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Random effects and extended generalized partial credit models

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In this paper it is shown that under the random effects generalized partial credit model for the measurement of a single latent variable by a set of polytomously scored items, the joint marginal probability distribution of the item scores has a closed-form expression in terms of item category location parameters, parameters that characterize the distribution of the latent variable in the subpopulation of examinees with a zero score on all items, and item-scaling parameters. Due to this closed-form expression, all parameters of the random effects generalized partial credit model can be estimated using marginal maximum likelihood estimation without assuming a particular distribution of the latent variable in the population of examinees and without using numerical integration. Also due to this closed-form expression, new special cases of the random effects generalized partial credit model can be identified. In addition to these new special cases, a slightly more general model than the random effects generalized partial credit model is presented. This slightly more general model is called the extended generalized partial credit model. Attention is paid to maximum likelihood estimation of the parameters of the extended generalized partial credit model and to assessing the goodness of fit of the model using generalized likelihood ratio tests. Attention is also paid to person parameter estimation under the random effects generalized partial credit model. It is shown that expected a posteriori estimates can be obtained for all possible score patterns. A simulation study is carried out to show the usefulness of the proposed models compared to the standard models that assume normality of the latent variable in the population of examinees. In an empirical example, some of the procedures proposed are demonstrated.

1. Introduction

A well-known item response model for the measurement of a single latent variable by a set of polytomously scored items with ordered response categories is the generalized partial credit model (Muraki, 1992; Muraki, 1993). In the generalized partial credit model, the probability distribution of an item score depends on a number of item category location parameters, an item-scaling parameter, and a single latent variable. The generalized partial credit model is a special case of the nominal response model for polytomously scored items with unordered response categories (Bock, 1972). Well-known special cases of the generalized partial credit model are the partial credit model (Masters, 1982), the rating scale model (Andrich, 1978), the two-parameter logistic model (Birnbaum, 1968) and the one-parameter logistic model (Rasch, 1960; Rasch, 1966).

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A distinction can be made between the fixed effects generalized partial credit model and the random effects generalized partial credit model. In the fixed effects model, both item and person parameters are treated as fixed. Maximum likelihood estimation of the item and person parameters of the fixed effects model is called joint maximum likelihood estimation. A disadvantage of the fixed effects model and joint maximum likelihood estimation of the parameters is that the item parameters are not consistently estimated if the items are finite in number. In the random effects model, the item parameters are also treated as fixed but the examinees in the sample are assumed to be randomly sampled from a population of examinees. Maximum likelihood estimation of the item parameters of the random effects model is called marginal maximum likelihood estimation. In marginal maximum likelihood estimation, the latent variable is integrated out and the item parameters are estimated in the joint marginal probability distribution of the item scores. An advantage of the random effects model and marginal maximum likelihood estimation of the item parameters is that the item parameters are consistently estimated if the items are finite in number (Bock & Aitkin, 1981; Bock & Lieberman, 1970). In practice, marginal maximum likelihood estimation is usually applied under the assumption that the latent variable is normally distributed in the population of examinees. Gauss–Hermite quadrature can then be used to numerically integrate out the latent variable.

It is known that under a random effects generalized partial credit model in which the item-scaling parameters are fixed to positive integers, the joint marginal probability distribution of the item scores has a closed-form expression in terms of item category location parameters and conditional non-central moments of the (positively scaled) latent variable given a zero score on all items (Agresti, 1993; Cressie & Holland, 1983; Maris, Bechger, & San Martin, 2015). In this paper, however, it is shown that under the random effects generalized partial credit model in general, the joint marginal probability distribution of the item scores has a closed-form expression in terms of item category location parameters, parameters that characterize the distribution of the latent variable in the subpopulation of examinees with a zero score on all items, and item-scaling parameters. A favourable consequence of this closed-form expression for the joint marginal probability distribution of the item scores is that all parameters of the random effects generalized partial credit model can be estimated using marginal maximum likelihood estimation without assuming a particular distribution of the latent variable in the population of examinees and without using numerical integration.

Since the parameters that characterize the distribution of the latent variable in the subpopulation of examinees with a zero score on all items are functions of non-central moments of the latent variable, these parameters satisfy complex inequalities that follow from the inequalities of a moment sequence. It is technically difficult to maximize the marginal likelihood function with respect to the parameters subject to the inequalities that follow from the moment inequalities. It is less complicated to maximize the likelihood function ignoring these inequalities, and this yields maximum likelihood estimates of the parameters of a slightly more general model called the extended generalized partial credit model. Well-known special cases of this extended generalized partial credit model are the extended partial credit model (Agresti, 1993) and the extended one-parameter logistic model (Follmann, 1988; Tjur, 1982).

In the following section the random effects and extended generalized partial credit models are presented. Subsequently, maximum likelihood estimation of the parameters of the extended generalized partial credit model is discussed. Next, generalizations of the extended generalized partial credit model are presented, which can each be used as an alternative hypothesis model in a generalized likelihood ratio test for the extended generalized partial credit model. One of these generalizations is an extended version of the nominal response model (Bock, 1972). Furthermore, attention is paid to person parameter estimation under the random effects generalized partial credit model. It is shown that expected a posteriori (EAP) estimates can be obtained for all possible score patterns. To show the usefulness of the models proposed compared to the standard models that assume normality of the latent variable in the population of examinees, a simulation study is carried out. Finally, some of the procedures proposed are demonstrated using an empirical example.

2. The models

Consider a situation in which a test of k polytomously scored items with ordered response categories is administered to a sample of examinees. It is assumed that the examinees are randomly sampled from an infinite population. It is also assumed that the items are measures of a single random latent variable Θ with realization θ. Furthermore, let $\boldsymbol{Y} = [Y_1 \ldots Y_k]'$ be the random vector of item scores, and let $\boldsymbol{y} = [y_1 \ldots y_k]'$ be a realization, where $y_i = 0,1,\ldots, m_i$, for $i = 1,2,\ldots, k$. Then the joint marginal probability distribution of $\mathbf{Y}$ can be written as

$$P(\boldsymbol{Y}=\boldsymbol{y}) = \int P(\boldsymbol{Y}=\boldsymbol{y}|\theta)f(\theta)d\theta, \tag{1}$$

where $P(\boldsymbol{Y} = \boldsymbol{y}|\theta)$ is the joint conditional probability distribution of $\boldsymbol{Y}$ given $\Theta = \theta$ and $f(\theta)$ is the probability density of Θ in the population of examinees. The elements of $\boldsymbol{Y}$ are assumed to be conditionally independent given $\Theta = \theta$. Conditional independence of the elements of $\boldsymbol{Y}$ given $\Theta = \theta$ is defined as

$$P(\boldsymbol{Y}=\boldsymbol{y}|\theta) = \prod_{i=1}^{k} P(Y_i = y_i|\theta), \tag{2}$$

where $P(Y_i = y_i|\theta)$ is the conditional probability distribution of Y_i given $\Theta = \theta$. Let $x_{is} = 1$ if $y_i = s$ and $x_{is} = 0$ otherwise, for $s = 1,\ldots,m_i$. Then the conditional probability distribution of Y_i given $\Theta = \theta$ can be written as

$$P(Y_i = y_i|\theta) = P(Y_i = 0|\theta) \prod_{s=1}^{m_i} \left\{ \prod_{a=1}^{s} V_{\mathrm{ia}}(\theta) \right\}, \tag{3}$$

where $V_{\mathrm{ia}}(\theta) = P(Y_i = a|\theta)/\mathrm{P}(Y_i = a - 1|\theta)$ is the odds of score a on item i relative to score $a - 1$ on item i as a function of θ. Substitution from equations (2) and (3) into equation (1) gives

$$P(\boldsymbol{Y}=\boldsymbol{y}) = \int P(\boldsymbol{Y}=\mathbf{0}|\theta) \left[\prod_{i=1}^{k} \prod_{s=1}^{m_i} \left\{ \prod_{a=1}^{s} V_{\mathrm{ia}}(\theta) \right\}^{x_{\mathrm{is}}} \right] f(\theta)d\theta, \tag{4}$$

where $P(\boldsymbol{Y}=\mathbf{0}|\theta) = \prod_{i=1}^{k} P(Y_i = 0|\theta)$.

In the generalized partial credit model (Muraki, 1992, 1993), it is assumed that

$$V_{ia}(\theta) = \exp\{\alpha_i(\theta - \delta_{ia})\}, \tag{5}$$

where α_i is a scaling parameter and δ_{ia} is an item category location parameter. The special case where $\alpha_i = 1$, for all i, is the partial credit model (Masters, 1982). In the case of dichotomously scored items ($m_i = 1$, for all i), the generalized partial credit model equals the two-parameter logistic model (Birnbaum, 1968) and the partial credit model equals the one-parameter logistic model (Rasch, 1960, 1966). Substitution from equation (5) into equation (4) and using the fact that $y_i = \sum_{s=1}^{m_i} x_{is}s$ gives, after some algebra,

$$P(\boldsymbol{Y}=\boldsymbol{y}) = \exp\left(\sum_{i=1}^{k}\sum_{s=1}^{m_i}\beta_{is}x_{is}\right)\int P(\boldsymbol{Y}=\boldsymbol{0}|\theta)\exp(\boldsymbol{\alpha}'\boldsymbol{y}\theta)f(\theta)\,d\theta, \tag{6}$$

where $\beta_{is} = -\alpha_i\sum_{a=1}^{s}\delta_{ia}$ is a transformed item category location parameter and $\boldsymbol{\alpha} = [\alpha_1\,\alpha_2\,\ldots\,\alpha_k]'$. Following Cressie and Holland (1983), we can write

