M. Boggi and E. Looijenga (2020) "Deforming a Canonical Curve Inside a Quadric,"
International Mathematics Research Notices, Vol. 2020, No. 2, pp. 367–377
Advance Access Publication March 2, 2018
doi:10.1093/imrn/rny027

Deforming a Canonical Curve Inside a Quadric

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Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded nonsingular nonhyperelliptic curve of genus g and let $X \subset \mathbb{P}^{g-1}$ be a quadric containing C. Our main result states among other things that the Hilbert scheme of X is at $[C \subset X]$ a local complete intersection of dimension $g^2 - 1$ and is smooth when X is. It also includes the assertion that the minimal obstruction space for this deformation problem is in fact the full associated Ext^1-group and that in particular the deformations of C in X are obstructed in case C meets the singular locus of X. Applications will be given in a forthcoming paper.

1. Statement of the Main Result

Throughout this paper we work over an algebraically closed field $\Bbbk$ of characteristic $\neq 2$. Let C be a smooth nonhyperelliptic projective curve of genus g (so that $g > 2$) and regard C as embedded in $\mathbb{P} := \check{\mathbb{P}}\left(H^0(C, \Omega_C)\right)$. It is well known that the Hilbert scheme of $\mathbb{P}$ is smooth at $C \subset \mathbb{P}$ of dimension $3g - 3 + g^2 - 1$ $\left(= \dim \mathscr{M}_g + \dim \mathrm{PGL}_g\right)$ and that the canonical embedding is unique modulo the action of the projective linear group $\mathrm{Aut}(\mathbb{P})$ of $\mathbb{P}$. It is also known that when $g \geq 4$ C is contained in a quadric hypersurface. Let $X \subset \mathbb{P}$ be one such quadric. Our main theorem helps us to understand what conditions are imposed on the deformation theory of the C in $\mathbb{P}$ by requiring that C stays inside X.

In order to state it we use the following notions and notation. For a variety Z, $\mathscr{T}_Z$ stands for its Zariski tangent sheaf, that is, the $\mathscr{O}_Z$-dual of Ω_Z. If $Y \subseteq Z$ is a closed

Received March 15, 2017; Revised January 30, 2018; Accepted January 31, 2018
Communicated by Prof. Enrico Arbarello

subscheme defined by the $\mathscr{O}_Z$-ideal $\mathscr{I}_Y \subseteq \mathscr{O}_Z$, then $\mathscr{C}_{Y/Z} := \mathscr{I}_Y/\mathscr{I}_Y^2$ (regarded as an $\mathscr{O}_Z$-module) is the *conormal sheaf of* Y *in* Z, and its $\mathscr{O}_Z$-dual, denoted $\mathscr{N}_{Y/Z}$, is the *normal sheaf of* Y *in* Z.

It is well known that the space $\mathrm{Ext}^0_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$ naturally identifies with the tangent space of the Hilbert scheme $\mathrm{Hilb}(X)$ at the point $[C \subset X]$. An obstruction theory for the embedding $C \subset X$ is instead provided by the vector space $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$ (cf. Proposition I.2.14 in [3]), but it is not always true that this space consists entirely of obstructions, that is, is a *minimal obstruction space* in the sense of Definition 5.5 in [6]. Our main theorem states, in particular, that, in our situation, this is so:

Theorem 1.1. Let $C \subset X \subset \mathbb{P}$ be as above (so C is nonhyperelliptic of genus $g \geq 4$). Then

(i) the Hilbert scheme $\mathrm{Hilb}(X)$ is a local complete intersection at $[C \subset X]$ of dimension $g^2 - 1$ with embedding dimension $g^2 - 1 + \dim \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$,

(ii) $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$ is a minimal obstruction space for deformations of C in X, and

(iii) when X is either nonsingular or has an isolated singularity disjoint from C, then $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C) = 0$ and $\mathrm{Hilb}(X)$ is smooth at $[C \subset X]$.

We shall also show (Corollary 2.9) that when the quadric X is singular and C meets its singular locus the obstruction space $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$ is nonzero.

