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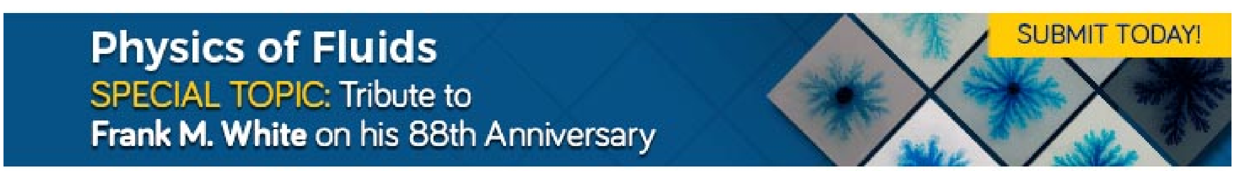

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ABSTRACT

Here, we point out on an intriguing mapping between the quantum harmonic oscillator ground state and the zero absolute vorticity plane Couette flow of a Boltzmann-like density distributed ideal gas in thermal equilibrium. The mapping is obtained when the gas is embedded in a rotating frame whose rotation rate is equal to half of the frequency of the quantum harmonic oscillator.

It has been shown numerically[1] and experimentally[2,3] that for pressure-driven turbulent channel flows, undergoing spanwise system rotation, the mean velocity profile may be deformed into a plane Couette flow in the central region of the channel. Furthermore, the absolute vorticity (that is, the sum of the averaged spanwise flow vorticity and the background vorticity due to the system rotation) of this Couette flow tends to zero, thus denoted as a "zero-absolute-vorticity state." Here, we address another fundamental aspect of the zero-absolute-vorticity state, in a much simpler setup of thermal equilibrium, which may serve as an analog of the ground state of a quantum harmonic oscillator.

Consider then a hypothetical experimental setup in which a quasi horizontal mono-atomic ideal gas in thermal equilibrium (e.g., immersed in a large enough heat bath) is overlaid on a platform that is rotating clockwise with the angular velocity $\Omega = \omega/2$ (where the choice of the clockwise rotation is made in order to obtain an analogy with a 1D harmonic oscillator in the x direction). Treating the gas as an inviscid fluid in a horizontal plane, the Euler momentum equation viewed from the rotating frame reads (e.g., Ref. 4)

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{\nabla p}{\rho_m} + \omega\hat{\mathbf{z}}\times\mathbf{u}, \tag{1}$$

where $\mathbf{u} = (\hat{\mathbf{x}}\,u + \hat{\mathbf{y}}v)$ is the horizontal velocity vector and $\nabla = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}\right)$ is the horizontal part of the nabla operator. $-\nabla p/\rho_m$ is the horizontal pressure gradient force (PGF), where p and ρ_m denote the gas pressure and mass density, respectively. The Coriolis force is expressed by $\omega\hat{\mathbf{z}}\times\mathbf{u}$, where ω is the Coriolis frequency.

The ideal gas equation of state is written as

$$p = \frac{\rho_m}{m}kT = \rho kT, \tag{2}$$

where m denotes the mass of each particle of the gas, so that ρ is the particle density. k is the Boltzmann constant and T denotes the temperature. In thermal equilibrium, T is constant; thus, the PGF can be written as the exact gradient,

$$-\frac{\nabla p}{\rho_m} = -\nabla(Q/m); \quad Q = kT\ln(\rho/\rho_s) \tag{3}$$

(where ρ_s is an arbitrary positive valued scaling factor). Therefore, the PGF cannot apply momentum torque to generate vorticity. Adding the continuity equation for a compressible gas,

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\mathbf{u}), \tag{4}$$

then for given constant values of T and ω, the system of Eqs. (1)–(4) solves for the variables $(\mathbf{u}, \rho)$.

