Algebra & Number Theory

Volume 14
2020
No. 9

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ALGEBRA AND NUMBER THEORY 14:9 (2020)
https://doi.org/10.2140/ant.2020.14.2295

The Brauer group of the moduli stack of elliptic curves

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We compute the Brauer group of $\mathcal{M}_{1,1}$, the moduli stack of elliptic curves, over $\operatorname{Spec}\mathbb{Z}$, its localizations, finite fields of odd characteristic, and algebraically closed fields of characteristic not 2. The methods involved include the use of the parameter space of Legendre curves and the moduli stack $\mathcal{M}(2)$ of curves with full (naive) level 2 structure, the study of the Leray–Serre spectral sequence in étale cohomology and the Leray spectral sequence in fppf cohomology, the computation of the group cohomology of S_3 in a certain integral representation, the classification of cubic Galois extensions of $\mathbb{Q}$, the computation of Hilbert symbols in the ramified case for the primes 2 and 3, and finding p-adic elliptic curves with specified properties.

1. Introduction

Brauer groups of fields have been considered since the times of Brauer and Noether; later Grothendieck generalized Brauer groups to the case of arbitrary schemes. Although both the definition via Azumaya algebras and the cohomological definition generalize to arbitrary Deligne–Mumford stacks, Brauer groups of stacks have so far mostly been neglected especially for stacks containing arithmetic information. Some exceptions are the use of Brauer groups of root stacks, as in the work of Chan and Ingalls [2005] on the minimal model program for orders on surfaces, the work of Auel, Bernardara and Bolognesi [Auel et al.

Benjamin Antieau was supported by NSF Grants DMS-1461847 and DMS-1552766. Lennart Meier was supported by DFG SPP 1786.
MSC2010: primary 14F22; secondary 14H52, 14K10.
Keywords: Brauer groups, moduli of elliptic curves, level structures, Hilbert symbols.

2014] on derived categories of families of quadrics, and the work of Lieblich [2011], who computed the Brauer group $\mathrm{Br}(\mathrm{B}\mu_n)$ over a field and applied it to the period-index problem. In this paper, we study the Brauer group $\mathrm{Br}(\mathcal{M})$ of the moduli stack of elliptic curves $\mathcal{M} = \mathcal{M}_{1,1}$.

The case of the Picard group has been considered before. Mumford [1965] showed that $\mathrm{Pic}(\mathcal{M}_k) = \mathrm{H}^1_{\text{ét}}(\mathcal{M}_k, \mathbb{G}_m) \cong \mathbb{Z}/12$ if k is a field of characteristic not dividing 6, and that the Picard group is generated by the Hodge bundle λ. The bundle λ is characterized by the property that $u^*\lambda \cong p_*\Omega^1_{E/T}$ when $u : T \to \mathcal{M}$ classifies a family $p : E \to T$ of elliptic curves. This calculation was extended by Fulton and Olsson who showed that $\mathrm{Pic}(\mathcal{M}_S) \cong \mathrm{Pic}(\mathbb{A}^1_S) \oplus \mathbb{Z}/12$ whenever S is a reduced scheme [Fulton and Olsson 2010].

In contrast, an equally uniform description of $\mathrm{Br}(\mathcal{M}_S)$ does not seem possible (even if we assume that S is regular noetherian); both the result and the proofs depend much more concretely on the arithmetic on S. The following is a sample of our results in ascending order of difficulty. We view (5) as the main result of this paper.

Theorem 1.1. (1) $\mathrm{Br}(\mathcal{M}_k) = 0$ *if k is an algebraically closed field of characteristic not* 2,[1]

(2) $\mathrm{Br}(\mathcal{M}_k) \cong \mathbb{Z}/12$ *if k is a finite field of characteristic not* 2,

(3) $\mathrm{Br}(\mathcal{M}_\mathbb{Q}) \cong \mathrm{Br}(\mathbb{Q}) \oplus \bigoplus_{p\not\equiv 3 \bmod 4} \mathbb{Z}/4 \oplus \bigoplus_{p\equiv 3 \bmod 4} \mathbb{Z}/2 \oplus \mathrm{H}^1(\mathbb{Q}, C_3)$, *where p runs over all primes and* -1,

(4) $\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/2]}) \cong \mathrm{Br}\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$, *and*

(5) $\mathrm{Br}(\mathcal{M}) = 0$.

In all cases the nontrivial classes can be explicitly described via cyclic algebras (see Lemma 7.2, Proposition 9.6 and Remark 9.8). In general, the p-primary torsion for $p \geq 5$ is often easy to control via the following theorem.

Theorem 1.2. *Let S be a regular noetherian scheme and $p \geq 5$ prime. Assume that $S[1/(2p)] = S_{\mathbb{Z}[1/(2p)]}$ is dense in S and that $\mathcal{M}_S \to S$ has a section. Then, the natural map ${}_p\mathrm{Br}'(S) \to {}_p\mathrm{Br}'(\mathcal{M}_S)$ is an isomorphism.*

Here, $\mathrm{Br}'(\mathcal{M}_S)$ denotes the cohomological Brauer group, which agrees with $\mathrm{Br}(\mathcal{M}_S)$ whenever S is affine and at least one prime is invertible on S.

Let us now explain how to compute the 2- and 3-primary torsion in the example of $\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$. We use the S_3-Galois cover $\mathcal{M}(2) \to \mathcal{M}_{\mathbb{Z}[1/2]}$, where $\mathcal{M}(2)$ is the moduli stack of elliptic curves with full (naive) level 2-structure. The Leray–Serre spectral sequence reduces the problem of computing $\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ to understanding the low-degree $\mathbb{G}_m$-cohomology of $\mathcal{M}(2)$ *together* with the action of S_3 on these cohomology groups and to the computation of differentials.

To understand the groups themselves, it is sufficient to use the fact that $\mathcal{M}(2) \cong \mathrm{B}C_{2,X}$, the classifying stack of cyclic C_2-covers over X, where X is the parameter space of Legendre curves. Explicitly, X is

[1] Minseon Shin [2019] has proved that when k is algebraically closed of characteristic 2 one has $\mathrm{Br}(\mathcal{M}_k) \cong \mathbb{Z}/2$.

the arithmetic surface

$$X = \mathbb{P}^1_{\mathbb{Z}[1/2]} - \{0, 1, \infty\} = \operatorname{Spec} \mathbb{Z}\big[\tfrac{1}{2}, t^{\pm 1}, (t-1)^{-1}\big],$$

and the universal Legendre curve over X is defined by the equation

$$y^2 = x(x-1)(x-t).$$

The Brauer group of $\mathrm{B}C_{2,X}$ can be described using a Leray–Serre spectral sequence as well, but this description is not S_3-equivariant, which causes some complications, and we have to use the S_3-equivariant map from $\mathcal{M}(2)$ to X, its coarse moduli space, to get full control. Knowledge about the Brauer group of X leads for the 3-primary torsion to the following conclusion.

Theorem 1.3. *Let S be a regular noetherian scheme. If 6 is a unit on S, then there is an exact sequence*

$$0 \to {}_3\operatorname{Br}'(S) \to {}_3\operatorname{Br}'(\mathcal{M}_S) \to \mathrm{H}^1(S, C_3) \to 0,$$

which is noncanonically split.

There is a unique cubic Galois extension of $\mathbb{Q}$ which is ramified at most at (2) and (3), namely $\mathbb{Q}(\zeta_9 + \bar{\zeta}_9)$. This shows that the cokernel of $\operatorname{Br}\big(\mathbb{Z}\big[\tfrac{1}{6}\big]\big) \hookrightarrow \operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/6]})$ is $\mathbb{Z}/3$. The proof that this extra class does not extend to $\mathcal{M}_{\mathbb{Z}[1/2]}$, which is similar to the strategy discussed below for the 2-torsion, uses the computation of cubic Hilbert symbols at the prime 3. Putting these ingredients together, we conclude that $\operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ is a 2-group, and further computations first over $\mathbb{Z}\big[\tfrac{1}{2}, i\big]$ and then over $\mathbb{Z}\big[\tfrac{1}{2}\big]$ let us deduce the structure in Theorem 1.1(4). The corresponding general results on 2-torsion are contained in Proposition 9.3 and Theorem 9.1. They are somewhat more complicated to state so we omit them from this introduction.

To show that $\operatorname{Br}(\mathcal{M}) = 0$, we need an extra argument since all our arguments using the Leray–Serre spectral sequence presuppose at least that 2 is inverted. Note first that the map $\operatorname{Br}(\mathcal{M}) \to \operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ is an injection. Thus, we have only to show that the nonzero classes in $\operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ do not extend to $\mathcal{M}$. Our general method is the following. For each nonzero class α in the Brauer group of $\mathcal{M}_{\mathbb{Z}[1/2]}$, we exhibit an elliptic curve over $\operatorname{Spec} \mathbb{Z}_2$ such that the restriction of α to $\operatorname{Spec} \mathbb{Q}_2$ is nonzero. Such an elliptic curve defines a morphism $\operatorname{Spec} \mathbb{Z}_2 \to \mathcal{M}$ and we obtain a commutative diagram

$$\begin{tikzcd} \operatorname{Br}(\mathbb{Q}_2) & \operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]}) \arrow[l] \\ \operatorname{Br}(\mathbb{Z}_2) \arrow[u] & \operatorname{Br}(\mathcal{M}) \arrow[l] \arrow[u] \end{tikzcd} \tag{1.4}$$

Together with the fact that $\operatorname{Br}(\mathbb{Z}_2) = 0$, this diagram implies that the class α cannot come from $\operatorname{Br}(\mathcal{M})$. This argument requires us to understand explicit generators for $\operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ and the computation of Hilbert symbols again.

We remark that the computation of $\operatorname{Br}(\mathcal{M})$ — while important in algebraic geometry and in the arithmetic of elliptic curves — was nevertheless originally motivated by considerations in chromatic homotopy theory, especially in the possibility in constructing twisted forms of the spectrum TMF of topological modular forms. The Picard group computations of Mumford and Fulton and Olsson are primary inputs into the computation of the Picard group of TMF due to Mathew and Stojanoska [2016].

Conventions. We will have occasion to use Zariski, étale, and fppf cohomology of schemes and Deligne–Mumford stacks. We will denote these by $\mathrm{H}^i_{\mathrm{Zar}}(X, F)$, $\mathrm{H}^i(X, F)$, and $\mathrm{H}^i_{\mathrm{pl}}(X, F)$ when F is an appropriate sheaf on X. Note in particular that without other adornment, $\mathrm{H}^i(X, F)$ or $\mathrm{H}^i(R, F)$ (when $X = \operatorname{Spec} R$) always denotes étale cohomology. If G is a group and F is a G-module, we let $\mathrm{H}^i(G, F)$ denote the group cohomology.

For all stacks $\mathcal{X}$ appearing in this paper, we will have $\mathrm{Br}'(\mathcal{X}) \cong \mathrm{H}^2(\mathcal{X}, \mathbb{G}_m)$. Thus, we will use the two groups interchangeably. When working over a general base S, we will typically state our results in terms of $\mathrm{Br}'(S)$ or $\mathrm{Br}'(\mathcal{M}_S)$. However, when working over an affine scheme, such as $S = \operatorname{Spec} \mathbb{Z}\left[\frac{1}{2}\right]$, we will write $\mathrm{Br}(S)$ or $\mathrm{Br}(\mathcal{M}_S)$. There should be no confusion as in all of these cases we will have $\mathrm{Br}(S) = \mathrm{Br}'(S)$ and $\mathrm{Br}(\mathcal{M}_S) = \mathrm{Br}'(\mathcal{M}_S)$ and so on by [de Jong 2005].

For an abelian group A and an integer n, we will denote by ${}_nA$ the subgroup of n-primary torsion elements: ${}_nA = \{x \in A : n^k x = 0 \text{ for some } k \geq 1\}$.

2. Brauer groups, cyclic algebras, and ramification

We review here some basic facts about the Brauer group, with special attention to providing references for those facts in the generality of Deligne–Mumford stacks. For more details about the Brauer group in general; see [Grothendieck 1968a].

Any Deligne–Mumford stack has an associated étale topos, and we can therefore consider étale sheaves and étale cohomology [Laumon and Moret-Bailly 2000].

Definition 2.1. If X is a quasicompact and quasiseparated Deligne–Mumford stack, the *cohomological Brauer group* of X is defined to be $\mathrm{Br}'(X) = \mathrm{H}^2(X, \mathbb{G}_m)_{\mathrm{tors}}$, the torsion subgroup of $\mathrm{H}^2(X, \mathbb{G}_m)$.

Because of its definition as the torsion in a cohomology group, the cohomological Brauer group is amenable to computation via Leray–Serre spectral sequences, long exact sequences, and so on, as we will see in the next sections. However, our main interest is in the *Brauer group* of Deligne–Mumford stacks.

Definition 2.2. An *Azumaya algebra* over a Deligne–Mumford stack X is a sheaf of quasicoherent $\mathcal{O}_X$-algebras $\mathcal{A}$ such that $\mathcal{A}$ is étale-locally on X isomorphic to $\mathcal{M}_n(\mathcal{O}_X)$, the sheaf of $n \times n$-matrices over $\mathcal{O}_X$, for some $n \geq 1$.

In particular, an Azumaya algebra $\mathcal{A}$ is a locally free $\mathcal{O}_X$-module, and the degree n appearing in the definition is a locally constant function. If the degree n is in fact constant, then $\mathcal{A}$ corresponds to a unique PGL_n-torsor on X because the group of k-algebra automorphisms of $\mathrm{M}_n(k)$ is isomorphic to $\mathrm{PGL}_n(k)$ for fields k. The exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$ gives a boundary map

$$\delta : \mathrm{H}^1(X, \mathrm{PGL}_n) \to \mathrm{H}^2(X, \mathbb{G}_m).$$

For an Azumaya algebra $\mathcal{A}$ of degree n, we write $[\mathcal{A}]$ for the class $\delta(\mathcal{A})$ in $\mathrm{H}^2(X, \mathbb{G}_m)$. In general, when X has multiple connected components, its invariant $[\mathcal{A}] \in \mathrm{H}^2(X, \mathbb{G}_m)$ is computed on each component.

Example 2.3. (1) If E is a vector bundle on X of rank $n > 0$, then $\mathcal{A} = \mathcal{E}\mathrm{nd}(E)$, the sheaf of endomorphisms of E, is an Azumaya algebra on X. Indeed, in this case, $\mathcal{A}$ is even Zariski-locally equivalent to $\mathcal{M}_n(\mathcal{O}_X)$. The class of $\mathcal{A}$ in $\mathrm{H}^1(X, \mathrm{PGL}_n)$ is the image of E via $\mathrm{H}^1(X, \mathrm{GL}_n) \to \mathrm{H}^1(X, \mathrm{PGL}_n)$, so the long exact sequence in nonabelian cohomology implies that $[\mathcal{A}] = 0$ in $\mathrm{H}^2(X, \mathbb{G}_m)$.

(2) If $\mathcal{A}$ and $\mathcal{B}$ are Azumaya algebras on X, then $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}$ is an Azumaya algebra.

Definition 2.4. Two Azumaya algebras $\mathcal{A}$ and $\mathcal{B}$ are *Brauer equivalent* if there are vector bundles E and F on X such that $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{E}\mathrm{nd}(E) \cong \mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{E}\mathrm{nd}(F)$. The *Brauer group* $\mathrm{Br}(X)$ of a Deligne–Mumford stack X is the multiplicative monoid of isomorphism classes of Azumaya algebras under tensor product modulo Brauer equivalence.

In terms of Azumaya algebras, addition is given by the tensor product $[\mathcal{A}] + [\mathcal{B}] = [\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}]$, and $-[\mathcal{A}] = [\mathcal{A}^{\mathrm{op}}]$. Here are the basic structural facts we will use about the Brauer group.

Proposition 2.5. (i) *The Brauer group of a quasicompact and quasiseparated Deligne–Mumford stack X is the subgroup of* $\mathrm{Br}'(X)$ *generated by* $[\mathcal{A}]$ *for $\mathcal{A}$ an Azumaya algebra on X.*

(ii) *The Brauer group* $\mathrm{Br}(X)$ *is a torsion group for any quasicompact and quasiseparated Deligne–Mumford stack X.*

(iii) *If X is a regular and noetherian Deligne–Mumford stack, then* $\mathrm{H}^2(X, \mathbb{G}_m)$ *is torsion, so in particular* $\mathrm{Br}'(X) \cong \mathrm{H}^2(X, \mathbb{G}_m)$.

(iv) *If X is a regular and noetherian Deligne–Mumford stack, and if $U \subseteq X$ is a dense open subset, then the restriction map* $\mathrm{H}^2(X, \mathbb{G}_m) \to \mathrm{H}^2(U, \mathbb{G}_m)$ *is injective.*

(v) *If X is a scheme with an ample line bundle, then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

(vi) *If X is a regular and noetherian scheme with p invertible on X, then the morphism* ${}_p\mathrm{H}^i(X, \mathbb{G}_m) \to {}_p\mathrm{H}^i(\mathbb{A}^1_X, \mathbb{G}_m)$ *on p-primary torsion is an isomorphism for all $i \geq 0$.*

Proof. See [Grothendieck 1968a, Section 2] for points (i) and (ii). The proof of (iv) is analogous to that of [Lieblich 2008, Lemma 3.1.3.3], using an analogue of [Laumon and Moret-Bailly 2000, Proposition 15.4] to generalize [Lieblich 2008, Lemma 3.1.1.9] to algebraic stacks. See also [Auel et al. 2014, Proposition 1.26]. For (v), see [de Jong 2005].

For (iii), see for instance [Grothendieck 1968b, Proposition 1.4] in the case of schemes. We must generalize it to the case of a regular noetherian Deligne–Mumford stack X. We can assume that X is connected and hence irreducible as X is normal. Pick $U \subseteq X$ a dense open such that U admits a *finite* étale map $V \to U$ of degree n where V is a scheme. The composition $\mathrm{H}^2(U, \mathbb{G}_m) \to \mathrm{H}^2(V, \mathbb{G}_m) \to \mathrm{H}^2(U, \mathbb{G}_m)$ of restriction and transfer is multiplication by n. Since $\mathrm{H}^2(V, \mathbb{G}_m)$ is torsion, this implies that $\mathrm{H}^2(U, \mathbb{G}_m)$ is torsion. By (iv), $\mathrm{H}^2(X, \mathbb{G}_m)$ is torsion as well.

Finally, we have to prove (vi). For $i \le 1$, the $\mathbb{A}^1$-invariance is even true before taking p-power torsion (see e.g., [Hartshorne 1977, II.6.6 and II.6.15]). Because p is invertible on S, the maps $\mathrm{H}^i(X, \mu_{p^m}) \to \mathrm{H}^i(\mathbb{A}^1_X, \mu_{p^m})$ are isomorphisms for all i and m. This is the $\mathbb{A}^1$-invariance of étale cohomology and a proof can be found in [Milne 1980, Corollary VI.4.20]. The short exact sequence

$$1 \to \mu_{p^m} \to \mathbb{G}_m \xrightarrow{p^m} \mathbb{G}_m \to 1$$

induces short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^{i-1}(X, \mathbb{G}_m)/p^m \arrow[r] \arrow[d] & \mathrm{H}^i(X, \mu_{p^m}) \arrow[r] \arrow[d, "\cong"] & \mathrm{H}^i(X, \mathbb{G}_m)[p^m] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{H}^{i-1}(\mathbb{A}^1_X, \mathbb{G}_m)/p^m \arrow[r] & \mathrm{H}^i(\mathbb{A}^1_X, \mu_{p^m}) \arrow[r] & \mathrm{H}^i(\mathbb{A}^1_X, \mathbb{G}_m)[p^m] \arrow[r] & 0
\end{tikzcd}$$

Inductively, using the $\mathbb{A}^1$-invariance of $\mathrm{H}^{i-1}(X, \mathbb{G}_m)$ if $i \le 2$ or of ${}_p\mathrm{H}^{i-1}(X, \mathbb{G}_m)$ if $i > 2$ as well as the fact that $\mathrm{H}^{i-1}(X, \mathbb{G}_m)$ is torsion for $i \ge 3$ since X is regular and noetherian [Grothendieck 1968b, Proposition 1.4], we see by the five lemma that $\mathrm{H}^i(X, \mathbb{G}_m)[p^m] \to \mathrm{H}^i(\mathbb{A}^1_X, \mathbb{G}_m)[p^m]$ is an isomorphism for all i and m and hence we also get an isomorphism ${}_p\mathrm{H}^i(X, \mathbb{G}_m) \to {}_p\mathrm{H}^i(\mathbb{A}^1_X, \mathbb{G}_m)$. □

Remark 2.6. At a couple points, we use another important fact, due to Gabber, which says that if $p : Y \to X$ is a surjective finite locally free map, if $\alpha \in \mathrm{Br}'(X)$, and if $p^*\alpha \in \mathrm{Br}(Y)$, then $\alpha \in \mathrm{Br}(X)$. This is already proved in [Gabber 1981, Chapter II, Lemma 4] for locally ringed topoi with strict hensel local rings, so we need to add nothing further in our setting.

