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Campana points of bounded height on vector group compactifications

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Abstract

We initiate a systematic quantitative study of subsets of rational points that are integral with respect to a weighted boundary divisor on Fano orbifolds. We call the points in these sets Campana points. Earlier work of Campana and subsequently Abramovich shows that there are several reasonable competing definitions for Campana points. We use a version that delineates well different types of behavior of points as the weights on the boundary divisor vary. This prompts a Manin-type conjecture on Fano orbifolds for sets of Campana points that satisfy a klt (Kawamata log terminal) condition. By importing work of Chambert-Loir and Tschinkel to our setup, we prove a log version of Manin's conjecture for klt Campana points on equivariant compactifications of vector groups.

Contents

1. *Introduction*

Manin's conjecture for rational points, extensively studied now for more than three decades, predicts an asymptotic formula for the counting function of rational points of bounded height on rationally connected algebraic varieties over number fields. The class of equivariant compactifications of homogeneous spaces has proved to be a particularly fertile testing ground for the conjecture [**6, 8, 23, 36, 38–40, 61, 62, 66**]. The related problem of counting integral points on homogeneous spaces has received much attention as well, both classically (see, for example, [**33, 35**]), and recently, as attested by [**10, 25–27, 64, 65**]. By choosing a suitable

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compactification, one can identify the set of integral points on the original variety with the set of rational points on the compactification that are integral with respect to the boundary divisor. Hence, this latter body of work represents progress toward a "logarithmic version" of Manin's conjecture for integral points. Regrettably, subtleties of a mostly geometric nature have so far prevented a general formulation of a Manin-type conjecture for integral points.

In this paper, we focus on an intermediate notion: sets of rational points that are *integral with respect to a weighted boundary divisor* [**21**], which we call *Campana points*. Such sets depend on the choice of weights and "interpolate" between the set of integral points and the set of rational points, which can both be recovered as sets of Campana points for suitable choices of weights. If the weighted boundary divisor is *Kawamata log terminal* (klt for short), we say that the Campana points are klt. The set of rational points is a set of klt Campana points, while the set of integral points is not. However, the set of integral points can be written as an infinite intersection of sets of klt Campana points.

To date, Manin-type problems for sets of Campana points have not been well studied. The only results we are aware of are to be found in [**16, 17, 71**] and we believe that this research direction is relatively new.

The purpose of this paper is to propose a Manin-type conjecture for the distribution of klt Campana points on Fano orbifolds. We show that the conjecture holds for all smooth vector group compactifications with a strict normal crossings boundary divisor for the weighted log-anticanonical height and for many more choices of heights. We investigate also the case of non-klt Campana points, and we observe that all the difficulties that one encounters when dealing with integral points appear also in this setting.

1.1. *Campana points*

There are several ways to "interpolate" between the classical notions of rational and integral points. Keeping Manin's conjecture in mind, this article argues in favor of a compelling option that arises from Campana's theory of pairs, which he baptized orbifoldes géométriques.† There are various competing notions of Campana points in the literature [**1, 3**], and they all agree with the original definition of Campana [**19, 21**] on curves. On higher dimensional varieties, the various notions can lead to significantly different sets of points, manifestly affecting the counting problems addressed in this paper, as we explain in § 3.2.1. We choose to work with Campana's original definition [**21**] because it best allows us to formulate a Manin-type conjecture which shares many characteristics with the now classical conjectures for rational points [**5, 54**]. Our study of local height integrals and Euler products for vector group compactifications shows that the notion considered in this paper interacts well with the tools from harmonic analysis: the regularization of the Euler product of local height integrals looks similar to the one used for the study of Manin's conjecture for rational points (see Proposition 7.4 and Corollary 7.5).

The notion of Campana points appearing in [**3**] is different from the one considered here. That notion enjoys good functoriality properties, but it seems ill-suited to the study of points of bounded height: for example, if one were to use the height zeta function method to count points of bounded height on vector group compactifications, then the regularization of the Euler product of local height integrals for the main term would require a newfound set of ideas. We consider this clarification an important contribution of this paper.

1.2. *A log Manin conjecture*

Let (X, D_ϵ) be a Campana orbifold (see § 3.1) over a number field F. Assume moreover that X is projective and that $-(K_X + D_\epsilon)$ is ample; a pair (X, D_ϵ) with this additional property

†Unlike the name suggests, such objects are not stacks, but simply pairs consisting of a variety equipped with a $\mathbb{Q}$-divisor of a specific type.

is called a Fano orbifold. Recall that the effective cone $\mathrm{Eff}^1(X)$ is finitely generated by [**14**]. Fix a finite set S of places of F containing all archimedean places, as well as a good integral model $(\mathcal{X}, \mathcal{D}_\epsilon)$ of (X, D_ϵ) over the ring of S-integers $\mathcal{O}_{F,S}$ of F (see § 3.1). Write $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ for the set of $\mathcal{O}_{F,S}$-Campana points of $(\mathcal{X}, \mathcal{D}_\epsilon)$ (see Definition 3.4), and assume that $\lfloor D_\epsilon \rfloor = 0$, that is, every weight ϵ_α is strictly smaller than 1. This condition is equivalent to saying that (X, D_ϵ) is klt in the sense of birational geometry (see [**49**, Definition 2.34] for a definition of klt singularities, and [**49**, Lemma 2.30] for a characterization). Let

$$\mathsf{H}_{\mathcal{L}} : X(F) \to \mathbb{R}_{>0}$$

be the height function determined by an adelically metrized big line bundle $\mathcal{L} = (L, \|\cdot\|)$ on X as in [**54**, § 1.3]. For any subset $U \subset X(F)$ and positive real number T, we consider the counting function

$$\mathsf{N}(U, \mathcal{L}, T) = \#\{P \in U \mid \mathsf{H}_{\mathcal{L}}(P) \leqslant T\}.$$

Conjecture 1.1 (Manin-type conjecture for Fano orbifolds). Suppose that in addition to being big, the divisor L is nef, and that the set of klt Campana points $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ is not thin. Then there exists a thin set $Z \subset (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ as in § 3.4 such that

$$\mathsf{N}((\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}) \setminus Z, \mathcal{L}, T) \sim c(F, S, (\mathcal{X}, \mathcal{D}_\epsilon), \mathcal{L}, Z) T^{a((X, D_\epsilon), L)} (\log T)^{b(F,(X,D_\epsilon),L)-1} \tag{1.1}$$

as $T \to \infty$, where

$$a((X, D_\epsilon), L) = \inf\{t \in \mathbb{R} \mid tL + K_X + D_\epsilon \in \mathrm{Eff}^1(X)\}$$

is the Fujita invariant of (X, D_ϵ) with respect to L, $b(F, (X, D_\epsilon), L)$ is the codimension of the minimal supported face of $\mathrm{Eff}^1(X)$ that contains the class $a((X, D_\epsilon), L)[L] + [K_X + D_\epsilon]$ (cf. [**42**, Definition 2.1]), and the leading constant $c(F, S, (\mathcal{X}, \mathcal{D}_\epsilon), \mathcal{L}, Z)$ is a positive Tamagawa constant, described in § 3.3.

The definition of the exponents $a((X, D_\epsilon), L)$ and $b(F, (X, D_\epsilon), L)$ in the conjecture above is analogous to the case of rational points [**5**]. This is the main reason for our choice among various possible definitions of Campana points.

Although $a((X, D_\epsilon), L)$ and $b(F, (X, D_\epsilon), L)$ do not depend on the choice of an integral model for (X, D_ϵ), the leading constant does depend on such a choice. The description of the leading constant is analogous to Peyre's constant in [**9, 54**].

The removal of a thin subset of rational points in order to get a count that is not dominated by accumulating subvarieties is a natural assumption, which is already present in the case of Manin's conjecture for rational points (see, for example, [**52, 55**]). In § 3.5 we explain why a recent example of Browning and Yamagishi [**17**] whose exceptional set cannot be a proper closed subset is still compatible with Conjecture 1.1.

While the geometric properties of klt singularities are not used in this paper, we believe that they will play a prominent role in the analysis of the exceptional sets for Conjecture 1.1. Indeed, in the classical case of rational points, one of the key ingredients in the proof of thinness of the conjectural exceptional set in [**52**] is the BAB conjecture, which holds for klt log Fano varieties (more precisely in the ϵ-klt setting), proved in [**13**] and [**12**], but fails in the dlt case. This is one of the main reasons for expecting that klt Campana points are easier to deal with compared to integral points.

In attempting to formulate a conjecture for sets of Campana points that are not klt, we encounter the same difficulties that have prevented the formulation of a conjecture in the much more extensively studied case of integral points. For example, the exponents appearing in the asymptotics of the counting functions in these results depend heavily on the *divisor* chosen for the counting function, and not only on its numerical class (see, for example, [**26**] for integral points and § 10 for Campana points). It seems sensible to study explicit examples of sets of Campana points that are "barely" non klt, for example, when exactly one of the weights ϵ_α is equal to 1, as a step toward a better understanding of the distribution of integral points on Fano varieties.

1.3. *Evidence*

We prove Conjecture 1.1 for equivariant compactifications of vector groups. This important class of varieties satisfies Manin's conjecture for rational points [**23**] and analogous asymptotics for integral points [**26**]. It has also been studied for the motivic version of Manin's conjecture in [**11, 22**]. Hence, it provides an ideal testing ground for Conjecture 1.1.

Let F be a number field and let $G = \mathbb{G}_a^n$ be the n-dimensional vector group. Let X be a smooth, projective, equivariant compactification of G defined over F, such that the boundary divisor $D = X \setminus G$ is a strict normal crossings divisor on X, with irreducible components $(D_\alpha)_{\alpha\in\mathcal{A}}$. Let S be a finite set of places of F, containing all archimedean places, such that there is a good integral model $(\mathcal{X}, \mathcal{D})$ for (X, D) over the ring of S-integers $\mathcal{O}_{F,S}$ of F in the sense of § 3.2. We choose a weight vector $\epsilon = (\epsilon_\alpha)_{\alpha\in\mathcal{A}}$, where

$$\epsilon_\alpha \in \left\{ 1 - \frac{1}{m} \,\middle|\, m \in \mathbb{Z}_{\geqslant 1} \right\} \cup \{1\}$$

for all α, and we set

$$D_\epsilon = \sum_{\alpha\in\mathcal{A}} \epsilon_\alpha D_\alpha, \quad \mathcal{D}_\epsilon = \sum_{\alpha\in\mathcal{A}} \epsilon_\alpha \mathcal{D}_\alpha,$$

where $\mathcal{D}_\alpha$ denotes the closure of D_α in $\mathcal{X}$. Let L be a big line bundle on X, and let $\mathcal{L}$ denote L equipped with a smooth adelic metrization.

Our first main result addresses the situation where all ϵ_α are strictly smaller than 1; we refer to this case as the *klt case*. In this situation, we get a precise result for "many" L. We recall that a divisor is said to be rigid if it has Iitaka dimension zero; see [**50**, Section 2.1] for a definition of Iitaka dimension.

THEOREM 1.2. *With the notation above, assume that (X, D_ϵ) is klt. Let $a = a((X, D_\epsilon), L)$ be defined as in Conjecture 1.1. If $aL + K_X + D_\epsilon$ is rigid, then the asymptotic formula in Conjecture 1.1 holds for $(\mathcal{X}, \mathcal{D}_\epsilon, \mathcal{L})$ with exceptional set*

$$Z = (X \setminus G) \cap (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}).$$

REMARKS 1.3. (i) The asymptotic (1.1) holds for a pair (X, D_ϵ) in Theorem 1.2 even if the pair is not a Fano orbifold. See Theorem 9.4.

(ii) If $L = -(K_X + D_\epsilon)$, the rigidity condition in the statement is trivially satisfied, since in that case $a = 1$. In this case, b is the Picard rank of X.

(iii) We prove the conclusion of Theorem 1.2 also when the adjoint divisor is not rigid, under additional technical assumptions. See Theorem 9.5.

The more general case where some of the weights ϵ_α are allowed to be equal to 1 — to which we refer as the *dlt case* — is more subtle. In this case, we have to restrict our attention to the case where L is the "orbifold anticanonical line bundle," due to subtleties arising in the formulation of the main term.

Theorem 1.4. *With notation as above, let L be the line bundle $-(K_X + D_\epsilon)$, and let $\mathcal{L}$ denote L equipped with a smooth adelic metrization as above. There exists a geometric invariant $b = b(F, S, (X, D_\epsilon), L) > 0$, defined in § 10, such that*

$$\mathsf{N}((\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}) \cap G(F), \mathcal{L}, T) \sim \frac{c}{(b-1)!} T(\log T)^{b-1} \text{ as } T \to \infty,$$

for some positive constant c that depends on $F, S, (\mathcal{X}, \mathcal{D}_\epsilon)$ and $\mathcal{L}$.

It is important to observe that the logarithmic exponent b in Theorem 1.4 for dlt points depends on the choice of S; this was not the case in Theorem 1.2 for klt Campana points. In essence, when $\epsilon_\alpha = 1$ for at least one index α, the local zeta functions associated to places in S can contribute positively to b. This is a typical feature observed in the literature about integral points of bounded height. Moreover, if $\epsilon_\alpha \in \{0, 1\}$ for all α, our result recovers [**26**].

We note that the pair (X, D_ϵ) in the statement of Theorem 1.4 is not required to be a Fano orbifold. In particular, Theorem 1.4 holds for all smooth compactifications of vector groups with strict normal crossings boundary, and there are numerous such compactifications: indeed, blowing-up invariant points always produces new examples. See § 5 for more details.

1.4. *Methods*

To prove Theorems 1.2 and 1.4, we use the height zeta function method, as in the foundational papers [**23, 26**]. Let

$$G(F)_\epsilon = G(F) \cap (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$$

be the set of rational points in G which extend to Campana $\mathcal{O}_{F,S}$-points on $(\mathcal{X}, \mathcal{D}_\varepsilon)$ in the sense of § 3.2. Even though the notation may suggest otherwise, the set $G(F)_\epsilon$ *does* depend on the choice of S and the $\mathcal{O}_{F,S}$-model $(\mathcal{X}, \mathcal{D})$, which we have fixed once and for all. Then the height zeta function is given by

$$\mathsf{Z}_\epsilon(\mathbf{s}) = \sum_{\mathbf{x} \in G(F)_\epsilon} \mathsf{H}(\mathbf{x}, \mathbf{s})^{-1} = \sum_{\mathbf{x} \in G(F)} \mathsf{H}(\mathbf{x}, \mathbf{s})^{-1} \delta_\epsilon(\mathbf{x}),$$

where $\delta_\epsilon(\mathbf{x})$ is the indicator function detecting whether a given point in $G(F)$ belongs to $G(F)_\epsilon$. Our goal is to obtain a meromorphic continuation of this analytic function, and to apply a Tauberian theorem. To this end, we consider the Fourier transform over the adèles:

$$\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}) = \int_{G(\mathbb{A}_F)} \mathsf{H}(\mathbf{x}, \mathbf{s})^{-1} \delta_\epsilon(\mathbf{x}) \psi_{\mathbf{a}}(\mathbf{x}) \, \mathrm{d}\mathbf{x},$$

and we use the Poisson summation formula

$$\sum_{\mathbf{x} \in G(F)} \mathsf{H}(\mathbf{x}, \mathbf{s})^{-1} \delta_\epsilon(\mathbf{x}) = \sum_{\mathbf{a} \in G(F)} \widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s})$$

to obtain a meromorphic continuation of $\mathsf{Z}_\epsilon(\mathbf{s})$. To prove the absolute convergence of the right-hand side, we estimate $\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s})$ by combining work from [**23, 24, 26**] on height integrals with oscillating phase.

1.5. *Structure of the paper*

After setting up the notation in § 2, we start § 3.1 by recalling the notion of Campana orbifold. We discuss different notions of Campana points that appear in the literature in § 3.2 — this is crucial, since only one of these works well for our purposes. We include an example in § 3.2.1 that shows how different notions lead to different asymptotics for point counts on a single orbifold. In § 3.2, we discuss a Peyre-type description of the leading constant in Conjecture 1.1, then we introduce a notion of thin set in the context of Campana points in § 3.4; in § 3.5 we discuss the compatibility of Browning and Yamagishi's example [**17**] with Conjecture 1.1. Finally, in § 3.6 we discuss the functoriality properties of Campana points under birational transformations.

In § 4 we review a type of simplicial complex, called the *Clemens complex*, which helps to keep track, in the presence of integrality conditions, of the contribution of local height integrals to the rightmost pole of the height zeta function. We then use these complexes to give birational invariance results (Lemmas 4.1 and 4.2) for the a and b-invariants that appear in the asymptotic formula of the counting function for Campana points.

In § 5, we specialize to Campana orbifolds that are equivariant compactifications of vector groups. We recall basic facts about their geometry such as their Picard groups and effective cones of divisors, as well as results from harmonic analysis. After a discussion on local and global heights, in § 6.3 we define the height zeta function of an equivariant compactification of a vector group, and explain how to reduce the Poisson summation formula to the convergence of a sum of Fourier transforms of local height functions (local height integrals). Sections 7 and 8 contain the necessary estimates of local height integrals; before carrying on these technical estimates, we have included an interlude with a detailed explanation of the calculations in dimension 1, for the benefit of readers new to this type of analysis.

Theorems 1.2 and 1.4 are established, respectively, in § 9 and § 10.

2. *Notation*

2.1. *Number fields, completions, and zeta functions*

Let F be an arbitrary number field. Denote by $\mathcal{O}_F$ its ring of integers, by Ω_F its set of places, by $\Omega_F^{<\infty}$ the set of all finite (non-archimedean) places, and by Ω_F^{∞} the set of all infinite (archimedean) places. For any finite set $S \subset \Omega_F$ containing Ω_F^{∞}, we denote by $\mathcal{O}_{F,S}$ the ring of S-integers of F. For each $v \in \Omega_F$, we denote by F_v the completion of F with respect to v. If v is non-archimedean, we denote by $\mathcal{O}_v$ the corresponding ring of integers, with maximal ideal $\mathfrak{m}_v$ and residue field k_v of size q_v. We write $\mathbb{A}_F$ for the ring of adèles of F.

For each $v \in \Omega_F$, the additive group F_v is locally compact, and carries a self-dual Haar measure $\mathrm{d}x_v = \mu_v$ that we normalize as follows:

- $\mathrm{d}x_v$ is the ordinary Lebesgue measure on the real line if v is real,
- $\mathrm{d}x_v$ is twice the ordinary Lebesgue measure on the plane if v is complex,
- $\mathrm{d}x_v$ is the measure for which $\mathcal{O}_v$ has volume $N(\mathfrak{D})^{-1/2}$ if v is a nonarchimedean place, where $\mathfrak{D}$ denotes the absolute different of F_v, with norm $N(\mathfrak{D})$.

These Haar measures satisfy $\mu_v(\mathcal{O}_v) = 1$ for all but finitely many non-archimedean places v; they induce a self-dual measure $\mathrm{d}x = \mu$ on $\mathbb{A}_F$. We denote by $\mathrm{d}\mathbf{x}_v$ the induced Haar measure on F_v^n. We also denote the product measure on $\mathbb{A}_F^n$ by $\mathrm{d}\mathbf{x}$.

We define the absolute value $|\cdot|_v$ by requiring that

$$\mu_v(xB) = |x|_v \cdot \mu_v(B)$$

for any Borel set $B \subset F_v$. When v is real, $|\cdot|_v$ is the usual absolute value. When v is complex, $|\cdot|_v$ is the square of the usual norm on the complex numbers. For any prime number p, we have $|p|_p = 1/p$. For any finite extension $F_v/\mathbb{Q}_p$, we have

$$|x|_v = |N_{F_v/\mathbb{Q}_p}(x)|_p.$$

We define the local zeta function by

$$\zeta_{F_v}(s) = \begin{cases} s^{-1} & \text{if } F_v = \mathbb{R} \text{ or } \mathbb{C}, \\ (1 - q_v^{-s})^{-1} & \text{if } v \text{ is non-archimedean.} \end{cases}$$

For non-archimedean places, the local zeta functions fit together to give the Dedekind zeta function

$$\zeta_F(s) = \prod_{v \in \Omega_F^{<\infty}} \zeta_{F_v}(s).$$

2.2. *Varieties and divisors*

Let F be a field with fixed algebraic closure $\bar{F}$. An F-variety X is a geometrically integral separated F-scheme of finite type. We denote by $\bar{X}$ the base change of X to $\bar{F}$. If F is a number field and $v \in \Omega_F$, we write X_v for the base change of X to F_v. Given a Weil $\mathbb{R}$-divisor $D = \sum_i a_i D_i$ on X, we denote by $\lfloor D \rfloor = \sum_i \lfloor a_i \rfloor D_i$ its "integral part." We denote the reduced divisor $\sum_{a_i \neq 0} D_i$ by D_{red}. Given a scheme $\mathcal{X}$ defined over a ring A, we denote by $\mathcal{X} \otimes_A B$ the base change of $\mathcal{X}$ under a ring extension $A \to B$.

2.3. *Conventions for complex numbers*

We denote the real part of a complex number s by $\Re(s)$, and the absolute value by $|s|$. Given $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{C}^n$ and $c \in \mathbb{R}$, by the expression $\Re(\mathbf{s}) > c$ we mean that $\Re(s_i) > c$ for all $i \in \{1, \ldots, n\}$. We also write $|\mathbf{s}| := \max_{i=1}^n |s_i|$.

3. *Campana orbifolds, Campana points, and the conjecture*

In this section we recall two notions of Campana points, we discuss the leading constant and the exceptional sets in Conjecture 1.1, and we investigate the functoriality properties of the sets of Campana points.

