13–47; and Matthew Jones, *Reckoning with Matter: Calculating Machines, Innovation, and Thinking about Thinking from Pascal to Babbage* (Chicago: Univ. Chicago Press, 2016), Ch. 6.

would resist the trend toward analysis and instead aim for slow-moving but steady and methodical knowledge.[29]

The valuation and promotion of this mathematical self bore crucially on the choices of conceptual tools and textual practices that these mathematicians elected to use in the course of their geometrical research. In his lectures, Chasles would craft new notations and linguistic devices in order to structure the whole of geometry around a few central concepts and to systematize the writing of geometrical proofs and propositions. One such line of inquiry he pursued from the early 1850s would be of particular importance for the theory of characteristics: the shaping of a new mode of description of curves through what he sometimes called "geometrical equations." In particular, Chasles studied procedures to construct certain curves determined by some of their points that were perfectly general—that is to say, the instructions involved would be applicable to any possible configuration of given points (e.g., aligned or coinciding points). Such procedures, Chasles claimed, had to rest on properties of these curves that are so absolutely fundamental as to characterize them completely. In turn, such properties would act as the "true equations" of these curves. Unlike traditional Cartesian equations, however, they involved no algebraic symbols or variables. In his 1865 *Traité des sections coniques* (the content of which had been written and taught much earlier), Chasles built on this framework to form two central propositions—namely, Pascal's and Brianchon's theorems—from which he thought the entire theory of these curves derived. Of these theorems, Chasles would later say that they were the "punctual and tangential equations of conics."[30] In both instances, Chasles would emphasize the search for a fundamental and characteristic property of a class of curves as the geometer's main task. Such properties then act as geometrical equations, which involve no extrinsic elements (unlike Cartesian equations, with their artificial coordinate systems), while displaying the same level of generality as analytical equations. From these geometrical equations, the pure geometer could effortlessly, and with no need for shrewd computations, derive an infinity of higher-level truths.[31]

Chasles's presentation of the theory of characteristics made explicit the connection with his past research on geometrical equations.[32] He first explained why analytical methods cannot solve the general problem of enumerating conics: the computations required by the procedure of elimination that this would entail are simply intractable. They require the combination of five algebraic equations (of potentially high degrees) in six unknowns, which is, in general, more than the human mind can handle. His own theory had no such problem. Chasles considered what he called systems of conics—that is to say, infinite collections of conics satisfying four conditions (see Figure 1). In a given system, for any condition Z, there is a finite number of conics satisfying Z. He then defined, for each of these systems, two numbers μ and ν, which he named the "characteristics" of these systems.[33] This terminology was not arbitrary: Chasles's central observation was that all properties of systems of conics can be expressed through a number obtained by adding

[29] For a suggestive analysis of the epistemology of early nineteenth-century French pure geometry in empiricist terms see Lorraine Daston, "The Physicalist Tradition in Early Nineteenth Century French Geometry," *Studies in History and Philosophy of Science*, 1985, *17*:269–295.

[30] Michel Chasles, "Construction de la courbe du troisième ordre déterminée par neuf points," *Compt. Rend. Acad. Sci. Paris*, 1853, *36*:943–952, on p. 945 ("true equations"); and Chasles, *Rapport sur les progrès de la géométrie* (Paris: Hachette, 1870), p. 269 ("punctual and tangential equations of conics"). This means that one is the equation whose roots are the points of a conic, while the roots of the other yield the tangent lines to the conic.

[31] As argued in Chemla, "Value of Generality in Michel Chasles' Historiography of Geometry" (cit. n. 22), p. 60, Chasles's ideal of geometrical practice is one "with almost no proof."

[32] Michel Chasles, "Considérations générales sur la méthode générale etc.," *Compt. Rend. Acad. Sci. Paris*, 1864, *58*:1167–1175.

[33] μ is the number of conics in the system passing through a given point, while ν is the number of conics in the system touching a given straight line. See Kleiman, "Chasles' Enumerative Theory of Conics" (cit. n. 2), for the mathematical details.

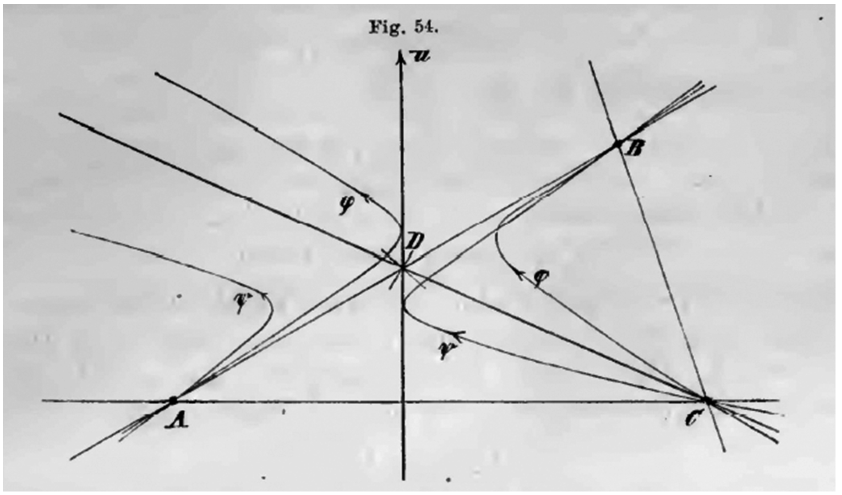

Figure 1. A system of conics touching one line and passing through three points. From Ferdinand von Lindemann, *Vorlesungen über Geometrie von Alfred Clebsch* (Leipzig: Teubner, 1876), p. 395.

these two numbers a certain number of times—that is to say, through a number of the form $\alpha\mu + \beta\nu$ (where α and β depend solely on Z). In other words, Chasles had found a systematic method that, to every geometrical condition, was able to associate two coefficients α and β, so that in any system of conics of characteristics (μ, ν) the number of conics satisfying this condition Z was $\alpha\mu + \beta\nu$; this latter expression being called the "module" of the condition. To showcase this remarkable regularity, Chasles then drew up long, monotonous lists of geometrical properties, each of which matched a geometrical condition. To enumerate conics satisfying five conditions, one had only to refer to these lists, find the properties corresponding to these five conditions, and carry out a series of simple additions and multiplications of these α's and β's. For instance, knowing that in a system of characteristics (μ, ν) there are $2\mu + 2\nu$ conics touching one given conic (that is to say, for this condition α and β are equal to 2), a series of elementary arithmetical operations given by the general procedure allowed Chasles to enumerate the 3,264 conics in a plane that touch five given conics, a result for which he is still remembered.[34] Thus, for Chasles, characteristics provided a natural representation of the properties of systems of curves, thereby bringing their enumeration within reach of simple additions of numbers. By contrast, algebraic representations, based on artificial coordinate systems and variables, yielded only intractable computations.

The claim that all properties of systems of conics can be expressed in such a compact form is what would be identified as Chasles's theorem. However, for his method to function and for it to rest on solid epistemic ground, Chasles did not need to prove this general formula: all he needed was to establish the lists of propositions covering any conceivable condition and to do so in a systematic manner. And this is exactly what he did: in the archives preserved at the Paris Académie des Sciences can be found thousands of leaflets on which Chasles sketched proofs of such propositions in a highly systematic and condensed form (see Figure 2), undated but most likely produced between 1864 and 1876. That a mathematician might consider this a worthwhile use of his time shows how active a role Chasles's normative ideal of the mathematician played in

[34] For instance, a popular textbook in algebraic geometry was named after this result: David Eisenbud and Joe Harris, *3264 and All That: A Second Course in Algebraic Geometry* (Cambridge: Cambridge Univ. Press, 2016).

