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# Dwork Crystals II

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We give a generalization of $p$-adic congruences for truncated period functions that were originally discovered for a class of hypergeometric functions by Bernard Dwork.

## 1 Introduction

This paper is a continuation of [3], which we will refer to as Part I.

In Part I, we considered $p$-adic limit formulas to matrices of the so-called Cartier action. As an example, consider the elliptic curve $f(x,y) = y^2 - x(x-1)(x-z) = 0$. Let $G_m(z)$ be the coefficient of $(xy)^{m-1}$ in $f(x,y)^{m-1}$. Let $z_0 \in \mathbb{Z}_p$ and we denote its residue modulo $p$ by $\bar{z}_0 \in \mathbb{F}_p$. Then it was shown in Part I that, if $G_p(\bar{z}_0) \neq 0$, the quotients $G_{p^s}(z_0)/G_{p^{s-1}}(z_0)$ form a $p$-adic Cauchy sequence tending to the unit root $\lambda(\bar{z}_0) \in \mathbb{Z}_p^\times$ of the zeta function of $y^2 = x(x-1)(x-\bar{z}_0)$ as $s \to \infty$. Furthermore, when $z$ is a variable, the quotients $G_{p^s}(z)/G_{p^{s-1}}(z^p)$ form a $p$-adic Cauchy sequence as $s \to \infty$. The limit of this sequence can be identified as $(-1)^{\frac{p-1}{2}} F(z)/F(z^p)$, where $F(z)$ denotes the hypergeometric function $F(1/2, 1/2, 1|z) = \sum_{k=0}^{\infty} \frac{(1/2)_k^2}{k!^2} z^k$. This computation was done in Example 5.5 of Part I. It then follows from the results in Part I that the ratio $F(z)/F(z^p) \in \mathbb{Z}_p[\![z]\!]$ can be approximated $p$-adically by rational functions whose denominators are powers of $G_p(z)$. This property was observed earlier by Bernard Dwork, who used a different kind

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of $p$-adic approximation [4, 5]. In this particular case, we can show (Remark 3.3) that

$$F(z)/F(z^p) \equiv F_{p^s}(z)/F_{p^{s-1}}(z^p) \pmod{p^s}, \tag{1}$$

where $F_m(z) = \sum_{k=0}^{m-1} \frac{(1/2)_k^2}{k!^2} z^k$ are truncations of $F(z)$. This congruence is a version of [4, (12)].

Here, $F_p(z) \equiv G_p(z) (\bmod\ p)$. We will also see that if $z_0 \in \mathbb{Z}_p$ and $G_p(\overline{z}_0) \neq 0$, the sequence $(-1)^{\frac{p-1}{2}} F_{p^s}(z_0)/F_{p^{s-1}}(z_0)$ tends to the unit root $\lambda(\overline{z}_0)$. This is clarified in Remark 4.5.

In this paper, we will give a vast generalization and explain the underlying mechanism of congruences of the above type. For a generic Laurent polynomial $f$, it turns out that the corresponding generalization of $F(z), F_m(z)$ is given by A-hypergeometric series and their truncations.

We now recall the notations and definitions from Part I.

Let $p$ be a prime and $R$ a $p$-adically complete characteristic zero domain such that $\cap_s p^s R = \{0\}$. Let $f \in R[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent polynomial and $\Delta \subset \mathbb{R}^n$ be its Newton polytope. A subset $\mu \subset \Delta$ is said to be *open* if its complement $\Delta \setminus \mu$ is a union of faces of any dimensions. For such a subset, we consider the $R$-module of rational functions

$$\Omega_f(\mu) = \left\{ (k-1)!\, \frac{g(\mathbf{x})}{f(\mathbf{x})^k} \;\middle|\; k \geq 1,\ g \in R\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right],\ \mathrm{supp}(g) \subset k\mu \right\}.$$

When $\mu = \Delta$ we tend to omit it from the notation, for example, $\Omega_f(\Delta)$ is simply $\Omega_f$. The submodule of derivatives $d\Omega_f \subset \Omega_f$ is defined as the $R$-span of all $x_i \frac{\partial}{\partial x_i} \omega$ with $\omega \in \Omega_f$ and $1 \leq i \leq n$. In Part I we constructed, for every Frobenius lift $\sigma$ on $R$, an $R$-linear Cartier operator on the $p$-adic completions

$$\mathscr{C}_p : \widehat{\Omega}_f(\mu) \to \widehat{\Omega}_{f^\sigma}(\mu).$$

This operator commutes with the derivations of $R$ and satisfies $\mathscr{C}_p \circ x_i \frac{\partial}{\partial x_i} = p\, x_i \frac{\partial}{\partial x_i} \circ \mathscr{C}_p$ for $1 \leq i \leq n$. It is then immediate that the Cartier operator preserves $d\Omega_f$. We consider submodules

$$U_f(\mu) = \{\omega \in \widehat{\Omega}_f(\mu) \mid \mathscr{C}_p^s(\omega) \equiv 0 \pmod{p^s \widehat{\Omega}_{f^{\sigma^s}}(\mu)} \text{ for all } s \geq 1\}.$$

It follows from the above-mentioned commutation relations that $d\Omega_f \cap \Omega_f(\mu) \subset U_f(\mu)$. Denote by $\mu_{\mathbb{Z}} = \mu \cap \mathbb{Z}^n$ the set of integral points in $\mu$. The main result of Part I states that if the *Hasse–Witt matrix*

$$\beta_p(\mu) = \left(\text{coefficient of } \mathbf{x}^{p\mathbf{v}-\mathbf{u}} \text{ in } f(\mathbf{x})^{p-1}\right)_{\mathbf{u},\mathbf{v}\in\mu_{\mathbb{Z}}}$$

is invertible then the quotient

$$Q_f(\mu) = \widehat{\Omega}_f(\mu)/U_f(\mu)$$

is a free $R$-module of rank $h = \#\mu_{\mathbb{Z}}$ where the images of

$$\omega_{\mathbf{u}} = \frac{\mathbf{x}^{\mathbf{u}}}{f(\mathbf{x})}, \qquad \mathbf{u} \in \mu_{\mathbb{Z}}$$

can be taken as a basis. In this case, for every Frobenius lift $\sigma$ and every derivation $\delta$ on $R$, we define matrices $\Lambda_\sigma, N_\delta \in R^{h\times h}$ by the conditions

$$\mathscr{C}_p(\omega_{\mathbf{u}}) \equiv \sum_{\mathbf{v}\in\mu_{\mathbb{Z}}} (\Lambda_\sigma)_{\mathbf{u},\mathbf{v}}\, \omega_{\mathbf{v}}^{\sigma} \pmod{U_{f^\sigma}(\mu)},$$
$$\delta(\omega_{\mathbf{u}}) \equiv \sum_{\mathbf{v}\in\mu_{\mathbb{Z}}} (N_\delta)_{\mathbf{u},\mathbf{v}}\, \omega_{\mathbf{v}} \pmod{U_f(\mu)}.$$

One has $\Lambda_\sigma \equiv \beta_p(\mu) \pmod{p}$, and hence $\mathscr{C}_p : Q_f(\mu) \to Q_{f^\sigma}(\mu)$ is invertible. In this paper, we shall give explicit formulas for the matrices $\Lambda_\sigma, N_\delta$ in a number of situations. One $p$-adic approximation was already given in Part I:

$$\Lambda_\sigma \equiv \beta_{p^s}(\mu) \cdot \sigma\left(\beta_{p^{s-1}}(\mu)\right)^{-1} \pmod{p^s},$$
$$N_\delta \equiv \delta\left(\beta_{p^s}(\mu)\right) \cdot \beta_{p^s}(\mu)^{-1} \pmod{p^s}, \tag{2}$$

where $\beta_m(\mu) \in R^{h\times h}$ is given by the same formula as the above Hasse–Witt matrix with $p$ replaced by a positive integer $m$.

