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Dynamical Belyi maps and arboreal Galois groups

Received: 14 March 2019 / Accepted: 9 May 2020 / Published online: 8 June 2020

Abstract. We consider a large class of so-called dynamical Belyi maps and study the Galois groups of iterates of such maps. From the combinatorial invariants of the maps, we construct a useful presentation of the geometric Galois groups as subgroups of automorphism groups of regular trees, in terms of iterated wreath products. Using results on the reduction of dynamical Belyi maps modulo certain primes, we obtain results on the corresponding arithmetic Galois groups of iterates. These lead to results on the behavior of the arithmetic Galois groups under specialization, with applications to dynamical sequences.

Introduction

Let $f : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ be a rational map defined over a number field K . The Galois theory of the iterates $f^n = f \circ \dots \circ f : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ was first studied in the work of Odoni [13], and has applications both in number theory and in arithmetic dynamics. Specializing the iterates f^n at a K -rational place $a \in \mathbb{P}_K^1$, we obtain a tower of number fields $(K_{n,a})_{n \geq 1}$. Many recent papers study the question of the distribution of the primes of K that ramify in this tower.

We denote the Galois group of f^n by $G_{n,K} = G_{n,K}(f)$ and the Galois group of its specialization at a by $G_{n,a} = G_{n,a,K}(f)$. For places a avoiding a finite subset of \mathbb{P}_K^1 , the limit $G_{\infty,a} = \varprojlim_n G_{n,a}$ acts on the infinite d -ary regular tree T_∞ , where $d = \deg(f)$. We obtain an *arboreal representation*, and hence a map of $G_{\infty,a}$ to the automorphism group $\text{Aut}(T_\infty)$ of the tree; this is why we call the group $G_{\infty,a}$ an *arboreal Galois group*. A paper by Jones [9] provides a survey of the theory of arboreal representations. A central question is to characterize when $G_{\infty,a}$ embeds in $\text{Aut}(T_\infty)$ as a subgroup of finite index [9, Question 1.1]. As Jones discusses, this question may be considered as an analog of Serre's open image theorem. We take a different perspective and study a class of rational maps for which the image of $G_{\infty,a}$ has infinite index in $\text{Aut}(T_\infty)$.

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Mathematics Subject Classification: Primary: 11G32 · 12F10; Secondary: 37P05 · 37P15

The maps we study in this paper are Belyi maps $f : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$ with exactly 3 ramification points, which we assume to be 0, 1, ∞ . We normalize the coordinate on the target projective line such that the ramification points are fixed by f . A map f satisfying these properties is called a *normalized (dynamical) Belyi map* and is completely determined by its (combinatorial) type (Definition 1.1.2), which tabulates the ramification indices. In particular, all maps may be defined over $K = \mathbb{Q}$ (Proposition 1.1.3). All normalized Belyi maps are so-called post-critically finite (PCF) maps, since the forward orbit of each ramification point is preperiodic. A result of Jones and Pink [9, Theorem 3.1] states that the index $[\text{Aut}(T_\infty) : G_{\infty,a}]$ is infinite for PCF maps, and hence for ours, as well.

The arboreal Galois group $G_{\infty,a}$ of a specialization of a PCF map is a mysterious group, which is hard to describe in general. The case that f is a polynomial of degree $\deg(f) = 2$ has been extensively studied, see e.g. [12, 15]. In [3], the authors consider the case of the cubic polynomial $f(x) = -2x^3 + 3x^2$, providing the first completely explicit result on its Galois theory. This cubic polynomial is the easiest example of a normalized Belyi map.

A related problem, which is very interesting but difficult in general, is to determine the primes that ramify in the tower of number fields $(K_{n,a})_{n \geq 1}$. For PCF maps f , it is known that only finitely many primes ramify in the whole tower $(K_{n,a})_{n \geq 1}$. This was proven in [1, Theorem 1.1] in the case that f is a polynomial and in [4, Theorem 1] in the general case; see also [10, Theorem 3.2].

The goal of this paper is to study the Galois group $G_{\infty,a}$ and the primes ramifying in the corresponding tower of number fields for normalized Belyi maps. The class of normalized Belyi maps we study in this paper contains maps of arbitrarily large degree, which are not necessarily polynomial. Explicit expressions exist for these maps; Proposition 1.1.4 gives a sample. However, we do not rely on these to prove general results for this class of maps.

Instead, we systematically use the Galois-theoretical characterization of these maps (Sect. 1) and exploit the description of their Galois groups as an iterated wreath product. In particular, the combinatorial description of the monodromy of f (in terms of its combinatorial type) yields a combinatorial description of the monodromy of its iterates, which enables us to explicitly determine the geometric Galois groups $G_{n,\overline{\mathbb{Q}}}$ for all $n \geq 1$ in Theorem 2.3.1. Furthermore, the product discriminant that we introduce in Sect. 2.4 allows us to study the arithmetic Galois groups $G_{n,\mathbb{Q}}$ and to make a comparison between the arithmetic and geometric Galois groups in Sect. 2.4. Finally, in analyzing the properties of the specialized Galois groups $G_{n,a}$ (Theorem 3.2.2) our full understanding of the ramification structure of iterates of dynamical Belyi maps plays a key role. Another important ingredient is that the reduction behavior of a normalized Belyi map f , like its monodromy, can also be completely expressed in terms of its combinatorial type, often yielding explicit and easy to apply criteria for good and bad reduction. Normalized Belyi maps were already studied in the context of arithmetic dynamics in our previous paper [2], where we proved a first result on the reduction of normalized Belyi maps (Proposition 1.2.4). In the current paper, we exploit this approach more fully to study the Galois theory of the towers $(K_{n,a})_{n \geq 1}$ for normalized Belyi maps.

Normalized Belyi maps form a very rich source of PCF maps. The techniques for studying their iterates presented here illustrate that the whole class of maps may be analyzed by general methods, which makes the class accessible for future applications in arithmetic dynamics. Just to mention one possible application: Remark 3.1.6 sketches how our results may be used to explicitly construct infinite towers of number fields branched over an explicit finite set of primes. Our knowledge on the reduction of a normalized Belyi map yields a much smaller set containing the branched primes than could be expected from previous work.

We now describe our results more precisely. Our first main result completely describes the geometric groups $G_{n, \overline{\mathbb{Q}}}$ as subgroups of $\text{Aut}(T_n)$ for any $n \geq 1$, excluding two exceptional cases in small degree. The geometric group $G_{n, \overline{\mathbb{Q}}}$ is the Galois group of $f^n : \mathbb{P}_{\overline{\mathbb{Q}}}^1 \rightarrow \mathbb{P}_{\overline{\mathbb{Q}}}^1$ considered as map defined over $\overline{\mathbb{Q}}$.

Theorem 1. (Theorem 2.3.1) *Let f be a normalized Belyi map and let $E_n \subseteq \text{Aut}(T_n)$ be the subgroup defined in Definition 2.1.2. With two exceptions the group $G_{n, \overline{\mathbb{Q}}}$ is either isomorphic to E_n or to the n -fold iterated wreath product of A_d with itself. The case distinction only depends on the type of f and is independent of n .*

The two possibilities for $G_{n, \overline{\mathbb{Q}}}$ can be described totally explicitly. For example, there is an easy formula for its index in $\text{Aut}(T_n)$ (Lemma 2.1.3(1)).

In general, the Galois group $G_{n, \mathbb{Q}}$ of f^n , considered as map defined over \mathbb{Q} , is strictly larger than $G_{n, \overline{\mathbb{Q}}}$. In Corollaries 2.4.4 and 2.4.6 and Remark 2.4.7 we describe exactly when $G_{n, \mathbb{Q}} = G_{n, \overline{\mathbb{Q}}}$ for all $n \geq 1$.

Hilbert's Irreducibility Theorem implies that $G_{n, a} = G_{n, \mathbb{Q}}$ for a in a dense open subset of $\mathbb{P}_{\mathbb{Q}}^1$. We give an explicit criterion on a and the type of the normalized dynamical Belyi map f that guarantees that $G_{n, a} = G_{n, \mathbb{Q}} = G_{n, \overline{\mathbb{Q}}} = E_n$ for all $n \geq 1$:

Theorem 2. (Theorem 3.2.2) *Let $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ be chosen such that Conditions 3.1.5 are satisfied for some choice of distinct rational primes p, q_1, q_2, q_3 . Assume that $G_{n, \mathbb{Q}} = G_{n, \overline{\mathbb{Q}}}$ for all $n \geq 1$. Then $G_{n, a} \simeq G_{n, \mathbb{Q}}$ for all $n \geq 1$.*

While the conditions we impose are somewhat involved, it is not hard to find instances where all conditions are satisfied; Remark 3.1.6 gives a sample.

Theorem 2 is a generalization of the main result (Theorem 1.1) of [3], in which the authors consider the special case that $f(x) = -2x^3 + 3x^2$ is a concrete polynomial of degree 3. Our Conditions 3.1.5 generalize Condition (†) in that paper. The general strategy for proving our results is similar to that of [3]. However, the details of the proofs are quite different. The authors of [3] can rely on ad hoc arguments in terms of the explicit polynomial $f(x)$ that do not extend directly. In our much more general setting, we exploit the Galois-theoretic properties of normalized Belyi maps and the group-theoretic properties of arboreal Galois groups as iterated wreath products instead.

Our last main result is an application to arithmetic dynamics. Let f be a normalized Belyi map of degree $d \geq 3$. For any $c \in \mathbb{P}^1(\mathbb{Q})$, we may construct the

dynamical sequence $\{f^i(c)\}_{i \geq 0}$, where $f^0(c) = c$, and study the primes dividing at least one non-zero finite term of the sequence. (Note that since normalized Belyi maps fix $0, 1, \infty$, the only interesting dynamical sequences are those avoiding these three points.)

Theorem 3. (Corollary 4.2.1(2)) *Let K be the splitting field of f and consider the non-zero preimages of zero under f , denoted $c_j \in K$. Suppose that for each c_j the analogues of Conditions 3.1.5 are satisfied for distinct K -primes $\mathfrak{p}_j, \mathfrak{q}_{1,j}, \mathfrak{q}_{2,j}, \mathfrak{q}_{3,j}$. Define a dynamical sequence $\{b_i\}_{i \geq 0}$ by $b_0 \in \mathbb{Q} \setminus \{0, 1\}$ and $b_i = f(b_{i-1})$ for $i \geq 1$. Then the set of prime divisors of the entries of this sequence has natural density zero.*

Outline of the paper In Sect. 1, we introduce normalized (dynamical) Belyi maps and formulate known facts on their Galois groups. We also recall some results from [2] on reduction of normalized Belyi maps. In Sect. 2 we consider the automorphism group $\text{Aut}(T_\infty)$ of an infinite regular tree. We prove our first main result, Theorem 2.3.1, which describes the groups $G_{n,\overline{\mathbb{Q}}}$ as subgroups of $\text{Aut}(T_n)$, and prove Corollary 2.4.4, which compares $G_{n,\mathbb{Q}}$ with $G_{n,\overline{\mathbb{Q}}}$.

Section 3 treats the specializations $G_{n,a}$. Our second main result, Theorem 3.2.2, proves sufficient conditions guaranteeing that $G_{n,a} \simeq G_{n,\overline{\mathbb{Q}}}$ for all $n \geq 1$. In Sect. 4, we show in Theorem 4.1.2 that the proportion of elements of G_n (and hence of $G_{n,a}$, when Conditions 3.1.5 are satisfied) that fix a leaf tends to zero as n tends to infinity. This is applied to derive consequences for prime divisors of dynamical sequences in Corollary 4.2.1.

1. Dynamical Belyi maps

1.1. Normalized Belyi maps of type $(d; e_1, e_2, e_3)$

In this section we introduce the class of Belyi maps that we will study in this paper. Recall that a *Belyi map* is a finite cover $f : X \rightarrow \mathbb{P}^1$ of smooth projective curves over \mathbb{C} that is branched exactly over $x_1 = 0, x_2 = 1$, and $x_3 = \infty$. A Belyi map has genus zero if X has genus $g(X) = 0$.

Remark 1.1.1. A *dynamical Belyi map* is a genus-0 Belyi map f such that $f(\{0, 1, \infty\}) \subseteq \{0, 1, \infty\}$. This notion was introduced in [18]. In this case the n th iterate of f , which we denote by f^n , is also a Belyi map of genus zero. All dynamical Belyi maps are *post-critically finite*; a map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is post-critically finite (PCF) if each of its ramification points has finite forward orbit.

Definition 1.1.2. (1) A Belyi map f is called *single cycle* if there is a unique ramification point over each of the three branch points.

(2) A single-cycle genus-0 Belyi map is called *normalized* if its ramification points are $0, 1, \infty$, and moreover $f(0) = 0, f(1) = 1$, and $f(\infty) = \infty$.

(3) The (*combinatorial*) *type* of a single-cycle genus-0 Belyi map f is the tuple $(d; e_1, e_2, e_3)$, where $d = \deg(f)$ and e_i is the ramification index of the unique ramification point above x_i .

Note that normalized (single-cycle genus-0) Belyi maps f are dynamical Belyi maps, and hence PCF maps. They even satisfy the stronger condition of being *conservative*, which means that they are PCF maps such that each of their ramification points is a fixed point.

The Riemann–Hurwitz formula implies that the type of a single-cycle genus-0 Belyi map of type $(d; e_1, e_2, e_3)$ satisfies

$$2d + 1 = e_1 + e_2 + e_3. \quad (1.1.1)$$

A Belyi map of type $(d; e_1, e_2, e_3)$ is automatically single cycle. All Belyi maps considered in this paper are assumed to be normalized (single-cycle genus-0 dynamical) Belyi maps from now on.

To exclude trivial cases we always assume that all normalized Belyi maps of type $(d; e_1, e_2, e_3)$ have exactly three ramification points, i.e., $e_i > 1$ for all $1 \leq i \leq 3$. For simplicity we will moreover always assume that $e_1 \leq e_2 \leq e_3$. Permuting the e_i corresponds to changing coordinates on both projective lines simultaneously, therefore these inequalities pose no restriction. An *abstract type* is a tuple $(d; e_1, e_2, e_3)$ such that

$$2 \leq e_1 \leq e_2 \leq e_3 \leq d$$

and such that (1.1.1) holds.

The following result is classical; a proof can be found in [2, Proposition 1]. Note that the normalization condition completely fixes the coordinates on both projective lines.

Proposition 1.1.3. *For each abstract type $\underline{C} := (d; e_1, e_2, e_3)$ there is a unique normalized Belyi map f of type \underline{C} . Moreover, this Belyi map is defined over \mathbb{Q} .*

Proposition 1.1.3 implies that a normalized Belyi map f is completely determined by its type. All proofs in this paper only depend on the type of f and do not use the explicit equation for f . However, it is not too difficult to explicitly calculate the normalized Belyi map of a given type, as the following result illustrates.

