Journal of Complex Networks (2020) **00**, 1–12
doi: 10.1093/comnet/cnaa045

Networks with degree–degree correlations are special cases of the edge-coloured random graph

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Edited by: Ernesto Estrada

[Received on 18 August 2020; editorial decision on 9 November 2020; accepted on 10 November 2020]

In complex networks, the degrees of adjacent nodes may often appear dependent—which presents a modelling challenge. We present a working framework for studying networks with an arbitrary joint distribution for the degrees of adjacent nodes by showing that such networks are a special case of edge-coloured random graphs. We use this mapping to study bond percolation in networks with assortative mixing and show that, unlike in networks with independent degrees, the sizes of connected components may feature unexpected sensitivity to perturbations in the degree distribution. The results also indicate that degree–degree dependencies may feature a vanishing percolation threshold even when the second moment of the degree distribution is finite. These results may be used to design artificial networks that efficiently withstand link failures and indicate the possibility of super spreading in networks without clearly distinct hubs.

Keywords: degree correlated networks; coloured random graph; percolation.

1. Introduction

The random graph with an arbitrary degree distribution is a widely discussed network model in which the links between the nodes obey the maximum entropy principle and the degree distribution is fixed as the only input parameter [1]. Such a model is often used as a null-model or ‘baseline’ allowing to detect deviations from the maximum entropy principle in empirical networks, and thus the presence of useful information [1].

One trivial consequence of the maximum entropy principle is that the degrees of the nodes in such models are independent random variables, whereas in many empirical networks this is not believed to be the case, and significant mutual dependencies between the degrees of adjacent nodes are frequently

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reported [2, 3]. A paradigmatic example of positive degree–degree correlation is a network of social contacts where hubs, that is, nodes with many neighbours, connect to other hubs more frequently than a random choice would dictate [4]. Conversely, an optimal packing of hard spheres in a finite volume, for example, is expected to feature negative degree–degree correlation in the corresponding network of the *ad hoc* contacts [5, 6]. Networks with degree–degree dependence are discussed in the context World Wide Web [7], co-authorship networks [8], neural networks [9] and dynamical processes on networks [10]. Other features that are not usually attributed to the classical infinite configuration model but are observed in empirical networks include clustering [11], cliques [12], small cycles [13], multiple edges [14], hidden embedding space [15], modular structure [16] and edges of multiple types or colours. The latter case has been recently addressed in the random graph model with coloured edges [17]. In this model, the degree of a node is not a scalar quantity but a vector counting number of edges of each colour and therefore the model is defined by a multivariate degree distribution. It turns out that having the freedom to choose such a multivariate distribution may allow one to impose structural constrains on the network by solely manipulating the degree distribution. For example, specifically chosen distribution may constrain a network to be modular [17] or to have a predefined volume growth trend [18]. Even in the case of a simple directed random graph, which can be regarded as a random graph with two types of half edges, manipulating the bivariate degree distribution counting in- and out-edges allows to manipulate the location of the percolation transition [19–21]. Interdependent percolation in multiplex networks was shown to correspond to edge percolation in branching cell complexes [22].

In this article, we explore the connection between random graphs with arbitrary degree-degree distributions and the random graphs with coloured edges and show that the former is a special case of the latter. In other words, if multiple types of edges are permitted in a random graph, the edges can absorb structural information about the degree–degree dependencies and thus allow us defining a random graph with a given joint degree-degree distribution. We exploit this connection to study the sizes of connected components and the location of the percolation transitions that occur during random removal of edges in degree–degree correlated networks. Such a mapping to edge-coloured graphs employs two-dimensional colour labels.

The rest of the article is organized as follows: first, we establish the mapping between random networks with a joint degree–degree distribution and networks that have edges of different types, which we refer to as the edge colour. Such coloured networks are defined by specifying multivariate degree distributions. We then establish the analytical expression for the critical percolation threshold, size of the giant component and typical sizes of sub-extensive components and illustrate several unexpected qualitative phenomena that emerge during bond percolation as a consequence of degree dependence. Namely, we show that, if strong degree dependence is present, perturbed degree distribution may result in peculiar behaviour of connected components during bond percolation, which is reminiscent to degenerate percolation transitions. Second, we demonstrate that the percolation threshold may be vanishing even if the degree distribution has a finite second moment. The latter indicates a manner to construct networks without clearly distinct hubs that are nevertheless super robust during random failure of links.