In a forthcoming paper, we will apply the above result to prove that, for a very general complex smooth projective curve C with automorphism group G such that the quotient curve C/G has genus at least three, the algebra of rational endomorphisms of the Jacobian $\mathrm{End}_{\mathbb{Q}}(J(C))$ is naturally isomorphic to the group algebra $\mathbb{Q}G$. In its turn, this result has applications to the theory of virtual linear representations of mapping class groups.

In a different direction, Theorem 1.1 may be useful in the study of the so-called Petri divisor on the moduli space $\overline{M}_g$ of stable curves of genus g. This divisor is defined to be the closure in $\overline{M}_g$ of the locus in the moduli space M_g of smooth projective curves of genus g, which parameterizes curves whose canonical model lies on a rank 3 quadric (cf. Section 6 of [2]).

2. Proof of the Theorem

Let us write $\mathfrak{aut}(\mathbb{P})$ for the Lie algebra of the projective linear group $\mathrm{Aut}(\mathbb{P})$ of $\mathbb{P}$, in other words, the Lie algebra of vector fields on $\mathbb{P}$. We first observe the following:

Lemma 2.1. The natural map $\mathfrak{aut}(\mathbb{P}) \to H^0(C, \mathscr{O}_C \otimes \mathscr{T}_{\mathbb{P}})$ is an isomorphism, and the cohomology spaces $H^1(C, \mathscr{O}_C \otimes \mathscr{T}_{\mathbb{P}})$, $\mathrm{Ext}^1_C(\mathscr{C}_{C/\mathbb{P}}, \mathscr{O}_C) = H^1(C, \mathcal{N}_{C/\mathbb{P}})$ all vanish (so that the 1st-order deformations of C in $\mathbb{P}$ are unobstructed). Furthermore, the sequence

$$0 \to \mathfrak{aut}(\mathbb{P}) \to H^0(C, \mathcal{N}_{C/\mathbb{P}}) \to H^1(C, \mathscr{T}_C) \to 0$$

is exact, where the middle term can be regarded as the tangent space of $\mathrm{Hilb}(\mathbb{P})$ at $[C]$ (and has dimension $\left(g^2-1\right) + (3g-3)$).

Proof. The Euler sequence, which describes the tangent sheaf $\mathscr{T}_{\mathbb{P}}$ of the projective space $\mathbb{P} = \check{\mathbb{P}}\left(H^0(C, \Omega_C)\right)$, tensored with $\mathscr{O}_C$ is

$$0 \to \mathscr{O}_C \to \Omega_C \otimes H^0(C, \Omega_C)^* \to \mathscr{O}_C \otimes \mathscr{T}_{\mathbb{P}} \to 0.$$

Consider its cohomology exact sequence:

$$0 \to H^0(C, \mathscr{O}_C) \to \mathrm{End}\left(H^0(C, \Omega_C)\right) \to H^0(C, \mathscr{O}_C \otimes \mathscr{T}_{\mathbb{P}}) \to$$
$$\to H^1(C, \mathscr{O}_C) \to \mathrm{Hom}\left(H^0(C, \Omega_C), H^1(C, \Omega_C)\right) \to H^1(C, \mathscr{O}_C \otimes \mathscr{T}_{\mathbb{P}}) \to 0.$$

The 1st nonzero map of the 1st line is just the inclusion of the scalars in $\mathrm{End}\left(H^0(C, \Omega_C)\right)$ (whose cokernel is $\mathfrak{aut}(\mathbb{P})$) and the 1st map of the 2nd line is readily verified to be the isomorphism provided by (Serre) duality. So $\mathfrak{aut}(\mathbb{P}) \to H^0(C, \mathscr{O}_C \otimes \mathscr{T}_{\mathbb{P}})$ is an isomorphism and $H^1(C, \mathscr{O}_C \otimes \mathscr{T}_{\mathbb{P}}) = 0$. If we use the last observation as input for the exact cohomology sequence of the short exact sequence

$$0 \to \mathscr{T}_C \to \mathscr{O}_C \otimes \mathscr{T}_{\mathbb{P}} \to \mathcal{N}_{C/\mathbb{P}} \to 0,$$

we obtain the stated exact sequence (where we use that $H^0(C, \mathscr{T}_C) = 0$). ■