$$\begin{aligned} P(\boldsymbol{Y}=\boldsymbol{y}) &= P(\boldsymbol{Y}=\boldsymbol{0})\exp\left(\sum_{i=1}^{k}\sum_{i=1}^{k}\beta_{is}x_{is}\right)\int\exp(\boldsymbol{\alpha}'\boldsymbol{y}\theta)g(\theta|\boldsymbol{0})d\theta \\ &= P(\boldsymbol{Y}=\boldsymbol{0})\exp\left(\sum_{i=1}^{k}\sum_{s=1}^{m_i}\beta_{is}x_{is}\right)M_{\Theta|\boldsymbol{0}}(\boldsymbol{\alpha}'\boldsymbol{y}), \end{aligned} \tag{7}$$

where $g(\theta|\boldsymbol{0}) = P(\mathrm{Y}=\boldsymbol{0}|\theta)f(\theta)/P(\boldsymbol{Y}=\boldsymbol{0})$ is the conditional density of Θ given $\boldsymbol{Y}=\boldsymbol{0}$ and $M_{\Theta|\boldsymbol{0}}(\boldsymbol{\alpha}'\boldsymbol{y}) = E\{\exp(\boldsymbol{\alpha}'\boldsymbol{y}\Theta)|\boldsymbol{0}\}$ is the conditional moment generating function of Θ given $\boldsymbol{Y}=\boldsymbol{0}$. Note that $M_{\Theta|\boldsymbol{0}}(\boldsymbol{\alpha}'\boldsymbol{y})$ depends on $\boldsymbol{y}$ as a function of $\boldsymbol{\alpha}'\boldsymbol{y}$ only. In the following theorem, a closed-form expression for $M_{\Theta|\boldsymbol{0}}(\boldsymbol{\alpha}'\boldsymbol{y})$ is presented. The closed-form expression for $M_{\Theta|\boldsymbol{0}}(\boldsymbol{\alpha}'\boldsymbol{y})$ is inferred from a closed-form expression for the conditional moment generating function of Θ given $\boldsymbol{Y}=\boldsymbol{0}$ obtained under the random effects one-parameter logistic model.

Theorem 1. *Under the random effects generalized partial credit model, the conditional moment generating function of* Θ *given* $\boldsymbol{Y}=\boldsymbol{0}$ *is given by*

$$M_{\Theta|\boldsymbol{0}}(\boldsymbol{\alpha}'\boldsymbol{y}) = \exp\left\{\sum_{r=1}^{k}\gamma_r(r!)^{-1}\prod_{u=0}^{r-1}(\boldsymbol{\alpha}'\boldsymbol{y}-u)\right\}, \tag{8}$$

where γ_r is a common rth-order interaction parameter that characterizes the distribution of Θ given $\boldsymbol{Y}=\boldsymbol{0}$.

Proof 1. In the case of dichotomously scored items, $\mathrm{y_i} = \mathrm{x_{i1}}$, for all i. The saturated model for dichotomously scored items is given by

$$P(\boldsymbol{Y}=\boldsymbol{y}) = P(\boldsymbol{Y}=\boldsymbol{0})\exp\left(\sum_{i=1}^{k}\lambda_i y_i + \sum_{i<j}\lambda_{ij}y_i y_j + \sum_{i<j<l}\lambda_{ijl}y_i y_j y_l + \ldots + \lambda_{1\ldots k}\prod_{i=1}^{k}y_i\right), \tag{9}$$

where λ_i is a main effect parameter, for all i, λ_{ij} is a two-way interaction parameter, for all $i < j$, λ_{ijl} is a three-way interaction parameter, for all $i < j < l$, and so on. Under the one-parameter logistic model, interaction parameters of the same order are equal (Hessen, 2011), that is,

$$\begin{aligned}\lambda_{ij} &= \gamma_2, \text{ for all } i<j,\\ \lambda_{ijl} &= \gamma_3, \text{ for all } i<j<l,\\ &\vdots\\ \lambda_{1\ldots k} &= \gamma_k.\end{aligned}$$

Substitution into equation (9) gives

$$P(\boldsymbol{Y}=\boldsymbol{y})=P(\boldsymbol{Y}=\boldsymbol{0})\exp\left(\sum_{i=1}^{k}\lambda_i y_i+\gamma_2\sum_{i<j}y_iy_j+\gamma_3\sum_{i<j<l}y_iy_jy_l+\ldots+\gamma_k\prod_{i=1}^{k}y_i\right). \tag{10}$$

Next, it can be readily verified that

$$\begin{aligned}\sum_{i<j}y_iy_j &= 1'y(1'y-1)/2!,\\ \sum_{i<j<l}y_iy_jy_l &= 1'y(1'y-1)(1'y-2)/3!,\\ \vdots\quad &\quad\ \vdots\\ \prod_{i=1}^{k}y_i &= 1'y(1'y-1)(1'y-2)\ldots(1'y-k+1)/k!\end{aligned}$$

by multiplying out $\prod_{u=0}^{r-1}(\mathbf{1}'\boldsymbol{y}-u)$, for $r = 2,\ldots,k$. Substitution from these equations and $\lambda_i=\beta_i+\gamma_1$ into equation (10) gives.

$$P(\boldsymbol{Y}=\boldsymbol{y})=P(\boldsymbol{Y}=\boldsymbol{0})\exp\left\{\sum_{i=1}^{k}\beta_i y_i+\sum_{r=1}^{k}\gamma_r(r!)^{-1}\prod_{u=0}^{r-1}(1'y-u)\right\}. \tag{11}$$

From equations (7) and (11), it then follows that

$$M_{\Theta|0}(1'y)=\exp\left\{\sum_{r=1}^{k}\gamma_r(r!)^{-1}\prod_{u=0}^{r-1}(1'y-u)\right\}. \tag{12}$$

Now, since the right-hand side of equation (8) depends on $\boldsymbol{y}$ as a function of $\boldsymbol{\alpha}'\boldsymbol{y}$ only and specializes to the moment generating function in equation (12) if $m_i = 1$ and $\alpha_i = 1$, for all i, the right-hand side of equation (8) must be the conditional moment generating function of Θ given $\boldsymbol{Y} = \mathbf{0}$ under the random effects generalized partial credit model. This completes the proof.

$M_{\Theta|0}(\boldsymbol{\alpha}'\boldsymbol{y})$ can be used to determine the first $k + 1$ non-central moments of Θ given $\boldsymbol{Y} = \mathbf{0}$, that is, $M^{(t)}_{\Theta|\mathbf{0}}(0)=\mu_{t|\mathbf{0}}=E(\Theta^t|\mathbf{0})$, for $t = 0,1,\ldots,k$, where $M^{(t)}_{\Theta|\mathbf{0}}(0)$ is the tth derivative of $M_{\Theta|0}(\boldsymbol{\alpha}'\boldsymbol{y})$ with respect to $\boldsymbol{\alpha}'\boldsymbol{y}$ evaluated at $\boldsymbol{\alpha}'\boldsymbol{y} = 0$. First, however, $M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y})$ is rewritten as

$$M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y}) = \exp\left\{\sum_{r=1}^{k}\gamma_r p_r(\boldsymbol{\alpha}'\boldsymbol{y})\right\}, \tag{13}$$

where

$$p_r(\alpha'\boldsymbol{y}) = \sum_{u=0}^{r-1}(-1)^u C_{r(r-u)}(\alpha'\boldsymbol{y})^{r-u} \tag{14}$$

is an rth-order polynomial function of $\boldsymbol{\alpha}'\boldsymbol{y}$. The first five polynomials are given in the Appendix. The constants in the coefficients of the rth-order polynomial are given by

$$\begin{aligned}
C_{rr} &= (r!)^{-1}, \text{ for all } r,\\
C_r(r-1) &= (r!)^{-1}\sum_{u=1}^{r-1}u = \frac{1}{2(r-2)!}, \text{ for } r>1,\\
C_r(r-2) &= (r!)^{-1}\sum_{u=1}^{r-2}\sum_{v=u+1}^{r-1}uv = \frac{3r-1}{24(r-3)!}, \text{ for } r>2,\\
C_r(r-3) &= (r!)^{-1}\sum_{u=1}^{r-3}\sum_{v=u+1}^{r-2}\sum_{w=v+1}^{r-1}uwv = \frac{(r-1)r}{48(r-4)!} \text{ for } r>3,\\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned}$$

Note that $C_{r1} = r^{-1}$, for all r. The mean of Θ given $\boldsymbol{Y} = \mathbf{0}$ can now be obtained by evaluating the first derivative of $M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y})$ at $\boldsymbol{\alpha}'\boldsymbol{y} = 0$. The first derivative equals

$$M^{(1)}_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y}) = M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y})\sum_{r=1}^{k}\gamma_r p_r^{(1)}(\boldsymbol{\alpha}'\boldsymbol{y}),$$

where

$$p_r^{(1)}(\alpha'\boldsymbol{y}) = \begin{cases} 1, & \text{for } r = 1,\\ \displaystyle\sum_{u=0}^{r-2}(-1)^u(r-u)C_{r(r-u)}(\alpha'\boldsymbol{y})^{r-u-1} + (-1)^{r-1}C_{r1}, & \text{for } r > 1, \end{cases}$$

is the first derivative of $p_r(\boldsymbol{\alpha}'\boldsymbol{y})$ with respect to $\boldsymbol{\alpha}'\boldsymbol{y}$. The first derivatives of the first five polynomials are given in the Appendix. Consequently, the mean of Θ given $\boldsymbol{Y} = \mathbf{0}$ is given by

$$M^{(1)}_{\Theta|\mathbf{0}}(0) = \mu_{1|\mathbf{0}} = \sum_{r=1}^{k}\gamma_r r^{-1}(-1)^{r-1}.$$