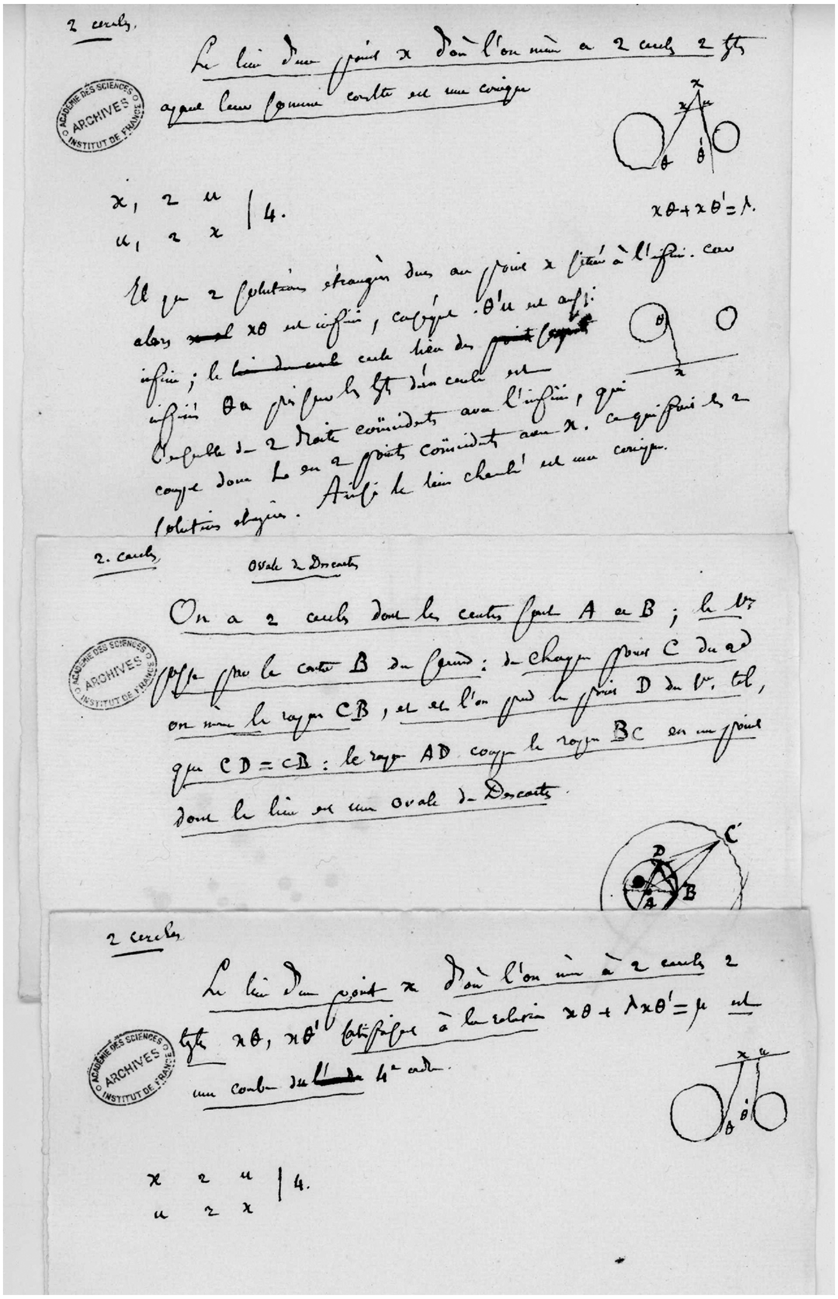

Figure 2. Chasles's lists of propositions derived from the uniform application of the "principle of correspondence." Courtesy of the Académie des Sciences de Paris, Archives et Patrimoine Historique, Dossier Chasles, 35J/4.

his scientific practice, up to the very identification of what should constitute valuable output and what proves the value of a theory.

The $\alpha\mu + \beta\nu$ formula was for Chasles less a theorem than the sign that, through the notion of characteristics, he had captured an essential, natural, and fundamental property of conic sections. For this very reason, he would liken the writing of characteristics (μ, ν) to that of the geometrical equation of a system of conics. It is not surprising, therefore, that Chasles later showed skepticism or even a lack of interest when he was confronted with the new generation's attempts at proving this formula.[35] This endeavor, in Chasles's view, amounted to proving something as fundamental as the adequacy of the concept of a degree of an equation and was not part of what he, as a mathematician and a geometer, identified as a valuable or meaningful epistemic task. The theory of characteristics, in Chasles's understanding, was the crowning achievement that displayed in full the worth of his way of acting *qua* geometer: a geometer who refused the help of artificial and computational machineries in favor of slower, pedestrian, but deeper and richer surveys of the fundamental properties of a theory and was thus rewarded with a method that would solve any problem related to conics. Of this, the simplicity and systematicity of his thousands of leaflets was a surer sign to Chasles than the complicated proofs later published would ever be.

HALPHEN: THE FALL FROM GEOMETRICAL GRACE

As Chasles's theory circulated across national, mathematical, and cultural boundaries, much was lost in translation—even at the Paris-based Société Mathématique de France (SMF). Chasles himself was the first president of the SMF, which had been formed in November 1872 partly after his lament, expressed in his 1870 *Rapport*, that French mathematics was doomed to lag behind its German, English, and Italian counterparts unless such a society was created immediately.[36] A mathematical journal, the *Bulletin de la Société Mathématique de France*, was created alongside this society. A glance at the papers published therein between 1873 and 1876 shows the dominating influence of Chasles, as the theory of characteristics and related geometrical problems form a much larger proportion of the publications than in any other major mathematical journal in Europe, aside from the *Comptes-Rendus de l'Académie des Sciences*, where Chasles's influence was equally strong.

And yet, many of the papers published in the *Bulletin* that seem to tackle the problems opened by Chasles preserved little of his notations, his mathematical style, or the central tasks he identified as motivating his work. Throughout the 1870s Halphen was, by far, the most prolific author on enumerative questions in both the *Comptes-Rendus* and the *Bulletin*. His very first paper, published in 1869, consisted in a successful attempt to replicate Chasles's theory of characteristics for a different geometrical object—namely, straight lines in space.[37] The first paragraph of the paper, however, illustrated a crucial difference in their approaches. Halphen's strategy was to express and prove a general formula for the number of straight lines in space satisfying four conditions, which, in his notations, is written $\alpha M + \beta N$. While the form of the main result is similar to Chasles's formula, the theorem-oriented structure of Halphen's paper thus contrasts starkly with that of Chasles's work. In lieu of lists of particular propositions exemplifying his $\alpha M + \beta N$ formula, Halphen's paper begins with the introduction of definitions and notations, moves on to a complete, algebraic proof of said formula, and concludes with an exploration of the theoretical consequences thereof—an inferential move seemingly of no interest to Chasles. Having identified different key epistemic goals for their work, Chasles and Halphen crafted different textual and literary resources to achieve them.

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[35] Georges-Henri Halphen, *Notices sur les travaux mathématiques de M. Georges-Henri Halphen* (Paris: Gauthier-Villars, 1885), p. 7.

[36] Chasles, *Rapport sur les progrès de la géométrie* (cit. n. 30), pp. 378–379. On the early history of this society see Hélène Gispert, *La France mathématique: La société mathématique de France, 1870–1914* (Paris: Société Française d'Histoire des Sciences et des Techniques, Société Mathématique de France, 1991).

[37] Georges-Henri Halphen, "Sur le nombre des droites qui satisfont à quatre conditions données," *Compt. Rend. Acad. Sci. Paris*, 1869, 68:142–145.

Unlike Chasles, Halphen maintained a constant engagement with German mathematics: not only did he read and communicate with German mathematicians; he also sent some of his work to German institutions and journals.[38] In 1882 he and Max Noether shared an award given by the Berlin Academy for work on skew curves; and some of his work on the theory of characteristics would be republished in the *Mathematische Annalen* at Klein's express insistence. By the time the SMF was created, Halphen was in possession of what he thought to be the first and definitive proof of Chasles's formula for conics. In 1873 he published it in the form of three short memoirs in the very first installment of the *Bulletin de la Société Mathématique de France*, only to discover that he had been beaten to it by an earlier paper written by Clebsch.

Clebsch, just like Halphen, had identified Chasles's formula as an important theorem that remained to be proven. Clebsch's intent, as outlined in the programmatic statement that opened his lectures, which Ferdinand von Lindemann edited in 1876, was the "use of simple auxiliary means for the clothing of geometrical problems by algebraic forms."[39] His 1872 posthumous paper, consequently, consists to a large extent of an attempt at extracting the algebraic content of Chasles's notions, so that the $\alpha\mu + \beta\nu$ formula would be tractable to and provable by the new theory of invariants. The first two sections aim to produce algebraic equations for systems of curves in a way less crude than mere Cartesian coordinates would allow for, while the third section is a discussion of the geometrical concept of the satisfaction of a condition, which Clebsch aims to show is equivalent to the vanishing of an invariant, a key concept in the new approaches to algebraic geometry.[40]

Halphen read Clebsch's memoir closely and made his own the project of an investigation into the algebraic content of geometrical notions. However, he soon came to believe that this project had been insufficiently pursued by Clebsch and that a more rigorous twist ought to be brought to it. In later recollections, written as part of an application for membership in the Académie des Sciences, Halphen wrote:

> I immediately noticed that I still had to make precise a notion which had until then remained vague, namely that of the *independence* of, on the one hand, the system of conics, and on the other hand, the extra condition that is imposed on the conics of this system. Often M. Chasles had neglected to mention it, but everyone restored it effortlessly. In each example, indeed, nothing is simpler. In the general theory, however, it is not clear at first how to make this independence precise.