Let us say that a formal series $q(t) = \sum_{k\geq 0} b_k t^k \in \mathbb{Z}_p[[t]]$ with $b_0 = 1$ satisfies *Dwork's congruences* if one has

$$\frac{q(t)}{q(t^p)} \equiv \frac{\sum_{k=0}^{p^s-1} b_k t^k}{\sum_{k=0}^{p^{s-1}-1} b_k t^{pk}} \bmod p^s\mathbb{Z}_p[[t]]$$

for every $s \geq 1$. In [5], Dwork proved this congruence for a class of hypergeometric series. His result was generalized in [6] for the generating series of sequences

$$b_k = \text{ constant term of } g(\mathbf{x})^k,$$

where $g(\mathbf{x})$ is a multivariable Laurent polynomial such that its Newton polytope $\Delta$ contains $\mathbf{0}$ as its only internal integral point. In Sections 2, 3, and 4, we shall apply our methods to give an alternative proof of the main result of [6]. Namely, with $f(\mathbf{x}) = 1 - tg(\mathbf{x})$ and $\mu = \Delta^\circ$, the module $Q_f(\mu)$ has rank 1 and we will see that $\Lambda_\sigma = q(t)/q(t^p)$. Dwork's congruence then follows from a $p$-adic approximation similar to (2), where $\beta_{p^s} = \sum_{k=0}^{p^s-1} (-1)^k \binom{p^s-1}{k} b_k t^k$ are substituted with the truncations $\gamma_{p^s} = \sum_{k=0}^{p^s-1} b_k t^k$. In Section 4, we explore the relation between truncations and *periods modulo m* used in Part I; this relation is the key fact in our proof of Dwork's congruences. The main result of this paper is Theorem 5.3. It generalizes Dwork's congruences to the A-hypergeometric setting.

At the end of this introduction, we would like to recall a detail from Part I that will be also useful for us here. When there is a vertex $\mathbf{b} \in \Delta$ such that the coefficient of $f$ at $\mathbf{b}$ is a unit in $R$, one can give the following description of our Cartier operator. By expanding rational functions into formal power series supported in the cone $C(\Delta - \mathbf{b})$, we embed $\Omega_f$ into $\Omega_{\text{formal}} = \{\sum_{\mathbf{k} \in C(\Delta - \mathbf{b})} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \mid a_{\mathbf{k}} \in R\}$. The Cartier operation on formal expansions is simply given by

$$\mathscr{C}_p : \sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}} \mapsto \sum_{\mathbf{k}} a_{p\mathbf{k}} \mathbf{x}^{\mathbf{k}}$$

and $U_f(\mu)$ coincides with the submodule of formal derivatives $\widehat{\Omega}_f(\mu) \cap d\Omega_{\text{formal}}$, see [3, Proposition 4.2].

## 2 Periods

In Part I, we introduced the Cartier operator as operator on infinite Laurent series. However, the image of a rational function under the Cartier operator is again rational. Consider the rational function $\omega = \frac{g(\mathbf{x})}{f(\mathbf{x})^k} \in \Omega_f$. We assert that the image of $\omega$ under $\mathscr{C}_p$ is given by

$$\frac{1}{p^n} \sum_{\mathbf{y}:\mathbf{y}^p = \mathbf{x}} \frac{g(\mathbf{y})}{f(\mathbf{y})^k},$$

where the summation is over all $\mathbf{y} = (\zeta_p^{r_1} x_1^{1/p}, \ldots, \zeta_p^{r_n} x_n^{1/p})$ with $0 \le r_1, \ldots, r_n < p$, with $\zeta_p$ a primitive $p$-th root of unity. This is again a rational function but with denominator $\prod_{\mathbf{y}:\mathbf{y}^p=\mathbf{x}} f(\mathbf{y})^k$. Choose a vertex $\mathbf{b}$ of the Newton polytope $\Delta$ of $f$ and expand in a Laurent series with respect to $\mathbf{x}^{\mathbf{b}}$. The result is a Laurent series with support in the cone $C(\Delta - \mathbf{b})$. Suppose it reads $\sum_{\mathbf{k}} a_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}$. Then application of $\mathscr{C}_p$ yields

$$\mathscr{C}_p(\omega) = \frac{1}{p^n} \sum_{\mathbf{k}} a_{\mathbf{k}} \left( \sum_{r_1,\ldots,r_n=0}^{p-1} \zeta_p^{r_1 k_1 + \cdots + r_n k_n} \right) \mathbf{x}^{\mathbf{k}/p}.$$

The summation over the integers $r_1, \ldots, r_n$ yields something non-zero if and only if $p$ divides $k_i$ for $i = 1, \ldots, n$. The summation value then equals $p^n$. Replacing $\mathbf{k}$ by $p\mathbf{k}$ then yields

$$\mathscr{C}_p(\omega) = \sum_{\mathbf{k}} a_{p\mathbf{k}} \mathbf{x}^{\mathbf{k}},$$

which is precisely the Cartier operator defined in Part I.

There are also other ways to produce Laurent series expansions of $\omega$. This happens in the case when $R$ has another non-archimedean valuation, let us call it the $t$-adic valuation, and one coefficient of $f$ that dominates all the others $t$-adically. So let us write $f = \sum_{\mathbf{w} \in \Delta_{\mathbb{Z}}} v_{\mathbf{w}} \mathbf{x}^{\mathbf{w}}$ and suppose that there exists $\mathbf{v}$ such that $v_{\mathbf{v}}$ is a unit in $R$ and $|v_{\mathbf{v}}|_t > |v_{\mathbf{w}}|_t$ for all $\mathbf{w} \neq \mathbf{v}$. We can then expand $\omega$ in a $t$-adically converging Laurent series via

$$\omega = \frac{g(\mathbf{x})}{\left(v_{\mathbf{v}} \mathbf{x}^{\mathbf{v}} + \sum_{\mathbf{w} \neq \mathbf{v}} v_{\mathbf{w}} \mathbf{x}^{\mathbf{w}}\right)^k} = \frac{g(\mathbf{x}) \mathbf{x}^{-k\mathbf{v}}}{v_{\mathbf{v}}^k \left(1 + \sum_{\mathbf{w} \neq \mathbf{v}} (v_{\mathbf{w}}/v_{\mathbf{v}}) \mathbf{x}^{\mathbf{w}-\mathbf{v}}\right)^k} \tag{3}$$

$$= \frac{1}{v_{\mathbf{v}}^k} g(\mathbf{x}) \mathbf{x}^{-k\mathbf{v}} \sum_{r \ge 0} \binom{-k}{r} \left( \sum_{\mathbf{w} \neq \mathbf{v}} (v_{\mathbf{w}}/v_{\mathbf{v}}) \mathbf{x}^{\mathbf{w}-\mathbf{v}} \right)^r. \tag{4}$$

The series expansion is $t$-adically convergent, but when $\mathbf{v}$ is not a vertex of $\Delta$ we may end up with a Laurent series in $\mathbf{x}$ whose support is not a cone. It could possibly be all of $\mathbb{Z}^n$. The coefficients are then in the completion of $R$ with respect to $|.|_t$. We denote this completion by $S$ and assume that $v_{\mathbf{v}} \in S^\times$. Suppose we get

$$\omega = \sum_{\mathbf{k} \in \mathbb{Z}^n} c_{\mathbf{k}} \mathbf{x}^{\mathbf{k}}, \quad c_{\mathbf{k}} \in S.$$