Lemma 2.2. We have a natural isomorphism $\mathscr{O}_C \otimes \mathcal{N}_{X/\mathbb{P}} \cong \Omega_C^{\otimes 2}$ and an exact sequence

$$0 \to H^0(C, \mathcal{N}_{C/X}) \to H^0(C, \mathcal{N}_{C/\mathbb{P}}) \xrightarrow{\phi_X} H^0(C, \Omega_C^{\otimes 2}) \to \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C) \to 0.$$

Proof. Consider the standard exact sequence of conormal sheaves associated to the chain of embeddings $C \subset X \subset \mathbb{P}$:

$$\mathscr{O}_C \otimes \mathscr{C}_{X/\mathbb{P}} \to \mathscr{C}_{C/\mathbb{P}} \xrightarrow{q} \mathscr{C}_{C/X} \to 0.$$

Since X is a hypersurface in $\mathbb{P}$ of degree 2, the embedding $X \subset \mathbb{P}$ is regular with conormal sheaf $\mathscr{C}_{X/\mathbb{P}}$ isomorphic to $\mathscr{O}_X(-2)$. Recalling that $\mathscr{O}_C(1) = \Omega_C$, this yields an isomorphism $\mathscr{O}_C \otimes \mathscr{C}_{X/\mathbb{P}} \cong \left(\Omega_C^{\otimes 2}\right)^\vee$. Now $\mathscr{C}_{C/\mathbb{P}}$ is the conormal sheaf of a regular embedding and so locally free. It follows that $\ker q$ is torsion free and (hence) locally free of rank 1. So $\mathscr{O}_C \otimes \mathscr{C}_{X/\mathbb{P}} \cong \left(\Omega_C^{\otimes 2}\right)^\vee \to \ker q$, being a surjective morphism of invertible sheaves, must be an isomorphism. We thus obtain a locally free resolution of $\mathscr{C}_{C/X}$:

$$0 \to \left(\Omega_C^{\otimes 2}\right)^\vee \to \mathscr{C}_{C/\mathbb{P}} \xrightarrow{q} \mathscr{C}_{C/X} \to 0. \tag{1}$$

Applying $\mathrm{Hom}_C(-, \mathscr{O}_C)$ to this resolution gives the exact sequence

$$0 \to H^0(C, \mathcal{N}_{C/X}) \to H^0(C, \mathcal{N}_{C/\mathbb{P}}) \to H^0(C, \Omega_C^{\otimes 2}) \to \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C) \to H^1(C, \mathcal{N}_{C/\mathbb{P}}).$$

The exact sequence of the lemma now follows from the vanishing of $H^1(C, \mathcal{N}_{C/\mathbb{P}})$. ■

Corollary 2.3. Assume that either X is nonsingular or $g \geq 5$ and X has an isolated singularity disjoint from C. Then the cohomology spaces $H^1(C, \mathscr{O}_C \otimes \mathscr{T}_X)$ and $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$ vanish. In particular, the deformations of C in X are unobstructed and $\mathrm{Hilb}(X)$ is nonsingular of dimension $\dim H^0(C, \mathcal{N}_{C/\mathbb{P}}) - \dim H^0\left(C, \Omega_C^{\otimes 2}\right) = g^2 - 1$ at the point $[C \subset X]$.

Proof. We prove that the restriction of ϕ_X to the subspace $\mathfrak{aut}(\mathbb{P}) \subset H^0(C, \mathcal{N}_{C/\mathbb{P}})$ is surjective; this will clearly imply that $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C) = 0$. This map is defined as follows: regard $A \in \mathfrak{aut}(\mathbb{P})$ as a vector field on $\mathbb{P}$, restrict it to X so that we get a normal vector field to X in $\mathbb{P}$, and then restrict this normal vector field to C and take its image in $H^0(C, \mathscr{O}_C \otimes \mathcal{N}_{X/\mathbb{P}}) \cong H^0\left(C, \Omega_C^{\otimes 2}\right)$. This map factors through $\mathrm{Sym}^2 H^0(C, \Omega_C) \to H^0\left(C, \Omega_C^{\otimes 2}\right)$: if we identify $\mathfrak{aut}(\mathbb{P})$ with $\mathfrak{sl}\left(H^0(C, \Omega_C)\right) \subset \mathrm{End}\left(H^0(C, \Omega_C)\right)$, then the lift in question is given by

$$A \in \mathrm{End}\left(H^0(C, \Omega_C)\right) \mapsto (A \otimes 1 + 1 \otimes A)(Q) \in \mathrm{Sym}^2 H^0(C, \Omega_C).$$