By far the most important class of Azumaya algebras arising in arithmetic applications is the class of cyclic algebras. For a treatment over fields, see [Gille and Szamuely 2006, Section 2.5]. These algebras give a concrete realization of the cup product in fppf cohomology

$$\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X}, C_n) \times \mathrm{H}^1_{\mathrm{pl}}(\mathcal{X}, \mu_n) \to \mathrm{H}^2_{\mathrm{pl}}(\mathcal{X}, \mu_n) \to \mathrm{H}^2_{\mathrm{pl}}(\mathcal{X}, \mathbb{G}_m)$$

for an algebraic stack $\mathcal{X}$. Given that C_n and $\mathbb{G}_m$ are smooth and that the image of such a cup product is torsion, we can rewrite this as $\mathrm{H}^1(\mathcal{X}, C_n) \times \mathrm{H}^1_{\mathrm{pl}}(\mathcal{X}, \mu_n) \to \mathrm{H}^2_{\mathrm{pl}}(\mathcal{X}, \mu_n) \to \mathrm{Br}'(\mathcal{X})$. Given $\chi \in \mathrm{H}^1(\mathcal{X}, C_n)$ and $u \in \mathrm{H}^1_{\mathrm{pl}}(\mathcal{X}, \mu_n)$, we write $[(\chi, u)_n]$ or $[(\chi, u)]$ for the image of the cup product in $\mathrm{Br}'(\mathcal{X})$.

Fix an algebraic stack $\mathcal{X}$. Let $p(\chi) : \mathcal{Y} \to \mathcal{X}$ be the cyclic Galois cover defined by $\chi \in \mathrm{H}^1(\mathcal{X}, C_n)$. Then, $p(\chi)_*\mathcal{O}_\mathcal{Y}$ is a locally free $\mathcal{O}_\mathcal{X}$-algebra of finite rank which comes equipped with a canonical C_n-action. There is a natural isomorphism $\mathbf{Spec}_\mathcal{X}(p(\chi)_*\mathcal{O}_\mathcal{Y}) \cong \mathcal{Y} \to \mathcal{X}$.

The group $\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X}, \mu_n)$ fits into a short exact sequence

$$0 \to \mathbb{G}_m(\mathcal{X})/n \to \mathrm{H}^1_{\mathrm{pl}}(\mathcal{X}, \mu_n) \to \mathrm{Pic}(\mathcal{X})[n] \to 0. \tag{2.7}$$

It is helpful to have a more concrete description of $\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X}, \mu_n)$, which will also show that the exact sequence (2.7) is noncanonically split. Let $\mathrm{H}(\mathcal{X}, n)$ be the abelian group of equivalence classes of pairs $(\mathcal{L}, s)$ where $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})[n]$ and s is a choice of trivialization $s : \mathcal{O}_\mathcal{X} \to \mathcal{L}^{\otimes n}$. Two pairs $(\mathcal{L}, s)$ and $(\mathcal{M}, t)$

are equivalent if there is an isomorphism $g:\mathcal{L}\to\mathcal{M}$ and a unit $v\in\mathbb{G}_m(\mathcal{X})$ such that $g(s)=v^nt$. The group structure is given by tensor product of line bundles and of trivializations. The following construction is part of Kummer theory and is well-known in the scheme case (see e.g., [Milne 1980, page 125], which also shows part of Proposition 2.9 below).

Construction 2.8. Let $\mathcal{X}$ be an algebraic stack and fix a class $[u]\in\mathrm{H}(\mathcal{X},n)$ with $u=(\mathcal{L},s)$ for a line bundle $\mathcal{L}$ on $\mathcal{X}$ with a trivialization $s\colon\mathcal{O}_\mathcal{X}\to\mathcal{L}^{\otimes n}$. Define $\mathcal{O}_\mathcal{X}(\sqrt[n]{u})$ as $\bigoplus_{i\in\mathbb{Z}}\mathcal{L}^{\otimes i}/(s-1)$ and $\mathcal{X}(\sqrt[n]{u})\to\mathcal{X}$ as the affine morphism $\mathbf{Spec}_\mathcal{X}\,\mathcal{O}_\mathcal{X}(\sqrt[n]{u})\to\mathcal{X}$. It is easy to see that $\mathcal{X}(\sqrt[n]{u})\to\mathcal{X}$ is an fppf μ_n-torsor and that this construction defines a group homomorphism $\mathrm{H}(\mathcal{X},n)\to\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X},\mu_n)$. Indeed, if $\mathcal{X}=\operatorname{Spec}A$ and $\mathcal{L}$ is trivial, then $\mathcal{X}(\sqrt[n]{u})\cong\operatorname{Spec}A(\sqrt[n]{s^{-1}})\cong\operatorname{Spec}A(\sqrt[n]{s})$.

Proposition 2.9. *The map* $\mathrm{H}(\mathcal{X},n)\to\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X},\mu_n)$ *is an isomorphism. In particular, there is a noncanonical splitting*

$$\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X},\mu_n)\cong\mathbb{G}_m(\mathcal{X})/n\oplus\operatorname{Pic}(\mathcal{X})[n].$$

Proof. We claim that the line bundle associated with the μ_n-torsor $\mathcal{X}(\sqrt[n]{u})\to\mathcal{X}$ (via the map $\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X},\mu_n)\to\mathrm{H}^1(\mathcal{X},\mathbb{G}_m)$ induced by the inclusion $\mu_n\to\mathbb{G}_m$) is exactly $\mathcal{L}$. Indeed, the obvious map from $\mathcal{X}(\sqrt[n]{u})$ to $\mathbf{Spec}_\mathcal{X}\big(\bigoplus_{i\in\mathbb{Z}}\mathcal{L}^{\otimes i}\big)$ is equivariant along the inclusion $\mu_n\to\mathbb{G}_m$ and the target is the $\mathbb{G}_m$-torsor associated with $\mathcal{L}$. Thus, the composition $\mathrm{H}(\mathcal{X},n)\to\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X},\mu_n)\to\operatorname{Pic}(\mathcal{X})[n]$ is surjective. By (2.7), to prove that $\mathrm{H}(\mathcal{X},n)\to\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X},\mu_n)$ is an isomorphism, it suffices to prove that the induced map from the kernel of this map to $\mathbb{G}_m(\mathcal{X})/n$ is an isomorphism. But, this follows immediately from the definition of $\mathrm{H}(\mathcal{X},n)$.

It remains to construct the splitting. By Prüfer's theorem [Fuchs 1970, Theorem 17.2], $\operatorname{Pic}(\mathcal{X})[n]$ is a direct sum of cyclic groups. Thus, we have only to show that for every divisor k of n and each k-torsion element $[\mathcal{L}]$ in $\operatorname{Pic}(\mathcal{X})$, there exists a preimage in $\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X},\mu_n)$ that is k-torsion. This preimage can be constructed as follows: choose a trivialization $s\colon\mathcal{O}_\mathcal{X}\to\mathcal{L}^{\otimes k}$ and take the μ_n-torsor associated with the μ_k-torsor $\mathcal{X}(\sqrt[k]{v})\to\mathcal{X}$ with $v=(\mathcal{L},s)$. $\square$

Now, given χ and u as above, we let $\tilde{\mathcal{A}}_{\chi,u}$ be the coproduct

$$\Big(p(\chi)_*\mathcal{O}_\mathcal{Y}\coprod_{\mathcal{O}_\mathcal{X}}\mathcal{O}_\mathcal{X}(\sqrt[n]{u})\Big)$$

in the category of sheaves of quasicoherent (associative and unital) $\mathcal{O}_\mathcal{X}$-algebras. Finally, we let

$$\mathcal{A}_{\chi,u}=\tilde{\mathcal{A}}_{\chi,u}/(ab-ba^{g(\chi)})$$

be the quotient of $\tilde{\mathcal{A}}_{\chi,u}$ by the two-sided ideal generated by terms $ab-ba^{g(\chi)}$ where a is a local section of $p(\chi)_*\mathcal{O}_\mathcal{Y}$, b is a local section of $\mathcal{O}_\mathcal{X}(\sqrt[n]{u})$, and $a^{g(\chi)}$ denotes the action of $g(\chi)$ on $p(\chi)_*\mathcal{O}_\mathcal{Y}$ for $g(\chi)$ a fixed generator of the action of C_n on $p(\chi)_*\mathcal{O}_\mathcal{Y}$.

Lemma 2.10. *Given an algebraic stack* $\mathcal{X}$ *and classes* $\chi\in\mathrm{H}^1(\mathcal{X},C_n)$ *and* $u\in\mathrm{H}^1_{\mathrm{pl}}(\mathcal{X},\mu_n)$, *the algebra* $\mathcal{A}_{\chi,u}$ *is an Azumaya algebra on* $\mathcal{X}$.

Proof. It suffices to check this fppf-locally, and in particular we can assume that in fact $\mathcal{X}$ is a scheme X, that $\mathcal{L} \cong \mathcal{O}_{\mathcal{X}}$, and that χ classifies a C_n-Galois cover $p(\chi)\colon Y \to X$. In this case, we can write the $\mathcal{O}_X$-algebra $p(\chi)_*\mathcal{O}_Y$ Zariski locally as a quotient $\mathcal{O}_X[x]/f(x)$ for some monic polynomial $f(x)$ of degree n. Then, locally, we have that

$$\mathcal{A}_{\chi,u} \cong \mathcal{O}_X\langle x, y\rangle/(f(x), y^n - u, xy - yx^{g(x)}),$$

a quotient of the free algebra over $\mathcal{O}_X$ on generators x and y. Note that the sections $x^i y^j$ for $0 \le i, j \le n-1$ form a basis of $\mathcal{A}_{\chi,u}$ as an $\mathcal{O}_X$-module and in particular that $\mathcal{A}_{\chi,u}$ is (locally) a free $\mathcal{O}_X$-module. Examining the fibers of $\mathcal{A}_{\chi,u}$ over X, we obtain the usual definition of a cyclic algebra given in [Gille and Szamuely 2006, Proposition 2.5.2]. So, $\mathcal{A}_{\chi,u}$ is locally free with central simple fibers and [Grothendieck 1968a, Théorème 5.1] implies that $\mathcal{A}_{\chi,u}$ is Azumaya. □

The following proposition is well-known, but we do not know an exact reference. However, in the case of quaternion algebras, it is given in [Parimala and Srinivas 1992, Lemma 8].

Proposition 2.11. *Let X be a regular noetherian scheme and suppose that $\chi \in \mathrm{H}^1(X, C_n)$ and $u \in \mathrm{H}^1_{\mathrm{pl}}(X, \mu_n)$ are fixed classes. In the notation above, we have $[(\chi, u)_n] = [\mathcal{A}_{\chi,u}]$ in $\mathrm{Br}'(X)$. In particular $[(\chi, u)_n] \in \mathrm{Br}(X)$.*

Proof. We can assume that X is connected. As $\mathrm{Br}'(X) \to \mathrm{Br}'(K)$ is injective (see [Milne 1980, IV.2.6] or Proposition 2.5(iv)), it is enough to check this on a generic point $\operatorname{Spec} K$ of X. By definition and the previous example, $\mathcal{A}_{\chi,u}$ is a standard cyclic algebra over K as defined in [Gille and Szamuely 2006, Chapter 2]. They check in [loc. cit., Proposition 4.7.3] that $\mathcal{A}_{\chi,u}$ does indeed have Brauer class given by the cup product. See also the remark at the beginning of the proof of [loc. cit., Proposition 4.7.1]. □

Remark 2.12. The reader may notice that $\mathcal{A}_{\chi,u}$ is defined in complete generality, but that we only prove the equality $[(\chi, u)_n] = [\mathcal{A}_{\chi,u}]$ for regular noetherian schemes. In fact, this equality extends to arbitrary algebraic stacks, but a different argument is necessary. It is given at the end of Section 3.

We will abuse notation and write $(\chi, u)_n$ or even just (χ, u) for $\mathcal{A}_{\chi,u}$. This is called a *cyclic algebra*. If there is a primitive n-th root of unity $\omega \in \mu_n(\mathcal{X})$ and the cyclic Galois cover $\mathcal{Y} \to \mathcal{X}$ is obtained by adjoining an n-th root of an element $a \in \mathbb{G}_m(\mathcal{X})$, we write $(a, u) = (a, u)_\omega$ for the corresponding cyclic algebra, where we need the choice of ω to fix an isomorphism $\mathrm{H}^1(\mathcal{X}, C_n) \cong \mathrm{H}^1(\mathcal{X}, \mu_n)$. For $n = 2$, we obtain the classical notion of a quaternion algebra.

For us, the key point about the cyclic algebra is that it allows us to compute the ramification of a Brauer class explicitly. Before explaining this, we mention that by the Gabber–Česnavičius purity theorem the Brauer group of a regular noetherian scheme X is insensitive to throwing away high codimension subschemes.

Proposition 2.13 (purity [Gabber 1981, Chapter I; Česnavičius 2019]). *Let X be a regular noetherian scheme. If $U \subseteq X$ is a dense open subscheme with complement of codimension at least 2, then the restriction map $\mathrm{Br}'(X) \to \mathrm{Br}'(U)$ is an isomorphism.*

Let X be a regular noetherian scheme and let η be the scheme of generic points in X. The purity theorem reduces the problem of computing $\mathrm{Br}(X)$ from $\mathrm{Br}(\eta)$ to the problem of extending Brauer classes $\alpha \in \mathrm{Br}(\eta)$ over divisors in X. This is controlled by ramification theory. The following proposition is basically well-known, but we include a proof for the reader's convenience.

Proposition 2.14. *Let X be a regular noetherian scheme, $i : D \subseteq X$ a Cartier divisor that is regular with complement U, and n an integer. There is an exact sequence*

$$0 \to {}_n\mathrm{Br}'(X) \to {}_n\mathrm{Br}'(U) \xrightarrow{\mathrm{ram}_D} {}_n\mathrm{H}^3_D(X, \mathbb{G}_m) \to {}_n\mathrm{H}^3(X, \mathbb{G}_m) \to {}_n\mathrm{H}^3(U, \mathbb{G}_m). \tag{2.15}$$

If n is prime to the residue characteristics of X, we have ${}_n\mathrm{H}^3_D(X, \mathbb{G}_m) \cong \mathrm{H}^1(D, {}_n\mathbb{Q}/\mathbb{Z})$.

Proof. By [Grothendieck 1968c, 6.1] or [Milne 1980, III.1.25] there is a long exact sequence

$$\mathrm{H}^2(X, \mathbb{G}_m) \to \mathrm{H}^2(U, \mathbb{G}_m) \to \mathrm{H}^3_D(X, \mathbb{G}_m) \to \mathrm{H}^3(X, \mathbb{G}_m) \to \mathrm{H}^3(U, \mathbb{G}_m).$$

By [Grothendieck 1968b, Propsition 1.4] all occurring groups are torsion so that the sequence is still exact after taking n-primary torsion. Furthermore, by Proposition 2.5 the first map is an injection.

We may assume that $n = p$ is a prime, in which case ${}_p\mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}_p/\mathbb{Z}_p$. We have to show that ${}_p\mathrm{H}^3_D(X, \mathbb{G}_m) \cong \mathrm{H}^1(D, \mathbb{Q}_p/\mathbb{Z}_p)$. By either the relative cohomological purity theorem of Artin [SGA 4_3 1973, Théorème XVI.3.7 and 3.8] (when both X and D are smooth over some common base scheme S) or the absolute cohomological purity theorem of Gabber [Fujiwara 2002, Theorem 2.1], we have the following identifications of local cohomology sheaves: $\mathcal{H}^t_D(\mu_{p^\nu}) = 0$ for $t \neq 2$ and $\mathcal{H}^2_D(\mu_{p^\nu}) \cong i_*\mathbb{Z}/p^\nu(-1)$. It follows from the long exact sequence of local cohomology sheaves associated to the exact sequence $1 \to \mu_{p^\nu} \to \mathbb{G}_m \xrightarrow{p^\nu} \mathbb{G}_m \to 1$ that p acts invertibly on $\mathcal{H}^t_D(\mathbb{G}_m)$ for $t \neq 1, 2$. Moreover, since X is regular and noetherian, for every open $V \subset X$, the map $\mathrm{Pic}(V) \to \mathrm{Pic}(U \cap V)$ is surjective with kernel $(i_*\mathbb{Z})(V)$ by [Hartshorne 1977, II.6.5] and $\mathrm{Br}'(V) \to \mathrm{Br}'(U \cap V)$ is injective; thus $\mathcal{H}^2_D(\mathbb{G}_m) = 0$ and $\mathcal{H}^1_D(\mathbb{G}_m) \cong i_*\mathbb{Z}$. Therefore, the only contribution to p-primary torsion in $\mathrm{H}^3_D(X, \mathbb{G}_m)$ in the local to global spectral sequence

$$\mathrm{H}^s(X, \mathcal{H}^t_D(\mathbb{G}_m)) \Rightarrow \mathrm{H}^{s+t}_D(X, \mathbb{G}_m)$$

is ${}_p\mathrm{H}^2(X, i_*\mathbb{Z})$. We obtain

$${}_p\mathrm{H}^3_D(X, \mathbb{G}_m) \cong {}_p\mathrm{H}^2(X, \mathcal{H}^1_D(\mathbb{G}_m)) \cong {}_p\mathrm{H}^2(D, \mathbb{Z}) \cong \mathrm{H}^1(D, \mathbb{Q}_p/\mathbb{Z}_p)$$

as desired, where the last isomorphism holds because $\mathrm{H}^i(D, \mathbb{Q}) = 0$ for $i > 0$ since D is normal (see for example [Deninger 1988, 2.1]). □

Note that in all cases where we use Proposition 2.14, the easier relative cohomological purity theorem of Artin is applicable, so that in the end our paper does not rely on the more difficult results of Gabber and Česnavičius.

We will need to know a special case of the ramification map $\mathrm{ram}_D : \mathrm{Br}(U) \to \mathrm{H}^1(D, \mathbb{Q}/\mathbb{Z})$.

Proposition 2.16. *Let R be a discrete valuation ring with fraction field K and residue field k. Set $X = \operatorname{Spec} R$, $U = \operatorname{Spec} K$, and $x = \operatorname{Spec} k$. Let $(\chi, \pi)_n$ be a cyclic algebra over K, where χ is a degree n cyclic character of K, π is a uniformizing parameter of R (viewed as an element of $\mathbb{G}_m(K)/n$), and n is prime to the characteristic of k. Finally, let L/K be the cyclic Galois extension defined by χ. If the integral closure S of R in L is a discrete valuation ring with uniformizing parameter π_S, then $\operatorname{ram}_{(\pi)}(\chi, \pi)$ is the class of the cyclic extension $S/(\pi_S)$ over k.*

Proof. See [Saltman 1999, Lemma 10.2]. □

Finally, we discuss cyclic algebras over local fields and some implications for global calculations. Let K be a local field containing a primitive n-th root of unity ω. Then there is a pairing

$$\left(\frac{-,-}{\mathfrak{p}}\right) : \mathbb{G}_m(K)/n \times \mathbb{G}_m(K)/n \to \mu_n(K),$$

called the *Hilbert symbol* (where $\mathfrak{p}$ stands for the maximal ideal of the ring of integers of K). Our standard reference for this pairing is [Neukirch 1999, Section V.3]. If $\mathfrak{p}$ is generated by an element π, we will also write $\left(\frac{a,b}{\pi}\right)$. We will use Hilbert symbols to check whether explicitly defined cyclic algebras are zero in the Brauer group.