Proposition 1.1.4. *Let $d \geq 3$ and $k \geq 1$.*

(1) *The unique normalized Belyi map of type $(d; d - k, k + 1, d)$ is*

$$f(x) = cx^{d-k}(a_0x^k + \cdots + a_{k-1}x + a_k),$$

where

$$a_i := \frac{(-1)^{k-i}}{(d-i)} \binom{k}{i} \text{ and } c = \frac{1}{k!} \prod_{j=0}^k (d-j).$$

(2) *The unique normalized Belyi map of type $(d; d - k, 2k + 1, d - k)$ is*

$$f(x) = x^{d-k} \left(\frac{a_0x^k - a_1x^{k-1} + \cdots + (-1)^k a_k}{(-1)^k a_k x^k + \cdots - a_1 x + a_0} \right),$$

where

$$a_i := \binom{k}{i} \prod_{k+i+1 \leq j \leq 2k} (d-j) \prod_{0 \leq j \leq i-1} (d-j) = k! \binom{d}{i} \binom{d-k-i-1}{k-i}.$$

Proof. This is [2, Propositions 2 and 3]. \square

Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3)$. Associated with f is a *generating system* (g_1, g_2, g_3) , which describes the quotient of the topological fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, *)$ corresponding to f . In our situation, a generating system consists of three permutations $g_i \in S_d$ for $1 \leq i \leq 3$, where g_i is a single cycle of length e_i , that satisfy the relation $g_1 g_2 g_3 = 1$. This motivates the terminology *single cycle*. The generating system for f is unique up to simultaneous conjugacy by elements of S_d . The group $G(f) := \langle g_1, g_2, g_3 \rangle$ is the Galois group of the Galois closure of the cover $f : \mathbb{P}_{\mathbb{Q}}^1 \rightarrow \mathbb{P}_{\mathbb{Q}}^1$, i.e., the Galois group of the splitting field of $f(x) - t$ over $\overline{\mathbb{Q}}(t)$. Liu–Osserman [11, Lemma 2.1] show that the triple (g_1, g_2, g_3) is *weakly rigid* in the sense that it is unique up to uniform conjugacy in S_d . However, in the case that $G(f) \subsetneq S_d$ the generating system is not unique up to uniform conjugacy in $G(f)$, i.e., the triple is not rigid. In this case $G(f)$ is not necessarily the Galois group of the cover $f : \mathbb{P}_{\mathbb{Q}}^1 \rightarrow \mathbb{P}_{\mathbb{Q}}^1$, even though the map f is defined over \mathbb{Q} . We discuss this phenomenon in more detail in Sect. 2.4. We refer to [16, Chapter 2] for a general introduction to this topic.

Lemma 1.1.5. *Let f be a normalized Belyi map of type $\underline{C} = (d; e_1, e_2, e_3)$.*

- (1) *Assume $\underline{C} \neq (6; 4, 4, 5)$. Then the Galois group $G(f)$ of the Galois closure of $f : \mathbb{P}_{\mathbb{Q}}^1 \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ is isomorphic to S_d if at least one e_i is even, and to A_d otherwise.*
(2) *Define*

$$\begin{aligned} g_1 &= (d, d-1, \dots, e_3, 1, 2, \dots, d-e_2-1, d-e_2), \\ g_2 &= (d-e_2+1, d-e_2+2, \dots, d-1, d), \\ g_3 &= (e_3, e_3-1, \dots, 2, 1). \end{aligned} \tag{1.1.2}$$

Then (g_1, g_2, g_3) is a generating system for f .

Proof. The first statement is proved in [11, Theorem 5.3]. The second statement is [11, Lemma 2.1]. \square

1.2. Reduction of normalized Belyi maps

In this section we recall from [2, Section 4] the definition of and some results on the reduction of normalized Belyi maps.

We identify the cover f with the rational function that defines it. Without loss of generality we may assume the following holds:

- (1) We may write $f(x) = x^{e_1} f_1(x)/f_2(x)$, with $f_1, f_2 \in \mathbb{Z}[x]$ such that $f_1(0) \neq 0$ and $f_2(0) \neq 0$;

- (2) The polynomials f_1, f_2 satisfy $\gcd(f_1, f_2) = 1$ and have coprime leading coefficients;
- (3) The greatest common divisor of all coefficients of f is 1.

Definition 1.2.1. Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3)$ and let p be a prime. The *reduction* \bar{f} of f at p is defined as $\bar{f} = x^{e_1} \bar{f}_1 / \bar{f}_2$, where \bar{f}_i is the reduction of f_i modulo p (as a polynomial with coefficients in \mathbb{Z}).

Definition 1.2.1 is a simplification of the definition in [2, Section 4], taking into account the result of [2, Proposition 4].

The following proposition lists some properties of the reduction of a normalized Belyi map. The proof uses the assumption that f is a normalized Belyi map of type \underline{C} in an essential way, see [2, Example 4].

Proposition 1.2.2. *Let f be a normalized Belyi map of type $\underline{C} = (d; e_1, e_2, e_3)$ and let p be a prime.*

- (1) *The rational function $\bar{f} \in \mathbb{F}_p(x)$ is non-constant.*
- (2) *We have $\bar{f}(0) = 0$, $\bar{f}(1) = 1$, and $\bar{f}(\infty) = \infty$.*

Proof. The statement is not stated explicitly in [2] in this form, but it follows immediately from the proof of [2, Proposition 4] and [2, Remark 6]. \square

- Definition 1.2.3.** (1) A normalized Belyi map f has *bad reduction* at a prime p if \bar{f} has strictly smaller degree than f . Otherwise, f has *good reduction* at p .
- (2) A normalized Belyi map f has *good monomial reduction* at p if it has good reduction at p and its reduction is $\bar{f}(x) = x^{\deg(f)}$.
 - (3) A normalized Belyi map f has *good separable reduction* at p if it has good reduction at p and its reduction is a separable rational map. A rational map in $\mathbb{F}_p(x)$ is separable if and only if it is not contained in $\overline{\mathbb{F}}_p(x^p)$, if and only if it defines a separable map $\mathbb{P}_{\mathbb{F}_p}^1 \rightarrow \mathbb{P}_{\mathbb{F}_p}^1$.

If a normalized Belyi map f has good separable reduction at p then its reduction \bar{f} is also a normalized Belyi map and \bar{f} has the same type as f . This follows from the proof of [2, Proposition 5]. Good monomial reduction is a special case of good inseparable reduction. In Proposition 3.1.2 we use it to characterize the irreducibility of specializations of f and its iterates.

The following result allows us to prescribe the reduction of a normalized Belyi map by purely combinatorial conditions on its ramification indices.

Proposition 1.2.4. *Let f be a normalized Belyi map of combinatorial type $(d; e_1, e_2, e_3)$.*

- (1) *Assume that $p > d$. Then f has good separable reduction at p .*
- (2) *Write $d = p^n d'$ where $p \nmid d'$. Then f has good monomial reduction at p if and only if $e_2 \leq p^n$.*

Proof. Statement (1) follows from [2, Corollary 2]. Statement (2) is [2, Theorem 1]. \square

2. Arboreal Galois groups

2.1. Automorphisms of the d -ary regular tree

Let T_n be the regular d -ary rooted tree of level n (cf. Fig. 1).

We label the vertices of the tree as follows: the root of the tree corresponds to the level 0, and has the empty label $()$. The vertices at level i are labeled as (ℓ_1, \dots, ℓ_i) with $\ell_j \in \{1, \dots, d\}$. Here $(\ell_1, \dots, \ell_{i-1})$ is the unique vertex at level $i-1$ which is connected to (ℓ_1, \dots, ℓ_i) by an edge. For the vertices of T_n at level n , also called the leaves, we additionally use the numbering

$$1 + \sum_{k=1}^n (\ell_k - 1)d^{k-1} \in \{1, 2, \dots, d^n\} \quad \text{instead of } (\ell_1, \dots, \ell_n). \quad (2.1.1)$$

Since $\text{Aut}(T_n)$ acts faithfully on the leaves of the tree T_n , the choice of the numbering induces an injective group homomorphism

$$\iota_n : \text{Aut}(T_n) \hookrightarrow S_{d^n}. \quad (2.1.2)$$

In this paper we use the convention that permutations act from the right.

Rather than considering $\text{Aut}(T_n)$ as a subgroup of S_{d^n} it is more convenient for our purposes to view $\text{Aut}(T_n)$ as a subgroup of the n -fold iterated wreath product of S_d by itself. The structure of $\text{Aut}(T_n)$ as n -fold iterated wreath product is

$$\text{Aut}(T_n) \simeq \text{Aut}(T_{n-1}) \wr \text{Aut}(T_1) \text{ for } n \geq 2. \quad (2.1.3)$$

This isomorphism is induced by the decomposition of T_n as the subtree T_1 (consisting of the levels 0 and 1), and d copies of T_{n-1} each consisting of the complete subtree of T_n with root (j) for $j \in \{1, \dots, d\}$. We remark that $\text{Aut}(T_1) \simeq S_d$, but that the iterated wreath product $\text{Aut}(T_n)$ is a strict subgroup of S_{d^n} for $n \geq 2$.

Equation (2.1.3) allows us to write the elements of $\text{Aut}(T_n)$ as tuples

$$(\sigma, \tau) = ((\sigma_1, \dots, \sigma_d), \tau) \text{ with } \sigma_j \in \text{Aut}(T_{n-1}) \text{ and } \tau \in \text{Aut}(T_1) \simeq S_d. \quad (2.1.4)$$

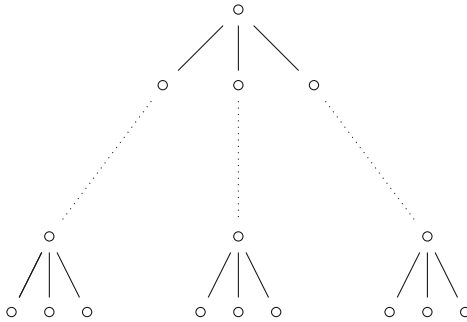


Fig. 1. The regular 3-ary tree

Using this identification, an element of $\text{Aut}(T_n)$ acts on vertices of T_n as

$$(\ell_1, \dots, \ell_n) \cdot ((\sigma_1, \dots, \sigma_d), \tau) = ((\ell_1)\tau, (\ell_2, \dots, \ell_n)\sigma_{\ell_1}). \quad (2.1.5)$$

In other words, τ permutes the d complete subtrees isomorphic to T_{n-1} , and σ_j acts on the complete subtree with root (j) .

Let (ℓ_1, \dots, ℓ_m) be a vertex of T_n at level $m \leq n-1$ and let $i = 1 + \sum_{k=1}^m (\ell_k - 1)d^{k-1}$. We denote the complete subtree of T_n with root (ℓ_1, \dots, ℓ_m) by

$$T_n(\ell_1, \dots, \ell_m) = T_n^i. \quad (2.1.6)$$

Correspondingly, we denote the group of permutations that only permute the leaves of T_n^i and fix all other leaves of T_n by

$$\mathcal{S}(T_n^i) = \{(\sigma, \tau) \in \text{Aut}(T_n) \mid (\sigma, \tau) \text{ acts trivially outside } T_n^i\}. \quad (2.1.7)$$

We may view $\mathcal{S}(T_n^i)$ as a subgroup of S_{d^n} via ι_n .

For future reference, we note that

$$\begin{aligned} ((\sigma'_1, \dots, \sigma'_d), \tau') \cdot ((\sigma_1, \dots, \sigma_d), \tau) &= ((\sigma'_1\sigma_{(1)\tau'}, \dots, \sigma'_d\sigma_{(d)\tau'}), \tau'\tau), \\ ((\sigma_1, \dots, \sigma_d), \tau)^{-1} &= ((\sigma_{(1)\tau^{-1}}^{-1}, \dots, \sigma_{(d)\tau^{-1}}^{-1}), \tau^{-1}), \\ ((\sigma_1, \dots, \sigma_d), \tau)^{-1}((\xi_1, \dots, \xi_d), -) &= ((\sigma_1, \dots, \sigma_d), \tau) \\ &= ((\sigma_{(1)\tau^{-1}}^{-1}\xi_{(1)\tau^{-1}}\sigma_{(1)\tau^{-1}}, \dots), -). \end{aligned} \quad (2.1.8)$$

Here $-$ denotes the trivial permutation.

For every $m \leq n$ we write π_m for the natural projection

$$\pi_m : \text{Aut}(T_n) \rightarrow \text{Aut}(T_m),$$

which corresponds to restricting the action of an element of $\text{Aut}(T_n)$ to the subtree T_m consisting of the levels $0, 1, \dots, m$.

Definition 2.1.1. (1) Define $\text{sgn}_2 : \text{Aut}(T_2) \rightarrow \{\pm 1\}$ by setting

$$\text{sgn}_2(((\sigma_1, \dots, \sigma_d), \tau)) = \text{sgn}(\tau) \prod_{i=1}^d \text{sgn}(\sigma_i). \quad (2.1.9)$$

Here sgn is the usual sign on $\text{Aut}(T_1)$ via the identification $\text{Aut}(T_1) \simeq S_d$ induced by the choice of labeling of the vertices.

(2) For $n > 2$ we define

$$\text{sgn}_2 := \text{sgn}_2 \circ \pi_2 : \text{Aut}(T_n) \rightarrow \{\pm 1\}. \quad (2.1.10)$$

We call sgn_2 the *wreath-product sign*.

Using (2.1.8), one may check that sgn_2 is a group homomorphism. We define a subgroup E_n of $\text{Aut}(T_n)$ using the wreath-product sign. We show in Corollary 2.2.4 below that the Galois group of f^n is a subgroup of E_n .

Definition 2.1.2. Define the subgroup $E_n \subseteq \text{Aut}(T_n)$ by

$$E_n = \begin{cases} \text{Aut}(T_1) & \text{if } n = 1, \\ (E_{n-1} \wr E_1) \cap \ker(\text{sgn}_2) \subseteq \text{Aut}(T_{n-1}) \wr \text{Aut}(T_1) = \text{Aut}(T_n) & \text{otherwise.} \end{cases}$$

Lemma 2.1.3. (1) For all $d \geq 2$ and $n \geq 2$ we have

$$[\text{Aut}(T_n) : E_n] = 2^{d^{n-2} + d^{n-3} + \dots + d + 1}.$$

(2) Assume that d is odd. Then the wreath-product sign on $\text{Aut}(T_n)$ agrees with the restriction of the usual sign on S_{d^2} via the embedding ι_n from (2.1.2):

$$\text{sgn}_2 : \text{Aut}(T_2) \rightarrow \{\pm 1\}, \quad \text{sgn}_2((\sigma, \tau)) \mapsto \text{sgn} \circ \iota_2((\sigma, \tau)).$$

Proof. The definition of E_n (Definition 2.1.2) implies that

$$|E_n| = |E_{n-1}|^d \cdot |E_1|/2.$$

Since $E_1 = \text{Aut}(T_1)$, the definition of the wreath-product sign (2.1.10) implies that

$$[\text{Aut}(T_n) : E_n] = 2([\text{Aut}(T_{n-1}) : E_{n-1}])^d.$$

Statement (1) follows from this by induction.