Paradoxically, his reading of a paper that supposedly agreed with his own work stirred up doubt in Halphen's mind. According to his later retelling of the story, in investigating the analytical expression of this independence further, Halphen discovered that there were not two, as Chasles, Clebsch, and many others previously thought, but, rather, three kinds of degenerate conics, and

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[38] Halphen was fully aware of the recent developments in algebraic geometry, and his work builds extensively on that of Charles Hermite and Arthur Cayley—but also on that of German mathematicians such as Max Noether and Clebsch, who had been colleagues in Giessen during the late 1860s. See Laurent Gruson, "Un aperçu des travaux mathématiques de G.-H. Halphen," in *Complex Projective Geometry*, ed. G. Ellingsrud, C. Peskine, G. Sacchiero, and S. A. Stromme (Cambridge: Cambridge Univ. Press, 1992), pp. 189–198.

[39] Lindemann, *Vorlesungen über Geometrie von Alfred Clebsch* (cit. n. 5), p. 1. For more on the nuanced interplay between algebra and geometry in Clebsch's work see François Lê, "Alfred Clebsch's 'Geometrical Clothing' of the Theory of the Quintic Equation," *Arch. Hist. Exact Sci.*, 2017, *71*:39–70, https://doi.org/10.1007/s00407-016-0180-5.

[40] See Karen Hunger Parshall, "Toward a History of Nineteenth-Century Invariant Theory," in *The History of Modern Mathematics*, Vol. 1: *Ideas and Their Reception*, ed. David E. Rowe and John McClearly (San Diego: Academic, 1989), pp. 155–206.

this discovery became the foundation on which he constructed his counterexamples to Chasles's formula.[41]

Looking back at this turn of events, the French artillery officer immediately framed the irruption of this refutation using military vocabulary:

> This theory, which led to so many controversies, seems today to be fixed. But, one must admit, what a strange fate it's had! Where to find the source of these vicissitudes? Too much imagination, perhaps, prematurely led geometers into an ill-prepared campaign. How much uncertainty, fumbling, how many mistakes even, soon to be corrected, were seen in this century's attempts at a general Geometry, which mingles with the *theory of algebraic functions*!

This juxtaposition of mathematical rigor and strategic preparation is not entirely unique, especially among a generation of *polytechniciens* who graduated right before, or during, the bitter defeat at the hands of Prussia in 1870.[42] In a notice written after Halphen's early death in 1889, Henri Poincaré quoted the following assessment by Charles Hermite:

> Halphen, Faidherbe, after so many others, have been faithful to the double mission of the École Polytechnique, and have continued its glorious traditions. Isn't there indeed, in the habits of intelligence, in this particular nature which the teaching of our great School creates, a normal link, a concordance with the soldier's qualities? A rigorous discipline of the mind prepares one for military duties, and doubtlessly mathematical studies contribute to form this faculty of abstraction which proves indispensable to the chief who needs to form an interior representation, an image of the action by which he leads himself, forgetting danger, into the tumult and obscurity of combat.

Poincaré, himself a Polytechnique alumnus, concurred. Among the other notices that appeared after Halphen's death, most of which were written by leading mathematicians from the same generation, there emerges a rather precise description of the kind of mathematician that Halphen supposedly was. Émile Picard, for instance, in his own obituary of Halphen, distinguished between two "tendencies of mind" one can find among mathematicians: there were those who "busy themselves mainly with widening the perimeter of known notions" and those who "prefer to remain in the purview of more developed notions, to deal with them in depth."[43] Halphen would then be characterized as an extreme example of this second tendency. His *modus operandi*, Picard tells us, was to leave no question incompletely solved, to persist in investigating a matter until absolute precision and rigor had been attained. Halphen's counterexamples, in this narrative, became emblematic of a certain mathematical frame of mind, a certain way of acting *qua* mathematician. The very focus on counterexamples as a threat to generality shows well how this self translates into a specific mathematical practice. Of course, at a superficial level, no mathematician views his or her theorems as true in some cases only. However, betwixt the strict logical

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[41] Halphen, *Notices sur les travaux mathématiques* (cit. n. 35), pp. 9–10. A degenerate conic is a curve that can be decomposed into lower-degree curves—such as, for instance, a pair of lines.

[42] *Ibid.*, p. 14. On the aftermath of the defeat by Prussia in French scientific communities see Robert Fox, *The Savant and the State* (Baltimore: Johns Hopkins Univ. Press, 2012), pp. 227–273.

[43] Henri Poincaré, "Notice sur Halphen," *Journal de l'École Polytechnique*, 1890, *60*:137–161, on p. 138; and Émile Picard, "Notice sur la vie et les travaux de Georges-Henri Halphen," *Compt. Rend. Acad. Sci. Paris*, 1890, *110*:489–497, on p. 489. This portrait of Halphen's mathematical mind was also partly the result of his own self-styling, in particular in his *Notices sur les travaux mathématiques*.

interpretation of this statement and actual mathematical practice, some leeway exists: unlike Halphen, some other contemporary algebraic geometers, from a different milieu or of a different mindset, practiced a form of generic reasoning, wherein expressions that explicitly include every possible counterexample are not needed—nor perhaps even wanted.[44]

While the epistemic virtue of rigor has long been a feature of mathematicians' representations of their craft, its presence in the discourse of Halphen and his colleagues had a peculiar flavor. In a context of growing anxiety, after a military defeat that had been largely attributed to an imbalance in scientific achievements between France and Germany, as well as rising internal tensions within the body of mathematical knowledge itself, Halphen's work was viewed as the salvation brought by a new kind of mathematician: an analyst whose scientific ethos, mathematical methods, and epistemic ends were foreign to Chasles's pure geometer.[45] Whereas the latter strove for the naturalization of theoretical settings, and set his mind to seeking the simplest and most fundamental properties of geometrical figures, the analyst investigated the domain of validity of each theorem with utmost precision, discussed every possible counterexample, and used analytical means to expunge all possible vagueness from mathematical language. To the authority of the genial *académicien* that Chasles represented, Halphen opposed the "specialized disciplinary expertise" of those with a hard-earned mastery of the modern techniques of algebraic analysis.[46] Between these two figures, a fall from geometrical grace had occurred: Halphen would think of mathematics not as a domain of knowledge in which Nature provides simple and general formulas to the acute observer but, rather, as a set of hidden truths to be coldly besieged and eventually conquered. His alternative theory of characteristics leads to no neat and concise formula for the enumeration of conics; it even rejects the possibility for any finite number of terms to express a general solution to Chasles's problem. For someone who evaluates a theory on the ground of the ease and systematicity of its use, this would be a major setback. To Halphen, for whom such lofty hopes of simplicity were unfounded, it was nothing more than another sign of the deceptive character of naive intuition.

SCHUBERT: HUMAN, ALL TOO HUMAN MATHEMATICS

Halphen's refutation built on intricate algebraic computations, which several of his colleagues admittedly struggled to understand. Even after the delayed publication of Halphen's memoirs in three of Europe's most famous mathematical journals, the explanations and arguments for his alternative formulas met a mixed reaction. While no one contested the mathematical skill displayed in these texts, several geometers elected to keep on using a formula that had been so fruitful—and seemingly correct—for years, sometimes merely adding a footnote or a passing remark mentioning Halphen's criticism to memoirs or papers that fully depended on Chasles's formula. This was the case of Hermann Schubert, a *Gymnasium* teacher in Hamburg, who had

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[44] Thomas Hawkins, "Hesse's Principle of Transfer and the Representation of Lie Algebras," *Arch. Hist. Exact Sci.*, 1988, 39:41–73, https://doi.org/10.1007/BF00329985. In another obituary for Halphen, written by Camille Jordan, comparisons were drawn between his mathematical style and that of Niels Henrik Abel, another important figure in the emergence of the notion of mathematical counterexamples; see Henrik K. Sørensen, "Exceptions and Counterexamples: Understanding Abel's Comment on Cauchy's Theorem," *Hist. Math.*, 2005, 32:453–480, https://doi.org/10.1016/j.hm.2004.11.010. The production of counterexamples, and the shift in epistemic norms it betrays, can be compared to what Daston and Galison presented as the passage from "truth-to-nature objectivity" to "mechanical objectivity" in the Cajal–Golgi dispute; see Daston and Galison, *Objectivity* (cit. n. 17), pp. 115–125.

[45] Fox, *Savant and the State* (cit. n. 42), pp. 259–273; and Jeremy Gray, "Anxiety and Abstraction in Nineteenth-Century Mathematics," *Science in Context*, 2004, 17:23–47, https://doi.org/10.1017/S0269889704000043. On the epistemic value of rigor in mathematicians' representations of what they do see Judith Grabiner, *The Origins of Cauchy's Rigorous Calculus* (Cambridge, Mass.: MIT Press, 1981).