The second derivative equals

$$M^{(2)}_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y}) = M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y})\left[\left\{\sum_{r=1}^{k}\gamma_r p_r^{(1)}(\boldsymbol{\alpha}'\boldsymbol{y})\right\}^2 + \sum_{r=1}^{k}\gamma_r p_r^{(2)}(\boldsymbol{\alpha}'\boldsymbol{y})\right],$$

where

$$p_r^{(2)}(\alpha'\boldsymbol{y}) = \begin{cases} 0, & \text{for } r = 1, \\ 1, & \text{for } r = 2, \\ \sum_{u=0}^{r-3} (-1)^u (r-u)(r-u-1) C_{r(r-u)} (\alpha'\boldsymbol{y})^{r-u-2} + (-1)^{r-2} 2C_{r2}, & \text{for } r > 2, \end{cases} \tag{15}$$

is the second derivative of $p_r(\boldsymbol{\alpha}'\boldsymbol{y})$ with respect to $\boldsymbol{\alpha}'\boldsymbol{y}$. The second derivatives of the first five polynomials are given in the Appendix. The second non-central moment of Θ given $\boldsymbol{Y} = \mathbf{0}$ is then given by

$$M_{\Theta|\mathbf{0}}^{(2)}(0) = \mu_{2|\mathbf{0}} = \left\{ \sum_{r=1}^{k} \gamma_r r^{-1} (-1)^{r-1} \right\}^2 + \sum_{r=1}^{k} \gamma_r (-1)^{r-2} 2C_{r2}. \tag{16}$$

As a consequence the variance of Θ given $\boldsymbol{Y} = \mathbf{0}$ is given by

$$\text{var}(\Theta|\mathbf{0}) = \mu_{2|\mathbf{0}} - \mu_{1|\mathbf{0}}^2 = \sum_{r=1}^{k} \gamma_r (-1)^{r-2} 2C_{r2}. \tag{17}$$

Since $\mu_{0|\mathbf{0}}, \mu_{1|\mathbf{0}}, \mu_{2|\mathbf{0}}, \ldots, \mu_{k|\mathbf{0}}$ satisfy inequalities that follow from the general solution to the Hamburger moment problem (Karlin & Studden, 1966) and are functions of the parameters $\gamma_1, \gamma_2, \ldots, \gamma_k$ the parameters $\gamma_1, \gamma_2, \ldots, \gamma_k$ also satisfy certain inequalities.

Special cases of the random effects generalized partial credit model can be obtained by setting some of $\gamma_1, \gamma_2, \ldots, \gamma_k$ equal to zero. A reasonable special case is one where $\gamma_{r+1}, \ldots, \gamma_k$ are all equal to zero and therefore only contain interactions up to rth order. The following theorem provides an interesting result for the special case that only contains second-order interactions.

Theorem 2. *If, under the random effects generalized partial credit model, $\gamma_3, \ldots, \gamma_k$ are all equal to zero, then the distribution of Θ given $\boldsymbol{Y} = \boldsymbol{y}$ is normal with mean $\gamma_2 \boldsymbol{\alpha}'\boldsymbol{y} + \gamma_1 - \frac{1}{2}\gamma_2$ and variance γ_2.*

Proof 2. If $\gamma_3, \ldots, \gamma_k$ are all equal to zero, then

$$M_{\Theta|0}(\boldsymbol{\alpha}'\boldsymbol{y}) = \exp\left\{ \gamma_1 \boldsymbol{\alpha}'\boldsymbol{y} + \gamma_2 \frac{1}{2} \alpha'\boldsymbol{y}(\alpha'\boldsymbol{y} - 1) \right\} = \exp\left\{ \left(\gamma_1 - \frac{1}{2}\gamma_2 \right) \alpha'\boldsymbol{y} + \frac{1}{2}\gamma_2 (\alpha'\boldsymbol{y})^2 \right\},$$

which is the moment generating function of a normal random variable with mean $\gamma_1 - \frac{1}{2}\gamma_2$ and variance γ_2. Furthermore,

$$g(\theta|\boldsymbol{y}) = \frac{P(\boldsymbol{Y} = \boldsymbol{y}|\theta) f(\theta)}{P(\boldsymbol{Y} = \boldsymbol{y})} = \frac{\exp(\boldsymbol{\alpha}'\boldsymbol{y}\theta) P(\boldsymbol{Y} = \mathbf{0}|\theta) f(\theta)}{M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y}) P(\boldsymbol{Y} = \mathbf{0})} = \frac{\exp(\boldsymbol{\alpha}'\boldsymbol{y}\theta)}{M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y})} g(\theta|\mathbf{0}),$$

where $g(\theta|0) = (2v)^{-1/2} \gamma_2^{-1/2} \exp\left[-\frac{1}{2} \left\{ \theta - \left(\gamma_1 - \frac{1}{2}\gamma_2 \right) \right\}^2 / \gamma_2 \right]$, so that after some algebra it follows that

$$g(\theta|\boldsymbol{y}) = (2v)^{-1/2}\gamma_2^{-1/2}\exp\left[-\frac{1}{2}\left\{\theta - \left(\gamma_2\boldsymbol{\alpha}'\boldsymbol{y} + \gamma_1 - \frac{1}{2}\gamma_2\right)\right\}^2 /\gamma_2\right],$$

which can be recognized as a normal density with mean $\gamma_2\boldsymbol{\alpha}'\boldsymbol{y} + \gamma_1 - \frac{1}{2}\gamma_2$ and variance γ_2 .

The special case in Theorem 2 is called the conditional normal generalized partial credit model. For this conditional normal generalized partial credit model, the moment inequalities simply imply that the variance γ_2 is positive. A consequence of conditional normality of Θ given $\boldsymbol{Y} = \boldsymbol{y}$ is that Θ has a mixture distribution of normals in the total population.

In the case of dichotomously scored items, the conditional normal generalized partial credit model is called the conditional normal two-parameter logistic model. The conditional normal two-parameter logistic model is a special case of the Ising model (Ising, 1925) given by

$$P(\boldsymbol{Y} = \boldsymbol{y}) = P(\boldsymbol{Y} = \boldsymbol{0})\exp\left(\sum_{i=1}^{k}\lambda_i y_i + \sum_{i<j}\lambda_{\text{ij}} y_i y_j\right), \tag{18}$$

where $y_i = x_{i1}$, for all i. The Ising model is the special case of the model in equation (9) where all interaction parameters of third order and higher are set to zero. The Ising model specializes to the conditional normal two-parameter logistic model if $\lambda_i = \beta_i + \gamma_1\alpha_i + \gamma_2\alpha_i(\alpha_i - 1)/2$, for all i, and $\lambda_{ij} = \gamma_2\alpha_i\alpha_j$, for all $i < j$.

A slightly more general model than the random effects generalized partial credit model in which moment inequalities are ignored is given by

$$P(\boldsymbol{Y} = \boldsymbol{y}) = P(\boldsymbol{Y} = \boldsymbol{0})\exp\left\{\sum_{i=1}^{k}\sum_{s=1}^{m_i}\beta_{\text{is}}x_{\text{is}} + \sum_{r=1}^{k}\nu_r p_r(\boldsymbol{\alpha}'\boldsymbol{y})\right\}, \tag{19}$$

where $\nu_r$is an unconstrained rth-order common interaction parameter, β_{is} is an item category main effect parameter, and α_i is a parameter that expresses the extent to which item i contributes to all common interactions. The model in equation (19) is called the extended generalized partial credit model. In the case of dichotomously scored items, the model is called the extended two-parameter logistic model. Fixing the parameters $\alpha_1,\ldots,\alpha_k$ to specific real numbers, the extended generalized partial credit model specializes to an exponential family model in which $p_r(\boldsymbol{\alpha}'\boldsymbol{y})$ is a sufficient statistic for ν_r, for all r. Special cases of this exponential family model are the extended partial credit model (Agresti, 1993), the extended Rasch model (Follmann, 1988; Tjur, 1982), and the models in which $\alpha_1,\ldots,\alpha_k$ are prespecified positive integers (Maris *et al.*, 2015; Verhelst & Glas, 1995).

Both the random effects and the extended generalized partial credit model have $\sum_{i=1}^{k} m_i + 2k$ parameters. However, both models have two indeterminacies. To show the first indeterminacy, the exponent on the right-hand side of equation (19) is rewritten using $p_1(\boldsymbol{\alpha}'\boldsymbol{y}) = \sum_{i=1}^{k}\alpha_i y_i = \sum_{i=1}^{k}\sum_{s=1}^{m_i}\alpha_i s x_{\text{is}}$. The exponent on the right-hand side of equation (19) then becomes $\sum_{i=1}^{k}\sum_{s=1}^{m_i}(\beta_{\text{is}} + \nu_1\alpha_i s)x_{\text{is}} + \sum_{r=2}^{k}\nu_r p_r(\boldsymbol{\alpha}'\boldsymbol{y})$, from which it follows that β_{is} and ν_1 cannot be identified, because if $\beta_{\text{is}} = \beta_{\text{is}}^* + \alpha_i sb$ and $\nu_1 = \nu_1^* - b$, for some constant b, then $\beta_{\text{is}} + \nu_1\alpha_i s = \beta_{\text{is}}^* + \nu_1^*\alpha_i s$. A convenient way to solve this first indeterminacy is to fix ν_1 to 0. To show the second indeterminacy it is assumed without loss of generality that $\nu_3 = 0$, for $r = 3,\ldots,$ k. Now, using $\nu_r = 0$ and $p_2(\boldsymbol{\alpha}'\boldsymbol{y}) = \boldsymbol{\alpha}'\boldsymbol{y}(\boldsymbol{\alpha}'\boldsymbol{y} - 1)/2 = \sum_i\sum_{s=1}^{m_i}\frac{1}{2}\alpha_i s(\alpha_i s - 1)x_{\text{is}} + \sum_{i<j}\alpha_i\alpha_j y_i y_j$, the exponent on the

right-hand side of equation (19) can be written as $\sum_{i=1}^{k}\sum_{s=1}^{m_i}\{\beta_{is}+\nu_2\alpha_i s(\alpha_i s-1)/2\}x_{is}+\nu_2\sum_{i<j}\alpha_i\alpha_j y_i y_j$, from which it follows that ν_2 and α_i cannot be identified, because if $\nu_2=\nu_2^*a^2$, $\alpha_i=\alpha_i^*a^{-1}$ and $\beta_{is}=\beta_{is}^*+\frac{1}{2}\nu_2^*\alpha_i^*as-\frac{1}{2}\nu_2^*\alpha_i^*s$, for some constant a and all i and s, then this exponent equals $\sum_{i=1}^{k}\sum_{s=1}^{m_i}\{\beta_{is}^*+\nu_2^*\alpha_i^*s(\alpha_i^*s-1)/2\}x_{is}+\nu_2^*\sum_{i<j}\alpha_i^*\alpha_j^*y_i y_j$. A convenient way to solve this second indeterminacy is to fix either one of $\alpha_1,\ldots,\alpha_k$ or ν_2 to 1. Consequently, the number of independent parameters under both models is $\sum_{i=1}^{k}m_i+2k-2$.

