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RESEARCH ARTICLE

Verlinde formulae on complex surfaces: K-theoretic invariants

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Abstract
We conjecture a Verlinde type formula for the moduli space of Higgs sheaves on a surface with a holomorphic 2-form. The conjecture specializes to a Verlinde formula for the moduli space of sheaves. Our formula interpolates between K-theoretic Donaldson invariants studied by Göttsche and Nakajima-Yoshioka and K-theoretic Vafa-Witten invariants introduced by Thomas and also studied by Göttsche and Kool. We verify our conjectures in many examples (for example, on K3 surfaces).

Contents

1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$. The classical θ-functions at level $k \geq 1$ for C are defined as follows. The map $C \to \mathrm{Pic}^1(C)$, $p \mapsto [\mathcal{O}_C(p)]$ gives rise to the Abel-Jacobi map on the symmetric product

$$\mathrm{Sym}^{g-1}(C) \to \mathrm{Pic}^{g-1}(C),$$

and the image Θ, which has codimension one, is known as the *theta divisor*. Denote by $\mathcal{L}$ the corresponding line bundle. The θ-functions of level k are defined as the elements of $H^0(\mathrm{Pic}^{g-1}(C), \mathcal{L}^{\otimes k})$. Since $H^{>0}(\mathrm{Pic}^{g-1}(C), \mathcal{L}^{\otimes k}) = 0$, the Riemann-Roch theorem gives the dimension of the space of θ-functions of level k as the degree of $\exp(k\Theta)$. Since $\Theta^g/g! = 1$, one obtains

$$\dim H^0(\mathrm{Pic}^{g-1}(C), \mathcal{L}^{\otimes k}) = k^g.$$

The Verlinde formula extends this equation to moduli spaces of rank 2 (and higher) stable vector bundles on C, as follows. See [6] for a survey.

Denote by $M := M_C(2,0)$ the moduli space of rank 2 semistable vector bundles E on C with $\det E \cong \mathcal{O}_C$. Then $\mathrm{Pic}(M)$ is generated by the determinant line bundle $\mathcal{L}$ [2, 9]. The Verlinde formula (for rank 2 and trivial determinant), originating from conformal field theory [44], is the following expression:

$$\dim H^0(M, \mathcal{L}^{\otimes k}) = \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \sin\left(\frac{\pi j}{k+2}\right)^{2-2g}.$$

This formula has been proved by several people [39, 5, 41, 32, 8, 38, 7, 46] (for rank 2) and [11, 3] (for general rank). Numerical aspects of this formula were studied by D. Zagier [45].

Let $N := N_C(2,0)$ be the moduli space of rank 2 semistable Higgs bundles (E, ϕ) on C with $\det E \cong \mathcal{O}_C$. Here, E is a rank 2 vector bundle, and $\phi : E \to E \otimes K_C$ is called the *Higgs field*. The moduli space N is non-compact. It has a $\mathbb{C}^*$-action defined by scaling the Higgs field. The determinant line bundle $\mathcal{L}$ on N is $\mathbb{C}^*$-equivariant; therefore $H^0(N, \mathcal{L}^{\otimes k})$ is a $\mathbb{C}^*$-representation. Recently, Halpern-Leistner [29] and Andersen-Gukov-Du Pei [1] found a formula for $\dim H^0(N, \mathcal{L}^{\otimes k})$, which can be seen as a Verlinde formula for Higgs bundles. In physics, this formula is related to complex Chern-Simons theory of the (three-dimensional) Seifert manifold $C \times S^1$ embedded into string theory [28].

In this paper, we study Verlinde type formulae on the moduli space of rank 2 Gieseker stable (Higgs) sheaves on S, where S is a smooth projective surface satisfying $p_g(S) > 0$ and $b_1(S) = 0$.

1.1. Verlinde Formula for Moduli of Sheaves

Denote by $M := M_S^H(2, c_1, c_2)$ the moduli space of rank 2 Gieseker H-stable torsion free sheaves on S with Chern classes $c_1 \in H^2(S, \mathbb{Z})$ and $c_2 \in H^4(S, \mathbb{Z})$. We assume there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Then M is a projective scheme with perfect obstruction theory of virtual dimension

$$\mathrm{vd} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S). \tag{1}$$

When a universal sheaf $\mathbb{E}$ exists on $M \times S$, the virtual tangent bundle is given by $T_M^{\mathrm{vir}} = R\mathcal{H}om_{\pi_M}(\mathbb{E}, \mathbb{E})_0[1]$, where $\pi_M : M \times S \to M$ denotes projection and $(\cdot)_0$ denotes a trace-free part.

In general, $\mathbb{E}$ exists only étale locally. Nevertheless, $R\mathcal{H}om_{\pi_M}(\mathbb{E},\mathbb{E})_0[1]$ exists globally on $M\times S$, essentially because this expression is invariant under replacing $\mathbb{E}$ by $\mathbb{E}\otimes\mathcal{L}$ for any $\mathcal{L}\in\mathrm{Pic}(M\times S)$ [30, Section 10.2]. Algebraic Donaldson invariants are defined by integrating polynomial expressions in slant products over $[M]^{\mathrm{vir}}$. These were studied in detail, for any rank, in T. Mochizuki's remarkable monograph [35].

Let $\alpha\in H^*(S,\mathbb{Q})$. When a universal sheaf $\mathbb{E}$ exists on $M\times S$, we consider the μ-insertion defined by the slant product

$$\begin{aligned} /\colon H^p(S\times M,\mathbb{Q})\times H_q(S,\mathbb{Q}) &\to H^{p-q}(M,\mathbb{Q}),\\ \mu(\alpha) := \big(c_2(\mathbb{E})-\frac{1}{4}c_1(\mathbb{E})^2\big)/\mathrm{PD}(\alpha) &\in H^*(M,\mathbb{Q}), \end{aligned} \tag{2}$$

where $\mathrm{PD}(\cdot)$ denotes a Poincaré dual. Note that

$$c_2(\mathbb{E})-\frac{1}{4}c_1(\mathbb{E})^2=-\frac{1}{4}\,\mathrm{ch}_2(\mathbb{E}\otimes\mathbb{E}\otimes\det(\mathbb{E})^*),$$

where the sheaf $\mathbb{E}\otimes\mathbb{E}\otimes\det(\mathbb{E})^*$ always exists globally on $M\times S$, again, essentially because this expression is invariant under replacing $\mathbb{E}$ by $\mathbb{E}\otimes\mathcal{L}$. Therefore, (2) is always defined. When $L\in\mathrm{Pic}(S)$ satisfies $c_1(L)c_1\in 2\mathbb{Z}$, there exists a line bundle $\mu(L)\in\mathrm{Pic}(M)$ whose class in cohomology is (2) for $\alpha=c_1(L)$ [30, Chapter 8]. One refers to $\mu(L)$ as a *Donaldson line bundle*. The first conjecture concerns

$$\chi^{\mathrm{vir}}(M,\mu(L)) := \chi(M,\mathcal{O}_M^{\mathrm{vir}}\otimes\mu(L)),$$

known as a *K-theoretic Donaldson invariant* [23].[1] Here, $\mathcal{O}_M^{\mathrm{vir}}$ denotes the virtual structure sheaf of M [12, Section 3.2]. Göttsche, H. Nakajima, and K. Yoshioka determined their wall-crossing behaviour, when S is a toric surface, using the K-theoretic Nekrasov partition function [23]. For rational surfaces Göttsche and Y. Yuan established structure formulae for these invariants and relations to strange duality [26, 17].

We denote intersection numbers such as $\int_S c_1(L)c_1(\mathcal{O}(K_S))$ by $c_1(L)c_1(\mathcal{O}(K_S))$ or simply LK_S. Denote by $\mathrm{SW}(a)$ the Seiberg-Witten invariant of $a\in H^2(S,\mathbb{Z})$.[2]

Conjecture 1.1. *Let S be a smooth projective surface with $p_g(S)>0$, $b_1(S)=0$, and $L\in\mathrm{Pic}(S)$. Let H,c_1,c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1,c_2. Then $\chi^{\mathrm{vir}}(M_S^H(2,c_1,c_2),\mu(L))$ equals the coefficient of x^{vd} of*

$$\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\frac{(L-K_S)^2}{2}+\chi(\mathcal{O}_S)}}\sum_{a\in H^2(S,\mathbb{Z})}\mathrm{SW}(a)\,(-1)^{ac_1}\left(\frac{1+x}{1-x}\right)^{\left(\frac{K_S}{2}-a\right)(L-K_S)}.$$

In Section 2, we verify this conjecture in many cases for S: a K3 surface, an elliptic surface, a Kanev surface, a double cover of $\mathbb{P}^2$ branched along a smooth octic curve, a quintic surface, and blow-ups thereof. Our strategy is similar to [24, 19, 20, 21]. We first express $\chi^{\mathrm{vir}}(M,\mu(L))$ in terms of algebraic Donaldson invariants. Using Mochizuki's formula [35, Theorem 1.4.6], the latter can be written in terms of integrals on Hilbert schemes of points. We show that these integrals can be combined into a generating series that is a cobordism invariant and hence determined on $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$. On $\mathbb{P}^2$ and $\mathbb{P}^1\times\mathbb{P}^1$, we determine this generating series (to some order) by localization.

Finally, in Section 4, we discuss interesting special cases of Conjecture 1.1.

[1] When the Donaldson line bundle does not exist, we *define* $\chi^{\mathrm{vir}}(M,\mu(L))$ by the virtual Hirzebruch-Riemann-Roch formula [12, Corollary 3.4]: that is, $\int_{[M]^{\mathrm{vir}}} e^{\mu(c_1(L))}\,\mathrm{td}(T_M^{\mathrm{vir}})$.

[2] We use Mochizuki's convention: $\mathrm{SW}(a)=\widetilde{\mathrm{SW}}(2a-K_S)$, with $\widetilde{\mathrm{SW}}(b)$ the usual Seiberg-Witten invariant in class $b\in H^2(S,\mathbb{Z})$. Moreover, there are finitely many $a\in H^2(S,\mathbb{Z})$ such that $\mathrm{SW}(a)\neq 0$. Such classes are called *Seiberg-Witten basic classes*.

1.2. Verlinde Formula for Moduli of Higgs Sheaves

Let H be a polarization on S. Recently, Y. Tanaka and R. P. Thomas [40] proposed a mathematical definition of $\mathrm{SU}(r)$ Vafa-Witten invariants of S. We consider the case $r = 2$. Their definition involves the moduli space of Higgs sheaves (E, ϕ)

$$N := N_S^H(2, c_1, c_2) = \big\{(E, \phi) \,:\, \operatorname{tr}\phi = 0,\ c_1(E) = c_1,\ c_2(E) = c_2\big\},$$

where E is a rank 2 torsion free sheaf, $\phi : E \to E \otimes K_S$ is a morphism, and the pair (E, ϕ) satisfies a (Gieseker) stability condition with respect to H. Tanaka and Thomas show that N admits a symmetric perfect obstruction theory in the sense of [4]. As in the curve case, one can scale a Higgs sheaf by sending (E, ϕ) to $(E, t\phi)$ for any $t \in \mathbb{C}^*$. This defines an action of $\mathbb{C}^*$ on N. As in the previous section, we assume stability and semistability coincide. Then the fixed locus $N^{\mathbb{C}^*}$ is projective, and the Vafa-Witten invariants are defined as

$$\int_{[N^{\mathbb{C}^*}]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})} \in \mathbb{Q},$$

where N^{vir} denotes the virtual normal bundle and $e(\cdot)$ is the equivariant Euler class [27]. The fixed locus $N^{\mathbb{C}^*}$ has two types of connected components:

- Components containing (E, ϕ) with $\phi = 0$, which we refer to as the *instanton branch*. This branch is isomorphic to the Gieseker moduli space $M := M_S^H(2, c_1, c_2)$. The $\mathbb{C}^*$-localized perfect obstruction theory on M coincides with the one from the previous section.
- Components containing (E, ϕ), where $E = E_0 \oplus E_1 \otimes \mathfrak{t}^{-1}$ is the decomposition of E into rank 1 eigensheaves, and $\phi : E_0 \to E_1 \otimes K_S$. Here, $\mathfrak{t}$ denotes the weight one character of $\mathbb{C}^*$. These components constitute the *monopole branch*, which we collectively denote by M^{mon}. Denote by $S^{[n]}$ the Hilbert scheme of n points on S and by $|\beta|$ the linear system of an algebraic class $\beta \in H^2(S, \mathbb{Z})$. A. Gholampour and Thomas [15, 16] prove that the monopole components are isomorphic to incidence loci[3]

  $$S_\beta^{[n_0,n_1]} := \{(Z_0, Z_1, C) \,:\, I_{Z_0}(-C) \subset I_{Z_1}\} \subset S^{[n_0]} \times S^{[n_1]} \times |\beta|,$$

  for certain n_0, n_1, β, where $I_Z \subset \mathcal{O}_S$ is the ideal sheaf corresponding to $Z \subset S$. Moreover, they show that the $\mathbb{C}^*$-localized perfect obstruction theory on $S_\beta^{[n_0,n_1]}$ is naturally obtained by realizing this space as a degeneracy locus inside the smooth space $S^{[n_0]} \times S^{[n_1]} \times |\beta|$ and reducing the perfect obstruction theory coming from this description (Section 3).

Let $M' \subset M^{\mathrm{mon}}$ be a connected component of the monopole branch and let $\alpha \in H^*(S, \mathbb{Q})$. Similar to the previous section, we define

$$\mu(\alpha) := \big(c_2^{\mathbb{C}^*}(\mathbb{E}) - \frac{1}{4} c_1^{\mathbb{C}^*}(\mathbb{E})^2\big)/\mathrm{PD}(\alpha) \in H^*_{\mathbb{C}^*}(M', \mathbb{Q}), \tag{3}$$

where the Chern classes are $\mathbb{C}^*$-equivariant, M' and S carry the trivial torus action, and $\mathbb{E}$ is the universal sheaf on $M' \times S$.

Vafa-Witten invariants can also be seen as reduced Donaldson-Thomas invariants counting two-dimensional sheaves on $X = \mathrm{Tot}(K_S)$—the total space of the canonical bundle on S [14]. From this perspective, it is more natural to work with the Nekrasov-Okounkov twist of $\mathcal{O}_N^{\mathrm{vir}}$ [37], which is defined as

$$\widehat{\mathcal{O}}_N^{\mathrm{vir}} := \sqrt{K_N^{\mathrm{vir}}} \otimes \mathcal{O}_N^{\mathrm{vir}},$$

[3]For fixed $r = 2$, c_1, c_2, the virtual dimension of $M^{\mathrm{mon}} \subset N^{\mathbb{C}^*}$ is in general *not* given by (1). In fact, M^{mon} can have components of different virtual dimension (see Remark 3.3).

where $\sqrt{K_N^{\mathrm{vir}}}$ is a choice of square root of $K_N^{\mathrm{vir}} = \det(\Omega_N^{\mathrm{vir}})$. Over the fixed locus $N^{\mathbb{C}^*}$, this choice of square root is canonical [42, Proposition 2.6]. For any (possibly infinite-dimensional) graded vector spaces, set

$$\chi\Big(\bigoplus_i \mathfrak{t}^{a_i} - \bigoplus_j \mathfrak{t}^{b_j}\Big) := \sum_i y^{a_i} - \sum_j y^{b_j}.$$

The K-theoretic Vafa-Witten invariants are [42, (2.12), Proposition 2.13]

$$\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}}) := \chi(R\Gamma(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}})) = \chi\Big(N^{\mathbb{C}^*}, \frac{\mathcal{O}^{\mathrm{vir}}_{N^{\mathbb{C}^*}}}{\Lambda_{-1}(N^{\mathrm{vir}})^\vee} \otimes \sqrt{K_N^{\mathrm{vir}}}\Big|_{N^{\mathbb{C}^*}}\Big).$$

Here, $\Lambda_{-1}(\cdot)$ is introduced in Section 2 and y is related to $t := c_1^{\mathbb{C}^*}(\mathfrak{t})$ by $y = e^t$. The Nekrasov-Okounkov twist ensures that these invariants are unchanged under $y \leftrightarrow y^{-1}$ [42, Proposition 2.27]. Our next two conjectures concern

$$\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}} \otimes \mu(L)) := \chi(R\Gamma(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}} \otimes \mu(L))),$$

where $L \in \mathrm{Pic}(S)$.[4] This expression has instanton and monopole contributions corresponding to the decomposition $N^{\mathbb{C}^*} = M \sqcup M^{\mathrm{mon}}$. By the argument in [42, Section 2.5], the instanton contribution equals

$$(-1)^{\mathrm{vd}} y^{-\frac{\mathrm{vd}}{2}} \chi_{-y}^{\mathrm{vir}}(M, \mu(L)) := (-1)^{\mathrm{vd}} y^{-\frac{\mathrm{vd}}{2}} \sum_p (-y)^p \chi^{\mathrm{vir}}(M, \Lambda^p \Omega_M^{\mathrm{vir}} \otimes \mu(L)),$$

where vd is given by (1) and $\chi_y^{\mathrm{vir}}(M, \cdot)$ is the twisted virtual χ_y-genus [12].