[46] Fox, *Savant and the State*, pp. 236–237.

begun working on the theory of characteristics at around the same time as Halphen. In 1870 Schubert had written for his doctoral thesis a faithful and competent adaptation of Chasles's theory to second-order surfaces, with little in the way of notational or conceptual innovation.[47] Things would change drastically toward the end of 1873. In a seemingly anodyne paper, Halphen had made an observation to which he would never return and to which he attributed no particular importance. To Schubert, however, this observation would mark the birth of an entirely new way of writing, proving, and understanding enumerative properties of figures.

In 1864 Chasles had already obtained a complex expression for the number of conics satisfying five given conditions with associated coefficients (α_1, β_1), (α_2, β_2), ... , (α_5, β_5) by applying the $\alpha\mu + \beta\nu$ formula to them five times in a row. This expression was too unwieldy for theoretical use but allowed for moderately faster computations when five concrete geometrical conditions were given. In 1873, at a time when he still believed in the validity of the $\alpha\mu + \beta\nu$ theorem, Halphen noticed that the final expression could be expressed through the much simpler formula

$$(\alpha_1 p + \beta_1 d)(\alpha_2 p + \beta_2 d) \ldots (\alpha_5 p + \beta_5 d),$$

where the letters p and d are to be understood as variables, like the x's and y's of a polynomial equation. Halphen showed that the expression above yielded the desired number of conics, provided that it be developed, and that each symbol $p^i d^{5-i}$ be respectively replaced by the number of conics passing through i points and touching 5-i lines. For Halphen, this was no more than a symbolic manipulation, helpful to make the general formula easier to handle. In particular, the above compact expression had no intrinsic meaning, and the letters p and d denoted empty variables, to be instantiated only at the end of a symbolic computation via concrete numbers.[48]

Schubert, however, saw something far more general and powerful in Halphen's paper and immediately started publishing a series of articles making very creative and fruitful use of this observation. In the first of these papers, he immediately restored the symbols for characteristics μ and ν in Halphen's general formula, instead of using the empty variables p and d. He read this formula as expressing the fact that the product of five $(\alpha_1\mu + \beta_1\nu)$ modules of five given conditions does indeed represent the number of conics satisfying these five conditions, for the symbols $\mu^i\nu^{5-i}$ represent a composed condition of dimension 5—that is to say, a condition that can only be satisfied by a finite number of conics. In Schubert's view, one could simply combine symbols for conditions, as if they were algebraic entities, and factors of maximal dimension would simply represent finite numbers of solutions.[49] For instance, if the symbols P and G denoted, respectively, the conditions "passing through a given point" and "touching a given line," then the symbol PG would denote the compound condition "passing through a given point and touching a given line." When turning to the enumerative geometry of a certain figure, Schubert would enumerate basic conditions that can be imposed on such curves, represent them with symbols, and proceed to give elementary algebraic formulas governing the use of these symbols. The juxtaposition of two such symbols would stand for the conjunction of the two initial conditions, while additions of symbols would denote the disjunction of the corresponding conditions; and the symbolism of algebra would provide a way to compute with geometrical conditions.

[47] Hermann Schubert, "Zur Theorie der Charakteristiken," *Journal für die Reine und Angewandte Mathematik*, 1870, *71*:368–382. For biographical information on Schubert see Werner Burau, "Der Hamburger Mathematiker Hermann Schubert," *Mitteilungen der Mathematischen Gesellschaft in Hamburg*, 1966, 9:10–19.

[48] Georges-Henri Halphen, "Sur les caractéristiques, dans la théorie des coniques, sur le plan et dans l'espace, et des surfaces du second ordre," *Compt. Rend. Acad. Sci. Paris*, 1873, 76:1074–1077.

[49] Hermann Schubert, "Die Characteristiken der ebenen Curven dritter Ordnung im Raume," in *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen* (Göttingen, 1874), pp. 267–283. For a more detailed exposition of Schubert's calculus see Steven Kleiman and Dan Laksov, "Schubert Calculus," *Amer. Math. Mon.*, 1972, *19*:1061–1082.

The wildly alogical nature of Schubert's mathematical act is striking. The validity of symbolic manipulations in Halphen's memoir rested on the truth of the $\alpha\mu + \beta\nu$ formula and constituted little more than a rewriting of this formula, with algebra being used to prove, delineate, and explore geometrical truths. Schubert, however, had turned this elementary observation into a full-fledged, autonomous symbolic apparatus, the justification of which would remain lacking in the eyes of the majority of his colleagues.[50] From a fruitful rereading of a consequence of Chasles's formula for conics, Schubert developed a symbolism whose stated goal was to combine geometrical conditions of all kinds and to deal with all sorts of geometrical figures, with algebra now serving as a model for a new geometrical language, rather than as a conceptual tool for stating and proving theorems.

The radical novelty of Schubert's approach cannot be overstated. Not only did Schubert obtain a plethora of new results—some of which were deemed extremely impressive, such as the enumeration of 666,841,088 quadrics tangent to nine others; his was also an entirely new way of researching, writing, and presenting the results of geometrical investigations. Unlike Halphen's theorem-oriented memoirs, and more like Chasles's texts, Schubert found value in the production of long lists of formulas. These lists, however, differed significantly from those of Chasles. They were not the systematic and voluntarily monotone enumeration of properties of geometrical conditions but, rather, tables of symbolic expressions to be used in the course of enumerative computations (see Figure 3).

Schubert's construction of the *Kalkül der abzählenden Geometrie* was contemporary with and partly shaped by an epistolary exchange with Halphen and Zeuthen that began in 1876, shortly after he published his first complete presentation of enumerative geometry and Halphen offered his first counterexamples to Chasles's formula. The tone of the discussion was at first cordial, as Halphen helped Schubert become a member of the SMF, and as the Royal Danish Academy, of which Zeuthen was a leading member, had just awarded him a gold medal for his work on cubics.[51] As Halphen published his counterexamples, however, the exchange turned polemical. Schubert continued to write papers deploying the methods of his *Kalkül* on various geometrical figures, systematically searching for analogues of Chasles's formula (of which he himself coauthored a proof with Adolf Hurwitz in the immediate wake of Halphen's refutation).[52] Halphen viewed this enterprise as doomed, since he had just shown that such a problem could not be solved even in the simple case of conics; but he failed to convince Schubert either to renounce his project or to take the necessary precautions to account for these new counterexamples. Schubert's letters, to which the replies are not extant, show the German mathematician maintaining a friendly and even disciple-like tone, even at some points acknowledging the merits

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[50] The power of Schubert's methods, combined with their apparent lack of theoretical justification, led two prominent modern-day mathematicians to describe some of Schubert's results as "landing a jumbo jet blindfolded!"; see Eisenbud and Harris, *3264 and All That* (cit. n. 34), p. 2.

[51] For Schubert's first complete presentation of enumerative geometry see Hermann Schubert, "Beiträge zur abzählende Geometrie," *Math. Ann.*, 1876, *10*:1–116. This text also marks the first occurrence of the term "enumerative geometry." For Schubert's contributions to the epistolary exchange see Hermann Schubert to Georges-Henri Halphen, 12 Aug., 5 Nov., 4 Dec. 1876, Bibliothèque de l'Institut, Paris, Ms 5624, nos. 168–170. The prizewinning paper was eventually published; see Schubert, "Die 13 Ausartungen und die Fundamentalzahlen der ebenen Curven dritter Ordnung mit Spitze," in *Nachrichten von der Königl. Gesellschaft der Wissenschaften zu Göttingen* (Göttingen, 1875), pp. 359–387.

[52] Adolf Hurwitz and Hermann Schubert, "Ueber den Chasles'schen Satz $\alpha\mu + \beta\nu$," in *Nachrichten von der Königl. Gesellschaft und der Georg-Augusts-Universität zu Göttingen* (Göttingen,1876), pp. 503–517. Adolf Hurwitz, who would become a well-known mathematician in his own right, was then Schubert's pupil in a high school in Hildesheim. See Nicola Oswald and Jörn Steuding, "Complex Continued Fractions: Early Work of the Brothers Adolf and Julius Hurwitz," *Arch. Hist. Exact Sci.*, 2014, 68:499–528, https://doi.org/10.1007/s00407-014-0135-7. In a letter to Halphen dated 21 May 1876, Schubert attributes the main idea of the proof to Hurwitz: Cod Ms 5624, no. 166.