To prove Statement (2), we first note that for any $(\sigma, \tau) \in \text{Aut}(T_2)$ we may write

$$(\sigma, \tau) = (\sigma, -) \circ ((-), \tau).$$

Since sgn_2 is a group homomorphism, it suffices to treat the elements on the right hand side separately.

From (2.1.5) it follows that $((-), \tau)$ acts as the product of d disjoint permutations of the same cycle type as τ . We conclude that the sign in S_{d^2} of the image of $((-), \tau)$ under ι_2 is $\text{sgn}(\tau)^d$. This equals the wreath-product sign $\text{sgn}_2(((-), \tau)) = \text{sgn}(\tau)$ if d is odd.

Now let $(\sigma, -) = ((\sigma_1, \dots, \sigma_d), -) \in \text{Aut}(T_2)$ with $\sigma_i \in E_1 = \text{Aut}(T_1)$. This element acts as the product $\prod_i \sigma_i$, where each σ_i acts on the subtree of T_2^i , i.e., as element of $\mathcal{S}(T_2^i)$. (See (2.1.7).) We may therefore identify $(\sigma, -)$ with its image under ι_2 in S_{d^2} . We conclude that

$$\text{sgn}_2((\sigma, -)) = \prod_j \text{sgn}(\sigma_j) = \text{sgn}(\iota_2(\sigma_1, \dots, \sigma_d)).$$

This finishes the proof of Statement (2). □

Note that it follows from the proof of Lemma 2.1.3(2) that for d even the wreath-product sign (2.1.9) is not compatible with the natural sign on S_{d^2} .

2.2. A generating system for f^n

Let f be a normalized Belyi map. Recall that $f^n = f \circ \dots \circ f$ denotes the n th iterate of f , which is again a normalized Belyi map. We start by defining the Galois groups that are the central object of study in this paper. We then determine a generating system for f^n in terms of a generating system for f .

Let K be a field of definition of f . Since f is a Belyi map, we may assume that K is a number field. Moreover, since f is normalized, it follows from Proposition 1.1.3 that we may take $K = \mathbb{Q}$. Write $F_0 := K(t)$ for the function field of \mathbb{P}_K^1 . Since K is a field of definition for f^n for all n , the field K is integrally closed in the extension of function fields corresponding to the map $f^n : \mathbb{P}_K^1 \rightarrow \mathbb{P}_K^1$, which we denote by F_n/F_0 . We choose a normal closure M_n/F_0 of F_n/F_0 such that M_n contains M_{n-1} for any $n \geq 1$.

The extension of function fields $(F_n \otimes_K \overline{\mathbb{Q}}) / \overline{\mathbb{Q}}(t)$ corresponds to $f^n : \mathbb{P}_{\overline{\mathbb{Q}}}^1 \rightarrow \mathbb{P}_{\overline{\mathbb{Q}}}^1$ considered as map over the algebraic closure $\overline{\mathbb{Q}}$ of the number field K . Note that $(M_n \otimes_K \overline{\mathbb{Q}}) / \overline{\mathbb{Q}}(t)$ is a normal closure of $(F_n \otimes_K \overline{\mathbb{Q}}) / \overline{\mathbb{Q}}(t)$.

Definition 2.2.1. For arbitrary $n \geq 1$ we define

$$G_{n,\mathbb{Q}} = \text{Gal}(M_n/F_0), \quad G_{n,\overline{\mathbb{Q}}} = \text{Gal}((M_n \otimes_K \overline{\mathbb{Q}}) / \overline{\mathbb{Q}}(t)).$$

Note that $G_{1,\overline{\mathbb{Q}}} = G(f)$ as defined in Sect. 1.1.

It follows from the definitions that

$$G_{n,\overline{\mathbb{Q}}} \subseteq G_{n,\mathbb{Q}}. \quad (2.2.1)$$

In general it is not true that $G_{n,\overline{\mathbb{Q}}} = G_{n,\mathbb{Q}}$; see Remark 2.4.1. In the case that we have equality in (2.2.1) we say that *the Galois extension $(M_n \otimes_K \overline{\mathbb{Q}}) / \overline{\mathbb{Q}}(t)$ descends to $\mathbb{Q}(t)$* or in short that the group $G_{n,\overline{\mathbb{Q}}}$ descends to \mathbb{Q} .

Our convention that the normal closure M_n contains M_m for all $m \leq n$ implies that $G_{n,\mathbb{Q}}$, and hence $G_{n,\overline{\mathbb{Q}}}$, naturally has the structure of a wreath product. Identifying the sheets of f^n above a chosen base point in $\mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ with the leaves of the tree T_n yields an inclusion

$$G_{n,\mathbb{Q}} \hookrightarrow \text{Aut}(T_n) \simeq \text{Aut}(T_{n-1}) \wr \text{Aut}(T_1). \quad (2.2.2)$$

In the rest of the paper we fix this inclusion.

Our next goal is to determine a generating system $(g_{1,n}, g_{2,n}, g_{3,n})$ for f^n for all n in terms of a fixed generating system $(g_{1,1}, g_{2,1}, g_{3,1})$ for f . Since $G_{n,\overline{\mathbb{Q}}} = \langle g_{1,n}, g_{2,n} \rangle$, we can use this to determine the group $G_{n,\overline{\mathbb{Q}}}$. We refer to Theorem 2.3.1 for the precise result.

Remark 2.2.2. By the single-cycle condition, the fiber above 0 in f^n contains a unique point with ramification index e_1^n . (This is the point $x = 0$.) Additionally, for each $0 \leq i \leq n-1$ there are exactly $(d - e_1)d^{n-1-i}$ ramification points with ramification index $(e_1)^i$. (These are exactly the points in the inverse image

$f^{-n+i}(0)$ which are not in $f^{-n-1+i}(0)$.) The analogous statements with 0 replaced by 1 or ∞ and e_1 by e_2 or e_3 , respectively, also hold. This description determines the cycle type of the group elements of a generating system of f^n considered as elements of S_{d^n} . For our purposes we need more precise information, which the following proposition supplies.

A number is said to be *in the support* of a permutation if it appears in one of the cycles of the permutation, i.e., if it is not fixed by the permutation.

Proposition 2.2.3. *Let $(g_{1,1}, g_{2,1}, g_{3,1})$ be a generating system of f such that 1 is in the support of $g_{1,1}$. For any $n \geq 2$, inductively define*

$$g_{1,n} = ((g_{1,n-1}, -, \dots, -), g_{1,1});$$

$$g_{2,n} = ((-, \dots, -, g_{2,n-1}, -, \dots, -), g_{2,1}), \text{ where } g_{2,n-1} \text{ is in position } (1)g_{1,1};$$

$$g_{3,n} = ((-, \dots, -, g_{3,n-1}, -, \dots, -), g_{3,1}), \text{ where } g_{3,n-1} \text{ is in position } (1)g_{1,1}g_{2,1}.$$

Here again $-$ denotes the trivial permutation. Then $(g_{1,n}, g_{2,n}, g_{3,n})$ is a generating system for f^n .

Proof. This follows by considering the image of the sheets of the Belyi map under $f^n = f \circ f^{n-1}$, using the notation introduced in (2.1.4). \square

A generating system for f was given in Lemma 1.1.5(2).

Corollary 2.2.4. *For all $n \geq 2$ we have that*

$$G_{n,\overline{\mathbb{Q}}} \subseteq E_n.$$

Proof. We have already seen that $G_{n,\overline{\mathbb{Q}}} \subseteq \text{Aut}(T_n)$. Since $G_{n,\overline{\mathbb{Q}}}$ is generated by the $g_{j,n}$, it suffices to check that $g_{j,n} \in E_n$ for $j = 1, 2, 3$.

The inductive definition of $g_{j,n}$ given in Proposition 2.2.3 implies that $\pi_2(g_{j,n}) = g_{j,2}$ and that $\text{sgn}_2(g_{j,2}) = 1$ for $j \in \{1, 2, 3\}$. The statement for $n = 2$ follows. The statement for arbitrary $n \geq 2$ follows by induction from the definition of E_n (Definition 2.1.2). \square

Definition 2.2.5. Define

$$N_{n,\overline{\mathbb{Q}}} = \ker(\pi_{n-1} : G_{n,\overline{\mathbb{Q}}} \rightarrow G_{n-1,\overline{\mathbb{Q}}}) \subseteq G_{n,\overline{\mathbb{Q}}}.$$

Note that $(\sigma, \tau) \in G_{n,\overline{\mathbb{Q}}}$ is contained in $N_{n,\overline{\mathbb{Q}}}$ if and only if (σ, τ) fixes all vertices of the tree T_n at the levels $1, \dots, n-1$. For any $1 \leq i = 1 + \sum_{k=1}^{n-1} (\ell_k - 1)d^{k-1} \leq d^{n-1}$, the subtree T_n^i with root $(\ell_1, \dots, \ell_{n-1})$ contains exactly d leaves.

Definition 2.2.6. For $1 \leq i \leq d^{n-1}$, we define

$$N_{n,\overline{\mathbb{Q}}}^i := N_{n,\overline{\mathbb{Q}}} \cap \mathcal{S}(T_n^i).$$

The group $N_{n, \overline{\mathbb{Q}}}^i$ is naturally a permutation group on the d letters $(\ell_1, \dots, \ell_{n-1}, s)$ for $s = 1, \dots, d$; recall from (2.1.1) that $i = 1 + \sum_{k=1}^{n-1} (\ell_k - 1)d^{k-1}$. We therefore obtain an identification

$$N_{n, \overline{\mathbb{Q}}}^i \subseteq N_{n, \overline{\mathbb{Q}}}^1 \oplus \dots \oplus N_{n, \overline{\mathbb{Q}}}^{d^{n-1}} = N_{n, \overline{\mathbb{Q}}} \subseteq (S_d)^{d^{n-1}}.$$

With this identification, we may write

$$N_{n, \overline{\mathbb{Q}}} = \{(\rho^1, \dots, \rho^{d^{n-1}}) \mid \rho^i \in \mathcal{S}(T_n^i) \text{ for all } 1 \leq i \leq d^{n-1}\} \subseteq (S_d)^{d^{n-1}}. \quad (2.2.3)$$

Lemma 2.2.7. *The group $N_{n, \overline{\mathbb{Q}}}^i$ is a subgroup of A_d for all $i \in \{1, \dots, d^{n-1}\}$.*

Proof. We prove this for $i = 1$; the proof for other i is identical. First let $n = 2$. If $\sigma = (\rho^1, -, \dots, -)$ is an element of $N_{2, \overline{\mathbb{Q}}}^1$, then $\text{sgn}_2(\sigma) = \text{sgn}(\rho^1) = 1$ by Corollary 2.2.4. Now let $n \geq 2$. If $\sigma = (\rho^1, \dots, \rho^{d^{n-1}})$ is in $N_{n, \overline{\mathbb{Q}}}$, then $(\rho^1, \dots, \rho^{d^{n-2}})$ is an element of $N_{n-1, \overline{\mathbb{Q}}}$, since G_n can be identified with a subgroup of $G_{n-1} \wr G_1$ for any $n \geq 2$. Hence the proof follows by induction on n . \square

In the rest of this section, we fix a normalized Belyi map f of type $\underline{C} = (d; e_1, e_2, e_3)$. We use Proposition 2.2.3 to construct suitable elements in $N_{n, \overline{\mathbb{Q}}}$. This is a first step towards determining $G_{n, \overline{\mathbb{Q}}}$ in Theorem 2.3.1.

Lemma 2.2.8. *Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3)$ and let $(g_{1,1}, g_{2,1}, g_{3,1})$ be a generating system of f . For any $n \geq 2$ and $j \in \{1, 2, 3\}$ we define*

$$\alpha_{j,n} := (g_{j,n})^{\ell_j^{n-1}}.$$

(1) *Then*

$$\alpha_{j,n} \in N_{n, \overline{\mathbb{Q}}}.$$

(2) *Write ρ_j^i for the component of $\alpha_{j,n}$ in $\mathcal{S}(T_n^i)$ using the notation from (2.2.3). Then*

$$\rho_j^i = \begin{cases} g_{j,1} & \text{if } i = 1 + \sum_{k=1}^{n-1} (\ell_k - 1)d^{k-1} \text{ with } \ell_k \in \text{supp}(g_{j,1}) \text{ for all } k, \\ - & \text{otherwise.} \end{cases}$$

(3) *Conjugation by the elements of $G_{n, \overline{\mathbb{Q}}}$ acts transitively on $N_{n, \overline{\mathbb{Q}}}^1, \dots, N_{n, \overline{\mathbb{Q}}}^{d^{n-1}}$ for all $n \geq 2$.*

Proof. Let $i \in \{1, 2, 3\}$ and $n \geq 2$ arbitrary. Using (2.1.8) one computes that

$$\alpha_{j,n} = ((\sigma_1, \dots, \sigma_d), -), \quad \text{where } \sigma_i = \begin{cases} \alpha_{i,n-1} & \text{if } i \in \text{supp}(g_{j,1}), \\ - & \text{otherwise.} \end{cases} \quad (2.2.4)$$

Expression (2.2.4) for $\alpha_{j,n}$ implies that $\alpha_{j,n}$ fixes all vertices on level one. If $\alpha_{j,n-1} \in N_{n-1,\overline{\mathbb{Q}}}$ then $\alpha_{j,n}$ fixes all vertices of the tree on levels $2, \dots, n-1$, as well. It follows that $\alpha_{j,n} \in N_{n,\overline{\mathbb{Q}}}$. Statement (1) of the proposition is vacuous for $n = 1$. The statement therefore follows by induction on n .

Moreover, $\alpha_{j,n}$ acts as $\alpha_{j,n-1}$ on the subtree $T_n(\ell_1)$ of T_n with root (ℓ_1) if $\ell_1 \in \text{supp}(g_{j,1})$ and acts trivially otherwise. Statement (2) therefore also follows by induction on n .

Statement (3) follows by induction, as well, since $G_{1,\overline{\mathbb{Q}}} = G(f)$ is a transitive subgroup of S_d . \square

In the following proposition we exclude two types in small degree. For $\underline{C} = (6; 4, 4, 5)$ we have that $G_{1,\overline{\mathbb{Q}}} \cong S_5$ (embedded as a transitive group in S_6), hence is isomorphic to neither S_6 nor A_6 . For $\underline{C} = (4; 3, 3, 3)$ we have that $G_{1,\overline{\mathbb{Q}}} \cong A_4$. However in this case Proposition 2.2.9 fails. (In the case that $\underline{C} = (4; 3, 3, 3)$ the group $N_{n,\overline{\mathbb{Q}}}^i$ is the Klein 4-group.)

