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Reductivity properties over an affine base

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Dedicated to the memory of T.A. Springer

Abstract

When the base ring is not a field, power reductivity of a group scheme is a basic notion, intimately tied with finite generation of subrings of invariants. Geometric reductivity is weaker and less pertinent in this context. We give a survey of these properties and their connections.

1. Power reductivity as a basic notion

1.1. Invariants

Throughout let $\mathbf{k}$ be a commutative ring and let G be a flat affine group scheme over $\mathbf{k}$. We simply refer to G as a group. Flatness of G is always needed, because one wants taking invariants to be left exact [8, I.2.10(4)]. The present paper is an addendum to our joint paper with Franjou [5]. In that paper we had a specific situation in mind, but now we care about the proper generality. For instance, we no longer assume that G is algebraic, *i.e.* that $\mathbf{k}[G]$ is a finitely generated $\mathbf{k}$-algebra. We view the ground ring $\mathbf{k}$ also as a G-module with trivial action. If M is a G-module [8, I.2.7, I.2.8] then its submodule of invariants M^G is isomorphic to $\mathrm{Hom}_G(\mathbf{k}, M)$.

1.2. Conventions

Rings and algebras are unitary. A ring A is called a $\mathbf{k}$-algebra if one is given a ring map $\mathbf{k} \to A$. Commutative algebras need not be finitely generated. They may have nilpotent elements and other zero-divisors. We say that G acts on the $\mathbf{k}$-algebra A (through algebra

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automorphisms) if the multiplication map $A \otimes_{\mathbf{k}} A \to A$ is a map of G-modules. If A is a commutative ring and N is an A-module, then $S_A^*(N)$ denotes the symmetric algebra over A on the module N. Thus $S_A^d(N)$ is the dth symmetric power of N over A. If $A = \mathbf{k}$ then we drop the subscript from the notation. We write $\mathrm{Hom}_{\mathbf{k}}(M, \mathbf{k})$ as $M^\vee$. If M is a G-module which is finitely generated and projective as a $\mathbf{k}$-module, then $M^\vee$ is also a G-module. Any map induced by evaluation at an element v is denoted eval@v.

Definition 1. The group G is *power reductive* over $\mathbf{k}$ if the following property holds.

Property (*Power Reductivity*)**.** *Let $\varphi : M \twoheadrightarrow \mathbf{k}$ be a surjective map of G-modules. Then there is a positive integer d such that the dth symmetric power of φ is a split surjection of G-modules*

$$S^d\phi : S^d M \overset{\curvearrowleft}{\twoheadrightarrow} S^d\mathbf{k}.$$

In other words, one requires that the kernel of $S^d\phi$ has a G-stable complement in $S^d M$.

Note that $S^*\mathbf{k}$ is better known as the polynomial ring $\mathbf{k}[x]$. And the G-module $S^d\mathbf{k}$ is isomorphic to $\mathbf{k}$, so a splitting of $S^d\phi$ gives an invariant in $S^d M$.

1.3. Mumford

Mumford conjectured in the introduction to the first edition of his GIT book [9] that a semisimple algebraic group defined over a field of positive characteristic p is power reductive. We have adapted his phrasing and introduced the terminology *power reductive* in [5] (with Vincent Franjou) in order to have a clear concept that also makes sense and is worth having over arbitrary commutative base rings. Mumford further required d to be a power of p, but it turns out that this makes no difference (Lemma 15).

1.4. Haboush

When Haboush proved the Mumford conjecture [7] he also used the dual concept known nowadays as *geometric reductivity*.

Definition 2 (*Geometric Reductivity Over a Field*)**.** Let $\mathbf{k}$ be field. The group G is called geometrically reductive if the following holds. Given an injective map $\varphi : \mathbf{k} \hookrightarrow M$ of finite dimensional G-modules, there is a positive integer d such that some invariant homogeneous polynomial f of degree d on M restricts to a nonzero function on $\mathbf{k}$. In other words, such that the restriction map $S^d(M^\vee)^G \to S^d(\mathbf{k}^\vee)$ is nonzero.