Assuming that for $v_1, v_2 \in R$ inequality $|v_1|_t > |v_2|_t$ implies $|\sigma(v_1)|_t > |\sigma(v_2)|_t$, one can do analogous expansion in $\Omega_{f^\sigma}$. Then the same argument as above yields

$$\mathscr{C}_p(\omega) = \sum_{\mathbf{k}\in\mathbb{Z}^n} c_{p\mathbf{k}}\mathbf{x}^{\mathbf{k}}.$$

**Definition 2.1.** Let $\mathbf{v} \in \Delta_{\mathbb{Z}}$ be such that $|v_{\mathbf{v}}|_t > |v_{\mathbf{w}}|_t$ for all $\mathbf{w} \in \Delta$ distinct from $\mathbf{v}$ and $v_{\mathbf{v}} \in S^\times$. Then define the period map $p_{\mathbf{v}} : \Omega_f \to S$ given by $p_{\mathbf{v}}(\omega) = c_{\mathbf{0}}$, the constant term in the Laurent series expansion of $\omega$ with respect to $\mathbf{v}$.

For a differential ring $S$ with a homomorphism $R \to S$ that extends the derivations of $R$, a *period map* is an $R$-linear map $p : \Omega_f \to S$ that vanishes on $d\Omega_f$ and commutes with derivations of $R$. Values of a period map on elements of $\Omega_f$ are called *periods*. All period maps considered in this paper satisfy an extra condition of vanishing on the submodule of formal derivatives $U_f = \Omega_f \cap d\Omega_{\text{formal}}$.

It follows almost from the definition that $p_{\mathbf{v}}$ vanishes on $d\Omega_f$. It is slightly less trivial to see that $p_{\mathbf{v}}$ vanishes on the formal derivatives.

**Proposition 2.2.** Let notation be as above. Then for all $\eta \in U_f$, we have $p_{\mathbf{v}}(\eta) = 0$.

**Proof.** First of all, notice that the constant term of $\eta$ equals the constant term of $\mathscr{C}_p^s(\eta)$ for all $s \geq 0$. Since $\eta \in U_f$, we also know that the $\mathscr{C}_p^s(\eta) \equiv 0 (\text{mod } p^s)$. In particular, the constant term of $\eta$ is divisible by $p^s$ for all $s \geq 0$, hence equals 0. We conclude that $p_{\mathbf{v}}(\eta) = 0$. ■

**Theorem 2.3.** Let $\mu \subseteq \Delta$ be an *open* set and $h = \#\mu_{\mathbb{Z}}$. Consider the column vector $\mathbf{p}_{\mathbf{v}} \in S^h$ with components $p_{\mathbf{v}}(\omega_{\mathbf{u}})$ for $\mathbf{u} \in \mu_{\mathbb{Z}}$.

Assume that $R$ is $p$-adically complete and the Hasse–Witt matrix $\beta_p(\mu)$ is invertible in $R$. For any Frobenius lift $\sigma$ and any derivation $\delta$ of $R$, we have

$$\mathbf{p}_{\mathbf{v}} = \Lambda_\sigma\, \sigma(\mathbf{p}_{\mathbf{v}}) \tag{5}$$

and

$$\delta(\mathbf{p}_{\mathbf{v}}) = N_\delta\, \mathbf{p}_{\mathbf{v}}. \tag{6}$$

**Proof.** Consider the equality

$$\mathscr{C}_p(\omega_{\mathbf{u}}) = \sum_{\mathbf{w}\in\mu_{\mathbb{Z}}} \lambda_{\mathbf{u},\mathbf{w}}\omega_{\mathbf{w}}^{\sigma} (\text{mod } U_f(\mu)).$$

Expand all terms in a Laurent series with respect to the vertex $\mathbf{v}$ and determine the constant coefficient. Using the fact that the constant term of elements in $U_f$ vanish (Proposition 2.2), we get the 1st statement. In a similar vein, starting with

$$\delta(\omega_{\mathbf{u}}) \equiv \sum_{\mathbf{w}\in\mu_{\mathbb{Z}}} \nu_{\mathbf{u},\mathbf{w}}\omega_{\mathbf{w}} (\mathrm{mod}\ U_f(\mu))$$

we get the 2nd statement again by taking the constant term of the Laurent series expansions with respect to $\mathbf{v}$. ■

## 3 Example

Let $g(\mathbf{x})$ be a Laurent polynomial in $x_1,\ldots,x_n$ with coefficients in $\mathbb{Z}_p$. Suppose that $\mathbf{0}$ is the only lattice point in the interior of the Newton polytope $\Delta$ of $g$. We introduce another variable $t$ and define $f(\mathbf{x}) = 1 - tg(\mathbf{x})$. We apply Theorem 2.3 to $f(\mathbf{x})$ with $\mu = \Delta^\circ$ and $\mathbf{u} = \mathbf{v} = \mathbf{0}$. In this case, $\beta_m$ has only one entry, the constant coefficient of $f(\mathbf{x})^{m-1}$. Let $R = \mathbb{Z}_p[t, \beta_p(t)^{-1}]\hat{}$ be the $p$-adic completion of $\mathbb{Z}_p[t, \beta_p(t)^{-1}]$. The $t$-adic closure of $R$ is $S = \mathbb{Z}_p[[t]]$. The period

$$q(t) := p_{\mathbf{0}}\left(\frac{1}{f(\mathbf{x})}\right)$$

reads $\sum_{k\geq 0} b_k t^k$ with $b_k$ equal to the constant term of $g(\mathbf{x})^k$. Take the Frobenius lift given by $t \mapsto t^p$. Then we obtain as a consequence of Theorem 2.3.

**Corollary 3.1.** We have $\frac{q(t)}{q(t^p)} = \Lambda$ where $\Lambda \in \mathbb{Z}[t, \beta_p(t)^{-1}]\hat{}$ is the (single entry) matrix of the Cartier operation $\mathscr{C}_p : \mathcal{O}_f(\Delta^\circ) \to \mathcal{O}_{f^\sigma}(\Delta^\circ)$.

One easily checks that

$$\beta_m(t) = \sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} b_k t^k.$$

Define

$$\gamma_m(t) = \sum_{k=0}^{m-1} b_k t^k.$$

These can be interpreted as truncated version of the power series $q(t)$. In [6], it is shown that

**Theorem 3.2** (Mellit–Vlasenko, 2016). For all $s \geq 1$, we have $\frac{q(t)}{q(t^p)} \equiv \frac{\gamma_{p^s}(t)}{\gamma_{p^{s-1}}(t^p)} (\mathrm{mod}\ p^s)$.

Note that Theorem 3.2 with $\gamma_m$ replaced by $\beta_m$ is simply Corollary 3.1. We shall prove Theorem 3.2 in the next section. It will follow from our proof that in fact

$$\frac{q(t)}{q(t^p)} \equiv \frac{\gamma_m(t)}{\gamma_{m/p}(t^p)} (\text{mod } p^{\text{ord}_p(m)}) \tag{7}$$

with any $m \geq 1$, and a similar congruence holds for the derivatives:

$$\frac{q'(t)}{q(t)} \equiv \frac{\gamma_m'(t)}{\gamma_m(t)} \ (\text{mod } p^{\text{ord}_p(m)}).$$

It is a curious fact that when $g(\mathbf{x})$ has coefficients in $\mathbb{Z}$ then the series $q'(t)q(t)^{-1} \in \mathbb{Z}[[t]]$ is a $p$-adic analytic element for each $p$.