Proposition 2.2.9. *Let f be a normalized Belyi map of type \underline{C} . Assume that $\underline{C} \notin \{(6; 4, 4, 5), (4; 3, 3, 3)\}$. Then*

$$N_{n,\overline{\mathbb{Q}}}^i \cong A_d$$

for all $1 \leq i \leq d^{n-1}$ and all $n \geq 2$.

Proof. Lemma 2.2.8(3) states that $G_{n,\overline{\mathbb{Q}}}$ acts transitively on the set of subgroups $N_{n,\overline{\mathbb{Q}}}^i$ for $i = 1, \dots, d^{n-1}$. Therefore it suffices to prove the proposition for a specific value of i . We prove that $N_{n,\overline{\mathbb{Q}}}^{i_0}$ is a non-trivial normal subgroup of $G_{1,\overline{\mathbb{Q}}}$ for some value i_0 , by showing it is normal and contains a non-trivial element β_n . Since A_d is simple for $d \geq 5$ and $N_{n,\overline{\mathbb{Q}}}^i \leq A_d$ for all i by Lemma 2.2.7, it follows that $N_{n,\overline{\mathbb{Q}}}^{i_0} = A_d$. In the remaining cases $d \in \{3, 4\}$ the statement can be shown by treating each type separately.

Claim 1 Let $i_0 := 1 + \sum_{k=1}^{n-1} (e_3 - 1)d^{k-1}$. There exists a non-trivial element $\beta_n \in N_{n,\overline{\mathbb{Q}}}^{i_0}$, hence $N_{n,\overline{\mathbb{Q}}}^{i_0}$ is non-trivial. Define

$$\beta_n := [\alpha_{1,n}, [\alpha_{2,n}, \alpha_{3,n}]] = \alpha_{1,n}[\alpha_{2,n}, \alpha_{3,n}]\alpha_{1,n}^{-1}[\alpha_{2,n}, \alpha_{3,n}]^{-1}, \quad (2.2.5)$$

where $[\alpha_{2,n}, \alpha_{3,n}] = \alpha_{2,n}\alpha_{3,n}\alpha_{2,n}^{-1}\alpha_{3,n}^{-1}$ is the commutator. Recall that

$$\alpha_{j,n} = ((\sigma_1, \dots, \sigma_d), -) \in G_{n-1} \wr G_1,$$

where $\sigma_i = \alpha_{j,n-1} \in G_{n-1}$ for i in the support of $g_{j,1}$ and trivial otherwise. Since the generating system is weakly rigid, to prove the claim it suffices to prove that the element β_n is a non-trivial element in $N_n^{i_0}$ for the generating system of Lemma 1.1.5(2). We use this generating system for the rest of this proof.

With this choice we have

$$\text{supp}(g_1) \cap \text{supp}(g_2) \cap \text{supp}(g_3) = \{e_3\}.$$

By induction we find that the component ρ^i of β_n in $N_{n, \overline{\mathbb{Q}}}^i$ satisfies

$$\rho^i = \begin{cases} [g_1, [g_2, g_3]] & \text{if } i = i_0, \\ - & \text{otherwise.} \end{cases}$$

Hence $\beta_n \in N_{n, \overline{\mathbb{Q}}}^{i_0}$ as claimed.

To show that β_n is non-trivial, it suffices to show that $[g_{1,1}, [g_{2,1}, g_{3,1}]]$ is non-trivial. This is an explicit calculation using the generating system in Lemma 1.1.5(2). For instance, one checks that e_3 is not sent to itself. We conclude that β_n is non-trivial, and Claim 1 follows.

Claim 2 $N_{n, \overline{\mathbb{Q}}}^{i_0}$ is a normal subgroup of A_d .

By Lemma 2.2.7, $N_{n, \overline{\mathbb{Q}}}^{i_0}$ is a subgroup of A_d . It follows from (2.1.8) that the conjugates of β_n by the elements $\alpha_{j,n} \in N_{n, \overline{\mathbb{Q}}}$ are also in $N_{n, \overline{\mathbb{Q}}}^{i_0}$ for $j \in \{1, 2, 3\}$. The group $G_{1, \overline{\mathbb{Q}}}$ is generated by the $g_{j,1}$. Since $g_{j,1}$ is the component of $\alpha_{j,n}$ in $N_{n, \overline{\mathbb{Q}}}^{i_0}$ for $j \in \{1, 2, 3\}$, the group $N_{n, \overline{\mathbb{Q}}}^{i_0}$ contains the element $\sigma^{-1} \rho^{i_0} \sigma$ for all $\sigma \in G_{1, \overline{\mathbb{Q}}}$. Since $A_d \subseteq G_{1, \overline{\mathbb{Q}}}$ we conclude that $N_{n, \overline{\mathbb{Q}}}^{i_0}$ is a normal subgroup of A_d . This proves Claim 2.

As explained in the beginning of the proof, the statement for $d \geq 5$ follows from Claims 1 and 2. The remaining cases can be checked separately. \square

2.3. Determination of $G_{n, \overline{\mathbb{Q}}}$ for normalized Belyi maps

Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3)$. In this section, we completely determine the group structure of the groups $G_{n, \overline{\mathbb{Q}}}$ (Definition 2.2.1) as a subgroup of $\text{Aut}(T_n)$. We refer to Lemma 1.1.5 for the description of $G_{1, \overline{\mathbb{Q}}}$.

Theorem 2.3.1. *Let f be a normalized Belyi map of type $\underline{C} = (d; e_1, e_2, e_3) \notin \{(4; 3, 3, 3), (6; 4, 4, 5)\}$.*

(1) *Assume that $G_{1, \overline{\mathbb{Q}}} \simeq S_d$. Then*

$$G_{n, \overline{\mathbb{Q}}} \simeq E_n.$$

(2) *Assume that $G_{1, \overline{\mathbb{Q}}} \simeq A_d$, i.e., that all e_j are odd. Then $G_{n, \overline{\mathbb{Q}}}$ is the n -fold iterated wreath product of A_d with itself.*

The key step in the proof of Theorem 2.3.1 is determining the size of the group $G_{n, \overline{\mathbb{Q}}}$.

Lemma 2.3.2. *Let f be a normalized Belyi map. Fix an integer $n \geq 2$ and assume that $E_{n-1} \simeq G_{n-1, \overline{\mathbb{Q}}}$. Then*

$$[(S_d)^{d^{n-1}} : N_{n, \overline{\mathbb{Q}}}] = 2^{d^{n-2}}.$$

Note that the assumption that $E_{n-1} \simeq G_{n-1, \overline{\mathbb{Q}}}$ for $n = 1$ states that $G_{1, \overline{\mathbb{Q}}} = E_1 \simeq S_d$.

Proof. Let χ denote the following homomorphism, induced by the sgn function on each of the components $N_{n,\overline{\mathbb{Q}}}^i$:

$$\chi : N_{n,\overline{\mathbb{Q}}} \rightarrow \mathbb{F}_2^{d^{n-1}}, \quad (\rho^i)_{1 \leq i \leq d^{n-1}} \mapsto (\text{dlog}_{-1}(\text{sgn}(\rho^i)))_{1 \leq i \leq d^{n-1}},$$

where the discrete logarithm dlog_{-1} sends $1 \mapsto 0$ and $-1 \mapsto 1$. Proposition 2.2.9 implies that the kernel of χ equals $N_{n,\overline{\mathbb{Q}}}^1 \oplus \cdots \oplus N_{n,\overline{\mathbb{Q}}}^{d^{n-1}} \simeq (A_d)^{d^{n-1}}$. Therefore, χ induces an injection

$$\overline{\chi} : \overline{N}_{n,\overline{\mathbb{Q}}} := N_{n,\overline{\mathbb{Q}}}/\left(\bigoplus_{1 \leq i \leq d^{n-1}} N_{n,\overline{\mathbb{Q}}}^i\right) \hookrightarrow \mathbb{F}_2^{d^{n-1}}.$$

For the remainder of the proof we identify $\overline{N}_{n,\overline{\mathbb{Q}}}$ with its image in $\mathbb{F}_2^{d^{n-1}}$.

Since $e_1 + e_2 + e_3 = 2d + 1$ is odd and we assume that $G_{1,\overline{\mathbb{Q}}} \simeq S_d$, at least one of the e_j is even. It follows that there is a unique j such that e_j is odd. Let $s \in \{1, 2, 3\}$ be one of the indices such that e_s is even. Since at most one of the e_j equals d , we may assume that $e_s \neq d$.

Lemma 2.2.8(2) implies that exactly e_s^{n-1} entries of the element $\alpha_{s,n} \in N_{n,\overline{\mathbb{Q}}}$ are e_s -cycles, and all other entries are trivial. It follows that exactly e_s^{n-1} entries of the corresponding element $\overline{\alpha}_{s,n} \in \overline{N}_{n,\overline{\mathbb{Q}}}$ are non-trivial, i.e., equal to 1. Moreover, dividing the indices once more into d^{n-2} blocks of d indices, where the i th block corresponds to the vertices of T_{n-1}^i , we see that, for each $\overline{\alpha}_{s,n}$, exactly e_s^{n-2} of these blocks contain exactly e_s non-trivial entries. In this proof we denote the i th block by $B(i)$. Our choice of numbering implies that

$$B(i) = \{1 + (i-1)d, 2 + (i-1)d, \dots, d + (i-1)d\},$$

but we do not need this in what follows. A vector $x \in \mathbb{F}_2^{d^{n-2}}$ with exactly e_s^{n-1} non-trivial entries distributed in this way among the blocks is said to *satisfy the block condition*.

Equation (2.1.8) implies that conjugation by $G_{n,\overline{\mathbb{Q}}}$ yields a well-defined action on $\overline{N}_{n,\overline{\mathbb{Q}}}$, by permutation of the coordinates of $x = (x_1, \dots, x_{d^{n-1}}) \in \overline{N}_{n,\overline{\mathbb{Q}}}$ respecting the block structure. The assumption that $G_{n-1,\overline{\mathbb{Q}}} \simeq E_{n-1}$, together with the observation that the projection $G_{n,\overline{\mathbb{Q}}} \rightarrow G_{n-1,\overline{\mathbb{Q}}}$ is surjective, implies that the orbit of $\overline{\alpha}_{s,n}$ under this action consists of all $x \in \mathbb{F}_2^{d^{n-1}}$ that have exactly e_s^{n-1} entries equal to 1 and satisfy the block condition.

For every subset $\mathcal{S} \subseteq \{1, 2, \dots, d^{n-1}\}$ with $|\mathcal{S}| = e_s^{n-1}$, we denote by $\xi_{\mathcal{S}}$ the element of $\mathbb{F}_2^{d^{n-1}}$ such that $(\xi_{\mathcal{S}})_i = 1$ if $i \in \mathcal{S}$, and $(\xi_{\mathcal{S}})_i = 0$ otherwise. With this notation, the elements in the orbit of $\overline{\alpha}_s$ can be described as $\xi_{\mathcal{S}}$ for some \mathcal{S} satisfying the following properties:

- (1) For $1 \leq i \leq d^{n-2}$, let $\mathcal{S}(i) := \mathcal{S} \cap B(i)$. Then $\mathcal{S}(i) \neq \emptyset$ for exactly e_s^{n-2} values of i , and
- (2) if $\mathcal{S}(i) \neq \emptyset$ for some i , then $|\mathcal{S}(i)| = e_s$.

Let $\mathcal{H} = \bigcap_{i=1}^{d^{n-2}} \mathcal{H}_i$ denote the intersection of all hyperplanes

$$\mathcal{H}_i := \left\{ x \in \mathbb{F}_2^{d^{n-1}} \mid \sum_{b \in B(i)} x_b = 0 \right\}.$$

Claim The elements $\xi_{\mathcal{J}}$ for \mathcal{J} satisfying properties (1) and (2) above generate \mathcal{H} .

Fix k, ℓ and i such that $1 \leq k < \ell \leq d$ and $1 \leq j \leq d^{n-2}$. Let $\xi_{k,\ell;j}$ denote the vector whose entries are 1 in the positions $k + (j-1)d$ and $\ell + (j-1)d$ and 0 otherwise. Note that $\xi_{k,\ell;j} \in \mathcal{H}$.

To prove the claim it suffices to show that for all $1 \leq j \leq d^{n-2}$ and for all $1 \leq k < \ell \leq d$, the vectors $\xi_{k,\ell;j}$ are in the linear hull of the $\xi_{\mathcal{J}}$. Using that $G_{n,\overline{\mathbb{Q}}}$ acts transitively on the blocks, it suffices to prove this for $j = 1$. Since $N_{n,\overline{\mathbb{Q}}}^1 \subseteq G_{n,\overline{\mathbb{Q}}}$ acts transitively on $B(1)$ by Proposition 2.2.9, we may moreover assume that $(k, \ell) = (1, 2)$. In other words, it suffices to show that $\xi_{1,2;1} \in \mathcal{H}$.

Define

$$\mathcal{J}(1) = \{1, \dots, e_{s+1}\} - \{2\} \quad \text{and} \quad \mathcal{J}'(1) = \{1, \dots, e_{s+1}\} - \{1\}.$$

For any $2 \leq i \leq e_s^{n-2}$, we define the set $\mathcal{J}(i) = \{b + (i-1)d \mid 1 \leq b \leq e_s\}$. Then letting $\mathcal{J} = (\bigcup_{i=2}^{e_s^{n-2}} \mathcal{J}(i)) \cup \mathcal{J}(1)$ and $\mathcal{J}' = (\bigcup_{i=2}^{e_s^{n-2}} \mathcal{J}'(i)) \cup \mathcal{J}'(1)$, we see that $\xi_{1,2;1} = \xi_{\mathcal{J}} + \xi_{\mathcal{J}'} \in \mathcal{H}$. This proves the claim.

The claim inductively implies that

$$[(S_d)^{d^{n-1}} : N_{n,\overline{\mathbb{Q}}}] = \frac{2^{d^{n-1}}}{|N_{n,\overline{\mathbb{Q}}}|} = 2^{d^{n-2}}.$$

This proves the lemma. □

Now we are ready to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. In Corollary 2.2.4 we have shown that $G_{n,\overline{\mathbb{Q}}}$ is a subgroup of E_n for all $n \geq 1$.

Case 1 Assume that $G_{1,\overline{\mathbb{Q}}} = E_1 = \text{Aut}(T_1) \simeq S_d$.

To prove Statement (1), it suffices to show that the groups $G_{n,\overline{\mathbb{Q}}}$ and E_n have the same cardinality. Since $G_{n,\overline{\mathbb{Q}}}$ is a subgroup of $\text{Aut}(T_n)$ for all $n \geq 1$, the definition of $N_{n,\overline{\mathbb{Q}}}$ implies that

$$\frac{[\text{Aut}(T_n) : G_{n,\overline{\mathbb{Q}}}]}{[\text{Aut}(T_{n-1}) : G_{n-1,\overline{\mathbb{Q}}}]} = [(S_d)^{d^{n-1}} : N_{n,\overline{\mathbb{Q}}}] = 2^{d^{n-2}}.$$

The last equality is the statement of Lemma 2.3.2. The expression for $[\text{Aut}(T_n) : E_n]$ in Lemma 2.1.3(1) implies that

$$\frac{[\text{Aut}(T_n) : E_n]}{[\text{Aut}(T_{n-1}) : E_{n-1}]} = 2^{d^{n-2}},$$

as well. It follows that $|G_{n,\overline{\mathbb{Q}}}| = |E_n|$. This proves the theorem in the first case.