2. The mapping

Consider a network model in which at the ends of a uniformly at random chosen edge one finds nodes of degree $j, k > 0$ with probability $e_{j,k} = e_{k,j}$, with $\sum_{j,k=1}^{D} e_{k,j} = 1$, where D is the maximum degree. Therefore, $e_{j,k}$ is a bivariate probability mass function providing the only input information to the model. In all other respects, the network is regarded as random. We will also use two quantities that are directly

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related to $e_{j,k}$:

$$p_k = \frac{\sum_{j=1}^{D} e_{j,k}}{k \sum_{j,k'=1}^{D} k'^{-1} e_{j,k'}} \tag{2.1}$$

is the degree distribution, that is the probability that a randomly chosen node has degree k, and

$$p_{j|k} = \frac{e_{j,k}}{\sum_{j'=1}^{D} e_{j',k}},$$

is the probability that a node at the end of a randomly chosen edge has degree j given that the node on the other side has degree k.

We will now present a mapping between networks with arbitrary $e_{j,k}$ and a coloured random graph. In the coloured random graph, each edge is assigned one of n colours, so that a randomly chosen node bears c_1 edges of colour one, c_2 edges of colour two, and so on. Thus the coloured degree of each node can be described by a vector of colour counts denoted by $\mathbf{c} = (c_1, c_2, ..., c_n)$, and the degree distribution is the probability $p(\mathbf{c})$ that a uniformly at random chosen node has configuration $\mathbf{c}$. The actual degree k of a node with configuration $\mathbf{c}$ is given by the sum of all colour counts:

$$k = |\mathbf{c}| := \sum_{i=1}^{n} c_i.$$

Here again, $p(\mathbf{c})$ provides the only information about the model.

Let D be the maximum degree in the network. We consider an edge colouring in which an edge colour $i \in \{1, \ldots, D^2\}$ encodes the degrees j and k of the incident nodes to this edge, as given by the lexicographic mapping:

$$(j, k) \to i, \tag{2.2}$$

where

$$j = (i-1) \operatorname{div} D + 1,$$
$$k = (i-1) \bmod D + 1.$$

Therefore, we say that a node has c_i edges of colour i if, in accordance with (2.2), it has degree k and it is connected to exactly c_i nodes of degree j. As an alternative notation, we write $c_{j,k} := c_i$, where $i = (j-1)D + k$. By using the above notations we write the probability that a randomly selected node has configuration $\mathbf{c}$ (i.e. c_1 edges of colour 1, c_2 edges of colour 2, and so on up to c_{D^2}) as a multinomial:

$$p(\mathbf{c}) = \mathbb{P}[c_1, \ldots, c_{D^2}] = \begin{cases} 0, & \exists i \notin \Omega(|\mathbf{c}|), \text{ and } c_i > 0, \\ p_k \frac{|\mathbf{c}|!}{\prod_{i=1}^{D^2} c_i!} \prod_{i=1}^{D^2} p_{j|k}^{c_i}, & \forall i \notin \Omega(|\mathbf{c}|), \text{ and } c_i = 0, \end{cases} \tag{2.3}$$

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where

$$\Omega(k) := \{i : i \geq (k-1)D \text{ and } i < kD\} \tag{2.4}$$

is the set of colours that may reside on a node of degree k.

The expectations of c_i are given by:

$$\mathbb{E}[c_{j,k}] = \mathbb{E}[c_i] = \sum_{\mathbf{c}} c_i p(\mathbf{c}) = \sum_{|\mathbf{c}|=k} c_i p_k \frac{k!}{\prod_{i=1}^{D^2} c_i!} \prod_{i=1}^{D^2} p_{j|k}^{c_i} = k p_k p_{j|k}, \tag{2.5}$$

for $j, k = 1, \ldots, D$, and to compute the second moments, $\mathbb{E}[c_{i_1} c_{i_2}] = \mathbb{E}[c_{j_1,k_1} c_{j_2,k_2}]$, we distinguish three cases:

1. If $k_1 \neq k_2$, then $\mathbb{E}[c_{j_1,k_1} c_{j_2,k_2}] = 0$.
2. If $j_1 \neq j_2$ and $k_1 = k_2 = k$, then

$$\mathbb{E}[c_{j_1,k_1} c_{j_2,k_2}] = k p_k \left(k p_{j_1|k} p_{j_2|k} - p_{j_1|k} p_{j_2|k} \right).$$