**Remark 3.3.** Theorem 3.2 is a generalization of the famous congruence of Dwork [4, (12)]. The latter can be obtained using $g(\mathbf{x}) = \frac{1}{4}(x+1/x)(y+1/y)$. In "$p$-adic cycles" Dwork also proved a generalization of his congruence for a class of hypergeometric functions (see [5, §1, Corollary 2 and §2, Theorem 2]).

In that particular case, the constant term of $g(\mathbf{x})^k$ equals $\binom{k}{k/2}^2 4^{-k}$ if $k$ is even and 0 if $k$ is odd. Thus, we get

$$q(t) = \sum_{k\geq 0} \binom{2k}{k}^2 (t/4)^{2k} = F(1/2, 1/2, 1|t^2).$$

Application of Theorem 2.3 and Corollary 3.1 now shows that $F(t^2)/F(t^{2p})$, hence $F(t)/F(t^p)$ is a $p$-adic analytic element. Here, $F(t)$ is the hypergeometric function $F(1/2, 1/2, 1|t)$. One can put $m = 2p^s$ in (7) to obtain congruence (1) mentioned in the Introduction.

## 4 Truncations

In this section, we consider periods mod $m$ which, in a number of relevant cases, turn out to be truncations of the Laurent series solutions of a system of linear differential equations. But first, we turn to general $f(\mathbf{x})$ with coefficients in a $p$-adic ring $R$.

By a *period map mod m*, we mean an $R$-linear map $\rho : \Omega_f \to R$ such that $\rho(d\Omega_f) \subset mR$ and $\rho \circ \delta \equiv \delta \circ \rho (\text{mod } mR)$ for every derivation $\delta$ on $R$. All period maps mod $m$ considered in this paper will satisfy the condition $\rho(U_f) \subset mR$ of "vanishing" on formal derivatives.

Choose a vertex $\mathbf{b} \in \Delta$ and consider Laurent series expansions with respect to $\mathbf{b}$. We assume its coefficient $f_{\mathbf{b}}$ in $f$ to be a unit in $R$. For an integer $m \geq 1$ and a Laurent polynomial $g(\mathbf{x}) \in R[x_1^{\pm}, \ldots, x_n^{\pm}]$, the functional

$$\rho_{m,g} : \omega \mapsto \text{constant term of } g(\mathbf{x})^m \omega$$

is a period map mod $m$. It is clear that on formal derivatives, we also have $\rho_{m,g}(U_f) \subset mR$. These properties follow easily if one observes that, modulo $m$, $m$th powers behave like constants under derivations (see Part I, Lemma 5.1). In Part I, we already used two particular instances of these period maps: $\tau_{m\mathbf{v}} = \rho_{m,\mathbf{x}^{-\mathbf{v}}f(\mathbf{x})}$ for $\mathbf{v} \in \Delta_{\mathbb{Z}}$ and $\alpha_{m\mathbf{k}} = \rho_{m,\mathbf{x}^{-\mathbf{k}}}$ for $\mathbf{k} \in C(\Delta - \mathbf{b})_{\mathbb{Z}}$. We now describe their behaviour under the Cartier operator and relevant congruences in this more general context:

**Proposition 4.1.** For a Laurent polynomial $g = \sum g_{\mathbf{w}} \mathbf{x}^{\mathbf{w}}$ denote $g^\sigma = \sum g_{\mathbf{w}}^\sigma \mathbf{x}^{\mathbf{w}}$. For any $m \geq 1$ divisible by $p$, we have $\rho_{m,g} \equiv \rho_{m/p,g^\sigma} \circ \mathscr{C}_p \pmod{p^{\mathrm{ord}_p(m)}}$.

**Proof.** Similar to the proof of Proposition 5.2 in Part I. ■

**Theorem 4.2.** Let $\mu \subseteq \Delta$ be an *open* set and $h = \#\mu_{\mathbb{Z}}$. For $m \geq 1$ consider column vectors $\boldsymbol{\rho}_m \in R^h$ with components $\rho_{m,g}(\omega_{\mathbf{u}})$ for $\mathbf{u} \in \mu_{\mathbb{Z}}$. If $R$ is $p$-adically complete and the Hasse–Witt matrix $\beta_p(\mu)$ is invertible, then for any Frobenius lift $\sigma$ and any derivation $\delta$ of $R$, we have

$$\boldsymbol{\rho}_m \equiv \Lambda_\sigma \, \sigma(\boldsymbol{\rho}_{m/p}) \pmod{p^{\mathrm{ord}_p(m)}} \tag{8}$$

and

$$\delta(\boldsymbol{\rho}_m) \equiv N_\delta \, \boldsymbol{\rho}_m \pmod{p^{\mathrm{ord}_p(m)}} \tag{9}$$

for all $m \geq 1$.

**Proof.** Similar to the proof of Theorem 5.3 in Part I. ■

Let us choose a tuple of elements $\phi_{\mathbf{v}} \in R$ for $\mathbf{v} \in \Delta_{\mathbb{Z}}$ and consider matrices of periods mod $m$ given by

$$(\gamma_m)_{\mathbf{u},\mathbf{v} \in \Delta_{\mathbb{Z}}} = \text{constant term of } \left(\phi_{\mathbf{v}}^m - \left(\phi_{\mathbf{v}} - f(\mathbf{x})/\mathbf{x}^{\mathbf{v}}\right)^m\right) \omega_{\mathbf{u}}. \tag{10}$$

Observe that the entries of $\gamma_m$ do not depend on the choice of $\mathbf{b}$ since they are constant terms of Laurent polynomials that are independent of $\mathbf{b}$. For a subset $\mu \subset \Delta$, we denote by $\gamma_m(\mu)$ the submatrix given by $(\gamma_m)_{\mathbf{u},\mathbf{v}\in\mu_{\mathbb{Z}}}$. We can rewrite these matrices via $\beta$-matrices as

$$(\gamma_m)_{\mathbf{u},\mathbf{v}} = \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}\phi_{\mathbf{v}}^{m-k}(\beta_k)_{\mathbf{u},\mathbf{v}},$$

from which the following congruence follows trivially.

**Lemma 4.3.** We have $\beta_p(\mu) \equiv \gamma_p(\mu) (\mathrm{mod}\ p)$. In particular, $\beta_p(\mu)$ is invertible if and only if this holds for $\gamma_p(\mu)$.

Application of Theorem 4.2 to the period map given by $\rho_{m,\phi_{\mathbf{v}}}$ minus $\rho_{m,\phi_{\mathbf{v}}-f/\mathbf{x}^{\mathbf{v}}}$ yields the following.

**Corollary 4.4.** Let $\gamma_m(\mu)$ be as above and suppose $\gamma_p(\mu)$ is invertible. Then for any Frobenius lift $\sigma$ and any derivation $\delta$ of $R$, we have

$$\gamma_m(\mu) \equiv \Lambda_\sigma\ \sigma\left(\gamma_{m/p}(\mu)\right)\ (\mathrm{mod}\ p^{\mathrm{ord}_p(m)}),$$

$$\delta(\gamma_m(\mu)) \equiv N_\delta\ \gamma_m(\mu)\ (\mathrm{mod}\ p^{\mathrm{ord}_p(m)})$$

for all $m \geq 1$.