Consider the following two theta functions and the normalized Dedekind eta function

$$\theta_3(x,y) = \sum_{n \in \mathbb{Z}} x^{n^2} y^n, \quad \theta_2(x) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} x^{n^2} y^n, \quad \overline{\eta}(x) = \prod_{n=1}^\infty (1 - x^n). \tag{4}$$

We also use the following notation. For any $a, b \in H^2(S, \mathbb{Z})$, define

$$\delta_{a,b} = \#\left\{\gamma \in H^2(S,\mathbb{Z}) \ : \ a - b = 2\gamma\right\}. \tag{5}$$

Conjecture 1.2. *Let S be a smooth projective surface with $p_g(S) > 0$, $b_1(S) = 0$, and $L \in \mathrm{Pic}(S)$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Let* vd *be defined by* (1)*. Then $y^{-\frac{\mathrm{vd}}{2}} \chi_{-y}^{\mathrm{vir}}(M_S^H(2, c_1, c_2), \mu(L))$ equals the coefficient of x^{vd} of*

$$\begin{aligned}
&4\left(\frac{1}{2}\prod_{n=1}^\infty \frac{1}{(1-x^{2n})^{10}(1-x^{2n}y)(1-x^{2n}y^{-1})}\right)^{\chi(\mathcal{O}_S)} \left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x, y^{\frac{1}{2}})}\right)^{K_S^2} \\
&\cdot \left(\prod_{n=1}^\infty \left(\frac{(1-x^{2n})^2}{(1-x^{2n}y)(1-x^{2n}y^{-1})}\right)^{n^2}\right)^{\frac{L^2}{2}} \left(\prod_{n=1}^\infty \left(\frac{1-x^{2n}y^{-1}}{1-x^{2n}y}\right)^n\right)^{LK_S} \\
&\cdot \sum_{a \in H^2(S,\mathbb{Z})} (-1)^{c_1 a}\, \mathrm{SW}(a) \left(\frac{\theta_3(x, y^{\frac{1}{2}})}{\theta_3(-x, y^{\frac{1}{2}})}\right)^{aK_S} \\
&\cdot \left(\prod_{n=1}^\infty \left(\frac{(1-x^{2n-1}y^{\frac{1}{2}})(1+x^{2n-1}y^{-\frac{1}{2}})}{(1-x^{2n-1}y^{-\frac{1}{2}})(1+x^{2n-1}y^{\frac{1}{2}})}\right)^{2n-1}\right)^{\frac{L(K_S-2a)}{2}}.
\end{aligned}$$

[4]If the line bundle $\mu(L)$ does not exist on N (or $N^{\mathbb{C}^*}$), then we *define* these invariants by virtual $\mathbb{C}^*$-localization combined with the virtual HRR formula as before.

Conjecture 1.3. *Let S be a smooth projective surface with $p_g(S) > 0$, $b_1(S) = 0$, and $L \in \operatorname{Pic}(S)$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable Higgs sheaves on S with Chern classes c_1, c_2. Let $N := N_S^H(2, c_1, c_2)$, and let* vd *be defined by* (1). *Then the monopole contribution to $\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}} \otimes \mu(L))$ equals the coefficient of $(-x)^{\mathrm{vd}}$ of*

$$\left(\prod_{n=1}^{\infty} \frac{1}{(1-x^{8n})^{10}(1-x^{8n}y^2)(1-x^{8n}y^{-2})}\right)^{\chi(\mathcal{O}_S)} \left(\frac{\overline{\eta}(x^4)^2}{\theta_2(x^4,y)}\right)^{K_S^2}$$
$$\cdot \left(\prod_{n=1}^{\infty} \left(\frac{(1-x^{8n})^2}{(1-x^{8n}y^2)(1-x^{8n}y^{-2})}\right)^{n^2}\right)^{2L^2} \left(\prod_{n=1}^{\infty} \left(\frac{1-x^{4n}y^{-1}}{1-x^{4n}y}\right)^{n}\right)^{2LK_S}$$
$$\cdot \sum_{a \in H^2(S,\mathbb{Z})} \delta_{c_1, K_S - a} \, \mathrm{SW}(a) \, k_a \left(\frac{\theta_2(x^4,y)}{\theta_3(x^4,y)}\right)^{aK_S} \left(\prod_{n=1}^{\infty} \left(\frac{1+x^{8n-4}y^{-1}}{1+x^{8n-4}y}\right)^{2n-1}\right)^{2aL}$$
$$\cdot \left(\prod_{n=1}^{\infty} \left(\frac{1+x^{8n}y^{-1}}{1+x^{8n}y}\right)^{n}\right)^{4L(K_S-a)} \cdot \left(\prod_{n=1}^{\infty} \left(\frac{1+x^{4n}y^{-1}}{1+x^{4n}y}\right)^{n}\right)^{LK_S},$$

where $k_a := x^{-3\chi(\mathcal{O}_S)}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{-\chi(\mathcal{O}_S)} y^{\frac{1}{2}L(a-K_S)}$.

Together, these two conjectures give a Verlinde-type formula for the moduli space of Higgs sheaves on a surface S satisfying $b_1(S) = 0$ and $p_g(S) > 0$. Moreover, our formulae interpolate between the following two invariants:

- **K-theoretic Donaldson invariants.** After replacing x with $xy^{\frac{1}{2}}$ in the formula of Conjecture 1.2, we can set $y = 0$. This replacement provides a formula for $\chi_{-y}^{\mathrm{vir}}(M, \mu(L))$, and setting $y = 0$ implies the formula for K-theoretic Donaldson invariants of Conjecture 1.1.
- **K-theoretic Vafa-Witten invariants.** Putting $L = \mathcal{O}_S$ in Conjectures 1.2 and 1.3, we obtain the conjectural formulae for K-theoretic Vafa-Witten invariants of [21, Remark 1.3, 1.7].

In [19, Appendix], Göttsche and Nakajima conjectured a formula interpolating between Donaldson invariants and virtual Euler numbers of $M := M_S^H(2, c_1, c_2)$. Conjecture 1.2 also implies this formula (Section 4).

Using the same strategy as for Conjecture 1.1, we verify Conjecture 1.2 in many examples. On the other hand, for Conjecture 1.3, we *prove* the universal dependence by presenting a variation on an argument of T. Laarakker [34], which in turn is an application of Gholampour-Thomas's description of the monopole virtual class in terms of nested Hilbert schemes [15, 16].

Theorem 1.4. *There exist universal series*

$$C_1(y,q), \ldots, C_6(y,q) \in 1 + q\,\mathbb{Q}[y^{\frac{1}{2}}][[q]]$$

with the following property. Let S be a smooth projective surface with $p_g(S) > 0$, $b_1(S) = 0$, and $L \in \operatorname{Pic}(S)$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable Higgs sheaves on S with Chern classes c_1, c_2. Let $N := N_S^H(2, c_1, c_2)$, and let vd *be defined by* (1). *Then the monopole contribution to $\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}} \otimes \mu(L))$ equals the coefficient of $(-x)^{\mathrm{vd}}$ of*

$$C_1(y,x^4)^{\chi(\mathcal{O}_S)} \, C_2(y,x^4)^{K_S^2} \, C_3(y,x^4)^{L^2} \, C_4(y,x^4)^{LK_S}$$
$$\cdot \sum_{a \in H^2(S,\mathbb{Z})} \delta_{c_1, K_S - a} \, \mathrm{SW}(a) \, \ell_a \, C_5(y,x^4)^{aK_S} \, C_6(y,x^4)^{aL},$$

where $\ell_a := x^{aK_S - K_S^2 - 3\chi(\mathcal{O}_S)}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})^{aK_S - K_S^2 - \chi(\mathcal{O}_S)} y^{\frac{1}{2}L(a-K_S)}$.

For $L = \mathcal{O}_S$, this was proved in [34] (actually, for $L = \mathcal{O}_S$, the analog of this theorem is proved in any rank [34]). Universality on the instanton branch is still open. The universal series C_i can be expressed in terms of intersection numbers on products of Hilbert schemes of points on surfaces. Again, these intersection numbers are determined on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, where we calculate using localization. This way, we determine $C_i \mod q^{15}$, and we find a match with Conjecture 1.2 (Section 3).

For $L = \mathcal{O}_S$, physicists [43] predict that the instanton and monopole generating functions of Conjectures 1.2 and 1.3 get swapped under the S-duality transformation $\tau \to -1/\tau$, where $q = \exp(2\pi i\tau)$. See [31] for a recent proof of S-duality for K3 surfaces (for any prime rank). For general L, the connection between instanton and monopole contribution is less clear. However, the series depending on L^2 are related by $x \mapsto x^4$, $y \mapsto y^2$, $L \mapsto L^{\otimes 2}$ (and similarly for any rank in Section 3.6).

1.3. K3 Surfaces

By adapting an argument from [23] combined with a new formula for twisted elliptic genera of Hilbert schemes of points on surfaces, Göttsche proves Conjecture 1.2 for K3 surfaces in [18]. By adapting an argument of [34] combined with the above-mentioned formula for twisted elliptic genera of Hilbert schemes of points on surfaces, we prove the following (where the formula for C_1 was previously determined in [42, 34]):

Theorem 1.5. *The universal functions $C_1(y,q), C_3(y,q)$ are given by*

$$C_1(y,q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})^{10}(1-q^{2n}y^2)(1-q^{2n}y^{-2})},$$

$$C_3(y,q) = \prod_{n=1}^{\infty} \left(\frac{(1-q^{2n})^2}{(1-q^{2n}y^2)(1-q^{2n}y^{-2})} \right)^{2n^2}.$$

In particular, Conjectures 1.2 and 1.3 hold for K3 surfaces.[5]

2. Instanton Contribution and Donaldson Invariants

In this section, we gather evidence for Conjectures 1.1 and 1.2 as follows:

- **Reduction to Donaldson invariants.** Express the invariants of Conjectures 1.1 and 1.2 in terms of Donaldson invariants of S.
- **Reduction to Hilbert schemes.** Use Mochizuki's formula [35, Theorem 1.4.6] to express these invariants as intersection numbers on Hilbert schemes of points on S.
- **Reduction to toric surfaces.** Show that the intersection numbers of the previous step are determined on $S = \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, where they can be calculated using localization.

The final step allows us to calculate the invariants of Conjectures 1.1 and 1.2 and compare to our conjectured formulae. This strategy has been used by Göttsche and Kool in the determination of the instanton contribution to rank 2 and 3 Vafa-Witten invariants and various refinements thereof [19, 20, 21]. Mochizuki's formula was also used by Göttsche and Nakajima-Yoshioka in their proof of the Witten conjecture for algebraic surfaces, which expresses (primary, rank 2) Donaldson invariants in terms of Seiberg-Witten invariants [24].

[5]The statement that Conjecture 1.3 holds for K3 surfaces has less content than initially meets the eye. On a K3 surface, $\delta_{c_1,a}$ is only non-zero only when c_1 is even. Assuming $\gcd(2, c_1H, \frac{1}{2}c_1^2 - c_2) = 1$, which guarantees 'stable=semistable', implies c_2 is odd. Hence the coefficient of $(-1)x^{\mathrm{vd}}$ of the conjectured expression is always zero. Indeed, 'stable=semistable' implies that the monopole branch is empty [40, Proposition 7.4].

2.1. Donaldson Invariants

Let S be a smooth projective complex surface such that $b_1(S) = 0$. Let H be a polarization on S, and let $M := M_S^H(r, c_1, c_2)$.[6] We assume there exist no rank r strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. For the moment, we also assume there exists a universal family $\mathbb{E}$ on $M \times S$, although we get rid of this assumption in Remark 2.3. For any $\alpha \in H^*(S, \mathbb{Q})$ and $k \geq 0$, define $\mu(\alpha) \in H^*(M, \mathbb{Q})$ as in (2), and

$$\tau_k(\alpha) := \mathrm{ch}_{k+2}(\mathbb{E})/\mathrm{PD}(\alpha) \in H^*(M, \mathbb{Q}).$$

We refer to $\tau_k(\alpha)$ as a *descendent insertion* and call it *primary* when $k = 0$. As mentioned in the introduction, if $L \in \mathrm{Pic}(S)$ satisfies $c_1(L)c_1 \in 2\mathbb{Z}$, then there exists a line bundle on M, denoted by $\mu(L)$ and called a *Donaldson line bundle*, whose class in cohomology is (2) for $\alpha = c_1(L)$.

Consider the K-group $K^0(M)$ generated by locally free sheaves on M. For any rank r vector bundle on M, define

$$\Lambda_y V := \sum_{i=0}^{r} [\Lambda^i V] y^i \in K^0(M)[y], \qquad \mathrm{Sym}_y V := \sum_{i=0}^{\infty} [\mathrm{Sym}^i V] y^i \in K^0(M)[[y]].$$

These expressions can be extended to complexes in $K^0(M)$ by setting $\Lambda_y(-V) = \mathrm{Sym}_{-y} V$ and $\mathrm{Sym}_y(-V) = \Lambda_{-y} V$. For any complex $E \in K^0(M)$, we define

$$\mathsf{X}_y(E) := \mathrm{ch}(\Lambda_y E^\vee)\, \mathrm{td}(E). \tag{1}$$

Since $\Lambda_y(E \oplus F) = \Lambda_y E \otimes \Lambda_y F$, we obtain

$$\mathsf{X}_y(E \oplus F) = \mathsf{X}_y(E)\, \mathsf{X}_y(F).$$

Furthermore, for any $L \in \mathrm{Pic}(M)$

$$\mathsf{X}_y(L) = \frac{L(1 + y e^{-L})}{1 - e^{-L}}.$$

Lemma 2.1. *Let S, H, r, c_1, c_2, and $M := M_S^H(r, c_1, c_2)$ be as above. Let $L \in \mathrm{Pic}(S)$. Then there exists a polynomial expression $P(\mathbb{E})$ in y and certain descendent insertions $\tau_k(\alpha)$ and $\mu(c_1(L))$ such that*

$$\chi_y^{\mathrm{vir}}(M, \mu(L)) = \int_{[M]^{\mathrm{vir}}} \mathsf{X}_y(T_M^{\mathrm{vir}})\, e^{\mu(c_1(L))} = \int_{[M]^{\mathrm{vir}}} P(\mathbb{E}).$$

Proof. The first equality is the virtual Hirzebruch-Riemann-Roch theorem [12, Corollary 3.4] (or the definition of our invariants when the Donaldson line bundle $\mu(L)$ does not exist on M). The second equality was proved for $L = \mathcal{O}_S$ in [19, Proposition 2.1] by applying Grothendieck-Riemann-Roch and the Künneth formula to

$$\mathrm{ch}(T_M^{\mathrm{vir}}) = \mathrm{ch}(R\mathcal{H}om_{\pi_M}(\mathbb{E}, \mathbb{E})_0[1]).$$

The argument for any L is the same, with $P(\mathbb{E})$ now involving $\mu(c_1(L))$. □

2.2. Mochizuki's Formula

We recall Mochizuki's formula [35, Theorem 1.4.6].