Proposition 2.17. *For $a, b \in K^\times$, the cyclic algebra $(a, b)_\omega$ is trivial in $\operatorname{Br}(K)$ if and only if $\left(\frac{a,b}{\mathfrak{p}}\right) = 1$.*

Proof. By Proposition V.3.2 of [Neukirch 1999] the Hilbert symbol $\left(\frac{a,b}{\mathfrak{p}}\right)$ equals 1 if and only if a is a norm from the extension $K(\sqrt[n]{b})|K$. By [Gille and Szamuely 2006, Corollary 4.7.7], this happens if and only if $(a, b)_\omega$ splits, i.e., defines the trivial class in $\operatorname{Br}(K)$. □

More generally, local class field theory calculates $\operatorname{Br}(\eta)$ when $\eta = \operatorname{Spec} K$ where K is a (nonarchimedean) local field. Let $X = \operatorname{Spec} R$ and $x = \operatorname{Spec} k$, where R is the ring of integers in K and k is the residue field of R. As $\mathrm{H}^3(X, \mathbb{G}_m) \cong \mathrm{H}^3(x, \mathbb{G}_m) = 0$ (for instance by [Grothendieck 1968c, Théorème 1.1]), we find from [loc. cit., Corollaire 2.2] that there is an exact sequence

$$0 \to \operatorname{Br}(X) \to \operatorname{Br}(\eta) \to \mathrm{H}^1(x, \mathbb{Q}/\mathbb{Z}) \to 0.$$

The idea is similar to that of Proposition 2.14, but here the proof is easier as $0 \to \mathbb{G}_m \to j_*\mathbb{G}_m \to i_*\mathbb{Z} \to 0$ is exact where $j : \eta \to X$ and $i : x \to X$. Since K is local, k is finite, so that $\mathrm{H}^1(x, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$. However, since R is Henselian, $\operatorname{Br}(X) = \operatorname{Br}(x)$ (see [Grothendieck 1968a, Corollaire 6.2]), and $\operatorname{Br}(x) = 0$ by a theorem of Wedderburn (see [loc. cit., Proposition 1.5]).

Now, let K be a number field, and let R be a localization of the ring of integers of K. Set $\eta = \operatorname{Spec} K$ and $X = \operatorname{Spec} R$. In this case, by [loc. cit., Proposition 2.1], there is an exact sequence

$$0 \to \operatorname{Br}(X) \to \operatorname{Br}(\eta) \to \bigoplus_{\mathfrak{p} \in X^{(1)}} \operatorname{Br}(\operatorname{Spec} K_{\mathfrak{p}}),$$

where $X^{(1)}$ denotes the set of codimension 1 points of X. This exact sequence is compatible with (2.15) and with the exact sequence

$$0 \to \mathrm{Br}(\eta) \to \bigoplus_{\mathfrak{p}} \mathrm{Br}(\mathrm{Spec}\, K_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{2.18}$$

of class field theory (see [Neukirch et al. 2000, Theorem 8.1.17]). The sum ranges over the finite and the infinite places of K, and the map $\mathrm{Br}(\mathrm{Spec}\, K_{\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is the isomorphism described above when $\mathfrak{p}$ is a finite place, the natural inclusion $\mathbb{Z}/2 \to \mathbb{Q}/\mathbb{Z}$ when $K_{\mathfrak{p}} \cong \mathbb{R}$, and the natural map $0 \to \mathbb{Q}/\mathbb{Z}$ when $K_{\mathfrak{p}} \cong \mathbb{C}$. Using these sequences, we can compute the Brauer group of X.

The two fundamental observations we need about (2.18) are that a class $\alpha \in \mathrm{Br}(\eta)$ is ramified at no fewer than 2 places and that if K is purely imaginary, then $\alpha \in \mathrm{Br}(\eta)$ is ramified at no fewer than 2 *finite* places. The reader can easily verify the following examples.

Example 2.19. (1) $\mathrm{Br}(\mathbb{Z}) = 0$.

(2) $\mathrm{Br}\big(\mathbb{Z}[\frac{1}{p}]\big) \cong \mathbb{Z}/2$.

(3) $\mathrm{Br}\big(\mathbb{Z}[\frac{1}{pq}]\big) \cong \mathbb{Z}/2 \oplus \mathbb{Q}/\mathbb{Z}$.

(4) $\mathrm{Br}\big(\mathbb{Z}[\frac{1}{p}, \zeta_p]\big) = 0$.

We will use these computations and those like them throughout the paper, often without comment.

3. The low-dimensional $\mathbb{G}_m$-cohomology of $\mathrm{B}C_m$

Let S be a scheme. Write $C_{n,S}$ for the constant étale group scheme on the cyclic group C_n of order $n \geq 2$ over S. We will often suppress the base in the notation and simply write C_n when the base is clear from context. The purpose of this section is to make a basic computation of the $\mathbb{G}_m$-cohomology of the Deligne–Mumford stack $\mathrm{B}C_n = \mathrm{B}C_{n,S}$. In fact, we are only interested in the cases $n = 2$ and $n = 4$, but the general case is no more difficult.

The first tool for our computations of the étale cohomology of an étale sheaf $\mathcal{F}$ is the convergent Leray–Serre spectral sequence. If $\pi : Y \to X$ is a G-Galois cover where X and Y are Deligne–Mumford stacks, then this spectral sequence has the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(G, \mathrm{H}^q(Y, \pi^*\mathcal{F})) \Rightarrow \mathrm{H}^{p+q}(X, \mathcal{F}),$$

with differentials d_r of bidegree $(r, 1-r)$.

We will use the spectral sequence in this section for the C_n-Galois cover $\pi : S \to \mathrm{B}C_n$, where C_n acts trivially on S. In this case it is of the form

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(C_n, \mathrm{H}^q(S, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}(\mathrm{B}C_{n,S}, \mathbb{G}_m)$$

and Figure 1 displays the low-degree part of its E_2-page. The fact that C_n acts trivially on the cohomology of S implies that the left-most column is simply the $\mathbb{G}_m$-cohomology of S. For F a constant C_n-module,

$$\begin{tikzcd}[row sep=small]
\mathrm{H}^2(S,\mathbb{G}_m) & & & \\
\mathrm{Pic}(S) & \mathrm{Pic}(S)[n] \arrow[drr] & \mathrm{Pic}(S)/n & \\
\mathbb{G}_m(S) & \mu_n(S) & \mathbb{G}_m(S)/n & \mu_n(S)
\end{tikzcd}$$

Figure 1. The E_2-page of the Leray–Serre spectral sequence computing $\mathrm{H}^i(\mathrm{B}C_n,\mathbb{G}_m)$.

we use the standard isomorphisms $\mathrm{H}^i(C_n,F)\cong F[n]$ when $i>0$ odd, and $\mathrm{H}^i(C_n,F)\cong F/n$ when $i>0$ is even. We are only interested in $\mathrm{H}^i(\mathrm{B}C_n,\mathbb{G}_m)$ for $0\le i\le 2$.

Recall Grothendieck's theorem that the natural morphism $\mathrm{H}^i(X,G)\to\mathrm{H}^i_{\mathrm{pl}}(X,G)$ is an isomorphism for $i\ge 0$ when G is a smooth group scheme on X (such as C_n or $\mathbb{G}_m$). See [Grothendieck 1968c, Section 5, Thèoréme 11.7]. Note that this implies the agreement of étale and fppf cohomology on a Deligne–Mumford stack. For example, Grothendieck's theorem implies that the morphism from the Leray–Serre spectral sequence above to the analogous Leray–Serre spectral sequence

$$\mathrm{E}_2^{p,q}=\mathrm{H}^p(C_n,\mathrm{H}^q_{\mathrm{pl}}(S,\mathbb{G}_m))\Rightarrow\mathrm{H}^{p+q}_{\mathrm{pl}}(\mathrm{B}C_n,\mathbb{G}_m)$$

for the fppf cohomology is an isomorphism; thus the comparison map

$$\mathrm{H}^i(\mathrm{B}C_n,\mathbb{G}_m)\to\mathrm{H}^i_{\mathrm{pl}}(\mathrm{B}C_n,\mathbb{G}_m)$$

is also an isomorphism. For $\mathbb{G}_m$-coefficients, we will thus not distinguish between étale and fppf cohomology in what follows.

We use these observations to compute the Picard and Brauer groups of $\mathrm{B}C_{n,S}$ via a Leray spectral sequence. The idea is borrowed from [Lieblich 2011, Section 4.1]. Consider the map to the coarse moduli space $c\colon \mathrm{B}C_{n,S}\to S$. We claim that

$$\begin{aligned}
\mathrm{R}^0_{\mathrm{pl}}c_*\mathbb{G}_m&=\mathbb{G}_m\\
\mathrm{R}^1_{\mathrm{pl}}c_*\mathbb{G}_m&=\mu_n\\
\mathrm{R}^2_{\mathrm{pl}}c_*\mathbb{G}_m&=0.
\end{aligned}$$

Indeed, $\mathrm{R}^i_{\mathrm{pl}}c_*\mathbb{G}_m$ is the fppf-sheafification of $U\mapsto\mathrm{H}^i(\mathrm{B}C_{n,U},\mathbb{G}_m)$ and in the Leray–Serre spectral sequences all classes in $\mathrm{H}^p(C_n,\mathrm{H}^q(U,\mathbb{G}_m))$ for $q>0$ are killed by some fppf cover of U. Furthermore every unit has an n-th root fppf-locally so that the fppf-sheafification of the presheaf $\mathbb{G}_m/n$ vanishes. This implies the claim.

It follows that the fppf-Leray spectral sequence

$$\mathrm{E}_2^{p,q}=\mathrm{H}^p_{\mathrm{pl}}(S,\mathrm{R}^q_{\mathrm{pl}}c_*\mathbb{G}_m)\Rightarrow\mathrm{H}^{p+q}_{\mathrm{pl}}(\mathrm{B}C_{n,S},\mathbb{G}_m)\tag{3.1}$$

for c takes the form given in Figure 2 in low degrees. As $\pi^*c^*=\mathrm{id}$, we see that the edge homomorphisms $\mathrm{H}^i(S,\mathbb{G}_m)\to\mathrm{H}^i(\mathrm{B}C_{n,S},\mathbb{G}_m)$ from the bottom line in the Leray spectral sequence are all split injections. In particular, the displayed differentials $d_2^{0,1}$ and $d_2^{1,1}$ are zero.

$$\begin{tikzcd}
0 & & & \\
\mu_n(S) \arrow[drr] & \mathrm{H}^1_{\mathrm{pl}}(S, \mu_n) \arrow[drr] & & \\
\mathbb{G}_m(S) & \mathrm{Pic}(S) & \mathrm{H}^2(S, \mathbb{G}_m) & \mathrm{H}^3(S, \mathbb{G}_m)
\end{tikzcd}$$

Figure 2. The E_2-page of the Leray spectral sequence (3.1) computing $\mathrm{H}^i(\mathrm{B}C_n, \mathbb{G}_m)$.

Proposition 3.2. *There is an isomorphism*

$$\mathbb{G}_m(\mathrm{B}C_{n,S}) \cong \mathbb{G}_m(S)$$

and short exact sequences

$$0 \to \mathrm{Pic}(S) \xrightarrow{c^*} \mathrm{Pic}(\mathrm{B}C_{n,S}) \to \mu_n(S) \to 0$$

and

$$0 \to \mathrm{H}^2(S, \mathbb{G}_m) \xrightarrow{c^*} \mathrm{H}^2(\mathrm{B}C_{n,S}, \mathbb{G}_m) \xrightarrow{r} \mathrm{H}^1_{\mathrm{pl}}(S, \mu_n) \to 0,$$
$$0 \to \mathrm{Br}'(S) \xrightarrow{c^*} \mathrm{Br}'(\mathrm{B}C_{n,S}) \xrightarrow{r} \mathrm{H}^1_{\mathrm{pl}}(S, \mu_n) \to 0,$$
$$0 \to \mathrm{Br}(S) \xrightarrow{c^*} \mathrm{Br}(\mathrm{B}C_{n,S}) \xrightarrow{r} \mathrm{H}^1_{\mathrm{pl}}(S, \mu_n) \to 0,$$

which are split. The isomorphism and the short exact sequences are functorial in $\mathrm{B}C_{n,S}$ (*i.e.*, *endomorphisms of* $\mathrm{B}C_{n,S}$ *induces endomorphisms of exact sequences in a functorial manner*), *but the splittings are only functorial in S.*

Proof. By the discussion above, the Leray spectral sequence proves everything except for the split exactness of the last two sequences. For the sequence involving the cohomological Brauer group, we just apply the torsion subgroup functor to the split exact sequence involving $\mathrm{H}^2(-, \mathbb{G}_m)$. By Remark 2.6 we furthermore see that $a = \pi^* c^* a \in \mathrm{H}^2(S, \mathbb{G}_m)$ is in $\mathrm{Br}(S)$ if and only if $c^* a \in \mathrm{Br}(\mathrm{B}C_{n,S})$, implying split exactness for the last exact sequence. □

Later on we will need not only the computation of the Brauer group of $\mathrm{B}C_n$, but also a description of the classes coming from the inclusion $\mathbb{G}_m(S)/n \hookrightarrow \mathrm{Br}(\mathrm{B}C_n)$, which is either defined via the Leray–Serre spectral sequence or using the splitting in Proposition 3.2 (the proof of the following lemma will, in particular, show that these two maps differ at most by a unit). These classes are described via the classical cyclic algebra construction from the previous section.

Lemma 3.3. *Let X be an algebraic stack and n a positive integer. Let $\sigma \in \mathrm{H}^1(\mathrm{B}C_{n,X}, C_n)$ be the class of the universal C_n-torsor $X \to \mathrm{B}C_{n,X}$. Then there is an integer k prime to n* (*that only depends on n*) *such that the map*

$$s : \mathrm{H}^1_{\mathrm{pl}}(X, \mu_n) \to \mathrm{H}^2(\mathrm{B}C_{n,X}, \mathbb{G}_m)$$

defined by $s(u) = k[(\sigma, u)_n]$ is a section to the map r from Proposition 3.2.

Proof. It suffices to consider the universal case of $X = \mathrm{B}_{\mathrm{pl}}\mu_n$ over $\operatorname{Spec}\mathbb{Z}$, the stack classifying fppf μ_n torsors. Note that $\mathrm{B}_{\mathrm{pl}}\mu_n$ is indeed a stack by [Stacks 2018, Tag 04UR] and is an algebraic stack by [Stacks 2018, Tag 06DC] with fppf atlas $\operatorname{Spec}\mathbb{Z} \to \mathrm{B}_{\mathrm{pl}}\mu_n$. Let $d : \mathrm{B}_{\mathrm{pl}}\mu_n \to \operatorname{Spec}\mathbb{Z}$ denote the structure map, and let $\mathrm{R}^q_{\mathrm{pl}}d_*\mu_n$ denote the derived functors of the push-forward in the fppf topos. Then, it is easy to see that $\mathrm{R}^0_{\mathrm{pl}}d_*\mu_n \cong \mu_n$ and we claim that $\mathrm{R}^1_{\mathrm{pl}}d_*\mu_n \cong C_n$. To see the latter isomorphism, consider the natural transformations

$$\mathrm{Hom}^{\mathrm{gp}}_{\operatorname{Spec} R}(\mu_{n,R}, \mathbb{G}_{m,R}) \to \mathrm{H}^1(\mathrm{B}_{\mathrm{pl}}\mu_{n,R}, \mathbb{G}_m)[n] \leftarrow \mathrm{H}^1_{\mathrm{pl}}(\mathrm{B}_{\mathrm{pl}}\mu_{n,R}, \mu_n) \tag{3.4}$$

of presheaves on affine schemes over $\mathbb{Z}$; the first map sends a homomorphism $f\colon \mu_{n,R} \to \mathbb{G}_{m,R}$ to the image of the canonical class $\mathrm{H}^1(\mathrm{B}_{\mathrm{pl}}\mu_{n,R}, \mu_n)$ under f_* and the second map is part of the Kummer sequence. The leftmost term is a sheaf and it is a standard fact that it is represented by the constant étale group scheme C_n. See [Cornell et al. 1997, Section V.2.10] for example. The fppf-sheafification of the rightmost term is $\mathrm{R}^1_{\mathrm{pl}}d_*\mu_n$. To see that the induced map of sheaves are isomorphisms, it is sufficient to check on stalks in the fppf topology [Gabber and Kelly 2015, Remark 1.8, Theorem 2.3] and in particular if R is a Henselian local ring with algebraically closed residue field [loc. cit., Lemma 3.3]. If R is such a local ring, then $\mathbb{G}_m(R)/n = 0$ so that $\mathrm{H}^1_{\mathrm{pl}}(\mathrm{B}_{\mathrm{pl}}\mu_{n,R}, \mu_n) \cong \mathrm{H}^1_{\mathrm{pl}}(\mathrm{B}_{\mathrm{pl}}\mu_{n,R}, \mathbb{G}_m)[n]$. Using that $\mathrm{Pic}(R) = 0$, the Leray–Serre spectral sequence for the cover $\operatorname{Spec} R \to \mathrm{B}_{\mathrm{pl}}\mu_{n,R}$ shows that

$$\mathrm{H}^1_{\mathrm{pl}}(\mathrm{B}_{\mathrm{pl}}\mu_{n,R}, \mathbb{G}_m) \cong \mathrm{H}^1_{\mathrm{group}}(\mu_{n,R}, \mathbb{G}_{m,R}) \cong \mathrm{Hom}^{\mathrm{gp}}_{\operatorname{Spec} R}(\mu_{n,R}, \mathbb{G}_{m,R}),$$

where $\mathrm{H}^1_{\mathrm{group}}(\mu_{n,R}, \mathbb{G}_{m,R})$ is the first cohomology of the cobar complex

$$\mathbb{G}_m(S) \to \mathbb{G}_m(\mu_{n,R}) \to \mathbb{G}_m(\mu_{n,R} \times_{\operatorname{Spec} R} \mu_{n,R}) \to \cdots$$

with differentials as in the usual definition of group cohomology. This shows that the morphisms in (3.4) are isomorphisms on fppf-stalks and thus that $\mathrm{R}^1_{\mathrm{pl}}d_*\mu_n \cong C_n$.

Now, the fppf-Leray spectral sequence for $d : \mathrm{B}_{\mathrm{pl}}\mu_n \to \operatorname{Spec}\mathbb{Z}$ yields an exact sequence

$$0 \to \mathrm{H}^1_{\mathrm{pl}}(\operatorname{Spec}\mathbb{Z}, \mu_n) \to \mathrm{H}^1_{\mathrm{pl}}(\mathrm{B}_{\mathrm{pl}}\mu_n, \mu_n) \to \mathrm{H}^0(\operatorname{Spec}\mathbb{Z}, C_n) \to 0. \tag{3.5}$$

The right-hand term is isomorphic to $\mathbb{Z}/n$, and the sequence is split by applying the pullback map along $\operatorname{Spec}\mathbb{Z} \to \mathrm{B}_{\mathrm{pl}}\mu_n$. We denote by $\tau \in \mathrm{H}^1_{\mathrm{pl}}(\mathrm{B}_{\mathrm{pl}}\mu_n, \mu_n)$ the class of the universal μ_n-torsor over $\mathrm{B}_{\mathrm{pl}}\mu_n$. This is (exactly) of order n and pulls back to zero on $\operatorname{Spec}\mathbb{Z}$.

Consider $c\colon \mathrm{B}C_{n,\mathrm{B}_{\mathrm{pl}}\mu_n} \to \mathrm{B}_{\mathrm{pl}}\mu_n$ and the class $\alpha = [(\sigma, c^*\tau)_n]$. The class of α has order (exactly) n as there are cyclic algebras of order n over fields, for example by [Gille and Szamuely 2006, Lemma 5.5.3]. As $\pi^*\alpha = 0$ for $\pi : \mathrm{B}_{\mathrm{pl}}\mu_n \to \mathrm{B}C_{n,\mathrm{B}_{\mathrm{pl}}\mu_n}$ the projection, it follows from Proposition 3.2 that $r(\alpha)$ in $\mathrm{H}^1_{\mathrm{pl}}(\mathrm{B}_{\mathrm{pl}}\mu_n, \mu_n)$ has order n as well. On the other hand, $r(\alpha)$ pulls back to zero over $\operatorname{Spec}\mathbb{Z}$ so it is a nonzero multiple of τ (using the split-exact sequence (3.5)). Thus, $r(\alpha) = m\tau$ for some m prime to n. This completes the proof if we set k to be a number such that $km \equiv 1 \bmod n$. □

Corollary 3.6. *Suppose that $\chi : X \to Y$ is a C_n-torsor for some positive integer n. Let $u \in \mathbb{G}_m(Y)/n$ be the class of a unit, and write α_u for the corresponding class in $\mathrm{Br}'(Y)$ (defined via the Leray–Serre* *spectral sequence). Then we have $\alpha_u = k[(\chi, u)]$ in $\mathrm{Br}'(Y)$, where k is some number prime to n which only depends on n.*

We do not know the value of k in the corollary. Perhaps it is always ± 1, as is the case in similar computations, such as the result of Lichtenbaum (see [Gille and Szamuely 2006, Theorem 5.4.10]), which computes the exact value of the map $\mathrm{Pic}(X_{\bar{k}})^G \cong \mathbb{Z} \to \mathrm{Br}(k)$ when X is a Severi–Brauer variety of a field with Galois group G, or the computation of [Gille and Quéguiner-Mathieu 2011] of the sign of the Rost invariant.