6. Die Bestimmung der Zahl der Curven, bei denen von den Schnittpunkten mit einer beliebigen Geraden zwei zusammenfallen, giebt:

$$4\nu = \varrho + 6\mu + 2b + \alpha_6.$$

7. Die Bestimmung der Zahl der Curven, bei denen von den vier Tangenten aus einem beliebigen Punkte zwei zusammenfallen, giebt:

$$6\varrho = \nu + 3f + \chi + \alpha_7.$$

8. Die Bestimmung der Zahl der Curven, bei denen zwei Wendetangenten zusammenfallen, giebt:

$$4f = 6\mu + 2u + 2\chi + \gamma + \alpha_8.$$

9. Die Bestimmung der Zahl der Curven, bei denen zwei Wendepunkte zusammenfallen, giebt:

$$4v = 6s + 2\chi + 3\gamma + \alpha_9.$$

10. Die Bestimmung der Zahl der Curven, bei denen zwei Eckpunkte des Wendetangentendreiseits zusammenfallen, giebt:

$$4u = 2f + 2\chi + \gamma + \alpha_{10}.$$

11. Die Bestimmung der Zahl der Curven, bei denen die beiden Doppelpunktstangenten zusammenfallen, giebt:

$$2p = 2\mu + 2b + \gamma + \alpha_{11}.$$

Figure 3. Another list-making practice, displayed in Hermann Schubert, *Kalkül der abzählenden Geometrie* (Leipzig: Teubner, 1879), p. 150.

of Halphen's work (and half-admitting that he could not fully understand the proofs), while making no changes whatsoever to his own publications. In turn, contemporary letters from Zeuthen to Halphen (to which the replies are not extant either, save for a few that were transcribed in Halphen's *Oeuvres complètes*) indicate that Halphen was becoming progressively annoyed with his interlocutor's reluctance to take note of the newly established falsity of Chasles's formula. Toward the end of 1879 Halphen grew restless and demanded that he and Schubert make their disagreement public and present their views for discussion during one of the biweekly meetings of the SMF (which Schubert could not attend). Having revisited his previous work on conics, Halphen was now able to produce counterexamples for each of Schubert's general formulas almost on command, whatever the geometrical figure at hand. Schubert would initially attempt to save his formulas by arguing that they were simply meaningless in the problematic cases that Halphen was pointing to.[53] Elsewhere, he would propose reinterpretations of Halphen's counterexamples to turn them into "interesting verifications" of his own formulas, without ever engaging with the Frenchman's intricate analytical discussion of what it means to satisfy a geometrical condition.[54] Eventually, in January 1880, Schubert gave up on defending his formulas.[55] While a retraction was soon published in the *Bulletin*, Schubert pushed ahead with his mathematical practice, entirely unaltered, in articles published at the same time in German

[53] "Now, you said that these formulas are inaccurate (*ungenau*), because they do not work in those cases, but I said that the formulas are even meaningless (*sinnlos*) then, and that their nonapplicability [in these cases] is self-evident from the context of my book": Schubert to Halphen, Dec. 1879, Cod Ms 5624.

[54] Hermann Schubert, "Sur le principe concernant la constance des nombres géométriques," a note intended for publication in the *Bulletin de la Société Mathématique de France*, is appended to Schubert to Halphen, 31 Nov. 1879, Cod Ms 5624.

[55] For the full sequence see Georges-Henri Halphen, "Observations sur la théorie des caractéristiques," *Bull. Soc. Math. France*, 1880, 8:31–34; and Hermann Schubert, "Réponse aux observations de M. Halphen sur la théorie des caractéristiques" and "Note sur l'évaluation du nombre des coniques faisant partie d'un système et satisfaisant à une condition simple," *ibid.*, pp. 60–61.

journals.[56] The concession made on French soil was largely born out of a desire to maintain peaceful scientific communications with a prestigious society—which was of paramount importance to a *Gymnasiumlehrer* in Hamburg, isolated from the main German mathematical communities—rather than a sincere renunciation.[57]

In letters he wrote to Zeuthen at the same time, Halphen explained his gripes with Schubert's persistence in looking for formulas such as Chasles's $\alpha\mu + \beta\nu$:

> Of all the reasons one can enlist against the allegedly *general* theorems, the best is the following: the arguments with which they can be covered disappear when the two beings (C), (Σ) [(C), (Σ) here refer respectively to a figure which solves a problem, and a system of such figures] are each defined by more than one equation. In these circumstances, we must abandon *intuition* and come back to Analysis. By this term I mean true reasoning; I demand no equation, of course. M. Schubert absolutely wants to alter nature to accommodate it to his formulas. We deal with a problem that admits one solution: *One! Are you joking? The formula yields two: therefore there are two!* Do you know how I replied? I took the question to be a particular case of another, wherein the formula yields 5, and then of another, wherein the same formula gives one.[58]

Halphen's criticism focused on Schubert's supposed belief that formulas can give rise to their own meaning. Schubert, according to Halphen, had inverted the proper epistemological order: he accepted the results produced by his symbols and left no room for critical appraisal of these results. And yet, Halphen thought he had uncovered and revealed the insufficiently precise determination of what this unchecked symbolism represents: to one formula of Schubert's Halphen had associated two possible, yet contradictory, analytical representations. For this reason, he disparaged Schubert's *Kalkül* as mere "intuition," despite there being little that one would spontaneously describe as intuitive in this highly formal and symbolic geometrical practice: "intuition," for Halphen, served as a generic term to refer pejoratively to any mathematical practice not based on the careful analysis of its equations or definitions of its concepts. A few days prior to this letter, Halphen had asserted to Zeuthen that he "knew the meaning of these formulas much better than Schubert, without a doubt."[59] There again, he asserted his ability to gain insight into the correspondence between complex formulas and geometrical configurations through his mastery of analysis against whatever loose, potentially fruitful, combination of symbols Schubert had devised. This line of defense could not have contrasted more strongly with Schubert's own depiction and understanding of mathematical practice.

Indeed, Schubert's enumerative geometry can be advantageously read against the backdrop of his own philological, ethnographical, and philosophical interests, thus casting new light on his mathematical work and his refusal to engage fully with Halphen's criticism. In the wake of his 1879 book and after a decade of intense work on enumerative geometry, Schubert began authoring articles and books on a wider range of subjects and in a wider range of journals. Conspicuous

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[56] See, e.g., Hermann Schubert, "Anzahlgeometrische Behandlung des Dreiecks," *Math. Ann.*, 1880, *17*:153–212. In a short passage (pp. 157–159), Schubert simply mentions "Halphen's degenerations" as a possible counterexample to his results and excludes them without discussing how one is supposed to detect their presence. Schubert's strategy may be compared to what Imre Lakatos called the "monster-barring strategy": Imre Lakatos, *Proofs and Refutations* (Cambridge: Cambridge Univ. Press, 1976), Ch. 1, Sect. 4.b.

[57] In his letters to both Halphen and Klein, Schubert complains frequently of the price and difficulty of having mathematical journals delivered to him or to the relatively small *Gesellschaft* of Hamburg-based mathematicians.

[58] Halphen to Hieronymus Zeuthen, 7 Dec. 1879, quoted in *Oeuvres de G.-H. Halphen*, ed. Camille Jordan and Henri Poincaré, Vol. 4 (Paris: Gauthier-Villars, 1924), p. 637.

[59] Halphen to Zeuthen, 29 Nov. 1879, *ibid.*, p. 636.

among these are works on the philology and ethnography of numbers, recreational mathematics, elementary textbooks on algebra, and popular essays on the nature of mathematical knowledge.