It is easy to verify that the image of this map consists precisely of the space of quadrics containing the singular locus of X. Thus, when X is nonsingular, the map is surjective and, when X has an isolated singularity $P \in X \setminus C$, its image has codimension one. Since, following Max Noether, the map $\mathrm{Sym}^2 H^0(C, \Omega_C) \to H^0(C, \Omega_C^{\otimes 2})$ is surjective, with kernel the space of quadrics containing the curve C, and, under our hypothesis, there is a quadric containing C but not P, it follows that, in both cases, the composition $\mathrm{End}\left(H^0(C, \Omega_C)\right) \to H^0\left(C, \Omega_C^{\otimes 2}\right)$ is surjective. Now $\mathrm{End}\left(H^0(C, \Omega_C)\right)$ is the direct sum of $\mathfrak{sl}\left(H^0(C, \Omega_C)\right)$ and the scalars. But the scalars map under the above map to the multiples of Q and hence vanish when restricted to C. It follows that $\phi_X|\,\mathfrak{aut}(\mathbb{P})$ is surjective as asserted.

Let us then show that $H^1(C, \mathscr{O}_C \otimes \mathscr{T}_X) = 0$. The normal sheaf to the quadric X in $\mathbb{P}$ is described by the short exact sequence

$$0 \to \mathscr{T}_X \to \mathscr{O}_X \otimes \mathscr{T}_\mathbb{P} \to \mathscr{O}_X(2) \to 0,$$

which on C restricts to the short exact sequence

$$0 \to \mathscr{O}_C \otimes \mathscr{T}_X \to \mathscr{O}_C \otimes \mathscr{T}_\mathbb{P} \to \Omega_C^{\otimes 2} \to 0.$$

By Lemma 2.1, there is then a long exact sequence

$$0 \to H^0(C, \mathscr{O}_C \otimes \mathscr{T}_X) \to \mathfrak{aut}(\mathbb{P}) \to H^0(C, \Omega_C^{\otimes 2}) \to H^1(C, \mathscr{O}_C \otimes \mathscr{T}_X) \to 0,$$

which, by the previous part of the proof, implies the claim. ■

Remark 2.4. In the above proof, we have shown that $H^1(C, \mathscr{T}_X \otimes \mathscr{O}_C) = 0$. This implies that, in the hypotheses of Theorem 1.1, the forgetful map $H^0(C, \mathscr{N}_{C/X}) \to H^1(C, \mathscr{T}_C)$, which sends an embedded deformation of C in X to the deformation of C obtained disregarding the embedding, is surjective. In other words, deformations of the curve C in X have general moduli. But this also shows that the argument of the proof of Corollary 2.3 cannot be extended to quadrics of rank 3. In fact, the locus of smooth projective curves of genus g whose canonical model lies on a rank-3 quadric defines a divisor of M_g (cf. Proposition 6.1 in [2]). On the other hand, (cf. Exercise F in Ch. VI of [1]) the quadrics containing a nontrigonal curve of genus 5 have no singularities in the points of the curve. Therefore, for an embedding $C \subset X$ of a genus-5 nontrigonal curve in a rank-3 quadric, we have that $H^1(C, \mathscr{T}_X \otimes \mathscr{O}_C) \neq 0$, providing a counterexample to the conclusion (and then to the proof) of Corollary 2.3 for quadrics of corank ≥ 2.

In order to complete the proof of part (iii) of Theorem 1.1, we have to deal with the case when $g = 4$ and X has an isolated singularity (which is always disjoint from C). From the short exact sequence (1) and the exact sequence (3.16) in Example 3.2.5 of [5], it follows that $\mathscr{C}_{C/X} = \mathscr{T}_C(-2)$ and then that $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C) = H^1(C, \Omega_C(2)) = 0$.