The spanwise (z direction) vorticity component of the flow is $\zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$. A zero absolute vorticity state is defined when the total vorticity, viewed from a frame of rest, is zero. As the tank rotates clockwise, the zero absolute vorticity must, therefore, satisfy the condition $\zeta = \omega$. A straightforward stationary zero absolute vorticity solution of systems (1)–(4) is obtained for the plane Couette flow: $u = 0$ and $v_c = \omega x$ (where the subscript "c" denotes the Couette profile). For $\rho = \rho(x)$, the flow is geostrophically balanced between the x components of the Coriolis force and the PGF,

$$\omega v_c = \omega^2 x = -\frac{1}{m}\frac{\partial Q}{\partial x}, \tag{5}$$

yielding

$$\rho(x) = \hat{\rho}\, e^{-\frac{V(x)}{kT}}, \quad \hat{\rho} = \omega\sqrt{\frac{m}{2\pi kT}}, \quad V(x) = \frac{m(\omega x)^2}{2} = \frac{mv_c^2}{2}. \tag{6}$$

Here, ρ was normalized [normalization does not affect the correctness of (5)] to satisfy $L_y \int_{-\infty}^{\infty} \rho dx = 1$, where L_y is a length scale in the y direction, taken hereafter as the unity without loss of generality. It obeys a Boltzmann-like probability density function (PDF). Alternatively, as $dv_c = \omega dx$, we may write the Couette velocity PDF, satisfying $\int_{-\infty}^{\infty} \tilde{\rho}(v_c) dv_c = 1$ as

$$\tilde{\rho}(v_c) = \sqrt{\frac{m}{2\pi kT}}\, e^{-\frac{mv_c^2/2}{kT}}, \tag{7}$$

suggesting that v_c distributes according to the 1D Maxwell–Boltzmann distribution,[5] where v_c plays the role of the thermal velocity in ideal gas.

Recalling that the momentum advection term can be written as: $\mathbf{u}\cdot\nabla\mathbf{u} = \zeta\hat{\mathbf{z}}\times\mathbf{u} + \nabla(\mathbf{u}^2/2)$, we rearrange (1) into

$$\frac{\partial \mathbf{u}}{\partial t} + (\zeta - \omega)\hat{\mathbf{z}}\times\mathbf{u} = -\nabla\left(\frac{\mathbf{u}^2}{2} + \frac{Q}{m}\right), \tag{8}$$

whose LHS vanishes for stationary zero absolute vorticity state solutions. Consequently, (8) is reduced to the Bernoulli equation,

$$\frac{\mathbf{u}^2}{2} + \frac{Q}{m} = Be, \tag{9}$$

where Be is the Bernoulli potential constant. For the Couette flow solution, (9) can be written as

$$\frac{mv_c^2}{2} + Q(x) = V(x) + Q(x) = m\,Be. \tag{10}$$

Hence, $Be = Q(x=0)/m = \frac{kT}{m}\ln(\hat{\rho}/\rho_s)$ is an undetermined constant since the scaling factor, ρ_s, in Q is arbitrary. This by itself is not surprising as the dynamics is governed by the geostrophic balance of (5), which does not depend on the precise value of Q. The mean (expectation) value $\langle f\rangle \equiv \int_{-\infty}^{\infty} \rho f(x) dx$ yields $\langle V\rangle = kT$, as expected in thermal equilibrium. Hence, if we choose to define $mBe = kT \equiv E_0$ (so that the density scaling factor $\rho_s = \hat{\rho}/e$, where e is Euler's number), then we obtain from (6) and (10),

$$\rho_0(x) = \omega\sqrt{\frac{m}{2\pi E_0}}\, e^{-V(x)/E_0}, \quad V(x) + Q(x) = E_0. \tag{11}$$