3. If $i = i_1 = i_2$ and $k_1 = k_2 = k$ then

$$\mathbb{E}\left[c_{j,k}^2\right] = k p_k \left(k p_{j|k}^2 + p_{j|k}(1 - p_{j|k}) \right).$$

Combining the above-stated cases together gives:

$$\mathbb{E}[c_{i_1} c_{i_2}] = \mathbb{E}[c_{j_1,k_1} c_{j_2,k_2}] = \delta_{k_1,k_2} k_1 p_{k_1} p_{j_1|k_1} \left((k_1 - 1) p_{j_2|k_1} + \delta_{j_1,j_2} \right), \; j_1, j_2, k_1, k_2 = 1, \ldots, D \tag{2.6}$$

3. Size of the giant component

Let $\mathbf{P}$ be a permutation matrix with all elements zero, except $P_{i_1,i_2} = 1$ when colour i_1 is identified with colour i_2, that is when $i_1 = (j,k)$ and $i_2 = (k,j)$. By using the multi-index notation this is written as:

$$P_{(j_1,k_1),(j_2,k_2)} = \delta_{j_1,k_2} \delta_{j_2,k_1}.$$

The size of a giant component in a coloured directed network [17] is given by:

$$s = 1 - \mathbb{E}[\mathbf{P}\mathbf{x}^{\mathbf{c}}], \tag{3.1}$$

where $\mathbf{x} = \{x_1, \ldots, x_{D^2}\}^\top, \; x_i \in (0,1]$, is the solution of the system

$$\mathbf{x} = \mathbf{P}\mathbf{F}(\mathbf{x}), \tag{3.2}$$

with

$$F(\mathbf{x})_i = \frac{\mathbb{E}[c_i \mathbf{x}^{\mathbf{c}-\mathbf{e}_i}]}{\mathbb{E}[c_i]}, \; i = 1, \ldots, D^2$$

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and $\mathbf{e}_i$ being standard basis vectors of size D^2. By expanding the expectation values used in the above-introduced equations for our particular choice of the coloured degree distribution, we find that

$$\mathbb{E}[\mathbf{x}^{\mathbf{c}}] = \sum_{k=1}^{D} p_k \left(\sum_{j=1}^{D} p_{j|k} x_{j,k} \right)^k$$

and

$$\mathbb{E}[c_{j,k}\mathbf{s}^{\mathbf{c}-\mathbf{e}_{j,k}}] = \frac{1}{\mathbb{E}[c_{j,k}]}\frac{\partial}{\partial x_{j,k}}\mathbb{E}[\mathbf{x}^{\mathbf{c}}] = \frac{1}{\mathbb{E}[c_{j,k}]}\frac{\partial}{\partial x_{j,k}}\sum_{k_1=1}^{D} p_{k_1}\left(\sum_{j_1=1}^{D} p_{j_1|k_1} x_{j_1,k_1}\right)^{k_1} =$$
$$\frac{1}{\mathbb{E}[c_{j,k}]} k p_{j|k} p_k \left(\sum_{j_1=1}^{D} p_{j_1|k} x_{j_1,k}\right)^{k-1} = \left(\sum_{j_1=1}^{D} p_{j_1|k} x_{j_1,k}\right)^{k-1},$$

which allows us to replace (3.1)–(3.2) with:

$$\begin{aligned} s &= 1 - \sum_{k=0}^{D} p_k \left(\sum_{j=1}^{D} p_{j|k} x_{j,k}\right)^k, \\ x_{k,j} &= \left(\sum_{l=1}^{D} p_{l|k} x_{l,k}\right)^{k-1}, \; j,k = 1,\dots,D. \end{aligned} \tag{3.3}$$

Since the right-hand side of the latter equation does not depend on j, we conclude that:

$$x_{k,1} = x_{k,2} = \cdots = x_{k,D}.$$

Let $y_{j,k}^{k-1} := x_{j,k}$, we may then rewrite (3.3) as

$$s = 1 - \sum_{k=0}^{D} p_k y_k^k, \; y_k = \sum_{j=1}^{D} p_{j|k} y_j^{j-1}, \; y_k \in (0,1]. \tag{3.4}$$

Therefore, we have expressed the size of the giant component s in terms the solution of a system with D non-linear equations. Equation (3.4) was first presented in [4] without derivation.