Note that in the conditional normal generalized partial credit model, moment inequalities are immediately satisfied if γ_1 is set to 0 and γ_2 is set to 1 for identification. The number of free parameters in this special case is $\sum_{i=1}^{k}m_i+k$.

3. Maximum likelihood estimation

It follows from equation (19) that the joint probability distribution of $\boldsymbol{Y}$ under the extended generalized partial credit model is given by

$$P(\boldsymbol{Y}=\boldsymbol{y})=\frac{\exp\left\{\sum_{i=1}^{k}\boldsymbol{\beta}_i'\boldsymbol{x}_i+\sum_{r=1}^{k}\nu_r p_r(\boldsymbol{\alpha}'\boldsymbol{y})\right\}}{\sum_{y}\exp\left\{\sum_{i=1}^{k}\boldsymbol{\beta}_i'\boldsymbol{x}_i+\sum_{r=1}^{k}\nu_r p_r(\boldsymbol{\alpha}'\boldsymbol{y})\right\}}, \tag{20}$$

where $\boldsymbol{\beta}_i=[\beta_{i1}\ldots\beta_{im_i}]'$ and $\boldsymbol{x}_i=[x_{i1}\ \ldots\ x_{im_i}]'$. Taking all possible score patterns into account and assuming independence of observations, it follows that the likelihood function is given by

$$L(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_k,\boldsymbol{\alpha},\boldsymbol{\nu})=\frac{\exp\left\{\sum_{i=1}^{k}\boldsymbol{\beta}_i'\boldsymbol{n}_i+\sum_{r=1}^{k}\nu_r\sum_{\mathbf{y}}\mathbf{n}_{\mathbf{y}}p_r(\boldsymbol{\alpha}'\mathbf{y})\right\}}{\left[\sum_{\mathbf{y}}\exp\left\{\sum_{i=1}^{k}\boldsymbol{\beta}_i'\mathbf{x}_i+\sum_{r=1}^{k}\nu_r p_r(\boldsymbol{\alpha}'\mathbf{y})\right\}\right]^n}, \tag{21}$$

where $\boldsymbol{\nu}=[\nu_1\ldots\nu_k]'$ $\mathbf{n}_i=[n_{i1}\ \ldots\ n_{im_i}]'$ is the vector that contains the numbers of individuals in the sample with scores $1,\ldots,m_i$ on item i, $n_{\boldsymbol{y}}$ is the number of individuals in the sample with response pattern $\boldsymbol{y}$, and n is the total sample size.

To find the estimates of the parameters that maximize the likelihood function in equation (21), the log-likelihood function given by $l(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_k,\boldsymbol{\alpha},\boldsymbol{\nu})=lnL(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_k,\boldsymbol{\alpha},\boldsymbol{\nu})$ can be maximized with respect to the parameters subject to the constraints that$\nu_1=0$ and $\nu_2=1$. To solve this unconstrained nonlinear optimization problem, the Broyden—Fletcher–Goldfarb–Shanno (BFGS) algorithm can be used (Fletcher, 1987). The BFGS algorithm is a quasi-Newton method, in which the Hessian matrix of second derivatives is not computed. Instead, the Hessian matrix is approximated using updates specified by (approximate) evaluations of the first derivatives. The first derivatives of the log-likelihood are given in the Appendix. Unfortunately, the likelihood function in equation (21) is not in general concave and may have multiple extreme points. This means that the BFGS algorithm can only find a local extreme point in the vicinity of the starting point. However, to find the global maximum the BFGS algorithm can be combined with a multistart method (Nash, 2014) or evolutionary algorithm methods (Mebane & Sekhon, 2011).

Note that the extended generalized partial credit model can be rewritten as the non-standard log-linear model

$$\ln e_{\mathbf{y}} = \rho + \sum_{i=1}^{k} \boldsymbol{\beta}_i' \mathbf{x}_i + \sum_{r=1}^{k} \nu_r p_r(\boldsymbol{\alpha}'\boldsymbol{y}) \tag{22}$$

where $e_y = nP(\boldsymbol{Y} = \boldsymbol{y})$ and $\rho = \ln\{nP(\boldsymbol{Y} = \mathbf{0})\}$. If $\alpha_1,\ldots,\alpha_k$ are prespecified real numbers, then the non-standard log-linear model in equation (22) turns into a standard log-linear model that can be fitted to data using an iterative proportional fitting algorithm (Kelderman, 1992) or iterative weighted least squares (Charnes, Frome, & Yu, 1976).

Let $\hat{\nu}_1, \ldots, \hat{\nu}_k$ be the maximum likelihood estimates of $\nu_1, \ldots, \nu_k$ and let

$$h(\boldsymbol{\alpha}'\boldsymbol{y}) = \exp\left\{ \sum_{r=1}^{k} \hat{\nu}_r p_r(\boldsymbol{\alpha}'\boldsymbol{y}) \right\}$$

Also let $h^{(1)}(0), h^{(2)}(0),\ldots, h^{(k)}$ be the first, second,..., kth derivative of $h(\boldsymbol{\alpha}'\boldsymbol{y})$ with respect to $\boldsymbol{\alpha}'\boldsymbol{y}$ evaluated at $\boldsymbol{\alpha}'\boldsymbol{y} = 0$. Now if $h(0), h^{(1)}(0), h^{(2)}(0),\ldots, h^{(k)}(0)$ satisfy the constraints of a moment sequence, then the maximum likelihood estimates of the parameters of the extended generalized partial credit model are also the maximum likelihood estimates of the parameters of the random effects generalized partial credit model. It follows from the solution to the Hamburger moment problem that if the matrix.

$$\begin{bmatrix} h(0) & h^{(1)}(0) & h^{(2)}(0) & \cdots & h^{(c)}(0) \\ h^{(1)}(0) & h^{(2)}(0) & h^{(3)}(0) & \cdots & h^{(c+1)}(0) \\ h^{(2)}(0) & h^{(3)}(0) & h^{(4)}(0) & \cdots & h^{(c+2)}(0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h^{(c)}(0) & h^{(c+1)}(0) & h^{(c+2)}(0) & \cdots & h^{(2c)}(0) \end{bmatrix}$$

where $c = k/2$ when k is even and $c = k/2 - 1/2$ when k is odd, is not positive definite, then $h(0), h^{(1)}(0), h^{(2)}(0),\ldots, h^{(k)}(0)$ do not satisfy the constraints of a moment sequence (Karlin & Studden, 1966).

4. Goodness-of-fit test

When, in practice, all possible response patterns are observed frequently enough, then the goodness of fit of the model in equation (19) can be assessed by testing the model against the saturated multinomial model using Pearson's asymptotic chi-square test or a generalized likelihood ratio test. Most of the time in practice, however, many possible response patterns are not observed and then these asymptotic tests are not appropriate. In such cases, the goodness of fit of the model in equation (19) can instead be assessed by testing the model against a less general alternative than the saturated multinomial model, using a generalized likelihood ratio test. One such less general alternative hypothesis model is given by

$$P(\boldsymbol{Y} = \boldsymbol{y}) = P(\boldsymbol{Y} = \mathbf{0}) \exp\left\{ \sum_{i=1}^{k} \sum_{s=1}^{m} \eta_{is} x_{is} + \sum_{i<j} \lambda_{\mathrm{ij}} y_i y_j + \sum_{r=3}^{k} \nu_r p_r(\boldsymbol{\alpha}'\boldsymbol{y}) \right\} \tag{23}$$

which specializes to the extended generalized partial credit model in equation (19) if $\eta_{is} = \beta_{is} + \nu_1 \alpha_i s + \nu_2 \alpha_i s(\alpha_i s - 1)/2$ and $\lambda_{ij} = \nu_2 \alpha_i \alpha_j$. The number of independent parameters under this model is $\sum_{i=1}^{k} m_i + k(k-1)/2 + 2k - 2$. So the number of degrees of freedom for the generalized likelihood ratio test of the extended generalized partial credit model against this alternative hypothesis model equals $k(k - 1)/2$. Note that the model in equation (23) specializes to the Ising model if $m_i = 1$, for all i, and $\nu_r = 0$, for $r = 3,\ldots,k$.

Another generalization of the extended generalized partial credit model that can be used as the alternative hypothesis model in a generalized likelihood ratio test is given by

$$P(\mathbf{Y}=\mathbf{y}) = P(\mathbf{Y}=\mathbf{0})\exp\left\{\sum_{i=1}^{k}\boldsymbol{\beta}_i'\mathbf{x}_i + \sum_{r=1}^{k}\nu_r p_r\left(\sum_{i=1}^{k}\boldsymbol{\alpha}_i'\mathbf{x}_i\right)\right\}, \tag{24}$$

where $\boldsymbol{\alpha}_i = [\alpha_{i1} \ldots \alpha_{im_i}]'$. The model in equation (24) is called the extended nominal response model (Bock, 1972). This extended nominal response model equals the random effects nominal response model if the parameters $\nu_1, \ldots, \nu_k$ satisfy the complex inequalities that follow from the inequalities of a moment sequence. The random effects nominal response model is a more general item response model than the random effects generalized partial credit model. In practice, however, the random effects nominal response model is often of less interest than the random effects generalized partial credit model because the response categories of the items are usually ordered. Note that the extended nominal response model specializes to the extended generalized partial credit model if $\alpha_{is} = s\alpha_i$, for all i and s. The number of independent parameters under this model is equal to $2\sum_{i=1}^{k} m_i + k - 2$. Also note that if $\boldsymbol{\alpha}_1,\ldots, \boldsymbol{\alpha}_k$ are vectors of prespecified real numbers, then the extended nominal response model specializes to an exponential family model in which $p_r\left(\sum_{i=1}^{k}\boldsymbol{\alpha}_i'\boldsymbol{x}_i\right)$ is a sufficient statistic for ν_r , for all r. Special cases of this exponential family model are the partial credit model (Masters, 1982) and the model in which $\boldsymbol{\alpha}_1,\ldots, \boldsymbol{\alpha}_k$ are vectors of prespecified positive integers (Maris *et al.*, 2015).