[6] In this paragraph, $r > 0$ is arbitrary, and we do not require $p_g(S) > 0$.

Let $S^{[n]}$ be the Hilbert scheme of n points on S. On $S^{[n_1]} \times S^{[n_2]} \times S$, we have (pull-backs of) the universal ideal sheaves $\mathcal{I}_1$ and $\mathcal{I}_2$ from both factors. For any $M \in \mathrm{Pic}(S)$, on $S^{[n_1]} \times S^{[n_2]}$, we have (pull-backs of) the tautological bundles $M^{[n_1]}$ and $M^{[n_2]}$ from both factors. We endow $S^{[n_1]} \times S^{[n_2]}$ with the trivial $\mathbb{C}^*$-action and denote the positive generator of the character group of $\mathbb{C}^*$ by $\mathfrak{s}$. Define $s := c_1^{\mathbb{C}^*}(\mathfrak{s})$; then

$$H^*(B\mathbb{C}^*, \mathbb{Q}) = H^*_{\mathbb{C}^*}(\mathrm{pt}, \mathbb{Q}) \cong \mathbb{Q}[s].$$

Fix $L \in \mathrm{Pic}(S)$, and let $P(\mathbb{E})$ be any polynomial in $\mu(c_1(L))$ and descendent insertions $\tau_k(\alpha)$. We assume $P(\mathbb{E})$ arises from a polynomial expression in $\mu(c_1(L))$ and the Chern classes of T_M^{vir} (for example, such as in Proposition 2.1). Let $A^1(S)$ be the Chow group of codimension 1 cycles up to linear equivalence; then for any $a_1, a_2 \in A^1(S)$ and $n_1, n_2 > 0$, we define (following Mochizuki)

$$\begin{aligned}
&\Psi(L, a_1, a_2, n_1, n_2) := \\
&\mathrm{Coeff}_{s^0}\left(\frac{P(\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2(a_2) \otimes \mathfrak{s})}{Q(\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1}, \mathcal{I}_2(a_2) \otimes \mathfrak{s})} \frac{e(\mathcal{O}(a_1)^{[n_1]})\, e(\mathcal{O}(a_2)^{[n_2]} \otimes \mathfrak{s}^2)}{(2s)^{n_1+n_2-\chi(\mathcal{O}_S)}} \right).
\end{aligned} \tag{2}$$

We explain the notation. Here, $\mathcal{I}_i(a_i)$ stands for $\mathcal{I}_i \otimes \pi_S^* \mathcal{O}(a_i)$, considered as a sheaf on $S^{[n_1]} \times S^{[n_2]} \times S$ pulled back along projection to $S^{[n_i]} \times S$. Similarly, $\mathcal{O}(a_i)^{[n_i]}$ is viewed as a vector bundle on $S^{[n_1]} \times S^{[n_2]}$ pulled back along projection to $S^{[n_i]}$. Since $S^{[n_1]} \times S^{[n_2]}$ has a trivial $\mathbb{C}^*$-action, we can view $\mathcal{O}(a_i)^{[n_i]}$ as endowed with the trivial $\mathbb{C}^*$-equivariant structure. Moreover,

$$\mathcal{O}(a_2)^{[n_2]} \otimes \mathfrak{s}^2$$

denotes $\mathcal{O}(a_2)^{[n_2]}$ with $\mathbb{C}^*$-equivariant structure given by tensoring with character $\mathfrak{s}^2$. Similarly, we endow $S^{[n_1]} \times S^{[n_2]} \times S$ with trivial $\mathbb{C}^*$-action, give $\mathcal{I}_i(a_i)$ the trivial $\mathbb{C}^*$-equivariant structure, and denote by

$$\mathcal{I}_1(a_1) \otimes \mathfrak{s}, \qquad \mathcal{I}_2(a_2) \otimes \mathfrak{s}^{-1}$$

the $\mathbb{C}^*$-equivariant sheaves obtained by tensoring with the characters $\mathfrak{s}$ and $\mathfrak{s}^{-1}$, respectively. We denote the $\mathbb{C}^*$-equivariant Euler class by $e(\cdot)$. Moreover, $P(\cdot)$ stands for the expression obtained from $P(\mathbb{E})$ by formally replacing $\mathbb{E}$ with $\cdot$ and all Chern classes with $\mathbb{C}^*$-equivariant Chern classes.[7] For any $\mathbb{C}^*$-equivariant sheaves E_1, E_2 on $S^{[n_1]} \times S^{[n_2]} \times S$ flat over $S^{[n_1]} \times S^{[n_2]}$,

$$Q(E_1, E_2) := e(-R\mathcal{H}om_\pi(E_1, E_2) - R\mathcal{H}om_\pi(E_2, E_1)),$$

where $\pi : S^{[n_1]} \times S^{[n_2]} \times S \to S^{[n_1]} \times S^{[n_2]}$ denotes projection. Finally, $\mathrm{Coeff}_{s^0}(\cdot)$ takes the coefficient of s^0. We define $\widetilde{\Psi}(L, a_1, a_2, n_1, n_2, s)$ by expression (2) *without* $\mathrm{Coeff}_{s^0}(\cdot)$. Let c_1, c_2 be a choice of Chern classes. For any decomposition $c_1 = a_1 + a_2$, we define (again following Mochizuki)

$$\mathcal{A}(L, a_1, a_2, c_2) := \sum_{n_1+n_2=c_2-a_1a_2} \int_{S^{[n_1]} \times S^{[n_2]}} \Psi(L, a_1, a_2, n_1, n_2). \tag{3}$$

Let $\widetilde{\mathcal{A}}(L, a_1, a_2, c_2, s)$ be defined by the same expression, with Ψ replaced by $\widetilde{\Psi}$.

Theorem 2.2 (Mochizuki)**.** *Let S be a smooth projective surface satisfying $b_1(S) = 0$, $p_g(S) > 0$, and let $L \in \mathrm{Pic}(S)$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable*

[7]The replacement of $\mathbb{E}$ with $\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2(a_2) \otimes \mathfrak{s}$ comes from Mochizuki's wall-crossing on the master space [35].

sheaves on S with Chern classes c_1, c_2 and such that a universal sheaf $\mathbb{E}$ on $M_S^H(2, c_1, c_2) \times S$ exists. Assume the following hold:

(i) $\chi(\mathrm{ch}) > 0$, *where* $\chi(\mathrm{ch}) := \int_S \mathrm{ch} \cdot \mathrm{td}(S)$ *and* $\mathrm{ch} = (2, c_1, \frac{1}{2}c_1^2 - c_2)$.

(ii) $p_{\mathrm{ch}} > p_{K_S}$, *where* $p_{\mathrm{ch}} = \chi(e^{mH} \cdot \mathrm{ch})/2$ *and* $p_{K_S} = \chi(e^{mH} \cdot e^{K_S})$ *are the reduced Hilbert polynomials of* ch *and* K_S.

(iii) *For all SW basic classes a_1 satisfying $a_1 H \leq (c_1 - a_1)H$, the inequality is strict.*

Let $P(\mathbb{E})$ be any polynomial in $\mu(c_1(L))$ and descendent insertions arising from a polynomial in $\mu(c_1(L))$ and Chern classes of T_M^{vir} (for example, as in Proposition 2.1). Then

$$\int_{[M_S^H(2,c_1,c_2)]^{\mathrm{vir}}} P(\mathbb{E}) = -2^{1-\chi(\mathrm{ch})} \sum_{\substack{c_1 = a_1 + a_2 \\ a_1 H < a_2 H}} \mathrm{SW}(a_1)\, \mathcal{A}(L, a_1, a_2, c_2). \tag{4}$$

Remark 2.3. *Assuming the existence of a universal sheaf $\mathbb{E}$ on $M \times S$, where $M := M_S^H(2, c_1, c_2)$, is unnecessary. As remarked in the introduction, T_M^{vir} and $\mu(c_1(L))$ always exist, so the left-hand side of Mochizuki's formula is always defined. Mochizuki [35] works over the Deligne-Mumford stack of oriented sheaves, which has a universal sheaf. This can be used to show that the global existence of $\mathbb{E}$ on $M \times S$ can be dropped from the assumptions. In fact, when working on the stack, P can be* any *polynomial in descendent insertions defined using the universal sheaf on the stack. Also, since Mochizuki works on the stack, his formula and our version differ by a factor of 2.*

Remark 2.4. *Conjecturally, assumptions (ii) and (iii) can be dropped from Theorem 2.2 [24, 19, 20, 21]. Moreover, also conjecturally, in the sum in Mochizuki's formula, the inequality $a_1 H < a_2 H$ can be dropped. Assumption (i) is necessary.*

Suppose the assumptions of Theorem 2.2 are satisfied. Combining with Lemma 2.1, we find that $y^{-\frac{\mathrm{vd}}{2}} \chi_{-y}^{\mathrm{vir}}(M, \mu(L))$ is given by (4) with

$$P(\mathbb{E}) = y^{-\frac{\mathrm{vd}}{2}} \mathsf{X}_{-y}(-R\mathcal{H}om_\pi(\mathbb{E}, \mathbb{E})_0)\, e^{\mu(c_1(L))}, \tag{5}$$

where $\mathbb{E}$ is replaced by

$$\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2(a_2) \otimes \mathfrak{s}.$$

We note that the rank of

$$-R\mathcal{H}om_\pi(\mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2(a_2) \otimes \mathfrak{s}, \mathcal{I}_1(a_1) \otimes \mathfrak{s}^{-1} \oplus \mathcal{I}_2(a_2) \otimes \mathfrak{s})_0$$

equals the rank of $T_M^{\mathrm{vir}} = -R\mathcal{H}om_\pi(\mathbb{E}, \mathbb{E})_0$.

2.3. Universal Series

In this paragraph, S is *any* smooth projective surface, so we allow $p_g(S) = 0$. We want to study the intersection numbers (3) with $P(\mathbb{E})$ given by (5). Let $\mathsf{X}_y^{\mathbb{C}^*}(\cdot)$ denote the same expression as in (1), but with Chern character and Todd class replaced by $\mathbb{C}^*$-*equivariant* Chern character and Todd class (recall that we endow $S^{[n_1]} \times S^{[n_2]}$ with trivial $\mathbb{C}^*$-action). Define

$$f(s, y) := y^{-\frac{1}{2}} \mathsf{X}_{-y}^{\mathbb{C}^*}(\mathfrak{s}^2) = y^{-\frac{1}{2}} \frac{2s(1 - ye^{-2s})}{1 - e^{-2s}}$$

where the second equality follows from the properties listed in Section 2.1. We write $\chi(a) := \chi(\mathcal{O}_S(a))$ for any $a \in A^1(S)$. For *any* $L, a, c_1 \in A^1(S)$, we define

$$\mathsf{Z}_S^{\mathrm{inst}}(L,a,c_1,s,y,q) := (2s)^{-\chi(\mathcal{O}_S)}\left(\frac{2s}{f(s,y)}\right)^{-\chi(c_1-2a)}\left(\frac{-2s}{f(-s,y)}\right)^{-\chi(2a-c_1)} e^{(c_1-2a)Ls}$$
$$\cdot \sum_{n_1,n_2} q^{n_1+n_2} \int_{S^{[n_1]}\times S^{[n_2]}} \widetilde{\Psi}(L,a,c_1-a,n_1,n_2,s).$$

The first line of this expression is just a normalization factor, so

$$\mathsf{Z}_S^{\mathrm{inst}}(L,a,c_1,s,y,q) \in 1 + q\,\mathbb{Q}[y^{\pm\frac{1}{2}}]((s))[[q]].$$

We note that the definition of $\mathsf{Z}_S^{\mathrm{inst}}(L,a,c_1,s,y,q)$ makes sense for any *possibly disconnected* smooth projective surface S, and $L, a, c_1 \in A^1(S)$.

Lemma 2.5. *Let $S = S' \sqcup S''$, where S', S'' are (possibly disconnected) smooth projective surfaces. Let $L, a, c_1 \in A^1(S)$, and define $L' := L|_{S'}$, $a' := a|_{S'}$, $c_1' := c_1|_{S'}$, $L'' := L|_{S''}$, $a'' := a|_{S''}$, and $c_1'' := c_1|_{S''}$. Then*

$$\mathsf{Z}_S^{\mathrm{inst}}(L,a,c_1,s,y,q) = \mathsf{Z}_{S'}^{\mathrm{inst}}(L',a',c_1',s,y,q)\,\mathsf{Z}_{S''}^{\mathrm{inst}}(L'',a'',c_1'',s,y,q).$$

Proof. The case $L = \mathcal{O}_S$ was established in [19, Proposition 3.3]. The only new feature of the present case is the following.

Define $S_2 = S \sqcup S$. As shown in [19, Proposition 3.3], the integrals over $S^{[n_1]} \times S^{[n_2]}$ occurring in the coefficients of $\mathsf{Z}_S^{\mathrm{inst}}(L,a,c_1,s,y,q)$ can be written as integrals on $S_2^{[n]}$ by using the decomposition

$$S_2^{[n]} = \bigsqcup_{n_1+n_2=n} S^{[n_1]} \times S^{[n_2]}.$$

Since $S = S' \sqcup S''$, we have a further decomposition

$$S^{[n_1]} \times S^{[n_2]} = \bigsqcup_{l_1+l_2=n_1,\, m_1+m_2=n_2} S'^{[l_1]} \times S''^{[l_2]} \times S'^{[m_1]} \times S''^{[m_2]}.$$

Then the insertion $e^{\mu(c_1(L))}$ restricted to $S'^{[l_1]} \times S''^{[l_2]} \times S'^{[m_1]} \times S''^{[m_2]}$ equals

$$p'^* e^{\mu(c_1(L'))}\, p''^* e^{\mu(c_1(L''))},$$

where p', p'' are the projections in the diagram

$$\begin{tikzcd}[column sep=small] & S'^{[l_1]} \times S''^{[l_2]} \times S'^{[m_1]} \times S''^{[m_2]} \arrow[dl, "p'"'] \arrow[dr, "p''"] & \\ S'^{[l_1]} \times S'^{[m_1]} & & S''^{[l_2]} \times S''^{[m_2]} \end{tikzcd}$$

and $S'^{[l_1]} \times S'^{[m_1]}$ is seen as a connected component of $S_2'^{[l_1+m_1]}$ and $S''^{[l_2]} \times S''^{[m_2]}$ as a connected component of $S_2''^{[l_2+m_2]}$. The rest of the proof proceeds exactly as in [19, Proposition 3.3]. □

Lemma 2.6. *There exist universal functions*

$$A_1(y,q), \ldots, A_{11}(y,q) \in 1 + q\,\mathbb{Q}[y^{\pm\frac{1}{2}}][[q]]$$

such that for any smooth projective surface S and $L, a, c_1 \in A^1(S)$, we have

$$\mathsf{Z}_S^{\mathrm{inst}}(L, a, c_1, s, y, q) = A_1^{L^2} A_2^{La} A_3^{a^2} A_4^{ac_1} A_5^{c_1^2} A_6^{Lc_1} A_7^{LK_S} A_8^{aK_S} A_9^{c_1K_S} A_{10}^{K_S^2} A_{11}^{\chi(\mathcal{O}_S)}.$$

Proof. By [10], tautological integrals on Hilbert schemes of points on surfaces are universal. We are dealing with integrals over products of Hilbert schemes, which were handled in [22, Lemma 5.5]. By [22, Lemma 5.5] (see also [19, Proposition 3.3]), there exists a universal power series

$$G \in \mathbb{Q}[x_1, \cdots, x_{11}][[q]]$$

such that for any smooth projective surface S and $L, a, c_1 \in A^1(S)$, we have

$$\mathsf{Z}_S^{\mathrm{inst}}(L, a, c_1, s, y, q) = e^{G(L^2, La, a^2, ac_1, c_1^2, Lc_1, LK_S, aK_S, c_1K_S, K_S^2, \chi(\mathcal{O}_S))}. \tag{6}$$

Here, we use the fact that $\mathsf{Z}_S^{\mathrm{inst}}(L, a, c_1, s, y, q)$ starts with 1.