1.5. Geometric reductivity over arbitrary base ring

When $\mathbf{k}$ is not a field the definition of geometric reductivity gets more technical. Following Seshadri [14] we then say that G is geometrically reductive if the following holds. Let us be given a G-module M that is finitely generated and free as a $\mathbf{k}$-module. Let F be a field and also a $\mathbf{k}$-algebra. We let G act trivially on F. Let $v \in (F \otimes_{\mathbf{k}} M)^G$ be a nonzero invariant vector. (A geometer may consider a nonzero invariant vector at a geometric point $\mathrm{Spec}(F)$ of $\mathrm{Spec}(\mathbf{k})$.) Then geometric reductivity stipulates that there is a positive integer d such that some invariant homogeneous polynomial f of degree d on M does not vanish at v. In other words, such that the evaluation map eval@$v : S^d(M^\vee)^G \to F$ is nonzero.

1.6. Contrast

Notice that power reductivity is much cleaner. It does not require any discussion of M as a $\mathbf{k}$-module. While geometric reductivity needs free $\mathbf{k}$-modules, power reductivity allows all comodules [8, I.2.8] that support a ϕ as in the definition. This important difference makes power reductivity more powerful and easier to work with. Working only with free modules (or only with flat $\mathbf{k}$-modules) gives an obstructed view of representation theory. We do not know of any example where geometric reductivity is easier to prove than power reductivity, so one may as well prove the latter. It is stronger (Lemma 12).

1.7. Locally finite

Recall that if the coordinate algebra $\mathbf{k}[G]$ is a projective $\mathbf{k}$-module, then any G-module M is a union of submodules that are finitely generated over $\mathbf{k}$ [14, Proposition 3]. Also, the intersection of G-submodules is then a G-submodule, even if one intersects infinitely many submodules.

Similarly, suppose $\mathbf{k}$ is noetherian. Again any G-module M is a union of submodules that are finitely generated over $\mathbf{k}$ [13, Proposition 2]. In the definition of power reductivity it would now suffice to consider M that are finitely generated over $\mathbf{k}$. On the other hand, an infinite intersection of G-submodules need not be a G-submodule [3, Exposé VI, Édition 2011, Remarque 11.10.1], despite the claim in [8, I.2.13] that we know this.

We do not know if local finiteness holds in general.

1.8. Our present definition of power reductivity is consistent with the one in [5]. Indeed if $\mathbf{k} = L$ in the following Lemma then the splitting of $S^d\phi : S^d M \to S^d L$ is of course equivalent to the surjectivity of $(S^d M)^G \to S^d L$.

Lemma 3. *Let L be a cyclic $\mathbf{k}$-module with trivial G-action. Let M be a G-module, and let φ be a G-module map from M onto L. If G is power reductive, then there is a positive integer d such that the dth symmetric power of φ induces a surjection:*

$$(S^d M)^G \twoheadrightarrow S^d L.$$

Proof. Choose a surjective map $\psi : \mathbf{k} \twoheadrightarrow L$. Let $P \twoheadrightarrow \mathbf{k}$ be the pullback of ϕ along ψ and choose a positive integer d such that $S^d P \twoheadrightarrow S^d\mathbf{k}$ splits. □

Definition 4. A morphism of $\mathbf{k}$-algebras $\phi : S \to R$ is *power surjective* if for every element r of R there is a positive integer n such that the power r^n lies in the image of ϕ.

Definition 5. Let p be a prime number. A morphism of $\mathbf{k}$-algebras $\phi : S \to R$ is *p-power surjective* if for every element r in R there is a non-negative integer n such that the power r^{p^n} lies in the image of ϕ.