The expression for the expected size of a sub-extensive connected component in coloured random graphs is given by [17]:

$$w = \frac{\mathbf{xD}(\mathbf{I} - \mathbf{H}(\mathbf{x})\mathbf{P})^{-1}\mathbf{x}}{1-s} + 1, \tag{3.5}$$

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where

$$\mathbf{D} = \mathrm{diag}\{\mathbb{E}[c_1], \mathbb{E}[c_2], \ldots, \mathbb{E}[c_{D^2}]\}$$

and $\mathbf{H}$ has elements:

$$H_{i_1,i_2}(\mathbf{x}) = H_{(j_1,k_1),(j_2,k_2)}(\mathbf{x}) = \frac{\mathbb{E}[(c_{i_1}c_{i_2} - \delta_{i_1,i_2}c_{i_1})\mathbf{x}^{\mathbf{c}-\mathbf{e}_{i_1}-\mathbf{e}_{i_2}}]}{\mathbb{E}[c_i]} =$$
$$\frac{1}{\mathbb{E}[c_{j_1,k_1}]}\frac{\partial^2}{\partial x_{j_1,k_1}\partial x_{j_2,k_2}}\mathbb{E}[\mathbf{x}^{\mathbf{c}}] = \frac{\partial}{\partial x_{j_2,k_2}}\left(\sum_{l=1}^{D} p_{l|k_1}x_{l,k_1}\right)^{k_1-1} =$$
$$(k_1-1)\left(\sum_{l=1}^{D} p_{l|k_1}x_{l,k_1}\right)^{k_1-2}\sum_{l=1}^{D} p_{l|k_1}\delta_{l,j_2}\delta_{k_1,k_2} = \delta_{k_1,k_2}(k_1-1)p_{j_2|k_1}\left(\sum_{l=1}^{D} p_{l|k_1}x_{l,k_1}\right)^{k_1-2}.$$

4. Bond percolation with degree–degree dependence

From the theory for edge-coloured random graphs [17], we know that such networks percolate when

$$\det(\mathbf{PM} - \mathbf{I}) = 0,$$

where

$$M_{i_1,i_2} = \frac{\mathbb{E}[c_{i_1}c_{i_2}]}{\mathbb{E}[c_{i_2}]} - \delta_{i_1,i_2}, \quad i_1, i_2 = 1, \ldots, D^2, \tag{4.1}$$

and, after plugging the moments expressions (2.5)–(2.6) into (4.1), we obtain

$$M_{i_1,i_2} = M_{(j_1,k_1),(j_2,k_2)} = \delta_{k_1,k_2}p_{j_1|k_1}(k_1-1). \tag{4.2}$$

The elements of the product are given by:

$$(\mathbf{PM})_{(j_1,k_1),(j_2,k_2)} = \delta_{j_1,k_2}(k_2-1)p_{k_1|k_2}. \tag{4.3}$$

Even though $\mathbf{PM}$ is a square matrix of size D^2, one can see from the definition (4.2) that this matrix has at most D unique columns and therefore the spectrum of $\mathbf{PM} - \mathbf{I}$ contains eigenvalue $\lambda = 1$ with multiplicity of at least $D^2 - D$. Since the determinant can be written as the product of all eigenvalues, there exists a smaller matrix of size at most D containing all the eigenvalues of $\mathbf{PM} - \mathbf{I}$ apparat of $\lambda = 1$ and therefore having the same determinant. Let $\mathbf{I}$ be $D \times D$ identity matrix, $\mathbf{e} = (1, 1, \ldots, 1)$ vector of length D and $\mathbf{S} = D^{-\frac{1}{2}}\mathbf{I} \otimes \mathbf{e}$, then

$$\mathbf{C} = \mathbf{SPMS}^{\top},$$

with

$$\det(\mathbf{C} - \mathbf{I}) = \det(\mathbf{PM} - \mathbf{I}), \tag{4.4}$$

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where **C** has elements

$$C_{j,k} = (k-1)p_{j|k},$$

and size D which is computationally more favourable than size of **PM** when detecting phase transitions.