As it follows from the 1st congruence in this corollary, we have

$$\gamma_{p^s}(\mu) \equiv \gamma_p(\mu)\cdot\sigma\left(\gamma_p(\mu)\right)\cdot\ldots\cdot\sigma^{s-1}\left(\gamma_p(\mu)\right)\ (\mathrm{mod}\ p).$$

Hence, all $\gamma_{p^s}(\mu)$ are invertible and we obtain $p$-adic limit formulas

$$\Lambda_\sigma \equiv \gamma_{p^s}(\mu)\cdot\sigma\left(\gamma_{p^{s-1}}(\mu)\right)^{-1},\quad N_\delta \equiv \delta(\gamma_{p^s}(\mu))\cdot\gamma_{p^s}(\mu)^{-1}\ (\mathrm{mod}\ p^s).$$

**Proof of Theorem 3.2.** We apply Corollary 4.4 in the case $f(\mathbf{x}) = 1 - tg(\mathbf{x}), \phi = 1$ and $\mu = \Delta^\circ$. Then $\gamma_m(\mu)$ is the polynomial $\sum_{k=0}^{m-1} b_k t^k$. It follows from Corollary 4.4 with $\sigma(t) = t^p$ that $\gamma_{p^s}(t) \equiv \Lambda\gamma_{p^{s-1}}(t^p)(\mathrm{mod}\ p^s)$ for all $s \geq 1$. Theorem 3.2 then follows from Corollary 3.1 that says that $\Lambda = q(t)/q(t^p)$. ■

**Remark 4.5.** Here is a small variation on the proof of Theorem 3.2. We take $t_0 \in \mathbb{Z}_p$ and consider $f(\mathbf{x}) = 1 - t_0 g(\mathbf{x}) \in \mathbb{Z}_p[x_1^{\pm1},\ldots,x_n^{\pm1}]$. Choose again $\mu = \Delta^\circ$ and suppose that $\gamma_p(t_0) \in \mathbb{Z}_p^\times$. Then we find that $\lim_{s\to\infty}\gamma_{p^s}(t_0)/\gamma_{p^{s-1}}(t_0)$ equals the unit root of the zeta-function of $f = 0$ (by the results in the Appendix to Part I). In the Dwork example, see

Remark 3.3, this means that $F_{p^s}(t_0)/F_{p^{s-1}}(t_0)$ tends to the unit root of the zeta function of the corresponding elliptic curve. This deviates from what one usually sees in the literature where one takes the limit $F_{p^s}(t_0)/F_{p^{s-1}}(t_0^p)$ and $t_0$ a Teichmüller lift, see for example [5, (6.29)]. In the 1st limit, we can take any $t_0$ in its residue class and the limit will not depend on it.

## 5 A-Hypergeometric Periods

We continue the calculation of periods following the idea in Section 2. Let $f(\mathbf{x}) = \sum_{i=1}^{N} v_i \mathbf{x}^{\mathbf{a}_i}$, where the $v_i$ are independent variables. This is the A-hypergeometric setting. Let $\Delta \subset \mathbb{R}^n$ be the Newton polytope of $f(\mathbf{x})$, which is now the convex hull of the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\} \subset \mathbb{Z}^n$. Pick some integer exponent vector $\mathbf{u} \in k\Delta$, expand $\mathbf{x}^{\mathbf{u}} f(\mathbf{x})^{-k}$ with respect to $\mathbf{a}_i \in \Delta_{\mathbb{Z}}$, and take the constant term. We get

$$p_{\mathbf{a}_i}\left(\mathbf{x}^{\mathbf{u}} f(\mathbf{x})^{-k}\right) := \text{ constant term of } \frac{\mathbf{x}^{\mathbf{u}-k\mathbf{a}_i}}{v_i^k} \sum_{d \geq 0} \binom{-k}{d} \left( \sum_{r \neq i} \frac{v_r}{v_i} \mathbf{x}^{\mathbf{a}_r - \mathbf{a}_i} \right)^d . \tag{11}$$

Before we proceed, we like to make a remark that considerably simplifies our calculation. Denote by $\tilde{\mathbf{a}}_r \in \mathbb{Z}^{n+1}$, the exponent vector $\mathbf{a}_r$ preceded by an extra component 1. We call the set $A = \{\widetilde{\mathbf{a}}_1, \ldots, \widetilde{\mathbf{a}}_N\} \subset \mathbb{Z}^{n+1}$ *saturated* when

$$\left( \sum_{j=1}^{N} \mathbb{R}_{\geq 0}\, \widetilde{\mathbf{a}}_j \right) \cap \mathbb{Z}^{n+1} = \sum_{j=1}^{N} \mathbb{Z}_{\geq 0}\, \widetilde{\mathbf{a}}_j .$$

When $A$ is saturated, the following Proposition can be applied to any exponent vector $\mathbf{u}$:

**Proposition 5.1.** For an integral point $\mathbf{u} \in k\Delta$, we denote $\tilde{\mathbf{u}} = (k, \mathbf{u})$. Assume that there exist $\alpha_1, \ldots, \alpha_N \in \mathbb{Z}_{\geq 0}$ such that $\sum_{r=1}^{N} \alpha_r \tilde{\mathbf{a}}_r = \tilde{\mathbf{u}}$. Then $p_{\mathbf{a}_i}(\mathbf{x}^{\mathbf{u}} f(\mathbf{x})^{-k})$ is equal to the application of the differential operator $\frac{(-1)^{k-1}}{(k-1)!} \prod_{r=1}^{N} \partial_r^{\alpha_r}$ where $\partial_r = \frac{\partial}{\partial v_r}$ to the universal series

$$p_{\mathbf{a}_i}(\log f) := \text{ constant term of } \left( \log v_i + \sum_{d \geq 1} \frac{(-1)^{d-1}}{d} \left( \sum_{r \neq i} \frac{v_r}{v_i} \mathbf{x}^{\mathbf{a}_r - \mathbf{a}_i} \right)^d \right).$$

The proof is straightforward with induction on $k$.

We proceed with the calculation of $p_{\mathbf{a}_i}(\log f)$ and get

$$\log v_i + \sum_{\boldsymbol{\ell}} \frac{(-1)^{\ell_1+\cdots\vee\cdots+\ell_N-1}}{\ell_1+\cdots\vee\cdots+\ell_N} \binom{\ell_1+\cdots\vee\cdots+\ell_N}{\ell_1,\ldots,\vee,\ldots,\ell_N} \prod_{r\neq i} (v_r/v_i)^{\ell_r},$$

where the sum is over all non-negative $\ell_1,\ldots\vee\ldots\ell_N$, not all zero, such that $\sum_{r\neq i}\ell_r(\mathbf{a}_r - \mathbf{a}_i) = \mathbf{0}$. Here, the $\vee$ in the summation range and the sum itself means that $\ell_i$ is to be omitted. Introduce $\ell_i = -\sum_{r\neq i}\ell_r$. Recall our notation $\tilde{\mathbf{a}}_r = (1, \mathbf{a}_r)$. Then the definition of $\ell_i$ sees to it that the support of the resulting Laurent series (aside from the constant $\log v_i$) is contained in the set

$$L_i := \left\{ \boldsymbol{\ell} = (\ell_1,\ldots,\ell_N) \in \mathbb{Z}^N \,\middle|\, \sum_{r=1}^{N} \ell_r \tilde{\mathbf{a}}_r = \mathbf{0}, \ell_r \geq 0 \text{ if } r \neq i \right\}.$$