Case 2 Assume that $G_{1, \overline{\mathbb{Q}}} \simeq A_d$.

In this case $G_{n, \overline{\mathbb{Q}}}$ is a subgroup of the n -fold iterated wreath product of A_d with itself. The statement of the theorem in this case therefore follows by induction on n using Proposition 2.2.9.

This finishes the proof of the theorem. \square

2.4. Descent

In this section we determine the groups $G_{n, \mathbb{Q}}$ (Definition 2.2.1) under certain conditions. Recall from (2.2.1) that

$$G_{n, \overline{\mathbb{Q}}} \subseteq G_{n, \mathbb{Q}} \subseteq \text{Aut}(T_n), \quad \text{for all } n \geq 1.$$

Theorem 2.3.1 implies that

$$[\text{Aut}(T_n) : G_{n, \overline{\mathbb{Q}}}] = \begin{cases} 2^{d^{n-2} + d^{n-3} + \dots + d + 1} & \text{if } G_{1, \overline{\mathbb{Q}}} \simeq S_d, \\ 2^{d^{n-1} + d^{n-2} + \dots + d + 1} & \text{if } G_{1, \overline{\mathbb{Q}}} \simeq A_d \text{ and } \underline{C} \neq (4; 3, 3, 3). \end{cases}$$

If we can show for some normalized Belyi map f that $G_{n, \overline{\mathbb{Q}}} = G_{n, \mathbb{Q}}$ for all $n \geq 1$, then we have found an explicit description of $G_{\infty, \mathbb{Q}} := \varprojlim_n G_{n, \mathbb{Q}}$, and we have shown that this group satisfies $[\text{Aut}(T_{\infty}) : G_{\infty, \mathbb{Q}}] = \infty$.

Remark 2.4.1. It is not true in general that $G_{n, \overline{\mathbb{Q}}} = G_{n, \mathbb{Q}}$. In fact, this may already fail for $n = 1$ when $G_{1, \overline{\mathbb{Q}}} \subsetneq S_d$. Consider a normalized Belyi map f of type $(d; e_1, e_2, e_3)$ with all e_i odd. Recall from Lemma 1.1.5(1) that then $G_{1, \overline{\mathbb{Q}}} \simeq A_d$. It follows that

$$A_d \simeq G_{1, \overline{\mathbb{Q}}} \subseteq G_{1, \mathbb{Q}} \subseteq \text{Aut}(T_1) \simeq S_d.$$

Using the notation from Sect. 1.2, we have that $G_{1, \mathbb{Q}} \simeq A_d$ if and only if the discriminant $\Delta(f - t)$ of $x^{e_1} f_1 - t f_2$, considered as polynomial in x over $\mathbb{Q}(t)$, is a square in $\mathbb{Q}(t)$, and $G_{1, \mathbb{Q}} \simeq S_d$ otherwise. For simplicity, we restrict to the case that f is a polynomial, i.e., that $e_3 = d$. In this case, a formula for the discriminant of f is given in [1, Proposition 3.1]:

$$\Delta(f - t) = (-1)^{(d-1)(d-2)/2} d^d \ell(f)^{d-1} t^{e_1-1} (t-1)^{e_2-1},$$

where $\ell(f)$ denotes the leading coefficient of f . Proposition 1.1.4(1) gives an explicit expression for $\ell(f)$. The formula for the discriminant, together with the expression for $\ell(f)$, implies that if f is a polynomial and d is odd, then $\Delta(f - t)$ is never a square in $\mathbb{Q}(t)$. We note that a formula for $\Delta(f - t)$ in the non-polynomial case may be deduced from [4, Proposition 1].

The structure of $\text{Aut}(T_n)$ as an iterated wreath product implies that the case $n = 2$ plays a key role in determining $G_{\infty, \mathbb{Q}}$. Recall that f^n denotes the n th iterate of f .

Proposition 2.4.2. *Let f be a normalized Belyi map of type $\underline{C} = (d; e_1, e_2, e_3)$. If $G_{2, \mathbb{Q}} = G_{2, \overline{\mathbb{Q}}}$, then $G_{n, \mathbb{Q}} = G_{n, \overline{\mathbb{Q}}}$ for all $n \geq 2$.*

Proof. We first assume that $G_{1, \overline{\mathbb{Q}}} = S_d$. We prove the statement by induction on $n \geq 2$. Assume that the statement holds for $n - 1$, i.e., we have $G_{n-1, \mathbb{Q}} = G_{n-1, \overline{\mathbb{Q}}}$. Theorem 2.3.1(1) implies that

$$E_n = G_{n, \overline{\mathbb{Q}}} \subseteq G_{n, \mathbb{Q}} \subseteq \text{Aut}(T_n).$$

The induction hypothesis implies that

$$G_{n, \overline{\mathbb{Q}}} \subseteq G_{n, \mathbb{Q}} \subseteq G_{n-1, \mathbb{Q}} \wr G_{1, \mathbb{Q}} = G_{n-1, \overline{\mathbb{Q}}} \wr G_{1, \overline{\mathbb{Q}}} = E_{n-1} \wr E_1.$$

The second inclusion follows from the structure of $G_{n, \mathbb{Q}}$ as a wreath product induced by the decomposition $f^n = f \circ f^{n-1}$. Lemma 2.1.3(1) implies that $[E_{n-1} \wr E_1 : E_n] = 2$. We conclude that $G_{n, \mathbb{Q}}$ equals either $G_{n, \overline{\mathbb{Q}}} = E_n$ or $E_{n-1} \wr E_1$. The definition of the wreath-product sign (Definition 2.1.1) implies that we can distinguish the two possible groups by considering their images under π_2 . Since $\pi_2(G_{n, \mathbb{Q}}) = G_{2, \mathbb{Q}}$ and $\pi_2(G_{n, \overline{\mathbb{Q}}}) = G_{2, \overline{\mathbb{Q}}}$, the result follows in this case.

Now assume that $G_{1, \overline{\mathbb{Q}}} = A_d$. Theorem 2.3.1(2) implies that $G_{n, \overline{\mathbb{Q}}}$ is the n -fold iterated wreath product of A_d with itself. We denote this group by E_n^+ . As in the previous case, we conclude from the induction hypothesis that

$$G_{n, \overline{\mathbb{Q}}} \subseteq G_{n, \mathbb{Q}} \subseteq G_{n-1, \mathbb{Q}} \wr G_{1, \mathbb{Q}} = G_{n-1, \overline{\mathbb{Q}}} \wr G_{1, \overline{\mathbb{Q}}} = E_{n-1}^+ \wr E_1^+.$$

Note that the wreath product sign restricted to E_n^+ is trivial. The statement follows as in the first case. \square

We write $f(x) = g(x)/h(x)$, where $g, h \in \mathbb{Z}[x]$ are relatively prime as polynomials in $\mathbb{Z}[x]$. We define the discriminant of the rational map $f(x) - t$ as the discriminant of the polynomial $g(x) - th(x)$, viewed as a polynomial in x over $\mathbb{Q}(t)$. In other words, $\Delta(f - t)$ is in $\mathbb{Q}[t]$ and

$$\Delta(f - t) := \Delta(g(x) - th(x)).$$

Let L be the splitting field of $g(x) - th(x)$ over $\mathbb{Q}(t)$ and let t_i for $i = 1, \dots, d$ be the roots of $g(x) - th(x)$ in L .

Lemma 2.4.3. *Let $f(x)$ be a rational map in $\mathbb{Q}(x)$. The Galois group $G_{2, \mathbb{Q}}$ (attached to f) is a subgroup of $\ker(\text{sgn}_2)$ if and only if the product $\Delta(f(x) - t) \prod_i \Delta(f(x) - t_i)$ is a square in $\mathbb{Q}(t)$.*

Proof. Let $\sigma = ((\sigma_1, \dots, \sigma_k), \tau)$ be in $G_{2, \mathbb{Q}} \leq \text{Aut}(T_2)$ and let $D = \Delta(f(x) - t) \prod_i \Delta(f(x) - t_i)$. Then we will show that $\sigma(\sqrt{D}) = \text{sgn}_2(\sigma)\sqrt{D}$.

Let t_{ij} , for $1 \leq j \leq d$, denote the roots of $f(x) - t_i$ for all i . We can identify t_{ij} with the vertex (i, j) of the tree T_2 . By (2.1.5) the action of σ on t_{ij} is given by

$$\sigma(t_{ij}) = t_{\tau(i)\sigma_i(j)}.$$

We note here that, for $i = 1, \dots, d$, $\sqrt{\Delta(f - t_i)}$ lives in some quadratic extension of L . Hence

$$\sigma(\sqrt{\Delta(f - t_i)}) = \sigma \left(\prod_{j < k} (t_{ij} - t_{ik}) \right) = \prod_{j < k} (t_{\tau(i)\sigma_i(j)} - t_{\tau(i)\sigma_i(k)})$$

$$= \operatorname{sgn}(\sigma_i) \sqrt{\Delta(f - t_{\tau(i)})}.$$

Similarly, $\sigma(\sqrt{\Delta(f - t)}) = \operatorname{sgn}(\tau) \sqrt{\Delta(f - t)}$ and hence

$$\sigma(\sqrt{D}) = \operatorname{sgn}_2(\sigma) \sqrt{D}.$$

This concludes the proof. \square

Corollary 2.4.4. *Let f be a normalized Belyi map of type $\underline{C} = (d; e_1, e_2, e_3)$. Assume that $G_{1, \overline{\mathbb{Q}}} = S_d$. Then $G_{n, \overline{\mathbb{Q}}} = G_{n, \overline{\mathbb{Q}}}$ for all $n \geq 1$ if and only if the product $\Delta(f(x) - t) \prod_i \Delta(f(x) - t_i)$ is a square in $\mathbb{Q}(t)$.*

Next, we will compute the discriminant product in Lemma 2.4.3 for a dynamical Belyi map of type $\underline{C} = (d; e_1, e_2, e_3)$. Let us write $\Delta(f(x) - t) = a(f)t^{e_1-1}(1-t)^{e_2-1}$ with $a(f) \in \mathbb{Q}$.

Proposition 2.4.5. *Let $f(x) = g(x)/h(x)$ be a normalized Belyi map of type $\underline{C} = (d; e_1, e_2, e_3)$. Then*

$$\Delta(f(x) - t) \prod_i \Delta(f(x) - t_i) = u(1-t)^{2(e_2-1)} t^{2(e_1-1)},$$

where $u = (-1)^{(d+1)(e_1-1)} a(f)^{d+1} h(0)^{e_1-1} g(1)^{e_2-1} / \ell(g)^{e_1+e_2-2}$.

Proof. For $f(x) = g(x)/h(x)$ as above we have $\Delta(g(x) - t_i h(x)) = a(f)t_i^{e_1-1}(1-t_i)^{e_2-1}$ and hence

$$\prod_i \Delta(g(x) - t_i h(x)) = a(f)^d \prod_{i=1}^d (t_i)^{e_1-1} \prod_{i=1}^d (1-t_i)^{e_2-1}.$$

Using the fact that $g(x) - th(x) = \ell(g) \prod_{i=1}^d (x - t_i)$, substituting $x = 0$ and $x = 1$, we compute that

$$\prod_{i=1}^d t_i = \frac{(-1)^{d+1} h(0)}{\ell(g)} t \quad \text{and} \quad \prod_{i=1}^d (1-t_i) = \frac{g(1)}{\ell(g)} (1-t).$$

Therefore we find that

$$\begin{aligned} & \Delta(f(x) - t) \prod_i \Delta(f(x) - t_i) \\ &= a(f)^{d+1} t^{e_1-1} (1-t)^{e_2-1} \prod_{i=1}^d (t_i)^{e_1-1} \prod_{i=1}^d (1-t_i)^{e_2-1} \\ &= a(f)^{d+1} t^{e_1-1} (1-t)^{e_2-1} \left(\frac{(-1)^{d+1} h(0)}{\ell(g)} t \right)^{e_1-1} \left(\frac{g(1)}{\ell(g)} (1-t) \right)^{e_2-1}, \end{aligned}$$

as claimed. \square

Corollary 2.4.6. (1) *Let d be odd.*

- (a) Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3 = d)$ such that e_1 or e_2 is even. Then f is a polynomial such that $G_{1, \overline{\mathbb{Q}}} = S_d$ and $G_{n, \mathbb{Q}} = G_{n, \overline{\mathbb{Q}}}$ for all $n \geq 1$.
- (b) Let f be a normalized Belyi map of type $(d; d - k, 2k + 1, d - k)$ with k odd. Then $G_{1, \overline{\mathbb{Q}}} = S_d$ and $G_{n, \mathbb{Q}} = G_{n, \overline{\mathbb{Q}}}$ for all $n \geq 1$.
- (2) Let f be a normalized Belyi map such that $G_{1, \overline{\mathbb{Q}}} = G_{1, \mathbb{Q}} = A_d$. Then $G_{n, \mathbb{Q}} = G_{n, \overline{\mathbb{Q}}}$ for all $n \geq 1$.

Proof. (1) Proposition 2.4.5, together with the explicit expression for f in Proposition 1.1.4 and the assumption that d is odd, implies in both cases that $\Delta(f(x) - t) \prod_i \Delta(f(x) - t_i)$ is a square. (In the situation of Statement (a) one has that $h(0) = g(a) = 1$. In the situation of Statement (b) one has that $h(0) = \ell(g) = a_0$.)

The statement follows therefore from Corollary 2.4.4.

- (2) We have that $G_{n, \mathbb{Q}}$ is a subgroup of the n -fold iterated wreath product of A_d with itself, since it can be identified with a subgroup of $G_{n-1, \mathbb{Q}} \wr G_1$. Hence the result follows from Theorem 2.3.1 and the fact that $G_{n, \overline{\mathbb{Q}}} \subset G_{n, \mathbb{Q}}$. \square

Remark 2.4.7. (1) One may treat the case that d is even using the formula in Proposition 2.4.5, though the statements are not quite as nice in this case. For instance, let $f(x) = -(d-1)x^d + x^{d-1}$ for $d \geq 4$. Note that f has combinatorial type $(d; d-1, 2, d)$. Assume that d is even and $G_{1, \overline{\mathbb{Q}}} = S_d$. Then we find that $G_{2, \mathbb{Q}} = G_{2, \overline{\mathbb{Q}}}$ if and only if d is divisible by 4.