We claim that equation (6) and Lemma 2.5 together imply the result. This can be seen as follows (see also [22, Lemma 5.5]). Choose 11 quadruples $(S^{(i)}, L^{(i)}, a^{(i)}, c_1^{(i)})$ such that the corresponding vectors of Chern numbers

$$w_i := ((L^{(i)})^2, \ldots, \chi(\mathcal{O}_{S^{(i)}})) \in \mathbb{Q}^{11}$$

form a $\mathbb{Q}$-basis. Now consider any (S, L, a, c_1). Then we can decompose its vector of Chern numbers $w = (L^2, \ldots, \chi(\mathcal{O}_S))$ as $w = \sum_i n_i w_i$, for some $n_i \in \mathbb{Q}$. If all $n_i \in \mathbb{Z}_{\geq 0}$; then Lemma 2.5 implies that

$$\mathsf{Z}_S^{\mathrm{inst}}(L, a, c_1, s, y, q) = \prod_{i=1}^{11} \left(e^{G(w_i)}\right)^{n_i}. \tag{7}$$

Let W be the matrix with column vectors $w_1, \ldots, w_{11}$ and $M = (m_{ij})$ its inverse. Defining $A_j := \exp(\sum_i m_{ij} G(w_i))$, equation (7) implies

$$\mathsf{Z}_S^{\mathrm{inst}}(L, a, c_1, s, y, q) = A_1^{L^2} \cdots A_{11}^{\chi(\mathcal{O}_S)}.$$

Since the set of vectors w with all $n_i \in \mathbb{Z}_{\geq 0}$ is Zariski dense in $\mathbb{Q}^{11}$, the proposition holds for *any* (S, L, a, c_1). □

Theorem 2.2 and Lemma 2.6 at once imply the following result.

Proposition 2.7. *Let S be a smooth projective surface with $b_1(S) = 0$, $p_g(S) > 0$, and $L \in \mathrm{Pic}(S)$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Assume the following hold:*

(i) $\chi(\mathrm{ch}) > 0$, *where* $\chi(\mathrm{ch}) := \int_S \mathrm{ch} \cdot \mathrm{td}(S)$ *and* $\mathrm{ch} = (2, c_1, \frac{1}{2}c_1^2 - c_2)$.

(ii) $p_{\mathrm{ch}} > p_{K_S}$, *where* $p_{\mathrm{ch}} = \chi(e^{mH} \cdot \mathrm{ch})/2$ *and* $p_{K_S} = \chi(e^{mH} \cdot e^{K_S})$ *are the reduced Hilbert polynomials of* ch *and* K_S.

(iii) *For all SW basic classes a with $aH \leq (c_1 - a)H$, the inequality is strict.*

Then $y^{-\frac{\mathrm{vd}}{2}}\chi^{\mathrm{vir}}_{-y}(M^H_S(2,c_1,c_2),\mu(L))$ *is the coefficient of* $x^{\mathrm{vd}}s^0$ *of*

$$
\begin{aligned}
&-2\sum_{\substack{a\in H^2(S,\mathbb{Z})\\ aH<(c_1-a)H}} \mathrm{SW}(a)\,A_1(y,2x^4)^{L^2}\left(e^{2s}A_2(y,2x^4)\right)^{La}\\
&\cdot\left(2^{-1}\left(\frac{2s}{f(s,y)}\right)^2\left(\frac{-2s}{f(-s,y)}\right)^2x^{-4}A_3(y,2x^4)\right)^{a^2}\\
&\cdot\left(2\left(\frac{2s}{f(s,y)}\right)^{-2}\left(\frac{-2s}{f(-s,y)}\right)^{-2}x^4A_4(y,2x^4)\right)^{ac_1}\\
&\cdot\left(2^{-\frac12}\left(\frac{2s}{f(s,y)}\right)^{\frac12}\left(\frac{-2s}{f(-s,y)}\right)^{\frac12}x^{-1}A_5(y,2x^4)\right)^{c_1^2}\\
&\cdot\left(e^{-s}A_6(y,2x^4)\right)^{Lc_1}A_7(y,2x^4)^{LK_S}\\
&\cdot\left(\left(\frac{2s}{f(s,y)}\right)\left(\frac{-2s}{f(-s,y)}\right)^{-1}A_8(y,2x^4)\right)^{aK_S}\\
&\cdot\left(2^{\frac12}\left(\frac{2s}{f(s,y)}\right)^{-\frac12}\left(\frac{-2s}{f(-s,y)}\right)^{\frac12}A_9(y,2x^4)\right)^{c_1K_S}\\
&\cdot A_{10}(y,2x^4)^{K_S^2}\left(\frac{s}{2}\left(\frac{2s}{f(s,y)}\right)\left(\frac{-2s}{f(-s,y)}\right)x^{-3}A_{11}(y,2x^4)\right)^{\chi(\mathcal{O}_S)}.
\end{aligned}
$$

Remark 2.8. *By Remark 2.4, conjecturally, assumptions (ii) and (iii) in the previous proposition, as well as the inequality* $aH<(c_1-a)H$ *in the sum, can be dropped.*

2.4. Reduction to Toric Surfaces

We now present 11 choices of (S,L,a,c_1) for which the vectors of Chern numbers $(L^2,\dots,\chi(\mathcal{O}_S))$ are $\mathbb{Q}$-independent:

$$
\begin{aligned}
(S,L,a,c_1)=\ &(\mathbb{P}^2,\mathcal{O},\mathcal{O},\mathcal{O}),\\
&(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O},\mathcal{O},\mathcal{O}),\\
&(\mathbb{P}^2,\mathcal{O},\mathcal{O}(1),\mathcal{O}(2)),\\
&(\mathbb{P}^2,\mathcal{O},\mathcal{O},\mathcal{O}(1)),\\
&(\mathbb{P}^2,\mathcal{O},\mathcal{O}(1),\mathcal{O}(3)),\\
&(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O},\mathcal{O}(0,1),\mathcal{O}(0,2)),\\
&(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O},\mathcal{O},\mathcal{O}(0,1)),\\
&(\mathbb{P}^2,\mathcal{O}(1),\mathcal{O},\mathcal{O}),\\
&(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(0,1),\mathcal{O},\mathcal{O}),\\
&(\mathbb{P}^2,\mathcal{O}(1),\mathcal{O}(1),\mathcal{O}(2)),\\
&(\mathbb{P}^2,\mathcal{O}(1),\mathcal{O},\mathcal{O}(1)).
\end{aligned}
$$

Each of these surfaces S is toric and hence has an action of $T = \mathbb{C}^* \times \mathbb{C}^*$. Choose T-equivariant structures on the line bundles corresponding to L, a, c_1. Then we can calculate $\mathsf{Z}_S^{\mathrm{inst}}(L, a, c_1, s, y, q)$ by Atiyah-Bott localization. More precisely, consider one of the intersection numbers

$$\int_{S^{[n_1]} \times S^{[n_2]}} \widetilde{\Psi}(L, a, c_1 - a, n_1, n_2, s)$$

appearing in the definition of $\mathsf{Z}_S^{\mathrm{inst}}(L, a, c_1, s, y, q)$. The action of T lifts to $S^{[n_1]} \times S^{[n_2]}$, and its fixed locus is indexed by pairs

$$\left(\{\lambda^{(\sigma)}\}_{\sigma=1}^{e(S)}, \{\mu^{(\sigma)}\}_{\sigma=1}^{e(S)}\right),$$

where each $\lambda^{(\sigma)} = (\lambda_1^{(\sigma)} \geq \lambda_2^{(\sigma)} \geq \ldots)$ and $\mu^{(\sigma)} = (\mu_1^{(\sigma)} \geq \mu_2^{(\sigma)} \geq \ldots)$ are partitions such that

$$\sum_\sigma |\lambda^{(\sigma)}| = \sum_{\sigma,i} \lambda_i^{(\sigma)} = n_1, \qquad \sum_\sigma |\mu^{(\sigma)}| = \sum_{\sigma,i} \mu_i^{(\sigma)} = n_2.$$

The Euler number $e(S)$ equals the number of torus fixed points p_σ of S, and each partition $\lambda^{(\sigma)}, \mu^{(\sigma)}$ corresponds (in the usual way) to a monomial ideal on the maximal T-invariant affine open subset $\mathbb{C}^2 \cong U_\sigma \subset S$ containing p_σ. For example, see [19, 20] for more details.

For any pair $(\{\lambda^{(\sigma)}\}_\sigma, \{\mu^{(\sigma)}\}_\sigma)$ corresponding to zero-dimensional T-fixed subschemes $(Z, W) \in S^{[n_1]} \times S^{[n_2]}$, we are interested in the restriction

$$\widetilde{\Psi}(L, a, c_1 - a, n_1, n_2, s)\Big|_{(Z,W)}. \tag{8}$$

Let $\widetilde{T} := T \times \mathbb{C}^*$, where $\mathbb{C}^*$ is the torus acting trivially on $S^{[n_1]} \times S^{[n_2]}$ (as in Mochizuki's formula). Denote by $\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{s}$ positive primitive generators of the character group of each factor of $\widetilde{T}$. Then the $\widetilde{T}$-equivariant K-group of a point is given by the following ring of Laurent polynomials:

$$K^0_{\widetilde{T}}(\mathrm{pt}) \cong \mathbb{Z}[\mathfrak{t}_1^\pm, \mathfrak{t}_2^\pm, \mathfrak{s}^\pm].$$

To calculate (8) in terms of $\epsilon_1 := c_1^{\widetilde{T}}(\mathfrak{t}_1)$, $\epsilon_2 := c_1^{\widetilde{T}}(\mathfrak{t}_2)$, and $s := c_1^{\widetilde{T}}(\mathfrak{s})$, we must determine the classes of the following complexes in $K^0_{\widetilde{T}}(\mathrm{pt})$:

$$\begin{aligned}
&H^0(\mathcal{O}_Z(a)), \quad H^0(\mathcal{O}_W(c_1 - a)),\\
&R\operatorname{Hom}_S(\mathcal{O}_Z, \mathcal{O}_Z), \quad R\operatorname{Hom}_S(\mathcal{O}_W, \mathcal{O}_W),\\
&R\operatorname{Hom}_S(\mathcal{O}_Z, \mathcal{O}_W(c_1 - 2a) \otimes \mathfrak{s}^2), \quad R\operatorname{Hom}_S(\mathcal{O}_W(c_1 - 2a) \otimes \mathfrak{s}^2, \mathcal{O}_Z),
\end{aligned}$$

where $I_Z, I_W \subset \mathcal{O}_S$ are the ideal sheaves of Z, W. The expressions in the first line follow at once from the $\widetilde{T}$-representations of Z, W in terms of the partitions $\lambda^{(\sigma)}, \mu^{(\sigma)}$. The expressions in lines two and three can be calculated by using a T-equivariant resolution of I_Z, I_W. For explicit formulae, see [19, Proposition 4.1]. Finally, $\mu(L)$ leads to the insertion

$$\pi_*\left(c_1^{\widetilde{T}}(L) \cdot (\mathrm{ch}_2^{\widetilde{T}}(\mathcal{O}_Z) + \mathrm{ch}_2^{\widetilde{T}}(\mathcal{O}_W)) \cap [S]\right) = \sum_{\sigma=1}^{e(S)} a_\sigma \cdot \left(|\lambda^{(\sigma)}| + |\mu^{(\sigma)}|\right),$$

where $\pi_* : K^0_{\widetilde{T}}(S) \longrightarrow K^0_{\widetilde{T}}(\mathrm{pt})$ denotes equivariant push-forward and a_σ is the character corresponding to $L|_{U_\sigma}$.

The calculation of $\mathsf{Z}_S^{\mathrm{inst}}$ for each of the 11 cases above is now a purely combinatorial problem, which we implemented in the computer program Pari/GP. We determined the universal series $A_1, \ldots, A_{11}$ of Proposition 2.6 to the following orders:

- For $A_1(1,q), \ldots, A_{11}(1,q)$, we computed the coefficients of $s^{l-3n}q^n$ for all $n \leq 10$, $l \leq 49$. (Recall: $A_i(1,q), A_i(y,q)$ are *Laurent* series in s.)
- For $A_1(y,q), \ldots, A_{11}(y,q)$, we computed the coefficients of $s^{l-5n}y^m q^n$ for all $n \leq 6, m \leq 9$, $l \leq 30$.

2.5. *Verifications*

We verified Conjecture 1.1 in the following cases.[8] In each case, we fix S, c_1, c_2 as indicated, and we choose H such that the assumptions of Proposition 2.7 are satisfied. We use the explicit expansions of $A_1(1,q), \ldots, A_{11}(1,q)$ determined in the previous section by localization calculations on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ using Pari/GP.

1. S is a K3 surface, c_1 is such that $c_1^2 = 0, 2, \ldots, 20$, and $\mathrm{vd} < 14$.
2. S is the blow-up of a K3 surface in a point, $c_1 = \pi^*C + \epsilon E$ such that $C^2 = -4, -2, \ldots, 10$, $\epsilon = -2, -1, \ldots, 2$, and $\mathrm{vd} < 15$.
3. S is the blow-up of a K3 surface in two distinct points, $c_1 = \pi^*C + e_1E_1 + e_2E_2$ such that $C^2 = -2, 0, \ldots, 6$, $e_1, e_2 = 0, 1$, and $\mathrm{vd} < 13$.
4. $S \to \mathbb{P}^1$ is an elliptic surface of type $E(N)$,[9] $N = \chi(\mathcal{O}_S) = 3, 4, \ldots, 7$, $c_1 = mB + nF$ where B is the class of a section, F is the class of a fibre, $m = -1, 0, 1, 2$, $n = -2, -1, \ldots, 5$, and $\mathrm{vd} < 12$.
5. S is the blow-up of an elliptic surface of type $E(3)$ in a point, $c_1 = \pi^*C + \epsilon E$ such that $CK_S = -1, 0, \ldots, 4$, $C^2 = -4, -3, \ldots, 10$, $\epsilon = 0, 1$, and $\mathrm{vd} < 12$.
6. S is a minimal general type surface with $b_1(S) = 0$, $\chi(\mathcal{O}_S) = 2$, $K_S^2 = 1$ [33], c_1 is such that $c_1 \cdot K_S = 0, 1$, $c_1^2 = -2, -1, \ldots, 11$, and $\mathrm{vd} < 12$.
7. S is a double cover of $\mathbb{P}^2$ branched along a smooth octic surface, c_1 is such that $c_1 \cdot K_S = 0, 1, \ldots, 10$, $c_1^2 = 0, 1, \ldots, 30$, and $\mathrm{vd} < 12$.
8. S is the blow-up of a surface S' as in (7) in a point, $c_1 = \pi^*C + \epsilon E$ such that $CK_S = -2, -1, \ldots, 2$, $C^2 = -2, -1, \ldots, 8$, $\epsilon = 0, 1$, and $\mathrm{vd} < 11$.
9. S is a very general smooth quintic surface in $\mathbb{P}^3$ (then $\mathrm{Pic}(S) = \mathbb{Z}[H]$), $c_1 = 2H$ and $\mathrm{vd} < 8$, or $c_1 = 3H$ and $\mathrm{vd} < 7$.

Assuming the strong form of Mochizuki's formula holds (Remark 2.4), we also verified Conjecture 1.1 in the following cases:

(10) S is a smooth quintic surface in $\mathbb{P}^3$, c_1 such that $c_1 \cdot K_S = 0, 1, \ldots, 25$, $c_1^2 = -4, -3, \ldots, 20$, and $\mathrm{vd} < 11$.