In order to prove parts (i) and (ii) of Theorem 1.1, we need some preliminary results on Hilbert schemes of canonical curves in quadrics. The Hilbert scheme of quadrics in $\mathbb{P}$ is naturally identified with $\mathbb{P}\left(\mathrm{Sym}^2 H^0(C,\Omega_C)\right)$. It comes with a universal family of quadrics:

$$\begin{array}{ccc} \mathcal{U} & \subset & \mathbb{P}\times\mathbb{P}\left(\mathrm{Sym}^2 H^0(C,\Omega_C)\right) \\ & \searrow & \downarrow \\ & & \mathbb{P}\left(\mathrm{Sym}^2 H^0(C,\Omega_C)\right). \end{array}$$

Let $S \subset \mathbb{P}\left(\mathrm{Sym}^2 H^0(C,\Omega_C)\right)$ be the open subscheme parameterizing quadrics of rank ≥ 3 and $\mathcal{U}_S \to S$ the restriction of the universal family over S. Denote by $\mathrm{Hilb}^\circ(\mathcal{U}_S/S)$ the subscheme of the relative Hilbert scheme $\mathrm{Hilb}(\mathcal{U}_S/S)$ whose closed points parameterize the pairs consisting of a nonsingular canonically embedded genus g curve in $\mathbb{P}$ and a quadric of rank ≥ 3 in $\mathbb{P}$ containing that curve. Note that $\mathrm{Hilb}^\circ(\mathcal{U}_S/S)$ is an open subscheme of $\mathrm{Hilb}(\mathcal{U}_S/S)$, which comes with a surjective morphism:

$$p\colon\ \mathrm{Hilb}^\circ(\mathcal{U}_S/S) \to S.$$

Lemma 2.5. The scheme $\mathrm{Hilb}^\circ(\mathcal{U}_S/S)$ is irreducible. Moreover, $p\colon\ \mathrm{Hilb}^\circ(\mathcal{U}_S/S) \to S$ is a syntomic (i.e., a flat local complete intersection) morphism of relative dimension g^2-1 that is generically smooth. In particular, $\mathrm{Hilb}^\circ(\mathcal{U}_S/S)$ is of dimension $\dim(S)+g^2-1$.

Proof. Let $\mathrm{Hilb}^\circ(\mathbb{P})$ be the subscheme of $\mathrm{Hilb}(\mathbb{P})$ parameterizing nonsingular canonically embedded projective genus g curves in $\mathbb{P}$. It is well known that $\mathrm{Hilb}^\circ(\mathbb{P})$ is a smooth open subscheme of an irreducible component of $\mathrm{Hilb}(\mathbb{P})$. Since $\mathrm{Hilb}(\mathbb{P}\times S/S) \cong \mathrm{Hilb}(\mathbb{P})\times S$, the product $\mathrm{Hilb}^\circ(\mathbb{P})\times S$ is identified with a smooth open subscheme of an irreducible component of $\mathrm{Hilb}(\mathbb{P}\times S/S)$.

The S-embedding $\mathcal{U}_S \subset \mathbb{P}\times S$ induces a morphism

$$\mathrm{Hilb}^\circ(\mathcal{U}_S/S) \to \mathrm{Hilb}^\circ(\mathbb{P})\times S \subset \mathrm{Hilb}(\mathbb{P}\times S/S).$$

The morphism $\pi:\ \mathrm{Hilb}^\circ(\mathcal{U}_S/S) \to \mathrm{Hilb}^\circ(\mathbb{P})$ obtained by composition with the projection on $\mathrm{Hilb}^\circ(\mathbb{P})$ is a surjection, the fiber of π over a closed point $[C\subset\mathbb{P}] \in \mathrm{Hilb}^\circ(\mathbb{P})$ being naturally identified with the linear system of quadrics in $\mathbb{P}$ containing C (which consists of quadrics of rank ≥ 3). Therefore, it is irreducible and hence $\mathrm{Hilb}^\circ(\mathcal{U}_S/S)$ is irreducible as well.

We claim that a general nonsingular nonhyperelliptic canonically embedded curve $C\subset\mathbb{P}$ of genus ≥ 4 is contained in a nonsingular quadric. For $g=4$, this follows from the description of canonical curves in $\mathbb{P}^3$ as complete intersections of a quadric with an irreducible cubic surface, for $g \geq 5$, we need the following lemma that, for char $\Bbbk = 0$, is just a particular case of Bertini's theorem: ■

Lemma 2.6. For an algebraically closed field $\Bbbk$ of characteristic $\neq 2$, a linear system of quadrics in $\mathbb{P}^n_{\Bbbk}$ has the property that a general member has its singular locus contained in the base locus of the system.