Let us now consider a dynamic network in which $\mathbf{C}(\pi)$ is continuously dependent on some parameter π and $\mathbf{C}(0)$ corresponds to no edges, that is $\lim_{\pi\to 0}\mathbf{C}(\pi) = \mathbf{0}$. In that case $\det(\mathbf{C}(0)-\mathbf{I}) = \pm 1$ and for some small ε the determinant does not change its sign when $\pi \in (0,\varepsilon)$, that is $(-1)^{D-1}\det(\mathbf{C}(\pi)-\mathbf{I}) < 0$ for all $\pi < \varepsilon$. Then the percolation threshold is expressed as

$$\pi_c = \inf\{\pi : (-1)^{D-1}\det(\mathbf{C}(\pi)-\mathbf{I}) > 0\}, \tag{4.5}$$

that is the smallest π for which $(-1)^{D-1}(\mathbf{C}-\mathbf{I})$ has positive determinant. This provides a framework for studying resilience of networks with (dis-)assortative mixing that evolve according to a wide class of dynamical processes. One example of such dynamic process is bond percolation.

If $\mathbf{C}(\pi)$ is a non-linear function in π, it is not possible to express the percolation threshold π_c as the largest eigenvalue of **C**, and one must apply the criticality condition (4.5) to infer the value of the critical percolation threshold. As such, our equation (4.5), although it is harder to apply, improves previous results on percolation in degree correlated networks [23, 24], which approximate $\mathbf{C}(\pi)$ to be a linear function in π. Beyond percolation, non-linearity of $\mathbf{C}(\pi)$ is typical for dynamic networks that evolve due to some external forcing, for example, as in [25].

4.1 *Evolution of the joint degree–degree distribution under bond percolation*

Removing edges uniformly at random during bond percolation affects the degree–degree distribution in the following manner,

$$e_{j,k}(\pi) = \sum_{j_1=j,k_1=k}^{D^2} e_{j_1,k_1}\binom{j_1-1}{j-1}\pi^{j-1}(1-\pi)^{j_1-j}\binom{k_1-1}{j-1}\pi^{k-1}(1-\pi)^{k_1-k}. \tag{4.6}$$

The degree distribution $p_k(\pi)$ can be readily expressed from the degree–degree distribution (4.6) by applying (2.1) and additionally taking care of the isolated nodes:

$$p_0 = \sum_{k>0} p_k \pi^{k-1}.$$

The parameter-dependent degree–degree distribution (4.6) gives rise to $\mathbf{C}(\pi)$ which is generally a non-linear function of π, and therefore, one must apply equation (4.5) to detect the percolation threshold. One peculiar property of $e_{i,j}(\pi)$ is that the joint degrees are becoming less dependent on edge removal. This could be seen by studying the correlation coefficient.

4.2 *Decay of the Pearson correlation coefficient during percolation*

The Pearson correlation coefficient r for adjacent degrees depends on π:

$$r(\pi) = \frac{r_1}{1-a(1-\frac{1}{\pi})},$$

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where

$$r_1 = \frac{\mathbb{E}_e[jk] - \mathbb{E}_e^2[k]}{\mathbb{E}_e[k^2] - \mathbb{E}_e^2[k]}$$

is the correlation coefficient at $\pi = 1$ and

$$a = \frac{\mathbb{E}_e[k] - 1}{\mathbb{E}_e[k^2] - \mathbb{E}_e^2[k]},$$

characterizes how fast the correlation decays. Indeed, a simple analysis shows that, as long as $r_1 > 0$, $r(\pi)$ is a strictly increasing function in π, which means that uniform removal of edges will always decrease the correlation between adjacent degrees. Moreover, $r(\pi)$ is convex for $a > 1$, concave for $0 < a < 1$, and a linear function for $a = 1$. Note that the value of a is expressed solely in terms of non-mixed moments, and therefore, it is a property of degree distribution and not the copula that characterises dependency between joint degrees. When $\mathbb{E}[k^2] = \infty$ and $\mathbb{E}[k] < \infty$, the correlation coefficient vanishes $r(\pi) \equiv r_1 = 0$ for $\pi > 0$, however the degree dependency may be still characterized with different measures [2, 26].