Hypothetically tuning the platform rotation rate Ω to satisfy $kT = \hbar\Omega = \hbar\omega/2 \equiv E_0$ (where $\hbar$ is the reduced Planck constant), then (11) becomes the familiar 1D quantum oscillator ground state PDF to find a spinless quantum particle with mass m in position x, where the zero subscript refers to the zeroth order Hermite polynomial solution.[6] We now wish to transform the momentum and continuity Eqs. (1) and (4) in an equivalent form to the ones obtained from the 1D Schrödinger equation in the presence of the harmonic potential $V(x) = \frac{m}{2}(\omega x)^2$. Toward this end, we assume again that (u, v, ρ, Q) are only functions of (x, t) and $v = v_c = \omega x$. Then, the x component of the momentum equation can be written as

$$m\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{mu^2}{2} + Q + V\right), \tag{12}$$

where the y component is trivially satisfied. The continuity equation obtains the simple form,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u), \tag{13}$$

where the plane Couette zero absolute vorticity stationary solution can be recovered by setting $u = 0$.

Consider now the 1D time-dependent Schrödinger equation[6] for a spinless quantum particle, in the presence of the harmonic potential:

$$i\hbar\frac{\partial \Psi}{\partial t} = \hat{H}\Psi = \left(\frac{\hat{p}^2}{2m} + V\right)\Psi, \tag{14}$$

where $\Psi(x,t)$ is the wavefunction, and $\hat{H}$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$ are the Hamiltonian and momentum operators, respectively. Recall that now m denotes the mass of the quantum particle, rather than an atom in a gas. We write the wavefunction in a polar form,

$$\Psi(x,t) = \sqrt{\rho}(x,t)e^{iS(x,t)/\hbar}, \tag{15}$$

where now $\rho(x,t)$ is the probability density function to find the particle in position x at time t, and S is the phase (scaled by $\hbar$). Defining the velocity according to the de Broglie guiding equation,

$$u = \frac{1}{m}\frac{\partial S}{\partial x}. \tag{16}$$

Madelung[7] decomposed (14) to its amplitude and phase to obtain two equations. The first is the continuity Eq. (13) and the second is the quantum Hamilton–Jacobi, time-dependent Bernoulli-like equation:

$$\frac{\partial S}{\partial t} = -\left(\frac{mu^2}{2} + V + Q_q\right), \tag{17}$$

where

$$Q_q = -\frac{\hbar^2}{2m\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial x^2}, \tag{18}$$

is the quantum potential (denoted also as the Bohm potential). Taking now the x derivative of (17) we obtain (12), except that instead of Q in the RHS, we have Q_q.

For the quantum stationary states $S = -E_n t$ (where $E_n = \hbar\omega(n + 1/2)$ and n takes integer values); thus, S is not a function of x and, consequently, by (16), $u = 0$. The continuity Eq. (13) ensures then that the eigen-states PDF, ρ_n, are indeed stationary, where (17) gives

$$V(x) + Q_{qn}(x) = E_n \tag{19}$$

so that specifically for the ground state ($n = 0$), we obtain the second equation of (11) with $Q_{q0} = Q_q(\rho_0)$ in the position of

$Q_0 = kT\ln(\rho_0/\rho_s)$. Nonetheless, a quick check reveals that the two are the same; thus, the mapping is complete.

According to the quantum field theory, the harmonic oscillator ground state composes the zero point energy of vacuum fluctuations. We find it intriguing that (apart from the obvious scaling by $\hbar$) such a fundamental building block has a relatively straightforward macroscopic analog. It may also be worth noting that under this analogy, $E_0/\hbar = \Omega = \omega/2$; hence, the factor half does not appear when relating the ground state with the rotation of the tank itself, rather than with the background vorticity Coriolis frequency it induces. Furthermore, the fact that the Couette flow carries zero absolute vorticity seems adequate to describe spinless quantum particle phenomena.