(11) S is the blow-up of a quintic surface in $\mathbb{P}^3$ in a point, $c_1 = \pi^*C + \epsilon E$ such that $CK_S = -5, -4, \ldots, 5$, $C^2 = -4, -3, \ldots, 8$, $\epsilon = 0, 1$, and $\mathrm{vd} < 10$.

Applying the same method and using our explicit expansions of $A_1(y,q)$, …, $A_{11}(y,q)$ from the previous section, we verified Conjecture 1.2 in the following cases:

1. S is a K3 surface, c_1 is such that $c_1^2 = 0, 2, \ldots, 14$, and $\mathrm{vd} < 11$.
2. S is the blow-up of a K3 surface in a point, $c_1 = \pi^*C + \epsilon E$ such that $C^2 = -4, -2, \ldots, 14$, $\epsilon = -2, -1, \ldots, 2$, and $\mathrm{vd} < 10$.
3. S is the blow-up of a K3 surface in two distinct points, $c_1 = \pi^*C + e_1E_1 + e_2E_2$ such that $C^2 = -2, 0, \ldots, 6$, $e_1, e_2 = 0, 1$, and $\mathrm{vd} < 10$.

[8] In this list, $\pi : S \to S'$ always denotes the blow-up in a point, and the exceptional divisor is written as E (or E_1, E_2 in the case of a blow-up in two distinct points).

[9] That is, an elliptic surface $S \to \mathbb{P}^1$ with section, $12N$ rational 1-nodal fibres, and no other singular fibres.

4. S is an elliptic surface of type $E(N)$ with $N = 3, 4, 5$, $c_1 = mB + nF$ with $m = -1, 0, 1, 2$, $n = -2, -1, \ldots, 10$, and vd < 9.
5. S is the blow-up of an elliptic surface of type $E(3)$ in a point, $c_1 = \pi^* C + \epsilon E$ such that $CK_S = -1, 0, \ldots, 4$, $C^2 = -16, -15, \ldots, 0$, $\epsilon = 0, 1$, and vd < 9.
6. S is the double cover of $\mathbb{P}^2$ branched along a smooth octic surface, c_1 is such that $c_1 \cdot K_S = -2, -1, \ldots, 2$, $c_1^2 = -16, -15, \ldots, -6$, and vd < 9.
7. S is the blow-up of S' as in (6) in a point, $c_1 = \pi^* C + \epsilon E$ such that $CK_S = -2, -1, \ldots, 2$, $C^2 = -16, -15 \ldots, 8$, $\epsilon = 0, 1$, and vd < 7.

Assuming the strong form of Mochizuki's formula holds (Remark 2.4), we also verified Conjecture 1.2 in the following cases:

(8) S is a smooth quintic surface in $\mathbb{P}^3$, c_1 is such that $c_1 \cdot K_S = 2, 3, \ldots, 6$, $c_1^2 = -16, -15, \ldots, -3$, and vd < 7.

(9) S is the blow-up of a smooth quintic surface in $\mathbb{P}^3$ in a point, $c_1 = \pi^* C + \epsilon E$ such that $CK_S = 0$, $C^2 = -23, -22, \ldots, -14$, $\epsilon = 0, 1$, and vd < 4.

3. Monopole Contribution and Nested Hilbert Schemes

In this section, we study the contribution of the monopole branch to the invariants $\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}} \otimes \mu(L))$ defined in the introduction. We prove that it is determined by universal series $C_1, \ldots, C_6$ as stated in Theorem 1.4. Moreover, we express these universal functions in terms of integrals over products over Hilbert schemes of points on S. Much like in the previous section, these integrals are determined by their value on $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, where we calculate them, modulo q^{15}, by localization.

The methods of this section are a variation on Laarakker's work [34], which in turn relies on Gholampour-Thomas's work [15, 16]. For $L = \mathcal{O}_S$ and $r = 2$, Theorem 1.4 was previously proved in [34] (in fact, for $L = \mathcal{O}_S$, he proved the analog of Theorem 1.4 in any rank). Then $\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}})$ are the rank 2 K-theoretic Vafa-Witten invariants defined by Thomas [42] and determined by the universal series C_1, C_2, C_5. Closed formulae for these universal series were conjectured in [21] (refining Vafa-Witten's original formula [43, (5.38)]) and subsequently verified in [34] up to the following orders:

$$\begin{aligned} C_1(y,q) &= \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})^{10}(1-q^{2n}y^2)(1-q^{2n}y^{-2})} \quad \mod q^{15} \\ C_2(y,q) &= (y^{\frac{1}{2}} + y^{-\frac{1}{2}}) q^{\frac{1}{4}} \frac{\overline{\eta}(q)^2}{\theta_2(q,y)} \quad \mod q^{15} \\ C_5(y,q) &= \frac{1}{(y^{\frac{1}{2}} + y^{-\frac{1}{2}}) q^{\frac{1}{4}}} \frac{\theta_2(q,y)}{\theta_3(q,y)} \quad \mod q^{15}, \end{aligned} \tag{1}$$

where $\overline{\eta}(q), \theta_2(q,y), \theta_3(q,y)$ were introduced in (4). The universal power series C_3, C_4, C_6 are new. In accordance with Conjecture 1.3, we show

$$\begin{aligned} C_3(y,q) &= \prod_{n=1}^{\infty} \left(\frac{(1-q^{2n})^2}{(1-q^{2n}y^2)(1-q^{2n}y^{-2})} \right)^{2n^2} \quad \mod q^{15} \\ C_4(y,q) &= \prod_{n=1}^{\infty} \left(\frac{1-q^n y^{-1}}{1-q^n y} \right)^n \left(\frac{1-q^{2n}y^{-2}}{1-q^{2n}y^2} \right)^n \left(\frac{1+q^{2n}y^{-1}}{1+q^{2n}y} \right)^{4n} \quad \mod q^{15} \\ C_6(y,q) &= \prod_{n=1}^{\infty} \left(\frac{1-(-1)^n q^n y^{-1}}{1-(-1)^n q^n y} \right)^{2n} \left(\frac{1-q^{4n}y^2}{1-q^{4n}y^{-2}} \right)^{4n} \quad \mod q^{15}. \end{aligned} \tag{2}$$

3.1. Gholampour-Thomas's Formula

Let S be a smooth projective surface satisfying $b_1(S) = 0$ and $p_g(S) > 0$. Let $r = 2$ and c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable Higgs sheaves on S with Chern classes c_1, c_2. Let $N := N_S^H(2, c_1, c_2)$, and let $M^{\mathrm{mon}} \subset N^{\mathbb{C}^*}$ be the monopole branch discussed in the introduction. Gholampour-Thomas [15] (see also [13]) prove that the components of M^{mon} are isomorphic to

$$S_\beta^{[n_0,n_1]} := \{(Z_0, Z_1, C) \,:\, I_{Z_0}(-C) \subset I_{Z_1}\} \subset S^{[n_0]} \times S^{[n_1]} \times |\beta|,$$

for *certain* (see Remark 3.2 below) $n_0, n_1 \geq 0$ and algebraic $\beta \in H^2(S, \mathbb{Z})$. In particular, such n_0, n_1, β satisfy

$$\begin{aligned} &c_1 - \beta + K_S \in 2H^2(S, \mathbb{Z}), \\ &c_2 = n_0 + n_1 + \Big(\frac{c_1 - \beta + K_S}{2}\Big)\Big(\frac{c_1 + \beta - K_S}{2}\Big). \end{aligned} \tag{3}$$

Whenever we have n_0, n_1, β satisfying (3), it is convenient to define

$$L_0 := \frac{c_1 - \beta + K_S}{2}, \quad L_1 := \frac{c_1 + \beta - K_S}{2}.$$

Consider the inclusion

$$\iota : S_\beta^{[n_0,n_1]} \subset S^{[n_0]} \times S^{[n_1]} \times |\beta|,$$

where $|\beta|$ denotes the linear system determined by $\mathcal{O}_S(\beta)$. The universal sheaf $\mathbb{E}$ on $M^{\mathrm{mon}} \times S$ restricted to the component $S_\beta^{[n_0,n_1]} \times S$ is

$$\mathbb{E} \cong \mathcal{I}_0 \otimes L_0 \oplus \mathcal{I}_1 \otimes L_1(1) \otimes \mathfrak{t}^{-1}, \tag{4}$$

where $\mathfrak{t}$ is a positive primitive character of the trivial $\mathbb{C}^*$-action on $M^{\mathrm{mon}} \times S \subset N^{\mathbb{C}^*} \times S$. Moreover, $\mathcal{I}_0, \mathcal{I}_1$ are the universal ideal sheaves pulled back from the factors of $S^{[n_0]} \times S^{[n_1]} \times S \times |\beta|$ (and then along $\iota \times \mathrm{id}_S$ to $S_\beta^{[n_0,n_1]}$), L_0, L_1 are pulled back from S, and $\mathcal{O}(1)$ is pulled back from $|\beta|$. Consider $M^{\mathrm{mon}} \subset N^{\mathbb{C}^*}$ with its $\mathbb{C}^*$-localized perfect obstruction theory [27].

Theorem 3.1 (Gholampour-Thomas)**.** *The class $\iota_*[S_\beta^{[n_0,n_1]}]^{\mathrm{vir}}$ is given by*

$$\mathrm{SW}(\beta)\, e\big(R\Gamma(\beta) \otimes \mathcal{O} - R\mathcal{H}om_\pi(\mathcal{I}_0, \mathcal{I}_1(\beta))\big) \in H_{2n_0+2n_1}(S^{[n_0]} \times S^{[n_1]} \times |\beta|),$$

where $\pi : S^{[n_0]} \times S^{[n_1]} \times S \to S^{[n_0]} \times S^{[n_1]}$ denotes projection, $e(\cdot) = c_{n_0+n_1}(\cdot)$, [10] *and the LHS should be interpreted as the image under push-forward along the inclusion $S^{[n_0]} \times S^{[n_1]} \times \{\mathrm{pt}\} \hookrightarrow S^{[n_0]} \times S^{[n_1]} \times |\beta|$ for any point* $\mathrm{pt} \in |\beta|$.

Remark 3.2. *Not all n_0, n_1, β satisfying* (3) *correspond to spaces $S_\beta^{[n_0,n_1]}$ containing Gieseker H-stable Higgs sheaves on S with Chern classes c_1, c_2. However, such components still have a virtual class given by the formula of Theorem 3.1 (induced by realizing $S_\beta^{[n_0,n_1]}$ as an incidence locus inside $S^{[n_0]} \times S^{[n_1]} \times |\beta|$ [15]). Laarakker proves that components $S_\beta^{[n_0,n_1]}$, which are not part of M^{mon}, satisfy $[S_\beta^{[n_0,n_1]}]^{\mathrm{vir}} = 0$ [34]. This implies that we may as well consider* all *n_0, n_1, β satisfying* (3) *and their corresponding spaces $S_\beta^{[n_0,n_1]}$.*

[10] By Carlsson-Okounkov vanishing, $c_{>n_0+n_1}(R\Gamma(\beta) \otimes \mathcal{O} - R\mathcal{H}om_\pi(\mathcal{I}_0, \mathcal{I}_1(\beta))) = 0$ [15].

Remark 3.3. *Unlike the instanton branch, it may happen that the monopole branch* M^{mon} *of* $N_S^H(2,c_1,c_2)^{\mathbb{C}^*}$ *has components of different virtual dimension with respect to the* $\mathbb{C}^*$*-localized perfect obstruction theory. A component* $S_\beta^{[n_0,n_1]} \subset M^{\mathrm{mon}}$*, where* n_0, n_1, β *satisfy* (3)*, has virtual dimension* n_0+n_1*. As an example, take* $S \to \mathbb{P}^2$*, a double cover branched over a smooth curve of degree 10; then* $K_S = 2L$*, where* $L \subset S$ *is the pull-back of the line from* $\mathbb{P}^2$*. Let* $H = L$*,* $c_1 = K_S$*, and* $c_2 \geq 3$ *odd; then* $\gcd(2, c_1H, \frac{1}{2}c_1(c_1-K_S)-c_2) = 1$*, in which case there are no rank 2 strictly Gieseker* H*-semistable Higgs sheaves on* S *with Chern classes* c_1, c_2*. For* $\beta = 0$ *and any* $0 \leq n_1 \leq n_0$ *such that* $c_2 = n_0+n_1$*, we obtain a non-empty component of virtual dimension* c_2*. For* $\beta = K_S$ *and any* $0 \leq n_0 < n_1$ *such that* $c_2 = n_0+n_1+2$*, we obtain a non-empty component of virtual dimension* c_2-2*. In both cases, the elements of the component correspond to Gieseker* H*-stable Higgs sheaves. Also note that in this example,* $\beta = 0, K_S$ *are the Seiberg-Witten basic classes of* S*.*

Although the virtual dimension of the monopole branch is in general *not* given by (1), we still define

$$\mathrm{vd}(2,c_1,c_2) := \mathrm{vd} = 4c_2 - c_1^2 - 3\chi(\mathcal{O}_S)$$

and use x^{vd} as the formal variable of our generating series.

3.2. Virtual Normal Bundle and $\mu(\mathbf{c_1}(\mathbf{L}))$*-Insertion*

The (dual) Tanaka-Thomas perfect obstruction theory is given by [40]

$$E_{\mathrm{TT}}^{\bullet\vee} = R\mathcal{H}om_\pi(\mathbb{E}, \mathbb{E}\otimes K_S \otimes \mathfrak{t})_0 - R\mathcal{H}om_\pi(\mathbb{E},\mathbb{E})_0.$$

Using (4), the class of $E_{\mathrm{TT}}^{\bullet\vee}|_{S_\beta^{[n_0,n_1]}}$ in $K^0_{\mathbb{C}^*}(S_\beta^{[n_0,n_1]})$ equals the restriction of the following element of $K^0_{\mathbb{C}^*}(S^{[n_0]}\times S^{[n_1]}\times|\beta|)$:

$$\begin{aligned} V_{n_0,n_1,\beta} :=\, & R\mathcal{H}om_\pi(\mathcal{I}_0, \mathcal{I}_1(\beta)\otimes\mathcal{O}(1)) + R\Gamma(\mathcal{O}_S)\otimes\mathcal{O} \\ & - R\mathcal{H}om_\pi(\mathcal{I}_0,\mathcal{I}_0) - R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_1) \\ & + R\mathcal{H}om_\pi(\mathcal{I}_1(\beta)\otimes\mathcal{O}(1), \mathcal{I}_0\otimes K_S^2\otimes\mathfrak{t}^2) - R\Gamma(K_S\otimes\mathfrak{t})\otimes\mathcal{O} \\ & + R\mathcal{H}om_\pi(\mathcal{I}_0,\mathcal{I}_0\otimes K_S\otimes\mathfrak{t}) + R\mathcal{H}om_\pi(\mathcal{I}_1,\mathcal{I}_1\otimes K_S\otimes\mathfrak{t}) \\ & - R\mathcal{H}om_\pi(\mathcal{I}_0,\mathcal{I}_1(\beta)\otimes K_S^*\otimes\mathfrak{t}^{-1}) - R\mathcal{H}om_\pi(\mathcal{I}_1(\beta)\otimes K_S^*\otimes\mathfrak{t}^{-1},\mathcal{I}_0), \end{aligned}$$

where lines 1–2 have $\mathbb{C}^*$-weight zero and lines 3–5 have non-zero $\mathbb{C}^*$-weight. We denote by $(\cdot)^{\mathrm{mov}}$ the weight $\neq 0$ part of a complex and by $(\cdot)^{\mathbb{C}^*}$ the weight zero part. Therefore, on $S_\beta^{[n_0,n_1]}$, the virtual normal bundle N^{vir} and $\mathbb{C}^*$-localized perfect obstruction theory are given by

$$\begin{aligned} N^{\mathrm{vir}} := (E_{\mathrm{TT}}^{\bullet\vee})^{\mathrm{mov}} &= V^{\mathrm{mov}}_{n_0,n_1,\beta}|_{S_\beta^{[n_0,n_1]}}, \\ (E_{\mathrm{TT}}^{\bullet\vee})^{\mathbb{C}^*} &= V^{\mathbb{C}^*}_{n_0,n_1,\beta}|_{S_\beta^{[n_0,n_1]}}. \end{aligned}$$