5. Discussion and conclusions

In this article, instead of proposing yet another new model for complex networks we draw the attention to the fact that they can be treated as special cases of coloured random graphs, enabling the collection of many of the existing models under one umbrella. By using this framework, we derive several results for networks defined by their join degree–degree distributions, namely: percolation threshold (4.5), the size of the giant component (3.4), and the typical size of the sub-extensive connected components (3.5). In this section, we provide several examples where the link with coloured random graphs reveals unexpected qualitative phenomena in the behaviour of degree–degree correlated networks during random removal of edges.

5.1 *Degenerate percolation transitions*

Consider the degree–degree distribution given by

$$e_{j,k} = (1-t)f(j)f(k) + t\delta_{j,k}f(j), \tag{5.1}$$

where

$$f(k) = (1-\varepsilon)\delta_{k,3} + \varepsilon\delta_{k,9}. \tag{5.2}$$

with $\varepsilon = 10^{-5}$. Clearly, $t > 0$ implies dependency between join degrees, with extreme case $t = 1$ signifying that the network is composed of multiple disjoint regular graphs. The parameter ε is small enough not to induce large changes in the size of the giant component, see Fig. 1b. However, in the correlated case, when t is close to 1, we observe two substantial peaks in the typical sizes of the sub-extensive connected components, while in the uncorrelated case, when t is close to zero, there is only one peak, see Fig. 1a. Note that the degree distribution is not affected by the value of the coupling parameter t and is bimodal in all three cases studied in Fig. 1. For all t, there is only one singularity, of the type $O(\frac{1}{\pi-\pi_c})$, while the second peak is bounded. The example illustrates that in networks with degree

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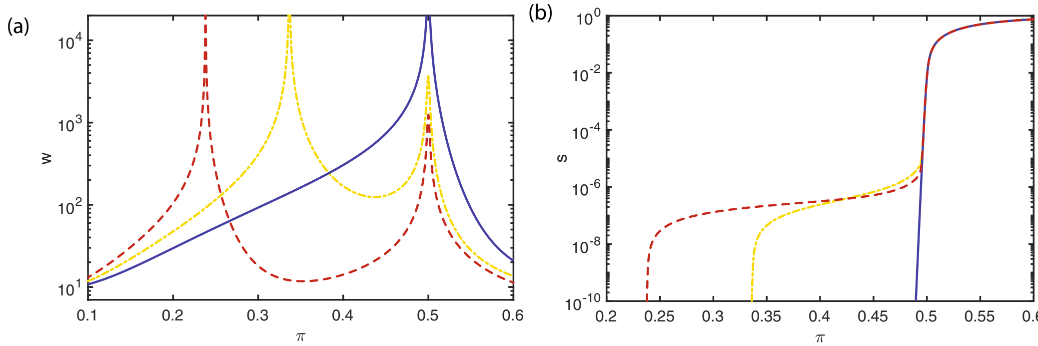

FIG. 1. (a) Typical size w of sub-extensive connected components. (b) The fraction s of nodes in the giant component. The *solid lines* correspond to uncorrelated networks with $t = 0$, *dash-dot lines* to $t = 0.5$, and *dashed lines* to $t = 1$. In this example, $D = 9$.

dependencies, small functional perturbations in the degree distribution may cause large changes in the sizes of connected components, a phenomenon that is not observed when degrees are independent [27]. One may speculate that such behaviour is enabled by the presence of multiple types of edges, as was shown in [28, 29], in this work, however, edge types are defined by the structure of the network itself.

5.2 *Super-robust networks*

Here, we consider a joint degree–degree distribution (5.1) where, as before, parameter t controls degree dependence. We analyse two functional forms of f_k: exponential distribution

$$f_k = C_1 e^{-k}$$

and distribution with a heavy tail

$$f_k = C_2 k^{-(\tau+1)},$$

where constants C_1 and C_2 provide normalization. In both cases, we exclude isolated nodes, $f_0 := 0$, and isolated doublets $e_{1,1} := 0$. Figure 2a shows that in the exponential case, percolation threshold π_c converges to a constant as maximum degree D increases. Here, stronger coupling t corresponds to smaller values of the threshold. This is in contrast with Fig. 2b and c where π_c is calculated for degree distribution with a heavy tail, showing a steady decrease of π_c with increasing maximum degree D. This tendency is maintained across different tail exponents and values of coupling constant t. To date, vanishing percolation threshold has been reported only for networks with diverging second moment of the degree distribution, that is $\tau \leq 2$, a property which is also frequently inherited by the spreading processes on such networks [30]. This example indicates that networks with degree–degree dependencies may feature a vanishing percolation threshold even when the second moment of the degree distribution is finite and therefore the nodes degrees are more homogeneously distributed featuring less pronounced hubs. A direct implication of this observation is that it indicates a way to construct networks with zero percolation threshold that nevertheless do not feature strong degree heterogeneity, and therefore, are