Proposition 3.7. *Let $\mathcal{X}$ be an algebraic stack and suppose that $\chi \in \mathrm{H}^1(\mathcal{X}, C_n)$ and $u \in \mathrm{H}^1_{\mathrm{pl}}(\mathcal{X}, \mu_n)$ are fixed classes. In the notation above, we have $[(\chi, u)_n] = [\mathcal{A}_{\chi,u}]$ in $\mathrm{Br}'(\mathcal{X})$.*

Proof. Both $[\mathcal{A}_{\chi,u}]$ and $[(\chi, u)_n]$ define classes in $\mathrm{H}^2_{\mathrm{pl}}(\mathrm{B}C_n \times \mathrm{B}_{\mathrm{pl}}\mu_n, \mathbb{G}_m)$. As at the end of the proof of Proposition 3.3, we see that $[\mathcal{A}_{\chi,u}] = k[(\chi, u)_n]$ for some k prime to n. We saw in Proposition 2.11 that they agree when pulled back to regular noetherian schemes. The result follows. □

4. A presentation of the moduli stack of elliptic curves

We will compute $\mathrm{Br}(\mathcal{M})$ using that it injects into $\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ by Proposition 2.5(iv) and using a specific presentation of $\mathcal{M}_{\mathbb{Z}[1/2]}$, which we now describe. This presentation is standard and we claim no originality in our presentation of it. For references, see [Deligne and Rapoport 1973] or [Katz and Mazur 1985],

Definition 4.1. A *full level* 2 *structure* on an elliptic curve E over a base scheme S is a fixed isomorphism $(\mathbb{Z}/2)^2_S \to E[2]$, where $(\mathbb{Z}/2)^2_S$ denotes the constant group scheme on $(\mathbb{Z}/2)^2$ over S and $E[2]$ is the subgroupscheme of order 2 points in E. If there exists an isomorphism $(\mathbb{Z}/2)^2_S \cong E[2]$, an equivalent way of specifying a level 2 structure is to order the points of exact order 2 in $E(S)$ (over each connected component of S).

Remark 4.2. These full level 2 structures are sometimes called *naive* to distinguish them from the level structures considered by Drinfeld, which allow one to extend $\mathcal{M}(2)$ to a stack supported over all of $\mathrm{Spec}\,\mathbb{Z}$. We will not need this generalization in this paper. It is the subject of [Katz and Mazur 1985].

The moduli stack $\mathcal{M}(2)$ of elliptic curves with fixed level 2 structures is a regular noetherian Deligne–Mumford stack. Moreover, since the existence of a full level 2 structure implies that 2 is invertible in S (by [Katz and Mazur 1985, Corollary 2.3.2] for example), the functor $\mathcal{M}(2) \to \mathcal{M}$ which forgets the level structure factors through $\mathcal{M}_{\mathbb{Z}[1/2]}$. This map is clearly equivariant for the right S_3-action on $\mathcal{M}(2)$ that permutes the nonzero 2-torsion points and the trivial S_3-action on $\mathcal{M}$. Note that in general, being G-equivariant for a map $f : \mathcal{X} \to \mathcal{Y}$ of stacks with G-action is extra structure: for every $g \in G$ one has to provide compatible 2-morphism $\sigma_g : gf \to fg$ (see [Romagny 2005] for details). In the case of $\mathcal{M}(2) \to \mathcal{M}_{\mathbb{Z}[1/2]}$ though the equivariance is strict in the sense that all σ_g are the identity 2-morphisms of $\mathcal{M}(2) \to \mathcal{M}_{\mathbb{Z}[1/2]}$.

We have the following well-known statement; to fix ideas, we will provide a proof.

Lemma 4.3. *The map $\mathcal{M}(2) \to \mathcal{M}_{\mathbb{Z}[1/2]}$ is an S_3-Galois cover.*

Proof. It is enough to show that for every affine scheme $\operatorname{Spec} R$ over $\operatorname{Spec} \mathbb{Z}\big[\frac{1}{2}\big]$ and every elliptic curve E over $\operatorname{Spec} R$, we can find a full level 2 structure étale locally. Indeed, if there is one full level 2 structure on E, the map

$$(S_3)_{\operatorname{Spec} R} = S_3 \times \operatorname{Spec} R \to \operatorname{Spec} R \times_{\mathcal{M}_{\mathbb{Z}[1/2]}} \mathcal{M}(2)$$

from the constant group scheme on S_3 is an isomorphism since we get every other full level 2 structure on E by permuting the nonzero 2-torsion points.

The elliptic curve E defines an R-point $E : \operatorname{Spec} R \to \mathcal{M}_{\mathbb{Z}[1/2]}$ of the moduli stack of elliptic curves. Zariski locally we can assume the pullback $E^*\lambda$ of the Hodge bundle to be trivial, in which case there exists a nowhere vanishing invariant differential ω. By [Katz and Mazur 1985, Section 2.2], we can then write E in Weierstrass form over $\operatorname{Spec} R$, which after a coordinate change takes the form

$$y^2 = x^3 + b_2x^2 + b_4x + b_6.$$

As a point (x, y) on E is 2-torsion if and only if $y = 0$, we have a full level 2 structure after adjoining the three roots e_1, e_2 and e_3 of $x^3 + b_2x^2 + b_4x + b_6$ to R. This defines an étale extension as the discriminant of this cubic polynomial does not vanish (because E is smooth). □

Definition 4.4. A *Legendre curve* with parameter t over S is an elliptic curve E_t with Weierstrass equation

$$y^2 = x(x-1)(x-t).$$

As the discriminant of this equation is $16t^2(t-1)^2$, such an equation defines an elliptic (and hence Legendre) curve if and only if 2, t and $t-1$ are invertible on S.

The points $(0, 0)$, $(1, 0)$, and $(t, 0)$ define three nonzero 2-torsion points on E_t. Taking them in this order fixes a full level 2 structure on E. This defines a morphism

$$\pi : X \to \mathcal{M}(2),$$

where X is the parameter space of Legendre curves. In fact, X is an affine scheme, given as

$$X = \operatorname{Spec} \mathbb{Z}\big[\tfrac{1}{2}, t^{\pm 1}, (t-1)^{-1}\big] = \mathbb{A}^1_{\mathbb{Z}[1/2]} - \{0, 1\}.$$

We will use X throughout this paper to refer specifically to this moduli space of Legendre curves. In particular, X is naturally defined over $\mathbb{Z}\big[\frac{1}{2}\big]$. In general, given a scheme S, we let $X_S = \mathbb{A}^1_S - \{0, 1\}$. Note that this is a slight abuse of notation as we do not assume that 2 is invertible on S.

We equip the map $\pi : X \to \mathcal{M}(2)$ with the structure of a C_2-equivariant map with the trivial C_2-action on X and $\mathcal{M}(2)$ by choosing $\sigma_g : g\pi \to \pi g$ to be $[-1]$ (i.e., multiplication by -1 on the universal elliptic curve) for $g \in C_2$ the nontrivial element. Note that $[-1]$ fixes the level 2 structure and so indeed defines a natural automorphism of $\mathrm{id}_{\mathcal{M}(2)}$. The structure of a C_2-equivariant map on π induces a map $[X/C_2] = \mathrm{B}C_{2,X} \to \mathcal{M}(2)$.

Proposition 4.5. *The C_2-equivariant map $\pi : X \to \mathcal{M}(2)$ is a C_2-torsor. Thus, the map $\mathrm{B}C_{2,X} \to \mathcal{M}(2)$ is an equivalence.*

Proof. First we will show that an elliptic curve $E \to \operatorname{Spec} R$ with full level 2 structure can étale locally be brought into Legendre form. Our proof will be along the lines of [Silverman 2009, Proposition III.1.7], but we have to take a little bit more care.

As in the proof of Lemma 4.3, Zariski locally over $\operatorname{Spec} R$, we can write E in the form

$$y^2 = x^2 + b_2x^2 + b_4x + b_6$$

and the full level 2 structure allows us to factor the right-hand side as

$$(x - e_1)(x - e_2)(x - e_3),$$

where $(e_1, 0)$, $(e_2, 0)$ and $(e_3, 0)$ are the nonzero 2-torsion points. We set $p = e_2 - e_1$ and $q = e_3 - e_1$. By a linear coordinate change, we get $y^2 = x(x - p)(x - q)$.

Since the equation $y^2 = x(x - p)(x - q)$ defines an elliptic curve, p, q and $p - q$ are nowhere vanishing. Thus, the extension $R \to R[\sqrt{p}]$ is étale so that we can (and will) assume étale locally to have a (chosen) square root $\sqrt{p}$. Now, E is isomorphic to $y^2 = x(x - 1)(x - t)$ for $t = q/p$, where the isomorphism is given by $x = px'$ and $y = p^{3/2}y'$. Thus our original E is indeed étale locally (on the base) isomorphic to a Legendre curve as an elliptic curve with level 2 structure. It is moreover an elementary check with coordinate transformations that there is at most one choice of $t \in R$ such that the Legendre curve E_t with parameter t is isomorphic to E in $\mathcal{M}(2)$.

Now assume that our elliptic curve E over R is in Legendre form and assume further that $\operatorname{Spec} R$ is connected. By definition, for a commutative R-algebra R', an element of $(X \times_{\mathcal{M}(2)} \operatorname{Spec} R)(R')$ consists of a Legendre curve E_t together with an isomorphism of E_t to $E_{R'}$ in $\mathcal{M}(2)$. By assumption this set is nonempty and it is indeed a torsor under the group of automorphisms of $E_{R'}$ in $\mathcal{M}(2)$. By [Katz and Mazur 1985, Corollary 2.7.2], the only nontrivial automorphism of $E_{R'}$ with level 2 structure is $[-1]$. Thus, the C_2-action exactly interchanges the two elements of $(X \times_{\mathcal{M}(2)} \operatorname{Spec} R)(R')$ and we obtain a C_2-equivariant equivalence

$$C_2 \times \operatorname{Spec} R \simeq X \times_{\mathcal{M}(2)} \operatorname{Spec} R.$$

As every E with full level 2 structure satisfies étale locally our assumptions, this implies that $X \to \mathcal{M}(2)$ is a C_2-torsor.

By the general fact that for a G-torsor $\mathcal{X} \to \mathcal{Y}$, the induced map $[\mathcal{X}/G] \to \mathcal{Y}$ is an equivalence, we obtain in our case the equivalence $\mathrm{B}C_{2,X} \simeq \mathcal{M}(2)$. □

Corollary 4.6. *The map $c\colon \mathcal{M}(2) \to X$ sending $y^2 = (x - e_1)(x - e_2)(x - e_3)$ to $y^2 = x(x - 1)(x - (e_3 - e_1)/(e_2 - e_1))$ exhibits X as the coarse moduli space of $\mathcal{M}(2)$.*

Proof. The set of maps from $\mathcal{M}(2) \simeq \mathrm{B}C_{2,X}$ to X is in bijection with C_2-equivariant maps $X \to X$. Thus, a map $\mathcal{M}(2) \to X$ exhibits X as the coarse moduli space if and only if the precomposition with π is the identity. This is clearly the case for c. □

It follows that the right S_3-action on $\mathcal{M}(2)$ induces a right S_3 action on X. We can describe this explicitly as follows. Consider the generators $\sigma = (1\,3\,2)$ and $\tau = (2\,3)$ of $\mathrm{GL}_2(\mathbb{Z}/2) \cong S_3$, of orders 3 and 2, respectively. Then,

$$\sigma(t) = \frac{t-1}{t}, \quad \text{and} \quad \tau(t) = \frac{1}{t}.$$

By a simple computation, the map $c\colon \mathcal{M}(2) \to X$ defined above is strictly S_3-equivariant.

In contrast, the map $\pi : X \to \mathcal{M}(2)$ described above is *not* S_3-equivariant, as one notes for example by checking that the elliptic curves $y^2 = x(x-1)(x-t)$ and $y^2 = x(x-1)\big(x - \frac{1}{t}\big)$ are generally not isomorphic. To actually explain the correct S_3-action on $\mathrm{B}C_{2,X}$, we have to fix some notation.

Consider again an elliptic curve E given by $y^2 = (x-e_1)(x-e_2)(x-e_3)$. Set again $p = e_2 - e_1$ and $q = e_3 - e_1$ so that we can write E as

$$y^2 = x(x-p)(x-q).$$

The only possible coordinate changes fixing the form of this equation are the transformations $y \mapsto u^3 y$ and $x \mapsto u^2 x$; such a coordinate change results in multiplying the standard invariant differential $\omega = -dx/2y$ by u^{-1} and sending p to $u^2 p$ and q to $u^2 q$. Thus, $p\omega^{\otimes 2}$ and $q\omega^{\otimes 2}$ define canonical sections of $\lambda^{\otimes 2}$ on $\mathcal{M}(2)$, not dependent on any choice of Weierstrass form. Note that these sections are nowhere vanishing. We can consider the C_2-torsor $\mathcal{M}(2)(\sqrt{p}) \to \mathcal{M}(2)$ defined as the cyclic cover $\mathbf{Spec}_{\mathcal{M}(2)}\big(\bigoplus_{i\in\mathbb{Z}} \lambda^{\otimes i}/(1-p)\big)$. Étale locally on some $\operatorname{Spec} R$, we can trivialize λ so that p becomes an element of R and the C_2-torsor becomes $\operatorname{Spec} R[\sqrt{p}] \to \operatorname{Spec} R$. The C_2-torsor $\mathcal{M}(2)(\sqrt{p}) \to \mathcal{M}(2)$ is equivalent to $X \to \mathcal{M}(2)$. Indeed, we have shown in the proof of Proposition 4.5 that the latter has a section as soon as we have a chosen square root of p.

As $g^*\lambda$ for $g \in S_3$ on $\mathcal{M}(2)$ is canonically isomorphic to λ (as this is pulled back from $\mathcal{M}$), we have an action of S_3 on $\mathrm{H}^0(\mathcal{M}(2), \lambda^{\otimes *})$. Consider the section

$$\frac{g(p)}{p} \in \mathrm{H}^0(\mathcal{M}(2), \mathcal{O}_{\mathcal{M}(2)}) \cong \mathrm{H}^0(X, \mathcal{O}_X),$$

which can for E as above be written as $(e_{g(2)} - e_{g(1)})/(e_2 - e_1)$. For example, we have $g(p)/p = q/p$ for $g = \tau$, which equals t on X. For a scheme S with a map $f\colon S \to X$ or $f\colon S \to \mathcal{M}(2)$, we denote the torsor adjoining the square root of $f^*g(p)/p$ by $T_{f,g} \to S$.

For the next lemma, we recall that an object in $\mathrm{B}C_{2,X}(S)$ corresponds to a C_2-torsor $T \to S$ and a C_2-equivariant map $T \to X$, where X has the trivial C_2-action. Equivalently, an object can be described as a C_2-torsor $T \to S$ with a map $f\colon S \to X$. Let S_3 act on $\mathrm{B}C_{2,X}$ in the following way: $g \in S_3$ acts (from the right) on $(T, f) \in \mathrm{B}C_{2,X}(S)$ by setting $g(T)$ to be $(T \times_S T_{f,g})/C_2$ and the map $g(f)$ to be the composition $S \xrightarrow{f} X \xrightarrow{g} X$.

Lemma 4.7. *The natural map* $\mathcal{M}(2) \to \mathrm{B}C_{2,X}$ *induces an* S_3*-equivariant equivalence* $\mathrm{B}C_{2,X} \simeq \mathcal{M}(2)$.

Proof. As noted above, the map $\mathcal{M}(2) \to \mathrm{B}C_{2,X}$ classifying the torsor $\mathcal{M}(2)(\sqrt{p}) \to \mathcal{M}(2)$ is an equivalence by Proposition 4.5 (as this torsor is equivalent to $X \to \mathcal{M}(2)$). We have only to check the S_3-equivariance of this map.

Given an $f\colon S \to \mathcal{M}(2)$, the corresponding object in $BC_{2,X}$ is the torsor $S(\sqrt{f^*p}) \to S$ together with $S \to \mathcal{M}(2) \to X$. The composition gf for $g \in S_3$ corresponds to the torsor $S(\sqrt{(gf)^*p}) \to S$ together with $S \to \mathcal{M}(2) \to X \xrightarrow{g} X$ as $\mathcal{M}(2) \to X$ is S_3-equivariant. As $(gf)^*p = f^*(g(p))$, we have $(gf)^*p = f^*p \cdot f^*(g(p)/p)$. Thus, we have a natural isomorphism $(S(\sqrt{p}) \times_S T_{f,g})/C_2 \xrightarrow{\cong} S(\sqrt{(gf)^*p})$. One can check that these isomorphisms are compatible (similarly to [Romagny 2005, Definition 2.1], although we do not have a strict S_3-action on $BC_{2,X}$) so that one actually gets the structure of an S_3-equivariant map. □

Of particular import will be the action of S_3 on the units of X. Let ρ be the tautological permutation representation of S_3 on $\mathbb{Z}^{\oplus 3}$ and let $\tilde{\rho}$ be the kernel of the morphism

$$\rho \cong \operatorname{ind}_{C_2}^{S_3} \mathbb{Z} \to \mathbb{Z}$$

to the trivial representation, the adjoint to the identity.

Lemma 4.8. *For any connected normal noetherian scheme S over $\mathbb{Z}[\frac{1}{2}]$, there is an S_3-equivariant exact sequence*

$$0 \to \mathbb{G}_m(S) \to \mathbb{G}_m(X_S) \to \tilde{\rho} \to 0,$$

where S_3 acts on $\mathbb{G}_m(S)$ trivially and $\tilde{\rho}$ is additively generated by the images of t and $t-1$. This exact sequence is nonequivariantly split.

Proof. Denote by $\pi: X_S \to S$ the structure map. We have a map $f\colon \mathbb{Z}^2 \oplus \mathbb{G}_m \to \pi_*\mathbb{G}_{m,X_S}$ of sheaves on S, where f takes the two $\mathbb{Z}$-summands to t and $(t-1)$, respectively. We claim that this map is an isomorphism. It is enough to check this on affine connected opens $\operatorname{Spec} R$, where it follows from R being an integral domain (as it is normal). The nonequivariant statement follows.

Moreover, the action of S_3 on $\mathbb{G}_m(S)$ is trivial by definition. Set $\sigma = (1\,3\,2)$ and $\tau = (2\,3)$. If we choose the basis vectors

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

for $\tilde{\rho}$, we obtain exactly the same S_3-representation as on $\mathbb{G}_m(X_S)/\mathbb{G}_m(S) \cong \mathbb{Z}\{t, t-1\}$, where the latter denotes the free $\mathbb{Z}$-module on t and $t-1$ with elements thought of as $t^k(t-1)^l$. □

5. Beginning of the computation

Let S be a connected regular noetherian scheme over $\mathbb{Z}[\frac{1}{2}]$, let $\mathcal{M}_S$ be the moduli stack of elliptic curves over S, and let $\mathcal{M}(2)_S$ be the moduli stack of elliptic curves with full level 2 structure over S. The Leray–Serre spectral sequence for $\mathcal{M}(2)_S \to \mathcal{M}_S$ takes the form

$$E_2^{p,q} : H^p(S_3, H^q(\mathcal{M}(2)_S, \mathbb{G}_m)) \Rightarrow H^{p+q}(\mathcal{M}_S, \mathbb{G}_m), \tag{5.1}$$

with differentials d_r of bidegree $(r, 1-r)$. In this section, we will collect the basic tools to compute the E_2-term. We start with two brief remarks about the cohomology of S_3.

Lemma 5.2. *Let M be a trivial S_3-module. Then,*

$$\begin{aligned}
\mathrm{H}^1(S_3, M) &\cong M[2],\\
\mathrm{H}^2(S_3, M) &\cong M/2,\\
\mathrm{H}^3(S_3, M) &\cong M[6],\\
\mathrm{H}^4(S_3, M) &\cong M/6.
\end{aligned}$$

Proof. We use the Lyndon–Hochschild–Serre spectral sequence for

$$1 \to \mathbb{Z}/3 \to S_3 \to \mathbb{Z}/2 \to 1.$$

A reference is [Weibel 1994, Example 6.7.10]. On the E_2-page, $\mathrm{E}_2^{pq} = 0$ whenever $p > 0$ and $q > 0$ because the cohomology of $\mathbb{Z}/3$ is 3-torsion. Moreover, $\mathbb{Z}/2$ acts on $\mathrm{H}^q(\mathbb{Z}/3, M)$ by multiplication by -1 for $q \equiv 1, 2 \mod 4$ and by 1 for $q \equiv 0, 3 \mod 4$. □

The next lemma is about the cohomology of the reduced regular representation $\tilde{\rho}$ of S_3 introduced in Section 4.

