A minimization problem involving a fractional Hardy–Sobolev type inequality

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Abstract In this work, we obtain existence of nontrivial solutions to a minimization problem involving a fractional Hardy–Sobolev type inequality in the case of inner singularity. Precisely, for $\lambda > 0$, we analyze the attainability of the optimal constant

$$\mu_{\alpha,\lambda}(\Omega) := \inf\left\{[u]^2_{s,\Omega} + \lambda \int_\Omega |u|^2\, dx : u \in H^s(\Omega), \int_\Omega \frac{|u(x)|^{2_{s,\alpha}}}{|x|^\alpha}\, dx = 1\right\},$$

where $0 < s < 1$, $n > 4s$, $0 \le \alpha < 2s$, $2_{s,\alpha} = \frac{2(n-\alpha)}{n-2s}$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain such that $0 \in \Omega$.

1. Introduction

Let $0 < s < 1$, $n > 4s$, $0 \le \alpha < 2s$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \Omega$. We introduce the fractional Sobolev space; see for instance [6]:

$$H^s(\Omega) := \left\{u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\Omega \times \Omega)\right\}, \tag{1.1}$$

endowed with the norm

$$\|u\|_{s,\Omega} := \left(\int_\Omega |u|^2\, dx + \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy\right)^{\frac{1}{2}}. \tag{1.2}$$

Let $\lambda > 0$ and $2_{s,\alpha} = \frac{2(n-\alpha)}{n-2s}$. This paper is devoted to analyzing the attainability of the optimal constant $C > 0$ for the following fractional Hardy–Sobolev inequality:

$$C\left(\int_\Omega \frac{|u(x)|^{2_{s,\alpha}}}{|x|^\alpha}\, dx\right)^{\frac{2}{2_{s,\alpha}}} \le \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy + \lambda \int_\Omega \left|u(x)\right|^2 dx,$$

for every $u \in H^s(\Omega)$. For the related Dirichlet problem, see the recent work [14].

In [18], Marano and Mosconi proved the existence of an extremal function u_0, solution to

$$\mu_\alpha := \inf\left\{[u]^2_s : u \text{ measurable, vanishing at infinity}, \int_{\mathbb{R}^N} \frac{|u(x)|^{2_{s,\alpha}}}{|x|^\alpha}\, dx = 1\right\}, \tag{1.3}$$

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where $2_{s,\alpha} = \frac{2(n-\alpha)}{n-2s}$ and

$$[u]_s^2 = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy.$$

See also [19]. Here, u vanishing at infinity means $|\{|u| > a\}| < \infty$ for every $a > 0$. Observe that $2_{s,2s} = 2$ and $2_{s,0} = 2_s^* = \frac{2n}{n-2s}$, the latter being related to the noncompact but continuous embedding $H^s(\mathbb{R}^n) \hookrightarrow L^{2_s^*}(\mathbb{R}^n)$. The constant μ_{2s} was calculated by Herbst [16]. In [18], for $p > 1$, the existence of extremal functions $u \in \dot{W}^{s,p}(\mathbb{R}^n)$ for the Hardy–Sobolev inequality was established through concentration-compactness. The authors also showed the asymptotic behavior of extremal functions: $u(x) \sim |x|^{-\frac{n-ps}{p-1}}$, as $|x| \to \infty$, and the summability information $u \in \dot{W}^{s,\gamma}(\mathbb{R}^n)$, for every $\frac{n(p-1)}{n-s} < \gamma < p$. Such properties turn out to be optimal when $s \to 1^-$, in which case optimizers are explicitly known. See [6] for the definitions of $\dot{W}^{s,p}(\mathbb{R}^n)$ and $\dot{W}^{s,\gamma}(\mathbb{R}^n)$.

In [10], the sharp constant in the Hardy inequality for fractional Sobolev spaces is calculated by using a nonlinear and nonlocal version of the ground state representation.

For unbounded domains, different from $\mathbb{R}^n$, in [8], it was proved a variant of the fractional Hardy–Sobolev–Maz'ya inequality for half spaces, applying a new version of the fractional Hardy–Sobolev inequality for general unbounded John domains. Frank and Seiringer gave an expression for the best constant in the half space in [11]; see also [1]. Concerning bounded domains, see [7, 17]. In [9], the authors considered domains with a uniformly fat complement.

In the local setting, in [12], the authors showed that the value and the attainability of the best Hardy–Sobolev constant on a smooth domain $\Omega \subset \mathbb{R}^n$,

$$\nu_\alpha(\Omega) := \left\{ \int_\Omega |\nabla u|^2\, dx : u \in H_0^1(\Omega), \int_\Omega \frac{|u(x)|^{2_\alpha}}{|x|^\alpha}\, dx = 1 \right\}$$

are closely related to the properties of the curvature of $\partial\Omega$ at 0, where $2_\alpha = \frac{2(n-\alpha)}{n-2}$, $n \geq 3$, $0 < \alpha < 2$, when $0 \in \partial\Omega$. For the nonsingular context with either $\alpha = 0$ or 0 belonging in the interior of the domain Ω, it is well-known that $\nu_\alpha(\Omega) = \nu_0(\mathbb{R}^n)$ for any domain Ω.