Along the same lines as for the extended generalized partial credit model, log-likelihood functions for the two models in equations (23) and (24) can be constructed and maximized. Both models can be rewritten as non-standard log-linear models and if $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_1,\ldots, \boldsymbol{\alpha}_k$ are vectors of prespecified real numbers, then the non-standard log-linear models turn into standard log-linear models that can be fitted to data using an iterative proportional fitting algorithm (Kelderman, 1992) or iterative weighted least squares (Charnes *et al.*, 1976).

The extended generalized partial credit model can in turn be used as the alternative hypothesis model in a generalized likelihood ratio test for one of its special cases, such as the conditional normal generalized partial credit model or the standard random effects generalized partial credit model in which the latent variable is assumed to be normally distributed in the population of examinees. As such, these tests provide possible checks for posterior normality and prior normality of the latent variable assuming the generalized partial credit model to be true. In both cases, the number of degrees of freedom equals $k - 2$.

Another alternative hypothesis model in a generalized likelihood ratio test for the conditional normal generalized partial credit model is the conditional normal nominal response model, which can be obtained from the model in equation (24) by setting $\nu_1 = 0$, $\nu_2 = 1$ and $\nu_3 = 0$, for $r = 3, \ldots, k$. A generalization of Theorem 2 is that under this conditional normal nominal response model, the latent variable Θ has a conditional normal distribution given $\boldsymbol{Y} = \boldsymbol{y}$ with mean $\sum_{i=1}^{k}\boldsymbol{\alpha}_i'\boldsymbol{x}_i - \frac{1}{2}$ and variance 1. The proof is similar to the proof of Theorem 2.

5. Person parameters

A natural way to obtain person parameter estimates under the random effects generalized partial credit model is to estimate the conditional expected value of the latent variable given response pattern $\boldsymbol{Y} = \boldsymbol{y}$, for all $\boldsymbol{y}$. These person parameter estimates are called expected a posteriori estimates. The calculation of these estimates requires a theoretical expression for the conditional expected value of the latent variable given $\boldsymbol{Y} = \boldsymbol{y}$. This theoretical expression can be obtained via the moment generating function of Θ given $\boldsymbol{Y} = \boldsymbol{y}$, which is presented in the following theorem.

Theorem 3. *Under the random effects generalized partial credit model, the conditional moment generating function of* Θ *given* $\boldsymbol{Y} = \boldsymbol{y}$ *is given by*

$$M_{\Theta|\mathbf{y}}(z) = \exp\left[\sum_{r=1}^{k} \gamma_r \{p_r(z + \boldsymbol{\alpha}'\boldsymbol{y}) - p_r(\boldsymbol{\alpha}'\boldsymbol{y})\}\right] \qquad (25)$$

Proof 3. The moment generating function of Θ given $\boldsymbol{Y} = \boldsymbol{y}$ is given by

$$M_{\Theta|\boldsymbol{y}}(z) = \int \exp(z\theta) g(\theta|\boldsymbol{y}) d\theta$$

Substitution of

$$g(\theta|\boldsymbol{y}) = \frac{\exp(\boldsymbol{\alpha}'\boldsymbol{y}\theta)}{M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y})} g(\theta|\mathbf{0}).$$

yields

$$\begin{aligned} M_{\Theta|\boldsymbol{y}}(z) &= \int \exp(z\theta)\exp(\boldsymbol{\alpha}'\boldsymbol{y}\theta) g(\theta|\mathbf{0})\, d\theta \{M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y})\}^{-1} \\ &= \int \exp\{(z + \boldsymbol{\alpha}'\boldsymbol{y})\theta\} g(\theta|\mathbf{0})\, d\theta \{M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y})\}^{-1} \\ &= M_{\Theta|\mathbf{0}}(z + \boldsymbol{\alpha}'\boldsymbol{y}) \{M_{\Theta|\mathbf{0}}(\boldsymbol{\alpha}'\boldsymbol{y})\}^{-1} \\ &= \exp\left\{ \sum_{r=1}^{k} \gamma_r p_r(z + \boldsymbol{\alpha}'\boldsymbol{y}) - \sum_{r=1}^{k} \gamma_r p_r(\boldsymbol{\alpha}'\boldsymbol{y}) \right\} \end{aligned}$$

and factoring $\sum_{r=1}^{k} \gamma_r$ yields the result in equation (25).

The theoretical expression for the expected value of Θ given $\boldsymbol{Y} = \boldsymbol{y}$ can now be obtained by taking the first derivative of $M_{\Theta|\mathbf{y}}(z)$ with respect to z and evaluating it at $z = 0$. The first derivative of $M_{\Theta|\mathbf{y}}(z)$ with respect to z is given by

$$M^{(1)}_{\Theta|\mathbf{y}}(z) = M_{\Theta|\mathbf{y}}(z) \sum_{r=1}^{k} \gamma_r p_r^{(1)}(z + \boldsymbol{\alpha}'\boldsymbol{y})$$

Consequently, the expected value of Θ given $\boldsymbol{Y} = \boldsymbol{y}$ is given by

$$M^{(1)}_{\Theta|y}(0) = \mu_{1|y} = \sum_{r=1}^{k} \gamma_r p_r^{(1)}(\alpha'\mathbf{y}) = \sum_{r=1}^{k} \gamma_r \sum_{u=0}^{r-1} (-1)^u (r-u) C_{r(r-u)} (\alpha'\mathbf{y})^{r-u-1}$$

An estimate of the expected value of Θ given $\boldsymbol{Y} = \boldsymbol{y}$ is then given by

$$\hat{\mu}_{1|\mathbf{y}} = \sum_{r=1}^{k} \hat{\gamma}_r p_r^{(1)}(\hat{\boldsymbol{\alpha}}'\mathbf{y}),$$

where $\hat{\gamma}_r$ is an estimate of γ_r, for all r, and $\hat{\boldsymbol{\alpha}}$ is an estimate of $\boldsymbol{\alpha}$.

6. A simulation study

The purpose of this simulation study is to investigate the usefulness of the extended and the standard two-parameter logistic models under normality and non-normality of the latent variable in the population of examinees. The standard two-parameter logistic model is the model in which the latent variable is assumed to be normal in the population of examinees. The usefulness of each model is assessed using rejection rates of generalized likelihood ratio tests and approximate parameter estimation bias and efficiency. Three generalized likelihood ratio tests are used. The first generalized likelihood ratio test concerns the comparison of the extended two-parameter logistic model to the saturated multinomial model (test 1). The second generalized likelihood ratio test concerns the comparison of the standard two-parameter logistic model to the saturated multinomial model (test 2). The third generalized likelihood ratio test concerns the comparison of the standard two-parameter logistic model to the extended two-parameter logistic model (test 3).

The R program (R Core Team, 2020) was used to generate binary data under the random effects two-parameter logistic model, for all combinations of three sample sizes and two latent variable distributions. Test length was not varied and was fixed to five items. The chosen sample sizes are 300, 500 and 700. The chosen latent variable distributions are a normal distribution and a mixture of two normal distributions. In the case of the normal distribution, the mean is zero and the standard deviation is 2. In the case of the mixture distribution, the mixing proportions are .55 and .45, the mean and standard deviation of the first normal distribution constituting the mixture are − 2 and 1, and the mean and standard deviation of the second normal distribution constituting the mixture are 2 and 3.5. The mean and standard deviation of the mixture distribution are − 0.2 and 3.17. The densities of both distributions are shown in Figure 1.

In each of the six conditions (3 sample sizes × 2 distributions), 1,000 data sets were randomly generated. For the generation of each data set, n latent values $\theta_1, \ldots, \theta_n$ were randomly drawn from the latent variable distribution. The histogram of the randomly drawn latent values for one of the mixture conditions is given in Figure 2.

Each binary data set was generated by randomly drawing a single sample from the Bernoulli distribution given by

$$P(Y_{vi} = y_{vi}|\theta_v) = \{\pi_i(\theta_v)\}^{y_{vi}}\{1-\pi_i(\theta_v)\}^{1-y_{vi}}, \text{ for } y_{vi} = 0, 1,$$

for all $v = 1, \ldots, n$ and $i = 1, \ldots, 5$, where

$$\pi_i(\theta_v) = \frac{\exp\{\alpha_i(\theta_v - \delta_i)\}}{1 + \exp\{\alpha_i(\theta_v - \delta_i)\}}$$

is the so-called item response function of item i, that is, the probability of score 1 as a function of θ_v. The item parameter values selected are given in Table 1. The item response functions, for $i = 1, \ldots, 5$, are shown in Figure 3.