In a booklet published in 1887 in a collection co-edited by Rudolf Virchow, Schubert expounded what he described as a "cultural-historical study" of the formation of numbers. This text consisted in a sketch of the developmental stages through which the formations of number words (*Zahlwortbildung*) and number signs (*Zahlzeichenbildung*) allegedly ought to pass. For Schubert, "the system of numbers that we take as self-explanatory in our childhood is not something that can be taken as self-explanatory, but rather the highest offshoot of a cultural-historical process that began when man became man, when he began to speak and write." Schubert went on to show how various peoples devised various ways to write numbers and to represent them within the context of both their cultural and their ecological landscape. Religions and mythologies, as well as surrounding seas or mountains, are cast as possible factors in the development of a people's number words and number signs. In a particularly striking passage, Schubert wrote:

> There is [in the oldest literature of the Brahmans] talk of a king who advanced his wealth to a hundred thousand trillion jewels, of a Monkey Prince who could confront his enemies with 10,000 sextillion monkeys in battle. And in Buddhist times one read of 24,000 trillion deities and of the 600,000 million sons of Buddha. . . . The Greeks were too friendly to the natural and the true, to love such exaggerations. Homer lets a wounded Ares scream like 9- or 10,000 men in the fifth book of the Iliad. In India, a god of war who could only scream like 10,000 men, would be considered asthmatic.[60]

Schubert goes on to explain why certain peoples (*Volk*) have a need and desire for large numbers, which in turn led them to devise ways of conveniently writing words for large numbers. To craft a word for the number ten thousand, the Greeks created the new word "μύριοι" (myriad), because they couldn't reasonably foresee a real need for many more such words. The Indians, on the contrary, yearned for ever-larger numbers, and so they devised a way to express them using number words that already existed, not unlike contemporary English does with the juxtaposition of the words "ten" and "thousand." Such systems of number words are ultimately classified by Schubert on a scale that goes from "*Natürliche Zahlreichen*" (numbers being represented by collections of points or other tokens) to the "*Prinzip des Stellenwerthes*," which corresponds to our modern way of writing the so-called Arabic numerals.[61]

In this book and further publications, Schubert displayed an exhaustive knowledge of the contemporary philological and ethnographical literature. He was in close contact with both explorers and philologists, publishing a summary of his views in the second edition of the German explorer Georg von Neumayer's *Guides to Scientific Observations on Travels* and participating in the Kongress Deutscher Philologen und Schulmänner in 1905.[62] Schubert's *kulturgeschichtliche* project also bears the mark of a larger German tradition of cultural history of mathematics, which is most famously associated with Moritz Cantor but actually goes back to Arthur Arneth. A professor of mathematics at the Heidelberg *Lyceum*, Arneth "viewed the abstraction process

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[60] Hermann Schubert, *Zählen und Zahl: Eine kulturgeschichtliche Studie* (Hamburg: Richter, 1887), pp. 1, 14–15. In other passages, ecological features such as the number of neighboring lakes are similarly taken as crucial determinants of a people's system of numbers.

[61] *Ibid.*, p. 36.

[62] Hermann Schubert, "Das Zählen," in *Anleitung zu wissenschaftlichen Beobachtung auf Reisen*, ed. Georg von Neumayer, 2nd ed., Vol. 2 (Berlin: Oppenheim, 1888), pp. 288–294. On the guides see Peter Monteath, "German Anthropology, Nationalism, and Imperialism: Georg von Neumayer's *Anleitung zu wissenschaftlichen Beobachtungen auf Reisen*," *History and Anthropology*, 2018, *31*:1–22, https://doi.org/10.1080/02757206.2018.1524758.

leading to mathematical content as being conditioned by cultural factors." In tying these locally and culturally rooted mathematics together into an ultimately universal science that surmounts national and regional characteristics, Schubert shifted this tradition closer to the cosmopolitan and Humboldtian historiography of Hermann Hankel, another German mathematician at the crossroads of philology, mathematics, and history of mathematics.[63]

Schubert's interest in philology and ethnography is crucial for understanding his epistemology of mathematics and the regulative ideal of mathematical activity that underlay his geometry.[64] Indeed, in a series of articles for the newly created journal the *Monist*, as well as in the very first chapter of Klein and Wilhelm Meyer's *Encyklopädie der mathematischen Wissenschaften*, Schubert built on this study of the (cultural) history of numbers.[65] From his study of "primitive" systems of numeration, and his understanding of the developmental stages of the path to ideal number systems, he attempted to derive a philosophical account of what numbers are, as well as what strings of symbols of numbers and operations represent.[66] "Counting a group of things," Schubert proposed, "is to regard the things as the same in kind and to associate ordinally, accurately, and singly with them other things. In writing, we associate with the things to be counted simple signs, like points, strokes, or circles." Philological and ethnographical studies reveal the original mathematician as a crafter of signs, words, and symbols, someone who progressively emancipates his science from the local cultural and ecological landscape it originated from. Once such emancipation has been achieved, the mathematician's numbers are pure cultural creations: "Observation of the world of actual facts, as revealed to us by our senses, can naturally lead us only to positive whole numbers, such only, and no others, being results of actual counting. All other kinds of numbers are nothing but artificial inventions of mathematicians."[67]

How, then, are we to know how to operate on these unnatural numbers? Schubert's solution to this question, while not completely unoriginal, borrows extensively from Hankel's work on systems of numbers, in particular his principle of permanence. Schubert renamed it the "principle of no exception" and summarized it as follows:

> In the construction of arithmetic every combination of two previously defined numbers by a sign for a previously defined operation (plus, minus, times, etc.) shall be invested with meaning, even where the original definition of the operation used excludes such a combination; and the meaning imparted is to be such that the combination considered shall obey the same formula of definition as a combination having from the outset a signification, so that the old laws of reckoning shall still hold good and may still be applied to.[68]

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[63] Ivahn Smadja, "Sanskrit *versus* Greek 'Proofs': History of Mathematics at the Crossroads of Philology and Mathematics in Nineteenth-Century Germany," *Revue d'Histoire des Mathématiques*, 2015, *21*:217–349, https://doi.org/10.24033/rhm.189, on p. 266; on Hankel more specifically see pp. 301–307.

[64] The crucial role played by the rise of philology as a science even in the seemingly remote province of mathematics (in the German context) offers more evidence for what has recently been argued in Lorraine Daston and Glenn Most, "History of Science and History of Philologies," *Isis*, 2015, *106*:378–390, https://doi.org/10.1086/681980.

[65] Hermann Schubert, "Grundlagen der Arithmetik," in *Encyklopädie der mathematischen Wissenschaften*, ed. Felix Klein and Wilhelm Meyer, 6 vols. (1898–1933), Vol. 1.A.1 (Leipzig: Teubner, 1898), pp. 1–27. On the creation of the *Monist* and Schubert's popular essays published there see Jemma Lorenat, "An Okapi Hypothesis," forthcoming.

[66] Of course, this was a concern shared by many German mathematicians at the time. Famous examples include Richard Dedekind, Gottlob Frege, and Edmund Husserl. See Moritz Epple, "The End of the Science of Quantity: Foundations of Analysis, 1860–1910," in *A History of Analysis*, ed. Hans Niels Jahnke (Providence, R.I.: American Mathematical Society, 2003), pp. 291–323.

[67] Hermann Schubert, "Notion and Definition of Number," *Monist*, 1894, *4*:396–402, on pp. 397, 402.

[68] Hermann Schubert, "Monism in Arithmetic," *Monist*, 1894, *4*:561–579, on p. 567. This principle closely resembles George Peacock's famous "principle of equivalent forms," whose shaping had also hinged on the author's interest in natural history. The

Crucial for both Schubert's and Hankel's understandings of what systems of numbers are is the latter's proof of the theorem that there can't possibly be any extension of the system of complex numbers that preserves basic algebraic laws, such as commutativity ($ab = ba$). Hamilton's quaternions, for instance, are an extension of complex numbers in which the order of multiplication matters. For Schubert, this shows that "the building up of arithmetic is thus completed" and that this science has reached absolute perfection because it derives from a single "monistic principle."[69]

The connection between these views and geometry would appear most clearly in Schubert's rebuttal of the spiritualist theses of the German astrophysicist Johann Zöllner.[70] Toward the end of his life, Zöllner had argued that the mathematics of four-dimensional spaces and its physical interpretation form a rational and scientific basis for spiritualism—that is, the study of the spirits of the dead. Rejecting any attempt to use pure mathematics to naturalize such phenomena, Schubert insisted on the purely artificial character of the numbers mathematicians freely construct in the course of their work. Dimensions are but one example of such artificial numbers:

> Is it permissible to extend the notion of space by the introduction of point-aggregates of more than three dimensions? . . . In mathematics, in fact, the extension of any notion is admissible, provided such extension does not lead to contradictions with itself or with results which are well established. Whether such extensions are necessary, justifiable, or important for the advancement of science is a different question. It must be admitted, therefore, that the mathematician is justified in the extension of the notion of space as a point-aggregate of three dimensions, and in the introduction of space or point-aggregates of more than three dimensions, and in the employment of them as means of research.

Schubert's views once more echoed those of Hankel, who had famously claimed that "number is no longer an object, a substance which exists outside the thinking subject and the objects giving rise to it, an independent principle, as it was for instance for the Pythagoreans. . . . Only that counts as impossible for the mathematician which is logically impossible, i.e. that which contradicts itself." For Schubert, the mathematician wields symbols and concepts with no intrinsic relation whatsoever to natural objects. The sole rules of such an activity are that it should preserve past discoveries and introduce no new contradictions. This is not to say that anything goes: mathematics, for Schubert, is always located on a path of progress, the end goal of which is the "[unification] under a high point of view of theories heretofore regarded as different."[71]

To view Schubert as a philosopher of mathematics is bound to lead to disappointments: his writings do not have the finesse and argumentative solidity to withstand assaults from the likes of Gottlob Frege, who harshly dismantled his views on numbers in an ironic review. There is, however, much to gain from reading Schubert's texts as depicting a regulative ideal of mathematical

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mathematical and social contexts of Peacock's and Hankel's work, however, are very different. See Kevin Lambert, "A Natural History of Mathematics: George Peacock and the Making of English Algebra," *Isis*, 2013, *104*:278–302, https://doi.org/10.1086/670948; and Joan Richards, "Augustus de Morgan, the History of Mathematics, and the Foundations of Algebra," *ibid.*, 1987, 78:6–30, https://doi.org/10.1086/354328.