Proof. First, observe that, by a standard argument, it suffices to treat the case of a pencil. Let us then suppose that we are given a pencil $\mathscr{X}$ of which each member is singular. For $n = 0$, the claim is empty and, for $n = 1$, is an easy exercise, so we proceed by induction and assume $n \geq 2$ and that the claim is verified for lower values of n.

The singular locus of each $X \in \mathscr{X}$ is a linear subspace. If for a general X this linear subspace has positive dimension, then the restriction of a general X to a general hyperplane $H \subset \mathbb{P}^n$ is still singular, hence, by the induction hypothesis, is contained in the fixed point set of $\mathscr{X}|H$. The claim then follows. Let us therefore assume that a general member of $\mathscr{X}$ has a unique singular point.

Let X_0, X_∞ be distinct members of $\mathscr{X}$, defined by linearly independent quadratic forms Q_0, Q_∞ of rank n. After a linear transformation, we may assume that $P_\infty := [1 : 0 : \ldots : 0]$ is the unique singular point of X_∞, so that we have $Q_\infty = Q_\infty(x_1, \ldots, x_n)$. Now write $Q_0 = x_0 f(x_0, \ldots, x_n) + R(x_1, \ldots, x_n)$ with f linear. If f is identically zero, then P_∞ is also a singular point of X_0 and hence of any member of $\mathscr{X}$ and we are done. If $a := f(1, 0, \ldots, 0) \neq 0$, then after modifying x_0 by adding to it a linear combination of the other variables, we can arrange that Q_0 has the form $ax_0^2 + R(x_1, \ldots, x_n)$. But then $Q_\infty + tQ_0$ is nonsingular for general t and so this cannot happen. After an appropriate linear transformation that leaves P_∞ invariant, we may then assume that Q_0 has the form $x_0x_1 + R(x_2, \ldots, x_n)$.

Thus, we have $Q_0 + tQ_\infty = x_0x_1 + R(x_2, \ldots, x_n) + tQ_\infty(x_1, \ldots, x_n)$. Hence, if $H \subset \mathbb{P}^n$ is the hyperplane defined by $x_1 = 0$, then for every $t \in \Bbbk$, the quadric X_t defined by $Q_0 + tQ_\infty$ has its singular locus contained in the singular locus of $X_t \cap H$. For generic t, the latter is, by our induction hypothesis, contained in the base locus of the pencil $\mathscr{X}|H$ and so X_t has its singular point in the base locus of $\mathscr{X}$. ■

Let now C be a nonsingular projective curve of genus ≥ 5, which is neither trigonal nor isomorphic to a plane quintic, then the general member of the linear system of quadrics of $\mathbb{P}$ containing C is nonsingular. Indeed, by Petri's Theorem, the linear system of quadrics containing C has for base locus the curve C and, by Lemma 2.6, a general member of this linear system has no singularities outside of C. In particular, it has at most corank 1. By Petri's Theorem, quadrics generate the canonical ideal, and so, by the Jacobian criterion for projective varieties, there are two quadrics Q_0 and Q_1 of corank ≤ 1 containing C with disjoint singular sets. Then either a general fiber of the pencil of quadrics $\lambda_0 Q_0 + \lambda_1 Q_1$ is nonsingular or the horizontal component of its singular locus is a (possibly singular) rational curve. Since, of course, C does not contain any rational curve, in the latter case, a general member of this pencil is a quadric of corank 1 containing C with singular point disjoint from C. But then a general quadric containing C has singular locus disjoint from C. Thus a general quadric containing C is nonsingular.

By Corollary 2.3 and Proposition 4.4.7 in [5], the restriction of the morphism p over the locus of S parameterizing nonsingular quadrics is therefore smooth of relative dimension $h^0(C, \mathcal{N}_{C/X}) = g^2 - 1$.