The extended and standard two-parameter logistic models were fitted to each data set generated. To fit the extended model self-written R code and the R package *rgenoud*

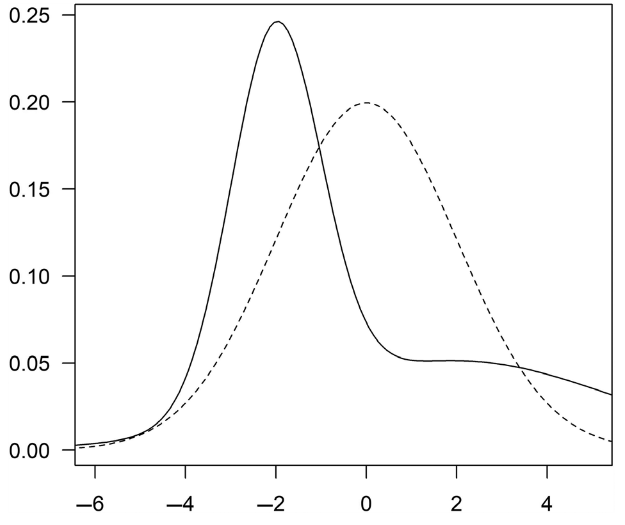

Figure 1. Densities of the normal distribution and the mixture of two normals used in the simulation study.

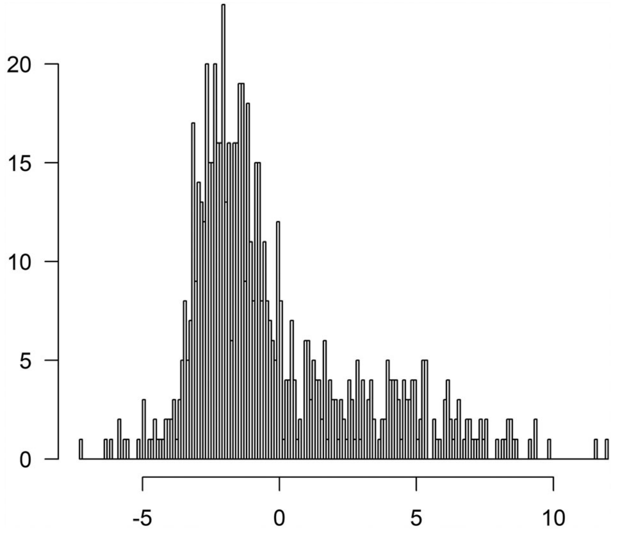

Figure 2. Histogram of sample latent values generated under the mixture of two normals.

Table 1. Item parameter values used to generate the data

	Item				
	1	2	3	4	5
δ_i	−1.50	−0.75	0.00	0.80	2.25
α_i	0.50	0.75	0.50	0.60	0.80

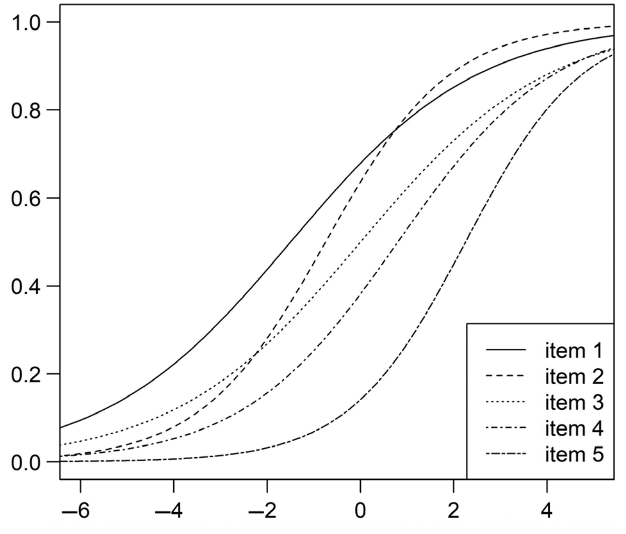

Figure 3. The five item response functions used to generate the data.

(Mebane & Sekhon, 2011) were used, and to fit the standard model the R package *ltm* (Rizopoulos, 2006) was used. The optimization algorithms did not show convergence for all data sets. For each condition, data sets continued to be generated until 1,000 proper data sets (for which the optimization algorithms showed convergence) were obtained. Information concerning the number of times the optimization algorithms did not converge is given in Table 2.

For each proper data set, the three generalized likelihood ratio tests (tests 1, 2 and 3) were carried out using the nominal level of significance of .05. In each condition, for each test a rejection rate was calculated. The rejection rate of a test is the number of times the

Table 2. Counts and proportions of non-convergence of the optimization algorithms

Distribution	n	Count	Proportion
Mixture	300	276	.216
	500	161	.139
	700	108	.097
Normal	300	307	.235
	500	144	.126
	700	87	.080

Table 3. Rejection rates of the three likelihood ratio tests (test 1, extended versus saturated; test 2, standard versus saturated; test 3, standard versus extended)

Distribution	n	Test 1	Test 2	Test 3
Mixture	300	.061	.480	.824
	500	.064	.729	.964
	700	.067	.884	.988
Normal	300	.073	.076	.075
	500	.051	.065	.054
	700	.069	.068	.070

null hypothesis model was rejected in favour of the alternative hypothesis model divided by 1,000. All calculated rejection rates are given in Table 3.

The results in Table 3 show that the proportion of times the extended two-parameter logistic model is rejected (test 1) is close to the nominal level of significance for all conditions, so irrespective of the sample size and the latent variable distribution. The results in Table 3 also show that the proportion of times the standard two-parameter logistic model is rejected (tests 2 and 3) is close to the nominal level of significance for the normal distribution but not for the mixture distribution. For the mixture distribution, the rejection rates of tests 2 and 3 are much higher than the nominal level of significance. This means that if the goodness of fit of the random effects two-parameter logistic model is assessed by fitting and testing the standard two-parameter logistic model, the random effects two-parameter logistic model will be rejected too often. For the mixture distribution, the rejection rates of tests 2 and 3 are approximate power values and the results in Table 3 show that likelihood ratio testing the standard two-parameter logistic model against the extended two-parameter logistic model (test 3) is more powerful in detecting non-normality than likelihood ratio testing the standard two-parameter logistic model against the saturated multinomial model (test 2). As expected, the approximate power of tests 2 and 3 increases with sample size. In conclusion, to assess the goodness of fit of the random effects two-parameter logistic model, test 1 should be used rather than test 2 or test 3, which means that the extended two-parameter logistic model should be tested instead of the standard two-parameter logistic model.

The scales of the item parameters used to generate the data differ from the scales of the estimates under the extended two-parameter logistic model. The estimates of the parameters of the extended model are obtained using the identification constraints $\nu_1 = 0$ and $\nu_2 = 1$. To study the bias and efficiency of the estimates, these scales must be equal. This can be achieved by generating data under the same parameterization as the one that is used in fitting the model to the data. For this reason a second simulation was carried out in which, for each of the six conditions, 1,000 proper data sets are randomly sampled from the multinomial distribution where the multinomial probabilities are given by equation (20) and the parameter values are chosen to be the mean estimates from the first simulation. These mean estimates are the true values of the parameters given in Table 4. After fitting the extended two-parameter logistic model to all newly generated data sets, for each parameter the mean over the new estimates, the approximate bias (the difference between the mean estimate and the true parameter value), and the standard deviation over the new estimates are calculated for each of the six conditions. The results are given in Table 4.

Table 4. Parameter estimation results under the extended two-parameter logistic model

Dist.	n		β_1	β_2	β_3	β_4	β_2	v_3	v_4	v_5	α_1	α_2	α_3	α_4	α_5
Mixture	300	Value	−0.381	−1.164	−1.129	−1.837	−3.694	0.755	−0.236	−1.854	0.624	1.006	0.608	0.727	1.004
		Mean	−0.425	−1.309	−1.193	−1.956	−4.001	0.301	0.169	−1.256	0.697	1.353	0.656	0.794	1.130
		Bias	−0.044	−0.145	−0.064	−0.119	−0.307	−0.454	0.405	0.598	0.073	0.347	0.048	0.067	0.126
		SD	0.344	1.044	0.323	0.462	1.021	1.423	3.076	6.350	0.326	0.799	0.226	0.283	0.446
	500	Mean	−0.401	−1.238	−1.166	1.882	−3.881	0.629	−0.372	−1.462	0.655	1.147	0.629	0.761	1.079
		Bias	−0.020	−0.074	−0.037	−0.045	−0.187	−0.0126	−0.136	0.392	0.031	0.141	0.021	0.034	0.075
		SD	0.318	0.587	0.228	0.324	0.837	1.016	2.267	5.598	0.191	0.522	0.151	0.198	0.346
	700	Mean	−0.393	−1.235	−1.144	−1.868	−3.811	0.684	−0.368	−2.241	0.636	1.114	0.617	0.739	1.043
		Bias	−0.012	−0.071	−0.015	−0.031	−0.117	−0.071	−0.132	−0.387	0.012	0.108	0.009	0.012	0.039
		SD	0.168	0.487	0.185	0.239	0.574	0.866	2.177	5.201	0.119	0.417	0.117	0.139	0.245
Normal	300	Value	−0.130	−0.830	−0.888	−1.582	−3.425	−0.122	0.097	−0.581	0.670	1.103	0.670	0.808	1.144
		Mean	−0.160	−0.966	−0.946	−1.716	−3.921	−0.155	−0.065	−0.682	0.736	1.311	0.718	0.882	1.374
		Bias	−0.030	−0.136	−0.058	−0.134	−0.496	−0.033	−0.162	−0.101	0.066	0.208	0.048	0.074	0.230
		SD	0.439	0.968	0.512	0.770	1.582	0.903	2.528	5.842	0.375	0.696	0.370	0.459	0.800
	500	Mean	−0.120	−0.925	−0.929	−1.664	−3.753	−0.219	0.063	−1.014	0.718	1.281	0.694	0.861	1.301
		Bias	0.010	−0.095	−0.041	−0.082	−0.328	−0.097	−0.034	−0.433	0.048	0.178	0.024	0.053	0.157
		SD	0.631	0.737	0.340	0.464	1.174	0.878	2.095	4.379	0.342	0.590	0.236	0.310	0.616
	700	Mean	−0.149	−0.934	−0.919	−1.650	−3.643	−0.180	0.004	−0.892	0.693	1.253	0.689	0.855	1.253
		Bias	−0.019	−0.104	−0.031	−0.068	−0.218	−0.058	−0.093	−0.311	0.023	0.150	0.019	0.047	0.109
		SD	0.329	0.683	0.260	0.447	0.986	0.714	1.835	3.725	0.195	0.517	0.171	0.250	0.500

The results in Table 4 show that the estimates of the parameters of the extended two-parameter logistic model are asymptotically unbiased irrespective of the latent variable distribution. Under both distributions, the calculated bias decreases and tends to zero as the sample size increases. The results also show that the efficiency of the estimates increases with sample size irrespective of the latent variable distribution. Under both distributions, the standard deviations of the estimates decrease with sample size.