69 Schubert, "Monism in Arithmetic," pp. 578–579. On the social history of Hamilton's quaternions and their reception see Josipa Petrunic, "Quaternion Engagements and Terrains of Knowledge (1858–1880)" (Ph.D. diss., Univ. Edinburgh, 2009).

70 See Diethard Sawicki, *Leben mit den Toten: Geisterglauben und die Entstehung des Spiritsmus in Deutschland 1770–1900*, 2nd ed. (Paderborn: Schöningh, 2016), pp. 299–310. Zöllner's theses were discussed by a range of philosophers and scientists, from Nietzsche to Helmholtz, and even caused a scandal in Leipzig in the early 1870s.

71 Hermann Schubert, "The Fourth Dimension: Mathematical and Spiritualistic," *Monist*, 1893, 3:402–449, on p. 410; Hermann Hankel, *Theorie der complexen Zahlensysteme* (Leipzig: Voss, 1867), pp. 6–7; and Schubert, "On the Nature of Mathematical Knowledge," *Monist*, 1896, 6:294–305, on p. 301.

activity, one that already ruled his geometrical research. Indeed, Schubert's enumerative geometry, as the name suggests, is a science of the numbers of geometry. In some instances, he even uses expressions such as "geometrical numbers" to refer to the symbols of his *Kalkül*.[72] Thus, the conclusions of his later philosophical papers are strongly tied to the way he envisioned and directed his geometrical practice. We can now understand why Halphen's criticism failed to elicit a strong reaction from Schubert. Halphen accused Schubert of "altering Nature," but this accusation could not sway the German geometer, for whom mathematicians were free to craft symbols and numbers as they saw fit, as long as no contradictions were thus introduced, in the hope of finding a path to a unitary formulation of the solution to a geometrical problem. The rhetorical recourse to Nature, whether in the form of Chasles's account of geometrical practice as the search for fundamental properties from which theories can be effortlessly derived or of Halphen's description of the analyst using his expert training and tools to track the traps and counterexamples inherent to our imprecise geometrical intuition, was ultimately meaningless for Schubert. Here again, a clash of geometers who shared little understanding of what mathematical activity consists in and what its goals are made constructive dialogue all but impossible.

STUDY: A MATTER OF POINT OF VIEW?

A subtle way out of Schubert and Halphen's impossible dialogue had been previously proposed by Hieronymus Zeuthen. In a long letter, written as a response to Halphen's announcement (in a private letter) that he "had no more doubts regarding the falsity of the $\alpha\mu + \beta\nu$ theorem," Zeuthen suggested that it might be possible to "preserve Chasles's theorem by adopting another point of view." Zeuthen then distinguished three points of view on what conics are and on what it means to enumerate them. The first one, which he attributed to de De Jonquières (Chasles's aforementioned student), consisted in "defining conics exclusively by their punctual properties"; the second, which he assumed to be Chasles's position, consisted in "regarding in an equal manner punctual and tangential properties." Halphen's point of view, which had led him to reject the $\alpha\mu + \beta\nu$ formula, was not characterized by Zeuthen in a similar manner, but only said to be "entirely clear and well-defined." These points of view were not equal: while the first one is "simple and very clear," it suffers from being altered by the principle of duality and leads to infinite numbers of solutions or other such meaningless results in some enumerative problems. As for the second, according to Zeuthen, "Chasles's theorem is such an intimate consequence [thereof], that its proof would present itself, were we only able to *define* it precisely."[73] The third viewpoint, Halphen's, was the only one deemed sufficiently clear and precise, while not presenting decisive geometrical flaws.

Halphen adopted this stance in early publications about his counterexamples but never fully committed to it.[74] His last letters to Zeuthen, such as the one quoted previously, completely break from it: in lieu of contrasting viewpoints, Halphen rooted the authority of his theory in the nature of geometrical objects and its examination through analysis. Zeuthen's solution, however, was not lost on everyone: Henri Poincaré, who acted as coeditor of Halphen's collected works after

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[72] For the review see Gottlob Frege, *Über die Zahlen des Herrn H. Schubert* (Jena: Pohle, 1889); there is a translation by Hans Kaal in *Gottlob Frege: Collected Papers on Mathematics, Logic, and Philosophy*, ed. Brian McGuiness (Oxford: Blackwell, 1984), pp. 249–272. For a use of the expression "geometrical numbers" see Schubert, "Sur le principe concernant la constance des nombres géométriques" (cit. n. 54).

[73] Halphen to Zeuthen, 29 July 1876, in *Oeuvres de G.-H. Halphen*, ed. Jordan and Poincaré (cit. n. 58), Vol. 4, pp. 629–635, on p. 629; and Zeuthen to Halphen, 11 Aug. 1876, Cod Ms 5624, no. 224.

[74] Georges-Henri Halphen, "Sur les caractéristiques des systèmes de coniques et de surfaces du second ordre," *Compt. Rend. Acad. Sci. Paris*, 1876, 83:886–888. This note is a summary of a memoir, written a few months after Zeuthen's letter, that Halphen sent to the Académie on 13 Nov. 1876.

his death, had initially invited Zeuthen to publish their correspondence. While only a small fraction of Halphen's letters were eventually published, Poincaré was able to survey the exchanges with Schubert and Zeuthen while editing Halphen's complete works. As he wrote his obituary for Halphen, Poincaré would reformulate Zeuthen's presentation of said exchanges with a twist of his own: "*points de vue*" had become "*conventions*," and Halphen was now credited with being the first to make explicit and perfectly precise the possible conventions one can adopt regarding the question of generality in enumerative geometry.[75] The notion that the validity of a theorem may depend on conventions or viewpoints, however, would be put to more critical use by Eduard Study, whom Klein had advised in 1884 to write his *Habilitationsschrift* on the disputes plaguing enumerative geometry. Indeed, Study's dissertation would display yet another understanding of the meaning of Chasles's formula, as well as another way to understand the nature of the epistemic task at hand for the enumerative geometer.

Not unlike Halphen, Study initially pitted the intuition of the geometers of the past against logical deductions and concepts, which alone could end the turmoil surrounding Chasles's formula: "If, however, one wants to settle with complete generality a problem which in special cases is treated in an intuitive manner, then one must move from intuition (*Anschauung*) to concepts (*Begriffe*), and put logical deductions in place of appeals to appearances. Often, in individual cases, the latter are merely one's silent confession of the insufficient awareness of the true reasons behind a result." Study viewed Clebsch as having taken a decisive step forward in that direction, and, like Halphen, he aimed to clarify the content of the vague geometrical concepts that had led Chasles to his formula. However, Study was not ready to give up on Chasles's formula. To defend it, he expounded a new way of conceptualizing generality in geometry:

> One has . . . to distinguish between the properties of the figures whose presence is regarded as the necessary and sufficient condition for the existence of those other properties which represent the geometric proposition, and the others which are regarded as consequences of said proposition, or only conditioned by the arbitrary manifestation of the general proposition. The former must be elevated to the rank of definitions and made the basis of proof. This operation is performed by anyone who makes a generalization, intentionally or not. Since it consists mainly in clarifying one's conception of what is essential for a proposition, it can be carried out without appearing to one's consciousness as a progress of thought.

Study put the emphasis on the clear delineation of these fundamental properties of a figure, which characterize the permanence of the other properties of said figure. His strategy, going forward, would be to search for the property that, in Chasles's theory, had implicitly characterized conics in enumerative context. Study claimed that this was also Clebsch's strategy but that the latter was misled in his search. Halphen's counterexamples, thus, only showed that Clebsch's algebraic characterization of conics (for enumerative purposes) was not adequate. However, Study added, the "definition of solutions is arbitrary"; and there was another interpretation of Chasles's formula that makes it absolutely precise and valid.[76] To expound it was the purpose of Study's dissertation.

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[75] Poincaré, "Notice sur Halphen" (cit. n. 43), p. 152. On conventions in Poincaré's philosophy of mathematics see Gerhard Heinzmann, "Hypotheses and Conventions in Poincaré," in *The Significance of the Hypothetical in the Natural Sciences*, ed. Michael Heidelberger and Gregor Schiemann (Berlin: De Gruyter, 2009), pp. 169–192, https://doi.org/10.1515/9783110210620.