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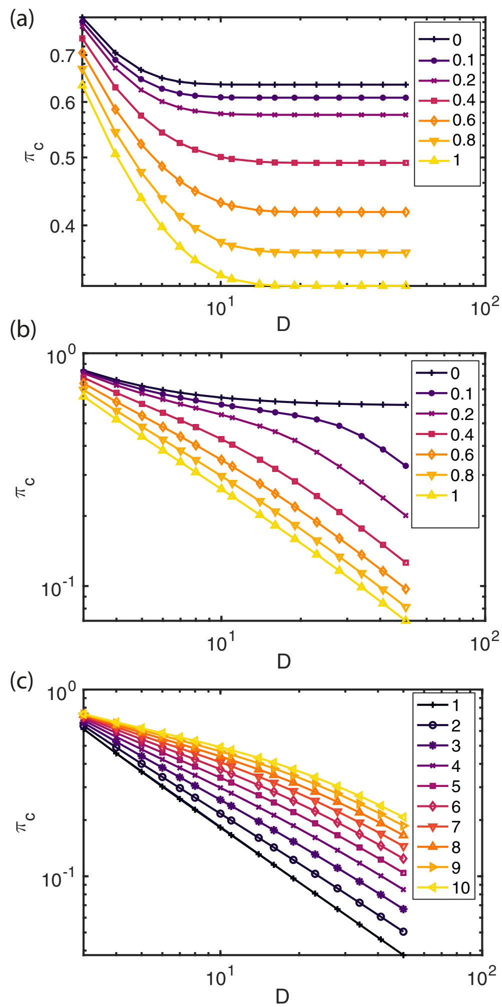

FIG. 2. Critical percolation threshold π_c plotted versus the maximum degree D for: (a) exponential degree distribution with parameter t as indicated, (b) degree distribution with tail exponent $\tau = 2.5$ and parameter t as indicated and (c) degree distribution with tail exponent τ as indicated and $t = 0.9$.

robust to random and hub-biased failures [31]. Vanishing percolation threshold also implies that some spreading processes that can be mapped to percolation, such as the susceptible-infected-recovered model with instantaneous transmissions, may feature no epidemic threshold on a wider range of networks than was previously acknowledged [30].

Acknowledgements

S.B. acknowledges kind hospitably of the Centre for Complex Systems Studies at Utrecht and the support from the Swaantje Mondt PhD Travel Fund.

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Funding

The Hungarian National Research, Development and Innovation Office (grants no. K128780, NVKP_16-1-2016-0004) and the Research Excellence Programme of the Ministry for Innovation and Technology in Hungary, within the framework of the Digital Biomarker thematic programme of the Semmelweis University to G.P., in part.