The scales of the item parameters used to generate the data in the first simulation and the scales of the estimates under the standard two-parameter logistic model were set equal to each other by linear transformations. After fitting the standard two-parameter logistic model to all data sets generated, for each parameter the mean estimate, the approximate bias, and the standard deviation over the estimates were again calculated for each of the six conditions. The results are given in Table 5.

The results in Table 5 show that the estimates of the parameters of the standard two-parameter logistic model are asymptotically unbiased for the normal distribution but not for the mixture distribution. This can be seen most clearly by comparing the results under the normal distribution with the results under the mixture distribution. Under the normal distribution, the calculated bias decreases and tends to zero as the sample size increases. Under the mixture distribution, the calculated bias is substantial compared to the calculated bias under the normal distribution. The efficiency results are similar to the results obtained by fitting the extended two-parameter logistic model.

7. A simple illustrative example

The data in this example are the scores of 1,310 toddlers on five dichotomously scored items that are supposed to measure the mastery of concepts of comparison such as most, least, higher and lower. All possible score patterns and their observed frequencies are given in Table 6.

Both the model in equation (19) (the extended two-parameter logistic model) and the standard two-parameter logistic model are fitted to the data in Table 6 using the same software as in the simulation study. In the first model the parameters ν_1 and ν_2 are respectively set to 0 and 1 for identification. In the second model the mean and variance of the normal latent variable are respectively set to 0 and 1 for identification. Maximum likelihood estimates of the parameters of both models, together with their standard errors, are given in Table 7. The standard errors in Table 7 were obtained by taking the square root of the reciprocals of the diagonal elements of the Hessian matrix produced by the BFGS algorithm (used in both R packages).

The value of the log-likelihood under the extended model is $-$ 2,540.066. The value of the log-likelihood under the standard model is $-$ 2,544.415. Testing the extended model against the saturated multinomial model using a generalized likelihood ratio test yields $-2\ln LR = 18.876$ on 18 degrees of freedom, and a p-value of .340. Testing the standard model against the saturated multinomial model using a generalized likelihood ratio test yields $-2\ln LR = 27.574$ on 21 degrees of freedom, and a p-value of .153. Testing the standard model against the extended model using a generalized likelihood ratio test yields $-2\ln LR = 8.698$ on 3 degrees of freedom, and a p-value of .034. So the standard two-parameter logistic model can be rejected in favour of the extended model at the .05 significance level. Since under the extended model, the matrix.

Table 5. Parameter estimation results under the standard 2PLM (assuming normality)

Distribution	n		δ_1	δ_2	δ_3	δ_4	δ_5	α_1	α_2	α_3	α_4	α_5
Mixture		Value	−1.500	−0.750	0.000	0.800	2.250	0.500	0.750	0.500	0.600	0.800
	300	Mean	−1.329	−0.234	0.362	1.188	2.294	0.437	0.681	0.500	0.661	1.489
		Bias	0.171	0.516	0.362	0.388	0.044	−0.063	−0.069	0.000	0.061	0.689
		SD	0.404	0.291	0.352	0.338	0.309	0.082	0.141	0.089	0.130	1.007
	500	Mean	−1.328	−0.254	0.355	1.171	2.293	0.435	0.666	0.494	0.658	1.276
		Bias	0.172	0.496	0.355	0.371	0.043	−0.065	−0.084	−0.006	0.058	0.476
		SD	0.302	0.239	0.268	0.263	0.250	0.062	0.098	0.069	0.101	0.586
	700	Mean	−1.323	−0.246	0.354	1.176	2.297	0.433	0.661	0.495	0.647	1.230
		Bias	0.177	0.504	0.354	0.376	0.047	−0.067	−0.089	−0.005	0.047	0.430
		SD	0.250	0.202	0.226	0.217	0.199	0.051	0.082	0.057	0.079	0.444
Normal		Value	−1.500	−0.750	0.000	0.800	2.250	0.500	0.750	0.500	0.600	0.800
	300	Mean	−1.542	−0.767	−0.003	0.803	2.284	0.513	0.787	0.520	0.628	0.871
		Bias	−0.042	−0.017	−0.003	0.003	0.034	0.013	0.037	0.020	0.028	0.071
		SD	0.447	0.246	0.286	0.284	0.422	0.129	0.219	0.135	0.160	0.417
	500	Mean	−1.542	−0.764	−0.016	0.808	2.275	0.509	0.771	0.504	0.616	0.824
		Bias	−0.042	−0.014	−0.016	0.008	0.025	0.009	0.021	0.004	0.016	0.024
		SD	0.323	0.180	0.220	0.221	0.320	0.098	0.154	0.096	0.116	0.183
	700	Mean	−1.514	−0.756	−0.006	0.804	2.268	0.507	0.760	0.504	0.607	0.819
		Bias	−0.014	−0.006	−0.006	0.004	0.018	0.007	0.010	0.004	0.007	0.019
		SD	0.270	0.161	0.188	0.185	0.264	0.085	0.125	0.079	0.094	0.146

Table 6. All possible score patterns and their observed frequencies for five items measuring the mastery of concepts of comparison in a sample of 1,310 toddlers, and EAP person parameter estimates under the extended two-parameter logistic model

y_1	y_2	y_3	y_4	y_5	n_y	$\hat{\mu}_{1\|y}$
0	0	0	0	0	15	−0.765
1	0	0	0	0	15	0.898
0	1	0	0	0	3	1.504
1	1	0	0	0	15	1.553
0	0	1	0	0	2	1.289
1	0	1	0	0	14	1.548
0	1	1	0	0	5	1.545
1	1	1	0	0	27	1.624
0	0	0	1	0	6	0.882
1	0	0	1	0	9	1.470
0	1	0	1	0	7	1.553
1	1	0	1	0	23	1.560
0	0	1	1	0	7	1.547
1	0	1	1	0	40	1.546
0	1	1	1	0	9	1.621
1	1	1	1	0	71	1.898
y_1	y_2	y_3	y_4	y_5	n_y	$\hat{\mu}_{1\|y}$
0	0	0	0	1	4	1.547
1	0	0	0	1	6	1.546
0	1	0	0	1	3	1.621
1	1	0	0	1	17	1.899
0	0	1	0	1	1	1.553
1	0	1	0	1	11	1.701
0	1	1	0	1	9	2.107
1	1	1	0	1	100	2.509
0	0	0	1	1	3	1.546
1	0	0	1	1	7	1.597
0	1	0	1	1	5	1.893
1	1	0	1	1	51	2.308
0	0	1	1	1	4	1.697
1	0	1	1	1	38	2.041
0	1	1	1	1	37	2.504
1	1	1	1	1	746	2.614

$$\begin{bmatrix} h(0) & h^{(1)}(0) & h^{(2)}(0) \\ h^{(1)}(0) & h^{(2)}(0) & h^{(3)}(0) \\ h^{(2)}(0) & h^{(3)}(0) & h^{(4)}(0) \end{bmatrix} = \begin{bmatrix} 1.000 & -0.765 & 2.435 \\ -0.765 & 2.435 & -5.743 \\ 2.435 & -5.743 & 20.689 \end{bmatrix}$$

is positive definite (its eigenvalues are 22.647, 0.810 and 0.667), the maximum likelihood estimates of the parameters of the extended two-parameter model are also the maximum likelihood estimates of the parameters of the random effects two-parameter logistic model. Consequently, the EAP person parameter estimate for response pattern $\boldsymbol{y}$ under the random effects two-parameter logistic model is given by.

Table 7. Parameter estimates and their estimated standard errors obtained from fitting the extended (E-2PLM) and standard two-parameter logistic models (N-2PLM) to the scores of 1,310 toddlers on five items measuring the mastery of concepts of comparison

	E-2PLM		N-2PLM	
Parameter	Estimate	SE	Estimate	SE
β_1	−0.476	0.409	2.818	0.166
β_2	−3.849	0.608	2.945	0.244
β_3	−2.406	0.485	2.501	0.172
β_4	−1.397	0.389	1.763	0.105
β_5	−5.440	0.793	2.431	0.244
v_3	−0.568	0.030		
v_4	0.254	0.010		
v_5	−0.059	0.000		
α_1	1.323	0.212	1.199	0.164
α_2	2.831	0.338	2.074	0.258
α_3	2.004	0.253	1.598	0.190
α_4	1.302	0.198	1.059	0.131
α_5	3.307	0.402	2.370	0.321

$$\hat{\mu}_{1|\mathbf{y}} = p_2^{(1)}(\hat{\boldsymbol{\alpha}}'\mathbf{y}) + \hat{\gamma}_3 p_3^{(1)}(\hat{\boldsymbol{\alpha}}'\mathbf{y}) + \hat{\gamma}_4 p_4^{(1)}(\hat{\boldsymbol{\alpha}}'\mathbf{y}) + \hat{\gamma}_5 p_5^{(1)}(\hat{\boldsymbol{\alpha}}'\mathbf{y}).$$

In Table 6, this EAP estimate is given for all $\mathbf{y}$.

8. Discussion

It does not seem to be feasible in practice to perform direct estimation of the parameters of the random effects generalized partial credit model by maximizing the marginal likelihood function with respect to the parameters subject to the complex inequalities that follow from the inequalities of a moment sequence. Estimation of the parameters of the slightly more general extended generalized partial credit model is more feasible and provides consistent estimates of the parameters of the random effects generalized partial credit model. If these estimates also satisfy the inequalities that follow from the inequalities of a moment sequence, then the estimates are also the maximum likelihood estimates of the parameters of the random effects generalized partial credit model. For a discussion of these estimation properties under the extended Rasch model, see De Leeuw and Verhelst (1986).

