Assumptive Sequent-Based Argumentation

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Abstract

In many expert and everyday reasoning contexts it is very useful to reason on the basis of defeasible assumptions. For instance, if the information at hand is incomplete we often use plausible assumptions, or if the information is conflicting we interpret it as consistently as possible. In this paper sequent-based argumentation, a form of logical argumentation in which arguments are represented by a sequent, is extended to incorporate defeasible assumptions. The resulting assumptive framework is general, in that several other approaches to reasoning with assumptions from the literature can adequately be represented in it. Moreover, assumptive sequent-based argumentation has many desirable properties. It will be shown that assumptive sequent-based argumentation can easily be extended to a prioritized setting, it satisfies rationality postulates and reasoning with maximally consistent subsets can be represented in it.

1 Introduction

Assumptions are an important concept in defeasible reasoning. Often, in both expert and everyday reasoning, the information provided is not complete or it is inconsistent. To derive conclusions in such cases, additional information can be assumed or only consistent subsets can be considered. There are many approaches to reasoning with assumptions within the artificial intelligence literature. One of the earlier and best-known formalisms is that of default logic [4, 59]. Intuitively, a default rule of the form $\phi : \phi_1, \ldots, \phi_n/\psi$, represents that the conclusion ψ can be derived, if ϕ is given and no inconsistencies arise when $\phi_1, \ldots, \phi_n$ hold.

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A well-known formal method for modeling defeasible reasoning is formal argumentation. The idea is that an argument can only be considered as accepted or warranted, when it is defended from all of its attackers. Argumentation frameworks in abstract argumentation theory, introduced by Dung [36], represent this idea by means of a directed graph. The nodes in the graph represent arguments (which are abstract entities) and the edges represent the attacks (the nature of which is unknown). Abstract argumentation can be instantiated in various ways, resulting in logical (also known as deductive or structured) argumentation. In these approaches the arguments have a specific structure and attacks depend on this structure [24, 25, 54]. For example, in [24] the argumentation machinery is combined with classical logic. In logical argumentation there is an explicit relationship among arguments (e.g., a sub-argument relation can be defined) and rationality postulates from e.g., [30], such as the consistency of the derived conclusions, can be studied, see also [55].

One such logical argumentation framework is *sequent-based argumentation* [10], in which arguments are represented by sequents, as introduced by Gentzen [39] and well-known in proof theory. Attacks between arguments are formulated by *sequent elimination rules*, which are special inference rules. The resulting framework is generic and modular, in that any logic, with a corresponding sound and complete sequent calculus, can be taken as the deductive base (the so-called *core logic*).

Several extensions and relations to other frameworks for nonmonotonic reasoning have been studied for sequent-based argumentation. A dynamic proof theory was introduced [11, 12] to study argumentation from a proof-theoretic perspective. Furthermore, the relation to reasoning with maximally consistent subsets, a common way to maintain consistency when given an inconsistent set of information [60], was investigated [7, 9]. Sequent-based argumentation was extended to incorporate priorities [8] and hypersequents [27]. The latter are a generalization of Gentzen's sequents [13] and allow to take logics such as the semi-relevance logic RM [3, 14] and the modal logic $\mathsf{S5}$ [38] as the core logic. However, in sequent-based argumentation or any of its generalizations, it is not possible to distinguish between facts and defeasible assumptions. This can result in attacks on arguments that are constructed only from facts. As facts represent knowledge that is known to be true, there should be no conflict between facts, nor should arguments constructed only from facts be attacked, since otherwise one could doubt the known information. Therefore, this paper, an extended version of [26], proposes a further generalization, that allows to distinguish between facts and defeasible assumptions.

The contribution of this paper is twofold. First, sequent-based argumentation is extended. To each sequent a component for assumptions is added, to distinguish between defeasible and strict premises. This way, in addition to the given information, assumptions can be made to reach further conclusions. An assumptive argument can only be attacked in its defeasible assumptions, thus assuring that the facts (the given information or strict premises) always hold. After introducing this assumptive sequent-based argumentation framework, we show how it can be generalized to include priorities, based on the approach from [8]. In human reasoning preferences are a common feature in the process of deriving conclusions. It is therefore beneficial if formal approaches to modeling defeasible reasoning can account for possible preferences. Including priorities in formal argumentation makes it possible to order arguments and accept only the most preferred ones. Then the rationality postulates from [30] are studied, which shows that the introduced framework satisfies some basic desirable properties. Furthermore, the representation of reasoning with maximally consistent subsets is investigated.

Second, instances of the obtained framework are studied. For this, three approaches to reasoning with assumptions from the literature are considered:

- *Assumption-based argumentation (ABA):* a structured argumentation framework which is also semi-abstract, in that there are only limited assumptions on the underlying deductive system [25, 64]. ABA was introduced to determine a set of assumptions that can be accepted as a conclusion from the given information. One of the aims of ABA is to provide a general framework that can incorporate other frameworks for nonmonotonic reasoning, such as default logic and other default reasoning frameworks.