[76] Eduard Study, "Ueber die Geometrie der Kegelschnitte, insbesondere deren Characteristikenproblem," *Math. Ann.*, 1886, 27:58–101, on pp. 61, 61–62, 64.

Later on, as Study traveled to Paris with Hilbert, he met Halphen in person, but neither of them was capable of changing the other's mind.[77] From the few letters they also exchanged around this period, it appears that their divergences were more than simply mathematical. Throughout this exchange, Study insisted that "Chasles did not have a sufficiently clear idea of the nature of the solutions which were to be counted; so that [Halphen's] conception of the theorem and [Study's] both should be regarded as interpretations, and indeed as equally valuable interpretations of the original formulation to be determined." Study wanted to frame the relation between their memoirs as that of two equally possible interpretations of Chasles's theory of characteristics, which led to two different truth-values for the $\alpha\mu + \beta\nu$ formula, thereby concluding that the validity of this formula is indeed a matter of convention. Study then attempted to convince Halphen that his interpretation was insufficiently faithful to what Chasles had in mind. But Halphen cared little for this new framing of his own work. In a brief reply, he "[persisted] in finding nothing new or useful in [Study's] interpretation of Chasles's theory."[78] The discussion between Study and Halphen was not just one between two mathematicians who disagreed on a technical issue: it was, yet again, the confrontation of two different figures of the mathematician.

Despite his defense of Chasles's formula and his emphasis on the arbitrariness of mathematical definitions, Study was no ally of Schubert but, rather, one of his staunchest critics. In a later paper he attacked one of the principles at the heart of Schubert's *Kalkül*, which he viewed as a symptom of another overarching problem plaguing contemporary geometry:

> In countless cases, the objects of geometrical investigations are so unclearly explained, that one has to guess the meaning (*Sinn*) of individual concepts (*Begriffe*) from the assertions made about them, whereby differences in opinion can naturally arise. . . . First of all, the concept of geometric figure, as defined by M. Schubert and explained by his examples, has such an unusual scope (*Umfang*), that it is most unlikely that anything universally valid could be said about it at this point. One is immediately forced to resort to an interpretation.[79]

Like Halphen, Study accused Schubert of inverting the proper epistemic order between definition and investigation, as the latter lets his symbols freely operate and never searches for the concept behind them. A staunch realist, even in the face of Einstein's introduction of non-Euclidean geometries in physics, Study railed against axiomatizers who never inquired about what objects fell under their definitions and those who, like Hilbert, equated the coherence of an axiomatics to the existence of its objects. Arbitrariness run amok, Study thought, could just as well lead to a state of generalized incomprehension among mathematicians, and the creation of mathematical concepts, while free, must always be "motivated" and critically assessed, lest "we let the creature (*das Geschöpf*) become the master (*zum Herren werden*)."[80] Against Schubert's concept-free

[77] On Hilbert and Study's visit to Paris see Constance Reid, *Hilbert-Courant* (New York: Springer, 1986), pp. 22–28. Two letters from Study to Halphen are kept in the Bibliothèque de l'Institut, Paris, Ms 5624, nos. 186–187. Two brief letters from Halphen to Study are in the archives of the Justus-Liebig Universität Giessen, as part of the Nachlassverzeichnis Friedrich Engel, NE090416–17.

[78] Eduard Study to Halphen, 21 Apr. 1886, Cod Ms 5624, no. 187; and Halphen to Study, 22 Apr. 1886, NE090416.

[79] Eduard Study, "Über das Prinzip der Erhaltung der Anzahl," in *Verhandlungen des Dritten Internationalen Mathematiker-Kongresses in Heidelberg 1904* (Leipzig: Teubner, 1905), pp. 388–395, on pp. 388, 391. This rebuttal is strikingly similar to the criticism of Schubert's philosophy of numbers expressed in Frege, *Über die Zahlen des Herrn H. Schubert* (cit. n. 72), p. 267: "If we did not know the sense of these words ['equal,' 'greater,' 'smaller' . . .], we would not know what thoughts were contained in these propositions and could not therefore prove these thoughts to be true."

[80] Eduard Study, "Ein neuer Zweig der Geometrie," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1902, *11*:97–123, on p. 99. On Study's epistemology of mathematics see Hartwich, "Eduard Study (1862–1930)" (cit. n. 2), pp. 130–159; and Gray, *Plato's Ghost* (cit. n. 12), pp. 293–296.

symbolism and Chasles's intuitive geometry, but also against Halphen's restrictive reliance on a natural, yet possibly irregular, theory of conics, Study reconciled arbitrary definitions with an intransigent emphasis on the importance of mathematicians' duty to motivate, measure, and delineate precisely the extension of the concepts they freely produce.

CONCLUSION: FROM TRUTH TO SIGNIFICANCE

As the disciplinary and cultural identity of the wielders of Chasles's formula differed, so did the definition of its terms and its status *qua* mathematical proposition.[81] Such were the shifts in epistemic ideals and norms to which the groups of mathematicians involved in this circulation were beholden that even the truth-value of the formula fluctuated. By the time Chasles's formula finally reached the Dutch geometer Van der Waerden, who produced the first widely accepted proof thereof in 1938, mathematics had largely gone through the so-called modernist transformation, which had been merely nascent in the writings of Schubert, Halphen, and Study. For Van der Waerden, mathematics was solely about the derivation of "flawless proofs" from rigorously defined frameworks: in the case of Chasles's formula, this included redefining every single term of its statement—even that of a "number" of solutions.[82] Not subject to Study's strict realist regimen, Van der Waerden did not expect concepts to capture a pretheoretical (and, indeed, preaxiomatic) meaning and thus did not feel obliged to measure the extension of said concepts: squabbles over which framework best captures Chasles's original intuition were of no interest to him. With Van der Waerden's proof, Chasles's formula had not only gained entry into the commonly accepted body of mathematical knowledge; it had finally been absorbed into modern mathematical practice.

As Daston and Galison have pointed out in *Objectivity*, however, epistemic ideals never disappear; and the succession of portraits presented here is a series not of replacements but, rather, of confrontations.[83] The tensions constitutive of the modernist transformation of mathematics can and indeed do still reappear, albeit with a decidedly novel ring to them. If concepts are freely postulated, how can one ensure that they indeed capture the original geometrical intuition and that the results they encapsulate are indeed those that were being sought? This classical philosophical critique of naive mathematical formalism (or, in Imre Lakatos's term, Euclideanism) largely motivated a somewhat recent attempt to reinvigorate Halphen's criticism of Chasles's formula. Indeed, while the formalism most commonly used in contemporary algebraic geometry is one in which Chasles's formula is true, some argue that frameworks in which the formula is false have a more profound "enumerative significance."[84] Beyond Chasles's formula, the interconnected values of naturalness and simplicity in mathematics feature prominently in the autobiography of the French mathematician Alexandre Grothendieck, while a reemerging anxiety about the uncertainty of a growing part of the body of mathematical knowledge has led many to call for new standards of proof and communication among practitioners.[85] To account for the return of these epistemic virtues, a new cultural history of the figure of the mathematician is needed.

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[81] For a case study in the *longue durée* history of the different readings a single theorem can elicit see Catherine Goldstein, *Un théorème de Fermat et ses lecteurs* (Saint-Denis: Presses Univ. Vincennes, 1995).

[82] Van der Waerden, "Zur algebraischen Geometrie, XV" (cit. n. 11), pp. 645–646. On Van der Waerden's geometrical works see Norbert Schappacher, "A Historical Sketch of B. L. Van der Waerden's Work on Algebraic Geometry, 1926–1946," in *Episodes in the History of Modern Algebra (1800–1950)*, ed. Gray and Parshall (cit. n. 28), pp. 245–284.

[83] Daston and Galison, *Objectivity* (cit. n. 17), p. 19.

[84] Eduardo Casas-Alvero and Sebastian Xambò-Descamps, *The Enumerative Theory of Conics after Halphen* (Berlin: Springer, 1986), p. iii. For Lakatos's critique of modern(ist) mathematics see Lakatos, *Proofs and Refutations* (cit. n. 56), Ch. 2, Sect. 1.

[85] Alexandre Grothendieck, *Récoltes et semailles: Réflexions et témoignage sur un passé de mathématicien* (undated, available online), e.g., pp. 50–52; and Alma Steingart, "A Group Theory of Group Theory: Collaborative Mathematics and the 'Uninvention' of a Thousand-Page Proof," *Social Studies of Science*, 2012, 42:185–213.