Lemma 6 (*[5, Prop 41]*)**.** *A morphism of commutative $\mathbb{F}_p$-algebras $\phi : S \to R$ is p-power surjective if and only if the induced map $S[x] \to R[x]$ between polynomial rings is power surjective.* □

1.9. As is common for a basic notion, there are several equivalent formulations of power reductivity.

Proposition 7. *Let G be a flat affine group scheme over* $\mathbf{k}$*. The following are equivalent*

(i) *G is power reductive,*
(ii) *For every power surjective G-homomorphism of commutative* $\mathbf{k}$*-algebras $f : A \to B$ the map $A^G \to B^G$ is power surjective,*
(iii) *For every surjective G-homomorphism of commutative* $\mathbf{k}$*-algebras $f : A \twoheadrightarrow B$ the ring B^G is integral over the image of A^G.*

Proof. The assumption that G is algebraic is not used in the proofs of [5, Proposition 10], [18, Proposition 4]. □

1.10. The main consequence of power reductivity is finite generation of subrings of invariants.

Theorem 8 (*Hilbert's Fourteenth Problem [5], cf. [1]*)**.** *Let* $\mathbf{k}$ *be a noetherian ring and let G be a flat affine group scheme over* $\mathbf{k}$*. Let A be a finitely generated commutative* $\mathbf{k}$*-algebra on which G acts through algebra automorphisms. If G is power reductive, then the subring of invariants A^G is a finitely generated* $\mathbf{k}$*-algebra.*

The proof follows Nagata [11] or rather the exposition of Springer [15, Theorem 2.4.9, Exercise 2.4.12]. See also Remark 9, Lemma 11 below. The proof does not need to touch upon the nontrivial topic of equivariant resolution by vector bundles [17]. It does not require further knowledge of G or $\mathbf{k}$. This is where power reductivity is more pertinent than geometric reductivity.

Remark 9. In the proof of finite generation of A^G by Nagata [11] the base ring $\mathbf{k}$ was a field. Nagata used at one point that a domain which is finitely generated over $\mathbf{k}$ has finite normalization. But that need no longer hold over our arbitrary commutative noetherian base ring $\mathbf{k}$. With the more elementary [15, Exercise 2.4.12] Springer avoided this step in the proof. His base ring was still a field but his audience did not know about normalizations. It is a happy accident that the modified proof goes through verbatim in our setting.

1.11. Necessary

The theorem has a converse showing that power reductivity is necessary if one seeks finite generation of invariants in the present setting, where algebras need not be domains. (In ancient Invariant Theory one considered invariants in a polynomial ring over $\mathbb{C}$ with a G-action that preserves the grading.)

Proposition 10. *Let* $\mathbf{k}$ *be a noetherian ring and let G be a flat affine group scheme over* $\mathbf{k}$*. Assume that the* $\mathbf{k}$*-algebra A^G is finitely generated for every finitely generated commutative* $\mathbf{k}$*-algebra A on which G acts through algebra automorphisms. Then G is power reductive.*

Proof. Let $f : A \twoheadrightarrow B$ be a surjective G-homomorphism of commutative $\mathbf{k}$-algebras, as in Proposition 7(iii). Let $b \in B^G$. We have to show b is integral over the image of A^G. As representations are locally finite, we may replace A with a finitely generated $\mathbf{k}$-subalgebra C

whose image D contains b. The symmetric algebra $S^*_C(D)$ is a finitely generated $\mathbf{k}$-algebra (a quotient of the polynomial ring $C[x]$), so $S^*_C(D)^G$ is finitely generated. We choose as our generators of $S^*_C(D)^G$ the homogeneous components of the elements of a finite generating set. The chosen generators in degree zero generate C^G and those in degree one generate D^G as a C^G-module. □

1.12. Graded

As a solution to [15, Exercise 2.4.12] we offer the following Lemma. It shows that in Theorem 8 one may assume that A is graded and generated over $\mathbf{k}$ by its degree one part.