In [15], a minimization problem involving a Hardy–Sobolev type inequality was solved, where the author analyzed both inner and boundary singularity. For further references in the local setting, see [3, 4] and the expository paper [13].

Our goal is analyzing the existence of solution to

$$\mu_{\alpha,\lambda}(\Omega) := \inf\left\{ [u]_{s,\Omega}^2 + \lambda \int_\Omega |u|^2\, dx : u \in H^s(\Omega), \int_\Omega \frac{|u(x)|^{2_{s,\alpha}}}{|x|^\alpha}\, dx = 1 \right\}. \tag{1.4}$$

THEOREM 1.1
Let $\lambda > 0$, $0 < s < 1$, $n > 4s$, $0 \leq \alpha < 2s$, $2_{s,\alpha} = \frac{2(n-\alpha)}{n-2s}$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain with $0 \in \Omega$. Then, there exists $\lambda_* \in (0, \infty]$ such that the constant $\mu_{\alpha,\lambda}(\Omega)$ is attained for every $0 < \lambda < \lambda_*$. Moreover, if $\lambda_* < \infty$, $\mu_{\alpha,\lambda}(\Omega)$ is not attained for every $\lambda > \lambda_*$.

The rest of the paper is organized as follows. In Section 2, we gather some preliminaries and features of the constant $\mu_{\alpha,\lambda}(\Omega)$. Section 3 is dedicated to the proof of Theorem 1.1. The crucial ingredients are the properties of $\mu_{\alpha,\lambda}(\Omega)$ seen as a function in λ and a fractional Hardy–Sobolev type inequality.

2. Preliminaries

From now on, we fix $0 < s < 1$, $n > 4s$, $0 \le \alpha < 2s$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain such that $0 \in \Omega$.

The relation between the global constant μ_α and $\mu_{\alpha,\lambda}(\Omega)$, defined in (1.3) and (1.4), respectively, will be a key element for the nonexistence result in Theorem 1.1. As mentioned, some features of $\mu_{\alpha,\lambda}(\Omega)$ seen as a function in the parameter λ play an important role as well. We start with the following basic lemma.

We denote $\dot{H}^s(\mathbb{R}^n)$ the space of measurable functions $u\colon \mathbb{R}^n \to \mathbb{R}$ such that $[u]_s$ is finite.

LEMMA 2.1
Let $\phi \in C_c^\infty(\Omega)$ and $u \in \dot{H}^s(\mathbb{R}^n)$ be such that $\|u\|_{s,\alpha} < \infty$. Then, $\phi u \in H^s(\Omega)$.

Proof
It is clear that $\phi u \in L^2(\Omega)$ since the embedding $L^{2_{s,\alpha}}(\Omega, |x|^{-\alpha}\, dx) \hookrightarrow L^2(\Omega)$ is continuous, as a consequence of Hölder's inequality with $p = \frac{2_{s,\alpha}}{2}$, $p' = \frac{n-\alpha}{2s-\alpha}$ and the boundedness of Ω.

To see $[\phi u]_{s,\Omega} < \infty$, observe that

$$\big|\phi(x)u(x) - \phi(y)u(y)\big| \le \big|u(x)\big|\big|\phi(x) - \phi(y)\big| + \big|\phi(y)\big|\big|u(x) - u(y)\big|. \tag{2.1}$$

Therefore, by Minkowski's inequality, we get

$$\begin{aligned}[\phi u]_{s,\Omega} &\le \Big(\int_\Omega \big|u(x)\big|^2 \int_\Omega \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}}\, dy\, dx\Big)^{\frac{1}{2}} \\ &\quad + \Big(\int_\Omega \big|\phi(x)\big|^2 \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dy\, dx\Big)^{\frac{1}{2}} \\ &=: I + C(\phi)[u]_{s,\Omega},\end{aligned}$$

where we have used $|\phi(x)|^2 \le \|\phi\|_\infty^2$ in the second term. For I, we proceed as in [18, Lemma 2.3] to get

$$\int_{\mathbb{R}^n} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}}\, dy \le C(\phi, n, s) \tag{2.2}$$

uniformly in $x \in \mathbb{R}^n$. Therefore,

$$[\phi u]_{s,\Omega}^2 \le C(\phi, n, s)\big(\|u\|_{L^2(\Omega)} + [u]_{s,\Omega}\big) < \infty$$

since $\|u\|_{s,\alpha,\Omega} \le \|u\|_{s,\alpha} < \infty$, $[u]_{s,\Omega} \le [u]_s < \infty$, and the embedding $L^{2_{s,\alpha}}(\Omega, |x|^{-\alpha}\, dx) \hookrightarrow L^2(\Omega)$ is continous. □

Now, we are able to establish the main result of this section, which gives useful properties of $\mu_{\alpha,\lambda}(\Omega)$ seen as a function in the parameter $\lambda > 0$. Part of the next lemma relies on the existence of an extremal function for the global constant μ_α, and its behavior for $|x| \geq 1$, given in [18].