3.1. *Orbifolds*

We recall Campana's notion of orbifolds ("orbifoldes géométriques"), as introduced in his foundational papers [**18, 20**]. In this article, we only consider those orbifolds which Campana calls "smooth"; in this section, we allow F to be any field.

Definition 3.1. A Campana orbifold over F is a pair (X, D) consisting of a smooth variety X and an effective Weil $\mathbb{Q}$-divisor D on X, both defined over F, such that

(i) we have

$$D = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha,$$

where the D_α are prime divisors on X, and ϵ_α belongs to the set of weights

$$\mathfrak{W} := \left\{ 1 - \frac{1}{m} \,\middle|\, m \in \mathbb{Z}_{\geqslant 1} \right\} \cup \{1\}$$

for all $\alpha \in \mathcal{A}$;

(ii) the support $D_{\mathrm{red}} = \sum_{\alpha \in \mathcal{A}} D_\alpha$ is a divisor with strict normal crossings on X.

Condition (2) in this definition implies that the irreducible components D_α of D_{red} are smooth; it is important to note, however, that they may well be geometrically reducible. We refer to [**69**, § 41.21] for the definition of strict normal crossings. The definition also implies that any Campana orbifold (X, D) is a dlt (divisorial log terminal) pair, in the sense of birational geometry (see [**49**, Definition 2.37] for this notion). We say that (X, D) is klt (Kawamata log terminal) if moreover $\epsilon_\alpha \neq 1$ for all $\alpha \in \mathcal{A}$, that is, if all weights are strictly smaller than 1.

Conversely, given a smooth F-variety X, a reduced divisor $D = \sum_{\alpha \in \mathcal{A}} D_\alpha$ on X with strict normal crossings and a weight vector $\epsilon = (\epsilon_\alpha)_{\alpha \in \mathcal{A}}$, where $\epsilon_\alpha \in \mathfrak{W}$ for all α, we obtain a Campana orbifold (X, D_ϵ) over F by setting $D_\epsilon = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha$.

In this paper, we consider only Campana orbifolds (X, D) with X proper over F.

3.2. *Two types of Campana points*

The notion of "orbifold rational point" is explored in Campana's papers [**18**, § 9], [**19**, § 4], [**20**, § 12], [**21**, § 7.6] and in Abramovich's survey [**1**, Lecture 2]. The adjective "rational" may create confusion, so we use the name Campana points here, to acknowledge that they are an intermediate notion between rational and integral points. In fact, [**1**] defines two different notions of Campana points, one more restrictive than the other. It is essential for us to separate the two notions, since the orbifold analog of Manin's conjecture seems to work well only for the more restrictive version; this is the one to which we will refer to simply as Campana points (Definition 3.4). The notion featuring in the recent paper [**3**] is (a slight variant of) the less restrictive version, and we will refer to it as weak Campana points (Definition 3.3); it seems to be ill-behaved for the problem studied in this paper (see § 3.2.1).

Remark 3.2. So far few results on the arithmetic of (weak) Campana points are available. Work on points of bounded height goes back to [**71**], followed immediately by [**16**] and more recently by [**17**]. Work of Schindler and the first author [**56**] investigates the distribution of Campana points on toric varieties. Recent work of Xiao [**72**] extends our results to biequivariant compactifications of the Heisenberg group.

In dimension 1, where both notions of Campana points coincide, the analog of Mordell's conjecture for Campana points has been proved over function fields, first in characteristic 0 by Campana himself [**19**], and only recently in arbitrary characteristic [**47**]. Over number fields, the only known result says that the *abc* conjecture implies Mordell's conjecture for Campana points; see [**63**, Appendix] for a detailed argument.

Let (X, D_ϵ) be a Campana orbifold with X proper over F, where $D_\epsilon = \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha D_\alpha$ and the ϵ_α belong to the usual set $\mathfrak{W}$. Let $S \subseteq \Omega_F$ be a finite set containing Ω_F^∞. We say that (X, D_ϵ) has a good integral model away from S if there exists a flat, proper model $\mathcal{X}$ over $\mathcal{O}_{F,S}$ such that $\mathcal{X}$ is regular. Given such a model, we denote by $\mathcal{D}_\alpha$ the Zariski closure of D_α in $\mathcal{X}$, and we write $(\mathcal{X}, \mathcal{D}_\epsilon)$ for the model, where $\mathcal{D}_\epsilon := \sum_{\alpha \in \mathcal{A}} \epsilon_\alpha \mathcal{D}_\alpha$.

Campana points can only be defined once a suitable model has been fixed, so let us choose a good integral model $(\mathcal{X}, \mathcal{D}_\epsilon)$ for (X, D_ϵ) over $\mathcal{O}_{F,S}$. Any rational point $P \in X(F)$ extends uniquely to an integral point $\mathcal{P} \in \mathcal{X}(\mathcal{O}_{F,S})$ by the valuative criterion for properness.

Let $\mathcal{A}_\epsilon = \{\alpha \in \mathcal{A} : \epsilon_\alpha \neq 0\}$. Let $X^\circ = X \setminus (\bigcup_{\alpha \in \mathcal{A}_\epsilon} D_\alpha)$. If $P \in X^\circ(F)$ and if $v \notin S$ is a place of F, then we get an induced point $\mathcal{P}_v \in \mathcal{X}(\mathcal{O}_v)$. For each $\alpha \in \mathcal{A}$ such that $\mathcal{P}_v \nsubseteq \mathcal{D}_\alpha$, the pullback of $\mathcal{D}_\alpha$ via $\mathcal{P}_v$ defines a non-zero ideal in $\mathcal{O}_v$. We denote its colength by $n_v(\mathcal{D}_\alpha, P)$; this is the intersection multiplicity of P and $\mathcal{D}_\alpha$ at v. When $P \in D_\alpha$ for some $\alpha \in \mathcal{A}_\epsilon$, we define $n_v(\mathcal{D}_\alpha, P)$ to be $+\infty$.

The total intersection number of P with $\mathcal{D}$ is then

$$n_v(\mathcal{D}_\epsilon, P) = \sum_{\alpha \in \mathcal{A}_\epsilon} \epsilon_\alpha n_v(\mathcal{D}_\alpha, P).$$

The following definition goes back to [**1**, § 2.1.7] and features in [**3**] as well.

Definition 3.3. With the notation introduced above, we say that $P \in X(F)$ is a weak Campana $\mathcal{O}_{F,S}$-point on $(\mathcal{X}, \mathcal{D}_\epsilon)$ if the following holds:

(i) for all α with $\epsilon_\alpha = 1$ and $v \notin S$, $n_v(\mathcal{D}_\alpha, P) = 0$, that is, $P \in (X \setminus \bigcup_{\epsilon_\alpha = 1} D_\alpha)(\mathcal{O}_{F,S})$ and

(ii) for $v \notin S$, if $n_v(\mathcal{D}_\epsilon, P) > 0$ then

$$n_v(\mathcal{D}_\epsilon, P) \leqslant \left(\sum_{\alpha\in\mathcal{A}_\epsilon} n_v(\mathcal{D}_\alpha, P)\right) - 1.$$

In particular, if $n_v(\mathcal{D}_\alpha, P) = +\infty$ for some $\alpha \in \mathcal{A}_\epsilon$, the inequality is trivially satisfied.

We denote the set of weak Campana $\mathcal{O}_{F,S}$-points on $(\mathcal{X}, \mathcal{D}_\epsilon)$ by $(\mathcal{X}, \mathcal{D}_\epsilon)_{\mathsf{w}}(\mathcal{O}_{F,S})$.

We obtain a more restrictive notion by imposing conditions for individual irreducible components of the support of D, in the spirit of [**1**, Definition 2.4.17]:

Definition 3.4. With the notation introduced above, we say that $P \in X(F)$ is a Campana $\mathcal{O}_{F,S}$-point on $(\mathcal{X}, \mathcal{D}_\epsilon)$ if the following hold:

(i) for all α with $\epsilon_\alpha = 1$ and $v \notin S$, $n_v(\mathcal{D}_\alpha, P) = 0$, that is, $P \in (X \setminus \bigcup_{\epsilon_\alpha=1} D_\alpha)(\mathcal{O}_{F,S})$ and
(ii) for $v \notin S$, and all $\alpha \in \mathcal{A}_\epsilon$ with both $\epsilon_\alpha < 1$ and $n_v(\mathcal{D}_\alpha, P) > 0$, we have

$$n_v(\mathcal{D}_\alpha, P) \geqslant \frac{1}{1-\epsilon_\alpha}.$$

In other words, writing $\epsilon_\alpha = 1 - \frac{1}{m_\alpha}$, we require $n_v(\mathcal{D}_\alpha, P) \geqslant m_\alpha$ whenever $n_v(\mathcal{D}_\alpha, P) > 0$.

Remark 3.5. Definition 3.4 implies that a point $P \in X(F)$ that lies in $D_\alpha(F)$ for some $\alpha \in \mathcal{A}_\epsilon$ is a Campana $\mathcal{O}_{F,S}$-point if it lies in the v-adic closure of $X^\circ(F_v) \cap ((\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}))$ for all places $v \notin S$.

We denote the set of Campana $\mathcal{O}_{F,S}$-points on $(\mathcal{X}, \mathcal{D}_\epsilon)$ by $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$. We have

$$X(F) \supseteq (\mathcal{X}, \mathcal{D}_\epsilon)_{\mathsf{w}}(\mathcal{O}_{F,S}) \supseteq (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}) \supseteq \mathcal{X}^\circ(\mathcal{O}_{F,S}),$$

where $\mathcal{X}^\circ = \mathcal{X} \setminus (\sum_{\alpha\in\mathcal{A}_\epsilon} \mathcal{D}_\alpha)$. The leftmost two inclusions are equalities if $\epsilon_\alpha = 0$ for all $\alpha \in \mathcal{A}$, and the rightmost inclusion is an equality if $\epsilon_\alpha = 1$ for all $\alpha \in \mathcal{A}_\epsilon$.

For $v \notin S$, we denote by $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v)$ the set of points $P_v \in X(F_v)$ such that $n_v(\mathcal{D}_\epsilon, P_v)$ satisfies the conditions in Definition 3.4. We also define the set of adelic Campana points by

$$(\mathcal{X}, \mathcal{D}_\epsilon)(\mathbb{A}_F) = \prod_{v\notin S} (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v) \times \prod_{v\in S} X(F_v).$$

By Remark 3.5, the space $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v)$ is a closed subspace of the topological space $X(F_v)$; in particular, it is compact.

3.2.1. *An instructive example.* The following example illustrates the difference between the two notions of Campana points introduced above. We show that these notions yield different asymptotics for counts of points of bounded height. Moreover, the difference is encoded not only in the leading constant, but also in the exponent of the logarithm. In § 3.6 we use this example to discuss functoriality of Campana points under birational transformations.

Let $X = \mathbb{P}^2_{\mathbb{Q}}$ with coordinates $(x_0 : x_1 : x_2)$, and let $D_i = \{x_i = 0\}$ for $i \in \{0,1,2\}$. Taking $\mathcal{X} = \mathbb{P}^2_{\mathbb{Z}}$ and $\epsilon_0, \epsilon_1, \epsilon_2 \in \mathfrak{W}$, the Campana orbifold $(X, \sum_{i=0}^2 \epsilon_i D_i)$ has the obvious good integral model $(\mathcal{X}, \sum_{i=0}^2 \epsilon_i \mathcal{D}_i)$ over $\mathbb{Z}$ in the sense of § 3.2. For $0 \leqslant i \leqslant 2$, we write $\epsilon_i = 1 - \frac{1}{m_i}$ with the convention that $\frac{1}{m_i} = 0$ if $\epsilon_i = 1$. A point in $\mathcal{X}(\mathbb{Z})$, represented by coprime integer coordinates $(x_0 : x_1 : x_2)$, is

- a weak Campana $\mathbb{Z}$-point if $x_i \in \{\pm 1\}$ for all $i \in \{0,1,2\}$ such that $\epsilon_i = 1$, and

$$p \mid \prod_{\substack{0\leqslant i\leqslant 2\\ \epsilon_i\neq 0}} x_i \quad \Rightarrow \quad \sum_{\substack{0\leqslant i\leqslant 2\\ \epsilon_i\neq 0}} \frac{1}{m_i} v_p(x_i) \geqslant 1$$

  for every prime p, or equivalently, if $x_0^{m_1m_2}x_1^{m_0m_2}x_2^{m_0m_1}$ is $m_0m_1m_2$-full (in the case $0 < \epsilon_0, \epsilon_1, \epsilon_2 < 1$);
- a Campana $\mathbb{Z}$-point if $x_i \in \{\pm 1\}$ for all $i \in \{0,1,2\}$ such that $\epsilon_i = 1$, and

$$p \mid x_i \quad \Rightarrow \quad \frac{1}{m_i} v_p(x_i) \geqslant 1$$

  for every prime p and every $i \in \{0,1,2\}$ such that $\epsilon_i \neq 1$, or equivalently, if x_i is m_i-full for all $i \in \{0,1,2\}$, assuming $\epsilon_0, \epsilon_1, \epsilon_2 < 1$.

Note how a point on the boundary divisor can be a Campana point: for example, if $\epsilon_0, \epsilon_1, \epsilon_2 < 1$ and $P = (0 : x_1 : x_2)$ with x_1, x_2 coprime integers, then P is a weak $\mathbb{Z}$-Campana point, although it is a $\mathbb{Z}$-Campana point only if for $i = 1, 2$, we have $p \mid x_i \Rightarrow v_p(x_i) \geqslant m_i$.

Let us specialize to the case where $m_0 = m_1 = m_2 = 2$. We set $X^\circ = X \setminus (\bigcup_{i=0}^{2} D_i)$.

To count (weak) Campana points of bounded height, we use the exponential Weil height

$$H \colon \mathbb{P}^2(\mathbb{Q}) \to \mathbb{R}$$

$$(x_0 : x_1 : x_2) \mapsto \max\{|x_0|, |x_1|, |x_2|\} \text{ whenever } x_0, x_1, x_2 \text{ are coprime integers.}$$

Proposition 3.6. *Let $\mathcal{X}, \mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2$ be as above and let $\mathcal{D}_\epsilon = \sum_{i=0}^{2} \frac{1}{2}\mathcal{D}_i$. Then for sufficiently large $T > 0$,*

$$\#\{x \in (\mathcal{X}, \mathcal{D}_\epsilon)(\mathbb{Z}) \cap X^\circ(\mathbb{Q}) : H(x) \leqslant T\} \ll T^{3/2}, \tag{3.1}$$

$$\#\{x \in (\mathcal{X}, \mathcal{D}_\epsilon)_{\mathrm{w}}(\mathbb{Z}) \cap X^\circ(\mathbb{Q}) : H(x) \leqslant T\} \gg T^{3/2}\log T. \tag{3.2}$$

Proof. In this setting, the set of Campana $\mathbb{Z}$-points on X° is in bijection with the set of triples $(x_0, x_1, x_2) \in \mathbb{Z}^3_{\neq 0}$ such that $\gcd(x_0, x_1, x_2) = 1$ and x_0, x_1 and x_2 are all squareful. The counting function of Campana $\mathbb{Z}$-points of Weil height bounded by T has an upper bound given by the cardinality of the set obtained by removing the coprimality condition, which grows asymptotically like $T^{\frac{3}{2}}$, up to multiplication by a positive constant, by [**34**] (see also [**4**]).

The set of weak Campana $\mathbb{Z}$-points on X° is in bijection with the set of triples $(x_0, x_1, x_2) \in \mathbb{Z}^3_{\neq 0}$ such that $\gcd(x_0, x_1, x_2) = 1$ and $x_0x_1x_2$ is squareful. To prove the lower bound in (3.2), we count points of bounded height in the subset A of coprime triples $(x_0, x_1, x_2) \in \mathbb{Z}^3_{>0}$ such that x_0 is a square and x_1x_2 is a square. The size of this subset is estimated by

$$\sum_{d\leqslant T} \mu(d) \cdot \#\{1 \leqslant x_0 \leqslant T : \, x_0 \text{ square}, d \mid x_0\} \cdot \#\{1 \leqslant x_1, x_2 \leqslant T : \, x_1x_2 \text{ square}, d \mid x_1, d \mid x_2\},$$

where μ denotes the Möbius function. The number of squares up to T that is divisible by a given squarefree integer d is $T^{1/2}/d + O(1)$. To estimate the cardinality of the set B of pairs $(x_1, x_2) \in (d\mathbb{Z}_{>0})^2$ such that $x_1, x_2 \leqslant T$ and x_1x_2 is a square, we write $u = \gcd(x_1/d, x_2/d)$ and $y_i = x_i/(du)$ for $i \in \{1, 2\}$. Then x_1x_2 is a square if and only if both y_1 and y_2 are squares. Writing $y_i = z_i^2$ for $i \in \{1, 2\}$, we get

$$\#B = \sum_{u\leqslant T/d} \sum_{\substack{z_1,z_2\leqslant (T/(du))^{1/2}\\ \gcd(z_1,z_2)=1}} 1 = \frac{T/d\log(T/d)}{\zeta_{\mathbb{Q}}(2)} + O(T/d).$$

Therefore, $\#A = (\zeta_{\mathbb{Q}}(2))^{-2}\, T^{3/2}\log T + O_\delta(T^{3/2}(\log T)^\delta)$ for all $\delta > 0$. $\square$

The upper bound (3.1) is in agreement with Conjecture 1.1. Indeed, for the line bundle $L = \mathcal{O}(1)$, we have $a((X, D_\epsilon), L) = 3/2$ and $b = b(F, (X, D_\epsilon), L) = 1$, so Conjecture 1.1 predicts a counting formula for Campana points of bounded height that grows like $cT^{3/2}$ as $T \to \infty$, which is correct. The upper bound is, in fact, sharp; see [**56**, Theorem 1.2]. The lower bound (3.2) shows that counting Campana points and weak Campana points of bounded height in the same setting can lead to different asymptotics. However, since the lower bound is based on counting points in a thin set (denoted by A in the proof), it does not show that Conjecture 1.1 fails when counting weak Campana points. We are unaware of any successful attempt to produce an asymptotic formula for the count of weak Campana points of bounded height in an example where the sets of Campana points and weak Campana points do not coincide.

3.3. *The leading constant*

We keep the notation introduced in § 1.2. In this section, we define the leading constant that appears in Conjecture 1.1, in the case when the divisor $a((X, D_\epsilon), L)L + K_X + D_\epsilon$ is $\mathbb{Q}$-linearly equivalent to a rigid effective divisor E. The construction here is analogous to [**9, 54**]. For simplicity, we assume that the boundary divisor D contains all components of E; we denote by $\mathcal{A}(L)$ the set of irreducible components of D that are not contained in the support of E.

Write $U = X \setminus \operatorname{Supp}(E)$, and let Λ be the image of $\operatorname{Eff}^1(X)$ under the projection map $\rho\colon \operatorname{Pic}(X) \to \operatorname{Pic}(U)$; this is a finitely generated, polyhedral cone since X is a Fano orbifold. Let

$$\chi_\Lambda(\rho([L])) = \int_{\Lambda^*} e^{-\langle \rho([L]), \mathbf{x}\rangle}\, \mathrm{d}\mathbf{x},$$

where $\Lambda^* \subset \operatorname{Pic}(U)^*_{\mathbb{R}}$ is the dual cone to Λ and $\mathrm{d}\mathbf{x}$ is the Lebesgue measure on $\operatorname{Pic}(U)^*_{\mathbb{R}}$, normalized by the dual lattice $\operatorname{Pic}(U)^* \subset \operatorname{Pic}(U)^*_{\mathbb{R}}$ (see [**9**, Definition 2.3.14]). The α-constant of the pair (X, D_ϵ) with respect to L is

$$\alpha((X, D_\epsilon), L) := \chi_\Lambda(\rho([L])) \prod_{\alpha \in \mathcal{A}(L)} (1 - \epsilon_\alpha),$$

and the β-constant of the pair (X, D_ϵ) with respect to L is

$$\beta((X, D_\epsilon), L) = \# \mathrm{H}^1(\Gamma, \operatorname{Pic}(\overline{U})).$$

The group $\mathrm{H}^1(\Gamma, \operatorname{Pic}(\overline{U}))$ is finite. Indeed, since X is a Fano orbifold, it follows from [**41**] that X is rationally connected. Hence, $\operatorname{Pic}(\overline{X})$ is a free $\mathbb{Z}$-module of finite rank. Furthermore since E is rigid, its geometric components generate a primitive lattice in $\operatorname{Pic}(\overline{X})$. Thus, its cokernel $\operatorname{Pic}(\overline{U})$ is torsion free. Hence we conclude that $\mathrm{H}^1(\Gamma, \operatorname{Pic}(\overline{U}))$ is finite.

The open set U can be endowed with a Tamagawa measure τ_U [**24**, Definition 2.8]; fixing an adelic metrization on each component of D and on K_X, we let $\tau_{U,D_\epsilon} = \mathsf{H}_{D_\epsilon}\tau_U$, where H_{D_ϵ} is the height function associated to the divisor D_ϵ. We define the Tamagawa constant by

$$\tau(F, S, (\mathcal{X}, \mathcal{D}_\epsilon), \mathcal{L}) := \int_{\overline{U(F)}_\epsilon} \mathsf{H}(x, a((X, D_\epsilon), L)L + K_X + D_\epsilon)^{-1}\, \mathrm{d}\tau_{U,D_\epsilon},$$

where $\overline{U(F)}_\epsilon$ denotes either

(1) the topological closure of $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}) \cap U(F)$ in $U(\mathbb{A}_F)$, or
(2) the Brauer set $U(\mathbb{A}_F)_\epsilon^{\mathrm{Br(U)}}$ defined as follows: for any subset $B \subset U(F_v)$, let B_ϵ denote the support of $\delta_{\epsilon,v}$ on B. The adelic Campana set is the restricted product

$$U(\mathbb{A}_F)_\epsilon = \prod_{v \in \Omega_F}{}' U(F_v)_\epsilon$$

with respect to $U(\mathcal{O}_v)_\epsilon$. The set $U(\mathbb{A}_F)_\epsilon^{\mathrm{Br(U)}}$ is the zero locus of the Brauer–Manin pairing. See [57, Chapter 8] for the definition of the Brauer–Manin pairing.