Lemma 11. *Let A be a commutative $\mathbf{k}$-algebra on which G acts through algebra automorphisms. Let V be a G-submodule of A that is finitely generated as a $\mathbf{k}$-module and that generates A as a $\mathbf{k}$-algebra. Assume $1 \in V$. Let R be the graded $\mathbf{k}$-subalgebra generated by xV in the polynomial ring $A[x]$. Substituting $x \mapsto 1$ defines a surjection $R^G \twoheadrightarrow A^G$.*

Proof. The component R_d of homogeneous degree d maps injectively into A, so R^G_d hits all invariants in the image of R_d. The union of the images of the R_d is A. □

Lemma 12. *Power reductivity implies geometric reductivity.*

Proof. If $\mathbf{k}$ is a field this is clear, when using Definition 2. In the situation of 1.5, factor $\mathbf{k} \to F$ as $\mathbf{k} \twoheadrightarrow D \hookrightarrow F$, where D is the image of $\mathbf{k}$ in F. Observe that $D \hookrightarrow F$ is flat, so that $S^d_F(F \otimes_{\mathbf{k}} M^\vee)^G = (D \otimes_{\mathbf{k}} S^d(M^\vee))^G \otimes_D F$ (exercise, cf. [8, I.2.10(3)]). Recall that we denote by eval@v any map defined by evaluation at v. Now eval@v : $S^*_F(F \otimes_{\mathbf{k}} M^\vee)^G \to S^*_F F$ is power surjective. First take a positive integer d such that eval@v : $S^d_F(F \otimes_{\mathbf{k}} M^\vee)^G \to S^d_F F \simeq F$ is nonzero. Then eval@v : $(D \otimes_{\mathbf{k}} S^d(M^\vee))^G \to F$ must also be nonzero. Say $f \in (D \otimes_{\mathbf{k}} S^d(M^\vee))^G$ satisfies $f(v) \neq 0$. Now $S^*(M^\vee) \to (D \otimes_{\mathbf{k}} S^*(M^\vee))$ is surjective. So by part (ii) of Proposition 7 some power of f lifts to $S^d(M^\vee)^G$. □

Lemma 13. *If $\mathbf{k}$ is a discrete valuation ring, then geometric reductivity implies power reductivity.*

Proof. Let F be the residue field of $\mathbf{k}$. Given $\phi : M \twoheadrightarrow \mathbf{k}$ as in Definition 1 choose $m \in M$ with $\phi(m) = 1$. Use [13, Proposition 2, Proposition 3] to find a G-module map $\psi : N \to M$ with $m \in \psi(N)$ and N finitely generated and free as a $\mathbf{k}$-module. Take for $v \in (N^\vee \otimes_{\mathbf{k}} F)^G$ the composite $N \to M \to \mathbf{k} \to F$. We find a positive integer d and $f \in S^d(N)^G$ with $f(v)$ nonzero. That means that f maps to a unit times the standard generator of $S^d\mathbf{k}$ (Exercise). So $S^d N \to S^d\mathbf{k}$ splits. □

Remark 14. More generally, if one has equivariant resolution [17] (and local finiteness 1.7), one may reason as in [5, 3.1] to show that geometric reductivity implies power reductivity.

Lemma 15. *Let $\mathbf{k}$ be an $\mathbb{F}_p$-algebra and G a power reductive flat affine group scheme over $\mathbf{k}$. If $\phi : M \twoheadrightarrow \mathbf{k}$ is a surjective map of G-modules, then there is a non-negative integer n so that $S^{p^n}\phi$ is split surjective.*

Proof. In view of Lemma 6 it suffices to show that $S^*(M)^G \to S^*\mathbf{k}$ is p-power surjective. Indeed $S^*(M)^G[x] \to S^*\mathbf{k}[x]$ is power surjective because $S^*(M)[x] \to S^*\mathbf{k}[x]$ is (power) surjective. □