LEMMA 2.2

Let $\lambda > 0$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain such that $0 \in \Omega$. Then,

(1) $\mu_{\alpha,\lambda}(\Omega) \leq \mu_\alpha$, *for every* $\lambda > 0$.
(2) $\mu_{\alpha,\lambda}(\Omega)$ *is continuous and nondecreasing with respect to* λ.
(3) $\lim_{\lambda\to 0} \mu_{\alpha,\lambda}(\Omega) = 0$,

where $\mu_{\alpha,\lambda}(\Omega)$ and μ_α are defined in (1.4), and (1.3), respectively.

Proof

(1) Let $\varepsilon > 0$, $R > 0$ and $\phi \in C_c^\infty(\Omega)$ be such that $0 \leq \phi \leq 1$, $\phi = 1$ in $B_R(0) \subset \Omega$, $\phi = 0$ in $\Omega \setminus B_{2R}(0)$.

Let u_0 be a positive minimizer of μ_α; see [18] for the existence of u_0. Consider

$$u_\varepsilon(x) := \varepsilon^{-\frac{n-2s}{2}} u_0\Big(\frac{x}{\varepsilon}\Big)\phi(x), \qquad v_\varepsilon(x) := \frac{1}{\|u_\varepsilon\|_{s,\alpha,\Omega}} u_\varepsilon(x).$$

Then, $v_\varepsilon \in H^s(\Omega)$, by Lemma 2.1. Moreover, $\|v_\varepsilon\|_{s,\alpha,\Omega} = 1$. Thus,

$$\mu_{\alpha,\lambda}(\Omega) \leq [v_\varepsilon]^2_{s,\Omega} + \lambda \int_\Omega v_\varepsilon^2(x)\,dx. \tag{2.3}$$

Observe that, after a change of variables,

$$\int_\Omega \frac{u_\varepsilon^{2_{s,\alpha}}(x)}{|x|^\alpha}\,dx = \int_{\varepsilon^{-1}\Omega} \phi^{2_{s,\alpha}}(\varepsilon y)\frac{u_0^{2_{s,\alpha}}(y)}{|y|^\alpha}\,dy.$$

Since $\phi = 1$ in $B_R(0) \subset \Omega$, $0 \leq \phi \leq 1$ and $\operatorname{supp}\phi \subset B_{2R}(0)$, we get

$$\int_{B_{\frac{R}{\varepsilon}}(0)} \frac{u_0^{2_{s,\alpha}}(y)}{|y|^\alpha}\,dy \leq \int_\Omega \frac{u_\varepsilon^{2_{s,\alpha}}(x)}{|x|^\alpha}\,dx \leq \int_{B_{\frac{2R}{\varepsilon}}(0)} \frac{u_0^{2_{s,\alpha}}(y)}{|y|^\alpha}\,dy;$$

therefore,

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{u_\varepsilon^{2_{s,\alpha}}(x)}{|x|^\alpha}\,dx = \int_{\mathbb{R}^n} \frac{u_0^{2_{s,\alpha}}(y)}{|y|^\alpha}\,dy = 1. \tag{2.4}$$

Moreover,

$$\int_\Omega v_\varepsilon^2(x)\,dx = \frac{\varepsilon^{2s}}{\|u_\varepsilon\|^2_{s,\alpha,\Omega}} \int_{B_{\frac{2R}{\varepsilon}}(0)} \phi^2(\varepsilon y)u_0(y)^2\,dy = O(\varepsilon^{2s}).$$

The last identity is due to (2.4), and the fact that

$$\int_{B_{\frac{2R}{\varepsilon}}(0)} \phi^2(\varepsilon y)u_0(y)^2\,dy \leq C. \tag{2.5}$$

Indeed, by [18, Theorem 1.1], we know that for

$$|u_0(y)| \le \frac{C}{|y|^{n-2s}}, \qquad \text{for every } |y| \ge 1. \tag{2.6}$$

Then, there exist $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, we have $\frac{2R}{\varepsilon} > 1$. Therefore, for every $0 < \varepsilon < \varepsilon_0$,

$$\int_{B_{\frac{2R}{\varepsilon}}(0)} \phi^2(\varepsilon y) u_0(y)^2 \, dy = \Big(\int_{\{|y|<1\}} + \int_{\{1 \le |y| \le \frac{2R}{\varepsilon}\}} \Big) \phi^2(\varepsilon y) u_0(y)^2 \, dy$$
$$=: I + II.$$

To manage I, recall $0 \le \phi \le 1$ and that $u_0 \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ by Lemma 2.1. To control II, we use $0 \le \phi \le 1$, (2.6), and the fact that $n > 4s$, to find

$$II \le C \int_{|y| \ge 1} \frac{1}{|y|^{2(n-2s)}} \, dy = C \int_1^{\infty} r^{-n-1+4s} \, dr = C.$$