In order to have a more compact notation, let us rewrite the multinomial coefficient as

$$\frac{(-1)^{\ell_1+\cdots\vee\cdots+\ell_N-1}}{\ell_1+\cdots\vee\cdots+\ell_N} \binom{\ell_1+\cdots\vee\cdots+\ell_N}{\ell_1,\ldots,\vee,\ldots,\ell_N} = \prod_{r=1}^{N} \frac{1}{\Gamma^*(\ell_r+1)},$$

where $\Gamma^*(n)$ with $n \in \mathbb{Z}$ is defined as $(n-1)!$ if $n \geq 1$ and $(-1)^n/|n|!$ if $n \leq 0$. Notice that the modified $\Gamma^*$ satisfies $\Gamma^*(n+1) = n\Gamma^*(n)$ for all integers $n \neq 0$. One also checks that $\Gamma^*(n)\Gamma^*(1-n) = \mathrm{sign}(n)(-1)^{n-1}$ for all integers $n$. Here, $\mathrm{sign}(n) = -1$ if $n \leq 0$ and 1 if $n \geq 1$. The period now takes the shape

$$p_{\mathbf{a}_i}\left(\frac{\mathbf{x}^{\mathbf{u}}}{f(\mathbf{x})^k}\right) = \frac{(-1)^{k-1}}{\Gamma(k)} \prod_{r=1}^{N} \partial_r^{\alpha_r} \left( \log v_i + \sum_{\boldsymbol{\ell}\in L_i^*} \prod_{r=1}^{N} \frac{v_r^{\ell_r}}{\Gamma^*(\ell_r+1)} \right), \tag{12}$$

where $L_i^* = L_i \setminus \{\mathbf{0}\}$. Although we do not need this in the rest of this paper, we like to notice that this period is a Laurent series solution of the A-hypergeometric system of equations with A-matrix the matrix with columns $\tilde{\mathbf{a}}_1,\ldots,\tilde{\mathbf{a}}_N$ and parameter vector $-\tilde{\mathbf{u}}$.

When we vary the different periods over $i$, we see that the supports of the Laurent series also vary. Fortunately, it turns out that their union also lies in a regular cone. The following result, as well as its proof, is taken from [1, Proposition 2.9]. We use a different formulation however.

**Lemma 5.2.** Let $L_i(\mathbb{R})$ be the real positive cone generated by $L_i$ and define $L^\circ(\mathbb{R}) = \sum_{i=1}^{N} L_i(\mathbb{R})$. Then $L^\circ(\mathbb{R})$ is a finitely generated cone with $\mathbf{0}$ as a vertex.

**Proof.** It suffices to show the following assertion. Let $\boldsymbol{\ell}^{(i)} \in L_i$ for $i = 1, \ldots, N$. Then $\sum_{i=1}^{N} \boldsymbol{\ell}^{(i)} = \mathbf{0}$ implies that $\boldsymbol{\ell}^{(i)} = \mathbf{0}$ for each $i$.

Denote the coordinates of $\boldsymbol{\ell}^{(i)}$ by $l_k^{(i)}$. Suppose that $\boldsymbol{\ell}^{(i)} \neq \mathbf{0}$. Then $l_i^{(i)} < 0$ and $l_k^{(i)} \geq 0$ for all $k \neq i$. In particular,

$$\tilde{\mathbf{a}}_i = \sum_{k \neq i} -\frac{l_k^{(i)}}{l_i^{(i)}} \tilde{\mathbf{a}}_k,$$

so we see that $\tilde{\mathbf{a}}_i$ is a (real) positive linear combination of some other $\tilde{\mathbf{a}}_k$. Define the set

$$C = \left\{ \tilde{\mathbf{a}}_k | \text{there exists } j \text{ such that } l_k^{(j)} \neq 0 \right\}.$$

So $C$ is the set of $\tilde{\mathbf{a}}_k$ that are non-trivially involved in some relation $\boldsymbol{\ell}^{(j)}$. Suppose $C$ is not empty. Let $\tilde{\mathbf{a}}_k$ be a vertex of the convex hull of $C$. Suppose that $l_k^{(k)} < 0$. Then $\tilde{\mathbf{a}}_k$, being a positive linear combination of other $\tilde{\mathbf{a}}_j \in C$ cannot be a vertex of the convex hull of $C$. So $l_k^{(k)} \geq 0$ and fortiori, $l_k^{(j)} \geq 0$ for all $j$. Their sum should be zero, contradicting the fact that $l_k^{(j)} \neq 0$ for some values of $j$. Hence, we conclude that $C$ is empty. In particular, $\boldsymbol{\ell}^{(j)} = \mathbf{0}$ for all $j$. ■

Due to Lemma 5.2, the set of formal power series supported in $L^\circ = L^\circ(\mathbb{R}) \cap \mathbb{Z}^N$ is a ring. Let us denote this ring by

$$\mathcal{R} = \left\{ \sum_{\boldsymbol{\ell} \in L^\circ} b_{\boldsymbol{\ell}} \mathbf{v}^{\boldsymbol{\ell}} | b_{\boldsymbol{\ell}} \in \mathbb{Z} \right\}.$$

We will also consider the bigger ring

$$\mathcal{S} = \mathcal{R}[v_1^{\pm 1}, \ldots, v_N^{\pm 1}].$$

Elements of $\mathcal{S}$ are power series supported in a finite number of integral translations of the cone $L^\circ$. It follows from Proposition 5.1 and formula (12) that $p_{\mathbf{a}_i}(\mathbf{x}^{\mathbf{u}} f(\mathbf{x})^{-k}) \in (\prod_{r=1}^{N} v_r^{-\alpha_r})\mathcal{R} \subset \mathcal{S}$. Note that when $A$ is saturated, this argument can be applied with any $k \geq 1$ and $\mathbf{u} \in k\Delta$. With a bit more effort, one can also show that $p_{\mathbf{a}_i}(\mathbf{x}^{\mathbf{u}} f(\mathbf{x})^{-k}) \in \mathcal{S}$ for any integral $\mathbf{u} \in k\Delta$ without the assumption. In what follows, we shall not assume that $A$ is a saturated set.

We shall be interested in the $N \times N$ matrix $\Psi$ with entries

$$\Psi_{ji} = p_{\mathbf{a}_i}(\omega_{\mathbf{a}_j}) = v_j^{-1} \left( \delta_{ij} + \sum_{\boldsymbol{\ell} \in L_i^*} \ell_j \prod_{r=1}^{N} \frac{v_r^{\ell_r}}{\Gamma^*(\ell_r + 1)} \right). \tag{13}$$

This formula follows from (12) with $\mathbf{u} = \mathbf{a}_j$ and $k = 1$. It will be convenient to work with the renormalized series $\tilde{\Psi}_{ji} := v_j \Psi_{ji} \in \mathcal{R}$. Let us now consider their truncated versions. Define for any $m \geq 1$ the $N \times N$-matrix $\psi_m$ with entries

$$(\psi_m)_{ji} = \text{ constant term of} \left(1 - \left(1 - \frac{f(\mathbf{x})}{v_i \mathbf{x}^{\mathbf{a}_i}}\right)^m\right) \omega_{\mathbf{a}_j}.$$

A straightforward calculation shows that this is equal to the series development (11) with $k = 1, \mathbf{u} = \mathbf{a}_j$ summed over $d = 0, 1, 2, \ldots, m-1$. Further calculation along the same lines as earlier shows that we get

$$v_j (\psi_m)_{ji} = \delta_{ij} + \sum_{\boldsymbol{\ell} \in L_i(m)^*} \ell_j \prod_{k=1}^{N} \frac{v_k^{\ell_k}}{\Gamma^*(\ell_k + 1)}, \tag{14}$$

where

$$L_i(m) = \left\{ \boldsymbol{\ell} \in \mathbb{Z}^N \middle| \sum_{k=1}^{N} \ell_k \tilde{\mathbf{a}}_k = \mathbf{0},\ \ell_k \geq 0 \text{ for all } k \neq i \text{ and } \ell_i > -m \right\}.$$