1.13. Restriction

Let S be a commutative $\mathbf{k}$-algebra. We get by base change a group G_S over S. Let M be a G_S-module. So M is in particular an S-module. Modules should not be confused with schemes. Nevertheless there is something similar to Weil restriction. Indeed M is also a $\mathbf{k}$-module, by restriction of scalars. Now the coaction $\Delta : M \to M \otimes_S S[G]$ has a target that may be identified with $M \otimes_{\mathbf{k}} \mathbf{k}[G]$. Thus, our G_S-module M may be viewed as a G-module (exercise) and $H^*(G_S, M) = H^*(G, M)$, because the Hochschild complexes [8, I.4.14] are isomorphic. In particular, $M^{G_S} = M^G$ and we usually write M^G.

1.14. Base change

Proposition 7 implies that power reductivity has marvelous base change properties.

Proposition 16. *Let $\mathbf{k} \to S$ be a map of commutative rings.*

(i) *If G is power reductive, then so is G_S.*
(ii) *If $\mathbf{k} \to S$ is faithfully flat and G_S is power reductive, then so is G.*
(iii) *If $G_{\mathbf{k}_\mathfrak{m}}$ is power reductive for every maximal ideal $\mathfrak{m}$ of $\mathbf{k}$, then G is power reductive.*

Proof. For the first part recall 1.13 that any G_S-module M may be viewed as a G-module with $M^{G_S} = M^G$. For the second part use that the integrality property in Proposition 7(iii) descends ([6, Proposition 2.7.1] or exercise). The last part holds for similar reasons [5, 3.1]. □

1.15. Reductive

An affine group scheme G over $\mathbf{k}$ is reductive in the sense of SGA3 [3] if G is smooth over $\mathbf{k}$ with geometric fibers that are connected reductive. Smooth implies algebraic.

Theorem 17 (*cf. [5, Theorem 12]*)**.** *Reductive group schemes (in the sense of SGA3) are power reductive.*

One exploits Proposition 16 and SGA3 [3], [2, §3, §5] to reduce to the case where the group is split and $\mathbf{k}$ is a local ring $\mathbb{Z}_{(p)}$. Then we are in the situation of [5, Theorem 12]. Or we may apply Lemma 13 and refer to Seshadri [14, Theorem 1].

Remark 18. Actually the proof of [5, Theorem 12] is overly complicated if $\mathbf{k} = \mathbb{Z}_{(p)}$. Let $\mathbf{k} = \mathbb{Z}_{(p)}$. As in the proof of Lemma 13 we may restrict attention to finitely generated free $\mathbf{k}$-modules in Definition 1. Then we need fewer arguments from section 3.4 of [5] (Exercise).

1.16. Finite

Recall that G is called a finite group scheme over $\mathbf{k}$ if the coordinate algebra $\mathbf{k}[G]$ is a finitely generated projective $\mathbf{k}$-module.

Theorem 19. *Finite group schemes are power reductive.*

In view of Proposition 7 this is an easy consequence of

Theorem 20 (*cf.* [12]). *If a finite group scheme G over a local ring* $\mathbf{k}$ *acts on a commutative* $\mathbf{k}$*-algebra A, then A is integral over A^G.*

Proof. Presumably the proofs in [4], [10, III 12, Thm 1] can be adapted to the present context. Theorem 20 is a special case of a more general result in the setting of groupoid schemes [3, Exposé V, Théorème 4.1]. That Theorem 20 fits in the setting of groupoid schemes is also explained at [16, Tag 03LK]. The proof of the theorem can then be found at [16, Tag 03BJ]. □

1.17. Reductive algebraic groups

Reductive algebraic groups defined over a field $\mathbf{k}$ are not assumed connected. They are of course power reductive. Indeed if G^0 is the identity component of a reductive G over a field, then both G^0 and G/G^0 are power reductive. Now see Proposition 7(ii). Or recall that Waterhouse [19] has shown that an algebraic affine group scheme G over a field is geometrically reductive if and only if the identity component G^0_{red} of its reduced subgroup G_{red} is reductive.

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