Lemma 5.3. *Let M be an abelian group, and let $\tilde{\rho} \otimes M$ be an S_3-module through the action on $\tilde{\rho}$. Then,*

$$\begin{aligned}
\mathrm{H}^0(S_3, \tilde{\rho} \otimes M) &\cong M[3],\\
\mathrm{H}^1(S_3, \tilde{\rho} \otimes M) &\cong M/3,\\
\mathrm{H}^2(S_3, \tilde{\rho} \otimes M) &\cong 0\\
\mathrm{H}^3(S_3, \tilde{\rho} \otimes M) &\cong 0.
\end{aligned}$$

Proof. There is a short exact sequence of S_3-modules

$$0 \to \tilde{\rho} \otimes M \to \rho \otimes M \to M \to 0.$$

In the associated long exact sequence in cohomology, note that $\mathrm{H}^i(S_3, \rho \otimes M) \cong \mathrm{H}^i(C_2, M)$ by Shapiro's lemma, as $\rho \otimes M \cong \mathrm{ind}_{C_2}^{S_3} M$. The map

$$\mathrm{H}^i(S_3, \rho \otimes M) \cong \mathrm{H}^i(C_2, M) \to \mathrm{H}^i(S_3, M)$$

is the transfer. We obtain short exact sequences

$$0 \to \operatorname{coker} \mathrm{tr}_{C_2}^{S_3}(\mathrm{H}^{i-1}) \to \mathrm{H}^i(S_3, \tilde{\rho} \otimes M) \to \ker \mathrm{tr}_{C_2}^{S_3}(\mathrm{H}^i) \to 0.$$

Because $C_2 \to S_3$ has a retraction, the restriction map $\mathrm{H}^i(S_3, M) \to \mathrm{H}^i(C_2, M)$ is the projection to a direct summand. The transfer equals 3 times the inclusion of this summand as can easily be deduced from the equation $\mathrm{tr}_{C_2}^{S_3} \mathrm{res}_{C_2}^{S_3} = 3$. Thus, the transfer is multiplication by 3 on H^0, an isomorphism on H^1 and H^2 and the inclusion $M[2] \to M[6]$ on H^3. The lemma follows. □

These computations allow us to compute the E_2-term of the Leray–Serre spectral sequence

$$\mathrm{E}_2^{p,q} : \mathrm{H}^p(S_3, \mathrm{H}^q(\mathcal{M}(2)_S, \mathbb{G}_m)) \Rightarrow \mathrm{H}^{p+q}(\mathcal{M}_S, \mathbb{G}_m)$$

in a range. Using the results of the last two sections, we can analyze $\mathrm{H}^q(\mathcal{M}(2)_S, \mathbb{G}_m)$ in terms of $\mathrm{H}^q(X_S, \mathbb{G}_m)$. Especially Proposition 3.2 turns out to be useful as the short exact sequences in it are S_3-equivariant by naturality. Using additionally Lemma 4.8 for the first one, we obtain the S_3-equivariant extensions

$$0 \to \mathbb{G}_m(S) \to \mathbb{G}_m(\mathcal{M}(2)_S) \cong \mathbb{G}_m(X_S) \to \tilde{\rho} \to 0, \tag{5.4}$$

$$0 \to \mathrm{Pic}(X_S) \cong \mathrm{Pic}(S) \to \mathrm{Pic}(\mathcal{M}(2)_S) \to \mu_2(S) \to 0, \tag{5.5}$$

and

$$0 \to \mathrm{Br}'(X_S) \to \mathrm{Br}'(\mathcal{M}(2)_S) \to \mathrm{H}^1(X_S, \mu_2) \to 0. \tag{5.6}$$

The only point needing justification is that the pullback map $\mathrm{Pic}(S) \to \mathrm{Pic}(X_S)$ is an isomorphism. It is injective because X_S has an S-point. It is surjective as it factors through the isomorphism $\mathrm{Pic}(S) \to \mathrm{Pic}(\mathbb{A}^1_S)$ and since $j^*\colon \mathrm{Pic}(\mathbb{A}^1_S) \to \mathrm{Pic}(X_S)$ is surjective, where j denotes the inclusion $X_S \subseteq \mathbb{A}^1_S$. Indeed, given a line bundle $\mathcal{L}$ on X_S, we take a coherent subsheaf $\mathcal{F}$ of $j_*\mathcal{L}$ with $j^*\mathcal{F} \cong \mathcal{L}$. The double dual of $\mathcal{F}$ is a reflexive sheaf $\mathcal{L}'$ with $j^*\mathcal{L}'$ still isomorphic to $\mathcal{L}$. By [Hartshorne 1980, Proposition 1.9], $\mathcal{L}'$ is a line bundle.

The sequence (5.5) is S_3-equivariantly split and thus consists only of S_3-modules with the trivial action. Indeed, the morphism $S \to \mathrm{Spec}\,\mathbb{Z}\big[\frac{1}{2}\big]$ induces by pullback a morphism from the exact sequence

$$0 \to 0 \to \mathrm{Pic}(\mathcal{M}(2)) \to \mu_2\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big) \to 0,$$

where the splitting is clearly S_3-equivariant. As $\mu_2\big(\mathbb{Z}\big[\frac{1}{2}\big]\big) \to \mu_2(S)$ is an isomorphism for S connected, the result follows. These observations allow us to compute the $q = 0, 1$ lines of the Leray–Serre spectral sequence (5.1).

Lemma 5.7. *If S is a connected regular noetherian scheme over $\mathbb{Z}\big[\frac{1}{2}\big]$, then there are natural extensions*

$$0 \to \mathrm{H}^p(S_3, \mathbb{G}_m(S)) \to \mathrm{H}^p(S_3, \mathbb{G}_m(\mathcal{M}(2)_S)) \to \mathrm{H}^p(S_3, \tilde{\rho}) \to 0$$

for $0 \le p \le 3$, and natural isomorphisms

$$\mathrm{H}^p(S_3, \mathrm{H}^1(\mathcal{M}(2)_S, \mathbb{G}_m)) \cong \mathrm{H}^p(S_3, \mathrm{Pic}(S)) \oplus \mathrm{H}^p(S_3, \mu_2(S))$$

for all $p \ge 0$.

Proof. The first exact sequence follows from Lemmas 5.2 and 5.3 using that $\mathrm{H}^1(S_3, \tilde{\rho})$ is 3-torsion and $\mathrm{H}^2(S_3, \mathbb{G}_m(S))$ is 2-torsion. The direct sum decomposition follows from the fact that (5.5) is S_3-equivariantly split. □

The only necessary remaining group we need to understand for our computations is $\mathrm{H}^2(\mathcal{M}(2)_S, \mathbb{G}_m)^{S_3}$, which we analyze using the short exact sequence (5.6).

Lemma 5.8. *If S is a regular noetherian scheme over $\mathbb{Z}\big[\frac{1}{2}\big]$, then there is a canonical isomorphism $\mathrm{H}^1(S, \mu_2) \cong \mathrm{H}^1(X_S, \mu_2)^{S_3}$.*

Proof. Using the S_3-equivariant short exact sequence

$$0 \to \mathbb{G}_m(X_S)/2 \to \mathrm{H}^1(X_S, \mu_2) \to \mathrm{Pic}(X_S)[2] \to 0,$$

we get a long exact sequence

$$0 \to (\mathbb{G}_m(X_S)/2)^{S_3} \to \mathrm{H}^1(X_S, \mu_2)^{S_3} \to \mathrm{Pic}(X_S)[2]^{S_3} \to \mathrm{H}^1(S_3, \mathbb{G}_m(X_S)/2) \to \cdots .$$

As the canonical map $X_S \to S$ is S_3-equivariant, we obtain a map into this from the exact sequence

$$0 \to \mathbb{G}_m(S)/2 \to \mathrm{H}^1(S, \mu_2) \to \mathrm{Pic}(S)[2] \to 0.$$

As the maps $\mathbb{G}_m(S)/2 \to (\mathbb{G}_m(X_S)/2)^{S_3}$ and $\mathrm{Pic}(S)[2] \to \mathrm{Pic}(X_S)[2]$ are isomorphisms (using the exact sequence (5.4) and Lemma 5.3), the five lemma implies that $\mathrm{H}^1(S, \mu_2) \to \mathrm{H}^1(X_S, \mu_2)^{S_3}$ is an isomorphism as well. □

From (5.6), we obtain a long exact sequence

$$0 \to \mathrm{Br}'(X_S)^{S_3} \to \mathrm{Br}'(\mathcal{M}(2))^{S_3} \to \mathrm{H}^1(S, \mu_2) \to \mathrm{H}^1(S_3, \mathrm{Br}'(X_S)) \to \cdots . \tag{5.9}$$

Lemma 5.10. *Let S be a regular noetherian scheme over* $\mathrm{Spec}\,\mathbb{Z}[1/p]$ *for some prime p, and let $X_S = \mathbb{A}^1_S - \{0, 1\}$ as before. There is a noncanonically split exact sequence*

$$0 \to {}_p\mathrm{Br}'(S) \to {}_p\mathrm{Br}'(X_S) \to {}_p\mathrm{H}^3_{\{0,1\}}(\mathbb{A}^1_S, \mathbb{G}_m) \to 0.$$

Proof. By Proposition 2.14 we have an exact sequence

$$0 \to {}_p\mathrm{Br}'(\mathbb{A}^1_S) \to {}_p\mathrm{Br}'(X_S) \to {}_p\mathrm{H}^3_{\{0,1\}}(\mathbb{A}^1_S, \mathbb{G}_m) \to {}_p\mathrm{H}^3(\mathbb{A}^1_S, \mathbb{G}_m) \to {}_p\mathrm{H}^3(X_S, \mathbb{G}_m).$$

Because p is invertible on S, Proposition 2.5 implies that ${}_p\mathrm{H}^i(S, \mathbb{G}_m) \cong {}_p\mathrm{H}^i(\mathbb{A}^1_S, \mathbb{G}_m)$ for all $i \geq 0$. But, since X_S has an S-point, it follows that ${}_p\mathrm{H}^i(\mathbb{A}^1_S, \mathbb{G}_m) \cong {}_p\mathrm{H}^i(S, \mathbb{G}_m) \to {}_p\mathrm{H}^i(X_S, \mathbb{G}_m)$ is split injective for all i. □

Lemma 5.11. *For any prime p and any regular noetherian scheme over* $\mathrm{Spec}\,\mathbb{Z}\left[\frac{1}{p}\right]$, *there is a canonical isomorphism*

$$\mathrm{H}^q_{\{0,1\}}(\mathbb{A}^1_S, \mathbb{G}_m) \cong \mathrm{H}^q_{\{0\}}(\mathbb{A}^1_S, \mathbb{G}_m) \oplus \mathrm{H}^q_{\{1\}}(\mathbb{A}^1_S, \mathbb{G}_m).$$

The action of S_3 on ${}_p\mathrm{Br}'(X_S)/{}_p\mathrm{Br}'(S)$ is isomorphic to

$$\tilde{\rho} \otimes {}_p\mathrm{H}^3_{\{0\}}(\mathbb{A}^1_S, \mathbb{G}_m) \cong \tilde{\rho} \otimes \mathrm{H}^1(S, \mathbb{Q}_p/\mathbb{Z}_p).$$

Proof. Given any étale sheaf $\mathcal{F}$, there is a canonical isomorphism

$$\mathrm{H}^0_{\{0,1\}}(\mathbb{A}^1_S, \mathcal{F}) \cong \mathrm{H}^0_{\{0\}}(\mathbb{A}^1_S, \mathcal{F}) \oplus \mathrm{H}^0_{\{1\}}(\mathbb{A}^1_S, \mathcal{F}),$$

as one sees by an easy diagram chase. By deriving this isomorphism, the first part of the lemma follows.

To prove the second statement, we compare the sequence of Lemma 5.10 with the long exact sequence for étale cohomology with supports coming from the open inclusion $X_S \subseteq \mathbb{P}^1_S$. Using the natural map of long exact sequences, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {}_p\operatorname{Br}'(\mathbb{P}^1_S) \arrow[r] \arrow[d, "\cong"] & {}_p\operatorname{Br}'(X_S) \arrow[r] \arrow[d, "="] & {}_p\mathrm{H}^3_{\{0,1,\infty\}}(\mathbb{P}^1_S, \mathbb{G}_m) \arrow[d] & \\
0 \arrow[r] & {}_p\operatorname{Br}'(\mathbb{A}^1_S) \arrow[r] & {}_p\operatorname{Br}'(X_S) \arrow[r] & {}_p\mathrm{H}^3_{\{0,1\}}(\mathbb{A}^1_S, \mathbb{G}_m) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where the left-hand vertical map is an isomorphism because it is injective (by Proposition 2.5(iv)), ${}_p\operatorname{Br}'(S) \to {}_p\operatorname{Br}'(\mathbb{A}^1_S)$ is an isomorphism, and there is an S-point of $\mathbb{A}^1_S \subseteq \mathbb{P}^1_S$.

Now, by Proposition 2.14,

$${}_p\mathrm{H}^3_{\{0,1,\infty\}}(\mathbb{P}^1_S, \mathbb{G}_m) \cong \bigoplus_{\{0,1,\infty\}} \mathrm{H}^1(S, \mathbb{Q}_p/\mathbb{Z}_p) \quad \text{and} \quad {}_p\mathrm{H}^3_{\{0,1\}}(\mathbb{A}^1_S, \mathbb{G}_m) \cong \bigoplus_{\{0,1\}} \mathrm{H}^1(S, \mathbb{Q}_p/\mathbb{Z}_p).$$

With this description, the right-hand vertical map above is the natural projection away from the factor of $\mathrm{H}^1(S, \mathbb{Q}_p/\mathbb{Z}_p)$ corresponding to ∞. Let χ_0 and χ_1 be p-primary characters of S, i.e., elements of $\mathrm{H}^1(S, \mathbb{Q}_p/\mathbb{Z}_p)$. Then, as χ_0, χ_1 vary, the Azumaya algebras $(\chi_0, t) \otimes (\chi_1, t-1)$ give elements of $\operatorname{Br}(X_S)$ whose ramification classes (χ_0, χ_1) span ${}_p\mathrm{H}^3_{\{0,1\}}(\mathbb{A}^1_S, \mathbb{G}_m)$. The ramification of such a class computed in $\mathrm{H}^3_{\{0,1,\infty\}}(\mathbb{P}^1_S, \mathbb{G}_m)$ is $(\chi_0, \chi_1, -\chi_0 - \chi_1)$. This follows from Proposition 2.16, the fact that $\operatorname{ram}_{(\pi)}(\chi, \pi^{-1}) = -\operatorname{ram}_{(\pi)}(\chi, \pi)$ in the notation of that proposition, and the fact that both t^{-1} and $(t-1)^{-1}$ are uniformizing parameters for the divisor at ∞ of $\mathbb{P}^1_S$. It follows that the image of ${}_p\operatorname{Br}'(X_S)$ inside ${}_p\mathrm{H}^3_{\{0,1,\infty\}}(\mathbb{P}^1_S, \mathbb{G}_m) \cong \bigoplus_{\{0,1,\infty\}} \mathrm{H}^1(S, \mathbb{Q}_p/\mathbb{Z}_p)$ can be identified with $\tilde{\rho} \otimes \mathrm{H}^1(S, \mathbb{Q}_p/\mathbb{Z}_p)$. □

We will analyze the implications for p-primary torsion for $p > 2$ in the next three sections. For the rest of this section, we will begin the study of the 2-primary torsion of $\operatorname{Br}'(\mathcal{M}_S)$ where S is a $\mathbb{Z}[\frac{1}{2}]$-scheme. By Lemmas 5.11 and 5.3, we know that ${}_2\mathrm{H}^3_{\{0,1\}}(\mathbb{A}^1_S, \mathbb{G}_m)^{S_3} = 0$. Thus from Lemma 5.10, we see that ${}_2\operatorname{Br}'(S) \to {}_2\operatorname{Br}'(X_S)^{S_3}$ is an isomorphism. If we tensor the sequence (5.9) with $\mathbb{Z}_{(2)}$, we obtain (using Lemmas 5.2 and 5.3) the exact sequence

$$0 \to {}_2\operatorname{Br}'(S) \to {}_2\operatorname{Br}'(\mathcal{M}(2))^{S_3} \to \mathrm{H}^1(S, \mu_2) \xrightarrow{\partial} \mathrm{H}^1(S_3, {}_2\operatorname{Br}'(X_S)) \cong \operatorname{Br}'(S)[2] \to \cdots. \tag{5.12}$$

Here we recall that we denote for an abelian group A by $A[2]$ its 2-torsion, while ${}_2A$ denotes its 2-primary torsion. We want to analyze the boundary map ∂.

Lemma 5.13. *If $u \in \mathrm{H}^1(S, \mu_2)$, then $\partial(u)$ equals the Brauer class of the cyclic (quaternion) algebra $(-1, u)$.*

Proof. We assume that S is connected. Denote by $\pi\colon X_S \to \mathrm{B}C_{2,X_S}$ the projection and by $c\colon \mathrm{B}C_{2,X_S} \to X_S$ the canonical map to the coarse moduli space. Denote by

$$r\colon \operatorname{Br}'(\mathrm{B}C_{2,X_S}) \to \mathrm{H}^1(X_S, \mu_2)$$

the map obtained from the Leray spectral sequence. Finally, let

$$s\colon \mathrm{H}^1(X_S, \mu_2) \to \mathrm{Br}'(\mathrm{B}C_{2,X_S})$$

be given by $s(u) = [(\chi, u)] = [(\chi, c^*u)]$, where $\chi \in \mathrm{H}^1(\mathrm{B}C_{2,X_S}, C_2)$ classifies π. We have $r(s(u)) = u$ by Lemma 3.3.

Using [Serre 1997, Section 5.4], we can compute a crossed homomorphism representing $\partial(u)$ thus as

$$g \mapsto \pi^*(g(s(u)) - s(u)) \in \mathrm{Br}'(X_S).$$

Consider the subgroup $C_2 = \langle(2\,3)\rangle \subset S_3$ and the C_2-equivariant morphism $z\colon S \to X_S$ classifying $y^2 = x(x-1)(x+1)$ (i.e., $t=-1$). It follows from Lemmas 5.2 and 5.3 that the morphism $z^*\,\mathrm{res}^{S_3}_{C_2}$ induces an isomorphism $\mathrm{H}^1(S_3, {}_2\mathrm{Br}'(X_S)) \to \mathrm{H}^1(C_2, {}_2\mathrm{Br}'(S))$. The isomorphism $\mathrm{H}^1(C_2, \mathrm{Br}'(S)) \to \mathrm{Br}'(S)[2]$ is given by evaluating the crossed homomorphism at the nontrivial element $(2\,3) \in C_2$. Thus, the coboundary map $\partial : \mathrm{H}^1(S, \mu_2) \to \mathrm{Br}'(S)[2]$ sends u to

$$z^*\pi^*((2\,3)(s(u)) - s(u)).$$

As the pullback of $X_S \to \mathrm{B}C_{2,X_S}$ along $\pi \circ z\colon S \to X_S \to \mathrm{B}C_{2,X_S}$ is the trivial C_2-torsor, $z^*\pi^* s(u) = (\pi z)^*(\chi, u)$ defines the trivial Brauer class. By Lemma 4.7, the action of $(2\,3)$ multiplies the torsor $X_S \to \mathrm{B}C_{2,X_S}$ with the torsor $\mathrm{B}C_{2,X_S}(\sqrt{t}) \to \mathrm{B}C_{2,X_S}$. Thus, $z^*\pi^*(2\,3)(s(u)) = (z^*t, u) = (-1, u)$. □

Summarizing, we obtain the following result.

Proposition 5.14. *Let S be a regular noetherian scheme over $\mathbb{Z}\left[\frac{1}{2}\right]$. We have an exact sequence*

$$0 \to {}_2\mathrm{Br}'(S) \to {}_2\mathrm{Br}'(\mathcal{M}(2)_S)^{S_3} \to \mathrm{H}^1(S, \mu_2) \xrightarrow{\partial} \mathrm{Br}'(S)[2]$$

with $\partial(u) = [(-1, u)]$. The map ${}_2\mathrm{Br}'(S) \to {}_2\mathrm{Br}'(\mathcal{M}(2)_S)^{S_3}$ is noncanonically split.

Proof. The exact sequence is exactly (5.12). The identification of $\partial(u)$ follows from the previous lemma. For the splitting, choose an S-point $S \to \mathcal{M}(2)_S$. Then the composition $\mathrm{Br}'(\mathcal{M}(2)_S)^{S_3} \to \mathrm{Br}'(\mathcal{M}(2)_S) \to \mathrm{Br}'(S)$ provides the splitting. □

6. The p-primary torsion in $\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ for primes $p \geq 5$

Before we proceed to study the 3-primary and 2-primary torsion, we will show in this section that for a large class of S there is no p-primary torsion for $p \geq 5$ in the Brauer group of $\mathcal{M}_S$. Lemma 5.3 implies the crucial fact that there are no S_3-invariant classes in ${}_p\mathrm{Br}'(X_S)$ ramified at $\{0, 1\}$ when $p \neq 3$ is invertible on S. The main point of the following theorem is that this is true for $p \geq 5$ even for certain regular noetherian schemes where p is not a unit.