Now, we have to estimate $[v_\varepsilon]^2_{s,\Omega} = \|u_\varepsilon\|^{-2}_{s,\alpha,\Omega} [u_\varepsilon]^2_{s,\Omega}$. Thanks to (2.4), it will be enough to analyze $[u_\varepsilon]^2_{s,\Omega}$. Similar to what we have done in Lemma 2.1 (Equation (2.1), Minkowski's inequality), but changing variables and recalling $0 \le \phi \le 1$, we get

$$[u_\varepsilon]_{s,\Omega} \le [u_0]_s + \Big(\int_{\varepsilon^{-1}\Omega \times \varepsilon^{-1}\Omega} \frac{u_0(x)^2 |\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x-y|^{n+2s}} \, dx \, dy \Big)^{\frac{1}{2}}.$$

Since u_0 is an extremal function for the constant μ_α, we obtain

$$[u_\varepsilon]_{s,\Omega} \le \mu_\alpha^{\frac{1}{2}} + \Big(\int_{\varepsilon^{-1}\Omega \times \varepsilon^{-1}\Omega} \frac{u_0(x)^2 |\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x-y|^{n+2s}} \, dx \, dy \Big)^{\frac{1}{2}}. \tag{2.7}$$

We will show that

$$\lim_{\varepsilon \to 0} \int_{\varepsilon^{-1}\Omega \times \varepsilon^{-1}\Omega} \frac{u_0(x)^2 |\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x-y|^{n+2s}} \, dx \, dy = 0. \tag{2.8}$$

That will be a consequence of the Lebesgue dominated convergence theorem. Clearly,

$$\lim_{\varepsilon \to 0} \chi_{\varepsilon^{-1}\Omega \times \varepsilon^{-1}\Omega}(x,y) \frac{u_0(x)^2 |\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x-y|^{n+2s}} = 0 \qquad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}^n.$$

To find the dominating function in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$, we split the domain and use (2.6). Indeed, for every $0 < \varepsilon < 1$,

$$\frac{u_0(x)^2 |\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x-y|^{n+2s}} \le C \psi(x,y) \Big(\chi_{\{|x|<1\}} u_0(x)^2 + \chi_{\{|x| \ge 1\}} \frac{1}{|x|^{2(n-2s)}} \Big)$$
$$=: \Psi(x,y),$$

where $\psi(x,y) = \frac{1}{|x-y|^{n+2s-2}} \chi_{\{|x-y|<1\}} + \frac{1}{|x-y|^{n+2s}} \chi_{\{|x-y| \ge 1\}}$. For the previous inequality, we used

$$\frac{|\phi(\varepsilon x) - \phi(\varepsilon y)|^2}{|x-y|^{n+2s}} \le \begin{cases} \frac{C\varepsilon^2}{|x-y|^{n+2s-2}} & \text{if } |x-y| < 1, \\ \frac{C}{|x-y|^{n+2s}} & \text{if } |x-y| \ge 1. \end{cases}$$

Let us see that $\Psi \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$:

$$\begin{aligned}\int_{\mathbb{R}^n\times\mathbb{R}^n} \Psi(x,y)\,dx\,dy &\le C\int_{|x|<1} u_0(x)^2 \int_{\mathbb{R}^n} \psi(x,y)\,dy\,dx \\ &\quad + C\int_{|x|\ge 1} \frac{1}{|x|^{2(n-2s)}} \int_{\mathbb{R}^n} \psi(x,y)\,dy\,dx \\ &\le C\int_{|x|<1} u_0(x)^2\,dx + C\int_{|x|\ge 1} \frac{1}{|x|^{2(n-2s)}}\,dx \le C.\end{aligned}$$

In the last step, we use that $u_0 \in L^2_{\text{loc}}(\mathbb{R}^n)$ by Lemma 2.1 and that $n > 4s$ in the second term. Hence, (2.8) holds. Consequently, from (2.7),

$$\limsup_{\varepsilon\to 0}[u_\varepsilon]^2_{s,\Omega} \le \mu_\alpha.$$

Then, (2.3) becomes

$$\mu_{\alpha,\lambda}(\Omega) \le \frac{1}{\|u_\varepsilon\|^2_{s,\alpha,\Omega}}[u_\varepsilon]^2_{s,\Omega} + O(\varepsilon^{2s}).$$

Taking the limit $\varepsilon \to 0$, we conclude $\mu_{\alpha,\lambda}(\Omega) \le \mu_\alpha$.