In Theorem 1.2, we use the latter definition of $\overline{U(F)}_\epsilon$; see Lemma 9.3. It is not known whether the two sets coincide; see Question 3.9 below. We recall that already in the classical case of rational points, it is not clear what domain should appear in the integral that defines the Tamagawa constant; see [58, Remarks 6.13 and 7.8]. This integral converges in the general setting of a Fano orbifold, by an analog of Denef's formula (7.3) in this setting. Finally, the leading constant for Conjecture 1.1 is

$$c(F,S,(\mathcal{X},\mathcal{D}_\epsilon),\mathcal{L}) = \frac{\alpha((X,D_\epsilon),L)\beta((X,D_\epsilon),L)\tau(F,S,(\mathcal{X},\mathcal{D}_\epsilon),\mathcal{L})}{a((X,D_\epsilon),L)(b(F,(X,D_\epsilon),L)-1)!}.$$

Our Theorem 1.2 agrees with Conjecture 1.1, including the prediction for the constant, as we show in § 9.1.

3.4. *Thin exceptional sets*

In the formulation of Conjecture 1.1, we expect that it is necessary to remove a thin set of Campana points from the count in order to obtain a formula that reflects the global geometry of the Campana orbifold; indeed, already for rational points, it has been understood for quite some time that a version of Manin's conjecture with only a closed — rather than thin — exceptional set admits counterexamples, see [7, 15, 51]. Meanwhile, several authors have recently built up evidence toward a version of Manin's conjecture with a thin exceptional set, see [52, 53, 55, 59]. While we do believe that the set of klt Campana points is itself *not* thin, we are unable at present to show this; however, we propose a problem that we hope will ameliorate this circumstance.

Let (X, D_ϵ) be a Fano orbifold over a number field F, that is, a Campana orbifold such that $-(K_X + D_\epsilon)$ is ample. Fix a finite set $S \subset \Omega_F$ containing all archimedean places of F, as well as a good integral model $(\mathcal{X}, \mathcal{D})$ of (X, D) over Spec $\mathcal{O}_{F,S}$, as in § 3.2. Write $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ for the set of $\mathcal{O}_{F,S}$-Campana points of $(\mathcal{X}, \mathcal{D}_\epsilon)$.

Definition 3.7. A **thin subset** of $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ is a subset of a finite union of

(i) type I sets: those of the form $Z \cap (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ for a proper Zariski closed subset $Z \subset X$;
(ii) type II sets: those of the form $f(Y(F)) \cap (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$, where $f\colon Y \to X$ is a generically finite cover of degree at least 2, with Y a projective, integral F-variety.

It is natural to ask whether $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ is itself *not* thin, possibly after a finite extension of the ground field. After all, if a version of Manin's conjecture with a thin exceptional set is to hold for Campana points on Fano orbifolds, we would like to have something left to count *after* the removal of a thin subset. We are thus forced to make what we hope is a superfluous hypothesis in Conjecture 1.1, namely, that $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ itself is *not* thin in our setting.

This shortcoming is already present in the traditional case of rational points on smooth Fano varieties, where we expect the set of rational points to be not thin if it is non-empty. This is known conditionally on Colliot-Thélène's conjecture predicting that the Brauer–Manin obstruction controls all failures of weak approximation on rationally connected varieties [29]. Indeed, this conjecture implies that smooth Fano varieties satisfy "weak weak approximation," which, in turn, implies that the set of rational points is not thin [60, Theorem 3.5.7].

On a positive note, Serre has shown that $\mathbb{P}^n(F)$ is not thin [60, § 3.4]. This prompts us to ask the following question.

QUESTION 3.8. Let F be a number field and let $D = \sum_{\alpha\in\mathcal{A}} D_\alpha$ be a divisor on $\mathbb{P}^n_F$ with strict normal crossings. For each $\alpha \in \mathcal{A}$, pick $\epsilon_\alpha \in \mathfrak{W}$ with $\epsilon_\alpha < 1$ and set $D_\epsilon = \sum_{\alpha\in\mathcal{A}} \epsilon_\alpha D_\alpha$, so that the Campana orbifold $(\mathbb{P}^n, D_\epsilon)$ is klt. Assume moreover that $-(K_{\mathbb{P}^n} + D_\epsilon)$ is ample. Fix a good integral model $(\mathcal{P}^n, \mathcal{D}_\epsilon)$ of $(\mathbb{P}^n, D_\epsilon)$, and a finite set S of places of F that includes all the archimedean places. Is the set $(\mathcal{P}^n, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ of klt Campana points non-thin?

For some partial results, we refer to the recent paper of Browning–Yamagishi [**17**, § 4]. A version of this question for integral points on a log K3 surface is addressed in [**28**].

In a different direction, if the set of Campana points $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ were thin, then there would exist a set of places T such that the image of this set in $\prod_{v\in T} X(F_v)$ is not dense, by [**60**, Theorem 3.5.3]. Since we expect $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ to be *not* thin, we ask the following question.

QUESTION 3.9. Is there a finite set $S_0 \subset \Omega_F$ containing S such that for any $T \subseteq \Omega_F$ a finite set of places such that $S_0 \cap T = \emptyset$, $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ is dense in $\prod_{v\in T}((\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v))$? In other words, does the set of Campana points satisfy weak weak approximation?

3.5. *Browning–Yamagishi's example*

In [**17**, Theorem 1.2], Browning and Yamagishi presented an illuminating example, which illustrates in particular that in the formulation of Conjecture 1.1, it is important to exclude a thin set to obtain the expected growth rate. We briefly recall the construction. We define divisors on $\mathbb{P}^2_{\mathbb{Q}} = \operatorname{Proj}\mathbb{Q}[x_0, x_1, x_2]$ by

$$D_i = \{x_i = 0\} \text{ for } i = 0, 1, 2, \quad \text{and} \quad D_3 = \{x_0 + x_1 + x_2 = 0\}.$$

We denote by H the hyperplane class, and we set $D = \bigcup_{i=0}^3 D_i$. Consider the Campana orbifold $(\mathbb{P}^2_{\mathbb{Q}}, D_\epsilon = \sum_{i=0}^3 \frac{1}{2} D_i)$, and extend it to the obvious good integral model $(\mathbb{P}^2_{\mathbb{Z}}, \mathcal{D}_\epsilon)$ over $\operatorname{Spec}(\mathbb{Z})$.

A computation shows that

$$a((\mathbb{P}^2, D_\epsilon), H) = 1, \quad b(\mathbb{Q}, (\mathbb{P}^2, D_\epsilon), H) = 1.$$

On the other hand, Browning and Yamagishi show that

$$\mathsf{N}((\mathbb{P}^2_{\mathbb{Z}}, \mathcal{D}_\epsilon)(\mathbb{Z}) \cap (\mathbb{P}^2 \setminus D)(\mathbb{Q}), H, T) \gg T \log T,$$

a computation at odds with a closed-set version of Conjecture 1.1. As we explain below, the unexpected rapid growth of the counting function is explained by a type II thin set.

Let $Q \subset \mathbb{P}^3 = \operatorname{Proj}\mathbb{Q}[w_0, w_1, w_2, w_3]$ be the smooth quadric defined by

$$w_0^2 - w_1^2 + w_2^2 = w_3^2,$$

and consider the finite morphism of degree 8 given by

$$f\colon Q \to \mathbb{P}^2_{\mathbb{Q}}$$

$$(w_0 : w_1 : w_2 : w_3) \mapsto (w_0^2 : -w_1^2 : w_2^2).$$

Note that

$$f(Q(\mathbb{Q})) \subset (\mathbb{P}^2_{\mathbb{Z}}, \mathcal{D}_\epsilon)(\mathbb{Z}),$$

and that, by the ramification formula, we have

$$K_Q = f^*(K_{\mathbb{P}^2} + D_\epsilon).$$

From this, it follows that

$$a(Q, f^*H) = 1, \quad b(\mathbb{Q}, Q, f^*H) = 2.$$

Therefore, the number of rational points on Q grows more quickly than the expected growth rate on $(\mathbb{P}^2_{\mathbb{Z}}, D_\epsilon)$.

There are in fact infinitely many twists $Q^\sigma/\mathbb{P}^2_{\mathbb{Q}}$ such that

$$a(Q^\sigma, H) = 1, \quad b(\mathbb{Q}, Q^\sigma, H) = 2,$$

so it is a priori unclear whether the combined images of their rational points on $\mathbb{P}^2_{\mathbb{Q}}$ form a thin set. This type of problem is already addressed in [**52**], using Hilbert's irreducibility theorem. We obtain the following auxiliary result.

Lemma 3.10. *The set*

$$Z = \bigcup_{\sigma} f^\sigma(Q^\sigma(\mathbb{Q})),$$

where the union is taken over all $\sigma \in \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Aut}(\overline{Q}/\mathbb{P}^2_{\overline{\mathbb{Q}}}))$ *with the property that*

$$b(\mathbb{Q}, Q^\sigma, (f^\sigma)^* H) = 2,$$

is thin.

The following proof is due to the referee.

Proof. The twists Q^σ are given by $Q_{a_0,a_1,a_2} = \{a_0 w_0^2 - a_1 w_1^2 + a_2 w_2^2 = w_3^2\} \subseteq \mathbb{P}^3$ for $a_0, a_1, a_2 \in \mathbb{Q}^\times$, and Q_{a_0,a_1,a_2} has Picard rank 2 if and only if $a_0a_1a_2$ is a square. The corresponding twists of f are

$$f_{a_0,a_1,a_2} : Q_{a_0,a_1,a_2} \to \mathbb{P}^2_{\mathbb{Q}}, \quad (w_0 : \cdots : w_3) \mapsto (a_0 w_0^2 : -a_1 w_1^2 : a_2 w_2^2).$$

We observe that for all $a_0, a_1, a_2 \in \mathbb{Q}^\times$ such that $a_0a_1a_2$ is a square, the images of the $\mathbb{Q}$-points on Q_{a_0,a_1,a_2} under f_{a_0,a_1,a_2} are contained in the set of points $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2(\mathbb{Q})$ such that $-x_0x_1x_2$ is a square, which is a thin set. $\square$

3.6. *Birational invariance and functoriality*

We conclude this section by exploring the functoriality properties of sets of Campana points under birational morphisms.

3.6.1. *An instructive example* (*continued*). To motivate our discussion, we appeal to the example of § 3.2.1: recall that $X = \mathbb{P}^2_{\mathbb{Q}}$ with coordinates $(x_0 : x_1 : x_2)$, $D_i = \{x_i = 0\}$ for $i \in \{0, 1, 2\}$, and consider the Campana orbifold $(X, \sum_{i=0}^2 (1 - \frac{1}{m_i}) D_i)$ with $\mathbb{Z}$-model $\mathcal{X} = \mathbb{P}^2_{\mathbb{Z}}$.

Let $\varphi : Y \to X$ be the blow-up with center the intersection point of D_1 and D_2. Then φ is an isomorphism over $X^\circ = X \setminus (\bigcup_{i=0}^2 D_i)$. Let $Y^\circ = \varphi^{-1}(X^\circ)$. Denote by E the exceptional divisor and by $\widetilde{D}_i$ the strict transform of D_i for $i \in \{0, 1, 2\}$. Then $Y^\circ = Y \setminus (E \cup (\bigcup_{i=0}^2 \widetilde{D}_i))$. The blow-up $\mathcal{Y}$ of $\mathcal{X}$ at the subvariety defined by $\{x_1 = x_2 = 0\}$ yields a smooth projective $\mathbb{Z}$-model of Y. We observe that given a point $P \in Y^\circ(\mathbb{Q})$, the point $\varphi(P)$ is

- a weak Campana $\mathbb{Z}$-point on $(\mathcal{X}, \sum_{i=0}^2 (1 - \frac{1}{m_i}) \mathcal{D}_i)$ if for every prime p, the sum

$$\frac{1}{m_0} n_p(\widetilde{D}_0, P) + \frac{1}{m_1} n_p(\widetilde{D}_1, P) + \frac{1}{m_2} n_p(\widetilde{D}_2, P) + \left(\frac{1}{m_1} + \frac{1}{m_2}\right) n_p(E, P)$$

  is either 0 or at least 1;

- a Campana $\mathbb{Z}$-point on $(\mathcal{X}, \sum_{i=0}^{2}(1-\frac{1}{m_i})\mathcal{D}_i)$ if for every prime p, the numbers
$$\frac{1}{m_0}n_p(\widetilde{D}_0, P), \quad \frac{1}{m_1}(n_p(\widetilde{D}_1, P) + n_p(E, P)), \quad \frac{1}{m_2}(n_p(\widetilde{D}_2, P) + n_p(E, P))$$
are either 0 or at least 1.

This description clearly shows that the set of (weak) Campana points is not invariant under birational morphisms, that is, for general m_0, m_1, m_2, there is no choice of positive integers $\widetilde{m}_0, \widetilde{m}_1, \widetilde{m}_2, \widetilde{m}_E$ such that the restriction of the blow-up φ to Y° would induce a bijection between the set of (weak) Campana points for $(\mathcal{Y}, (1-\frac{1}{\widetilde{m}_E})\mathcal{E} + \sum_{i=0}^{2}(1-\frac{1}{\widetilde{m}_i})\widetilde{\mathcal{D}}_i)$ on the open subset Y° and the set of (weak) Campana points for $(\mathcal{X}, \sum_{i=0}^{2}(1-\frac{1}{m_i})\mathcal{D}_i)$ on the isomorphic open subset X°, where $\mathcal{E}, \widetilde{\mathcal{D}}_0, \widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2$ denote the closures in $\mathcal{Y}$ of $E, \widetilde{D}_0, \widetilde{D}_1, \widetilde{D}_2$, respectively.

Not all is lost, however: if we define $\widetilde{m}_i = m_i$ for $i \in \{0, 1, 2\}$ and $\widetilde{m}_E = \max\{m_1, m_2\}$, then the set of (weak) Campana points on the resulting orbifold $(\mathcal{Y}, (1-\frac{1}{\widetilde{m}_E})\mathcal{E} + \sum_{i=0}^{2}(1-\frac{1}{\widetilde{m}_i})\widetilde{\mathcal{D}}_i)$ is mapped by φ into a *subset* of the set of (weak) Campana points on $(\mathcal{X}, \sum_{i=0}^{2}(1-\frac{1}{m_i})\mathcal{D}_i)$.

3.6.2. *The general picture.* Let X be a rationally connected smooth projective variety defined over a number field F and let $D = \sum_{\alpha\in\mathcal{A}} D_\alpha$ be a strict normal crossings divisor on X. Fix a weight vector $\epsilon = (\epsilon_\alpha)_{\alpha\in\mathcal{A}}$ where $\epsilon_\alpha \in \mathfrak{W}$ with $\epsilon_\alpha = 1 - 1/m_\alpha < 1$. Set $D_\epsilon = \sum_{\alpha\in\mathcal{A}} \epsilon_\alpha D_\alpha$ and consider the Campana orbifold (X, D_ϵ), which is a klt pair.

Let
$$\varphi\colon \widetilde{X} \to X$$
be a birational morphism from a smooth projective variety $\widetilde{X}$, such that $\widetilde{D} = (\varphi^* D)_{\mathrm{red}}$ is a strict normal crossings divisor. We assume for simplicity that φ is an isomorphism outside of D and that both $(\widetilde{X}, \widetilde{D})$ and (X, D) admit good integral models $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}})$ and $(\mathcal{X}, \mathcal{D})$ that are compatible. We assign a weight vector $\tilde{\epsilon}$ to $\widetilde{D}$ as follows. For the strict transform of a component D_α of D, we set $\tilde{\epsilon}_\alpha = \epsilon_\alpha$. If E_β is an exceptional divisor and if $e_{\beta,\alpha}$ denotes the coefficient of E_β in $\varphi^* D_\alpha$, then we define
$$\tilde{m}_\beta = \max\{\lceil m_\alpha / e_{\beta,\alpha} \rceil \mid e_{\beta,\alpha} > 0\} \quad \text{and} \quad \tilde{\epsilon}_\beta = 1 - 1/\tilde{m}_\beta.$$
Then $\varphi : (\widetilde{X}, \widetilde{D}_{\tilde{\epsilon}}) \to (X, D)$ is a “morphisme orbifolde” in the sense of [20, Définition 2.3].

By construction, we have
$$\varphi((\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}_{\tilde{\epsilon}})(\mathcal{O}_{F,S})) \subset (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}),$$
but this inclusion need not be an equality. On the other hand, the a- and b-invariants are well behaved for our choice of $\tilde{\epsilon}$, as we now explain. We observe that
$$K_{\widetilde{X}} + D_{\tilde{\epsilon}} \geqslant \varphi^*(K_X + D_\epsilon)$$
by [20, Corollaire 2.12]. Then the arguments of [42, § 2] show that
$$a((\widetilde{X}, \widetilde{D}_{\tilde{\epsilon}}), \varphi^* L) = a((X, D_\epsilon), L), \quad b(F, (\widetilde{X}, \widetilde{D}_{\tilde{\epsilon}}), \varphi^* L) = b(F, (X, D_\epsilon), L).$$

We end by remarking that $\tau(F, S, (\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}_\epsilon), \mathcal{L})$ and $\tau(F, S, (\mathcal{X}, \mathcal{D}_\epsilon), \mathcal{L})$ will be different in general because $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$ and $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{D}}_{\tilde{\epsilon}})(\mathcal{O}_{F,S})$ are different. Our overall conclusion is that our Manin-type conjecture for klt Campana points is quite sensitive to birational modifications. In particular, proving the asymptotic formula for the counting function after a birational modification need not easily yield an asymptotic formula for the original variety.

4. *Analytic Clemens complexes*

Clemens complexes are simplicial sets that keep track of containment relations between the intersections of components of a divisor in a variety. As in [**26**], Clemens complexes will be used in § 10 to keep track of the contribution of the local height integrals to the pole of the height zeta function when some integrality conditions appear, that is, when some component of the boundary has weight 1. For a more detailed treatment, we refer the reader to [**24**, § 3.1].

In this section, X is a smooth, proper variety over a number field F, and $D = \sum_{\alpha\in\mathcal{A}} D_\alpha$ is a reduced divisor on X with strict normal crossings. Let $v \in \Omega_F$, and fix an embedding $\bar{F} \subseteq \bar{F}_v$, so that $\Gamma_v := \mathrm{Gal}(\bar{F}_v/F_v)$ acts on $\bar{X}$ and $\bar{D}$. Write $\bar{\mathcal{A}}$ for the indexing set of $\bar{D}$, and $\mathcal{A}_v$ for the set of orbits of $\bar{\mathcal{A}}$ under the action of Γ_v. Recall that X_v denotes the base change of X to F_v; write $D_v := D \otimes_F F_v = \bigcup_{\beta\in\mathcal{A}_v} D_{v,\beta}$, where the $D_{v,\beta}$ are irreducible components.

Given a divisor D' on X such that $\bar{D}' = \bigcup_{\alpha\in\mathscr{A}} \bar{D}_\alpha$ for some $\mathscr{A} \subseteq \bar{\mathcal{A}}$, we denote by $\mathscr{A}_v$ the set of orbits of $\mathscr{A}$ under the action of Γ_v. As a *set*, the F_v-analytic Clemens complex associated to D' consists of irreducible components Z of intersections $\bigcap_{\beta\in B} D_{v,\beta}$ for $B \subseteq \mathscr{A}_v$ such that $Z(F_v) \neq \emptyset$. The complex enjoys additional structure, for example, as a poset; see [**24**, § 3.1] for details. The dimension of the Clemens complex of D' is

$$\max\left\{\#B : B \subseteq \mathscr{A}_v, \bigcap_{\beta\in B} D_{v,\beta}(F_v) \neq \emptyset\right\} - 1.$$

We may now define the a- and b-invariants of the pair (X, D) at v with respect to a linear combination of boundary components with positive coefficients. These invariants will come up in the calculation of the position and order of the rightmost pole of a local height integral of X at v, in the case where X is an equivariant compactification of $G = \mathbb{G}_a^n$.

Keeping the notation introduced above, we assume further that $-K_{X_v} \sim \sum_{\beta\in\mathcal{A}_v} \rho_\beta D_{v,\beta}$, with $\rho_\beta \in \mathbb{Z}$ for all β, and we set $L = \sum_{\beta\in\mathcal{A}_v} \lambda_\beta D_{v,\beta}$ with $\lambda_\beta > 0$ for all β. We define the $\tilde{a}$-invariant of the pair (X, D) at v with respect to L by

$$\tilde{a}((X, D), L) = \max_{\beta\in\mathcal{A}_v}\left\{\frac{\rho_\beta - 1}{\lambda_\beta}\right\}.$$

Let us denote the sum of the boundary components that do *not* appear in the support of $\tilde{a}((X, D), L)L + K_X + D$ by D'; in other words, we set

$$D' = D - (\tilde{a}((X, D), L)L + K_X + D)_{\mathrm{red}}.$$

Writing $\mathcal{C}^{\mathrm{an}}_{F_v}(D, L)$ for the F_v-analytic Clemens complex associated to D', we define the b-invariant of (X, D) at v with respect to L as follows:

$$b(F_v, (X, D), L) = 1 + \dim \mathcal{C}^{\mathrm{an}}_{F_v}(D, L).$$

We will now prove that the $\tilde{a}$- and b-invariants are birational invariants in a suitable sense. While this result is certainly of independent interest, we will use it to prove the meromorphic continuation of certain local height integrals in § 7.