Theorem 6.1. *Let S be a regular noetherian scheme over $\mathbb{Z}$ and $p \geq 5$ prime. Assume that $S[1/(2p)] = S_{\mathbb{Z}[1/(2p)]}$ is dense in S and that $\mathcal{M}_S \to S$ has a section. Then the natural map ${}_p\mathrm{Br}'(S) \to {}_p\mathrm{Br}'(\mathcal{M}_S)$ is an isomorphism.*

Proof. Assume first that 2 is invertible on S. The only contribution to ${}_p\operatorname{Br}'(\mathcal{M}_S)$ in the Leray–Serre spectral sequence (5.1) occurs as

$$ {}_p(\mathrm{H}^2(\mathrm{B}C_{2,X_S}, \mathbb{G}_m)^{S_3}) = ({}_p\mathrm{H}^2(\mathrm{B}C_{2,X_S}, \mathbb{G}_m))^{S_3} \tag{6.2}$$

because H^i of S_3 for $i \geq 1$ can never have p-primary torsion for $p \geq 5$.

We will argue that the p-group (6.2) is isomorphic to ${}_p\operatorname{Br}'(S)$ for all primes $p \geq 5$ if additionally p is invertible on S. To do so, note first that

$$ {}_p\mathrm{H}^2(\mathrm{B}C_{2,X_S}, \mathbb{G}_m) \cong {}_p\operatorname{Br}'(X_S)$$

for $p \neq 2$ by Proposition 3.2. By Lemmas 5.10, 5.11 and 5.3, we see that ${}_p\operatorname{Br}'(S) \to {}_p\operatorname{Br}'(X_S)^{S_3}$ is an isomorphism.

This shows the theorem if $2p$ is invertible on S. Let now S be arbitrary regular noetherian such that $S[1/(2p)] \subset S$ is dense and $\mathcal{M}_S$ has an S-point. Consider the commutative diagram

$$\begin{tikzcd} {}_p\operatorname{Br}'(\mathcal{M}_S) \arrow[r] \arrow[d] & {}_p\operatorname{Br}'(\mathcal{M}_{S[1/(2p)]}) \arrow[d, "\cong"] \\ {}_p\operatorname{Br}'(S) \arrow[r] & {}_p\operatorname{Br}'\big(S\big[\tfrac{1}{2p}\big]\big). \end{tikzcd}$$

induced by the choice of an S-point of $\mathcal{M}_S$. As $\mathcal{M}_S$ has a cover by a scheme that is fppf over S and fppf morphisms are open [EGA IV$_2$ 1965, Theorem 2.4.6], $\mathcal{M}_S \to S$ is open as well. Thus, $\mathcal{M}_{S[1/(2p)]} \subset \mathcal{M}_S$ is dense and hence $\operatorname{Br}'(\mathcal{M}_S) \to \operatorname{Br}'(\mathcal{M}_{S[1/(2p)]})$ is injective by Proposition 2.5. This implies that $\operatorname{Br}'(\mathcal{M}_S) \to \operatorname{Br}'(S)$ is injective as well. As it is also split surjective, we see that it is an isomorphism. □

Remark 6.3. In general, it is a subtle question to decide whether $\mathcal{M}_S$ has an S-point. For example for $S = \mathbb{Z}\big[\frac{1}{2}\big]$ or $S = \mathbb{Z}\big[\frac{1}{3}\big]$, there is such an S-point, but for $S = \mathbb{Z}\big[\frac{1}{5}\big]$ or $S = \mathbb{Z}\big[\frac{1}{29}\big]$ there is none (for this and other examples; see [Edixhoven et al. 1990, Corollary 1]). Nevertheless, sometimes one can still control the p-power torsion for $p \geq 5$ if there is no S-point, as the following corollary shows.

Corollary 6.4. *The Brauer groups* $\operatorname{Br}(\mathcal{M}) \subseteq \operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ *have only* 2 *and* 3*-primary torsion.*

Proof. Indeed, $\operatorname{Br}(\mathcal{M}) \subseteq \operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$, and there is no p-torsion in $\operatorname{Br}\big(\mathbb{Z}\big[\frac{1}{2}\big]\big) \cong \mathbb{Z}/2$ for $p \neq 2$. □

7. The 3-primary torsion in $\operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/6]})$

The next theorem describes the 3-primary torsion in $\operatorname{Br}'(\mathcal{M}_S)$ in many cases.

Theorem 7.1. *Let S be a regular noetherian scheme. If* 6 *is a unit on S, then there is an exact sequence*

$$0 \to {}_3\operatorname{Br}'(S) \to {}_3\operatorname{Br}'(\mathcal{M}_S) \to \mathrm{H}^1(S, C_3) \to 0,$$

which is noncanonically split. The map ${}_3\operatorname{Br}'(\mathcal{M}_S) \to \mathrm{H}^1(S, C_3)$ can be described as the composition of pullback to X_S and taking the ramification at the divisor $\{0\}$ in $\mathbb{A}^1_S$ defined by t (using Proposition 2.14).

Proof. The 3-primary torsion in $\mathrm{Br}'(\mathcal{M}(2)_S) \cong \mathrm{Br}'(BC_{2,X_S})$ is just the 3-primary torsion in $\mathrm{Br}'(X_S)$ by Proposition 3.2 as $\mathrm{H}^1_{pl}(X_S; \mu_2)_{(3)} = 0$. Similarly, since 2 is invertible in S, the Leray–Serre spectral sequence (5.1) together with the group cohomology computations of Lemmas 5.2 and 5.3 and the exact sequences (5.4)–(5.6) say that

$$ {}_3\,\mathrm{Br}'(\mathcal{M}_S) \cong ({}_3\mathrm{Br}'(X_S))^{S_3}. $$

Since 3 is invertible in S, we have a short exact sequence

$$ 0 \to {}_3\,\mathrm{Br}'(S) \to {}_3\,\mathrm{Br}'(X_S) \to {}_3\mathrm{H}^3_{\{0,1\}}(\mathbb{A}^1_S, \mathbb{G}_m) \to 0 $$

by Lemma 5.10.

The 0 and 1 sections are disjoint, so that there is an isomorphism of S_3-modules

$$ {}_3\mathrm{H}^3_{\{0,1\}}(\mathbb{A}^1_S, \mathbb{G}_m) \cong \bigoplus_{i=0,1} \mathrm{H}^1(S, \mathbb{Q}_3/\mathbb{Z}_3) \cong \tilde{\rho} \otimes \mathrm{H}^1(S, \mathbb{Q}_3/\mathbb{Z}_3) $$

by Proposition 2.14 and Lemma 5.11. Thus, Lemma 5.3 implies that the long exact sequence in S_3-cohomology takes the following form:

$$ 0 \to {}_3\,\mathrm{Br}'(S) \to {}_3\,\mathrm{Br}'(X_S)^{S_3} \to \mathrm{H}^1(\operatorname{Spec} S, C_3) \to \mathrm{H}^1(S_3, {}_3\,\mathrm{Br}'(S)). $$

However, the action of S_3 on ${}_3\,\mathrm{Br}'(S)$ is trivial, so the group on the right vanishes by Lemma 5.2. Since $\mathcal{M}_S$ has an S-point (because 2 is inverted), the splitting follows.

The map ${}_3\,\mathrm{Br}'(X_S) \to \bigoplus_{i=0,1} \mathrm{H}^1(S, \mathbb{Q}_3/\mathbb{Z}_3)$ takes the ramification at the divisors $\{0\}$ and $\{1\}$. At this point, we need to make the isomorphism $(\tilde{\rho} \otimes \mathrm{H}^1(S, \mathbb{Q}_3/\mathbb{Z}_3))^{S_3} \to \mathrm{H}^1(S, C_3)$ more explicit. By choosing the ordered basis $t, t-1$ of $\mathbb{G}_m(X_S)/\mathbb{G}_m(S) \cong \tilde{\rho}$, we see from the description of the action that $\mathbb{Z}/3 \cong (\tilde{\rho}/3)^{S_3} \subseteq \tilde{\rho}/3$ is generated by $t(t-1)$. Indeed, $\sigma(t(t-1)) = -t^{-2}(t-1)$ and $\tau(t(t-1)) = -t^{-2}(t-1)$ for $\sigma = (1\,3\,2)$ and $\tau = (2\,3)$ and thus

$$ \sigma(t(t-1)) \equiv t(t-1) \equiv \tau(t(t-1)) \in \mathbb{G}_m(X_S)/3. $$

This implies that $(\tilde{\rho} \otimes \mathrm{H}^1(S, \mathbb{Q}_3/\mathbb{Z}_3))^{S_3} \to \mathrm{H}^1(S, C_3)$ can be identified with projection onto the first coordinate. Thus, ${}_3\,\mathrm{Br}'(S_R)^{S_3} \to \mathrm{H}^1(S, C_3)$ takes the ramification at the divisor $\{0\}$. □

We want to be more specific about the Azumaya algebras arising from $\mathrm{H}^1(S, C_3)$. For that purpose consider the section $\Delta \in \mathrm{H}^0(\mathcal{M}, \lambda^{\otimes 12})$, which is defined as follows. Given an elliptic curve E over S, we can write it Zariski locally in Weierstrass form. Consider its discriminant $\Delta_E \in \mathcal{O}(S)$ and its invariant differential $\omega \in \Omega^1_{E/R}(E) \cong \lambda(R)$. It is easy to see by [Silverman 2009, Table 3.1] that $\Delta = \Delta_E \omega^{\otimes 12}$ is a section of $\lambda^{\otimes 12}$, which is invariant under coordinate changes. Thus, Δ defines a section of $\lambda^{\otimes 12}$ on $\mathcal{M}$.

Lemma 7.2. *By Construction 2.8, we can associate with the line bundle $\mathcal{L} = \lambda^{\otimes 4}$ and the trivialization $\Delta\colon \mathcal{O}_{\mathcal{M}} \to \mathcal{L}^{\otimes 3}$ the μ_3-torsor $\mathcal{M}(\sqrt[3]{\Delta}) \to \mathcal{M}$ whose class in $\mathrm{H}^1(\mathcal{M}, \mu_3)$ we denote by $[\Delta]_3$. If S is a regular noetherian scheme and* 6 *is a unit on S, then the composite*

$$ \mathrm{H}^1(S, C_3) \to \mathrm{H}^1(\mathcal{M}_S, C_3) \xrightarrow{\cup(-[\Delta]_3)} \mathrm{H}^2(\mathcal{M}_S, \mu_3) \to {}_3\,\mathrm{Br}'(\mathcal{M}_S) $$

is a section of the map ${}_3\,\mathrm{Br}'(\mathcal{M}_S) \to \mathrm{H}^1(S, C_3)$ of Theorem 7.1.

Remark 7.3. Informally, this section associates with $\chi \in \mathrm{H}^1(S, C_3)$ the symbol algebra $[(\chi, \Delta^{-1})_3]$.

Proof. The pullback of Δ to X is the discriminant of the universal Legendre curve, which is $16t^2(t-1)^2$ (using the standard trivialization of λ on X given by $dx/(2y)$). For $\chi \in \mathrm{H}^1(S, C_3)$ the pullback of $[(\chi, \Delta^{-1})_3]$ to $\mathrm{Br}'(X_S)$ is thus $[(\chi, 4t(t-1))_3]$. As $4t(t-1)$ is a uniformizer for the local ring of $\mathbb{A}^1_S$ at $t=0$, Proposition 2.16 implies the result. $\square$

Remark 7.4. While this map $\mathrm{H}^1(S, C_3) \to \mathrm{Br}'(\mathcal{M}_S)$ is defined whether or not 6 is a unit on S, without this assumption we do not know that $\mathrm{H}^1(S, C_3)$ is the cokernel of ${}_3\mathrm{Br}'(S) \to {}_3\mathrm{Br}'(\mathcal{M}_S)$.

Corollary 7.5. *When $S = \operatorname{Spec} \mathbb{Z}\left[\frac{1}{6}\right]$, there is an isomorphism ${}_3\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/6]}) \cong \mathbb{Q}_3/\mathbb{Z}_3 \oplus \mathbb{Z}/3$. The 3-torsion subgroup is generated by classes σ and θ, which can be described as follows. Let $\chi \in \mathrm{H}^1\left(\operatorname{Spec} \mathbb{Z}\left[\frac{1}{6}\right], C_3\right)$ be the character of the Galois extension $\mathbb{Q}(\zeta_9 + \bar{\zeta}_9)$ of $\mathbb{Q}$. Then $\sigma = [(\chi, 6)_3]$ and $\theta = [(\chi, 16\Delta^{-1})_3]$, which pulls back to $[(\chi, t(t-1))_3]$ on $X_{\mathbb{Z}[1/6]}$.*

Proof. We claim first that $\mathbb{Q}(\zeta_9 + \bar{\zeta}_9)$ is the only cyclic cubic extension L of $\mathbb{Q}$ that ramifies at most at 2 and 3. This can either be deduced from [Hasse 1948, I. Section 1.2] or shown as follows. By the Kronecker–Weber theorem, any cyclic cubic extension L of $\mathbb{Q}$ has to embed into a cyclotomic extension $\mathbb{Q}(\zeta_n)$ and is more precisely its fixed field under a normal subgroup $H \subset (\mathbb{Z}/n)^\times$ of index 3. As H contains all elements of 2-power order, we can assume that n is odd and thus L does not ramify at 2 but only at 3. Proposition 3.1 of [Lemmermeyer 2005] shows that L is unique and must be $\mathbb{Q}(\zeta_9 + \bar{\zeta}_9)$. This implies that $\mathrm{H}^1\left(\operatorname{Spec} \mathbb{Z}\left[\frac{1}{6}\right], C_3\right) \cong \mathbb{Z}/3$.

Using that ${}_3\mathrm{Br}\left(\mathbb{Z}\left[\frac{1}{6}\right]\right) \cong \mathbb{Q}_3/\mathbb{Z}_3$, the structure of the Brauer group $\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/6]})[3]$ follows from Theorem 7.1. The description of θ follows directly from the last lemma (where we have modified the section by an element of ${}_3\mathrm{Br}\left(\mathbb{Z}\left[\frac{1}{6}\right]\right)$ for convenience).

Last we need to show that $[(\chi, 6)_3]$ is nonzero in $\mathrm{Br}\left(\mathbb{Z}\left[\frac{1}{6}\right]\right)[3] \cong \mathbb{Z}/3$. It suffices to check that $(\chi, 6)$ is ramified at the prime (2). Note that the minimal polynomial of $\zeta_9 + \bar{\zeta}_9$ is $w^3 + w + 1$. By Proposition 2.16, the ramification at (2) in $\mathrm{H}^1(\mathbb{F}_2, C_3) \cong \mathbb{Z}/3$ is the class of the extension $w^3 + w + 1$ over $\mathbb{F}_2$. Since this polynomial is irreducible (it has no solutions in $\mathbb{F}_2$ and it has degree 3), it follows that the ramification is nonzero. $\square$

Corollary 7.6. *Let $R = \mathbb{Z}\left[\frac{1}{2}\right]$ or $R = \mathbb{Z}$. Then the order of ${}_3\mathrm{Br}(\mathcal{M}_R)$ is either 1 or 3.*

Proof. Using the injectivity of $\mathrm{Br}(\mathcal{M}) \to \mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$, it suffices to prove this when $R = \mathbb{Z}\left[\frac{1}{2}\right]$. Now, we claim that no nonzero class $\alpha \in {}_3\mathrm{Br}\left(\mathbb{Z}\left[\frac{1}{6}\right]\right) \subseteq {}_3\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/6]})$ extends to $\mathcal{M}_{\mathbb{Z}[1/2]}$. Indeed, we can take the Legendre curve $y^2 = x(x-1)(x-2)$, which defines a point $\operatorname{Spec} \mathbb{Z}\left[\frac{1}{2}\right] \to \mathcal{M}$. Using the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{Z}\left[\frac{1}{6}\right] \arrow[r] \arrow[d] & \mathcal{M}_{\mathbb{Z}[1/6]} \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}\left[\frac{1}{6}\right] \\
\operatorname{Spec} \mathbb{Z}\left[\frac{1}{2}\right] \arrow[r] & \mathcal{M}_{\mathbb{Z}[1/2]},
\end{tikzcd}$$

we see that if $\alpha \in {}_3\operatorname{Br}\big(\mathbb{Z}\big[\frac{1}{6}\big]\big)$ did extend to $\mathcal{M}_{\mathbb{Z}[1/2]}$, then it would be zero in the Brauer group of $\operatorname{Spec}\mathbb{Z}\big[\frac{1}{6}\big]$, as it would extend to ${}_3\operatorname{Br}\big(\mathbb{Z}\big[\frac{1}{2}\big]\big)=0$. However, these classes are all nonzero in $\operatorname{Br}\big(\mathbb{Z}\big[\frac{1}{6}\big]\big)$ since the composition at the top of the commutative diagram is the identity for any $\mathbb{Z}\big[\frac{1}{6}\big]$-point of the moduli stack.

Now, if $\alpha=\beta+m\theta$ extends, where $\beta\in{}_3\operatorname{Br}\big(\mathbb{Z}\big[\frac{1}{6}\big]\big)$, then $3\alpha=3\beta$ also extends. Hence, it must be that α has order at most 3. In particular, this means that every class of ${}_3\operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ is actually 3-torsion and hence this group is a subgroup of $(\mathbb{Z}/3)^2$. But, we have already seen that σ does not extend. So, it is a proper subgroup, and hence it has order at most 3. □

Proposition 7.7. *Suppose there are Legendre curves $E_i : y^2 = x(x-1)(x-t_i)$ over $\operatorname{Spec}\mathbb{Z}_3[\zeta_3]$ for $i=1,2$ such that*

$$[(\chi, t_1(t_1-1))] \neq 0 \quad \textit{and} \quad [(\chi, t_2(t_2-1))] = 0$$

in $\operatorname{Br}(\mathbb{Q}_3(\zeta_3))[3]=\mathbb{Z}/3$, where χ denotes the pullback of the Galois extension $\mathbb{Q}(\zeta_9+\bar{\zeta}_9)$ of $\mathbb{Q}$ to $\mathbb{Q}_3(\zeta_3)$. Then ${}_3\operatorname{Br}\big(\mathcal{M}\big[\frac{1}{2}\big]\big)=0$.

Proof. Suppose that $\alpha=a\sigma+b\theta$ is a linear combination of the classes found in Corollary 7.5 where we can assume that $b\in\{1,2\}$ since σ does not extend. Suppose that α extends to $\operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$. We can pull back α along the two $\mathbb{Q}_3(\zeta_3)$-points of $\mathcal{M}$ defined by E_i and compute the ramifications in $\operatorname{Br}(\mathbb{Q}_3(\zeta_3))$. Let $k=[(\chi, t_1(t_1-1))]$. For $i=1$, we get $a+bk$ and for $i=2$ we get a. Since k is nonzero, these cannot be simultaneously zero modulo 3. But the two maps $\operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})\to\operatorname{Br}(\mathbb{Q}_3(\zeta_3))$ factor over $\operatorname{Br}(\mathbb{Z}_3[\zeta_3])=0$, which is a contradiction. Thus, α cannot extend to $\operatorname{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$. □

8. The ramification of the 3-torsion

Our aim in this section is to show that ${}_3\operatorname{Br}\big(\mathcal{M}\big[\frac{1}{2}\big]\big)=0$ using Proposition 7.7.

Lemma 8.1. *The natural inclusion $\mathbb{Q}(\zeta_9+\bar{\zeta}_9,\zeta_3)\to\mathbb{Q}(\zeta_9)$ is an isomorphism.*

Proof. The left-hand side is a subfield of $\mathbb{Q}(\zeta_9)$ that is strictly larger than $\mathbb{Q}(\zeta_9+\bar{\zeta}_9)$ and thus is equal to $\mathbb{Q}(\zeta_9)$. □

We will need the following lemma to aid our Hilbert symbol calculations below.