Finally, we write

$$V_{n_0,n_1} := V_{n_0,n_1,\beta}|_{S^{[n_0]}\times S^{[n_1]}\times\{\mathrm{pt}\}} \in K^0_{\mathbb{C}^*}(S^{[n_0]}\times S^{[n_1]}).$$

This restriction essentially amounts to removing $\mathcal{O}(1)$ from the expression of $V_{n_0,n_1,\beta}$. Using Theorem 3.1, we conclude that the contribution of $S_\beta^{[n_0,n_1]}$ to $\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}}\otimes\mu(L))$ equals

$$\mathrm{SW}(\beta)\cdot\int_{S^{[n_0]}\times S^{[n_1]}} e\big(R\Gamma(\beta)\otimes\mathcal{O}-R\mathcal{H}om_\pi(\mathcal{I}_0,\mathcal{I}_1(\beta)\big)$$
$$\cdot\frac{\mathrm{ch}(\sqrt{\det(V_{n_0,n_1})^\vee})}{\mathrm{ch}(\Lambda_{-1}(V^{\mathrm{mov}}_{n_0,n_1})^\vee)}\, e^{\mu(c_1(L))}\,\mathrm{td}(V^{\mathbb{C}^*}_{n_0,n_1}).$$

Here, we used that $\mu(c_1(L))=\pi_*(\pi_S^*c_1(L)\cap(-\mathrm{ch}_2(\mathbb{E})+\frac{1}{4}c_1(\mathbb{E})^2)$ restricted to $S_\beta^{[n_0,n_1]}$ also pulls back from an expression on $S^{[n_0]}\times S^{[n_1]}\times|\beta|$. On

$$S^{[n_0]}\times S^{[n_1]}\times\{\mathrm{pt}\}\subset S^{[n_0]}\times S^{[n_1]}\times|\beta|,$$

this expression is given by

$$\begin{aligned}
\mu(c_1(L)) =&\,\pi_*\Big(\pi_S^*c_1(L)\cdot(-\mathrm{ch}_2(\mathcal{I}_0)-\mathrm{ch}_2(\mathcal{I}_1))\cap[S^{[n_0]}\times S^{[n_1]}\times S]\Big)\\
&-\frac{1}{4}\int_S L\cdot\Big(\frac{c_1-\beta+K_S}{2}\Big)^2-\frac{1}{4}\int_S L\cdot\Big(\frac{c_1+\beta-K_S}{2}-t\Big)^2\\
&+\frac{1}{2}\int_S L\cdot\Big(\frac{c_1-\beta+K_S}{2}\Big)\Big(\frac{c_1+\beta-K_S}{2}-t\Big)\\
=&\,\pi_*\Big(\pi_S^*c_1(L)\cdot(-\mathrm{ch}_2(\mathcal{I}_0)-\mathrm{ch}_2(\mathcal{I}_1))\cap[S^{[n_0]}\times S^{[n_1]}\times S]\Big)\\
&+\frac{t}{2}\int_S L\cdot(\beta-K_S),
\end{aligned}$$

where the equivariant integrals $\int_S(\cdots)\in K^0_{\mathbb{C}^*}(\mathrm{pt})=\mathbb{Z}[t^{\pm1}]$ are multiplied with the fundamental class $[S^{[n_0]}\times S^{[n_1]}]$, and we are suppressing some Poincaré duals. Exponentiating and using $y:=e^t$ gives

$$e^{\mu(c_1(L))}=y^{\frac{1}{2}L(\beta-K_S)}e^{\pi_*\left(\pi_S^*c_1(L)\cdot(-\mathrm{ch}_2(\mathcal{I}_0)-\mathrm{ch}_2(\mathcal{I}_1))\cap[S^{[n_0]}\times S^{[n_1]}\times S]\right)}.$$

3.3. Universal Series

Let S be any smooth projective surface not necessarily satisfying $b_1(S)=0$ and $p_g(S)>0$. For any $L,\beta\in\mathrm{Pic}(S)$ and n_0,n_1, the expressions

$$V_{n_0,n_1},\ \mu(c_1(L))\in K^0_{\mathbb{C}^*}(S^{[n_0]}\times S^{[n_1]})$$

are defined as in the previous paragraph. We define

$$\mathsf{Z}_S^{\mathrm{mon}}(L,\beta,y,q):=y^{-\frac{1}{2}L(\beta-K_S)}\left(\frac{-1}{y^{\frac{1}{2}}+y^{-\frac{1}{2}}}\right)^{-\chi(\beta-K_S)}(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^{-\chi(\beta)+\chi(\mathcal{O}_S)}$$
$$\cdot\sum_{n_0,n_1\geq0}q^{n_0+n_1}\int_{S^{[n_0]}\times S^{[n_1]}} e\big(R\Gamma(\beta)\otimes\mathcal{O}-R\mathcal{H}om_\pi(\mathcal{I}_0,\mathcal{I}_1(\beta)\big)$$
$$\cdot\frac{\mathrm{ch}(\sqrt{\det(V_{n_0,n_1})^\vee})}{\mathrm{ch}(\Lambda_{-1}(V^{\mathrm{mov}}_{n_0,n_1})^\vee)}\, e^{\mu(c_1(L))}\,\mathrm{td}(V^{\mathbb{C}^*}_{n_0,n_1}).$$

Here, the first line is a normalization factor ensuring that

$$\mathsf{Z}_S^{\mathrm{mon}}(L,\beta,y,q)\in 1+q\,\mathbb{Q}[y^{\pm\frac{1}{2}}][[q]].$$

The normalization factor can be computed as follows. Putting $n_0 = n_1 = 0$, the definition of V_{n_0,n_1} gives

$$\begin{aligned} V_{0,0} &= R\Gamma(\mathcal{O}_S(\beta)) - R\Gamma(\mathcal{O}_S) + R\Gamma(\mathcal{O}_S(-\beta+2K_S)\otimes \mathfrak{t}^2) \\ &\quad + R\Gamma(\mathcal{O}_S(K_S)\otimes \mathfrak{t}) - R\Gamma(\mathcal{O}_S(\beta-K_S)\otimes \mathfrak{t}^{-1}) - R\Gamma(\mathcal{O}_S(-\beta+K_S)\otimes \mathfrak{t}). \end{aligned} \tag{5}$$

Using

$$\frac{\mathrm{ch}(\sqrt{L^*})}{\mathrm{ch}(\Lambda_{-1}L^*)} = \frac{1}{e^{\frac{1}{2}c_1(L)} - e^{-\frac{1}{2}c_1(L)}}$$

combined with Serre duality and $y = e^t$, we obtain

$$\begin{aligned} &e^{\mu(c_1(L))}\,\frac{\mathrm{ch}(\sqrt{\det(V_{0,0})^\vee})}{\mathrm{ch}(\Lambda_{-1}(V_{0,0}^{\mathrm{mov}})^\vee)}\,\mathrm{td}(V_{0,0}^{\mathbb{C}^*}) \\ &= y^{\frac{1}{2}L(\beta-K_S)}\left(\frac{y^{-\frac{1}{2}}-y^{\frac{1}{2}}}{y-y^{-1}}\right)^{\chi(\beta-K_S)}(y^{\frac{1}{2}}-y^{-\frac{1}{2}})^{\chi(\beta)-\chi(\mathcal{O}_S)}. \end{aligned}$$

The generating series $\mathsf{Z}_S^{\mathrm{mon}}(L,\beta,y,q)$ has the following universal property.

Lemma 3.4. *There exist universal functions*

$$B_1(y,q),\ldots,B_7(y,q) \in 1 + q\,\mathbb{Q}[y^{\pm\frac{1}{2}}][[q]]$$

such that for any smooth projective surface S and $L,\beta \in A^1(S)$, we have

$$\mathsf{Z}_S^{\mathrm{mon}}(L,\beta,y,q) = B_1^{L^2} B_2^{L\beta} B_3^{\beta^2} B_4^{LK_S} B_5^{\beta K_S} B_6^{K_S^2} B_7^{\chi(\mathcal{O}_S)}.$$

Proof. The case $L = \mathcal{O}_S$ is proved (for any rank r) in [34, Section 8]. The strategy is similar to the proof of Proposition 2.6:

Step 1: Multiplicativity. Let $S = S' \sqcup S''$, where S', S'' are possibly disconnected smooth projective surfaces. Let $L,\beta \in A^1(S)$, and define $L' := L|_{S'}$, $\beta' := \beta|_{S'}$, $L'' := L|_{S''}$, and $\beta'' := \beta|_{S''}$. Then

$$\mathsf{Z}_S^{\mathrm{mon}}(L,\beta,y,q) = \mathsf{Z}_{S'}^{\mathrm{mon}}(L',\beta',y,q)\,\mathsf{Z}_{S''}^{\mathrm{mon}}(L'',\beta'',y,q).$$

The only new feature compared to [34, Section 8] is the insertion

$$\pi_*\left(\pi_S^* c_1(L)\cdot(-\mathrm{ch}_2(\mathcal{I}_0)-\mathrm{ch}_2(\mathcal{I}_1))\cap[S^{[n_0]}\times S^{[n_1]}]\right),$$

which we discussed in Lemma 2.5.

Step 2: Universality. This is proved as in Lemma 2.6. □

Lemma 3.5. *Let S be a smooth projective surface with $b_1(S) = 0$, $p_g(S) > 0$, and $L \in \mathrm{Pic}(S)$. Let H, c_1, c_2 be chosen such that there exist no rank 2 strictly Gieseker H-semistable Higgs sheaves on S*

with Chern classes c_1, c_2. For vd *given by* (1)*, the monopole contribution to $\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}} \times \mu(L))$ is given by the coefficient of $(-x)^{\mathrm{vd}}$ of*

$$\sum_{\beta \in H^2(S,\mathbb{Z})} \delta_{c_1, K_S-\beta}\, \mathrm{SW}(\beta)\, B_1(y,x^4)^{L^2} \left(y^{\frac{1}{2}} B_2(y,x^4) \right)^{L\beta}$$
$$\cdot \left(\left(\frac{-1}{y^{\frac{1}{2}} + y^{-\frac{1}{2}}} \right)^{\frac{1}{2}} (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{\frac{1}{2}} (-x)^{-1} B_3(y,x^4) \right)^{\beta^2}$$
$$\cdot \left(y^{-\frac{1}{2}} B_4(y,x^4) \right)^{LK_S} \left(\left(\frac{-1}{y^{\frac{1}{2}} + y^{-\frac{1}{2}}} \right)^{-\frac{3}{2}} (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{-\frac{1}{2}} (-x)^2 B_5(y,x^4) \right)^{\beta K_S}$$
$$\cdot \left(\left(\frac{-1}{y^{\frac{1}{2}} + y^{-\frac{1}{2}}} \right) (-x)^{-1} B_6(y,x^4) \right)^{K_S^2} \left(\left(\frac{-1}{y^{\frac{1}{2}} + y^{-\frac{1}{2}}} \right) (-x)^{-3} B_7(y,x^4) \right)^{\chi(\mathcal{O}_S)}.$$

Proof. By Remark 3.2, we sum the contributions to the invariant of $S_\beta^{[n_0,n_1]}$ for all $\beta \in H^2(S,\mathbb{Z})$, $n_0, n_1 \in \mathbb{Z}_{\geq 0}$ such that $c_1 + \beta - K_S \in 2H^2(S,\mathbb{Z})$, and

$$c_2 = n_0 + n_1 + \left(\frac{c_1 - \beta + K_S}{2} \right) \left(\frac{c_1 + \beta - K_S}{2} \right),$$

or, equivalently, $\mathrm{vd} = 4(n_0+n_1) - (\beta - K_S)^2 - 3\chi(\mathcal{O}_S)$. As shown in [34, Section 8], this gives $\sum_{\beta \in H^2(S,\mathbb{Z})} \delta_{c_1, K_S-\beta}\, \mathrm{SW}(\beta) \cdot (\cdots)$, where $\delta_{a,b}$ was defined in (5), and $(\cdots)$ equals the coefficient of $(-1)^{\mathrm{vd}} x^{\mathrm{vd}}$ of

$$(-1)^{\mathrm{vd}} y^{\frac{1}{2}L(\beta - K_S)} \left(\frac{y^{-\frac{1}{2}} - y^{\frac{1}{2}}}{y - y^{-1}} \right)^{\chi(\beta - K_S)} (y^{\frac{1}{2}} - y^{-\frac{1}{2}})^{\chi(\beta) - \chi(\mathcal{O}_S)} \mathsf{Z}_S^{\mathrm{mon}}(L, \beta, y, x^4).$$

Lemma 3.4 then gives $\mathsf{Z}_S^{\mathrm{mon}}(L,\beta,y,x^4)$ in terms of the universal series B_i. □

Proof of Theorem 1.4. There are finitely many $\beta \in H^2(S,\mathbb{Z})$ for which $\mathrm{SW}(\beta) \neq 0$. These classes satisfy $\beta^2 = \beta K_S$ [35, Proposition 6.3.1]. The theorem follows by defining $C_1 := B_7$, $C_2 := B_6$, $C_3 := B_1$, $C_4 := B_4$, $C_5 := B_3 B_5$, $C_6 := B_2$. □

3.4. *Reduction to Toric Surfaces*

Consider the following seven choices of (S, L, β) for which the corresponding vectors of Chern numbers $(L^2, \ldots, \chi(\mathcal{O}_S))$ are $\mathbb{Q}$-independent:

$$\begin{aligned} (S, L, \beta) = \; & (\mathbb{P}^2, \mathcal{O}, \mathcal{O}), \\ & (\mathbb{P}^2, \mathcal{O}(-3), \mathcal{O}), \\ & (\mathbb{P}^2, \mathcal{O}(-6), \mathcal{O}), \\ & (\mathbb{P}^2, \mathcal{O}, \mathcal{O}(6)), \\ & (\mathbb{P}^2, \mathcal{O}, \mathcal{O}(-6)), \\ & (\mathbb{P}^2, \mathcal{O}(-3), \mathcal{O}(-6)), \\ & (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}, \mathcal{O}). \end{aligned}$$

In each case, localization (as in Section 2.4) reduces the series $\mathsf{Z}_S^{\mathrm{mon}}(L,\beta,y,q)$ to a purely combinatorial expression. In this way, we determined the universal series $B_1,\ldots,B_7$ modulo q^{15}. For our calculations, we used (and slightly adapted) a SAGE program of Laarakker, which was used for the calculation of K-theoretic Vafa-Witten invariants in [34]. Using the definitions of $C_1,\ldots,C_6$ in terms of $B_1,\ldots,B_7$, we obtain (1) and (2).

3.5. K3 Surfaces

In this section, we consider $\mathsf{Z}_S^{\mathrm{mon}}(L,\beta,y,q)$ when S is a K3 surface and $\beta=0$. Note that 0 is the only Seiberg-Witten basic class of a K3 surface, and $\mathrm{SW}(0)=1$. Let $\iota : S_0^{[n_0,n_1]} \hookrightarrow S^{[n_0]}\times S^{[n_1]}$ be the natural inclusion. Laarakker [34, Section 10] observes that

$$\iota_*[S_0^{[n_0,n_1]}]^{\mathrm{vir}} = \begin{cases} \Delta_* S^{[n]} & \text{when } \; n_0=n_1=n \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where $\Delta : S^{[n]} \hookrightarrow S^{[n]}\times S^{[n]}$ is the diagonal embedding. In other words, only universally thickened nestings $Z_0=Z_1$ contribute to the invariants.[11] This fact is explained geometrically using cosection localization in [42, Section 5.3]. This gives a simplication of $V_{n,n,0}$ (derived in [34, Section 10] for any rank r)

$$\Delta^* V_{n,n} = T_{S^{[n]}} + T_{S^{[n]}}\otimes \mathfrak{t}^{-1} - T_{S^{[n]}}\otimes\mathfrak{t} - T_{S^{[n]}}\otimes \mathfrak{t}^2 + V_{0,0}, \tag{7}$$

where $V_{0,0}$ is the normalization term (5), which should be viewed as pulled back from $S^{[n]}\to \mathrm{pt}$. Using (6) and (7), Laarakker expresses the universal function C_1 of Theorem 1.4 in terms of

$$\chi_y(S^{[n]}) = \chi(S^{[n]}, \Lambda_y \Omega_{S^{[n]}}), \quad \text{where } \; S=\mathrm{K3}.$$

In turn, χ_y-genera of Hilbert schemes of points on K3 surfaces were calculated by Göttsche and W. Soergel [25].