Lemma 4.1. *Let X, D and L be as above. Let $(\widetilde{X}, \widetilde{D})$ be another pair satisfying the same hypotheses as (X, D), namely: (i) $\widetilde{D}$ is a reduced divisor with strict normal crossings on a smooth proper variety $\widetilde{X}$ over F and (ii) $-K_{\widetilde{X}_v}$ is a linear combination of irreducible components of $\widetilde{D}_v$. Assume that there is a birational morphism $\varphi\colon \widetilde{X} \to X$ with $\varphi^{-1}(D) = \widetilde{D}$ that is an isomorphism outside D. Then*

$$\tilde{a}((X, D), L) = \tilde{a}((\widetilde{X}, \widetilde{D}), \varphi^* L) \quad \text{and} \quad b(F_v, (X, D), L) = b(F_v, (\widetilde{X}, \widetilde{D}), \varphi^* L).$$

Proof. First, we observe that the birational invariance of the $\tilde{a}$-invariant follows from the fact that the pair (X, D) is log canonical, that is, we can write

$$\tilde{a}((X,D),L)\varphi^*L + K_{\widetilde{X}} + \widetilde{D} = \varphi^*(\tilde{a}((X,D),L)L + K_X + D) + E,$$

where $E \geqslant 0$ is an effective divisor supported on the exceptional locus of φ.

From now on, we denote $\tilde{a}((X,D),L)$ simply by a and we work over F_v, for a fixed place v. To prove birational invariance of the b-invariant, we first use [**2**, Theorem 0.3.1] to reduce to the case where the morphism φ is a blow-up of a smooth center having normal crossings with D. Let E be an exceptional divisor of φ.

First suppose that the image of E is not a component of the intersection of some of the boundary components. Then [**48**, (3.11.1)] shows that the log discrepancy of the exceptional divisor E is greater than -1, hence that E appears in the support of $a\varphi^*L + K_{\widetilde{X}} + \widetilde{D}$. Let Z be a maximal element in $\mathcal{C}^{\mathrm{an}}_{F_v}(D, L)$ such that $b(F_v, (X, D), L) = \operatorname{codim} Z$. Let Z be a component of $\bigcap_{i=1}^r D_{v,\beta_i}$ thus $\operatorname{codim} Z = r$. If the image T of E does *not* contain Z, then $b(F_v, (\widetilde{X}, \widetilde{D}), \varphi^*L) = \operatorname{codim} Z$. Thus our assertion follows in this case. If T contains Z, then by rearranging indices, we may assume that $T \subset D_{v,\beta_i}$ for $i \leqslant k$ and $T \not\subset D_{v,\beta_i}$ for $i > k$. Denoting the codimension of T by t, we have $k < t$; hence, the strict transforms of D_{v,β_i} for $i \leqslant k$ meet in $\varphi^{-1}(Z)$. On the other hand, the strict transforms of D_{v,β_i} for $i > k$ all contain $\varphi^{-1}(Z)$. Thus, $b(F_v, (\widetilde{X}, \widetilde{D}), \varphi^*L) = r = b(F_v, (X, D), L)$. Thus our assertion follows in this case too.

Next suppose that T is a component of the intersection of some of the boundary components. Then E does not appear in the support of the difference of $a\varphi^*L + K_{\widetilde{X}} + \widetilde{D}$ and $\varphi^*(aL + K_X + D)$. We further distinguish two cases. First, if E does not appear in the support of $\varphi^*(aL + K_X + D)$, we denote by Z a maximal element of $\mathcal{C}^{\mathrm{an}}_{F_v}(D, L)$ so that $b(F_v, (X, D), L) = \operatorname{codim} Z$ and we assume that Z is a component of $\bigcap_{i=1}^r D_{v,\beta_i}$. Either T and Z do not meet, or T contains Z; in the former case, we have $b(F_v, (\widetilde{X}, \widetilde{D}), \varphi^*L) = \operatorname{codim} Z$. In the latter case, we may assume that T is a component of $\bigcap_{i=1}^k D_{v,\beta_i}$ with $k \leqslant r$. Then the strict transforms of the divisors D_{v,β_i} do not meet in $\varphi^{-1}(Z)$, but E and the $r-1$ strict transforms of $D_{v,\beta_2}, \ldots, D_{v,\beta_r}$ intersect. Thus, we conclude that $b(F_v, (\widetilde{X}, \widetilde{D}), \varphi^*L) = r$. Second, if E does appear in the support of $\varphi^*(aL + K_X + D)$, then T does not contain Z, and therefore, T and Z do not meet. This implies that $b(F_v, (\widetilde{X}, \widetilde{D}), \varphi^*L) = b(F_v, (X, D), L)$. □

We will now introduce a version of the b-invariant for rational functions. If f is an arbitrary rational function on X, then for every $\alpha \in \mathcal{A}$, we denote by $d_\alpha(f)$ the coefficient of D_α in the principal divisor $\operatorname{div}(f)$. Let D'' be the sum of boundary components D_α such that D_α does not appear in the support of $aL + K_X + D$ and $d_\alpha(f) \leqslant 0$. We denote by $\mathcal{C}^{\mathrm{an}}_{F_v}(D, L, f)$ the F_v-analytic Clemens complex associated to D'', and we define the b-invariant by

$$b(F_v, (X, D), L, f) = 1 + \dim \mathcal{C}^{\mathrm{an}}_{F_v}(D, L, f).$$

Using the same methods, we obtain the following analog of Lemma 4.1.

Lemma 4.2. *Let X, D, L and f be as above. Let $(\widetilde{X}, \widetilde{D})$ be another pair satisfying the same hypotheses as (X, D), namely: (i) $\widetilde{D}$ is a reduced divisor with strict normal crossings on a smooth proper variety $\widetilde{X}$ over F and (ii) $-K_{\widetilde{X}_v}$ is a linear combination of irreducible components of $\widetilde{D}_v$. Assume that there is a birational morphism $\varphi\colon \widetilde{X} \to X$ with $\varphi^{-1}(D) = \widetilde{D}$ that is an isomorphism outside D. Then*

$$b(F_v, (X, D), L, f) = b(F_v, (\widetilde{X}, \widetilde{D}), \varphi^*L, f \circ \varphi).$$

5. *Geometry of equivariant compactifications of vector groups*

The geometry of vector group compactifications is worked out in [**43**], where equivariant compactifications of a vector group on $\mathbb{P}^n$ are classified. Surprisingly, there is more than one such compactification. There are classification results of equivariant compactifications that are del Pezzo surfaces and Fano 3-folds [**31, 32, 44**], but equivariant compactifications of vector groups need not be Mori dream spaces. Indeed, blow-ups of the standard equivariant compactification on $\mathbb{P}^n$ along a smooth center on the boundary hyperplane inherit the group compactification structure, so examples with a Cox ring that is not finitely generated can be constructed by blowing up suitable centers (see [**42**, Example 2.17]). This feature makes equivariant compactifications of vector groups difficult to study via universal torsors, showing once more the power of the height zeta function method. In addition, equivariant compactifications of vector groups admit deformations, whereas equivariant compactifications involving reductive groups typically do not; this feature also makes the former class of compactifications interesting objects from a geometric point of view.

We now recall some basic facts on the geometry of equivariant compactifications of vector groups from [**23, 43**]. Let X be a smooth equivariant compactification of $G = \mathbb{G}_a^n$ defined over a field F of characteristic 0. By definition, X contains G as a dense Zariski open, and its complement $D = X \setminus G$ is divisorial, that is, it is a union of prime divisors:

$$D = \bigcup_{\alpha \in \mathcal{A}} D_\alpha.$$

The irreducible divisors D_α need not be geometrically irreducible, so we also consider the decomposition of $\bar{D}$ into irreducible components:

$$\bar{D} = \bigcup_{\alpha \in \bar{\mathcal{A}}} \bar{D}_\alpha.$$

There is a natural action of the Galois group $\Gamma = \mathrm{Gal}(\bar{F}/F)$ on the index set $\bar{\mathcal{A}}$, and Galois orbits are in one-to-one correspondence with elements of $\mathcal{A}$.

5.1. *Picard groups and the anticanonical class*

PROPOSITION 5.1 [**23**, Proposition 1.1]. *With the above notation, the following hold.*

(i) *There are natural isomorphisms of Galois modules*

$$\mathrm{Pic}(\bar{X}) = \bigoplus_{\alpha \in \bar{\mathcal{A}}} \mathbb{Z}\bar{D}_\alpha, \quad \mathrm{Eff}^1(\bar{X}) = \bigoplus_{\alpha \in \bar{\mathcal{A}}} \mathbb{R}_{\geqslant 0}\bar{D}_\alpha,$$

where $\mathrm{Eff}^1(\bar{X})$ *is the cone of effective divisors on* $\bar{X}$*.*

(ii) *By taking* Γ*-invariant parts, we have*

$$\mathrm{Pic}(X) = \bigoplus_{\alpha \in \mathcal{A}} \mathbb{Z}D_\alpha, \quad \mathrm{Eff}^1(X) = \bigoplus_{\alpha \in \mathcal{A}} \mathbb{R}_{\geqslant 0}D_\alpha,$$

where $\mathrm{Eff}^1(X)$ *is the cone of* Γ*-invariant effective divisors on* X*.*

Let f be a non-zero linear form on $G = \mathbb{G}_a^n$, defined over F. Considering f as an element of the function field $F(X)$, we can write $\mathrm{div}(f)$ uniquely as

$$\mathrm{div}(f) = E(f) - \sum_{\alpha \in \mathcal{A}} d_\alpha(f) D_\alpha,$$

where $E(f)$ is the hyperplane along which f vanishes in G, and the $d_\alpha(f)$ are integers.

Proposition 5.2 [23, Lemma 1.4], [26, Before Lemma 3.4.1]. *We have $d_\alpha(f) \geqslant 0$ for all $\alpha \in \mathcal{A}$, and the set of integral vectors*

$$\{(d_\alpha(f))_{\alpha\in\mathcal{A}} \mid f \text{ is a non-zero linear form on } G\}$$

is finite.

Finally, the anticanonical divisor turns out to be linearly equivalent to an integral linear combination of boundary components: we have $-K_X \sim \sum_{\alpha\in\mathcal{A}} \rho_\alpha D_\alpha$ for certain integers ρ_α, and by [23, Lemma 2.4], we know that $\rho_\alpha \geqslant 2$ for all α.

Remark 5.3. With the above notation, if $(\epsilon_\alpha)_{\alpha\in\mathcal{A}}$ is any vector of weights chosen from the allowed set $\mathfrak{W} = \{1 - \frac{1}{m} \,|\, m \in \mathbb{Z}_{\geqslant 1}\} \cup \{1\}$, the orbifold anticanonical divisor $-(K_X + D_\epsilon)$ of the Campana orbifold (X, D_ϵ) is automatically big. This follows from the fact that the cone of big divisors is the interior of the pseudo-effective cone, together with Proposition 5.1.

5.2. *Harmonic analysis on vector groups*

In this section, we recall some of the basic elements of harmonic analysis on adelic vector groups as developed in [67]. Let $G = \mathbb{G}_a^n$.

For any non-archimedean place v such that the completion F_v is a finite extension of $\mathbb{Q}_p$, we define the local additive unitary character by

$$\psi_v(x) := \exp(2\pi i \cdot \mathrm{Tr}_{F_v/\mathbb{Q}_p}(x)).$$

When v is an archimedean place, we define the local additive character by

$$\psi_v(x) := \exp(-2\pi i \cdot \mathrm{Tr}_{F_v/\mathbb{R}}(x)).$$

The Euler product $\psi := \prod_v \psi_v$ is an automorphic character of $\mathbb{A}_F$.

Lemma 5.4 [23, Lemma 10.3], [26, Lemma 2.3.1]. *Let $v \in \Omega_F^{<\infty}$ and let us fix integers $d \geqslant 0$ and $i \geqslant 1$. Let j be an integer and $c = \log_{q_v} \#(\mathcal{O}_v/(d\mathfrak{D}))$. If $j = 0$, we have*

$$\frac{1}{\mu(\mathcal{O}_v)} \int_{\mathcal{O}_v^\times} \psi_v(\pi_v^{-id+j} x_v^d)\,\mathrm{d}x_v = \begin{cases} (1 - q_v^{-1}) & \text{if } d = 0, \\ -q_v^{-1} & \text{if } i = d = 1, \\ 0 & \text{otherwise.} \end{cases}$$

If $j \neq 0$ the integral above vanishes whenever $id - j \geqslant c + 2$.

To each adelic point $\mathbf{a} \in G(\mathbb{A}_F)$, we associate the linear functional $f_\mathbf{a} : G(\mathbb{A}_F) \to \mathbb{A}_F$ that sends an element $\mathbf{x}$ to the inner product $\mathbf{a}\cdot\mathbf{x}$, which is the sum of the coordinatewise products in the adelic ring. The composition $\psi_\mathbf{a} = \psi \circ f_\mathbf{a}$ defines a Pontryagin duality

$$G(\mathbb{A}_F) \to G(\mathbb{A}_F)^\vee, \quad G(F) \to (G(\mathbb{A}_F)/G(F))^\vee.$$

(Note that $G(F)$ is discrete and cocompact in $G(\mathbb{A}_F)$.)

Given an integrable function Φ on $G(\mathbb{A}_F)$, we define its Fourier transform by

$$\widehat{\Phi}(\mathbf{a}) = \int_{G(\mathbb{A}_F)} \Phi(\mathbf{x})\psi_\mathbf{a}(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Theorem 5.5 [67, Theorem 4.2.1], Poisson summation. *Let Φ be a continuous function on $G(\mathbb{A}_F)$. Assume that the series*

$$\sum_{\mathbf{x}\in G(F)} \Phi(\mathbf{x} + \mathbf{b})$$

converges absolutely and uniformly when $\mathbf{b}$ *belongs to a fundamental domain for the quotient* $G(\mathbb{A}_F)/G(F)$*, and that the infinite sum*

$$\sum_{\mathbf{a}\in G(F)} \widehat{\Phi}(\mathbf{a})$$

converges absolutely. Then we have

$$\sum_{\mathbf{x}\in G(F)} \Phi(\mathbf{x}) = \sum_{\mathbf{a}\in G(F)} \widehat{\Phi}(\mathbf{a}).$$

6. *Height zeta functions*

In this section, we will establish some basic properties of height zeta functions. Let $G = \mathbb{G}_a^n$ and let X be a smooth equivariant compactification of G defined over a number field F. We assume that the boundary $D = X \setminus G$ is a strict normal crossings divisor on X. Let $S \subseteq \Omega_F$ be a finite set containing all archimedean places, such that there exists a good integral model $(\mathcal{X}, \mathcal{D})$ of (X, D) over Spec $\mathcal{O}_{F,S}$ as in § 3.2.

6.1. *Height functions*

We first recall some of the basic properties of height functions, referring to [**24**, § 2] for more details. Let us consider the decomposition of the boundary into irreducible components:

$$D = \bigcup_{\alpha\in\mathcal{A}} D_\alpha.$$

For each $\alpha \in \mathcal{A}$, we fix a smooth adelic metrization on the line bundle $\mathcal{O}(D_\alpha)$, and let f_α be a section corresponding to D_α. For each place v, we define the local height pairing by

$$\mathsf{H}_v \colon G(F_v) \times \mathrm{Pic}(X)_{\mathbb{C}} \to \mathbb{C}^\times, \quad \left(\mathbf{x}, \sum_{\alpha\in\mathcal{A}} s_\alpha D_\alpha\right) \mapsto \prod_{\alpha\in\mathcal{A}} \|\mathsf{f}_\alpha(\mathbf{x})\|_v^{-s_\alpha}.$$

This pairing varies linearly on the factor $\mathrm{Pic}(X)_{\mathbb{C}}$ and continuously on the factor $G(F_v)$. We define the global height pairing H as the product of the local height pairings

$$\mathsf{H} = \prod_{v\in\Omega_F} \mathsf{H}_v \colon G(\mathbb{A}_F) \times \mathrm{Pic}(X)_{\mathbb{C}} \to \mathbb{C}^\times.$$

Again, this pairing varies continuously on the first factor and linearly on the second factor. The following lemma plays a crucial rôle in the analysis of height zeta functions in general.

LEMMA 6.1 [**23**, Proposition 4.2]. *For each non-archimedean place* $v \in \Omega_F$*, there exists a compact open subgroup* $K_v \subset G(\mathcal{O}_v)$ *such that* H_v *is* K_v*-invariant, that is, such that for any* $\mathbf{s} \in \mathrm{Pic}(X)_{\mathbb{C}}$*, any* $g_v \in G(F_v) \subset X(F_v)$ *and any* $k_v \in K_v$*, we have*

$$\mathsf{H}_v(g_v + k_v, \mathbf{s}) = \mathsf{H}_v(g_v, \mathbf{s}).$$

Moreover, if

(i) *the metric* $\|\cdot\|_v$ *is induced by our integral model* $(\mathcal{X}, \mathcal{D})$*,*
(ii) *our* $\mathcal{O}_v$*-model* $(\mathcal{X} \otimes_{\mathcal{O}_{F,S}} \mathcal{O}_v, \mathcal{D} \otimes_{\mathcal{O}_{F,S}} \mathcal{O}_v)$ *is a smooth, projective, and relative strict normal crossings pair over* $\mathcal{O}_v$ [**45**, § 2]*, and it comes equipped with an action of the* $\mathcal{O}_v$*-group scheme* $\mathbb{G}^n_{a,\mathcal{O}_v}$ *extending the given action of* G *on* X*, and if*
(iii) *the unique linearization on* $\mathcal{O}(D_\alpha)$ *extends to* $\mathcal{O}(\mathcal{D}_\alpha)$ *for every* $\alpha \in \mathcal{A}$*,*

then we can choose $K_v = G(\mathcal{O}_v)$*.*

In particular, for all but finitely many places $v \in \Omega_F$, we may simply take $K_v = G(\mathcal{O}_v)$.

6.2. *Intersection multiplicities*

With the notation introduced above, let $\mathcal{D} = \sum_{\alpha \in \mathcal{A}} \mathcal{D}_\alpha$, where $\mathcal{D}_\alpha$ denotes the closure of D_α in $\mathcal{X}$ for all α. Moreover, let $\epsilon = (\epsilon_\alpha)_{\alpha \in \mathcal{A}}$ be a weight vector as in § 3.1. Our object of study is

$$G(F)_\epsilon = G(F) \cap (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S}),$$

the set of F-rational points in G which extend to Campana $\mathcal{O}_{F,S}$-points on $(\mathcal{X}, \mathcal{D}_\varepsilon)$. For any $v \notin S$, the functions $n_v(\mathcal{D}_\alpha, \cdot)$ defined in § 3.2 extend naturally from $G(F)$ to $G(F_v)$. Hence we may define the analogous sets

$$G(F_v)_\epsilon = G(F_v) \cap (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_v).$$

For $v \notin S$, we denote by $\delta_{\epsilon,v}$ the indicator function detecting whether or not a given point in $G(F_v)$ belongs to the subset $G(F_v)_\epsilon$. For $v \in S$, we simply set $\delta_{\epsilon,v} = 1$. Let $\delta_\epsilon = \prod_{v \in \Omega_F} \delta_{\epsilon,v}$.

For $v \notin S$, we have the reduction map

$$\eta_v \colon G(F_v) \subset \mathcal{X}(\mathcal{O}_v) \to \mathcal{X}(k_v).$$

Given $\mathbf{x} \in G(F_v)$ and $\alpha \in \mathcal{A}$, we have $n_v(\mathcal{D}_\alpha, \mathbf{x}) > 0$ if and only if $\eta_v(\mathbf{x}) \in \mathcal{D}_\alpha(k_v)$. Let

$$D_\alpha \otimes_F F_v = \bigcup_{\beta \in \mathcal{A}_v(\alpha)} D_{v,\beta}$$

be the decomposition of $D_\alpha \otimes_F F_v$ into irreducible components, and let $\mathcal{D}_{v,\beta}$ be the Zariski closure of $D_{v,\beta}$ in $\mathcal{X}$.

Suppose that our integral model has good reduction at v in the sense of Lemma 6.1, conditions (ii) and (iii). Since $D_{v,\beta}$ is smooth, if $y \in \mathcal{D}_{v,\beta}(k_v)$, then Hensel's lemma implies that $D_{v,\beta}$ has an F_v-point, and therefore, it is geometrically irreducible over F_v. Using a standard argument in Arakelov geometry (see, for example, [**58**, Theorem 2.13] and its proof), we see that there exist analytic local coordinates $(z_1, \ldots, z_n)$ on $\eta_v^{-1}(y)$ mapping to $\mathbb{A}^n_{F_v}$ such that the following conditions are satisfied:

- these local coordinates induce an analytic isomorphism $\eta_v^{-1}(y) \cong \mathfrak{m}_v^n$;
- $\eta_v^{-1}(y) \cap D_{v,\beta}(F_v)$ is defined by $z_1 = 0$.

With this notation, we see that for any $\mathbf{x} \in \eta_v^{-1}(y)$, we have $n_v(\mathcal{D}_{v,\beta}, \mathbf{x}) = v(z_1(\mathbf{x}))$. Hence, the function $n_v(\mathcal{D}_\epsilon, \cdot) \colon G(F_v) \to \mathbb{Z}_{\geqslant 0}$ is locally constant for every $v \notin S$. Moreover, since condition (ii) in Lemma 6.1 is satisfied, the group action of $G(\mathcal{O}_v)$ preserves $v(z_1(\mathbf{x}))$ so that $n_v(\mathcal{D}_{v,\beta}, \mathbf{x})$ is invariant under the action of $G(\mathcal{O}_v)$.