Comparing (14) and (13), one sees that $(\tilde{\psi}_m)_{ji} := v_j (\psi_m)_{ji} \in \mathcal{R}$ is the truncation of the element $\tilde{\Psi}_{ji} = v_j \Psi_{ji} \in \mathcal{R}$. Let us consider the function $|\cdot| : L^\circ \to \mathbb{Z}_{\geq 0}$ given by

$$|\boldsymbol{\ell}| := \sum_{k:\ell_k > 0} \ell_k = -\sum_{k:\ell_k < 0} \ell_k \text{ for } \boldsymbol{\ell} \in L^\circ$$

and define truncations of elements of $\mathcal{R}$ by

$$r = \sum_{\boldsymbol{\ell} \in L^\circ} b_{\boldsymbol{\ell}} \mathbf{v}^{\boldsymbol{\ell}} \quad \rightsquigarrow \quad r(m) := \sum_{|\boldsymbol{\ell}| \leq m} b_{\boldsymbol{\ell}} \mathbf{v}^{\boldsymbol{\ell}}$$

for all $m \geq 0$. With this notation, the above computation shows that $\tilde{\Psi}(m) = \tilde{\psi}_m$. Note that the constant term of $\tilde{\Psi}$ is the identity matrix, and hence $\tilde{\Psi}$ and all its truncations $\tilde{\psi}_m$ are invertible over $\mathcal{R}$.

**Theorem 5.3.** Let $\mu \subseteq \Delta$ be an *open* set and denote $h = \#\mu_{\mathbb{Z}}$. Assume that $h \geq 1$ and $\#\{j : \mathbf{a}_j \in \mu\} = h$. Consider the $h \times h$ submatrices with entries in $\mathcal{R}$ given by

$$\tilde{\Psi} = (\tilde{\Psi}_{ji})_{\mathbf{a}_j, \mathbf{a}_i \in \mu},$$

where $\tilde{\Psi}_{ji} = v_j \Psi_{ji}$ are renormalized series (13). Let $\tilde{\psi}_m = \tilde{\Psi}(m)$ for $m \geq 1$ be the respective truncations. For the Frobenius lift $\sigma : \mathcal{R} \to \mathcal{R}$ that sends $v_j$ to $v_j^p$ for each $1 \leq j \leq N$ and any of the derivations $\delta = v_i \frac{\partial}{\partial v_i} : \mathcal{R} \to \mathcal{R}$, one has congruences

$$\tilde{\Psi} \cdot \sigma(\tilde{\Psi})^{-1} \equiv \tilde{\psi}_m \cdot \sigma(\tilde{\psi}_{m/p})^{-1} \pmod{p^{\mathrm{ord}_p(m)}} \tag{15}$$

and

$$\delta(\tilde{\Psi}) \cdot \tilde{\Psi}^{-1} \equiv \delta(\tilde{\psi}_m) \cdot \tilde{\psi}_m^{-1} \pmod{p^{\mathrm{ord}_p(m)}} \tag{16}$$

for all $m \geq 1$.

Let $V$ be the $h \times h$ diagonal matrix with the entries $v_j$ for $\mathbf{a}_j \in \mu$. Note that substituting $\tilde{\Psi} = V\Psi$ and $\tilde{\psi}_m = V\psi_m$ into (15) and (16) shows that these congruences are equivalent to

$$\Psi \cdot \sigma(\Psi)^{-1} \equiv \psi_m \cdot \sigma(\psi_{m/p})^{-1} \pmod{p^{\mathrm{ord}_p(m)}},$$
$$\delta(\Psi) \cdot \Psi^{-1} \equiv \delta(\psi_m) \cdot \psi_m^{-1} \pmod{p^{\mathrm{ord}_p(m)}}.$$

Matrices in the latter congruences have entries in the bigger ring $\mathcal{S}$. We preferred to state our theorem for the normalized matrices because truncations are more naturally defined on elements of $\mathcal{R}$ rather than $\mathcal{S}$.

**Proof.** Consider the matrices of periods mod $m$ given by (10) with $\phi_{\mathbf{a}_i} = v_i$:

$$(\gamma_m)_{j,i} = \text{ constant term of } (v_i^m - (v_i - f(\mathbf{x})/\mathbf{x}^{\mathbf{a}_i})^m)\frac{\mathbf{x}^{\mathbf{a}_j}}{f(\mathbf{x})} = v_i^m(\psi_m)_{ji}. \tag{17}$$

Their entries are in $\mathbb{Z}[v_1, \ldots, v_N]$, and we have $\gamma_m = V^{-1}\tilde{\psi}_m V^m$. In particular, the coefficient of the monomial $(\prod_{\mathbf{a}_j\in\mu} v_j)^{p-1}$ in $\det(\gamma_p)$ is 1. Let $R$ be the $p$-adic completion of $\mathbb{Z}[v_1^{\pm 1}, \ldots, v_N^{\pm 1}, \det(\gamma_p)^{-1}]$. Since $\det(\gamma_p)$ is not divisible by $p$, this ring satisfies our assumption $\cap_{s\geq 1} p^s R = \{0\}$ and hence one can apply Corollary 4.4. It follows that there are matrices $\Lambda_\sigma, N_\delta \in R^{h\times h}$ such that

$$\gamma_m \equiv \Lambda_\sigma \sigma(\gamma_{m/p}) \quad \text{and} \quad \delta(\gamma_m) \equiv N_\delta \gamma_m \pmod{p^{\mathrm{ord}_p(m)}}. \tag{18}$$

Observe that all matrices $\gamma_m$ are invertible over $\mathcal{S}$ because

$$\det(\gamma_m) = \left(\prod_{\mathbf{a}_j\in\mu} v_j\right)^{m-1} \det(\tilde{\psi}_m) \in \left(\prod_{\mathbf{a}_j\in\mu} v_j\right)^{m-1} \mathcal{R}^\times \subset \mathcal{S}^\times.$$

One of the consequences of this fact is that $R$ is a subring of the $p$-adic completion

$$S := \widehat{\mathcal{S}} \subset \mathbb{Z}_p\left[\left[v_1^{\pm 1}, \ldots, v_N^{\pm 1}\right]\right].$$

Working in the big ring $S$, we can invert matrices in (18) and conclude that

$$\gamma_m \cdot \sigma(\gamma_{m/p})^{-1} \equiv \Lambda_\sigma \quad \text{and} \quad \delta(\gamma_m) \cdot \gamma_m^{-1} \equiv N_\delta \quad (\mathrm{mod}\ p^{\mathrm{ord}_p(m)}).$$

Substituting $\gamma_m = V^{-1}\tilde{\psi}_m V^m$ in the left-hand sides yields

$$\begin{aligned} \tilde{\psi}_m \cdot \sigma(\tilde{\psi}_{m/p})^{-1} &\equiv V\Lambda_\sigma V^{-p} \ (\mathrm{mod}\ p^{\mathrm{ord}_p(m)}) \\ \delta(\tilde{\psi}_m) \cdot \tilde{\psi}_m^{-1} &\equiv VN_\delta V^{-1} + \delta(V)V^{-1} \ (\mathrm{mod}\ p^{\mathrm{ord}_p(m)}). \end{aligned} \tag{19}$$