(2) Assume that $G_{1, \overline{\mathbb{Q}}} = A_d$ and that $G_{1, \mathbb{Q}} = S_d$. By Proposition 2.4.5 and Lemma 2.4.3, the group $G_{2, \mathbb{Q}}$ is contained in $\ker(\text{sgn}_2)$ if and only if $a(f)^{d+1}$ is a square in \mathbb{Q} . Since $G_{1, \mathbb{Q}} = S_d$, we know that $a(f)$ is not a square. It follows that $G_{2, \mathbb{Q}}$ is contained in $\ker(\text{sgn}_2)$ if and only if d is odd.

3. Specialization

In this section we prove some explicit results on the specialization of normalized Belyi maps f . For any $n \geq 1$, Hilbert's Irreducibility Theorem implies that there exists a non-empty Zariski-open set $\mathcal{H}_n = \mathcal{H}_n(f) \subseteq \mathbb{P}^1(\mathbb{Q})$, called a *Hilbert set*, such that specializing the parameter t to $a \in \mathcal{H}_n$ does not change the Galois group. In this section we determine explicit elements $a \in \mathcal{H}_n(f)$ for all n . This means that we get an explicit tower of number fields $(K_{n,a})_{n \geq 1}$ with prescribed Galois groups (Definition 3.1.1) by specializing to these values of a .

These elements are determined by Conditions 3.1.5. Conditions 3.1.5 can be thought of as the analogue of Conditions (\dagger) of [3], and Proposition 3.1.3 as the analogue of [3, Proposition 3.4].

3.1. Irreducibility and ramification conditions

Throughout this section, we fix a normalized Belyi map f of type $\underline{C} := (d; e_1, e_2, e_3)$. Recall that we write f both for the rational function $f(x) \in \mathbb{Q}(x)$

and the corresponding map $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1 : x \mapsto t = f(x)$. As in the beginning of Sect. 1.2 we write $f(x) = x^{e_1} f_1(x)/f_2(x)$ and assume that the polynomials f_i satisfy (1)–(3) introduced there.

Recall that for any $n \geq 1$ we write $F_n/\mathbb{Q}(t)$ for the extension of function fields corresponding to the map $f^n : \mathbb{P}_{\mathbb{Q}}^1 \rightarrow \mathbb{P}_{\mathbb{Q}}^1$ and that $G_{n,\mathbb{Q}}$ denotes the Galois group of the Galois closure of this extension (Definition 2.2.1).

Definition 3.1.1. Let $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ such that the numerator $f(n, a)$ of $f^n - a$ is irreducible for all $n \geq 1$ and define $K_{n,a}$ as the extension of $K_{0,a} := \mathbb{Q}$ obtained by adjoining a root of $f(n, a)$. We denote by $G_{n,a}$ the Galois group of the normal closure of $K_{n,a}/K_{0,a}$.

Proposition 3.1.2. Let f be a normalized Belyi map of type $\underline{C} := (d; e_1, e_2, e_3)$. Assume f has good monomial reduction at p for some prime p . Choose $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ with $v_p(a) = 1$. Then

$$[K_{n,a} : \mathbb{Q}] = d^n \quad \text{for all } n \geq 1.$$

In particular, $G_{n,a}$ is a transitive subgroup of S_{d^n} for all $n \geq 1$.

Recall that an explicit criterion for good monomial reduction was given in Proposition 1.2.4(2).

Proof. Write $f(x) = x^{e_1} f_1(x)/f_2(x)$ with $f_1(x) = \sum_{i=0}^{d-e_1} a_i x^i$ and $f_2(x) = \sum_{i=0}^{d-e_3} b_i x^i$ both in $\mathbb{Z}[x]$. Since f satisfies (1)–(3) from Sect. 1.2 and we assume it has good monomial reduction at p , we have that

- $v_p(a_i) = v_p(b_j) = 0$ for $i = d - e_1$ and $j = 0$,
- $v_p(a_i) > 0$ for $i \neq d - e_1$,
- $v_p(b_j) > 0$ for $j \neq 0$.

We conclude that the numerator of

$$f_a(x) := f(x) - a = \frac{x^{e_1} f_1(x) - a f_2(x)}{f_2(x)}$$

is an Eisenstein polynomial for p , hence irreducible. Here we have also used that $f(0) = 0$ and $v_p(a) = 1$. Moreover, the numerator and the denominator of $f_a(x)$ are relatively prime. The statement follows for $n = 1$.

Similarly, for $n > 1$ arbitrary we find

$$f^n(x) = \frac{a_{d-e_1}^{1+d+\dots+d^n} x^{d^n} + p \cdot (\text{terms of degree } < d^n)}{p \cdot (\text{terms of degree } \geq 1) + b_0^{1+d+\dots+d^n}}.$$

The same argument as for $n = 1$ therefore also applies to the case of arbitrary n . \square

For $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ such that the numerator of $f^n - a$ is irreducible, i.e., such that $[K_{n,a} : \mathbb{Q}] = [F_n : \mathbb{Q}(t)] = d^n$, there exists an isomorphism between the field extensions $(K_{n,a} \otimes_{\mathbb{Q}} \mathbb{Q}(t))/\mathbb{Q}(t)$ and $F_n/\mathbb{Q}(t)$. This isomorphism induces an inclusion of Galois groups

$$G_{n,a} \hookrightarrow G_{n,\mathbb{Q}}. \tag{3.1.1}$$

In the rest of this section we fix these inclusions for all n . For more details we refer to [16, Sections 1.1.1 and 1.1.2].

Proposition 3.1.3 below provides partial information on the ramification of $K_{n,a}/\mathbb{Q}$ for suitable choices of a . Here we use the reduction of f at a prime q for which f has good separable reduction. In Proposition 1.2.4(1) we showed that this holds if $q > d$. The idea of the proof of Proposition 3.1.3 is similar to that of Proposition 3.1.2. What we show is that if $a \equiv 0 \pmod{q}$, then the ramification of q in $K_{n,a}/\mathbb{Q}$ is the same as the ramification above $t = 0$ in the iteration $f^n : \mathbb{P}^1 \rightarrow \mathbb{P}^1$, which was described in Remark 2.2.2. We give similar statements for the ramification above the other branch points $t = 1, \infty$. From this, we deduce the existence of concrete elements in $G_{n,a}$; see Lemma 3.1.8 below for the precise statement.

Proposition 3.1.3. *Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3)$ and let q be a prime of good separable reduction for f . Let $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ such that $[K_{n,a} : \mathbb{Q}] = d^n$ for all $n \geq 1$. Assume additionally that $v_q(a) > 0$.*

Then there is a unique sequence $\{q_n\}_{n \geq 0}$ such that $q_0 = q$ and q_n is a prime ideal of $K_{n,a}$ lying above q_{n-1} , satisfying the following properties: the ramification index of $q_n \in K_{n,a}/K_{n-1,a}$ is e_1 , and all other primes of $K_{n,a}$ lying above q are unramified in $K_{n,a}/K_{n-1,a}$.

The analogous statement with e_1 replaced by e_2 (resp. e_3) holds if we require $v_q(1-a) > 0$ (resp. $v_q(a) < 0$) instead.

Proof. We only prove the statement in the case that $v_q(a) > 0$; the other two cases are similar.

The assumption that f has good separable reduction at q implies that this reduction satisfies

$$\bar{f} = \frac{x^{e_1} \bar{f}_1}{\bar{f}_2},$$

where \bar{f}_1 and \bar{f}_2 are separable polynomials of degree $d - e_1$ and $d - e_3$, respectively, which are relatively prime. The ramification points of f are exactly $0, 1, \infty$. Since the reduction \bar{f} of f at q is assumed to be separable, \bar{f} is exactly ramified at $0, 1, \infty$. Proposition 1.2.2(2) implies that \bar{f} is also of type $(d; e_1, e_2, e_3)$. It follows that $\bar{f}_1(0) \neq 0$ and $\bar{f}_2(0) \neq 0$. We conclude that there is a unique prime q_1 of $K_{1,a}$ above $q_0 := q$ that is ramified. Moreover, the ramification index of this prime is e_1 .

The description of the ramification of the map \bar{f} implies that the Newton polygon of $x^{e_1} \bar{f}_1 - a \bar{f}_2$ has two q -adic slopes: a slope 0 with multiplicity $d - e_1$ and a slope $-v_q(a)/e_1$ with multiplicity e_1 . Here we use that $e_1 > d - e_3 = \deg(\bar{f}_2)$ and that the leading and the constant coefficients of \bar{f}_1 and \bar{f}_2 are both q -adic units.

The choice of q_1 implies that there exists a root $a_1 \in K_{1,a}$ of $f(x) - a$ that satisfies $v_{q_1}(a_1) = v_q(a) > 0$. We conclude that the prime q_1 and the rational function $f(x) - a_1 \in K_{1,a}(x)$ satisfy the hypothesis of the proposition, as well. By induction, we conclude that there exists a unique ramified prime q_n in $K_{n,a}/K_{n-1,a}$. Moreover, since we assume that $[K_{n,a} : K_{n-1,a}] = d$, the prime q_n lies above the prime q_{n-1} , and its ramification index is e_1 .

Let q'_1 of $K_{1,a}$ be an unramified prime above q_0 . In this case it follows that $v_{q'_1}(a_1) = 0$. We conclude that all primes above q'_1 in $K_{2,a}$ are unramified in $K_{2,a}/K_{1,a}$. By induction, we conclude that if a prime q'_n above q is unramified in $K_{n,a}/K_{n-1,a}$ then all primes above it in the tower of number fields are unramified. This concludes the proof. \square

Remark 3.1.4. In the proof of Proposition 3.1.3 we use that fact that if f has good separable reduction at a prime q then the reduction \bar{f} has the same type as the map f . This property is very specific to the case of normalized single-cycle genus-0 Belyi maps and does not hold without these assumptions on the map. We give an example to illustrate this.

Let $p > 3$ be a prime. The map $f(x) = x^2(x - p)$ has type $(3; 2, 2, 3)$, but it is not normalized in the sense of Definition 1.1.2(2). The branch points are also not normalized to $0, 1, \infty$. The two ramification points with ramification index 2 are 0 and p , which specialize to the same point modulo p , and this point has ramification index $3 > 2$. In our very special setting this cannot happen.

Propositions 3.1.2 and 3.1.3 suggest the following conditions.

Conditions 3.1.5. Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3)$. Choose $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ and distinct primes p, q_1, q_2, q_3 such that the following hold: f has good monomial reduction at p and good separable reduction at q_1, q_2, q_3 , and we have

$$v_p(a) = 1, \quad v_{q_1}(a) > 0, \quad v_{q_2}(1 - a) > 0, \quad v_{q_3}(a) < 0.$$

Remark 3.1.6. The results in this paper can be used to construct towers of number fields that are branched over an explicit finite set of primes. Fix a normalized Belyi map f of type $(d; e_1, e_2, e_3)$ and a value $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ such that $[K_{n,a} : \mathbb{Q}] = d^n$ for all $n \geq 1$. Construct the tower $\mathbb{Q} = K_{0,a} \subseteq K_{1,a} \subseteq K_{2,a} \subseteq \dots$ of number fields. We denote the set of rational primes in \mathbb{Q} by \mathbb{P} . Recall from the introduction that there is a finite set $\mathcal{P} \subseteq \mathbb{P}$ such that $K_{n,a}/K_{0,a}$ is unramified outside primes lying above \mathcal{P} . (This is [4, Theorem 1], using that normalized Belyi maps are post-critically finite.)

We sketch what we can say about the finite set \mathcal{P} in our situation. (This is a more precise version of [4, Section 5], using the results on the reduction of normalized Belyi maps from [2].) A subtle point is that there is a difference between the reduction of a rational function $f \in \mathbb{Z}[x]$ (defined by reducing the coefficients modulo p as in Definition 1.2.1) and reduction of the cover $f : \mathbb{P}^1_{\mathbb{Q}} \rightarrow \mathbb{P}^1_{\mathbb{Q}}$. A model over $\text{Spec}(\mathbb{Z}_p)$ of the cover $f \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is required to be finite and flat. Reducing the rational function $f \in \mathbb{Z}[x]$ modulo p by reducing its coefficients may yield a rational function of strictly smaller degree. This happens if and only if f has bad reduction to characteristic $p > 0$ in the sense of Definition 1.2.3.

The results of [2, Section 4] can be interpreted as saying that these notions are closely related for normalized Belyi maps in the single-cycle case. Namely, the rational function f has good separable reduction at p in the sense of Definition 1.2.3 if and only if the Galois closure of the map $f : \mathbb{P}^1_{\mathbb{Q}} \rightarrow \mathbb{P}^1_{\mathbb{Q}}$ has potentially good

reduction, meaning that there exist an extension L/\mathbb{Q}_p and a model of $f \otimes_{\mathbb{Q}_p} L$ over $\text{Spec}(\mathcal{O}_L)$ whose special fiber is a separable Galois cover of \mathbb{P}^1 branched over three points. This is very special to the case of normalized single-cycle Belyi maps.

Recall that we have chosen $f \in \mathbb{Z}(x)$ so that the conditions (1)–(3) in Sect. 1.2 are satisfied. As in [4, Section 5] we let \mathcal{P}_{bad} be the set of rational primes for which the fiber of f at p has degree strictly less than p or is inseparable. (The third case of [4, Section 5], in which the ramification points coalesce modulo p , does not occur in our case, by Proposition 1.2.2(2).) Proposition 1.2.4(1) implies that

$$\mathcal{P}_{\text{bad}} \subseteq \{p \in \mathbb{P} \mid p \leq d\}.$$

We can determine this set more precisely for a given type: $\deg(\bar{f}) < \deg(f)$ at p if and only if p divides the leading coefficient of f . If f has good inseparable reduction at p , then $p \mid \deg(f)$. In fact one can show that f has either bad or good inseparable reduction at the primes p dividing $\deg(f)$. (This may be deduced from [2, Proposition 5].)

It follows from [4, Theorem 2] that we may take $\mathcal{P} = \mathcal{P}_{\text{bad}} \cup \mathcal{P}_a$, where

$$\mathcal{P}_a = \{p \in \mathbb{P} \mid v_p(a) \neq 0 \text{ or } v_p(1-a) > 0\},$$

i.e., the set of primes p such that a is congruent modulo p to one of the branch points $\{0, 1, \infty\}$ of f .

As a concrete example, consider the polynomial Belyi map of type $(d; d-1, 2, d)$. Then $f(x) = -(d-1)x^d + dx^{d-1}$ has good monomial reduction at all primes dividing d and bad reduction exactly at the primes dividing $d-1$, i.e., $\mathcal{P}_{\text{bad}} = \{p \in \mathbb{P} \mid p \mid d(d-1)\}$. Choosing $a = p$ yields a tower of number fields only branched over \mathcal{P}_{bad} .