References

1. Newman, M. E. J. (2003) The structure and function of complex networks. *SIAM Rev.*, **45**, 167–256.
2. Litvak, N. & van der Hofstad, R. (2013) Uncovering disassortativity in large scale-free networks. *Phys. Rev. E*, **87**, 022801.
3. Estrada, E. (2011) Combinatorial study of degree assortativity in networks. *Phys. Rev. E*, **84**, 047101.
4. Newman, M. E. J. (2002) Assortative mixing in networks. *Phys. Rev. Lett.*, **89**, 208701.
5. Papadopoulos, L., Porter, M. A., Daniels, K. E. & Bassett, D. S. (2018) Network analysis of particles and grains. *J. Complex Netw.*, **6**, 485–565.
6. Torres-Knoop, A., Kryven, I., Schamboeck, V. & Iedema, P. D. (2018) Modeling the free-radical polymerization of hexanediol diacrylate (HDDA): a molecular dynamics and graph theory approach. *Soft Matter*, **14**, 3404–3414.
7. Olvera-Cravioto, M. (2019) Pagerank's behavior under degree-degree correlations. *arXiv preprint arXiv:1909.09744*.
8. Mondragón, R. J. (2020) estimating degree–degree correlation and network cores from the connectivity of high-degree nodes in complex networks. *Sci. Rep.*, **10**, 1–24.
9. Teller, S., Granell, C., De Domenico, M., Soriano, J., Gómez, S. & Arenas, A. (2014) Emergence of assortative mixing between clusters of cultured neurons. *PLoS Comput. Biol.*, **10**, e1003796.
10. Boguá, M., Pastor-Satorras, R. & Vespignani, A. (2003) Epidemic spreading in complex networks with degree correlations. *Statistical Mechanics of Complex Networks* (R. Pastor-Satorras, M. Rubi, & A. Diaz-Guilera, eds), Lecture Notes in Physics, vol. 625. Berlin, Heidelberg: Springer, pp. 127–147.
11. Watts, D. J. & Strogatz, S. H. (1998) Collective dynamics of 'small-world' networks. *Nature*, **393**, 440–442.
12. Palla, G., Derényi, I., Farkas, I. & Vicsek, T. (2005) Uncovering the overlapping community structure of complex networks in nature and society. *Nature*, **435**, 814–818.
13. Kryven, I. & Stegehuis, C. (2020) Contact tracing in configuration models. *arXiv preprint arXiv:2010.05590*.
14. Kivelä, M., Arenas, A., Barthelemy, M., Gleeson, J. P., Moreno, Y. & Porter, M. A. (2014) Multilayer networks. *J. Complex Netw.*, **2**, 203–271.
15. Angeles Serrano, M., Krioukov, D. & Bogunà, M. (2008) Self-similarity of complex networks and hidden metric spaces. *Phys. Rev. Lett.*, **100**, 078701.
16. Fortunato, S. Community detection in graphs. (2010) *Phys. Rep.*, **486**, 75–174.
17. Kryven, I. (2019) Bond percolation in coloured and multiplex networks. *Nat. Commun.*, **10**, 1–16, 2019.
18. Schamboeck, V., Kryven, I. & Iedema, P. D. (2020) Effect of volume growth on the percolation threshold in random directed acyclic graphs with a given degree distribution. *Phys. Rev. E*, **101**, 012303.
19. Kryven, I. (2016) Emergence of the giant weak component in directed random graphs with arbitrary degree distributions. *Phys. Rev. E*, **94**, 012315.
20. Kryven, I. (2017) Finite connected components in infinite directed and multiplex networks with arbitrary degree distributions. *Phys. Rev. E*, **96**, 052304.
21. Schamboeck, V., Iedema, P. D. & Kryven, I. (2019) Dynamic networks that drive the process of irreversible step-growth polymerization. *Sci. Rep.*, **9**, 1–18.
22. Bianconi, G., Kryven, I. & Ziff, R. M. (2019) Percolation on branching simplicial and cell complexes and its relation to interdependent percolation. *Phys. Rev. E*, 100:062311.
23. Goltsev, A. V., Dorogovtsev, S. N. & Mendes, J. F. F. (2008) Percolation on correlated networks. *Phys. Rev. E*, **78**, 051105.

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24. Boguná, M. & Pastor-Satorras, R. (2002) Epidemic spreading in correlated complex networks. *Phys. Rev. E*, **66**, 047104.
25. Schamboeck, V., Iedema, P. D. & Kryven, I. (2020) Coloured random graphs explain the structure and dynamics of cross-linked polymer networks. *Sci. Rep.*, **10**, 1–18.
26. Stegehuis, C. (2019) Degree correlations in scale-free random graph models. *J. Appl. Prob.*, **56**, 672–700.
27. Kryven, I. (2018) Analytic results on the polymerisation random graph model. *J. Math. Chem.*, **56**, 140–157.
28. Colomer-de Simón, P. & Boguñá, M. (2014) Double percolation phase transition in clustered complex networks. *Phys. Rev. X*, **4**, 041020.
29. Hackett, A., Cellai, D., Gómez, S., Arenas, A. & Gleeson, J. P. (2016) Bond percolation on multiplex networks. *Phys. Rev. X*, **6**, 021002.
30. Boguná, M., Pastor-Satorras, R. & Vespignani, A. (2003) Absence of epidemic threshold in scale-free networks with degree correlations. *Phys. Rev. Lett.*, **90**, 028701.
31. Shao, S., Huang, X., Eugene Stanley, H. & Havlin, S. (2015) Percolation of localized attack on complex networks. *N. J. Phys.*, **17**, 023049.

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