One particular consequence of these congruences is that the matrices in their right-hand sides have entries in $\mathcal{R}$. Secondly, they must coincide with the limits of the left-hand sides which, using the fact that $\tilde{\psi}_m$ is a truncation of $\tilde{\Psi}$, immediately implies that

$$V\Lambda_\sigma V^{-p} = \tilde{\Psi} \cdot \sigma(\tilde{\Psi})^{-1} \quad \text{and} \quad VN_\delta V^{-1} + \delta(V)V^{-1} = \delta(\tilde{\Psi}) \cdot \tilde{\Psi}^{-1}. \tag{20}$$

Substituting these values back into (19) proves our theorem. ■

The above proof is based on the ideas from Section 4. By Lemma 4.3, the Hasse–Witt matrix $\beta_p(\mu)$ is congruent modulo $p$ to the matrix $\gamma_p$ given in (17). (In the special case $\mu = \Delta^\circ$ this was observed in [1, Proposition 3.8].) Using this fact, we can conclude from the above proof that under the assumptions of Theorem 5.3 the determinant of the Hasse–Witt matrix is a polynomial not divisible by $p$ and there exist the respective matrices $\Lambda_\sigma, N_\delta \in R^{h\times h}$, where $R$ is the $p$-adic completion of the ring $\mathbb{Z}[v_1^{\pm 1}, \ldots, v_N^{\pm 1}, \det(\beta_p(\mu))^{-1}]$. These are the same ring $R$ and the same matrices that were used in the proof. In particular, $R$ is a subring of the $p$-adic completion $S = \widehat{\mathcal{S}}$ and we have

**Corollary 5.4.** $\Lambda_\sigma = \Psi \cdot \sigma(\Psi)^{-1}, N_\delta = \delta(\Psi) \cdot \Psi^{-1}$.

**Proof.** Substitute $\tilde{\Psi} = V\Psi$ into (20). ■

A special consequence of this corollary is that the matrices $V\Lambda_\sigma V^{-p}$ and $VN_\delta V^{-1} + \delta(V)V^{-1}$ have their entries in $\mathcal{R}$. Furthermore, it turns out that $N_\delta$ and, in a lesser way, $\Lambda_\sigma$, are independent of the choice of $p$.

Finally, we remark that in fact there are well-defined period maps

$$p_{\mathbf{a}_i} : \widehat{\Omega}_f \to S.$$

As we explained in Section 2, these period maps are invariant under the Cartier operator (we have $p_{\mathbf{a}_i} = p^{\sigma}_{\mathbf{a}_i} \circ \mathscr{C}_p$ where $p^{\sigma}_{\mathbf{a}_i}$ denotes the respective period map $\widehat{\Omega}_{f^\sigma} \to S$ ) and vanish on formal derivatives. Corollary 5.4 is then a direct consequence of Theorem 2.3.

Let us also mention the main result of [2], Theorem 1.4. It states that in the A-hypergeometric setting with the assumption that $\Delta$ has $\mathbf{a}_0$ as its unique interior lattice point the series $\Phi(\mathbf{v})/\Phi(\mathbf{v}^p)$, where $\Phi(\mathbf{v}) = \Psi_{00}(v_0,\ldots,v_N)$ is the unique entry of our matrix $\Psi$ for $\mu = \Delta^\circ$, is a $p$-adic analytic element with the set of poles determined by the Hasse invariant $\beta_p(\Delta^\circ)$. Hence, [2, Theorem 1.4] follows from Corollary 5.4.

## 6 Example

We continue the example from Part I, Section 7 with

$$f(x,y) = v_1y^2 + v_2x + v_3x^3 + v_4x^2 + v_5xy.$$

We determine the entries of the matrix $\tilde{\Psi}$. The vectors $\tilde{\mathbf{a}}_k$ are given by the columns of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 2 & 1 \\ 2 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The supports $L_i$ lie in the null space of this matrix that can be written as

$$(r+2s, s, s, r, -2r-4s), \qquad r, s \in \mathbb{Z}.$$

In $L_1$, we have the inequalities $s, r, -2r-4s \geq 0$. This is only possible when $r = s = 0$. The only non-trivial series $\Psi_{j,1}$ is $v_1\Psi_{1,1} = 1$.

In $L_2$, we have the inequalities $r+2s, s, r, -2r-4s \geq 0$ and we find $v_2\Psi_{2,2} = 1$ as non-trivial series.

In $L_3$, we again get $v_3\Psi_{3,3}$ as only non-trivial $\Psi_{j,3}$.

In $L_4$, we have the inequalities $r+2s, s, -2r-4s \geq 0$. Hence $r = -2s, s \geq 0$. So we get

$$v_j\Psi_{j,4} = \delta_{j,4} - \sum_{s\geq 1} m_j(s)\frac{(2s-1)!}{s!\,s!}(v_2v_3/v_4^2)^s,$$

where $m_j(s)$ is the $j$-th component of $(0, s, s, -2s, 0)$. The $m$-truncated version has the extra condition $m_4(s) = -2s > -m$, hence $s < m/2$.

In $L_5$, we have the inequalities $r+2s, s, r \geq 0$. So we get

$$v_j\Psi_{j,5} = \delta_{j,5} - \sum_{r,s\geq 0} m_j(r,s)\frac{(2r+4s-1)!}{(r+2s)!\,s!\,s!\,r!}(v_1v_4/v_5^2)^r(v_1^2v_2v_3/v_5^4)^s,$$

where $m_j(r,s)$ is the $j$-th component of $(r+2s, s, s, r, -2r-4s)$. The $m$-truncated version has the extra condition $m_5(r,s) = -2r-4s > -m$, hence $r+2s < m/2$.

If we restrict our matrix to the index set $\Delta_{\mathbb{Z}}^{\circ}$, a computation shows that we get the $1\times 1$-matrix with element

$$v_5\Psi_{5,5} = \sum_{r,s\geq 0}\frac{(2r+4s)!}{(r+2s)!\,s!\,s!\,r!}x^ry^s = \frac{1}{\sqrt{1-4x}}F\left(1/4, 3/4, 1\,\middle|\,\frac{64y}{(1-4x)^2}\right),$$

where $x = v_1v_4/v_5^2, y = v_1^2v_2v_3/v_5^4$. The other components $v_j\Psi_{j,5}$ are not so easy to express in terms of one-variable hypergeometric functions, if possible at all.

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## References

[1] Adolphson, A. and S. Sperber. "A-hypergeometric series and the Hasse–Witt matrix of a hypersurface." *Finite Fields Appl.* 41 (2016): 55–63.

[2] Adolphson, A. and S. Sperber. "A-hypergeometric series associated to a lattice polytope with a unique interior lattice point." arXiv:1308.4439 [math.AG].

[3] Beukers, F. and M. Vlasenko. "Dwork crystals I." *Int. Math. Res. Not. IMRN* to appear. doi: 10.1093/imrn/rnaa119.

[4] Dwork, B. "A Deformation Theory for the Zeta function of a Hypersurface." In *Proceedings of the International Congr*ess of Mathematicians, Stockholm, 247–59. Djursholm: Inst. Mittag-Leffler, 1963.

[5] Dwork, B. "p-Adic cycles." *Publ. Math. Inst. Hautes Études Sci.* 37 (1969): 27–115.

[6] Mellit, A. and M. Vlasenko. "Dwork's congruences for the constant terms of powers of a Laurent polynomial." *Int. J. Number Theory* 12 (2016): 313–21.