The Chinese Remainder Theorem implies that we may also choose a such that the Conditions 3.1.5 are satisfied. For example for $d = 9$ we may choose $a = 60/11$, and we find $\mathcal{P}_{\text{bad}} = \{2, 3\}$ and $\mathcal{P}_a = \{5, 7, 11\}$. The infinite tower of number fields $(K_{n,a})_{n \geq 1}$ corresponding to f only ramifies above $\{2, 3, 5, 7, 11\}$. Combining Corollary 3.2.3 from the next section with Corollary 2.4.4 and 2.4.6 yields that $G_{n,a,\mathbb{Q}} = E_n$ for all $n \geq 1$.

Using the explicit expressions in Proposition 1.1.4 it is easy to find many more results along these lines.

Lemma 3.1.8 below translates Conditions 3.1.5 into a statement on the existence of certain elements $h_{j,n,a} \in G_{n,a}$. We start by setting up some notation. Since $G_{n,a} \subseteq G_{n,\mathbb{Q}} \subseteq \text{Aut}(T_n)$, we may define subgroups of $G_{n,a}$ analogous to the subgroups $N_{n,\overline{\mathbb{Q}}}$ and $N_{n,\mathbb{Q}}^i$ defined in Definitions 2.2.5 and 2.2.6.

Definition 3.1.7. Define $N_{n,a} := \ker(G_{n,a} \rightarrow G_{n-1,a})$ and $N_{n,a}^i := N_{n,a}^i \cap \mathcal{S}(T_n^i)$.

Analogous to (2.2.3), we may write elements of $N_{n,a}$ as tuples $(\rho^1, \dots, \rho^{d^{n-1}})$, where $\rho^i \in N_{n,a}^i \subseteq \mathcal{S}(T_n^i)$.

Lemma 3.1.8. *Let f be a normalized Belyi map satisfying Conditions 3.1.5 for a choice of a, p, q_1, q_2, q_3 . For $n \geq 2$ and $j \in \{1, 2, 3\}$ there exist elements $h_{j,n} \in G_{n,a}$ such that the following conditions hold.*

- (1) The elements $h_{j,n} \in G_{n,a}$ are conjugate to $g_{j,n}$ in $G_{n,a}$.
- (2) The element $h_{j,1} \in G_{1,a}$ is a single cycle of length e_j .
- (3) For $n \geq m$, we have that $\pi_m(h_{j,n}) = h_{j,m}$.
- (4) We have

$$\alpha_{j,n,a} := h_{j,n}^{e_j^{n-1}} \in N_{n,a},$$

where the components $\alpha_{j,n,a}^i$ of $\alpha_{j,n,a}$ in $\mathcal{S}(T_n^i)$ for $1 \leq i \leq d^{n-1}$ are either single cycles of length e_i or trivial. The permutation $\alpha_{j,n,a}^i$ is non-trivial for exactly e_j^{n-1} values of i .

Proof. For $n \geq 1$ and $j \in \{1, 2, 3\}$ let $h_{j,n} \in G_{n,a}$ be a generator of the inertia group of q_j in $G_{n,a}$. Then Proposition 3.1.3 implies that $h_{j,n}$ is conjugate in $G_{n,a}$ to $g_{j,n}$. Hence Statement (1) holds. For $n = 1$, the $h_{j,1}$ are single e_j -cycles for $j = 1, 2, 3$, proving Statement (2). Moreover, it is clear that we may choose the $h_{j,n}$ for varying n consistently, guaranteeing that Statement (3) holds.

Arguing as in the proof of Lemma 2.2.8, we conclude that the elements $\alpha_{j,n,a}$ are contained in $N_{n,a}$ for $j \in \{1, 2, 3\}$. Statement (1) implies that $\alpha_{j,n,a}$ has the same cycle type as $\alpha_{j,n}$ (defined in Lemma 2.2.8) when considered as an element of S_{d^n} . Statement (4) follows therefore immediately from Lemma 2.2.8. \square

3.2. Comparing $G_{n,a}$ and $G_{n,\mathbb{Q}}$

In this section we compare the Galois groups $G_{n,a} \subseteq G_{n,\mathbb{Q}}$ and give sufficient conditions on a for these groups to be equal for all $n \geq 1$. The key step is to show that the geometric Galois group $G_{n,\overline{\mathbb{Q}}}$ is a subgroup of $G_{n,a}$ for all n if Conditions (3.1.5) are satisfied. We show this using the explicit elements of $G_{n,a}$ we produced in Lemma 3.1.8 and by arguing as in Sects. 2.2 and 2.3.

Proposition 3.2.1. *Let $f = x^{e_1} f_1/f_2$ be a normalized Belyi map of type $(d; e_1, e_2, e_3) \neq (6; 4, 4, 5)$. Assume f satisfies Conditions 3.1.5 for a choice of a, p, q_1, q_2, q_3 .*

- (1) We have that $G_{1,\overline{\mathbb{Q}}} \subseteq G_{1,a} \subseteq S_d$. In particular, $G_{1,a} \simeq S_d$ in the case that $G_{1,\overline{\mathbb{Q}}} \simeq S_d$.
- (2) For $n \geq 2$ and $1 \leq i \leq d^{n-1}$, the image of the projection map

$$N_{n,a} \rightarrow \mathcal{S}(T_n^i), \quad (\rho^1, \dots, \rho^{d^{n-1}}) \mapsto \rho^i$$

contains $G_{1,\overline{\mathbb{Q}}} \subseteq S_d$ as a subgroup.

Proof. The existence of the prime p of good monomial reduction implies that the Galois group $G_{1,a} \subseteq S_d$ of $K_{1,a}/\mathbb{Q}$ is a transitive group on d letters (Proposition 3.1.2). The conditions on a with respect to the primes q_i imply that $G_{1,a}$ contains elements $h_{1,1}, h_{2,1}, h_{3,1}$, which are pure cycles of length e_1, e_2, e_3 respectively, where $e_1 + e_2 + e_3 = 2d + 1$ (Lemma 3.1.8(2)).

We argue as in the proof of [11, Theorem 5.3] to show that $G_{1,a}$ contains a subgroup isomorphic to $G_{1,\overline{\mathbb{Q}}}$. We start by proving that $G_{1,a}$ is primitive. To reach a contradiction, suppose that $G_{1,a}$ is *not* primitive. Since $G_{1,a}$ is transitive on d letters, there exists a number $m|d$ (with $1 < m < d$) and a division of $\{1, 2, \dots, d\}$ into d/m blocks of length m such that every element of $G_{1,a}$ either has order strictly less than m and acts trivially on the blocks, or has order mk for some $k \geq 1$. Now we distinguish the following cases.

- If all e_i are strictly less than m , then $2d + 1 = e_1 + e_2 + e_3 < 3m \leq 3d/2$, since $m \leq d/2$, by assumption. We obtain a contradiction.
- If all e_i are a multiple of m , then $2d + 1 = e_1 + e_2 + e_3$ is divisible by m . Since $m|d$ we obtain a contradiction.
- Assume that exactly one of the e_i is strictly less than m ; this is necessarily e_1 , since $1 < e_1 \leq e_2 \leq e_3$. Write $e_2 = mk_2$ and $e_3 = mk_3$ for some $k_2, k_3 \geq 1$. We obtain $2d + 1 = e_1 + m(k_2 + k_3)$. Since $m|d$, we have $e_1 \equiv 1 \pmod{m}$. This implies that $e_1 > m$, and we obtain a contradiction.
- If exactly two of the e_i (that is, e_1 and e_2) are strictly less than m , then a similar argument shows that $e_1 + e_2 \equiv 1 \pmod{m}$ so $e_1 + e_2 = 1 + m \leq 1 + d/2$. But then $e_3 = (2d + 1) - (e_1 + e_2) \geq 3d/2 > d$, which again yields a contradiction.

Hence $G_{1,a}$ is primitive, as claimed. For $d > 10$, the group $G_{1,a}$ contains a cycle of length $e \leq (d - e)!$ (namely, $h_{1,1}$). Hence, by Williamson's Theorem [17], we have that $G_{1,a}$ is isomorphic to either A_d or S_d . If at least one of the e_i is even, then $G_{1,a}$ is isomorphic to S_d . In both cases we therefore have that $G_{1,a}$ contains $G_{1,\overline{\mathbb{Q}}}$ (Lemma 1.1.5(1)). For $d \leq 10$, the statement follows from the case-by-case analysis in the proof of [11, Theorem 5.3]. Statement (1) follows.

The group $N_{n,a}$ contains the elements $\alpha_{1,n,a}, \alpha_{2,n,a}, \alpha_{3,n,a}$ from Lemma 3.1.8(4). Recall that $G_{n,a}$ is a transitive group on d^n letters. In particular, it follows that conjugation by $G_{n,a}$ acts transitively on the d^{n-1} blocks of d indices, where the i th block corresponds to the vertices of T_n^i . It therefore suffices to prove Statement (2) for $i = 1$.

Replacing $\alpha_{j,n,a}$ by a conjugate under $G_{n,a}$, we may assume that the component $\alpha_{j,n,a}^1$ in $\mathcal{S}(T_n^1)$ is non-trivial. Note that the group elements we conjugate by may depend on $j \in \{1, 2, 3\}$. We conclude that the image of $N_{n,a}$ under projection to $\mathcal{S}(T_n^1)$ contains an e_1 -cycle, an e_2 -cycle, and an e_3 -cycle. We denote this image by $G_{n,a}^1$.

The group $G_{n,a}^1$ may be identified with the Galois group of the specialization of f at a point b with $f^{n-1}(b) = a$. The corresponding field extension $K_{1,b}$ of $K_{0,b} := \mathbb{Q}$ defined by f may therefore be identified with a subextension of $K_{n,a}/K_{n-1,a}$ (cf. Definition 3.1.1). Since $[K_{n,a} : K_{0,a}] = d^n$ (Proposition 3.1.2), it follows that $[K_{1,b} : K_{0,b}] = d$. We conclude that $G_{n,a}^1$ is a transitive group on d letters. The argument from the proof of Statement (1) shows that $G_{1,\overline{\mathbb{Q}}} \subseteq G_{n,a}^1 \subseteq S_d$. This proves Statement (2). \square

Theorem 3.2.2 below extends the conclusion of Proposition 3.2.1(1) that $G_{1,\overline{\mathbb{Q}}} \subseteq G_{1,a}$ under Conditions 3.1.5 to arbitrary n . Phrased differently, we show

that we do not need additional conditions on a in passing from $n = 1$ to arbitrary n . We exclude exactly the same types as in Theorem 2.3.1.

Theorem 3.2.2. *Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3) \notin \{(4; 3, 3, 3), (6; 4, 4, 5)\}$. Choose $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ and distinct prime numbers p, q_1, q_2, q_3 such that Conditions 3.1.5 hold. Then*

$$G_{n, \overline{\mathbb{Q}}} \subseteq G_{n, a} \quad \text{for all} \quad n \geq 2.$$

Proof. We argue as in the proofs of Sect. 2.2. The statement of the theorem for $n = 1$ holds by Proposition 3.2.1(1).

After replacing the $h_{j,1}$ by a conjugate under $G_{1,a} \supset A_d$ we may assume that

$$|\text{supp}(h_{1,1})| \cap |\text{supp}(h_{2,1})| \cap |\text{supp}(h_{3,1})| = 1. \quad (3.2.1)$$

Here we have also used that $2d + 1 = e_1 + e_2 + e_3$. (For the generating system from Lemma 1.1.5(2), the unique integer in the intersection is e_3 .) In the case that $G_{1, \overline{\mathbb{Q}}} \simeq S_d$ we have that $G_{1, \overline{\mathbb{Q}}} = G_{1, a} \simeq S_d$, and in this case we may even assume that $h_{j,1} = g_{j,1}$ for all $j \in \{1, 2, 3\}$. More precisely, from the fact that the generating system from Lemma 1.1.5(2) is weakly rigid it follows that we may assume that $h_{j,1} = \sigma^{-1} g_{j,1} \sigma$ for all $j \in \{1, 2, 3\}$ and some $\sigma \in S_d$ which is independent of j .

The proof of Lemma 2.2.8(2) may be applied to the elements $\alpha_{j,n,a}$ defined in Lemma 3.1.8(4). In the current situation we conclude that the component $\alpha_{j,n,a}^i$ of $\alpha_{j,n,a}$ in $\mathcal{S}(T_n^i)$ satisfies

$$\alpha_{j,n,a}^i = \begin{cases} \text{an } e_1\text{-cycle} & \text{if } i \in \text{supp}(h_{h,1}), \\ \text{trivial} & \text{otherwise.} \end{cases}$$

The following claim is analogous to Claim 1 in the proof of Proposition 2.2.9.

Claim 1 We have that $N_{n,a}^i \neq \emptyset$ for some $1 \leq i \leq d^n$.

Consider the commutator

$$\beta_{n,a} := [\alpha_{1,n,a}, [\alpha_{2,n,a}, \alpha_{3,n,a}]].$$

Since $\alpha_{2,n,a}, \alpha_{3,n,a} \in N_{n,a}$, it follows from Eq. (2.1.8) that the component $\beta_{n,a}^i$ of $\beta_{n,a}$ in $\mathcal{S}(T_n^i)$ is

$$\beta_{n,a}^i = [\alpha_{1,n,a}^i, [\alpha_{2,n,a}^i, \alpha_{3,n,a}^i]] \in N_{n,a}^i.$$

Arguing as in the proofs of Lemma 2.2.8 and Proposition 2.2.9 we conclude that the component $\beta_{n,a}^i$ for $i = 1 + \sum_{k=1}^{n-1} (\ell_k - 1)d^{k-1}$ is non-trivial if and only if $\ell_k \in \text{supp}(h_{1,1}) \cap \text{supp}(h_{2,1}) \cap \text{supp}(h_{3,1})$ for all $1 \leq k \leq n-1$. Equation (3.2.1) implies that $\beta_{n,a}^i$ is non-trivial for a unique i_0 , i.e., $\beta_{n,a} \in N_n^{i_0}$. Moreover, the i_0 -th component $\beta_{n,a}^{i_0}$ equals

$$[h_{1,1}, [h_{2,1}, h_{3,1}]] \neq \text{id}.$$

The last statement may for example be checked explicitly using the fact that $(h_{1,1}, h_{2,1}, h_{3,1})$ is uniformly conjugate under S_d to $(g_{1,1}, g_{2,1}, g_{3,1})$, together with the explicit expression for the $g_{j,1}$ given in Lemma 1.1.5(2). Claim 1 is now proved.

Claim 2 $N_{n,a}^i \simeq A_d$ for all $1 \leq i \leq d^{n-1}$.

Claim 2 follows from Claim 1 as in the proof of Proposition 2.2.9 (Claim 2).