(2) The decreasing property of $\mu_{\alpha,\cdot}(\Omega)$ is clear from the definition (1.4). To see the continuity of $\mu_{\alpha,\cdot}(\Omega)$, let $\{\lambda_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ be such that $\lambda_k \to \lambda \in (0,\infty)$ as $k\to\infty$. Then, for every $u \in H^s(\Omega)$ verifying $\|u\|_{s,\alpha,\Omega} = 1$,

$$\mu_{\alpha,\lambda_k}(\Omega) \le [u]^2_{s,\Omega} + \lambda_k \int_\Omega |u|^2\,dx.$$

By taking the limit $k\to\infty$ in the previous inequality, we get

$$\limsup_{k\to\infty} \mu_{\alpha,\lambda_k}(\Omega) \le [u]^2_{s,\Omega} + \lambda \int_\Omega |u|^2\,dx$$

for every $u \in H^s(\Omega)$ such that $\|u\|_{s,\alpha,\Omega} = 1$, implying

$$\limsup_{k\to\infty} \mu_{\alpha,\lambda_k}(\Omega) \le \mu_{\alpha,\lambda}(\Omega). \tag{2.9}$$

On the other hand, for every $u \in H^s(\Omega)$ such that $\|u\|_{s,\alpha,\Omega} = 1$, we have

$$\begin{aligned}\mu_{\alpha,\lambda}(\Omega) &\le [u]^2_{s,\Omega} + \lambda_k \int_\Omega |u|^2\,dx + (\lambda-\lambda_k)\int_\Omega |u|^2\,dx \\ &\le [u]^2_{s,\Omega} + \lambda_k \int_\Omega |u|^2\,dx + (\lambda-\lambda_k)C,\end{aligned}$$

where $C > 0$ is independent of u since $L^{2_{s,\alpha}}(\Omega, |x|^{-\alpha}\,dx) \hookrightarrow L^2(\Omega)$ is continuous and $\|u\|_{s,\alpha,\Omega} = 1$. By taking first the infimum in $u \in H^s(\Omega)$ such that $\|u\|_{s,\alpha,\Omega} = 1$, we get

$$\mu_{\alpha,\lambda}(\Omega) \le \mu_{\alpha,\lambda_k}(\Omega) + (\lambda-\lambda_k)C.$$

By taking the limit $k\to\infty$, we obtain

$$\mu_{\alpha,\lambda}(\Omega) \le \liminf_{k\to\infty} \mu_{\alpha,\lambda_k}(\Omega). \tag{2.10}$$

Combining (2.9) and (2.10), we get the continuity of $\mu_{\alpha,\cdot}(\Omega)$.

(3) Consider $c := (\int_\Omega \frac{1}{|x|^\alpha}\,dx)^{-\frac{1}{2_{s,\alpha}}} \in H^s(\Omega)$. Then,

$$\mu_{\alpha,\lambda}(\Omega) \le [c]^2_{s,\Omega} + \lambda \int_\Omega c^2\,dx = \lambda c^2|\Omega|.$$

Now, take the limit $\lambda \to 0$ to conclude (3). □

3. Existence of extremal function

We start this section with the second ingredient to prove Theorem 1.1, which is a fractional Hardy–Sobolev type inequality. We follow ideas from [15], where the local version was studied.

LEMMA 3.1
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \Omega$. Then, there exists a positive constant $C_1 = C_1(\Omega, n, s)$ such that

$$\mu_\alpha\Big(\int_\Omega \frac{|u(x)|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} \le [u]^2_{s,\Omega} + C_1 \int_\Omega |u|^2\,dx \tag{3.1}$$

for every $u \in H^s(\Omega)$.

Proof
Let $\Omega_1 \subset \Omega_2 \subset \Omega$ be bounded sets to be determined, such that $0 \in \Omega_1$. Let $\phi \in C_c^\infty(\Omega)$ be such that $0 \le \phi \le 1$ in Ω, $\phi = 1$ in Ω_1, $\phi = 0$ in $\Omega \setminus \Omega_2$. Consider

$$\eta_1 = \frac{\phi^2}{\phi^2 + (1-\phi)^2}, \qquad \eta_2 = \frac{(1-\phi)^2}{\phi^2 + (1-\phi)^2}.$$

Then, $\eta_1^{\frac{1}{2}} \in C_c^1(\Omega)$, $\eta_2^{\frac{1}{2}} \in C^1(\Omega)$, $\eta_1 + \eta_2 = 1$, $\operatorname{supp}\eta_1 \subset \Omega_2 \subset \Omega$, $\operatorname{supp}\eta_2 \subset \mathbb{R}^n \setminus \Omega_1$. Let $u \in H^s(\Omega)$. We consider $\eta_2^{\frac{1}{2}}u\colon \Omega \to \mathbb{R}$, by [6, Lemma 5.3], $\eta_2^{\frac{1}{2}}u \in H^s(\Omega)$ since $u \in H^s(\Omega)$ and $\eta_2^{\frac{1}{2}} \in C^{0,1}(\Omega)$. Moreover, $\|\eta_2^{\frac{1}{2}}u\|_{H^s(\Omega)} \le C(n,s,\Omega)\|u\|_{H^s(\Omega)}$. By using the auxiliary functions η_1, η_2, we can write

$$|u|^{2_{s,\alpha}} = \big(\eta_1|u|^2 + \eta_2|u|^2\big)^{\frac{2_{s,\alpha}}{2}}$$

and, by Minkowski's inequality in $L^{\frac{2_{s,\alpha}}{2}}(\Omega, |x|^{-\alpha}\,dx)$, split the main integral into two pieces and analyze them separately, as follows:

$$\mu_\alpha\Big(\int_\Omega \frac{|u(x)|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} \le \mu_\alpha\Big(\sum_{i=1}^{2}\Big(\int_\Omega \frac{|\eta_i^{\frac{1}{2}}u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}}\Big) =: I_1 + I_2.$$