In order to complete the proof of the lemma, it is enough to show that, given a closed point $[X] \in S$ and a closed point $[C \subset X] \in \mathrm{Hilb}^\circ(\mathcal{U}_S/S)$ over it, the morphism $p\colon \mathrm{Hilb}^\circ(\mathcal{U}_S/S) \to S$ is syntomic at $[C \subset X]$. Since $\mathrm{Hilb}^\circ(\mathcal{U}_S/S)$ is an open subscheme of $\mathrm{Hilb}(\mathcal{U}_S/S)$, we have

$$\dim_{[C\subset X]} \mathrm{Hilb}(\mathcal{U}_S/S) = g^2 - 1 + \dim S. \tag{2}$$

By (ii) of Theorem I.2.15 in [3], there is an inequality:

$$\dim_{[C\subset X]} \mathrm{Hilb}(\mathcal{U}_S/S) \geq h^0(C, \mathcal{N}_{C/X}) + \dim S - \dim \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C). \tag{3}$$

The exact sequence in Lemma 2.2 gives the identity:

$$h^0(C, \mathcal{N}_{C/X}) = h^0(C, \mathcal{N}_{C/\mathbb{P}}) - h^0(C, \Omega_C^{\otimes 2}) + \dim \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C) = \tag{4}$$

$$= g^2 - 1 + \dim \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C).$$

Combining the identities (1) and (3), we get that the inequality (2) is actually an identity. By (iv) of Theorem I.2.15 in [3], this implies that $p\colon \mathrm{Hilb}^\circ(\mathcal{U}_S/S) \to S$ is a syntomic morphism at $[C \subset X]$.

Proof of Theorem 1.1. It remains to prove parts (i) and (ii). An open neighborhood of the point $[C \subset X]$ in the Hilbert scheme $\mathrm{Hilb}(X)$ is naturally isomorphic to the fiber of the morphism $p\colon \mathrm{Hilb}^\circ(\mathcal{U}_S/S) \to S$ over the point $[X] \in S$. Since the fibers of a syntomic morphism are local complete intersections, Lemma 2.5 implies that Hilb(X) is a local complete intersection at the point $[C \subset X]$ of dimension $g^2 - 1$.

Let us denote by Obs(C/X) the minimal obstruction space in $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$ for deformations of C in X. By (iv) of Theorem I.2.8 in [3] and the identity (3), there is an inequality

$$\begin{aligned}\dim_{[C\subset X]} \mathrm{Hilb}(X) = g^2 - 1 \quad &\geq h^0(C, \mathcal{N}_{C/X}) - \dim \mathrm{Obs}(C/X) = \\ &= g^2 - 1 + \dim \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C) - \dim \mathrm{Obs}(C/X),\end{aligned}$$

which shows that $\dim \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C) - \dim \mathrm{Obs}(C/X) = 0$. Thus the theorem follows. ■

Remark 2.7. Let $\mathrm{Obs}_S(C/\mathcal{U}_S)$ be the minimal obstruction space in $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$ for deformations of C in $\mathcal{U}_S/S$. A corollary of the proof of Lemma 2.5 is that $\mathrm{Obs}_S(C/\mathcal{U}_S) = \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$. Since $\mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$ is also the obstruction space of an obstruction theory for the embedding $C \subset X$, it follows that there is a natural injective linear map $i\colon \mathrm{Obs}(C/X) \hookrightarrow \mathrm{Obs}_S(C/\mathcal{U}_S)$. By the functorial property of obstruction theories, the map i has a natural (not necessarily linear) section $s\colon \mathrm{Obs}_S(C/\mathcal{U}_S) \to \mathrm{Obs}(C/X)$ induced by specializing to $[X] \in S$ the tiny extensions from which the elements of the minimal obstruction space $\mathrm{Obs}_S(C/\mathcal{U}_S)$ arise. We have just proved that i is actually an isomorphism. However, this was not clear a priori. In fact, $\mathrm{Obs}_S(C/\mathcal{U}_S)$ could have contained obstructions coming from tiny extensions that became trivial when specialized to the point $[X] \in S$.