The way in which indeterminacies in the extended generalized partial credit model are solved seems to determine whether its estimates are proper estimates of the parameters of the random effects generalized partial credit model. If, under the conditional normal generalized partial credit model, γ_1 is set to 0 and γ_2 is set to 1 for identification, then marginal maximum likelihood estimation always yields a proper solution, whereas if γ_1 is set to 0 and one of $\alpha_1, \ldots, \alpha_k$ is set to 1, then the estimate of the variance $\gamma_2$might be negative. This example raises the interesting question whether there is in general a method of identification that yields proper estimates. Further study is needed to find a conclusive answer to this question.

The likelihood function of the extended model is not concave in general, which means that it can have multiple extreme points. The methods to find the global maximum of the likelihood function proposed in this paper are a combination of the BFGS algorithm and a multistart method or evolutionary algorithms. These methods have a limited guarantee of finding the globally optimal solution. However, the probability of finding the globally optimal solution increases with the number of runs of the BFGS algorithm in a multistart method, but also with the population size and/or the number of generations used by evolutionary algorithms (Nix & Vose, 1992).

The proposed maximum likelihood estimation procedures are useful in practice when the number of items is small. Unfortunately, the efficiency of the procedure in terms of computation time rapidly decreases with the number of items and the numbers of response categories. This decrease in efficiency is due to the fact that the denominator of the normalizing constant $P(\boldsymbol{Y} = \mathbf{0})$ is computed by direct summation of its $\prod_{i=1}^{k} m_i$ terms. To increase the practical applicability of the proposed maximum likelihood estimation procedures, more efficient algorithms should be devised for computing $P(\boldsymbol{Y} = \mathbf{0})$. Pseudo-likelihood methods (Besag, 1975) are alternatives for obtaining parameter estimates that do not suffer from this computational problem.

A limitation of the proposed maximum likelihood estimation procedures is that they can only by applied to complete data. The likelihood function in equation (21) can, however, be modified such that all available data are used in estimating the parameters of the extended generalized partial credit model. Such a full information maximum likelihood estimation procedure would be one possible way to deal with missing data.

Maris *et al.* (2015) developed a Markov chain Monte Carlo method for Bayesian inference for the random effects one-parameter logistic model that does not rely on data augmentation. By applying the Dutch identity (Holland, 1990) to the random effects one-parameter logistic model, they derived the posterior expectation of ability for different scores. Using their approach, however, the posterior expectation of ability for a person with the highest possible score on all items cannot be estimated under the random effects one-parameter logistic model (Maris *et al.*, 2015, p. 863). An additional advantage of the approach proposed in this paper over the approach proposed by Maris *et al.* (2015) is that it provides the possibility of estimating this posterior expectation of ability under any random effects generalized partial credit model, and thus also under the random effects one-parameter logistic model.

Author contribution

David J. Hessen (Conceptualization; Formal analysis; Investigation; Methodology; Project administration; Writing – original draft; Writing – review & editing).

Conflicts of interest

All authors declare no conflict of interest.

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Appendix 1:

The first five polynomials are

$$\begin{aligned}
p_1(\boldsymbol{\alpha}'\boldsymbol{y}) &= \boldsymbol{\alpha}'\boldsymbol{y}\\
p_2(\boldsymbol{\alpha}'\boldsymbol{y}) &= \frac{1}{2}(\boldsymbol{\alpha}'\boldsymbol{y})^2-\frac{1}{2}\boldsymbol{\alpha}'\boldsymbol{y}\\
p_3(\boldsymbol{\alpha}'\boldsymbol{y}) &= \frac{1}{6}(\boldsymbol{\alpha}'\boldsymbol{y})^3-\frac{1}{2}(\boldsymbol{\alpha}'\boldsymbol{y})^2+\frac{1}{3}\boldsymbol{\alpha}'\boldsymbol{y}\\
p_4(\boldsymbol{\alpha}'\boldsymbol{y}) &= \frac{1}{24}(\boldsymbol{\alpha}'\boldsymbol{y})^4-\frac{1}{4}(\boldsymbol{\alpha}'\boldsymbol{y})^3+\frac{11}{24}(\boldsymbol{\alpha}'\boldsymbol{y})^2-\frac{1}{4}\boldsymbol{\alpha}'\boldsymbol{y}\\
p_5(\boldsymbol{\alpha}'\boldsymbol{y}) &= \frac{1}{120}(\boldsymbol{\alpha}'\boldsymbol{y})^5-\frac{1}{12}(\boldsymbol{\alpha}'\boldsymbol{y})^4+\frac{7}{24}(\boldsymbol{\alpha}'\boldsymbol{y})^3-\frac{5}{12}(\boldsymbol{\alpha}'\boldsymbol{y})^2+\frac{1}{5}\boldsymbol{\alpha}'\boldsymbol{y}
\end{aligned}$$

The first derivatives of the first five polynomials are

$$\begin{aligned}
p_1^{(1)}(\boldsymbol{\alpha}'\mathbf{y}) &= 1,\\
p_2^{(1)}(\boldsymbol{\alpha}'\mathbf{y}) &= \boldsymbol{\alpha}'\mathbf{y}-\frac{1}{2},\\
p_3^{(1)}(\boldsymbol{\alpha}'\mathbf{y}) &= \frac{1}{2}(\boldsymbol{\alpha}'\mathbf{y})^2-\boldsymbol{\alpha}'\mathbf{y}+\frac{1}{3},\\
p_4^{(1)}(\boldsymbol{\alpha}'\mathbf{y}) &= \frac{1}{6}(\boldsymbol{\alpha}'\mathbf{y})^3-\frac{3}{4}(\boldsymbol{\alpha}'\mathbf{y})^2+\frac{11}{12}\boldsymbol{\alpha}'\mathbf{y}-\frac{1}{4},\\
p_5^{(1)}(\boldsymbol{\alpha}'\mathbf{y}) &= \frac{1}{24}(\boldsymbol{\alpha}'\mathbf{y})^4-\frac{1}{3}(\boldsymbol{\alpha}'\mathbf{y})^3+\frac{7}{8}(\boldsymbol{\alpha}'\mathbf{y})^2-\frac{5}{6}\boldsymbol{\alpha}'\mathbf{y}+\frac{1}{5}.
\end{aligned}$$

The second derivatives of the first five polynomials are

$$\begin{aligned}
p_1^{(2)}(\boldsymbol{\alpha}'\mathbf{y}) &= 0,\\
p_2^{(2)}(\boldsymbol{\alpha}'\mathbf{y}) &= 1,\\
p_3^{(2)}(\boldsymbol{\alpha}'\mathbf{y}) &= \boldsymbol{\alpha}'\mathbf{y}-1,\\
p_4^{(2)}(\boldsymbol{\alpha}'\mathbf{y}) &= \frac{1}{2}(\boldsymbol{\alpha}'\mathbf{y})^2-\frac{3}{2}\boldsymbol{\alpha}'\mathbf{y}+\frac{11}{12},\\
p_5^{(2)}(\boldsymbol{\alpha}'\mathbf{y}) &= \frac{1}{6}(\boldsymbol{\alpha}'\mathbf{y})^3-(\boldsymbol{\alpha}'\mathbf{y})^2+\frac{7}{4}\boldsymbol{\alpha}'\mathbf{y}-\frac{5}{6}.
\end{aligned}$$

The log-likelihood function for the model in equation (19) is given by

$$l(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_k,\boldsymbol{\alpha},\boldsymbol{\nu}) = \sum_{i=1}^{k}\boldsymbol{\beta}_i'\mathbf{n}_i + \sum_{r=1}^{k}\nu_r\sum_{\mathbf{y}} n_{\mathbf{y}} p_r(\boldsymbol{\alpha}'\boldsymbol{y}) - n\ln\left\{\sum_{\mathbf{y}} q(\mathbf{y})\right\},$$

where $q(\mathbf{y}) = \exp\left\{\sum_{i=1}^{k}\boldsymbol{\beta}_i'\mathbf{x}_i + \sum_{r=1}^{k}\nu_r p_r(\boldsymbol{\alpha}'\mathbf{y})\right\}$. The derivatives of $l(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_k,\boldsymbol{\alpha},\boldsymbol{\nu})$ with respect to β_{is}, ν_r and α_i are given by

$$\frac{\partial l(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_k,\boldsymbol{\alpha},\boldsymbol{\nu})}{\partial\beta_{\mathrm{is}}} = n_{\mathrm{is}} - n\frac{\sum_{\mathbf{y}} q(\mathbf{y})x_{\mathrm{is}}}{\sum_{\mathbf{y}} q(\mathbf{y})},$$

$$\frac{\partial l(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_k,\boldsymbol{\alpha},\boldsymbol{\nu})}{\partial\nu_r} = \sum_{\mathbf{y}} n_{\mathbf{y}} p_r(\boldsymbol{\alpha}'\mathbf{y}) - n\frac{\sum_{\mathbf{y}} q(\mathbf{y})p_r(\boldsymbol{\alpha}'\mathbf{y})}{\sum_{\mathbf{y}} q(\mathbf{y})},$$

$$\frac{\partial l(\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_k,\boldsymbol{\alpha},\boldsymbol{\nu})}{\partial\alpha_i} = \sum_{r=1}^{k}\nu_r\sum_{\mathbf{y}} n_{\mathbf{y}} p_r^{(1)}(\boldsymbol{\alpha}'\mathbf{y})y_i - n\frac{\sum_{\mathbf{y}} q(\mathbf{y})\sum_{r=1}^{k}\nu_r p_r^{(1)}(\boldsymbol{\alpha}'\mathbf{y})y_i}{\sum_{\mathbf{y}} q(\mathbf{y})}.$$