To estimate I_1, notice that we can use the fractional Hardy–Sobolev inequality given by μ_α for $\eta_1^{\frac{1}{2}}u$; see (1.3). Thus,

$$I_1 = \mu_\alpha\Big(\int_\Omega \frac{|\eta_1^{\frac{1}{2}}u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} = \mu_\alpha\Big(\int_{\mathbb{R}^n} \frac{|\eta_1^{\frac{1}{2}}u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} \le [\eta_1^{\frac{1}{2}}u]^2_s. \tag{3.2}$$

Notice that $\operatorname{supp}\eta_1 \subset \Omega$. Similarly to (2.7), we obtain

$$[\eta_1^{\frac{1}{2}} u]_s^2 \le \int_{\Omega\times\Omega} \frac{|\eta_1^{\frac{1}{2}}(x)u(x) - \eta_1^{\frac{1}{2}}(y)u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + 2\int_{(\mathbb{R}^n\setminus\Omega)\times\Omega} \frac{\eta_1(x)|u(x)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

For the first term, we use (2.1) for $\eta_1^{\frac{1}{2}}u$ and Minkowski's inequality. For the second term, we proceed similar to Lemma 2.1 (2.2), to get

$$[\eta_1^{\frac{1}{2}} u]_s^2 \le \int_{\Omega\times\Omega} \frac{\eta_1(y)|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + C(\phi,n,s)\int_\Omega |u|^2\,dx.$$

Therefore,

$$I_1 \le \int_{\Omega\times\Omega} \frac{\eta_1(y)|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + C(\phi,n,s)\int_\Omega \big|u(x)\big|^2\,dx. \tag{3.3}$$

To analyze I_2, notice that $\eta_2 = 0$ in Ω_1, so that

$$I_2 = \mu_\alpha\Big(\int_\Omega \frac{|\eta_2^{\frac{1}{2}}u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} = \mu_\alpha\Big(\int_{\Omega\setminus\Omega_1} \frac{|\eta_2^{\frac{1}{2}}u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}}.$$

Observe that $0 \notin \operatorname{supp}\eta_2$. Denote by $d_1 := \operatorname{dist}(0,\partial\Omega_1)$. Thus, by Hölder's inequality with $p = \frac{n}{n-\alpha}$, $p' = \frac{n}{\alpha}$,

$$\begin{aligned}
I_2 &\le \mu_\alpha d_1^{-\frac{2\alpha}{2_{s,\alpha}}}\Big(\int_{\Omega\setminus\Omega_1} |\eta_2^{\frac{1}{2}}u|^{2_{s,\alpha}}\,dx\Big)^{\frac{2}{2_{s,\alpha}}}\\
&\le \mu_\alpha d_1^{-\frac{2\alpha}{2^*_{s,\alpha}}}|\Omega\setminus\Omega_1|^{\frac{2\alpha}{n2_{s,\alpha}}}\Big(\int_{\Omega\setminus\Omega_1} |\eta_2^{\frac{1}{2}}u|^{2^*_s}\,dx\Big)^{\frac{2}{2^*_s}}\\
&\le \mu_\alpha d_1^{-\frac{2\alpha}{2_{s,\alpha}}}|\Omega\setminus\Omega_1|^{\frac{2\alpha}{n2_{s,\alpha}}}\kappa_{\Omega_1}^{-1}[\eta_2^{\frac{1}{2}}u]_{s,\Omega}^2,
\end{aligned}$$

where κ_{Ω_1} is given by

$$\kappa_{\Omega_1} := \inf\Big\{[v]_{s,\Omega}^2 \colon v \in H^s(\Omega), v = 0 \text{ in } \Omega_1, \int_\Omega |v|^{2^*_s}\,dx = 1\Big\}.$$

It will be enough to prove that

$$\mu_\alpha d_1^{-\frac{2\alpha}{2_{s,\alpha}}}|\Omega\setminus\Omega_1|^{\frac{2\alpha}{n2_{s,\alpha}}}\kappa_{\Omega_1}^{-1} \le 1. \tag{3.4}$$