Lemma 8.2. *Consider the cyclotomic field $\mathbb{Q}(\zeta)$ with $\zeta=\zeta_3$ and $\pi=1-\zeta$ and denote by Tr the trace for $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$. Then*

$$\operatorname{Tr}(\pi^{6k+l}) = \begin{cases} (-1)^{3k}\cdot 3^{3k}\cdot 2 & \text{if } l=0,\\ (-1)^{3k}\cdot 3^{3k+1} & \text{if } l=1,\\ (-1)^{3k}\cdot 3^{3k+1} & \text{if } l=2,\\ 0 & \text{if } l=3,\\ (-1)^{3k+1}\cdot 3^{3k+2} & \text{if } l=4,\\ (-1)^{3k+1}\cdot 3^{3k+3} & \text{if } l=5,\end{cases}$$

Proof. We have $\pi^2=(1-\zeta)^2=-3\zeta$ and thus $\pi^{6k}=(-3)^{3k}$. Therefore, we have just to compute $\operatorname{Tr}(\pi^l)$ for $l=0,\ldots,5$, which is easily done. □

We come to a key arithmetic point in our proof, where we compute the Hilbert symbol at the prime 3 of certain degree 3 cyclic algebras. By Proposition 2.17, this will allow us to check whether certain cyclic algebras are zero in the Brauer group.

Lemma 8.3. *Consider the cyclotomic field $\mathbb{Q}_3(\zeta)$ with $\zeta = \zeta_3$ and $\pi = 1-\zeta$ the uniformizer. Then we have*

$$\left(\frac{\zeta, t(t-1)}{\pi}\right) = \zeta^{1-b^2}$$

in $\mu_3(\mathbb{Q}_3(\zeta))$, where $t = 2 + b\pi$ with $b \in \mathbb{Z}_3$.

Proof. We use the formula of Artin and Hasse (see [Neukirch 1999, Theorem V.3.8]) to compute this Hilbert symbol. By this formula, we have

$$\left(\frac{\zeta, a}{\pi}\right) = \zeta^{\mathrm{Tr}(\log a)/3},$$

where $a \in 1+\mathfrak{p}$ (for $\mathfrak{p} \subset \mathbb{Z}_3[\zeta_3]$ the maximal ideal) and Tr the trace for $\mathbb{Q}_3(\zeta)$ over $\mathbb{Q}_3$.

This formula directly applies to $t-1 = 1+b\pi$. We have

$$\log(t-1) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(b\pi)^i}{i}.$$

Again, it follows easily from Lemma 8.2 that $\mathrm{Tr}(\pi^i)/i$ is divisible by 9 for $i \geq 3$. Thus,

$$\frac{\mathrm{Tr}(\log(t-1))}{3} \equiv \frac{\mathrm{Tr}(b\pi)}{3} - \frac{\mathrm{Tr}(b^2\pi^2)}{6} \equiv b - \frac{b^2}{2} \mod 3$$

and $\left(\frac{\zeta, t-1}{\pi}\right) = \zeta^{b-b^2/2}$.

To compute $\left(\frac{\zeta,t}{\pi}\right)$ note that $\left(\frac{\zeta,t}{\pi}\right) = \left(\frac{\zeta,-t}{\pi}\right)\left(\frac{\zeta,-1}{\pi}\right) = \left(\frac{\zeta,-t}{\pi}\right)$. Indeed, $\left(\frac{\zeta,-1}{\pi}\right)^2 = 1$ and hence also $\left(\frac{\zeta,-1}{\pi}\right) = 1$ (in $\mu_3(\mathbb{Q}_3(\zeta_3))$). We have $-t = 1 + (-3-b\pi)$. Thus,

$$\log(-t) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(-3-b\pi)^i}{i}.$$

It follows easily from Lemma 8.2 that $\mathrm{Tr}((-3-b\pi)^i))/i$ is divisible by 9 for $i \geq 3$. Thus,

$$\frac{\mathrm{Tr}(\log(-t))}{3} \equiv \frac{\mathrm{Tr}(-3-b\pi)}{3} - \frac{\mathrm{Tr}((-3-b\pi)^2)}{6} \equiv -2-b-\frac{b^2}{2} \mod 3.$$

Thus, $\left(\frac{\zeta,t}{\pi}\right) = \left(\frac{\zeta,-t}{\pi}\right) = \zeta^{-2-b-b^2/2}$. It follows that

$$\left(\frac{\zeta, t(t-1)}{\pi}\right) = \left(\frac{\zeta, t}{\pi}\right)\left(\frac{\zeta, t-1}{\pi}\right) = \zeta^{-2-b^2} = \zeta^{1-b^2},$$

as desired. □

Theorem 8.4. *We have*

$$_3\,\mathrm{Br}(\mathcal{M}) = {}_3\,\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/2]}) = 0.$$

$$\begin{tikzcd}
{}_2\mathrm{Br}'(S)\oplus \mathrm{Pic}(S)[2]\oplus G \arrow[drr] & & & \\
\mathrm{Pic}(S)_{(2)}\oplus\mu_2(S) \arrow[drr] & \mathrm{Pic}(S)[2]\oplus\mu_2(S) \arrow[drr] & \mathrm{Pic}(S)/2\oplus\mu_2(S) & \\
\mathbb{G}_m(S)_{(2)} & \mu_2(S) & \mathbb{G}_m(S)/2 & \mu_2(S)
\end{tikzcd}$$

Figure 3. The E_2-page of the Leray–Serre spectral sequence computing $\mathrm{H}^i(\mathcal{M}_S, \mathbb{G}_m)_{(2)}$ for $i \leq 2$.

Proof. By Proposition 7.7 and Proposition 2.17, it suffices to find two Legendre curves E_1 and E_2 over $\mathbb{Z}_3[\zeta_3]$ with corresponding classes $[(\chi, t_1(t_1-1))] \neq 0$ and $[(\chi, t_2(t_2-1))] = 0$. The associated condition on the Hilbert symbols is $\left(\frac{\zeta, t_1(t_1-1)}{\pi}\right) \neq 1$ and $\left(\frac{\zeta, t_2(t_2-1)}{\pi}\right) = 1$. (Recall here that χ is the character associated with adjoining $\zeta_9 + \bar{\zeta}_9$ which over $\mathbb{Q}_3(\zeta_3)$ is isomorphic to $\mathbb{Q}_3(\zeta_9)$ by Lemma 8.1.) Take $t_i = 2 + b_i\pi$, where $b_1 = 0$ and $b_2 = 1$. Consider the two elliptic curves

$$E_1 : y^2 = x(x-1)(x-2) \quad \text{and} \quad E_2 : y^2 = x(x-1)(x-(2+\pi)).$$

The previous lemma says that we have

$$\left(\frac{\zeta, t_1(t_1-1)}{\pi}\right) = \left(\frac{\zeta, 2(2-1)}{\pi}\right) = \zeta \neq 1 \quad \text{and} \quad \left(\frac{\zeta, t_2(t_2-1)}{\pi}\right) = \left(\frac{\zeta, (2+\pi)(1+\pi)}{\pi}\right) = \zeta^0 = 1.$$

This completes the proof. □

9. The 2-primary torsion in $\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$

Throughout this section, let S denote a connected regular noetherian scheme over $\operatorname{Spec}\mathbb{Z}\left[\frac{1}{2}\right]$. Given a stack X over S, let $\overline{\mathrm{Br}}'(X) = \operatorname{coker}(\mathrm{Br}'(S) \to \mathrm{Br}'(X))$.

Theorem 9.1. *Let S be a regular noetherian scheme over $\mathbb{Z}\left[\frac{1}{2}\right]$ with* $\mathrm{Pic}(S) = 0$. *There is a natural exact sequence*

$$0 \to \mathbb{G}_m(S)/2 \to {}_2\overline{\mathrm{Br}}'(\mathcal{M}_S) \to G \to 0,$$

where $G \subset \mathbb{G}_m(S)/2$ is the subgroup of all those u with $[(-1, u)] = 0 \in \mathrm{Br}(S)$.

We will prove the theorem after several preliminaries. Figure 3 shows a small part of the 2-local Leray–Serre spectral sequence (5.1) for the S_3-Galois cover $\mathcal{M}(2)_S \to \mathcal{M}_S$. The description follows from Lemmas 5.2, 5.3, 5.7 and Proposition 5.14.

From now on, we will localize everything in this section implicitly at 2.

Proposition 9.2. *Let S be a connected regular noetherian scheme over $\mathbb{Z}\left[\frac{1}{2}\right]$. The differential $d_2^{0,1}$ in the Leray–Serre spectral sequence of Figure 3 always vanishes and $d_2^{0,2}$ and $d_3^{0,2}$ vanish if* $\mathrm{Pic}(S) = 0$.

Proof. The map

$$\mathrm{Pic}(\mathcal{M}_S) \to \mathrm{E}_2^{0,1} \cong \mathrm{Pic}(S)_{(2)} \oplus \mu_2(S)$$

is surjective as $-1 \in \mu_2(S)$ can be realized as $\lambda^{\otimes 6}$. This implies that there can be no differential originating from $\mathrm{E}_2^{0,1}$. Moreover, ${}_2\operatorname{Br}'(S)$ splits off from $\operatorname{Br}'(\mathcal{M}_S)$, so the differentials $d_2^{0,2}$ and $d_3^{0,3}$ vanish on ${}_2\operatorname{Br}'(S)$.

Now assume $\operatorname{Pic}(S) = 0$. Then also $\operatorname{Pic}(S)[2] = 0$ and $\operatorname{Pic}(S)/2 = 0$. So, we are concerned with the vanishing of $d_2^{0,2} : G \to \mu_2(S)$ and $d_3^{0,2} : G \to \mu_2(S)$. However, by pulling back the spectral sequence to a geometric point $\bar{x}$ of S, we find $G = \mathbb{G}_m(\bar{x})/2 = 0$, while $\mu_2(\bar{x}) \cong \mathbb{Z}/2$. This implies that $d_2^{0,2}$ and $d_3^{0,2}$ vanish. □

To resolve the differential $d_2^{1,1}$ and solve possible extension issues we will first consider schemes S over $\mathbb{Z}\left[\frac{1}{2}, i\right]$. In this case we can compare the Leray–Serre spectral sequence considered above with the Leray–Serre spectral sequence for the C_2-Galois cover $\mathrm{B}C_{2,S} \to \mathrm{B}C_{4,S}$.

Proposition 9.3. *If S is a regular noetherian $\mathbb{Z}\left[\frac{1}{2}, i\right]$-scheme, then* $\operatorname{Br}'(\mathcal{M}_S) \cong \operatorname{Br}'(\mathrm{B}C_{4,S})$.

Proof. Consider the elliptic curve $E : y^2 = x(x-1)(x+1)$ over $\mathbb{Z}\left[\frac{1}{2}, i\right]$ with discriminant 64. It has an automorphism η of order 4 given by $y \mapsto iy$ and $x \mapsto -x$, which defines a map $\mathrm{B}C_{4,\mathbb{Z}[1/2,i]} \to \mathcal{M}_{\mathbb{Z}[1/2]}$. The 2-torsion points of E are $(0,0)$, $(1,0)$ and $(-1,0)$; taking them in this order defines a full level 2 structure. We can base change this elliptic curve together with its level structure to an arbitrary $\mathbb{Z}\left[\frac{1}{2}, i\right]$-scheme S. This results in pullback squares

$$\begin{tikzcd} \coprod_{S_3} S \arrow[r] \arrow[d] & \coprod_{S_3/C_2} \mathrm{B}C_{2,S} \arrow[r] \arrow[d] & \mathcal{M}(2)_S \arrow[d] \\ S \arrow[r] & \mathrm{B}C_{4,S} \arrow[r] & \mathcal{M}_S. \end{tikzcd} \tag{9.4}$$

Here we use that η acts on the scheme $\coprod_{S_3} S$ of level structures on E_S by multiplication with the cycle $(2\,3) \in S_3$ and in particular η^2 acts trivially. Thus, the stack quotient $\left(\coprod_{S_3} S\right)/C_4$ is equivalent to $\coprod_{S_3/C_2} \mathrm{B}C_{2,S}$. More precisely, the S_3-Galois cover $\coprod_{S_3/C_2} \mathrm{B}C_{2,S} \to \mathrm{B}C_{4,S}$ is induced along an inclusion $C_2 \to S_3$ from the C_2-Galois cover $\mathrm{B}C_{2,S} \to \mathrm{B}C_{4,S}$. The right square is indeed cartesian as can be checked after base change along the étale cover $S \to \mathrm{B}C_{4,S}$.

In the Leray–Serre spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p\left(S_3, \mathrm{H}^q\left(\coprod_{S_3/C_2} \mathrm{B}C_{2,S}, \mathbb{G}_m\right)\right) \Rightarrow \mathrm{H}^{p+q}(\mathrm{B}C_{4,S}, \mathbb{G}_m),$$

the S_3-modules $\mathrm{H}^q\left(\coprod_{S_3/C_2} \mathrm{B}C_{2,S}\right)$ are all induced up from C_2. Thus, the spectral sequence is isomorphic to the Leray–Serre spectral sequence for the C_2-Galois cover $\mathrm{B}C_{2,S} \to \mathrm{B}C_{4,S}$.

The Leray–Serre spectral sequence computing $\mathrm{H}^{p+q}(\mathrm{B}C_{4,S}, \mathbb{G}_m)$ from the $\mathbb{G}_m$-cohomology of $\mathrm{B}C_{2,S}$ is displayed in Figure 4. The computation follows from Proposition 3.2 together with the fact that C_2 acts trivially on the cohomology of $\mathrm{B}C_{2,S}$ (as indeed the morphism $t\colon \mathrm{B}C_{2,S} \to \mathrm{B}C_{2,S}$ for $t \in C_2$ the generator is the identity; only the natural transformation $\mathrm{id} \to t^2$ is not the identity).

By the considerations above, the pullback square (9.4) induces a map

$$\mathrm{H}^p(S_3, \mathrm{H}^q(\mathcal{M}(2)_S, \mathbb{G}_m)) \to \mathrm{H}^p(C_2, \mathrm{H}^q(\mathrm{B}C_{2,S}, \mathbb{G}_m)).$$

$$\begin{tikzcd}
\mathrm{Br}'(S) \oplus \mathrm{Pic}(S)[2] \oplus \mathbb{G}_m(S)/2 \arrow[rrd] & & & \\
\mathrm{Pic}(S) \oplus \mu_2(S) & \mathrm{Pic}(S)[2] \oplus \mu_2(S) \arrow[rrd] & \mathrm{Pic}(S)/2 \oplus \mu_2(S) & \\
\mathbb{G}_m(S) & \mu_2(S) & \mathbb{G}_m(S)/2 & \mu_2(S)
\end{tikzcd}$$

Figure 4. Part of the Leray–Serre spectral sequence for $\mathrm{B}C_{2,S} \to \mathrm{B}C_{4,S}$.

Note first that $G = \mathbb{G}_m(S)/2$ in our case as -1 is a square. If we identify $\mathcal{M}(2)_S$ with $\mathrm{B}C_{2,X_S}$ this map on cohomology groups is induced by the maps $S \to X_S$ (classifying the Legendre curve E_S) and $C_2 \to S_3$. This induces an isomorphism of spectral sequences for $p+q \le 3$ and $q \le 1$ for $p+q = 3$ by Figure 3. □

Corollary 9.5. *Let S be a regular noetherian scheme over $\mathbb{Z}\left[\frac{1}{2}\right]$. The restriction of the differential $d_2^{1,1}$ to $\mu_2(S)$ in the Leray–Serre spectral sequence for $\mathcal{M}(2)_S \to \mathcal{M}_S$ defines an isomorphism $\mu_2(S) \xrightarrow{\cong} \mu_2(S)$, while $d_3^{0,2} = 0$.*

Proof. Consider $S' = \operatorname{Spec} \mathbb{Z}\left[\frac{1}{2}, i\right]$. By Proposition 3.2, we see that

$$\mathrm{Br}'(\mathrm{B}C_{4,S'}) \cong \mathrm{Br}'(S') \oplus \mathrm{Pic}(S')[4] \oplus \mathbb{G}_m(S')/4 \cong \mathbb{G}_m(S')/4,$$

since the Brauer and Picard groups of $\mathbb{Z}\left[\frac{1}{2}, i\right]$ are zero. Hence, $\mathrm{Br}'(\mathrm{B}C_{4,S'}) \cong \mathbb{Z}/4 \oplus \mathbb{Z}/4$, with generators given as i and $1+i$. In Figure 4, we see that the only way to have a group of order 16 in the abutment $\mathrm{Br}'(\mathrm{B}C_{4,S'})$ is that $d_2^{1,1} : \mu_2(S') \to \mu_2(S')$ is an isomorphism, both in the Leray–Serre spectral sequence for $\mathrm{H}^*(\mathrm{B}C_{4,S'}, \mathbb{G}_m)$ and for $\mathrm{H}^*(\mathcal{M}_{S'}, \mathbb{G}_m)$. It follows that this differential is already an isomorphism in the Leray–Serre spectral sequence for $\mathrm{H}^*(\mathcal{M}_{\mathbb{Z}[\,unfrac12]}, \mathbb{G}_m)$ by naturality. This in turn implies by naturality that $d_2^{1,1}|_{\mu_2(S)} \colon \mu_2(S) \to \mu_2(S)$ in the Leray–Serre spectral sequence for $\mathrm{H}^*(\mathcal{M}_S, \mathbb{G}_m)$ is an isomorphism for any regular noetherian $\mathbb{Z}\left[\frac{1}{2}\right]$-scheme S. As the target of $d_3^{0,2}$ is already zero on E_3, the differential $d_3^{0,2}$ must vanish. □

Finally, we prove the theorem from the beginning of the section.

Proof of Theorem 9.1. The claim follows from the determination of the differentials in the range pictured in Figure 3. □

We want to be more specific about the Brauer group classes coming from $\mathbb{G}_m(S)/2$. Recall the section $\Delta \in \mathrm{H}^0(\mathcal{M}, \lambda^{\otimes 12})$ from Section 7. As in Construction 2.8, we can define the C_2-torsor $\mathcal{M}(\sqrt{\Delta}) = \mathbf{Spec}_{\mathcal{M}_{\mathbb{Z}[1/2]}}\left(\bigoplus_{i \in \mathbb{Z}} \lambda^{\otimes 6i}/(\Delta - 1)\right) \to \mathcal{M}_{\mathbb{Z}[1/2]}$ that adjoins a square root of Δ to $\mathcal{M}_{\mathbb{Z}[1/2]}$. For a unit $u \in \mathbb{G}_m\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)$, we denote by (Δ, u) the symbol (quaternion) algebra associated with this torsor.

Proposition 9.6. *Let S denote a connected regular noetherian scheme over* $\operatorname{Spec} \mathbb{Z}\left[\frac{1}{2}\right]$. *Then the map*

$$\mathbb{G}_m(S)/2 \to \mathrm{Br}'(\mathcal{M}_S)$$

from the Leray–Serre spectral sequence sends u to $[(u, \Delta)_2]$.

Proof. We consider the Leray–Serre spectral sequence

$$\mathrm{H}^p(C_3, \mathrm{H}^q(\mathcal{M}(2)_S, \mathbb{G}_m))_{(2)} \Rightarrow \mathrm{H}^{p+q}(\mathcal{M}(2)_S/C_3, \mathbb{G}_m)_{(2)},$$

where $\mathcal{M}(2)/C_3$ denotes the stack quotient by the subgroup $C_3 \subset S_3$. Its E_2-term is clearly concentrated in the column $p = 0$. From Lemma 5.7 and Proposition 5.14, it is easy to see that $\mathrm{H}^0(C_3, \mathrm{H}^q(\mathcal{M}(2), \mathbb{G}_m)_{(2)} \cong \mathrm{H}^0(S_3, \mathrm{H}^q(\mathcal{M}(2), \mathbb{G}_m)_{(2)})$. We can now consider the further Leray–Serre spectral sequence

$$\mathrm{H}^p(C_2, \mathrm{H}^q(\mathcal{M}(2)_S/C_3, \mathbb{G}_m))_{(2)} \Rightarrow \mathrm{H}^{p+q}(\mathcal{M}_S, \mathbb{G}_m)_{(2)},$$

and we see that it has the same E_2-term as the Leray–Serre spectral sequence for the S_3-cover $\mathcal{M}(2)_S \to \mathcal{M}_S$ in the range depicted in Figure 3.

We claim that the C_2-torsor $\mathcal{M}(2)_S/C_3 \to \mathcal{M}_S$ agrees with $\mathcal{M}_S(\sqrt{\Delta}) \to \mathcal{M}_S$. For this it suffices to show that Δ becomes a square on $\mathcal{M}(2)_S/C_3$. With $p, q \in \mathrm{H}^0(\mathcal{M}(2), \lambda^{\otimes 2})$ as in the discussion after Corollary 4.6, we have $\Delta = 16p^2q^2(p-q)^2$. The C_3-action permutes p, $(-q)$ and $(q-p)$ cyclically so that $4pq(p-q)$ is a C_3-invariant section of $\lambda^{\otimes 6}$ whose square is indeed Δ.