By the local–global spectral sequence for Ext and the vanishing of $E_2^{2,0} = H^2(C, \mathcal{N}_{C/X})$, there is a short exact sequence:

$$0 \to H^1(C, \mathcal{N}_{C/X}) \to \mathrm{Ext}^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C) \to H^0(C, \mathcal{E}xt^1(\mathscr{C}_{C/X}, \mathscr{O}_C)) \to 0.$$

Lemma 2.8. Let D be the (effective) divisor on C defined by the ideal defining the singular locus of X. Then there is a short exact sequence

$$0 \to \mathcal{N}_{C/X} \to \mathcal{N}_{C/\mathbb{P}} \to \Omega_C^{\otimes 2}(-D) \to 0$$

and the sheaf $\mathcal{E}xt^1_C(\mathscr{C}_{C/X}, \mathscr{O}_C)$ can be canonically identified with the direct image on C of the skyscraper sheaf $\mathscr{O}_D \otimes \Omega_C^{\otimes 2}$. In particular, $\dim H^0\left(C, \mathcal{E}xt^1(\mathscr{C}_{C/X}, \mathscr{O}_C)\right) = \deg D$.

Proof. In the proof of Lemma 2.2, we obtained the locally free resolution of $\mathscr{C}_{C/X}$:

$$0 \to \left(\Omega_C^{\otimes 2}\right)^\vee \to \mathscr{C}_{C/\mathbb{P}} \xrightarrow{p} \mathscr{C}_{C/X} \to 0.$$

The map $\mathscr{O}_C \otimes \mathscr{C}_{X/\mathbb{P}} \cong \left(\Omega_C^{\otimes 2}\right)^\vee \to \mathscr{C}_{C/\mathbb{P}}$ is best understood in terms of affine local coordinates. Let U be a standard affine open subset of $\mathbb{P}$ and q a generator of the ideal of the affine variety $U \cap X$. Then q is a local generator for the sheaf $\mathscr{O}_C \otimes \mathscr{C}_{X/\mathbb{P}}$ on $U \cap C$ and dq is a differential on U whose restriction to $U \cap C$ vanishes on the tangent fields to $U \cap C$ and then defines an element of the conormal sheaf $\mathscr{I}_C/\mathscr{I}_C^2 = \mathscr{C}_{C/X}$. The restriction of dq to C is just the image of q under the map $\mathscr{O}_C \otimes \mathscr{C}_{X/\mathbb{P}} \to \mathscr{C}_{C/\mathbb{P}}$.

Since the partial derivatives of q, with respect to a system of affine coordinates in U, define the ideal of the singular locus of X, the $\mathscr{O}_C$-dual $\mathcal{N}_{C/\mathbb{P}} \to \Omega_C^{\otimes 2}$ of the above map has $\Omega_C^{\otimes 2}(-D)$ as image and hence $\mathscr{O}_D \otimes \Omega_C^{\otimes 2}$ as cokernel. This proves all the statements of the lemma. ∎

An immediate consequence of Theorem 1.1 and Lemma 2.8 is then:

Corollary 2.9. Under the hypotheses of Theorem 1.1, $\dim \mathrm{Obs}(C/X) \geq \deg D$. So if the deformations of C in X are unobstructed, then C does not meet the singular locus of X.

In (iii) of Theorem 1.1, we proved that, if X has corank 1 and C does not contain the singular point of X, then $H^1(C, \mathcal{N}_{C/X}) = 0$ and the deformations of C in X are unobstructed. A natural question is whether this is true in general:

Question 2.10. Are the deformations of C in X unobstructed *if and only if* the curve C does not meet the singular locus of the quadric X? In other words, does the last property imply the vanishing of $H^1(C, \mathcal{N}_{C/X})$?

An interesting case of the above question is when the quadric X has rank 4. By a result of M. Green (see [4]), the degree 2 component of the canonical homogeneous ideal of a smooth projective curve of genus ≥ 4 is generated by quadrics of rank ≤ 4 and, for a general curve, it is generated by quadrics of rank 4. A Bertini's argument, similar to the one in the proof of Lemma 2.5, then implies that a general curve C of genus $g \geq 5$ is contained in a quadric X of rank 4 and is disjoint from its singular locus. For $g = 5$, we know, by (iii) of Theorem 1.1, that $H^1(C, \mathcal{N}_{C/X}) = 0$ but the question is open for $g > 5$.

Funding

This work was supported by the Chinese National Science Foundation to [E.L.]

Acknowledgments

We thank Eduardo Sernesi for a helpful conversation with one of us and two anonymous referees for suggesting various improvements and, in particular, the extension of item (iii) of Theorem 1.1 to quadrics of corank 1.

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