The proof of Lemma 2.3.2 applies in this case as well: in the proof of that lemma, we only use the fact that exactly $(e_j)^{n-1}$ components of the element $\alpha_{j,n} \in N_{n,\overline{\mathbb{Q}}}$ are non-trivial. In the current situation, this property follows from Lemma 3.1.8(4). We conclude that $N_{n,a}$ contains a subgroup isomorphic to $N_{n,\overline{\mathbb{Q}}}$. The statement of the theorem follows as in the proof of Theorem 2.3.1. \square

The following straightforward corollary of Theorem 3.2.2 summarizes the relation between the groups $G_{n,\overline{\mathbb{Q}}}, G_{n,a}, G_{n,\mathbb{Q}}$ in the case that all assumptions we have imposed at various places in this paper hold. Recall that we gave explicit conditions guaranteeing that $G_{n,\overline{\mathbb{Q}}} = G_{n,\mathbb{Q}}$ for all $n \geq 1$ in Corollaries 2.4.4 and 2.4.6.

Corollary 3.2.3. *Let f be a normalized Belyi map of type $(d; e_1, e_2, e_3) \notin \{(4; 3, 3, 3), (6; 4, 4, 5)\}$. Choose $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ and distinct primes p, q_1, q_2, q_3 such that Conditions 3.1.5 hold. Assume that $G_{n,\overline{\mathbb{Q}}} = G_{n,\mathbb{Q}}$ for all $n \geq 1$. Then $G_{n,a} = G_{n,\overline{\mathbb{Q}}}$ for all $n \geq 1$.*

Proof. We have $G_{n,\overline{\mathbb{Q}}} \subseteq G_{n,a} \subseteq G_{n,\mathbb{Q}} \subseteq \text{Aut}(T_n)$ for all $n \geq 1$: the first inclusion is Theorem 3.2.2, the second one is Eq. (3.1.1), and the third holds by definition. The assumption that $G_{n,\overline{\mathbb{Q}}} = G_{n,\mathbb{Q}}$ for all $n \geq 1$ therefore implies that all three groups are equal. The second statement on the index follows from Lemma 2.1.3(1). \square

Remark 3.2.4. Corollary 3.2.3 also immediately implies that, under the same assumptions,

$$[\text{Aut}(T_n) : G_{n,a}] \rightarrow \infty \text{ as } n \rightarrow \infty.$$

4. Applications to dynamical sequences

Let f be a normalized Belyi map of degree d such that $G_{1,\overline{\mathbb{Q}}} \simeq S_d$. By Theorem 2.3.1, the iterates f^n then have geometric monodromy groups $G_{n,\overline{\mathbb{Q}}} \simeq E_n$ for all $n \geq 1$, where the groups E_n are defined in Definition 2.1.2. We prove in Theorem 4.1.2 that when $E_1 \simeq S_d$ and for any $d \geq 3$, the proportion of elements of $E_n \simeq G_n$ that fix a leaf on level n tends to zero as n tends to infinity. This generalizes [3, Theorem 5.1], where the authors prove the same for the Belyi map $f(x) = -2x^3 + 3x^2$ of degree $d = 3$; our proof is inspired by theirs.

Choose $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ and distinct primes p, q_1, q_2, q_3 such that Conditions 3.1.5 hold. Then $G_{n,\overline{\mathbb{Q}}} \subseteq G_{n,a} \subseteq G_{n,\mathbb{Q}}$ for all $n \geq 1$, by Theorem 3.2.2. Further assume that $G_{n,\overline{\mathbb{Q}}} \simeq G_{n,\mathbb{Q}}$ for all $n \geq 1$, so that these inclusions all become equalities. For f of degree $d \geq 3$, Theorem 4.1.2 then shows that the proportion of elements of $G_{n,a}$ that fix a leaf tends to zero as n tends to infinity. In Corollary 4.2.1 we derive some arithmetic dynamical consequences from this theorem.

4.1. Proportion of elements fixing a leaf

Definition 4.1.1. Let $n \geq 1$. Define

$$\begin{aligned} E_{n,\text{fix}} &= \{ \text{elements of } E_n \text{ that fix a leaf } \}, \\ A_{n,\text{id}} &= |\{ \text{elements of } E_n \text{ that act as id on } T_1 \}|, \\ A'_{n,\text{id}} &= |\{ \text{elements of } E_{n,\text{fix}} \text{ that act as id on } T_1 \}|, \end{aligned}$$

and for $2 \leq k \leq d$, let

$$\begin{aligned} A_{n,k} &= |\{ \text{elements of } E_n \text{ that act as } \tau \text{ with } |\text{supp}(\tau)| = k \text{ on } T_1 \}|, \\ A'_{n,k} &= |\{ \text{elements of } E_{n,\text{fix}} \text{ that act as } \tau \text{ with } |\text{supp}(\tau)| = k \text{ on } T_1 \}|. \end{aligned} \quad (4.1.1)$$

We see that $|E_n| = A_{n,\text{id}} + \sum_{i=2}^d A_{n,i}$ and $|E_{n,\text{fix}}| = A'_{n,\text{id}} + \sum_{i=2}^{d-1} A'_{n,i}$. Recall from Sect. 2 that

$$|E_{n+1}| = \frac{|E_1|}{2} |E_n|^d = |E_n|^d d! / 2. \quad (4.1.2)$$

Finally, we note that for any $n \geq 1$, exactly half of the elements of $E_{n-1} \wr E_1$ are contained in E_n by Definition 2.1.2.

Theorem 4.1.2. Let T_n be the regular d -ary rooted tree of level n . Then

$$|E_{n,\text{fix}}| / |E_n| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Remark 4.1.3. Let f be a normalized Belyi map of degree $d \geq 3$, such that $G_{n,\mathbb{Q}} \simeq G_{n,\overline{\mathbb{Q}}} \simeq E_n$ for all $n \geq 1$. If we choose $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ and distinct primes p, q_1, q_2, q_3 such that Conditions 3.1.5 hold, then $G_{n,\overline{\mathbb{Q}}} \simeq G_{n,a} \simeq G_{n,\mathbb{Q}}$ for all $n \geq 1$, and Theorem 4.1.2 implies that

$$|G_{n,a,\text{fix}}| / |G_n| \rightarrow 0 \text{ as } n \rightarrow \infty,$$

where $G_{n,a,\text{fix}} = \{ \text{elements of } G_{n,a} \text{ that fix a leaf } \}$.

Proof. By inclusion–exclusion, this yields

$$A'_{n+1,\text{id}} = \frac{1}{2} \left(d |E_{n,\text{fix}}| |E_n|^{d-1} - \binom{d}{2} |E_{n,\text{fix}}|^2 |E_n|^{d-2} + \dots + (-1)^{d-1} |E_{n,\text{fix}}|^d \right). \quad (4.1.3)$$

Similarly, an element of $E_{n+1,\text{fix}}$ that acts as a non-trivial permutation τ on T_1 with $|\text{supp}(\tau)| = k$ is of the form $((\sigma_1, \dots, \sigma_d), \tau)$, where at least one of the σ_j with $j \notin \text{supp}(\tau)$ is in $E_{n,\text{fix}}$. Moreover, $\text{sgn}_2(\tau) \prod_{j=1}^d \text{sgn}_2(\sigma_j) = 1$. This yields

$$\begin{aligned} A'_{n+1,k} &\leq \frac{1}{2} |E_n|^k \cdot \left((d-k) |E_{n,\text{fix}}| |E_n|^{d-k-1} \right. \\ &\quad \left. - \binom{d-k}{2} |E_{n,\text{fix}}|^2 |E_n|^{d-k-2} + \dots + (-1)^{d-k-1} |E_{n,\text{fix}}|^{d-k} \right). \end{aligned} \quad (4.1.4)$$

Adding the contributions from (4.1.3) and (4.1.4), dividing out by $|E_{n+1}|$, and using Eq. (4.1.2) yields

$$\begin{aligned} \frac{|E_{n+1, \text{fix}}|}{|E_{n+1}|} &\leq \frac{1}{|E_1|} \left(\left(1 - \left(1 - \frac{|E_{n, \text{fix}}|}{|E_n|} \right)^d \right) \right. \\ &\quad \left. + \sum_{i=2}^{d-1} |\{\tau \in S_d : |\text{supp}(\tau)| = i\}| \cdot \left(1 - \left(1 - \frac{|E_{n, \text{fix}}|}{|E_n|} \right)^{d-i} \right) \right). \end{aligned} \quad (4.1.5)$$

For any $n \geq 1$, let x_n denote $|E_{n, \text{fix}}|/|E_n|$. Then Eq. (4.1.5) shows that $x_{n+1} \leq \phi(x_n)$, where

$$\phi : x \mapsto \frac{1}{d!} \sum_{i=0}^d |\{\tau \in S_d : |\text{supp}(\tau)| = i\}| \cdot \left(1 - (1-x)^{d-i} \right).$$

The function $g(x) = 1 - (1-x)^k$ is increasing on $[0, 1]$ for any $k \geq 1$, hence so is ϕ . Since $1 - (1-x)^k \geq 0$ for any $x \in [0, 1]$ and any $k \geq 1$, we have $\phi(x) \geq 0$ for any $x \in [0, 1]$.

Hence we find that

$$x_{n+1} \leq \phi(x_n) \leq \cdots \leq \phi^n(x_1) \leq \phi^n(1),$$

where $x_1 = |E_{1, \text{fix}}|/|E_1| \leq 1$. Let y_n denote $\phi^n(1)$. Then $\{y_n\}_{n \geq 1}$ is a non-increasing sequence in the interval $[0, 1]$. Hence it has a limit y which satisfies $\phi(y) = y$. Since the only solution to $\phi(y) = y$ in $[0, 1]$ is $y = 0$, the limit of the $y_n = \phi^n(1)$ is zero. Thus, the limit of $x_n = |E_{n, \text{fix}}|/|E_n|$ is also zero, as required. \square

4.2. Dynamical sequences

For a set S of prime numbers, let $\delta(S)$ denote its natural density if it exists. A *dynamical sequence* in a field K is a sequence $\{c_i\}_{i \geq 0}$ with $c_i \in K$ such that $c_i = f(c_{i-1})$ for some map $f : K \rightarrow K$. Prime divisors of entries of such sequences were first studied using Galois theory by Odoni in [14]. For four particular quadratic maps f , Jones [8] shows the density of prime divisors in the dynamical sequence for f is zero; Gottesman and Tang [6] show non-zero densities can also occur for quadratic maps. More general treatments of higher degree maps can be found in e.g. [5], [7].

Corollary 4.2.1. *Let f be a normalized Belyi map such that $G_{n, \mathbb{Q}} \simeq G_{n, \overline{\mathbb{Q}}} \simeq E_n$ for all $n \geq 1$.*

- (1) *Choose $a \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$ and distinct primes p, q_1, q_2, q_3 , such that Conditions 3.1.5 hold. Consider the sequence $\{a_i\}_{i \geq 0}$ where $a_0 \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty, a\}$ and $a_i = f(a_{i-1})$ for all $i \geq 1$. Then $\delta(S) = 0$, where S is the set of primes $q \in \mathbb{Q}$ such that $a_i \equiv a \pmod{q}$ for some $i \geq 0$.*

- (2) Let K be the splitting field of f and consider the $d - e_1$ non-zero preimages of zero under f , denoted $c_j \in K$ for $1 \leq j \leq d - e_1$. Suppose that for each c_j there exist primes $\mathfrak{p}_j, \mathfrak{q}_{1,j}, \mathfrak{q}_{2,j}, \mathfrak{q}_{3,j}$ of K such that the natural analogues of Conditions 3.1.5 hold. Now form the sequence $\{b_i\}_{i \geq 0} \subseteq \mathbb{Q}$ where $b_0 \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0, 1, \infty\}$, and $b_i = f(b_{i-1})$ for all $i \geq 1$. Then $\delta(T) = 0$, where T is the set of rational primes $q \in \mathbb{P}$ such that $q|b_i$ for some $i \geq 0$.

Proof. (1) For any $n \geq 1$, the set S_n of primes q not dividing a such that $a_i \equiv a \pmod{q}$ for some $0 \leq i \leq n - 1$ is finite. Further, if $a_i \equiv a \pmod{q}$ for some $i \geq n$, then the rational map $f^n(x) - a$ has a rational root over $\mathbb{Z}/q\mathbb{Z}$ (namely, a_{i-n}). It follows from the Chebotarev density theorem that

$$\delta(S) \leq \delta(\{q \in \mathbb{P} : q \notin S_n \text{ and } f^n(x) - a \text{ has a root modulo } q\}) = \frac{|E_{n,\text{fix}}|}{|E_n|},$$

so the result follows by letting $n \rightarrow \infty$ and using Theorem 4.1.2.

- (2) Consider the sequence as a subset of K . We may ignore the finitely many primes of bad or good inseparable reduction for f , as well as the finitely many primes dividing b_0 . Then $b_i \neq 0$ for any $i \geq 0$, i.e., $b_i \neq c_j$ for any $1 \leq j \leq d - e_1$, since by our assumption $G_{n,\mathbb{Q}} \simeq G_{n,\overline{\mathbb{Q}}} \simeq E_n$ none of the c_j are rational. Let \mathfrak{p} be a prime of K such that $b_i \equiv 0 \pmod{\mathfrak{p}}$ for some, without loss of generality minimal, value of $i \geq 1$. Writing $f(x) = x^{e_1} f_1(x)/f_2(x)$ as before, we see that $f_1(b_{i-1}) \equiv 0 \pmod{\mathfrak{p}}$ and hence that b_{i-1} is congruent modulo \mathfrak{p} (but not equal) to c_j for some $1 \leq j \leq d - e_1$.

Arguing as in Sect. 3, the assumptions on the c_j guarantee that $G_{n,c_j,K} \simeq G_{n,K} \simeq G_{n,\overline{\mathbb{Q}}} \simeq E_n$ for all $n \geq 1$. Therefore, we conclude by observing that

$$\begin{aligned} \delta(T) &= \delta(\{p \in \mathbb{P} : \exists \mathfrak{p} \mid p \text{ prime of } K \text{ such that } b_i \\ &\equiv c_j \pmod{\mathfrak{p}} \text{ for some } i \geq 1 \text{ and } 1 \leq j \leq d - e_1\}) \end{aligned}$$

and arguing as in (1). □

Remark 4.2.2. Corollary 4.2.1 generalizes [3, Proposition 6.1 and Corollary 6.2] to normalized Belyi maps of degree $d \geq 3$ whose Galois groups satisfy $G_{n,\mathbb{Q}} \simeq G_{n,\overline{\mathbb{Q}}} \simeq E_n$.

Acknowledgements Open Access funding provided by Projekt DEAL. This project started at the American Institute of Mathematics as part of an AIM SQuARE on dynamical Belyi maps. We thank AIM for providing a stimulating environment, and the other participants Jacqueline Anderson, Neslihan Girgin, and Michelle Manes for many useful discussions. Our previous paper [2], which evolved from work done at the conference *Women in Numbers Europe 2*, is a collaboration with them. We are grateful to Alexander Hulpke for his help with some group-theoretic computations. We gratefully acknowledge the support of the Mathematisches Forschungsinstitut Oberwolfach, where part of the work was carried out. The second and third authors were supported by an AMS – Simons Travel Grant. We thank the anonymous referees for their careful reading and their helpful suggestions which greatly improved the exposition.

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