One may ask for the physical basis of the analogy between the two phenomena. This stems from the Coriolis force acting as a restoring force to generate inertial oscillations in the presence of geostrophically balanced flows. Consider the solution where (u, v, ρ, Q) are only functions of (x, t), then the two components of the momentum equations read

$$\frac{Du}{Dt} = -\frac{\partial Q}{\partial x} - \omega v; \quad \frac{Dv}{Dt} = \omega u, \tag{20}$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$ is the 1D material derivative in the x direction. Suppose that we now perturb the geostrophic equilibrium of (5) by slightly displacing a fluid parcel in the x direction so that $u = D\chi/Dt$, where $\chi(t)$ denotes the small displacement. This will immediately results in a change in v, as can be seen from the second equation of (20),

$$\frac{Dv}{Dt} = \omega u \quad \Rightarrow \quad v = \omega(x + \chi) = v_c + \omega\chi. \tag{21}$$

Plugging v back in the first equation of (20), we recover the classical harmonic oscillator,

$$\frac{Du}{Dt} = \frac{D^2\chi}{Dt^2} = -\omega^2\chi. \tag{22}$$

Hence, the Coriolis force acting on the Couette flow, $v_c = \omega x$, becomes $-m\omega^2 x = -\frac{\partial V}{\partial x}$, where V is the harmonic potential. As for the quantum harmonic oscillator energy states, the gradient of this harmonic potential is locally balanced. For the quantum states, it is balanced by the gradient of the Bohm potential, whereas in the hydrodynamic analog, it is (geostrophically) balanced by the pressure gradient force. For the thermal Couette flow, the latter happens to have the same structure as the Bohm potential gradient in the quantum oscillator ground state. In both systems, the consequence of the balance is that the de Broglie velocity (16) remains zero. Only when the energy states are perturbed, u becomes non zero. Hence, while v_c defines the potential, u relates to the particle motion within this potential.

As pointed out by,[8] generally the Bohm potential plays an analog role of the enthalpy in a barotropic fluid. Then, the quantum Hamilton–Jacobi Eq. (17) reads as the corresponding time-dependent Bernoulli barotropic fluid equation. It is interesting that for the Couette flow solution in thermal equilibrium, this equivalency between the equations still holds formally, even though the ideal gas enthalpy in thermal equilibrium is constant. Furthermore, the analogy seems to hold solely for the quantum oscillator ground state. For the higher energy states ($n = 1, 2, 3, \ldots$):

$$\rho_n(x) = \frac{1}{2^n n!} H_n^2\left(\sqrt{\frac{V(x)}{E_0}}\right)\rho_0(x) \tag{23}$$

(where H_n denotes the Hermite polynomial of order n), one cannot find a temperature structure $T(x)$ that simultaneously allows writing the PGF as an exact gradient and satisfying the geostrophic balance of (5).

We mentioned that the suggested "experiment" is hypothetical, not only because of the technical challenges it poses, but mainly because the required rotation rate of $\Omega = (k/\hbar)T = O(10^{11})T$ is obviously unattainable in practice due to the smallness of the Planck constant. This may be expected as it is improbable to construct, in practice, a macroscopic system that quantitatively demonstrates a quantum phenomenon.[9]

It should also be stressed that this analog does not suggest a classical counterpart for the existence of a non zero ground state, as the Bernoulli potential can be chosen to be set to zero for the choice of $\rho_s = \hat{\rho}$. From the fluid dynamics perspective, it is interesting that although the Couette zero absolute vorticity state emerges as a consequence of highly nonlinear pressure driven turbulent dynamics; here, a simple version of it can be obtained by the linear Schrödinger equation.

This analog motivates the examination of further zero absolute vorticity geostrophic-like balances, embedded within the Schrödinger equation, in more complex setups such as in the presence of an electromagnetic field. In a follow-up paper, we will address such hydrodynamic balance shear flow analogs of the Landau levels and the integer quantum Hall effect.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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[9] Even to experimentally obtain the macroscopic density distribution of (6), $\rho \sim \exp[-2(\Omega x)^2/RT]$ (where $R = k/m$ is the specific gas constant), is not an easy task. For air at room temperature and a rotating tank of one meter radius, the tank should exhibit a rotation rate on the order of $O(10^3)$ RPM, which is still too high for standard experimental rotating tanks.