Even if our integral model has bad reduction at v, then one can define

$$H_{\mathcal{D}_{v,\beta}}(\mathbf{x}) = q_v^{n_v(\mathcal{D}_{v,\beta}, \mathbf{x})},$$

and one may interpret this as a local height function of $\mathcal{D}_{v,\beta}$ associated to this particular model $\mathcal{X}_v \to \operatorname{Spec} \mathcal{O}_v$. Thus from Lemma 6.1 we deduce the following result.

Lemma 6.2. *For each non-archimedean place $v \in \Omega_F$, there exists a compact open subgroup $K_v \subset G(\mathcal{O}_v)$ such that the indicator function $\delta_{\epsilon,v}$ is K_v-invariant. If we moreover assume that v satisfies conditions* (ii) *and* (iii) *in Lemma 6.1, then we can take $K_v = G(\mathcal{O}_v)$.*

For each non-archimedean place v, we denote by K_v a maximal compact open subgroup of $G(\mathcal{O}_v)$ satisfying the conclusions of Lemma 6.1 and Lemma 6.2, and we denote

$$\mathbf{K} = \prod_{v \in \Omega_F^{<\infty}} K_v.$$

Our discussion shows that both $\mathsf{H}(\cdot, \mathbf{s})$ and δ_ϵ are $\mathbf{K}$-invariant.

6.3. *Height zeta functions*

To understand the asymptotic formula for the counting function of Campana points of bounded height, we introduce the height zeta function:

$$\mathsf{Z}_\epsilon(\mathbf{s}) = \sum_{\mathbf{x} \in G(F)_\epsilon} \mathsf{H}(\mathbf{x}, \mathbf{s})^{-1} = \sum_{\mathbf{x} \in G(F)} \mathsf{H}(\mathbf{x}, \mathbf{s})^{-1} \delta_\epsilon(\mathbf{x}).$$

The proof of [**23**, Proposition 4.5] shows that $\mathsf{Z}_\epsilon(\mathbf{s})$ is holomorphic when $\Re(\mathbf{s}) \gg 0$. The existence of a meromorphic continuation of this zeta function, together with a standard Tauberian theorem, yields a proof of the desired asymptotic formula. We therefore consider the Fourier transform

$$\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}) = \int_{G(\mathbb{A}_F)} \mathsf{H}(\mathbf{x}, \mathbf{s})^{-1}\, \delta_\epsilon(\mathbf{x})\, \psi_{\mathbf{a}}(\mathbf{x})\, \mathrm{d}\mathbf{x},$$

in hopes of using the Poisson summation formula (Theorem 5.5)

$$\sum_{\mathbf{x} \in G(F)} \mathsf{H}(\mathbf{x}, \mathbf{s})^{-1}\, \delta_\epsilon(\mathbf{x}) = \sum_{\mathbf{a} \in G(F)} \widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s})$$

to obtain the desired meromorphic continuation of $\mathsf{Z}_\epsilon(\mathbf{s})$. The first two of the three conditions in Theorem 5.5 follow from the proof of [**23**, Lemma 5.2], assuming that $\Re(\mathbf{s})$ is sufficiently large. To verify the third condition, we recall the following result.

PROPOSITION 6.3 [**23**, Proposition 5.3]. *With the notation introduced above, for all characters $\psi_{\mathbf{a}}$ that are non-trivial on $\mathbf{K}$ and for all $\mathbf{s}$ such that $\mathsf{H}(\cdot, \mathbf{s})^{-1}$ is integrable, we have $\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}) = 0$.*

Let $\Lambda_X \subset G(F)$ be the set of $\mathbf{a}$ such that $\psi_{\mathbf{a}}$ is trivial on $\mathbf{K}$. Then Λ_X is a sub-$\mathcal{O}_F$-module of $G(F)$ of full rank n. Indeed, Λ_X is a sub-$\mathcal{O}_F$-module commensurable with $G(\mathcal{O}_F)$. To verify the third condition in Theorem 5.5, we will prove in § 9 that the sum

$$\sum_{\mathbf{a} \in \Lambda_X} \widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s})$$

is absolutely convergent whenever $\Re(\mathbf{s}) \gg 0$. Once this is established, we obtain

$$\mathsf{Z}_\epsilon(\mathbf{s}) = \sum_{\mathbf{a} \in \Lambda_X} \widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}), \tag{6.1}$$

for $\Re(\mathbf{s}) \gg 0$.

Interlude I: dimension 1

Let us first make our analysis explicit for $\mathbb{P}^1$ over $\mathbb{Q}$, considered as the natural equivariant compactification of $G = \mathbb{G}_a = \mathbb{A}^1$, with boundary $D = (1:0)$. We fix the standard integral models for $\mathbb{P}^1$ as well as D. Given $\epsilon \in \mathfrak{W}$, we consider the problem of counting Campana $\mathbb{Z}$-points on $(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{D}_\epsilon)$. Note that if $\epsilon < 1$, then $x \in G(\mathbb{Q}) = \mathbb{Q}$ is a Campana $\mathbb{Z}$-point if and only if the denominator of x is m-full, where $m = 1/(1-\epsilon)$; this means that any prime dividing the denominator of x occurs with exponent at least m in the prime factorization. If, on the other hand, $\epsilon = 1$, then x is a Campana $\mathbb{Z}$-point if and only if $x \in \mathbb{Z}$. Since the latter case is trivial, we will assume from now on that $\epsilon < 1$.

We fix a finite set of places S. Going back to the notation introduced in § 6, we see that we can take $\mathbf{K} = \prod_{p \text{ prime}} G(\mathbb{Z}_p)$ in this case, so that $\Lambda_X = \mathbb{Z}$. This yields

$$\mathsf{Z}_\epsilon(s) = \sum_{n\in\mathbb{Z}} \widehat{\mathsf{H}}_\epsilon(n,s).$$

We would like to compute $\widehat{\mathsf{H}}_\epsilon(n,s)$ explicitly. Using Fubini's theorem, we have

$$\widehat{\mathsf{H}}_\epsilon(n,s) = \int_{\mathbb{A}_F} \mathsf{H}(x)^{-s}\,\delta_\epsilon(x)\,\psi(nx)\,\mathrm{d}x = \prod_{v\in\Omega_\mathbb{Q}} \int_{\mathbb{Q}_v} \mathsf{H}_v(x_v)^{-s}\,\delta_{\epsilon,v}(x_v)\,\psi_v(nx_v)\,\mathrm{d}x_v.$$

Note that the inner function of each Euler factor is trivial on $\mathbb{Z}_p$ for almost all places p.

We fix metrizations as follows:

$$\begin{aligned} \mathsf{H}_v(x_v) &= \max\{1, |x_v|_v\} && \text{if } v \text{ is non-archimedean,}\\ \mathsf{H}_\infty(x_v) &= \sqrt{1+|x_v|_v^2} && \text{if } v \text{ is archimedean.}\end{aligned}$$

The trivial character. Here we compute $\widehat{\mathsf{H}}_\epsilon(0,s)$. For any prime $p \notin S$, we have

$$\widehat{\mathsf{H}}_{\epsilon,p}(0,s) = \int_{\mathbb{Q}_p} \max\{1,|x_p|_p\}^{-s}\,\delta_{\epsilon,p}(x_p)\,\mathrm{d}x_p = 1 + \left(1-\frac{1}{p}\right)\frac{p^{-(s-1)m}}{1-p^{-(s-1)}},$$

where $m = 1/(1-\epsilon)$. On the other hand, if $p \in S$ then

$$\widehat{\mathsf{H}}_{\epsilon,p}(0,s) = 1 + \left(1-\frac{1}{p}\right)\frac{1}{1-p^{-(s-1)}}.$$

Furthermore, we have

$$\widehat{\mathsf{H}}_{\epsilon,\infty}(0,s) = \frac{\Gamma((s-1)/2)}{\Gamma(s/2)}.$$

It follows that the rightmost pole of $\widehat{\mathsf{H}}_\epsilon(0,s)$ is at $s = 1 + 1/m = 2-\epsilon$, and that it has order 1.

Non-trivial characters. Let n be a non-zero integer. Our aim is to understand

$$\widehat{\mathsf{H}}_\epsilon(n,s) = \prod_{v\in\Omega_\mathbb{Q}} \widehat{\mathsf{H}}_{\epsilon,v}(n,s),$$

where the local factors are given by

$$\int_{\mathbb{Q}_v} \mathsf{H}_v(x_v)^{-s}\,\delta_{\epsilon,v}(x_v)\,\psi_v(nx_v)\,\mathrm{d}x_v.$$

Suppose first that $p \notin S$ and $p \nmid n$. The local factor then reduces to

$$\int_{\mathbb{Q}_p} \mathsf{H}_p(x_p)^{-s}\,\delta_{\epsilon,p}(x_p)\,\psi_p(x_p)\,\mathrm{d}x_p,$$

which equals

$$1 + \sum_{i=m}^{\infty}\left(1-\frac{1}{p}\right)p^{-i(s-1)}\int_{\mathbb{Z}_p^\times} \psi_p(p^{-i}x_p)\,\mathrm{d}x_p = \begin{cases} 1 & \text{if } \epsilon \neq 0,\\ 1 - \left(1-\frac{1}{p}\right)p^{-s} & \text{if } \epsilon = 0.\end{cases}$$

Let us now assume that $p \notin S$ and $p \mid n$, and let us denote the p-adic valuation of n by k. In this case, the local factor becomes

$$\begin{aligned}\widehat{\mathsf{H}}_{\epsilon,p}(n,s) &= 1 + \sum_{i=m}^{\infty} \left(1 - \frac{1}{p}\right) p^{-i(s-1)} \int_{\mathbb{Z}_p^\times} \psi_p(p^{-i+k} x_p)\, \mathrm{d}x_p \\ &= \begin{cases} 1 & \text{if } m \geqslant k+2, \\ 1 - \sum_{i=m}^{k+1} \left(1 - \frac{1}{p}\right) p^{-i(s-1)} \int_{\mathbb{Z}_p^\times} \psi_p(p^{-i+k} x_p)\, \mathrm{d}x_p & \text{if } m \leqslant k+1. \end{cases}\end{aligned}$$

When $p \in S$, we recover the formula above for $\epsilon = 0$.

Using these explicit formulae, we obtain the following lemma.

Lemma. *Let p be prime. The function $s \mapsto \widehat{\mathsf{H}}_{\epsilon,p}(n,s)$ is holomorphic everywhere. Moreover, the product $\prod_{p \text{ prime}} \widehat{\mathsf{H}}_{\epsilon,p}(n,s)$ is holomorphic for $\Re(s) > 1 - \epsilon$, and there exists positive constants ℓ and C such that*

$$\left| \prod_{p \text{ prime}} \widehat{\mathsf{H}}_{\epsilon,p}(n,s) \right| < C(1 + |s| + |n|)^{\ell}$$

for any s such that $\Re(s) > 1 - \epsilon$.

Finally, we analyze the archimedean place.

Lemma. *The function $s \mapsto \widehat{\mathsf{H}}_{\epsilon,\infty}(n,s)$ is holomorphic everywhere. Moreover, for any integer N, there exists positive constants ℓ and C such that*

$$\left| \widehat{\mathsf{H}}_{\epsilon,\infty}(n,s) \right| < C \frac{1 + |s|^{\ell}}{(1 + |n|)^N}$$

for all s.

Conclusion. Putting all the information together, we obtain that $\mathsf{Z}_\epsilon(s)$ has a unique pole located at $s = 1 + 1/m = 2 - \epsilon$, contributed by the trivial character. Applying a Tauberian theorem (see, for example, [68, II.7, Theorem 15]), for the line bundle $L = \mathcal{O}(1)$ metrized as above, we obtain

$$\mathsf{N}(G(\mathbb{Q})_\epsilon, \mathcal{L}, T) \sim c T^{1+1/m}$$

for some $c > 0$.

7. *Height integrals I: the trivial character*

In this section, we resume our general analysis and study the height integral

$$\widehat{\mathsf{H}}_\epsilon(0, \mathbf{s}) = \prod_{v \in \Omega_F} \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \mathrm{d}\mathbf{x}_v =: \prod_{v \in \Omega_F} \widehat{\mathsf{H}}_{\epsilon,v}(0, \mathbf{s}).$$

Note that the inner function of each Euler factor is trivial on $G(\mathcal{O}_v)$ for almost all places v. We begin by setting up some necessary notation. Each $c \in \mathbb{R}$ gives rise to a tube domain

$$\mathsf{T}_{>c} = \{\mathbf{s} \in \operatorname{Pic}(X)_{\mathbb{C}} : \Re(s_\alpha) > \rho_\alpha - \epsilon_\alpha + c \ \text{ for all } \alpha \in \mathcal{A}\},$$

where $(\rho_\alpha)_{\alpha\in\mathcal{A}}$ is the integer vector given by

$$-K_X \sim \sum_{\alpha\in\mathcal{A}} \rho_\alpha D_\alpha;$$

recall that $\rho_\alpha \geqslant 2$ for all $\alpha \in \mathcal{A}$.

We write

$$D \otimes_F F_v = \bigcup_{\beta\in\mathcal{A}_v} D_{v,\beta},$$

where the $D_{v,\beta}$ are irreducible components, and we write

$$D_\alpha \otimes_F F_v = \bigcup_{\beta\in\mathcal{A}_v(\alpha)} D_{v,\beta}$$

for an analogous decomposition of $D_\alpha \otimes_F F_v$ into irreducible components.

Given $\beta \in \mathcal{A}_v$, let us denote by $F_{v,\beta}$ the field of definition for one of the geometric irreducible components of $D_{v,\beta}$, that is, the algebraic closure of F_v inside the function field of $D_{v,\beta}$, and by $f_{v,\beta}$ the extension degree $[F_{v,\beta} : F_v]$.

Finally, for any subset $B \subseteq \mathcal{A}_v$, we define

$$D_{v,B} := \bigcap_{\beta\in B} D_{v,\beta}, \qquad D^\circ_{v,B} := D_{v,B} \setminus \bigcup_{B\subsetneq B'\subset\mathcal{A}_v} \left(\bigcap_{\beta\in B'} D_{v,\beta}\right),$$

with the convention that $D_{v,\emptyset} = X_{F_v}$ and $D^\circ_{v,\emptyset} = G_{F_v}$. The collection $(D^\circ_{v,B})_{B\subseteq\mathcal{A}_v}$ yields a stratification of the F_v-variety $X \otimes_F F_v$ into finitely many locally closed subsets. If $v \notin S$, then we denote by $\mathcal{D}_{v,B}$ the Zariski closure of $D_{v,B}$ in $\mathcal{X} \otimes_{\mathcal{O}_{F,S}} \mathcal{O}_v$. We define $\mathcal{D}^\circ_{v,B}$ as above.

7.1. *Places away from S*

We will now study the basic properties of

$$\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s}) = \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\mathrm{d}\mathbf{x}_v$$

in the case that $v \notin S$.

7.1.1. *Places of good reduction.* Here we assume that our model

$$(\mathcal{X}_v = \mathcal{X} \otimes_{\mathcal{O}_{F,S}} \mathcal{O}_v, \mathcal{D} \otimes_{\mathcal{O}_{F,S}} \mathcal{O}_v)$$

has good reduction over $\mathcal{O}_v$ in the sense of Lemma 6.1, conditions (i) and (ii). In this setting, we have the following formula which resembles Denef's formula in [**24**, Proposition 4.5].

Theorem 7.1. *We have*

$$\frac{1}{\mu_v(\mathcal{O}_v)^n}\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s}) = \sum_{B\subset\mathcal{A}_v} \frac{\#\mathcal{D}^\circ_{v,B}(k_v)}{q_v^{n-\#B}} \prod_{\beta\in B}\left(1-\frac{1}{q_v}\right)\frac{q_v^{-m_{\alpha(\beta)}(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)}}{1-q_v^{-(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)}}. \tag{7.1}$$

Proof. To avoid clutter, we first assume that $\mu_v(\mathcal{O}_v) = 1$. Set $\boldsymbol{\rho} = (\rho_\alpha)_{\alpha\in\mathcal{A}}$. Let ω be a gauge form on G, that is, a nowhere vanishing differential form of top degree. Considering ω as a rational section of $\mathcal{O}(K_X)$ equipped with the adelic metrization fixed in the previous section, we have the equality

$$\|\omega\|_v = \mathsf{H}_v(\mathbf{x}_v,\boldsymbol{\rho}).$$

Writing

$$\mathrm{d}\tau = \frac{\mathrm{d}\mathbf{x}_v}{\|\omega\|_v}$$

for the corresponding Tamagawa measure, we see that

$$\begin{aligned}\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s}) &= \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s})^{-1}\,\mathsf{H}_v(\mathbf{x}_v,\boldsymbol{\rho})\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\frac{\mathrm{d}\mathbf{x}_v}{\|\omega\|_v}\\ &= \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s}-\boldsymbol{\rho})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\mathrm{d}\tau.\end{aligned}$$

Breaking up this integral over the fibres of the reduction map $\eta_v \colon G(F_v) \to \mathcal{X}(k_v)$, we obtain

$$\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s}) = \sum_{B\subset\mathcal{A}_v}\ \sum_{y\in\mathcal{D}^\circ_{v,B}(k_v)} \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s}-\boldsymbol{\rho})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\mathrm{d}\tau.$$

We now compute the inner integral

$$\int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s}-\boldsymbol{\rho})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\mathrm{d}\tau. \tag{7.2}$$

If $B=\emptyset$, then there is a measure-preserving analytic isomorphism $\eta_v^{-1}(y)\cong\mathfrak{m}_v^n$. Since any $\mathbf{x}_v\in\eta_v^{-1}(y)$ is integral with respect to $\mathcal{D}$, we have

$$\mathsf{H}_v(\mathbf{x}_v,\mathbf{s}-\boldsymbol{\rho}) = \delta_{\epsilon,v}(\mathbf{x}_v) = 1$$

for all such $\mathbf{x}_v$, so that (7.2) simply evaluates to $1/q_v^n$.

If $B\neq\emptyset$, then every $\beta\in B$ lies above a unique $\alpha(\beta)\in\mathcal{A}$. If $\mathcal{D}^\circ_{v,B}(k_v)\neq\emptyset$, then $\mathcal{D}_{v,\beta}(k_v)\neq\emptyset$ for all $\beta\in B$. Using Hensel's lemma, we deduce that $D_{v,\beta}$ has an F_v-rational point, and hence is geometrically irreducible; in particular, $F_{v,\beta}=F_v$ for all $\beta\in B$. Writing $B=\{\beta_1,\dots,\beta_\ell\}$ and $\alpha_i=\alpha(\beta_i)$ for simplicity, we see as in § 6.2 that there exist analytic local coordinates $(z_1,\dots,z_n)$ on $\eta_v^{-1}(y)$ inducing a measure-preserving analytic isomorphism $\eta_v^{-1}(y)\cong\mathfrak{m}_v^n$, such that $D_{v,\beta_i}(F_v)\cap\eta_v^{-1}(y)$ is given by $z_i=0$, for $i=1,\dots,\ell$.

The integral (7.2) can now be rewritten as

$$\int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s}-\boldsymbol{\rho})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\mathrm{d}\tau = \int_{\mathfrak{m}_v^n}\prod_{i=1}^{\ell}\left(|z_i|_v^{s_{\alpha_i}-\rho_{\alpha_i}}\,\delta_{\epsilon,v}(z_i)\,\mathrm{d}z_i\right)\prod_{i>\ell}\mathrm{d}z_i,$$

where

$$\delta_{\epsilon,v}(z_i)=1 \iff \epsilon_{\alpha_i}\neq 1 \text{ and } \mathrm{val}_v(z_i)\geqslant m_i := \frac{1}{1-\epsilon_{\alpha_i}}$$

by definition of $\delta_{\epsilon,v}$ (see § 6.2).

Therefore, if $\Re(s_{\alpha_i})-\rho_{\alpha_i}+1>0$ for all $i\in\{1,\dots,\ell\}$, we obtain

$$\begin{aligned}\int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s}-\boldsymbol{\rho})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\mathrm{d}\tau &= \frac{1}{q_v^{n-\ell}}\prod_{i=1}^{\ell}\sum_{j=m_i}^{\infty} q_v^{-j(s_{\alpha_i}-\rho_{\alpha_i})}\cdot\mathrm{Vol}(\pi_v^j\mathcal{O}_v^\times)\\ &= \frac{1}{q_v^{n-\ell}}\prod_{i=1}^{\ell}\sum_{j=m_i}^{\infty} q_v^{-j(s_{\alpha_i}-\rho_{\alpha_i})}\cdot q_v^{-j}\left(1-\frac{1}{q_v}\right)\\ &= \frac{1}{q_v^{n-\ell}}\prod_{i=1}^{\ell}\left(1-\frac{1}{q_v}\right)\frac{q_v^{-m_i(s_{\alpha_i}-\rho_{\alpha_i}+1)}}{1-q_v^{-(s_{\alpha_i}-\rho_{\alpha_i}+1)}},\end{aligned}$$

where π_v denotes a choice of generator for $\mathfrak{m}_v$.

Summing the contributions coming from different subsets of $\mathcal{A}_v$, we obtain the equality

$$\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s}) = \sum_{B\subset\mathcal{A}_v} \frac{\#\mathcal{D}^\circ_{v,B}(k_v)}{q_v^{n-\#B}} \prod_{\beta\in B}\left(1-\frac{1}{q_v}\right)\frac{q_v^{-m_{\alpha(\beta)}(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)}}{1-q_v^{-(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)}}. \tag{7.3}$$

Here we interpret the term $q_v^{-m_{\alpha(\beta)}(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)}$ to be zero whenever $\epsilon_{\alpha(\beta)}=1$.