Recently, using Borisov-Libgober's proof of the Dijkgraaf-Moore-Verlinde-Verlinde formula [3], Göttsche found a formula for elliptic genera, with values in a line bundle, of Hilbert schemes of points on surfaces [18]. We briefly discuss this result. Let S be any smooth projective surface (not necessarily K3), and $L\in\mathrm{Pic}(S)$. The determinant line bundle on $S^{[n]}$ is $\mu(L):=\det((L-\mathcal{O}_S)^{[n]})$. Its first Chern class is described as follows. Consider projections from the universal subscheme $\mathcal{Z}\subset S^{[n]}\times S$

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S^{[n]} & & S. \end{tikzcd}$$

Then

$$c_1(\mu(L)) = \mu(c_1(L)) := p_* q^* c_1(L) \in H^2(S^{[n]},\mathbb{Z}). \tag{8}$$

Specialized to χ_y-genera, the results of [18] imply:

Theorem 3.6 (Göttsche)**.** *Let S be a smooth projective surface and $L\in\mathrm{Pic}(S)$. Then*

$$\sum_{n=0}^{\infty} \chi(S^{[n]}, \Lambda_{-y}\Omega_{S^{[n]}}\otimes\mu(L))\,(qy^{-1})^n =$$

[11] The case $n_0=n_1=n$ appears in [13].

$$\left(\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{10}(1-q^ny)(1-q^ny^{-1})}\right)^{\chi(O_S)}\left(\prod_{n=1}^{\infty}(1-q^n)\right)^{K_S^2}$$
$$\left(\prod_{n=1}^{\infty}\left(\frac{(1-q^n)^2}{(1-q^ny)(1-q^ny^{-1})}\right)^{n^2}\right)^{\frac{L^2}{2}}\prod_{n=1}^{\infty}\left(\left(\frac{1-q^ny^{-1}}{1-q^ny}\right)^n\right)^{\frac{LK_S}{2}}.$$

Just like Laarakker requires Göttsche-Soergel's result to determine the monopole contribution to $\chi(N,\widehat{O}_N^{\mathrm{vir}})$ for a K3 surface, we will require Theorem 3.6 to determine the monopole contribution to $\chi(N,\widehat{O}_N^{\mathrm{vir}}\otimes\mu(L))$ for a K3 surface.

Adapting an argument from [23] and combining with Theorem 3.6, Conjecture 1.2 (and hence Conjecture 1.1) are proved for K3 surfaces in [18]. We now use (6), (7), and Theorem 3.6 to prove Theorem 1.5.

Proof Theorem 1.5. Let S be a K3 surface. The case $L=O_S$ was done in [34] and gives C_1. Let $L\in\mathrm{Pic}(S)$ be arbitrary. It is useful to work with $V_{n,n}^{\circ}:=V_{n,n}-V_{0,0}$, where $V_{0,0}$ is the normalization factor (5) pulled back along $S^{[n]}\times S^{[n]}\to\mathrm{pt}$. For S a K3 surface and $\beta=0$, (6) and (7) imply

$$\mathsf{Z}_S^{\mathrm{mon}}(L,0,y,q)=\sum_n q^{2n}\int_{S^{[n]}}e^{\mu(2c_1(L))}\,\Delta^*\frac{\mathrm{ch}(\sqrt{\det(V_{n,n}^{\circ})^{\vee}})}{\mathrm{ch}(\Lambda_{-1}(V_{n,n}^{\circ})^{\mathrm{mov}\vee})}\,\mathrm{td}((V_{n,n}^{\circ})^{\mathbb{C}^*}),$$

where we used

$$\begin{aligned}\Delta^*\pi_*\left(\pi_S^*c_1(L)\cdot(-\mathrm{ch}_2(\mathcal{I}_0)-\mathrm{ch}_2(\mathcal{I}_1))\cap[S^{[n]}\times S^{[n]}\times S]\right)&=\pi_*(\pi_S^*c_1(L)\cap(2[\mathcal{Z}]))\\&=\mu(2c_1(L)),\end{aligned}$$

where $\mathcal{Z}\subset S^{[n]}\times S$ is the universal subscheme and $\mu(c_1(L))$ is defined by (8).

We require two identities from [42]. By [42, Proposition 2.6], the canonical square root is given by

$$\sqrt{\det(V_{n,n}^{\circ})^{\vee}}=\left(\det(V_{n,n}^{\circ})^{\vee}\right)^{\geq 0}\cdot\mathfrak{t}^{\frac{1}{2}r_{\geq 0}},$$

where $(\cdot)^{\geq 0}$ denotes the part with non-negative $\mathbb{C}^*$-weight, and $r_{\geq 0}$ is its rank. Moreover, for any complex E, we have [42, (2.28)]

$$\Lambda_{-1}E^{\vee}\cong(-1)^{\mathrm{rk}\,E}\Lambda_{-1}E\otimes\det E^{\vee}. \tag{9}$$

Pulling back along $\Delta:S^{[n]}\hookrightarrow S^{[n]}\times S^{[n]}$ and using (7) yields

$$\Delta^*\sqrt{(\det V_{n,n}^{\circ})^{\vee}}=\det\left(\Omega_{S^{[n]}}+\Omega_{S^{[n]}}\otimes\mathfrak{t}\right)\cdot\mathfrak{t}^{2n}=\det(\Omega_{S^{[n]}})\cdot\det(\Omega_{S^{[n]}}\otimes\mathfrak{t})\cdot\mathfrak{t}^{2n}.$$

Furthermore,

$$\Delta^*\frac{1}{\Lambda_{-1}(V_{n,n}^{\circ})^{\mathrm{mov}\vee}}=\frac{\Lambda_{-1}(\Omega_{S^{[n]}}\otimes\mathfrak{t}^{-1})}{\Lambda_{-1}(\Omega_{S^{[n]}}\otimes\mathfrak{t})}\cdot\Lambda_{-1}(\Omega_{S^{[n]}}\otimes\mathfrak{t}^{-2}).$$

Hence,

$$\Delta^* \frac{\sqrt{\det(V^\circ_{n,n})^\vee}}{\Lambda_{-1}(V^\circ_{n,n})^{\mathrm{mov}\vee}} = \det(\Omega_{S^{[n]}}) \cdot \frac{\det(\Omega_{S^{[n]}} \otimes \mathfrak{t})}{\Lambda_{-1}(\Omega_{S^{[n]}} \otimes \mathfrak{t})} \cdot \mathfrak{t}^{2n} \cdot \Lambda_{-1}(\Omega_{S^{[n]}} \otimes \mathfrak{t}^{-1}) \cdot \Lambda_{-1}(\Omega_{S^{[n]}} \otimes \mathfrak{t}^{-2})$$
$$= \det(\Omega_{S^{[n]}}) \cdot \mathfrak{t}^{2n} \cdot \frac{\Lambda_{-1}(\Omega_{S^{[n]}} \otimes \mathfrak{t}^{-1})}{\Lambda_{-1}(T_{S^{[n]}} \otimes \mathfrak{t}^{-1})} \cdot \Lambda_{-1}(\Omega_{S^{[n]}} \otimes \mathfrak{t}^{-2})$$
$$= \mathfrak{t}^{2n} \cdot \Lambda_{-1}(\Omega_{S^{[n]}} \otimes \mathfrak{t}^{-2}),$$

where the second equality uses (9), the third equality uses $T_{S^{[n]}} \cong \Omega_{S^{[n]}}$ (because $S^{[n]}$ is holomorphic symplectic), and the last equation uses $K_{S^{[n]}} \cong \mathcal{O}$. Using $y := e^t$ and Serre duality (see also [12, Remark 4.13]), we find

$$\mathsf{Z}^{\mathrm{mon}}_S(L, 0, y, q) = \sum_{n=0}^{\infty} y^{2n} \chi(S^{[n]}, \Lambda_{-1}(\Omega_{S^{[n]}} \otimes \mathfrak{t}^{-2}) \otimes \mu(L \otimes L))\, q^{2n}$$
$$= \sum_{n=0}^{\infty} y^{2n} \chi(S^{[n]}, \Lambda_{-y^{-2}}\Omega_{S^{[n]}} \otimes \mu(L \otimes L))\, q^{2n}$$
$$= \sum_{n=0}^{\infty} y^{-2n} \chi(S^{[n]}, \Lambda_{-y^{2}}\Omega_{S^{[n]}} \otimes \mu(L^* \otimes L^*))\, q^{2n}.$$

The result follows from Theorem 3.6 and Lemmas 3.4, 3.5. □

3.6. Higher Rank

The methods of Section 3.1–3.5 generalize to any rank r. Let S be any smooth projective surface with $b_1(S) = 0$ and $p_g(S) > 0$. Let $N := N^H_S(r, c_1, c_2)$. Suppose there are no rank r strictly Gieseker H-semistable Higgs sheaves on S with Chern classes c_1, c_2. Consider the components of N containing Higgs sheaves (E, ϕ) such that

$$E = E_0 \oplus E_1 \otimes \mathfrak{t}^{-1} \oplus \cdots \oplus E_{r-1} \otimes \mathfrak{t}^{-(r-1)}$$

and $\operatorname{rk} E_0 = \cdots = \operatorname{rk} E_{r-1} = 1$. We denote the union of such components by M_{1^r}. These components are described by Gholampour-Thomas in terms of nested Hilbert schemes [15, 16] (see also [34])

$$S^{[n_0,\ldots,n_r]}_{\beta_1,\ldots,\beta_{r-1}} \subset S^{[n_0]} \times \cdots \times S^{[n_{r-1}]} \times |\beta_1| \times \cdots \times |\beta_{r-1}|.$$

Let $L \in \operatorname{Pic}(S)$, and replace $c_2(\mathbb{E}) - \frac{1}{4}c_1(\mathbb{E})^2$ with $c_2(\mathbb{E}) - \frac{r-1}{2r}c_1(\mathbb{E})^2$ in definitions (2), (3). Let

$$\mathrm{vd} := 2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_S).$$

Then the contribution of M_{1^r} to $\chi(N, \widehat{\mathcal{O}}^{\mathrm{vir}}_N \otimes \mu(L))$ is given by the coefficient of $(-x)^{\mathrm{vd}}$ of

$$\widetilde{C}^{(r)}_1(y, x^{2r})^{\chi(\mathcal{O}_S)}\, \widetilde{C}^{(r)}_2(y, x^{2r})^{K_S^2}\, \widetilde{C}^{(r)}_3(y, x^{2r})^{L^2}\, \widetilde{C}^{(r)}_4(y, x^{2r})^{LK_S}$$
$$\cdot \sum_{(a_1,\ldots,a_{r-1}) \in H^2(S,\mathbb{Z})^{r-1}} \delta_{c_1, K_S - a_1, \ldots, K_S - a_{r-1}} \prod_{i=1}^{r-1} \mathrm{SW}(a_i)\, \widetilde{C}^{(r)}_{5i}(y, x^{2r})^{a_i K_S}\, \widetilde{C}^{(r)}_{6i}(y, x^{2r})^{a_i L}$$
$$\cdot \prod_{i<j} \widetilde{C}^{(r)}_{7ij}(y, x^{2r})^{a_i a_j},$$

where $\widetilde{C}_i^{(r)}$, $\widetilde{C}_{ij}^{(r)}$, $\widetilde{C}_{ijk}^{(r)}$ are universal series in $\mathbb{Q}(y^{\frac{1}{2}})((x))$ and

$$\delta_{a,b_1,\ldots,b_{r-1}} := \#\left\{\gamma \in H^2(S,\mathbb{Z}) \,:\, a - \sum_{i=1}^{r-1} i b_i = r\gamma\right\}.$$

We did *not* normalize the universal series to start with 1. Since [34] works in any rank, Sections 3.1–3.3 readily generalize to the above statement.

Equations (6) and (7) have analogs in any rank [34, Section 10]. Define

$$\widetilde{C}_1^{(r)}(y,q) = x^{-(r^2-1)}(y^{-\frac{r-1}{2}} + y^{-\frac{r-2}{2}} + \cdots + y^{\frac{r-1}{2}})^{-1} C_1^{(r)}(y,q),$$
$$\widetilde{C}_3^{(r)}(y,q) = C_3^{(r)}(y,q).$$

Then $C_3^{(r)}, C_5^{(r)} \in 1 + q\,\mathbb{Q}(y^{\frac{1}{2}})[[q]]$. Generalizing Section 3.5 accordingly yields

$$C_1^{(r)}(y,q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^{rn})^{10}(1-q^{rn}y^r)(1-q^{rn}y^{-r})},$$
$$C_3^{(r)}(y,q) = \prod_{n=1}^{\infty} \left(\frac{(1-q^{rn})^2}{(1-q^{rn}y^r)(1-q^{rn}y^{-r})}\right)^{\frac{r^2n^2}{2}},$$

where $C_1^{(r)}$ was previously derived in [34, 42] and $C_3^{(r)}$ is new.

Let $M := M_S^H(r,c_1,c_2)$, and assume there are no rank r strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. The instanton contribution to $(-1)^{\mathrm{vd}}\chi(N, \widehat{\mathcal{O}}_N^{\mathrm{vir}} \otimes \mu(L))$, which equals $y^{-\frac{\mathrm{vd}}{2}}\chi_{-y}^{\mathrm{vir}}(M,\mu(L))$, is determined in [18] for S, a K3 surface. It is derived by combining Theorem 3.6 with an adaptation of an argument of [23]. The result is the coefficient of $q^{\mathrm{vd}/2}$ of

$$\left(\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{20}(1-q^n y)^2(1-q^n y^{-1})^2}\right)\left(\prod_{n=1}^{\infty}\left(\frac{(1-q^n)^2}{(1-q^n y)(1-q^n y^{-1})}\right)^{n^2}\right)^{\frac{L^2}{2}}.$$

Unlike the monopole contribution, these universal series are independent of r.