Indeed, given $\delta > 0$, choose $\Omega_1 \subset \Omega$ such that $0 \in \Omega_1$ and $|\Omega\setminus\Omega_1| < \delta$. Let $\Omega_0 \subset \Omega$ be an open bounded set such that $0 \in \Omega_0 \subset \Omega_1$. Then, $d_1 \ge d_0 := \operatorname{dist}(0,\partial\Omega_0)$. Moreover, $\kappa_{\Omega_0} \le \kappa_{\Omega_1}$. Therefore,

$$\begin{aligned}
\mu_\alpha d_1^{-\frac{2\alpha}{2_{s,\alpha}}}|\Omega\setminus\Omega_1|^{\frac{2\alpha}{n2_{s,\alpha}}}\kappa_{\Omega_1}^{-1} &\le \mu_\alpha d_0^{-\frac{2\alpha}{2_{s,\alpha}}}|\Omega\setminus\Omega_1|^{\frac{2\alpha}{n2_{s,\alpha}}}\kappa_{\Omega_0}^{-1}\\
&\le C(\Omega_0)|\Omega\setminus\Omega_1|^{\frac{2\alpha}{n2_{s,\alpha}}} \le C(\Omega_0)\delta^{\frac{2\alpha}{n2_{s,\alpha}}}.
\end{aligned}$$

Let $\delta > 0$ be such that $C(\Omega_0)\delta^{\frac{2\alpha}{n 2_{s,\alpha}}} < 1$. Consequently, proceeding as for the estimate of $[\eta_1^{\frac{1}{2}} u]_s$, we obtain

$$I_2 \le [\eta_2^{\frac{1}{2}} u]^2_{s,\Omega} \le \int_{\Omega\times\Omega} \frac{\eta_2(y)|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + C(\phi,n,s)\int_\Omega |u(x)|^2\,dx. \tag{3.5}$$

By (3.3), (3.5), and the fact that $\eta_1 + \eta_2 = 1$, we conclude (3.1). □

The next corollary will be one of the main tools for proving Theorem 1.1.

COROLLARY 3.2

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \Omega$. Then,

$$\lim_{\lambda\to\infty} \mu_{\alpha,\lambda}(\Omega) = \mu_\alpha. \tag{3.6}$$

and one of the following statements holds:

(1) *For every $\lambda > 0$, we have the strict inequality $\mu_{\alpha,\lambda}(\Omega) < \mu_\alpha$.*
(2) *There exists $\bar\lambda > 0$ such that $\mu_{\alpha,\lambda}(\Omega) = \mu_\alpha$ for every $\lambda \ge \bar\lambda$.*

Proof

The statements (1) and (2) follow trivially from Lemma 2.2 (1). To see (3.6), again by Lemma 2.2 (1), we know that for every $\lambda > 0$, it holds that $\mu_{\alpha,\lambda}(\Omega) \le \mu_\alpha$. Therefore,

$$\limsup_{\lambda\to\infty} \mu_{\alpha,\lambda}(\Omega) \le \mu_\alpha.$$

By Lemma 3.1, there exists a positive constant $C_1 = C_1(\Omega, n, s)$ such that

$$\mu_\alpha \le [u]^2_{s,\Omega} + C_1 \int_\Omega |u|^2\,dx \le [u]^2_{s,\Omega} + \lambda \int_\Omega |u|^2\,dx$$

for every $u \in H^s(\Omega)$ verifying $\|u\|_{s,\alpha,\Omega} = 1$ and $\lambda \ge C_1$. By taking the limit $\lambda \to \infty$, we get

$$\mu_\alpha \le \liminf_{\lambda\to\infty} \mu_{\alpha,\lambda}(\Omega),$$

which finishes the proof of (3.6). □

Combining Lemmas 2.2 and 3.1, we get the next proposition which gives (non)existence of an extremal function for $\mu_{\alpha,\lambda}(\Omega)$, depending on the relation with the global constant in $\mathbb{R}^n$ (i.e., μ_α).

PROPOSITION 3.3

Let $\lambda > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain such that $0 \in \Omega$.

(1) *If $\mu_{\alpha,\lambda}(\Omega) < \mu_\alpha$, then $\mu_{\alpha,\lambda}(\Omega)$ is attained.*
(2) *If there exists a $\bar\lambda > 0$ such that $\mu_{\alpha,\bar\lambda}(\Omega) = \mu_\alpha$, then for every $\lambda > \bar\lambda$, $\mu_{\alpha,\lambda}(\Omega)$ is not attained.*

Proof

(i) Let $\{u_k\}_{k\in\mathbb{N}} \subset H^s(\Omega)$ be a minimizing sequence for $\mu_{\alpha,\lambda}(\Omega)$; that is,

$$\int_\Omega \frac{|u_k|^{2_{s,\alpha}}}{|x|^\alpha}\,dx = 1 \quad \text{for every } k \in \mathbb{N},$$

and

$$\lim_{k\to\infty}\Big([u_k]^2_{s,\Omega} + \lambda\int_\Omega |u_k|^2\,dx\Big) = \mu_{\alpha,\lambda}(\Omega).$$

Then, $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $H^s(\Omega)$. Therefore, up to a subsequence, we can assume that

$u_k \rightharpoonup u$ weakly in $H^s(\Omega)$,
$u_k \to u$ strongly in $L^p(\Omega)$ for $1 \le p < 2^*_s = \frac{2n}{n-2s}$; see [5, Theorem 4.54],
$u_k \to u$ a.e. in Ω.