When $\mu_v(\mathcal{O}_v)\neq 1$, the same arguments show our statement. □

7.1.2. *Places of bad reduction.* Here we still assume that $v\notin S$, but now our model has bad reduction at v, that is, at least one of the assumptions (i) and (ii) of Lemma 6.1 is not satisfied. We have the following proposition.

Proposition 7.2. *The function*

$$\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s}) = \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\mathrm{d}\mathbf{x}_v$$

is holomorphic in $\mathbf{s}$ *whenever* $\Re(s_\alpha)>\rho_\alpha-1$ *for all* $\alpha\in\mathcal{A}$ *such that* $\epsilon_\alpha<1$.

Proof. We observe that an application of [24, Lemma 4.1] with $\Phi=\delta_{\epsilon,v}$ gives holomorphy of $\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s})$ whenever $\Re(s_\alpha)>\rho_\alpha-1$ for all $\alpha\in\mathcal{A}$. Indeed, let ω be a G-invariant top form on G. Then we have

$$\begin{aligned}\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s}) &= \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\mathrm{d}\mathbf{x}_v\\ &= \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\|\omega\|_v\,\frac{\mathrm{d}\mathbf{x}_v}{\|\omega\|_v}\\ &= \int_{X(F_v)} \mathsf{H}_v(\mathbf{x}_v,\mathbf{s}-\boldsymbol{\rho})^{-1}\,\delta_{\epsilon,v}(\mathbf{x}_v)\,\mathrm{d}\tau,\end{aligned}$$

where $\boldsymbol{\rho}=(\rho_\alpha)_{\alpha\in\mathcal{A}}$ and τ is the local Tamagawa measure. Next, recall that

$$\mathsf{H}_v(\mathbf{x}_v,\mathbf{s}-\boldsymbol{\rho})^{-1} = \prod_{\alpha\in\mathcal{A}}\|\mathsf{f}_\alpha(\mathbf{x}_v)\|_v^{s_\alpha-\rho_\alpha},$$

so in the notation of [24, Lemma 4.1], we have

$$\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s}) = \mathscr{I}(\delta_{\epsilon,v};(s_\alpha-\rho_\alpha+1)_{\alpha\in\mathcal{A}}),$$

which is holomophic whenever $\Re(s_\alpha-\rho_\alpha+1)=\Re(s_\alpha)-\rho_\alpha+1>0$ for $\alpha\in\mathcal{A}$. Finally, observe that for all $\alpha\in\mathcal{A}$ such that $\epsilon_\alpha=1$, the set $D_\alpha(F_v)$ is disjoint from the support of $\delta_{\epsilon,v}$; hence, $\|\mathsf{f}_\alpha\|_v$ is a nowhere vanishing bounded function on $X(F_v)_\epsilon$. Thus the integral that defines $\widehat{\mathsf{H}}_{\epsilon,v}(0,\mathbf{s})$ is absolutely convergent also for all $\mathbf{s}$ that satisfy $\Re(s_\alpha)>\rho_\alpha-1$ only for $\alpha\in\mathcal{A}$ such that $\epsilon_\alpha<1$. □

7.2. *Places contained in S*

Assume now that $v\in S$. In this case, $\delta_{\epsilon,v}\equiv 1$ by definition. Therefore the local height integral for Campana points coincides with the usual local height integral, so that we do not need to do anything new.

Proposition 7.3. *The height integral* $\widehat{\mathsf{H}}_v(0,\mathbf{s})$ *is holomorphic when* $\Re(s_\alpha)>\rho_\alpha-1$ *for all* $\alpha\in\mathcal{A}$. *If* $L=\sum_{\alpha\in\mathcal{A}}\lambda_\alpha D_\alpha$ *is a big divisor on* X, *and if*

$$\widetilde{a} := \tilde{a}((X,D_{\mathrm{red}}),L) \quad \textit{and} \quad b := b(F_v,(X,D_{\mathrm{red}}),L)$$

(as in § 4), then the function

$$s \mapsto (\zeta_{F_v}(s-\widetilde{a}))^{-b} \cdot \widehat{\mathsf{H}}_v(0, sL)$$

admits a holomorphic continuation to the domain $\Re(s) > \widetilde{a} - \delta$ for some $\delta > 0$. Moreover, the function $s \mapsto \widehat{\mathsf{H}}_v(0, sL)$ has a pole at $s = \widetilde{a}$ of order b.

Proof. One may apply [**24**, Lemma 4.1, Proposition 4.3], taking $\Phi \equiv 1$ on $X(F_v)$. Note that in [**24**, Proposition 4.3], the main term of the local height integral is formed by the contributions of faces of maximal dimension in the analytic Clemens complex; however, these contribute to the pole at $\widetilde{a}$ all with the same order b. Also note that there is a typo in [**24**, Proposition 4.3]: each product of local zeta functions should be taken over $\alpha \in A$, not $\alpha \in \mathcal{A}$. This means that D_α contains an F_v-point, so one has $F_\alpha = F_v$ for all $\alpha \in A$. □

7.3. *Euler products*

Given $\alpha \in \mathcal{A}$, we denote by F_α the field of definition for one of the geometric irreducible components of D_α; in other words, F_α is the algebraic closure of F in the function field of D_α.

Proposition 7.4. *Let v be a place of F not contained in S and of good reduction for (X, D_ϵ). Let $\alpha \in \mathcal{A}$. Write*

$$D_\alpha \otimes_F F_v = \bigcup_{\beta \in \mathcal{A}_v(\alpha)} D_{v,\beta}$$

for the decomposition of $D_\alpha \otimes_F F_v$ into irreducible components.

(1) *For $\delta > 0$ sufficiently small, the function*

$$\mathbf{s} \mapsto \prod_{\alpha \in \mathcal{A}} \prod_{\beta \in \mathcal{A}_v(\alpha)} \zeta_{F_{v,\beta}}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1}\, \widehat{\mathsf{H}}_{\epsilon,v}(0, \mathbf{s})$$

is holomorphic on $\mathsf{T}_{>-\delta}$. (If $\epsilon_\alpha = 1$, we interpret $\zeta_{F_{v,\beta}}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1}$ to be 1.)

(2) *For $\delta > 0$ sufficiently small, there exists $\delta' > 0$ such that*

$$\prod_{\alpha \in \mathcal{A}} \prod_{\beta \in \mathcal{A}_v(\alpha)} \zeta_{F_{v,\beta}}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1}\, \widehat{\mathsf{H}}_{\epsilon,v}(0, \mathbf{s}) = 1 + O(q_v^{-(1+\delta')}),$$

for any $\mathbf{s} \in \mathsf{T}_{>-\delta}$.

Proof. We may safely assume that $\mu_v(\mathcal{O}_v) = 1$. We analyze the right-hand side of (7.3), separating the analysis into three cases.

- If $B = \emptyset$, then $\#\mathcal{D}^\circ_{v,B}(k_v) = \#G(k_v) = q_v^n$. Therefore the term corresponding to B in the right-hand side of expression (7.3) for $\widehat{\mathsf{H}}_{\epsilon,v}(0, \mathbf{s})$ is simply equal to 1.
- If $B = \{\beta\}$, define $\alpha(\beta) \in \mathcal{A}$ as in § 7.1. If $\mathcal{D}^\circ_{v,B}(k_v) = \emptyset$ or $\epsilon_{\alpha(\beta)} = 1$, then B does not contribute to the right-hand side of (7.3). If, on the other hand, $\mathcal{D}^\circ_{v,B}(k_v) \neq \emptyset$, then $\mathcal{D}_{v,B} \otimes_{\mathcal{O}_v} k_v$ is a geometrically irreducible k_v-variety of dimension $n-1$, so that

$$\#\mathcal{D}^\circ_{v,B}(k_v) = q_v^{n-1} + O(q_v^{n-1-\delta_1})$$

for some $\delta_1 > 0$, which may be chosen independently of β. Therefore by choosing $\delta > 0$ sufficiently small and $\mathbf{s} \in \mathsf{T}_{>-\delta}$, the term corresponding to $B = \{\beta\}$ contributes to the sum in the right-hand side of (7.3) by

$$q_v^{-m_{\alpha(\beta)}(s_{\alpha(\beta)} - \rho_{\alpha(\beta)} + 1)}(1 + O(q_v^{-\delta_2})),$$

for some $\delta_2 > 0$. Since $\delta > 0$, we have

$$\left| q_v^{-m_{\alpha(\beta)}(s_{\alpha(\beta)} - \rho_{\alpha(\beta)} + 1)} \right| \leqslant q_v^{-(1 - m_{\alpha(\beta)}\delta)}$$

whenever $\mathbf{s} \in \mathsf{T}_{>-\delta}$. It follows that if we choose δ sufficiently small and $\mathbf{s} \in \mathsf{T}_{>-\delta}$, then the contribution of the term corresponding to $B = \{\beta\}$ can be rewritten as

$$q_v^{-m_{\alpha(\beta)}(s_{\alpha(\beta)} - \rho_{\alpha(\beta)} + 1)} + O(q_v^{-(1+\delta')})$$

for some $\delta' > 0$.

- Finally, if $\#B \geqslant 2$, then $\#\mathcal{D}^\circ_{v,B}(k_v) = O(q_v^{n - \#B})$. Moreover, the product in the term in the right-hand side of (7.3) corresponding to B is $O(q_v^{-(1+\delta')})$, with δ' as above, assuming that we have chosen $\mathbf{s} \in \mathsf{T}_{>-\delta}$ for $\delta > 0$ sufficiently small. Indeed, each of the factors in the product is bounded from above by $q_v^{-(1-m\delta)}$ for some $m > 0$, as $\mathbf{s} \in \mathsf{T}_{>-\delta}$. There are at least two such factors, so the result is bounded from above by $q_v^{-2(1-m\delta)}$ for some $m > 0$, and hence certainly by $q_v^{-(1+\delta')}$ if δ is chosen small enough.

We conclude that for $\delta > 0$ small enough and $\mathbf{s} \in \mathsf{T}_{>-\delta}$, we have

$$\widehat{\mathsf{H}}_{\epsilon,v}(0, \mathbf{s}) = 1 + \sum_{\alpha \in \mathcal{A}} \sum_{\substack{\beta \in \mathcal{A}_v(\alpha) \\ f_{v,\beta} = 1}} q_v^{-m_\alpha(s_\alpha - \rho_\alpha + 1)} + O(q_v^{-(1+\delta')}),$$

where $f_{v,\beta} = [F_{v,\beta} : F_v]$, and therefore,

$$\widehat{\mathsf{H}}_{\epsilon,v}(0, \mathbf{s}) \prod_{\alpha \in \mathcal{A}} \prod_{\beta \in \mathcal{A}_v(\alpha)} \left(1 - q_v^{-f_{v,\beta} m_\alpha(s_\alpha - \rho_\alpha + 1)}\right) = 1 + O(q_v^{-(1+\delta')}).$$

This implies the proposition. □

Corollary 7.5. *The function*

$$\mathbf{s} \mapsto \left(\prod_{\alpha \in \mathcal{A}} \zeta_{F_\alpha}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1} \right) \prod_{v \notin S} \widehat{\mathsf{H}}_{\epsilon,v}(0, \mathbf{s})$$

is holomorphic on $\mathsf{T}_{>-\delta'}$ *for sufficiently small* $\delta' > 0$.

Proof. This follows immediately from Proposition 7.4 and Proposition 7.2, taking into account the fact that

$$F_\alpha \otimes_F F_v \simeq \prod_{\beta \in \mathcal{A}_v(\alpha)} F_{v,\beta}$$

for all $\alpha \in \mathcal{A}$. □

8. *Height integrals II: nontrivial characters*

In this section, we study the height integral

$$\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}) = \prod_{v \in \Omega_F} \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s})^{-1} \, \delta_{\epsilon,v}(\mathbf{x}_v) \psi_{\mathbf{a},v}(\mathbf{x}_v) \, \mathrm{d}\mathbf{x}_v =: \prod_{v \in \Omega_F} \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}).$$

Note that the inner function of each Euler factor is trivial on $G(\mathcal{O}_v)$ for almost all places v. We introduce some notation. For each $\mathbf{a} \in G(F)$ with $\mathbf{a} \neq 0$, we denote the linear functional $\mathbf{x} \mapsto \mathbf{a} \cdot \mathbf{x}$ by $f_{\mathbf{a}}$, where $\mathbf{a} \cdot \mathbf{x}$ is the standard inner product. Recall from § 5 that

$$\mathrm{div}(f_{\mathbf{a}}) = E(f_{\mathbf{a}}) - \sum_{\alpha \in \mathcal{A}} d_\alpha(f_{\mathbf{a}}) D_\alpha$$

with $d_\alpha(f_{\mathbf{a}}) \geqslant 0$. We define

$$\mathcal{A}^0(\mathbf{a}) = \{\alpha \in \mathcal{A} \mid d_\alpha(f_{\mathbf{a}}) = 0\},$$
$$\mathcal{A}^{\geqslant 1}(\mathbf{a}) = \{\alpha \in \mathcal{A} \mid d_\alpha(f_{\mathbf{a}}) \geqslant 1\}.$$

For any place $v \in \Omega_F$, we define

$$H_v(\mathbf{a}) = \max\{|a_1|_v, \ldots, |a_n|_v\}$$

and for any non-archimedean place v, we take

$$j_v(\mathbf{a}) = \min\{v(a_1), \ldots, v(a_n)\}$$

so that $H_v(\mathbf{a}) = q_v^{-j_v(\mathbf{a})}$. We also define

$$H_{\mathrm{fin}}(\mathbf{a}) = \prod_{v \in \Omega_F^{<\infty}} H_v(\mathbf{a}), \quad H_\infty(\mathbf{a}) = \prod_{v \in \Omega_F^\infty} H_v(\mathbf{a}).$$

Note that we have

$$H_\infty(\mathbf{a}) \gg H_{\mathrm{fin}}(\mathbf{a})^{-1}. \tag{8.1}$$

8.1. *Places away from S*

In this section, we assume that $v \notin S$ and we analyze

$$\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}) = \prod_{v \in \Omega_F} \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\mathbf{x}_v =: \prod_{v \in \Omega_F} \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}).$$

Since $\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}) = 0$ whenever $\mathbf{a} \notin \Lambda_X$ by Proposition 6.3, we may safely assume that $\mathbf{a} \in \Lambda_X$. We separate the analysis into the cases of good reduction and bad reduction.

8.1.1. *Places of good reduction.* We further assume that our model $(\mathcal{X}, \mathcal{D})$ has good reduction at v in the sense of Lemma 6.1, conditions (i) and (ii). We will distinguish two cases, depending on whether $j_v(\mathbf{a}) = 0$ or $j_v(\mathbf{a}) \neq 0$; we start with the former case.

To analyze the integral

$$\widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}) = \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\mathbf{x}_v$$

in the domain $\mathsf{T}_{>-\delta}$, we begin by stratifying $G(F_v)$ by the fibers of the reduction map:

$$\frac{1}{\mu_v(\mathcal{O}_v)^n}\widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}) = \sum_{B \subset \mathcal{A}_v} \sum_{y \in \mathcal{D}^\circ_{v,B}(k_v)} \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s} - \boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau.$$

- If $B = \emptyset$, then the inner sum is 1, since $\eta_v^{-1}(\mathcal{D}^\circ_{v,\emptyset}(k_v)) = G(\mathcal{O}_v)$ and $\mathbf{a} \in \Lambda_X$.
- If $B = \{\beta\}$, we define $\alpha(\beta)$ as in § 7.1. Without loss of generality, we may assume that $D_{v,\beta}$ is geometrically irreducible and that $\epsilon_{\alpha(\beta)} \neq 1$. We distinguish two cases: either $\alpha(\beta) \in \mathcal{A}^0(\mathbf{a})$, or $\alpha(\beta) \in \mathcal{A}^{\geqslant 1}(\mathbf{a})$.

If $\alpha(\beta) \in \mathcal{A}^0(\mathbf{a})$, then the character $\psi_{\mathbf{a},v}$ becomes trivial on $\eta_v^{-1}(\mathcal{D}^\circ_{v,B}(k_v))$. Arguing as in the proof of Proposition 7.4, the inner summation contributes

$$q_v^{-m_{\alpha(\beta)}(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)}(1+O(q_v^{-\delta_1})),$$

for some $\delta_1 > 0$, assuming that $\delta > 0$ is sufficiently small.
If, on the other hand, $\alpha(\beta) \in \mathcal{A}^{\geqslant 1}(\mathbf{a})$, we set $d := d_{\alpha(\beta)}(f_{\mathbf{a}})$. If $y \notin E(f_{\mathbf{a}})(k_v)$, we can use Lemma 5.4 to compute

$$\begin{aligned}
&\int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s}-\boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau \\
&\quad = \frac{1}{q_v^{n-1}} \int_{\mathfrak{m}_v} |x|_v^{s_{\alpha(\beta)}-\rho_{\alpha(\beta)}}\, \mathbf{1}_{\mathfrak{m}_v^{m_{\alpha(\beta)}}}(x)\, \psi_v\left(\frac{1}{x^d}\right) \mathrm{d}x \\
&\quad = \frac{1}{q_v^{n-1}} \sum_{i=m_{\alpha(\beta)}}^{+\infty} q_v^{-i(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)} \int_{\mathcal{O}_v^\times} \psi_v\left(\frac{\pi_v^{-id}}{x^d}\right) \mathrm{d}x \\
&\quad = \frac{1}{q_v^{n-1}} \sum_{i=m_{\alpha(\beta)}}^{+\infty} q_v^{-i(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)} \int_{\mathcal{O}_v^\times} \psi_v\left(\pi_v^{-id}x^d\right) \mathrm{d}x \\
&\quad = \begin{cases} -\frac{1}{q_v^n} q_v^{-(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)} & \text{if } d = m_{\alpha(\beta)} = 1 \\ 0 & \text{otherwise} \end{cases} \\
&\quad = O(q_v^{-(n+\delta_2)})
\end{aligned}$$

for some $\delta_2 > 0$, for sufficiently small $\delta > 0$.
If $y \in E(f_{\mathbf{a}})(k_v)$ and $\delta > 0$ is sufficiently small, then we have

$$\begin{aligned}
&\left| \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s}-\boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau \right| \\
&\quad \leqslant \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \Re(\mathbf{s})-\boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \mathrm{d}\tau \\
&\quad = O(q_v^{-(n-1+\delta_3)})
\end{aligned}$$

for some $\delta_3 > 0$.
Thus, using the Lang–Weil estimates

$$\#(\mathcal{D}^\circ_{v,B} \setminus E(f_{\mathbf{a}}))(k_v) = O(q_v^{n-1}), \quad \#E(f_{\mathbf{a}})(k_v) = O(q_v^{n-2}),$$

we obtain

$$\sum_{y \in \mathcal{D}^\circ_{v,B}(k_v)} \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s}-\boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau = O(q_v^{-(1+\delta_4)})$$

for some $\delta_4 > 0$.

- If $\#B \geqslant 2$, then arguing as in the proof of Proposition 7.4, one can show that

$$\sum_{y \in \mathcal{D}^\circ_{v,B}(k_v)} \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s}-\boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau = O(q_v^{-(1+\delta_5)})$$

for some $\delta_5 > 0$ assuming that $\delta > 0$ is sufficiently small.

Combining the estimates above, we obtain the following analogue of Proposition 7.4.

Proposition 8.1. *There exist real numbers $\delta, \delta' > 0$, independent of $\mathbf{a}$, such that the function*

$$\mathbf{s} \mapsto \left(\prod_{\alpha \in \mathcal{A}^0(\mathbf{a})} \prod_{\beta \in \mathcal{A}_v(\alpha)} \zeta_{F_{v,\beta}}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1} \right) \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s})$$

is holomorphic on $\mathsf{T}_{>-\delta}$, and such that

$$\left(\prod_{\alpha \in \mathcal{A}^0(\mathbf{a})} \prod_{\beta \in \mathcal{A}_v(\alpha)} \zeta_{F_{v,\beta}}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1} \right) \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}) = 1 + O(q_v^{-(1+\delta')})$$

for all $s \in \mathsf{T}_{>-\delta}$. Here we interpret $\zeta_{F_{v,\beta}}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1}$ to be 1 whenever $\epsilon_\alpha = 1$.

This finishes the analysis in the case $j_v(\mathbf{a}) = 0$. From now on, we assume that $j_v(\mathbf{a}) \neq 0$.

Proposition 8.2. *There exists a real number $\delta > 0$, independent of $\mathbf{a}$, such that the function*

$$\mathbf{s} \mapsto \left(\prod_{\alpha \in \mathcal{A}^0(\mathbf{a})} \prod_{\beta \in \mathcal{A}_v(\alpha)} \zeta_{F_{v,\beta}}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1} \right) \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}),$$

is holomorphic on the domain $\mathsf{T}_{>-\delta}$.

Moreover, there exists a real number $\kappa > 0$, independent of $\mathbf{a}$, such that

$$\left| \left(\prod_{\alpha \in \mathcal{A}^0(\mathbf{a})} \prod_{\beta \in \mathcal{A}_v(\alpha)} \zeta_{F_{v,\beta}}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1} \right) \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}) \right| \ll (1 + H_v(\mathbf{a})^{-1})^\kappa.$$

Here we interpret $\zeta_{F_{v,\beta}}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1}$ to be 1 whenever $\epsilon_\alpha = 1$.