Now the statement follows from Corollary 3.6. □

The following is one of our main results. We recall the convention that everything is implicitly 2-local so that $\mathrm{Br}(\mathcal{M}_R)$ for a ring R denotes really $\mathrm{Br}(\mathcal{M}_R)_{(2)}$.

Proposition 9.7. *Let P be a set of prime numbers including 2 and denote by $\mathbb{Z}_P \subset \mathbb{Q}$ the subset of all fractions where the denominator is only divisible by primes in P. Then*

$$\mathrm{Br}(\mathcal{M}_{\mathbb{Z}_P}) \cong \mathrm{Br}(\mathbb{Z}_P) \oplus \bigoplus_{\substack{p \in P \cup \{-1\}, \\ p \equiv 3 \bmod 4}} \mathbb{Z}/2 \oplus \bigoplus_{\substack{p \in P, \\ p \not\equiv 3 \bmod 4}} \mathbb{Z}/4.$$

Proof. First we have to compute the subgroup

$$G \subset \mathbb{G}_m(\mathbb{Z}_P)/2 \cong \bigoplus_{P \cup \{-1\}} \mathbb{F}_2.$$

By Proposition 2.17, a quaternion algebra (a, b) ramifies at p if and only if the Hilbert symbol $\left(\frac{a,b}{p}\right)$ equals -1. By [Neukirch 1999, Theorem V.3.6], $\left(\frac{-1,-1}{p}\right) = -1$ if and only if $p = 2, \infty$ and $\left(\frac{-1,q}{p}\right) = -1$ if and only if $q \equiv 3 \bmod 4$ and $p = 2, q$ (for q a prime number). We see that G has an $\mathbb{F}_2$-basis given by the primes not congruent to 3 mod 4.

We obtain a diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{G}_m(\mathbb{Z}_P)/2 \arrow[r] \arrow[d] & \overline{\mathrm{Br}}(\mathcal{M}_{\mathbb{Z}_P}) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{G}_m(\mathbb{Z}_P[i])/2 \arrow[r] & \overline{\mathrm{Br}}(\mathcal{M}_{\mathbb{Z}_P[i]}) \arrow[r] & \mathbb{G}_m(\mathbb{Z}_P[i])/2 \arrow[r] & 0
\end{tikzcd}$$

By Proposition 9.3, $\overline{\mathrm{Br}}(\mathcal{M}_{\mathbb{Z}[1/2,i]}) \cong \mathbb{G}_m(\mathbb{Z}_P[i])/4$. As the map $G \to \mathbb{G}_m(\mathbb{Z}_P[i])/2$ is injective, we see that none of the nonzero lifts of elements of G to $\overline{\mathrm{Br}}(\mathcal{M}_{\mathbb{Z}_P})$ are 2-torsion. The proposition follows. □

This shows the 2-local part of the computation of $\mathrm{Br}(\mathcal{M}_{\mathbb{Q}})$ and $\mathrm{Br}(\mathcal{M}_{\mathbb{Z}[1/2]})$ in Theorem 1.1, while the 3-local part was already contained in Theorems 7.1 and 8.4 and the p-local part for $p > 3$ in Theorem 6.1.

Remark 9.8. We can describe all the Brauer classes in $\overline{\mathrm{Br}}(\mathcal{M}_{\mathbb{Z}_P})$ explicitly when P is again a set of prime numbers including 2. We already saw in the last two propositions that $\overline{\mathrm{Br}}(\mathcal{M}_{\mathbb{Z}_P})[2]$ has an $\mathbb{F}_2$-basis given by $[(p, \Delta)_2]$, where $p \in P \cup \{-1\}$. When $p \in S$ and either $p = 2$ or $p \equiv 1 \bmod 4$, we will give explicit elements of order 4 in the Brauer group of $\mathcal{M}_{\mathbb{Z}_P}$ generating the $\mathbb{Z}/4$-subgroups of the proposition.

To describe the 4-torsion we start with a small observation. Given a cyclic algebra (χ, υ), where χ is a C_4-torsor and υ a μ_4-torsor, $2[(\chi, \upsilon)]$ is represented by (χ', υ'), where χ' and υ' are obtained from χ and υ via the morphisms $C_4 \to C_2$ and $\mu_4 \to \mu_2$. Concretely, this means that χ' are the C_2-fixed points of χ and that if υ is given by adjoining the 4-th root of a section u of $\mathcal{L}^{\otimes 4}$, then υ' is given by adjoining a square root of u, a section of $(\mathcal{L}^{\otimes 2})^{\otimes 2}$.

For primes $p \equiv 1 \bmod 4$, we construct a C_4-Galois extension L of $\mathbb{Q}$ whose C_2-fixed points are $\mathbb{Q}(\sqrt{p})$. As $\sqrt{p}$ is the Gauss sum $\sum_{a=1}^{p-1} \left(\frac{a}{p}\right)\zeta_p^a$, we see that $\mathbb{Q}(\sqrt{p}) \subset \mathbb{Q}(\zeta_p)$. The Galois group of the $\mathbb{Q}$-extension $\mathbb{Q}(\zeta_p)$ is cyclic of order $p-1$, which is divisible by 4. Thus, it has a unique cyclic subextension L of degree 4 whose C_2-fixed points are $\mathbb{Q}(\sqrt{p})$. Note that L is only ramified at p. Explicitly, L is generated by the Gauss sum $\sum_{a=1}^{p-1} \varphi(a)\zeta_p^a$, where $\varphi\colon (\mathbb{Z}/p)^\times \to \mu_4(\mathbb{C})$ is a surjective character.

For $p = 2$, we take $L = \mathbb{Q}(\zeta_{16} + \bar{\zeta}_{16})$ instead, which is the unique C_4-Galois subextension of $\mathbb{Q}(\zeta_{16})$ over $\mathbb{Q}$. If we denote for $p = 2$ or $p \equiv 1 \bmod 4$ the character of $L/\mathbb{Q}$ by χ, these define C_4-Galois covers of $\mathcal{M}_S$ by pullback and we abuse notation and write χ also for these covers. The cup product $[(\chi, \Delta)_4]$ in $\mathrm{Br}'(\mathcal{M}_{\mathbb{Z}_P})$ is a class such that $2[(\chi, \Delta)_4] = [(p, \Delta)_2]$ and thus has exact order 4. It follows that the classes $[(\chi, \Delta)_4]$ give a basis of the 4-torsion of $\overline{\mathrm{Br}}(\mathcal{M}_{\mathbb{Z}_P})$.

10. The Brauer group of $\mathcal{M}$

In this section, we will complete the computation of $\mathrm{Br}(\mathcal{M})$. In the last section, we saw that

$$\mathrm{Br}\big(\mathcal{M}[\tfrac{1}{2}]\big) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$$

with generators $\alpha = [(-1, -1)_2]$, $\beta = [(-1, \Delta)_2]$ and $\frac{1}{2}\gamma$ for $\gamma = [(2, \Delta)_2]$. We will study the ramification of these classes for certain elliptic curves and the reader can find more information about these curves in *The L-functions and modular forms database* [LMFDB 2013].

As above, we will write $\left(\frac{a,b}{2}\right) \in \mu_2(\mathbb{Q}_2)$ for the Hilbert symbol in $\mathbb{Q}_2$ at the prime 2. Hence, $\left(\frac{a,b}{2}\right) = \pm 1$. Recall from Proposition 2.17 that if $\chi \in \mathrm{H}^1(\mathbb{Q}_2, C_2) \cong \mathrm{H}^1(\mathbb{Q}_2, \mu_2)$ corresponds to a unit $v \in \mathbb{G}_m(\mathbb{Q}_2)/2$, then the degree 2 cyclic algebra (χ, u) has class $\left(\frac{u,v}{2}\right)$ in $\mathrm{Br}(\mathbb{Q}_2)[2] \cong \mathbb{Z}/2 \cong \mu_2(\mathbb{Q}_2)$. Since $\mathrm{Br}(\mathbb{Z}_2) = 0$, the Hilbert symbol measures the ramification along (2) in $\operatorname{Spec} \mathbb{Z}_2$.

Proposition 10.1. *Every nonzero linear combination of α, β, $\frac{1}{2}\gamma$ is ramified along* (2), *so these linear combinations are not in the image of* $\mathrm{Br}(\mathcal{M}) \to \mathrm{Br}\big(\mathcal{M}[\tfrac{1}{2}]\big)$.

Proof. It suffices to prove this for all seven nonzero linear combinations of α, β, γ. Indeed, if all these linear combinations are ramified, then any linear combination $r\alpha + s\beta + \frac{1}{2}\gamma$ is ramified as well. As explained in the introduction around the diagram (1.4), it suffices to construct for each nonzero linear combination ρ an elliptic curve $\operatorname{Spec}\mathbb{Z}_2 \to \mathcal{M}$ such that the pullback of ρ to $\operatorname{Spec}\mathbb{Q}_2$ is nonzero in $\operatorname{Br}(\mathbb{Q}_2)$. Indeed, $\operatorname{Br}(\mathbb{Z}_2) = 0$.

Let

$$E_1 : y^2 + y = x^3 - x^2,$$

the elliptic curve with Cremona label `11a3`. This curve has discriminant -11, which is a unit, so we get an elliptic curve over $\mathbb{Z}_2$, and two associated Brauer classes, $(2, -11)$ and $(-1, -11)$ over $\mathbb{Q}_2$. We can ask what the ramification is. The Hilbert symbol in this case is computed as follows [Serre 1973, Chapter III]. Given $a = 2^\alpha u$ and $b = 2^\beta v \in \mathbb{G}_m(\mathbb{Q}_2)$, where $u, v \in \mathbb{G}_m(\mathbb{Z}_2)$, we have

$$\left(\frac{a, b}{2}\right) = (-1)^{\epsilon(u)\epsilon(v)+\alpha\omega(v)+\beta\omega(u)},$$

where $\epsilon(u) \equiv (u-1)/2$ and $\omega(u) \equiv (u^2-1)/8$.

Hence,

$$\left(\frac{2, -11}{2}\right) = (-1)^{\omega(-11)} = (-1)^{15} = -1.$$

Hence, $(\Delta, 2)$ is ramified at 2. Similarly,

$$\left(\frac{-1, -11}{2}\right) = (-1)^{\epsilon(-1)\epsilon(-11)} = (-1)^{6} = 1.$$

The curve

$$E_2 : y^2 + xy = x^3 - 2x^2 + x$$

has Cremona label `15a8` and discriminant -15. This time, the Hilbert symbols are

$$\left(\frac{2, -15}{2}\right) = 1 \quad \text{and} \quad \left(\frac{-1, -15}{2}\right) = 1.$$

The curve

$$E_3 : y^2 + xy + y = x^3 - x^2$$

has Cremona label `53a1` and discriminant -53. In this case, the Hilbert symbols are

$$\left(\frac{2, -53}{2}\right) = -1 \quad \text{and} \quad \left(\frac{-1, -53}{2}\right) = -1.$$

Now, let $\rho = \alpha + k\beta + m\gamma$ with k and m integers. The elliptic curve E_2 gives a point $x : \operatorname{Spec}\mathbb{Q}_2 \to \mathcal{M}$ where $v_2(x^*\rho) = -1$. It follows that all classes of the form ρ are ramified along (2). Now, E_3 proves that β and γ are ramified along (2). Finally, E_1 proves that $\beta + \gamma$ is ramified along (2). $\square$

Theorem 10.2. $\operatorname{Br}(\mathcal{M}) = 0$.

Proof. This follows from Corollary 6.4, Theorem 8.4, and Propositions 9.7 and 10.1. $\square$

11. The Brauer group of $\mathcal{M}$ over $\mathbb{F}_q$ with q odd

As another application of our methods, we compute the Brauer group of $\mathcal{M}_{\mathbb{F}_q}$ when $q = p^n$ and p is an odd prime. There is remarkable regularity in this case, which is possibly surprising based on what happens for number fields.

Theorem 11.1. *Let $q = p^n$ where p is an odd prime. Then,* $\mathrm{Br}(\mathcal{M}_{\mathbb{F}_q}) \cong \mathbb{Z}/12$.

Proof. Recall that $\mathrm{Br}(\mathbb{F}_q) = 0$. Thus, by Theorem 9.1, there is an extension $0 \to \mathbb{F}_q^\times/2 \to {}_2\mathrm{Br}(\mathcal{M}_{\mathbb{F}_q}) \to \mathbb{F}_q^\times/2 \to 0$. Since q is odd, $\mathbb{F}_q^\times/2 \cong \mathbb{Z}/2$. The remainder of Section 9, especially Remark 9.8, implies that the extension is nonsplit so that in fact ${}_2\mathrm{Br}(\mathcal{M}_{\mathbb{F}_q}) \cong \mathbb{Z}/4$.

Now, let $\ell \geq 3$ be a prime, which we *do not* assume is different from p. The possible terms contributing to ${}_\ell\mathrm{Br}(\mathcal{M}_{\mathbb{F}_q})$ in the Leray–Serre spectral sequence for $\mathcal{M}(2)_{\mathbb{F}_q} \to \mathcal{M}_{\mathbb{F}_q}$ in $\mathbb{G}_m$ cohomology, besides ${}_\ell\mathrm{Br}(X_{\mathbb{F}_q})^{S_3}$, are $\mathrm{H}^1(S_3, {}_\ell\mathrm{Pic}(\mathcal{M}(2)_{\mathbb{F}_q})$ and $\mathrm{H}^2(S_3, {}_\ell\mathbb{G}_m(\mathcal{M}(2)_{\mathbb{F}_q})) \cong \mathrm{H}^2(S_3, {}_\ell\mathbb{G}_m(X_{\mathbb{F}_q}))$. The first of these is zero since ${}_\ell\mathrm{Pic}(\mathcal{M}(2)_{\mathbb{F}_q}) \cong {}_\ell\mathrm{Pic}(X_{\mathbb{F}_q}) = 0$ for ℓ odd. The second has no odd primary torsion. This follows from the exact sequence $0 \to \mathbb{G}_m(\mathbb{F}_q) \to \mathbb{G}_m(X_{\mathbb{F}_q}) \to \tilde{\rho} \otimes \mathbb{Z} \to 0$ together with Lemmas 5.2 and 5.3.

Thus, we see that ${}_\ell(\mathrm{Br}(\mathcal{M}_{\mathbb{F}_q})) \cong {}_\ell\mathrm{Br}(X_{\mathbb{F}_q})^{S_3}$ for ℓ odd. So, it suffices to compute the Brauer group of $X_{\mathbb{F}_q}$ as an S_3-module. In general, our argument in the rest of the paper relies fundamentally on Lemma 5.11, which requires ℓ to be invertible to analyze the ramification map as in Proposition 2.14. However, in this case, $X_{\mathbb{F}_q}$ is a curve over a finite field, so the ramification theory simplifies drastically. Consider the commutative diagram of exact ramification sequences due to [Grothendieck 1968c, Proposition 2.1]:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Br}(\mathbb{P}^1_{\mathbb{F}_q}) \arrow[r] \arrow[d] & \mathrm{Br}(\eta) \arrow[r] \arrow[d, "="] & \bigoplus_{x \in (\mathbb{P}^1_{\mathbb{F}_q})^{(1)}} \mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Br}(X_{\mathbb{F}_q}) \arrow[r] & \mathrm{Br}(\eta) \arrow[r] & \bigoplus_{x \in (X_{\mathbb{F}_q})^{(1)}} \mathbb{Q}/\mathbb{Z} & &
\end{tikzcd}$$

where η is the generic point of $X_{\mathbb{F}_q}$. Note that the exactness at the right in the top sequence is due to the fact (see [Gille and Szamuely 2006, Corollary 6.5.4]) that $\bigoplus_{x \in (\mathbb{P}^1_{\mathbb{F}_q})^{(1)}} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is given by summation in $\mathbb{Q}/\mathbb{Z}$. Since $\mathrm{Br}(\mathbb{P}^1_{\mathbb{F}_q}) = 0$ by [Grothendieck 1968c, Remarques 2.5.b], we see that $\mathrm{Br}(X_{\mathbb{F}_q}) \subseteq \mathrm{Br}(\eta)$ is the subgroup consisting of classes ramified only at $0, 1, \infty$. Using the top row of the diagram, it follows that $\mathrm{Br}(X_{\mathbb{F}_q})$ fits into an exact sequence

$$0 \to \mathrm{Br}(X_{\mathbb{F}_q}) \to \bigoplus_{0,1,\infty} \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

from which it follows that

$$\mathrm{Br}(X_{\mathbb{F}_q}) \cong \tilde{\rho} \otimes \mathbb{Q}/\mathbb{Z}.$$

By Lemma 5.3, we find that ${}_\ell\mathrm{Br}(X_{\mathbb{F}_q})^{S_3} = 0$ if $\ell \geq 5$ and ${}_3\mathrm{Br}(X_{\mathbb{F}_q})^{S_3} \cong \mathbb{Z}/3$. This proves the theorem. $\square$

Remark 11.2. We can again be more specific about the Azumaya algebras representing the classes in $\mathrm{Br}(\mathcal{M}_{\mathbb{F}_q})$ with q odd. Let T be a $\mathbb{F}_q$-scheme (or stack) with an elliptic curve E of discriminant Δ. Let χ_m be the pullback of a character of the Galois extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ to T. We claim that there is a generator $a \in \mathrm{Br}(\mathcal{M}_{\mathbb{F}_q})$ such that the pullback of a to $\mathrm{Br}(T)$ agrees with $[(\chi_{12}, \Delta)]$. Informally, $a = [(\chi_{12}, \Delta)]$ in the universal case $T = \mathcal{M}_{\mathbb{F}_q}$, where more precisely we should replace here Δ by the μ_{12}-torsor corresponding to $\Delta \in \Gamma(\lambda^{\otimes 12})$ via Construction 2.8.

First we consider the 3-torsion. The proof of Lemma 7.2 applies here to show that $[(\chi_3, \Delta)]$ is indeed the pullback of a generator of $\mathrm{Br}(\mathcal{M}_{\mathbb{F}_q})[3]$. Moreover, Proposition 9.6 implies that the unique 2-torsion element $6a \in \mathrm{Br}(\mathcal{M}_{\mathbb{F}_q})$ pulls back to $[(\chi_2, \Delta)]$ as $\mathbb{F}_{q^2}$ agrees with $\mathrm{F}_q[\sqrt{x}]$ for an arbitrary nonsquare x in $\mathbb{F}_q$. As in Remark 9.8 we see that $6[(\chi_{12}, \Delta)] = [(\chi_2, \Delta)] \neq 0$ and $4[(\chi_{12}, \Delta)] = [(\chi_3, \Delta)] \neq 0$. Thus, $[(\chi_{12}, \Delta)]$ is indeed a generator of $\mathrm{Br}(\mathcal{M}_{\mathbb{F}_q}) \cong \mathbb{Z}/12$.

Finally, we also treat the easier case of an algebraically closed base.

Proposition 11.3. *Let k be an algebraically closed field of characteristic not* 2. *Then* $\mathrm{Br}(\mathcal{M}_k) = 0$.

Proof. By Theorem 9.1, ${}_2\mathrm{Br}(\mathcal{M}_k) = 0$. As in the last proof we see that ${}_\ell(\mathrm{Br}(\mathcal{M}_k)) \cong {}_\ell\mathrm{Br}(X_k)^{S_3}$ for ℓ an odd prime. By Tsen's theorem, $\mathrm{Br}(\eta)$ vanishes for η the generic point of X_k. By [Grothendieck 1968c, Proposition 2.1] we obtain that $\mathrm{Br}(X_k) = 0$. □

Remark 11.4. In an earlier version of this paper we suggested using the $\mathrm{GL}_2(\mathbb{Z}/3)$-cover $\mathcal{M}(3) \to \mathcal{M}_{\mathbb{Z}[1/3]}$ to determine $\mathrm{Br}(\mathcal{M}_k)$ also for algebraically closed fields k of characteristic 2. Combining this approach with a new idea, Minseon Shin [2019] has proved in the meanwhile that $\mathrm{Br}(\mathcal{M}_k) \cong \mathbb{Z}/2$ for such fields k.

Acknowledgments

We thank the Hausdorff Research Institute for Mathematics, UIC, and Universität Bonn for hosting one or the other author in 2015 and 2016 while this paper was being written. Furthermore, we thank Asher Auel, Akhil Mathew, and Vesna Stojanoska for their ideas at the beginning of this project and Peter Scholze and Yichao Tian for the suggestion to use fppf-cohomology and the idea for the splitting in Proposition 2.9. Finally, we thank the anonymous referees for their helpful suggestions.

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Communicated by Martin Olsson
Received 2017-11-10 Revised 2020-03-26 Accepted 2020-05-04

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ANT peer review and production are managed by EditFlow® from MSP.

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Algebra & Number Theory

Volume 14 No. 9 2020