4. Applications

In this section, we discuss special cases of Conjectures 1.1 and 1.2: (1) minimal surfaces of general type, (2) surfaces with disconnected canonical divisor, (3) a blow-up formula, and (4) Vafa-Witten invariants with μ-classes. We denote the formula of Conjecture 1.1, after some slight rewriting, by

$$\psi_{S,L,c_1}(x) := \frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}} \sum_{a \in H^2(S,\mathbb{Z})} \mathrm{SW}(a)\,(-1)^{ac_1}(1+x)^{(K_S-a)(L-K_S)}(1-x)^{a(L-K_S)}. \tag{1}$$

4.1. Minimal Surfaces of General Type

Proposition 4.1. *Let S be a smooth projective surface satisfying $p_g(S) > 0$, $b_1(S) = 0$, and $K_S \neq 0$ and such that its only Seiberg-Witten basic classes are 0 and K_S. Let $L \in \mathrm{Pic}(S)$, and let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2.*

Suppose Conjecture 1.1 holds in this setting. Then $\chi^{\mathrm{vir}}(M_S^H(2,c_1,c_2),\mu(L))$ is given by the coefficient of x^{vd} of

$$2^{3-\chi(\mathcal{O}_S)+K_S^2}\frac{(1+x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}}.$$

Proof. Since $\mathrm{SW}(0)=1$, we have $\mathrm{SW}(K_S)=(-1)^{\chi(\mathcal{O}_S)}$ [35, Proposition 6.3.4]. By Conjecture 1.1, $\chi^{\mathrm{vir}}(M_S^H(2,c_1,c_2),\mu(L))$ is given by the coefficient of x^{vd} of (1), which simplifies to

$$\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}}\left[(1+x)^{K_S(L-K_S)}+(-1)^{c_1K_S+\chi(\mathcal{O}_S)}(1-x)^{K_S(L-K_S)}\right].$$

Varying over c_2, we put the coefficients of all terms x^{vd} of $\psi_{S,L,c_1}(x)$ into a generating series as follows. Suppose $\psi_{S,L,c_1}(x)=\sum_{n=0}^{\infty}\psi_n x^n$ and $i=\sqrt{-1}$. Then for vd given by (1), we have

$$\begin{aligned}
\sum_{c_2}\mathrm{Coeff}_{x^{\mathrm{vd}}}(\psi_{S,L,c_1}(x))\,x^{\mathrm{vd}} &= \sum_{n\equiv -c_1^2-3\chi(\mathcal{O}_S)\mod 4}\psi_n\,x^n\\
&=\sum_{k=0}^{3}\frac{1}{4}i^{k(c_1^2+3\chi(\mathcal{O}_S))}\psi(i^kx)\\
&=2^{1-\chi(\mathcal{O}_S)+K_S^2}\left[\frac{(1+x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}}+(-1)^{c_1^2+3\chi(\mathcal{O}_S)}\frac{(1-x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}}\right.\\
&\qquad\left.+i^{c_1^2+3\chi(\mathcal{O}_S)}\frac{(1+ix)^{K_S(L-K_S)}}{(1+x^2)^{\chi(L)}}+(-i)^{c_1^2+3\chi(\mathcal{O}_S)}\frac{(1-ix)^{K_S(L-K_S)}}{(1+x^2)^{\chi(L)}}\right],
\end{aligned}$$

where the third equality uses $c_1K_S\equiv c_1^2 \mod 2$. Now define

$$\phi_{S,L,c_1}(x):=2^{3-\chi(\mathcal{O}_S)+K_S^2}\frac{(1+x)^{K_S(L-K_S)}}{(1-x^2)^{\chi(L)}}.$$

Then

$$\begin{aligned}
\sum_{c_2}\mathrm{Coeff}_{x^{\mathrm{vd}}}(\phi_{S,L,c_1}(x))\,x^{\mathrm{vd}} &= \sum_{n\equiv -c_1^2-3\chi(\mathcal{O}_S)\mod 4}\phi_n\,x^n\\
&=\sum_{k=0}^{3}\frac{1}{4}i^{k(c_1^2+3\chi(\mathcal{O}_S))}\phi(i^kx)
\end{aligned}$$

is given by the same expression as above, which proves the proposition. □

Remark 4.2. *Examples of surfaces satisfying the conditions of Proposition 4.1 are (1) minimal surfaces of general type satisfying $p_g(S)>0$ and $b_1(S)=0$ [36, Theorem 7.4.1], and (2) smooth projective surfaces with $b_1(S)=0$ and containing an irreducible reduced curve $C\in|K_S|$ (for example, discussed in [19, Section 6.3]).*

Remark 4.3. *In general, the formula of Proposition 4.1 has integer coefficients only when $\chi(\mathcal{O}_S)-3\le K_S^2$. For minimal surfaces of general type, this inequality is implied by Noether's inequality, $\chi(\mathcal{O}_S)-3\le\frac{1}{2}K_S^2$.*

Corollary 4.4. *Let S be a smooth projective surface with $b_1(S)=0$ and containing a smooth connected curve $C\in|K_S|$ of genus g. Let $L\in\mathrm{Pic}(S)$, and let H,c_1,c_2 be chosen such that there are no rank 2* *strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Suppose Conjecture 1.1 holds in this setting. Then $\chi^{\mathrm{vir}}(M_S^H(2, c_1, c_2), \mu(L))$ is given by the coefficient of x^{vd} of*

$$2^{3-\chi(\mathcal{O}_C)-\chi(\mathcal{O}_S)} \frac{(1+x)^{\chi(L|_C)}}{(1-x^2)^{\chi(L)}}.$$

Proof. We have $g = K_S^2 + 1$ and $\chi(L|_C) = 1 - g + \deg L|_C$ by Riemann-Roch. □

4.2. Disconnected Canonical Divisor

Proposition 4.5. *Let S be a smooth projective surface with $b_1(S) = 0$, and suppose there exists $0 \neq C_1 + \cdots + C_m \in |K_S|$, where $C_1, \ldots, C_m$ are mutually disjoint irreducible reduced curves. Let $L \in \mathrm{Pic}(S)$, and let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Suppose Conjecture 1.1 holds in this setting. Then $\chi^{\mathrm{vir}}(M_S^H(2, c_1, c_2), \mu(L))$ is given by the coefficient of x^{vd} of*

$$\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}} \prod_{j=1}^{m} \left[(1+x)^{\chi(L|_{C_i})} + (-1)^{C_i c_1 + h^0(N_{C_i/S})}(1-x)^{\chi(L|_{C_i})}\right],$$

where $N_{C_i/S}$ denotes the normal bundle of $C_i \subset S$.

Proof. We describe the Seiberg-Witten basic classes and invariants for S in this setting [19, Lemma 6.14]. For any $I \subset M := \{1, \ldots, m\}$, define $C_I := \sum_{i \in I} C_i$; and we write $I \sim J$ when C_I and C_J are linearly equivalent. Also, $C_\varnothing := 0$. The Seiberg-Witten basic classes of S are precisely $\{C_I\}_{I \subset M}$, and

$$\mathrm{SW}(C_I) = \#[I] \prod_{i \in I} (-1)^{h^0(N_{C_i/S})},$$

where $\#[I]$ denotes the number of elements of equivalence class $[I]$. Therefore, (1) becomes

$$\begin{aligned}
&\frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}} \left(\sum_{[I]} \#[I] \prod_{i \in I} (-1)^{h^0(N_{C_i/S})} \right) (-1)^{C_I c_1} (1+x)^{C_{M \setminus I}(L-K_S)} (1-x)^{C_I(L-K_S)} \\
&= \frac{2^{2-\chi(\mathcal{O}_S)+K_S^2}}{(1-x^2)^{\chi(L)}} \sum_{I \subset M} \left(\prod_{i \in I} (-1)^{C_i c_1 + h^0(N_{C_i/S})} (1-x)^{C_i(L-C_i)} \right) \left(\prod_{i \in M \setminus I} (1+x)^{C_i(L-C_i)} \right),
\end{aligned}$$

where we used $K_S = C_M$ and the assumption that the curves C_i are mutually disjoint. The result follows from $\chi(L|_{C_i}) = 1 - g(C_i) + \deg L|_{C_i} = C_i(L - C_i)$ and expanding the product in the statement of the proposition. □

4.3. Blow-Up Formula

Proposition 4.6. *Let S be a smooth projective surface, $\pi : \widetilde{S} \to S$ the blow-up of S in a point, and E the exceptional divisor. Let $L, c_1 \in \mathrm{Pic}(S)$, $\widetilde{c}_1 = \pi^* c_1 - kE$, and $\widetilde{L} = \pi^* L - \ell E$. Then*

$$\psi_{\widetilde{S}, \widetilde{L}, \widetilde{c}_1}(x) = \frac{1}{2}(1-x^2)^{\binom{\ell+1}{2}} \left[(1+x)^{\ell+1} + (-1)^k (1-x)^{\ell+1}\right] \psi_{S, L, c_1}(x).$$

Proof. The Seiberg-Witten basic classes of $\widetilde{S}$ are $\pi^* a$ and $\pi^* a + E$ with corresponding Seiberg-Witten invariant $\mathrm{SW}(a)$, where a runs over all Seiberg-Witten basic classes of S [36, Theorem 7.4.6]. Using $\chi(\mathcal{O}_{\widetilde{S}}) = \chi(\mathcal{O}_S)$, $K_{\widetilde{S}} = \pi^* K_S + E$, $E^2 = -1$, and $\chi(\widetilde{L}) = \chi(L) - \binom{\ell+1}{2}$, the proposition follows at once from (1). □

4.4. Vafa-Witten Formula with μ-Classes

Let S be a smooth projective surface satisfying $b_1(S) = 0$ and $p_g(S) > 0$. In an appendix of [19], Göttsche and Nakajima gave a conjectural formula for

$$\sum_{k=0}^{\mathrm{vd}} \int_{[M]^{\mathrm{vir}}} e^{\mu(c_1(L))} \lambda^{\mathrm{vd}-k} c_k(T_M^{\mathrm{vir}}), \tag{2}$$

where $M := M_S^H(2, c_1, c_2)$, vd is given by (1), and we assume 'stable=semistable'. Here, λ is a formal parameter. Setting $\lambda = 0$ in (2) gives $e^{\mathrm{vir}}(M)$. Replacing λ with λ^{-1}, then multiplying by λ^{vd}, and finally setting $\lambda = 0$ gives Donaldson invariants $\int_{[M]^{\mathrm{vir}}} e^{\mu(c_1(L))}$. Therefore, (2) interpolates between Donaldson invariants and virtual Euler characteristics. Let $G_2(q)$ be the Eisenstein series of weight 2, and define

$$\overline{G}_2(q) = G_2(q) + \frac{1}{24} = \sum_{d=1}^{\infty} \sigma_1(d)\, q^d,$$

where $\sigma_1(d) = \sum_{d|n} d$. Furthermore, let $\theta_3(q) := \theta_3(q, 1)$ and $D := q\frac{d}{dq}$.

Conjecture 4.7 (Göttsche-Nakajima)**.** *Let S be a smooth projective surface with $p_g(S) > 0$, $b_1(S) = 0$, and let $L \in \mathrm{Pic}(S)$. Let H, c_1, c_2 be chosen such that there are no rank 2 strictly Gieseker H-semistable sheaves on S with Chern classes c_1, c_2. Let $M := M_S^H(2, c_1, c_2)$. Then*

$$\sum_{k=0}^{\mathrm{vd}} \int_{[M]^{\mathrm{vir}}} e^{\mu(c_1(L))} \lambda^{\mathrm{vd}-k} c_k(T_M^{\mathrm{vir}})$$

is given by the coefficient of x^{vd} of

$$4\left(\frac{1}{2\overline{\eta}(x^2)^{12}}\right)^{\chi(\mathcal{O}_S)} \left(\frac{2\overline{\eta}(x^4)^2}{\theta_3(x)}\right)^{K_S^2} \left(e^{DG_2(x^2)}\right)^{\frac{(\lambda L)^2}{2}} \left(e^{-2\overline{G}_2(x^2)}\right)^{\lambda L K_S}$$
$$\cdot \sum_{a \in H^2(S,\mathbb{Z})} (-1)^{c_1 a}\, \mathrm{SW}(a) \left(\frac{\theta_3(x, y^{\frac{1}{2}})}{\theta_3(-x, y^{\frac{1}{2}})}\right)^{aK_S} \left(e^{G_2(x) - G_2(-x)}\right)^{\frac{\lambda L(K_S - 2a)}{2}}.$$

Recall that specializing Conjecture 1.2 to $y = 0$ implies Conjecture 1.1 (after replacing x by $xy^{\frac{1}{2}}$; see Section 1). We show that specializing Conjecture 1.2 to $y = 1$ implies Conjecture 4.7 (after replacing x with $xy^{\frac{1}{2}}$ and L with $\lambda L(y^{-\frac{1}{2}} - y^{\frac{1}{2}})^{-1}$). In summary, the invariants of this paper interpolate between:

- Donaldson invariants
- Virtual Euler numbers of moduli spaces of sheaves
- K-theoretic Donaldson invariants
- K-theoretic Vafa-Witten invariants

Proposition 4.8. *Conjecture 1.2 implies Conjecture 4.7.*

Proof of Proposition 4.8. Recall the definition of $y^{-\frac{r}{2}}\mathsf{X}_{-y}(E)$ for any complex E of rank r on M from Section 2.1. Suppose $r \geq 0$, and denote by $\{\cdot\}_r$ the degree r part in $A^*(M)_{\mathbb{Q}}$. Then [12, Theorem 4.5]

$$\left\{y^{-\frac{r}{2}}\mathsf{X}_{-y}(E)\right\}_r = c_r(E) \mod (1-y).$$

For $D \in A^1(M)_{\mathbb{Q}}$, we are interested in

$$\left\{ \sum_{k=0}^{r} e^D \lambda^{r-k} c_k(E) \right\}_r ,$$

which is insertion (2) for $E = T_M^{\mathrm{vir}}$ and $D = \mu(c_1(L))$. We consider

$$e^{\lambda D (y^{-\frac{1}{2}} - y^{\frac{1}{2}})^{-1}} y^{-\frac{r}{2}} \mathsf{X}_{-y}(E).$$

Again using [12, Theorem 4.5], we find

$$\left\{ e^{\lambda D (y^{-\frac{1}{2}} - y^{\frac{1}{2}})^{-1}} y^{-\frac{r}{2}} \mathsf{X}_{-y}(E) \right\}_r = \left\{ \sum_k e^D \lambda^{r-k} c_k(E) y^{-\frac{k}{2}} \right\}_r \mod (1-y).$$

Hence,

$$\left\{ e^{\lambda D (y^{-\frac{1}{2}} - y^{\frac{1}{2}})^{-1}} y^{-\frac{r}{2}} \mathsf{X}_{-y}(E) \Big|_{y=1} \right\}_r = \left\{ \sum_{k=0}^{r} e^D \lambda^{r-k} c_k(E) \right\}_r . \tag{3}$$

Take $E = T_M^{\mathrm{vir}}$ and $D = \mu(c_1(L))$. Replacing L with

$$\frac{\lambda L}{y^{-\frac{1}{2}} - y^{\frac{1}{2}}}$$

in Conjecture 1.2 and setting $y = 1$ gives the invariants (2) by equation (3).

This reduces the proof to the following identities:

$$DG_2(x^2) = \lim_{y \to 1} \sum_{n=1}^{\infty} \frac{n^2}{(y^{-\frac{1}{2}} - y^{\frac{1}{2}})^2} \log \frac{(1-x^{2n})^2}{(1-x^{2n}y)(1-x^{2n}y^{-1})},$$

$$\overline{G}_2(x^2) = -\frac{1}{2} \lim_{y \to 1} \sum_{n=1}^{\infty} \frac{n}{(y^{-\frac{1}{2}} - y^{\frac{1}{2}})} \log \frac{(1-x^{2n}y^{-1})}{(1-x^{2n}y)},$$

$$G_2(x) - G_2(-x) = \lim_{y \to 1} \sum_{\substack{n>0 \\ \text{odd}}} \frac{n}{y^{-\frac{1}{2}} - y^{\frac{1}{2}}} \log \frac{(1-x^n y^{\frac{1}{2}})(1+x^n y^{-\frac{1}{2}})}{(1-x^n y^{-\frac{1}{2}})(1+x^n y^{\frac{1}{2}})}.$$

These identities follow from an elementary computation using repeatedly that

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n},$$

$$\lim_{y \to 1} \frac{y^{-\frac{n}{2}} - y^{\frac{n}{2}}}{y^{-\frac{1}{2}} - y^{\frac{1}{2}}} = n.$$

Therefore,

$$\begin{aligned} \lim_{y \to 1} \sum_{n=1}^{\infty} \frac{n^2}{(y^{-\frac{1}{2}} - y^{\frac{1}{2}})^2} \log \frac{(1-x^{2n})^2}{(1-x^{2n}y)(1-x^{2n}y^{-1})} &= \lim_{y \to 1} \sum_{n,l>0} \frac{n^2 x^{2nl}}{l} \left(\frac{y^{-\frac{l}{2}} - y^{\frac{l}{2}}}{y^{-\frac{1}{2}} - y^{\frac{1}{2}}} \right)^2 \\ &= \sum_{n,l>0} n^2 l \, x^{2nl} = DG_2(x^2). \end{aligned}$$

The other identities follow similarly. ◻

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