Let us prove that $u \not\equiv 0$. We proceed by contradiction. Assume $u \equiv 0$ a.e. in Ω. By (3.1), we get

$$\mu_\alpha = \mu_\alpha\Big(\int_\Omega \frac{|u_k|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} \le [u_k]^2_{s,\Omega} + C\int_\Omega |u_k|^2\,dx,$$

which implies

$$\mu_\alpha \le \mu_{\alpha,\lambda}(\Omega) + o(1) + (C-\lambda)\int_\Omega |u_k|^2\,dx. \tag{3.7}$$

By taking the limit in k, we obtain $\mu_\alpha \le \mu_{\alpha,\lambda}(\Omega)$ which is a contradiction. Therefore, $u \not\equiv 0$ in Ω. By the Brézis–Lieb theorem [2], we know that

$$\int_\Omega \frac{|u_k|^{2_{s,\alpha}}}{|x|^\alpha}\,dx = \int_\Omega \frac{|u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx + \int_\Omega \frac{|u_k-u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx + o(1);$$

therefore,

$$\begin{aligned}
1 &= \Big(\int_\Omega \frac{|u_k|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} = \Big(\int_\Omega \frac{|u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx + \int_\Omega \frac{|u_k-u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx + o(1)\Big)^{\frac{2}{2_{s,\alpha}}}\\
&\le \Big(\int_\Omega \frac{|u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} + \Big(\int_\Omega \frac{|u_k-u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} + o(1)\\
&\le \frac{1}{\mu_{\alpha,\lambda}(\Omega)}\Big([u]^2_{s,\Omega} + \lambda\int_\Omega |u|^2\,dx\Big)\\
&\quad + \frac{1}{\mu_{\alpha,\lambda}(\Omega)}\Big([u_k-u]^2_{s,\Omega} + \lambda\int_\Omega |u_k-u|^2\,dx\Big) + o(1)\\
&= \frac{1}{\mu_{\alpha,\lambda}(\Omega)}\Big([u_k]^2_{s,\Omega} + \lambda\int_\Omega |u_k|^2\,dx\Big) + o(1)\\
&= 1 + o(1).
\end{aligned}$$

Notice that we have used that

$$\big|(u_k-u)(x)-(u_k-u)(y)\big|^2 = \big|u_k(x)-u_k(y)\big|^2 + \big|u(x)-u(y)\big|^2 - 2\big(u_k(x)-u_k(y)\big)\big(u(x)-u(y)\big),$$

implying

$$\begin{aligned}[u]^2_{s,\Omega} + [u_k-u]^2_{s,\Omega} &\le [u_k]^2_{s,\Omega} + [u]^2_{s,\Omega} \\ &\quad - 2\int_{\Omega\times\Omega} \frac{(u_k(x)-u_k(y))(u(x)-u(y))}{|x-y|^{n+2s}}\,dx\,dy \\ &= [u_k]^2_{s,\Omega} + o(1),\end{aligned}$$

due to the weak convergence $u_k \rightharpoonup u$ in $H^s(\Omega)$. As a consequence, there exists the following limit:

$$\begin{aligned}1 &= \lim_{k\to\infty}\Big(\int_\Omega \frac{|u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx + \int_\Omega \frac{|u_k-u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} \\ &= \lim_{k\to\infty}\Big[\Big(\int_\Omega \frac{|u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}} + \Big(\int_\Omega \frac{|u_k-u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx\Big)^{\frac{2}{2_{s,\alpha}}}\Big].\end{aligned}$$

Since $u \not\equiv 0$, we conclude that $u_k \to u$ strongly in $L^{2_{s,\alpha}}(\Omega, |x|^{-\alpha}\,dx)$, and, by the strict subadditivity of $t \mapsto t^{\frac{2}{2_{s,\alpha}}}$,

$$\int_\Omega \frac{|u|^{2_{s,\alpha}}}{|x|^\alpha}\,dx = 1,$$

which implies that $\mu_{\alpha,\lambda}(\Omega)$ is attained by u.

(ii) Let $\lambda > \bar\lambda$. Assume that there exists a function $u \in H^s(\Omega)$ which is a minimizer for $\mu_{\alpha,\lambda}(\Omega)$. Then,

$$\begin{aligned}\mu_{\alpha,\lambda}(\Omega) &= [u]^2_{s,\Omega} + \lambda\int_\Omega |u|^2\,dx > [u]^2_{s,\Omega} + \bar\lambda\int_\Omega |u|^2\,dx \\ &\ge \mu_{\alpha,\bar\lambda}(\Omega) = \mu_\alpha \ge \mu_{\alpha,\lambda}(\Omega),\end{aligned}$$

where we have used (1) from Lemma 2.2 in the last inequality. This contradiction finishes the proof. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1
We define $\lambda_* = \inf\{\lambda > 0 : \mu_{\alpha,\lambda}(\Omega) = \mu_\alpha\} \in [0,\infty]$. By Lemma 2.2 (3), we deduce $\lambda_* > 0$. The proof follows from Corollary 3.2 and Proposition 3.3. □

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