Proof. As before we use the stratification of $G(F_v)$ by the fibers of the reduction map:

$$\frac{1}{\mu_v(\mathcal{O}_v)^n} \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}) = \sum_{B \subset \mathcal{A}_v} \sum_{y \in \mathcal{D}^\circ_{v,B}(k_v)} \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s} - \boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau.$$

- If $B = \emptyset$, the inner summation is holomorphic everywhere and equal to some constant as in § 8.1.
- If $B = \{\beta\}$, we define $\alpha(\beta)$ as in § 7.1. Without loss of generality, we may assume that $D_{v,\beta}$ is geometrically irreducible and that $\epsilon_{\alpha(\beta)} \neq 1$. We again distinguish two cases: either $\alpha(\beta) \in \mathcal{A}^0(\mathbf{a})$ or $\alpha(\beta) \in \mathcal{A}^{\geqslant 1}(\mathbf{a})$.

If $\alpha(\beta) \in \mathcal{A}^0(\mathbf{a})$, the character $\psi_{\mathbf{a},v}$ becomes trivial on $\eta_v^{-1}(\mathcal{D}^\circ_{v,B}(k_v))$. Hence, arguing as in the proof of Proposition 7.4, for a sufficiently small $\delta > 0$, the inner summation is holomorphic and bounded by

$$q_v^{-m_{\alpha(\beta)}(s_{\alpha(\beta)} - \rho_{\alpha(\beta)} + 1)}(c + O(q_v^{-\delta_1})),$$

for some constant c and $\delta_1 > 0$.

If, on the other hand, $\alpha(\beta) \in \mathcal{A}^{\geqslant 1}(\mathbf{a})$, we denote $d := d_{\alpha(\beta)}(f_{\mathbf{a}})$. If $y \notin E(f_{\mathbf{a}})(k_v)$, then we use Lemma 5.4 to compute

$$\int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s} - \boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau$$
$$= \frac{1}{q_v^{n-1}} \int_{\mathfrak{m}_v} |x|_v^{s_{\alpha(\beta)} - \rho_{\alpha(\beta)}}\, \mathbf{1}_{\mathfrak{m}_v^{m_{\alpha(\beta)}}}(x)\, \psi_v\left(\frac{\pi_v^{j_v(\mathbf{a})}}{x^d} \right) \mathrm{d}x$$

$$= \frac{1}{q_v^{n-1}} \sum_{i=m_{\alpha(\beta)}}^{+\infty} q_v^{-i(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)} \int_{\mathcal{O}_v^\times} \psi_v\left(\frac{\pi_v^{-id+j_v(\mathbf{a})}}{x^d}\right) \mathrm{d}x$$
$$= \frac{1}{q_v^{n-1}} \sum_{i=m_{\alpha(\beta)}}^{+\infty} q_v^{-i(s_{\alpha(\beta)}-\rho_{\alpha(\beta)}+1)} \int_{\mathcal{O}_v^\times} \psi_v\left(\pi_v^{-id+j_v(\mathbf{a})}x^d\right) \mathrm{d}x$$
$$= O\left(\frac{|j_v(\mathbf{a})|}{q_v^{n-1}}\right).$$

We note that the implied constant can be taken independent of $\mathbf{a}$; indeed, there are only finitely many possibilities for $d_\alpha(f_{\mathbf{a}})$ by Proposition 5.2. Finally, if $y \in E(f_{\mathbf{a}})(k_v)$, then for $\delta > 0$ sufficiently small we have

$$\left|\int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s}-\boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau\right|$$
$$\leqslant \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \Re(\mathbf{s})-\boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \mathrm{d}\tau$$
$$= O(q_v^{-(n-1+\delta')})$$

for some $\delta' > 0$. Thus, using the Lang–Weil estimates as in § 8.1, we obtain

$$\sum_{y\in\mathcal{D}^\circ_{v,A}(k_v)} \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s}-\boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau = O(|j_v(\mathbf{a})|).$$

- If $\#B \geqslant 2$, then as in the proof of Proposition 7.4, we have

$$\sum_{y\in\mathcal{D}^\circ_{v,B}(k_v)} \int_{\eta_v^{-1}(y)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s}-\boldsymbol{\rho})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\, \psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\tau = O(q_v^{-(1+\delta')}).$$

We conclude as in the proof of Proposition 7.4. □

8.1.2. *Places of bad reduction.* We still assume that $v \notin S$ but our model has bad reduction at v, that is, at least one of the assumptions (i) and (ii) of Lemma 6.1 is not satisfied.

PROPOSITION 8.3. *The function*

$$\widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}) = \int_{G(F_v)} \mathsf{H}_v(\mathbf{x}_v, \mathbf{s})^{-1}\, \delta_{\epsilon,v}(\mathbf{x}_v)\psi_{\mathbf{a},v}(\mathbf{x}_v)\, \mathrm{d}\mathbf{x}_v$$

is holomorphic in $\mathbf{s}$ *whenever* $\Re(s_\alpha) > \rho_\alpha - 1$ *for all* $\alpha \in \mathcal{A}^0(\mathbf{a})$ *such that* $\epsilon_\alpha < 1$. *Moreover, for any* $\delta > 0$ *there exists constants* $\kappa, \delta' > 0$ *and* $C(\delta) > 0$ *such that*

$$|\widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s})| < C(\delta)(1 + |\mathbf{s}|)^\kappa (1 + H_\infty(\mathbf{a}))^{\delta'}$$

whenever $\Re(s_\alpha) > \rho_\alpha - 1 + \delta$ *for all* $\alpha \in \mathcal{A}^0(\mathbf{a})$ *such that* $\epsilon_\alpha < 1$.

Proof. One may argue as in [**26**, Corollary 3.4.4 and Lemma 3.5.2]. □

8.2. *Places contained in S*

We now treat the remaining places.

PROPOSITION 8.4. *The following hold whenever $v \in S$.*

(i) *Let $\delta > 0$ be any positive real number. Then the function*

$$\mathbf{s} \mapsto \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}),$$

is holomorphic in the domain given by $\Re(s_\alpha) > \rho_\alpha - 1 + \delta$ for each $\alpha \in \mathcal{A}$. Moreover, there exists a real number $M_N > 0$, which does not depend on $\mathbf{a}$, such that

$$\left| \prod_{v \in S} \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}) \right| \ll \frac{(1 + |\mathbf{s}|)^{M_N}}{(1 + H_\infty(\mathbf{a}))^N}$$

in the above domain.

(ii) *Let $L = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha D_\alpha$ be a big divisor, and let*

$$a := \tilde{a}((X, D_{\mathrm{red}}), L) \quad \text{and} \quad b := b(F_v, (X, D_{\mathrm{red}}), L, f_{\mathbf{a}})$$

be the respective a- and b-invariants of X defined in § 4. Then the function

$$s \mapsto (\zeta_{F_v}(s - a))^{-b} \widehat{\mathsf{H}}_v(\mathbf{a}, sL)$$

admits a holomorphic continuation to $\Re(s) > a - \delta$ for some $\delta > 0$. Furthermore,

$$\left| \prod_{v \in S} (\zeta_{F_v}(s - a))^{-b} \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, sL) \right| \ll_N \frac{(1 + |s|)^{M_N}}{(1 + H_\infty(\mathbf{a}))^N}$$

in the above domain.

Proof. The first statement is simply [**23**, Proposition 8.1]. The second one follows from [**26**, Proposition 3.4.4 and Lemma 3.5.2] as well as the discussion in [**26**, § 3.3.3]. Note that [**26**, Proposition 3.4.4] is stated for a birational modification $Y_{\mathbf{a}}$ of X, but this does not matter because of Lemma 4.2. ☐

8.3. *Euler products*

Finally we analyze the product

$$\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}) = \prod_{v \in \Omega_F} \widehat{\mathsf{H}}_{\epsilon,v}(\mathbf{a}, \mathbf{s}).$$

We introduce some notation. For every $\alpha \in \mathcal{A}$ we set

$$\zeta_{F_\alpha, S^c}(s) = \prod_{v \notin S} \prod_{\beta \in \mathcal{A}_v(\alpha)} \zeta_{F_{v,\beta}}(s).$$

PROPOSITION 8.5. *Assume that $\lfloor D_\epsilon \rfloor = 0$. There is a real number $\delta > 0$, independent of $\mathbf{a}$, such that the function*

$$\mathbf{s} \mapsto \left(\prod_{\alpha \in \mathcal{A}^0(\mathbf{a})} \zeta_{F_\alpha, S^c}(m_\alpha(s_\alpha - \rho_\alpha + 1)) \right)^{-1} \widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s})$$

is holomorphic on $\mathsf{T}_{>-\delta}$.

Moreover, for any integer $N > 0$, there exists a real number $M_N > 0$ such that

$$\left| \left(\prod_{\alpha \in \mathcal{A}^0(\mathbf{a})} \zeta_{F_\alpha, S^c}(m_\alpha(s_\alpha - \rho_\alpha + 1)) \right)^{-1} \widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}) \right| \ll \frac{(1 + \|\mathbf{s}\|)^{M_N}}{(1 + H_\infty(\mathbf{a}))^N}.$$

Proof. This follows from Propositions 8.1, 8.2, 8.3, and 8.4, together with the estimate (8.1). The implied constant can be chosen independently of $\mathbf{a}$, since $\mathbf{a}$ belongs to the $\mathcal{O}_F$-module Λ_X. ◻

9. *Proof of the main result for klt Campana points*

In this section we prove our main result, Theorem 1.2. We work in the setting introduced in § 1.3, recalled here for the reader's convenience.

By X we mean a smooth, projective and equivariant compactification of $G = \mathbb{G}_a^n$, defined over a number field F. We assume that the boundary divisor $D = X \setminus G$ is a strict normal crossings divisor on X, with irreducible components $(D_\alpha)_{\alpha\in\mathcal{A}}$, so that $D = \sum_{\alpha\in\mathcal{A}} D_\alpha$. We denote by F_α the field of definition for one of the geometric irreducible components of D_α; in other words, F_α is the algebraic closure of F in the function field of D_α.

Let $S \subseteq \Omega_F$ be a finite set containing Ω_F^∞, such that there exists a good integral model $(\mathcal{X}, \mathcal{D})$ of (X, D) over Spec $\mathcal{O}_{F,S}$ as in § 3.2, and let $\mathcal{D} = \sum_{\alpha\in\mathcal{A}} \mathcal{D}_\alpha$. Having fixed $\epsilon_\alpha \in \mathfrak{W}$ for each $\alpha \in \mathcal{A}$, we let $D_\epsilon = \sum_{\alpha\in\mathcal{A}} \epsilon_\alpha D_\alpha$ and $\mathcal{D}_\epsilon = \sum_{\alpha\in\mathcal{A}} \epsilon_\alpha \mathcal{D}_\alpha$. In this section, we assume that the pair (X, D_ϵ) is Kawamata log terminal (klt for short), that is, $\epsilon_\alpha < 1$ for all $\alpha \in \mathcal{A}$.

Let $\mathcal{L}$ denote a big line bundle L on X, equipped with a smooth adelic metrization. Our goal is to understand the asymptotic behavior of the counting function

$$\mathsf{N}(G(F)_\epsilon, \mathcal{L}, T),$$

which records the number of points of $\mathcal{L}$-height at most T in $G(F)_\epsilon = G(F) \cap (\mathcal{X}, \mathcal{D}_\epsilon)(\mathcal{O}_{F,S})$. To do this, we apply a Tauberian theorem to the height zeta function

$$\mathsf{Z}_\epsilon(\mathbf{s}) = \sum_{\mathbf{x}\in G(F)} \mathsf{H}(\mathbf{x}, \mathbf{s})^{-1}\delta_\epsilon(\mathbf{x})$$

introduced in § 6.3. This function is a holomorphic function when $\Re(\mathbf{s}) \gg 0$; our first goal is to establish a meromorphic continuation of this function. Subsequently, knowledge of the location of the rightmost pole of $\mathsf{Z}_\epsilon(sL)$ along $\Re(s)$, its order, and its residue will serve as inputs to the Tauberian theorem that establishes the asymptotic formula we seek.

Recall that for any real number c, we defined

$$\mathsf{T}_{>c} = \{\mathbf{s} \in \mathrm{Pic}(X)_\mathbb{C} : \Re(s_\alpha) > \rho_\alpha - \epsilon_\alpha + c, \text{ for all } \alpha \in \mathcal{A}\},$$

where the ρ_α are integers satisfying $-K_X \sim \sum_{\alpha\in\mathcal{A}} \rho_\alpha D_\alpha$.

Proposition 9.1. *The function*

$$\mathbf{s} \mapsto \left(\prod_{\alpha\in\mathcal{A}} \zeta_{F_\alpha}(m_\alpha(s_\alpha - \rho_\alpha + 1))^{-1}\right)\mathsf{Z}_\epsilon(\mathbf{s})$$

is holomorphic in the region $\mathsf{T}_{\geqslant 0}$.

Proof. We begin by verifying that the Poisson summation formula

$$\mathsf{Z}_\epsilon(\mathbf{s}) = \sum_{\mathbf{a}\in\Lambda_X} \widehat{\mathsf{H}}_\epsilon(\mathbf{a}, \mathbf{s}) \tag{9.1}$$

holds for $\Re(\mathbf{s}) \gg 0$. The discussion in § 6.3 shows that all that remains to be done is checking that the right-hand side converges absolutely. This follows from Proposition 8.5, as

$$\sum_{\mathbf{a}\in\Lambda_X} \frac{1}{(1 + H_\infty(\mathbf{a}))^N}$$

converges for sufficiently large N. The result now follows from an application of Propositions 7.3 and 8.5 and Corollary 7.5 to the summands of the right-hand side of (9.1). ☐

REMARK 9.2. It is important to note that the local height integrals studied in §§ 7–8 have poles along $s_\alpha = \rho_\alpha - 1$; however, it follows from Proposition 9.1 that the rightmost pole of $\mathsf{Z}_\epsilon(\mathbf{s})$ occurs along some $s_\alpha = \rho_\alpha - \epsilon_\alpha > \rho_\alpha - 1$, because of the klt condition.

With a meromorphic continuation of $\mathsf{Z}_\epsilon(\mathbf{s})$ in hand, we turn to the case where $\mathbf{s} = sL$. We may write $L = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha D_\alpha$, where $\lambda_\alpha > 0$ for all $\alpha \in \mathcal{A}$, because L is big. Then $s_\alpha = s\lambda_\alpha$. Proposition 9.1 suggests that the rightmost pole along $\Re(s)$ of the zeta function $\mathsf{Z}_\epsilon(sL)$ is

$$a = a((X, D_\epsilon), L) = \max_{\alpha\in\mathcal{A}} \left\{ \frac{\rho_\alpha - \epsilon_\alpha}{\lambda_\alpha} \right\}.$$

Setting

$$\mathcal{A}_\epsilon(L) = \left\{ \alpha \in \mathcal{A} : \frac{\rho_\alpha - \epsilon_\alpha}{\lambda_\alpha} = a((X, D_\epsilon), L) \right\},$$

the order of this pole should be

$$b = b(F, (X, D_\epsilon), L) := \#\mathcal{A}_\epsilon(L);$$

see Remark 9.2. We shall establish these statements, separating our analysis into two cases, according to the Iitaka dimension of the adjoint divisor

$$aL + K_X + D_\epsilon.$$

9.1. *Rigid case*

In this subsection we assume that the adjoint divisor $aL + K_X + D_\epsilon$ has Iitaka dimension (see [**50**, § 2.1] for the definition) equal to zero; we say that $aL + K_X + D_\epsilon$ is rigid. Recall that $\Lambda_X \subset G(F)$ is the set of $\mathbf{a}$ such that the character $\psi_{\mathbf{a}}$ is trivial on the compact open $\mathbf{K}$ defined in § 6.2.

By the Poisson summation formula, we have

$$\mathsf{Z}_\epsilon(sL) = \sum_{\mathbf{a}\in\Lambda_X} \widehat{\mathsf{H}}_\epsilon(\mathbf{a}, sL).$$

We study the poles of $\mathsf{Z}_\epsilon(sL)$ by looking at the individual terms of the right-hand side. When $\mathbf{a} = 0$, it follows from Corollary 7.5 that $\widehat{\mathsf{H}}_\epsilon(0, sL)$ has a pole at $s = a$ of order b, provided that we show that the corresponding residue is not zero (we verify this last claim presently). On the other hand, Proposition 8.5 shows that if $\mathbf{a} \neq 0$, the term $\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, sL)$ has a pole of the highest order equal to that of $\widehat{\mathsf{H}}_\epsilon(0, sL)$ if and only if

$$\mathcal{A}^0(\mathbf{a}) \supset \mathcal{A}_\epsilon(L).$$

This condition means that whenever $(\rho_\alpha - \epsilon_\alpha)/\lambda_\alpha = a$, we must have $d_\alpha(f_{\mathbf{a}}) = 0$. Since

$$E(f_{\mathbf{a}}) \sim \sum_{\alpha\in\mathcal{A}} d_\alpha(f_{\mathbf{a}}) D_\alpha \quad \text{and} \quad aL + K_X + D_\epsilon = \sum_{\alpha\in\mathcal{A}} (a\lambda_\alpha - \rho_\alpha + \epsilon_\alpha) D_\alpha,$$

it follows that $E(f_{\mathbf{a}})$ is equivalent to a boundary divisor whose support is contained in that of the adjoint divisor $aL + K_X + D_\epsilon$. This is not possible. Indeed, $aL + K_X + D_\epsilon$ is rigid, and any positive linear combination of components of a rigid effective divisor has a unique effective divisor in its $\mathbb{Q}$-linear equivalence class. However, we showed that the effective divisor $E(f_{\mathbf{a}})$, which is not a boundary divisor, is linearly equivalent to an effective boundary divisor with support contained in the support of $aL + K_X + D$. This is a contradiction. Hence, if $\mathbf{a} \neq 0$, the summand $\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, sL)$ does not contribute to the residue of the pole of $\mathsf{Z}_\epsilon(sL)$ at $s = a$.

Our analysis shows that the main term of $\mathsf{Z}_\epsilon(sL)$ is furnished by $\widehat{\mathsf{H}}_\epsilon(0, sL)$, provided

$$c := \lim_{s \to a} (s-a)^b \, \widehat{\mathsf{H}}_\epsilon(0, sL)$$

is non-zero, that is, only the trivial character can contribute to the leading pole of $\mathsf{Z}_\epsilon(sL)$. Recall that

$$\widehat{\mathsf{H}}_\epsilon(0, sL) = \int_{G(\mathbb{A}_F)} \mathsf{H}(\mathbf{x}, sL)^{-1} \, \delta_\epsilon(\mathbf{x}) \, \mathrm{d}\mathbf{x} = \int_{G(\mathbb{A}_F)_\epsilon} \mathsf{H}(\mathbf{x}, sL + K_X)^{-1} \, \mathrm{d}\tau,$$

where $\tau = \prod_v \tau_v$ is the Tamagawa measure on G. Let $X^\circ = X \setminus (\bigcup_{\alpha \notin \mathcal{A}_\epsilon(L)} D_\alpha)$. Setting $\Gamma = \mathrm{Gal}(\bar{F}/F)$ and $\Gamma_{F_\alpha} = \mathrm{Gal}(\bar{F}/F_\alpha)$, we construct the virtual Artin representation

$$P(\bar{X}^\circ) = \mathrm{Pic}(\bar{X})_{\mathbb{C}} - \sum_{\alpha \notin \mathcal{A}_\epsilon(L)} \mathrm{Ind}_{\Gamma_{F_\alpha}}^{\Gamma} \mathbb{C}.$$

We denote the corresponding virtual Artin L-function by

$$L^S(P(\bar{X}^\circ), s) = \prod_{v \notin S} L_v(P(\bar{X}^\circ), s).$$

This function has a pole of order $\#\mathcal{A}_\epsilon(L)$ at $s = 1$ by [46, Corollary 5.47]. For $v \in S$ we define $L_v(P(\bar{X}^\circ), s) = 1$. Using this we define the Tamagawa measure

$$\tau_{X^\circ} = L_*^S(P(\bar{X}^\circ), 1) \prod_{v \in \Omega_F} L_v(P(\bar{X}^\circ), 1)^{-1} \tau_{X^\circ, v}, \tag{9.2}$$

where $L_*^S(P(\bar{X}^\circ), 1)$ is the leading constant of $L^S(P(\bar{X}^\circ), s)$. We also define

$$\tau_{X^\circ, D_\epsilon, v} = \mathsf{H}_v(\mathbf{x}, D_\epsilon)\tau_{X^\circ, v} \quad \text{and} \quad \tau_{X^\circ, D_\epsilon} = \mathsf{H}(\mathbf{x}, D_\epsilon)\tau_{X^\circ}.$$

Lemma 9.3. *With notation as above, we have*

$$c = \prod_{\alpha \in \mathcal{A}_\epsilon(L)} \frac{1}{m_\alpha \lambda_\alpha} \int_{X^\circ(\mathbb{A}_F)_\epsilon} \mathsf{H}(\mathbf{x}, aL + K_X + D_\epsilon)^{-1} \, \mathrm{d}\tau_{X^\circ, D_\epsilon} > 0,$$

where $X^\circ(\mathbb{A}_F)_\epsilon$ is defined in § 3.3.

Proof. First, we note that

$$\begin{aligned} c &= \lim_{s \to a} (s-a)^b \, \widehat{\mathsf{H}}_\epsilon(0, sL) \\ &= \lim_{s \to a} (s-a)^b \prod_{\alpha \in \mathcal{A}_\epsilon(L)} \zeta_{F_\alpha, S^c}(m_\alpha(\lambda_\alpha s - \rho_\alpha + 1)) \\ &\quad \times \int_{G(\mathbb{A}_F)_\epsilon} \left(\prod_{\alpha \in \mathcal{A}_\epsilon(L)} \zeta_{F_\alpha, S^c}(m_\alpha(\lambda_\alpha s - \rho_\alpha + 1)) \right)^{-1} \mathsf{H}(\mathbf{x}, sL + K_X)^{-1} \, \mathrm{d}\tau. \end{aligned}$$

For each $\alpha \in \mathcal{A}_\epsilon(L)$, we have $a = (\rho_\alpha - \epsilon_\alpha)/\lambda_\alpha$, where $\epsilon_\alpha = 1 - 1/m_\alpha$. Each of the b-many Dedekind zeta factors $\zeta_{F_\alpha,S^c}(m_\alpha(\lambda_\alpha s - \rho_\alpha + 1))$ has a simple pole at $s = a$, so that the limit

$$\lim_{s\to a} (s-a)\,\zeta_{F_\alpha,S^c}(m_\alpha(\lambda_\alpha s - \rho_\alpha + 1))$$

is equal to the residue at $s = a$ for the Dedekind zeta factor corresponding to α, which we denote by $\zeta^*_{F_\alpha,S^c}(1)/m_\alpha\lambda_\alpha$, where $\zeta^*_{F_\alpha,S^c}(1)$ is the residue of $\zeta_{F_\alpha,S^c}(s)$ at $s = 1$, the normalization $1/m_\alpha\lambda_\alpha$ being a consequence of the chain rule. With the notation

$$\zeta_{F_\alpha,S^c,v}(s) = \begin{cases} \prod_{\beta\in\mathcal{A}_v(\alpha)} \zeta_{F_{v,\beta}}(s) & \text{if } v \notin S, \\ 1 & \text{otherwise,} \end{cases}$$

we rewrite the integral

$$\int_{G(\mathbb{A}_F)_\epsilon} \left(\prod_{\alpha\in\mathcal{A}_\epsilon(L)} \zeta_{F_\alpha,S^c}(m_\alpha(\lambda_\alpha s - \rho_\alpha + 1)) \right)^{-1} \mathsf{H}(\mathbf{x}, sL + K_X)^{-1}\, \mathrm{d}\tau$$

as a product of local integrals

$$\prod_{v\in\Omega_F} \int_{G(F_v)_\epsilon} \left(\prod_{\alpha\in\mathcal{A}_\epsilon(L)} \zeta_{F_\alpha,S^c,v}(m_\alpha(\lambda_\alpha s - \rho_\alpha + 1) \right)^{-1} \mathsf{H}_v(\mathbf{x}, aL + K_X)^{-1}\, \mathrm{d}\tau_{X^\circ,v}$$

each of which is regular at $s = a$ (note that τ_v and $\tau_{X^0,v}$ coincide on G). We obtain

$$c = \prod_{\alpha\in\mathcal{A}_\epsilon(L)} \frac{1}{m_\alpha\lambda_\alpha} \zeta^*_{F_\alpha,S^c}(1) \prod_{v\in\Omega_F} \int_{G(F_v)_\epsilon} \left(\prod_{\alpha\in\mathcal{A}_\epsilon(L)} \zeta_{F_\alpha,S^c,v}(1) \right)^{-1} \mathsf{H}_v(\mathbf{x}, aL + K_X)^{-1}\, \mathrm{d}\tau_{X^\circ,v}.$$

Using the equality

$$\prod_{v\in\Omega_F} L_v(P(\bar{X}^\circ), 1) \left(\prod_{\alpha\in\mathcal{A}_\epsilon(L)} \zeta_{F_\alpha,S^c,v}(1) \right)^{-1} = L^S_*(P(\bar{X}^\circ), 1) \left(\prod_{\alpha\in\mathcal{A}_\epsilon(L)} \zeta^*_{F_\alpha,S^c}(1) \right)^{-1},$$

we may simplify the above expression for c to

$$\prod_{\alpha\in\mathcal{A}_\epsilon(L)} \frac{1}{m_\alpha\lambda_\alpha} L^S_*(P(\bar{X}^\circ), 1) \prod_{v\in\Omega_F} \int_{G(F_v)_\epsilon} \mathsf{H}_v(\mathbf{x}, aL + K_X + D_\epsilon)^{-1}\, L_v(P(\bar{X}^\circ), 1)^{-1}\, \mathrm{d}\tau_{X^\circ,D_\epsilon,v}. \tag{9.3}$$

Finally, (9.2) allows us to conclude that

$$c = \prod_{\alpha\in\mathcal{A}_\epsilon(L)} \frac{1}{m_\alpha\lambda_\alpha} \int_{X^\circ(\mathbb{A}_F)_\epsilon} \mathsf{H}(\mathbf{x}, aL + K_X + D_\epsilon)^{-1}\, \mathrm{d}\tau_{X^\circ,D_\epsilon} > 0.$$

Let us discuss the positivity of this constant. Recall that this integration is expressed as the Euler product (9.3). The integral at each place is positive as the inner function is positive over some open subset. Then a partial Euler product is also positive because of Proposition 7.4 (2). Thus our assertion follows. $\square$

Applying a Tauberian theorem (see, for example, [**68**, II.7, Theorem 15]), we obtain the following theorem.

THEOREM 9.4. *Let $\mathcal{X}$, $\mathcal{L}$, $\mathcal{D}$ and ϵ be as above. Assume that (X, D_ϵ) is klt and set*

$$\begin{aligned} a &= a((X, D_\epsilon), L), \\ b &= b(F, (X, D_\epsilon), L), \\ c &= c(F, S, (\mathcal{X}, \mathcal{D}_\epsilon), \mathcal{L}) \\ &= \prod_{\alpha \in \mathcal{A}_\epsilon(L)} \frac{1}{m_\alpha \lambda_\alpha} \int_{X^\circ(\mathbb{A}_F)_\epsilon} \mathsf{H}(\mathbf{x}, aL + K_X + D_\epsilon)^{-1} \, \mathrm{d}\tau_{X^\circ, D_\epsilon}. \end{aligned}$$

If $aL + K_X + D_\epsilon$ is rigid, then

$$\mathsf{N}(G(F)_\epsilon, \mathcal{L}, T) \sim \frac{c}{a(b-1)!} T^a (\log T)^{b-1} \text{ as } T \to \infty.$$

9.2. *Non-rigid case*

The analysis in this subsection is modeled on [70]. With notation as above, we now assume that the divisor $E := aL + K_X + D_\epsilon$ is not rigid, that is, that its Iitaka dimension is positive. Then some multiple mE defines the Iitaka fibration $\phi_m \colon X \dashrightarrow Y_m$. (See [50, § 2.2] for its definition.) Since mE admits a G-linearization, Y_m admits a natural G-action, and ϕ_m is G-equivariant. For the sake of simplicity, we assume that ϕ_m is a *morphism*. The variety Y_m contains an open orbit of the G-action, so it has the structure of an equivariant compactification of the quotient vector space G/G_L, where $G_L \subset G$ is a linear subspace of G.

As in § 9.1, the term $\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, sL)$ has a pole of the highest order equal to that of $\widehat{\mathsf{H}}_\epsilon(0, sL)$ if and only if $\mathcal{A}^0(\mathbf{a}) \supset \mathcal{A}_\epsilon(L)$. This condition is equivalent to having $f_{\mathbf{a}} = 0$ on G_L. Therefore, the rightmost pole of $\mathsf{Z}_\epsilon(sL)$ is furnished by the sum

$$\begin{aligned} \sum_{\{f_{\mathbf{a}}=0\} \supset G_L} \widehat{\mathsf{H}}_\epsilon(\mathbf{a}, sL) &= \sum_{\{f_{\mathbf{a}}=0\} \supset G_L} \int_{G(\mathbb{A}_F)} \mathsf{H}^{-1}(\mathbf{x}, sL) \, \delta_\epsilon(\mathbf{x}) \, \psi_{\mathbf{a}}(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{y} \in (G/G_L)(F)} \int_{G_L(\mathbb{A}_F)} \mathsf{H}^{-1}(\mathbf{x}+\mathbf{y}, sL) \, \delta_\epsilon(\mathbf{x}+\mathbf{y}) \, \mathrm{d}\mathbf{x}, \end{aligned}$$

where the last equality follows from the Poisson summation formula. Note that the equality holds for any s with $\Re(s) > a$ by the monotone and dominated convergence theorems.

Let $X_{\mathbf{y}}$ be the fiber of ϕ_m above $\mathbf{y}$. It is a smooth equivariant compactification of G_L, with boundary divisor $D|_{X_{\mathbf{y}}}$. Let $\mathcal{X}_{\mathbf{y}}$ be the closure of $X_{\mathbf{y}}$ inside $\mathcal{X}$. The restriction $(aL + K_X + D_\epsilon)|_{X_{\mathbf{y}}}$ is rigid, since ϕ_m is an Iitaka fibration. Applying the analysis of § 9.1, we conclude that the inner integral has a pole at $s = a((X_{\mathbf{y}}, D_\epsilon|_{X_{\mathbf{y}}}), L)$ of order $b(F, (X_{\mathbf{y}}, D_\epsilon|_{X_{\mathbf{y}}}), L)$. Now [42, Lemma 5.2] yields

$$a((X, D_\epsilon), L) = a((X_{\mathbf{y}}, D_\epsilon|_{X_{\mathbf{y}}}), L), \quad \text{and} \quad b(F, (X, D_\epsilon), L) = b(F, (X_{\mathbf{y}}, D_\epsilon|_{X_{\mathbf{y}}}), L).$$

We claim that

$$\lim_{s \to a} (s-a)^b \, \mathsf{Z}_\epsilon(sL) = \sum_{\mathbf{y} \in (G/G_L)(F)} c(F, S, (\mathcal{X}_{\mathbf{y}}, \mathcal{D}_\epsilon|_{\mathcal{X}_{\mathbf{y}}}), \mathcal{L}|_{X_{\mathbf{y}}}).$$

All we need to do is justify the interchange of limits: the right-hand side converges by Fatou's lemma, and the claim then follows from the Poisson summation formula (Theorem 5.5).

As before, applying a Tauberian theorem ([68, II.7, Theorem 15]), we obtain the following theorem.

THEOREM 9.5. *Let X, $\mathcal{L}$, D and ϵ be as above. Assume that (X, D_ϵ) is klt, and that m is an integer such that the Iitaka fibration $\phi_m\colon X \dashrightarrow Y_m$ defined by mE is a* morphism. *Set*

$$a = a((X, D_\epsilon), L),$$
$$b = b(F, (X, D_\epsilon), L),$$
$$c = \sum_{\mathbf{y}\in(G/G_L)(F)} c(F, S, (\mathcal{X}_{\mathbf{y}}, \mathcal{D}_\epsilon|_{\mathcal{X}_{\mathbf{y}}}), \mathcal{L}|_{X_{\mathbf{y}}}).$$

Then

$$\mathsf{N}(G(F)_\epsilon, \mathcal{L}, T) \sim \frac{c}{a(b-1)!} T^a (\log T)^{b-1} \text{ as } T \to \infty.$$

Interlude II: examples

As mentioned in the introduction, Theorem 9.4 for klt Campana points of bounded log-anticanonical height (that is, $L = -(K_X + D_\epsilon)$) applies to all smooth compactifications of vector groups with strict normal crossings boundary, as $aL + K_X + D_\epsilon$ is always rigid in that case. We recall that there are numerous such compactifications, as blowing up points that are invariant for the action of the vector group on a compactification always produces new examples.

For the convenience of the reader, we describe two explicit examples to which Theorem 9.4 applies with $L \neq -(K_X + D_\epsilon)$. Both can be described as blow-ups of a projective space. We describe the set of Campana points in terms of the projective coordinates to show what type of explicit counting problems can be solved using Theorem 9.4.

Blow-ups of $\mathbb{P}^n$

Let $f \in \mathbb{Z}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d such that the subscheme $\{x_0 = f = 0\}$ of $\mathbb{P}^n_{\mathbb{Z}}$ is regular over $\mathbb{Z}$. Let $\varphi\colon \mathcal{X} \to \mathbb{P}^n_{\mathbb{Z}}$ be the blow-up with center $\{x_0 = f = 0\}$. Let $\mathcal{D}_1$ be the exceptional divisor and $\mathcal{D}_2$ the strict transform of $\{x_0 = 0\}$. We set $\mathcal{X}^\circ = \mathcal{X} \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$.

Fix positive integers m_1 and m_2, and let $\epsilon_i = 1 - 1/m_i$ for $i \in \{1, 2\}$. Then $(\mathcal{X}, \mathcal{D}_\epsilon)$ is a good integral model of a klt Campana orbifold in the sense of § 3.2. By definition of blow-up, the restriction of the morphism φ to $\mathcal{X}^\circ$ is injective. Thus, φ induces a bijection between $(\mathcal{X}, \mathcal{D}_\epsilon)(\mathbb{Z}) \cap \mathcal{X}^\circ(\mathbb{Q})$ and the set A of $(n+1)$-tuples $(\tilde{x}_0, \ldots, \tilde{x}_n) \in \mathbb{Z}^n$ such that

$$\gcd(\tilde{x}_0, \ldots, \tilde{x}_n) = 1, \quad \tilde{x}_0 > 0, \quad \gcd(\tilde{x}_0, f(\tilde{x}_0, \ldots, \tilde{x}_n)) \text{ is } m_1\text{-full},$$
$$\tilde{x}_0/\gcd(\tilde{x}_0, f(\tilde{x}_0, \ldots, \tilde{x}_n)) \text{ is } m_2\text{-full}.$$

Indeed, given a point $\tilde{x} \in \mathbb{P}^n(\mathbb{Q}) \setminus \{x_0 = 0\}$, the first two conditions fix a representative for the projective coordinates of $\tilde{x}$, and given a linear form $\ell \in \mathbb{Z}[x_0, \ldots, x_n]$ such that $\ell(\tilde{x}) = 1$, we can describe explicitly the morphism φ over the neighborhood $U_\ell := \mathbb{P}^n_{\mathbb{Z}} \setminus \{\ell = 0\}$ of $\tilde{x}$. In particular, $\varphi^{-1}(U_\ell) = \{y_0 f \ell^{-d} = y_1 x_0 \ell^{-1}\} \subseteq U_\ell \times \mathbb{P}^1_{\mathbb{Z}}$, with coordinates $(y_0 : y_1)$ on $\mathbb{P}^1_{\mathbb{Z}}$, and the preimage of $\tilde{x}$ is the point $(\tilde{x}, (\tilde{x}_0/\gcd(\tilde{x}_0, f(\tilde{x})) : f(\tilde{x})/\gcd(\tilde{x}_0, f(\tilde{x})))) \in U_\ell \times \mathbb{P}^1_{\mathbb{Z}}$. In a neighborhood of $\varphi^{-1}(\tilde{x})$, the equations defining $\mathcal{D}_1$ as a subscheme of $U_\ell \times \mathbb{P}^1_{\mathbb{Z}}$ are $x_0 = f = 0$, the equations defining $\mathcal{D}_2$ are $x_0 = y_0 = 0$. So $\varphi^{-1}(\tilde{x}) \in (\mathcal{X}, \mathcal{D}_\epsilon)(\mathbb{Z})$ if and only if $\gcd(\tilde{x}_0, f(\tilde{x}))$ is m_1-full and $\gcd(\tilde{x}_0, \tilde{x}_0/\gcd(\tilde{x}_0, f(\tilde{x})))$ is m_2-full.

An application of Theorem 9.4 with $L = \pi^* \mathcal{O}_{\mathbb{P}^n}(1)$ shows that

$$\#\{(x_0, \ldots, x_n) \in A : \max\{|x_0|, \ldots, |x_n|\} \leqslant T\} \sim cT^{n+1/m_2} \text{ as } T \to \infty,$$

for some $c > 0$.

A singular del Pezzo surface

Let X be the minimal desingularization of a split quartic del Pezzo surface of type D_5 over $\mathbb{Q}$. Then X is an equivariant compactification of $\mathbb{G}_a^2$ by [**31**, Lemmas 4 and 6]. The irreducible components of the boundary on X are the divisors $E_1, \dots, E_6$ from [**30**, § 3.4 Type D_5]. We fix coordinates $(x_0 : x_1 : x_2)$ on $\mathbb{P}^2_{\mathbb{Q}}$ and we denote by $\varphi : X \to \mathbb{P}^2$ the morphism from [**30**, § 3.4 Type D_5] that contracts E_1, E_2, E_4, E_5, E_6 to the point $(0 : 0 : 1)$ and maps E_3 onto $\{x_0 = 0\}$. The morphism φ is a sequence of five successive blow-ups at $\mathbb{Q}$-points. Performing the same sequence of blow-ups over $\mathbb{Z}$ as in [**37**, Proposition 3.9] yields a smooth projective $\mathbb{Z}$-model $\mathcal{X}$ for X. For every $i \in \{1, \dots, 6\}$, we fix a positive integer m_i, we define $\epsilon_i = 1 - \frac{1}{m_i}$, and we denote by $\mathcal{E}_i$ the closure of E_i in $\mathcal{X}$. Then $(\mathcal{X}, \sum_{i=1}^6 \epsilon_i \mathcal{E}_i)$ is a good integral model for the klt Campana orbifold $(X, \sum_{i=1}^6 \epsilon_i E_i)$. Let $X^\circ = X \setminus \bigcup_{i=1}^6 E_i$.

We use the notation $f(\cdot) := \cdot/\gcd(\cdot, x_1)$ and $g(\cdot) := x_1/\gcd(\cdot, x_1)$, and we denote by $f^{(n)}$ the nth composition of f with itself. We write $h := f^{(3)}(x_0)x_2^2 + g(f^{(2)}(x_0))g(f(x_0))g(x_0)$. Reasoning as in the previous example for each of the five successive blow ups, we see that the set of $\mathbb{Z}$-Campana points $(\mathcal{X}, \sum_{i=1}^6 \epsilon_i \mathcal{E}_i)(\mathbb{Z}) \cap X^\circ(\mathbb{Q})$ is in bijection, via φ, with the set A of triples $(x_0, x_1, x_2) \in \mathbb{Z}^3$ such that $\gcd(x_0, x_1, x_2) = 1$, $x_0 > 0, x_1 \neq 0$ and

$$\gcd(f^{(2)}(x_0), g(h)) \text{ is } m_1\text{-full},$$

$$x_2^{m_2} \gcd(h, g(f(h))) \text{ is } m_2\text{-full},$$

$$f^{(3)}(x_0) \text{ is } m_3\text{-full}, \quad \gcd(f(x_0), g(f^{(2)}(x_0))) \text{ is } m_4\text{-full},$$

$$\gcd(x_0, g(f(x_0))) \text{ is } m_5\text{-full}, \quad x_2^{m_6} \gcd(x_1, f(h)) \text{ is } m_6\text{-full}.$$

Then an application of Theorem 9.4 with $L = \varphi^* \mathcal{O}_{\mathbb{P}^2}(1)$ shows that

$$\#\{(x_0, x_1, x_2) \in A : \max\{|x_0|, |x_1|, |x_2|\} \leqslant T\} \sim cT^{2+1/m_3} \text{ as } T \to \infty,$$

for some $c > 0$.

10. *Proof of the main result for dlt Campana points*

In this section, we sketch the proof of Theorem 1.4. We use the notation of § 9, but this time we assume that $\lfloor D_\epsilon \rfloor \neq 0$, so that (X, D_ϵ) is not a klt pair. We set

$$\mathcal{A}^{\mathrm{klt}} = \{\alpha \in \mathcal{A} \mid \epsilon_\alpha \neq 1\},$$

$$\mathcal{A}^{\mathrm{nklt}} = \{\alpha \in \mathcal{A} \mid \epsilon_\alpha = 1\}.$$

Let $L = -(K_X + D_\epsilon)$. Arguing as in the proof of Proposition 9.1, we obtain the following proposition.

Proposition 10.1. *The function*

$$s \mapsto \left(\prod_{\alpha \in \mathcal{A}^{\mathrm{klt}}} \zeta_{F_\alpha}(1 + m_\alpha(\rho_\alpha - \epsilon_\alpha)(s-1)) \right)^{-1} \left(\prod_{v \in S} \zeta_{F_v}(s-1)^{-b(F_v, (X, D_{\mathrm{red}}), L)} \right) \mathsf{Z}_\epsilon(sL)$$

is holomorphic in the region $\Re(s) \geqslant 1$.

This implies that the zeta function $\mathsf{Z}_\epsilon(sL)$ possibly has a pole at $s = 1$.

We define

$$b(F, S, (X, D_\epsilon), L) = \#\mathcal{A}^{\mathrm{klt}} + \sum_{v \in S} b(F_v, (X, D_{\mathrm{red}}), L),$$

where the summands on the right are the b-invariants defined in § 4. Proposition 7.3 and Corollary 7.5 together imply that $\widehat{\mathsf{H}}_\epsilon(0, sL)$ has a pole at $s = 1$ of order $b(F, S, (X, D_\epsilon), L)$.

Arguing as in [**26**, Lemma 3.5.4], we see that the order of the pole of the function $\widehat{\mathsf{H}}_\epsilon(\mathbf{a}, sL)$ at $s = 1$ is strictly less than $b(F, S, (X, D_\epsilon), L)$ when $\mathbf{a} \neq 0$. A final application of the Tauberian theorem [**68**, II.7, Theorem 15] then gives the following asymptotic formula for the counting function $N(G(F)_\epsilon, \mathcal{L}, T)$ in the dlt case when $L = -(K_X + D_\epsilon)$.

THEOREM 10.2. *Let X, D and ϵ be as above. Set*

$$L = -(K_X + D_\epsilon), \quad a = 1, \quad \text{and} \quad b = b(F, S, (X, D_\epsilon), L).$$

Then there exists a constant $c > 0$ that depends on $F, S, (\mathcal{X}, \mathcal{D}_\epsilon)$, and $\mathcal{L}$, such that

$$\mathsf{N}(G(F)_\epsilon, \mathcal{L}, T) \sim \frac{c}{a(b-1)!} T^a (\log T)^{b-1} \text{ as } T \to \infty.$$

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