- *Adaptive logics:* is a logical framework in which the goal is to interpret information as consistently or as normally as possible [21, 62]. What *as consistently* or *as normally as possible* means, depends on the lower limit logic, which can be understood as the core logic of the adaptive logic, and the application. In contrast to the other two approaches, the defeasible assumptions (called abnormalities) are assumed not to hold. A dynamic proof system provides a syntactic way to derive conclusions. Many forms of defeasible reasoning can be expressed by an adaptive logic, (see, e.g., [62], in particular page 86, for an overview).

- *Default assumptions:* were introduced as one of three ways to turn a monotonic consequence relation nonmonotonic [48]. Nonmonotonicty is obtained by varying the set of assumptions. Maximal sets of assumptions that are consistent with the given set of formulas are added to the consequence relation. A formula is then considered as derived if it is a consequence for each set of assumptions. Due to the maximality requirement on the sets of assumptions, it is a generalization of the consequence relations from [60].

Each of these three approaches covers instances of defeasible reasoning. Although they are related (see [43]), what makes them interesting to consider separately are their particular designs. For example, the type of framework (e.g., argumentation based or (supra-classical) logic based) and the different notions of assumptions, i.e., positive interpretations (the assumptions are assumed to hold) and negative interpretations (the assumptions are assumed not to hold). A general assumptive argumentation framework, of which the above three cases are instances, will therefore be beneficial in the search for a general framework for defeasible reasoning.

The introduced framework is general and modular. Any Tarskian logic with a corresponding sequent calculus can be taken as the core logic and, as will be shown in Section 4, it incorporates some well-known approaches to nonmonotonic reasoning with assumptions. Furthermore, the framework is well-behaved since, in most cases, the rationality postulates from [30] are satisfied. By means of the here introduced assumptive sequent-based argumentation framework, logics, such as intuitionistic logic, many of the well-known modal logics and several relevance logics, can be equipped with defeasible assumptions. Hence, the results of this paper generalize to many deductive core systems, as long as the Tarskian conditions are fulfilled.

As noted above, this paper is an extension of [26]. The results of [26] are part of this paper, now including full proofs. Additionally, this paper studies the properties of the proposed framework in more detail. That is, the incorporation of priorities and the rationality postulates from [30] are studied and it is shown how reasoning with maximally consistent subsets with assumptions can be represented in it. Moreover, the sections on adaptive logics and default assumptions are new.

The paper is organized as follows. In the next section, we provide preliminaries on the used notation and logical notions, a short introduction to abstract argumentation is given and the main definitions of sequent-based argumentation are recalled. Then, in Section 3, the general framework for assumptive sequent-based argumentation is introduced and generalized to a prioritized setting (Section 3.1), rationality postulates are studied (Section 3.2) and the representation of reasoning with maximally consistent subsets is investigated (Section 3.3). To demonstrate the expressiveness of the assumptive sequent-based framework and how it can be applied, in Section 4 it is shown how some well-known approaches to reasoning with defeasible assumptions can be represented in it: assumption-based argumentation (Section 4.1); adaptive logics (Section 4.2); and default assumptions (Section 4.3). Related work is discussed in Section 5 and we conclude in Section 6.

2 Preliminaries

In this section we review some basic notions that will be useful throughout the paper: the basic logical setting, abstract argumentation as introduced in [36] (Section 2.1) and sequent-based argumentation from [5, 10] (Section 2.2).

Throughout the paper only propositional languages are considered, denoted by $\mathcal{L}$. Atomic formulas are denoted by p, q, formulas are denoted by $\gamma, \delta, \phi, \psi$, sets of formulas are denoted by $\mathcal{S}, \mathcal{T}$, and finite sets of formulas are denoted by Γ, Δ. Later on sets of assumptions are denoted by $\mathcal{AS}, \mathcal{A}$ and finite sets of assumptions by A. All of these can be primed or indexed.

Definition 1. A *logic* for a language $\mathcal{L}$ is a pair $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, having the following properties:

- *reflexivity:* if $\phi \in \mathcal{S}$, then $\mathcal{S} \vdash \phi$;
- *transitivity:* if $\mathcal{S} \vdash \phi$ and $\mathcal{S}', \phi \vdash \psi$, then $\mathcal{S}, \mathcal{S}' \vdash \psi$; and
- *monotonicity:* if $\mathcal{S}' \vdash \phi$ and $\mathcal{S}' \subseteq \mathcal{S}$, then $\mathcal{S} \vdash \phi$.

Furthermore, the following property is assumed:

- *non-triviality*: there is a non-empty set of $\mathcal{L}$-formulas $\mathcal{S}$ and an $\mathcal{L}$-formula ϕ such that $\mathcal{S} \nvdash \phi$.

In this section and the next (Section 3) the following connectives will sometimes be considered:

- a negation operator ($\neg$): $p \nvdash \neg p$ and $\neg p \nvdash p$, for every atom p,
- a conjunction operator ($\wedge$): $\mathcal{S} \vdash \phi \wedge \psi$ iff $\mathcal{S} \vdash \phi$ and $\mathcal{S} \vdash \psi$,
- a disjunction operator ($\vee$): $\mathcal{S}, \phi \vee \psi \vdash \gamma$ iff $\mathcal{S}, \phi \vdash \gamma$ or $\mathcal{S}, \psi \vdash \gamma$,
- an implication operator ($\supset$): $\mathcal{S}, \phi \vdash \psi$ iff $\mathcal{S} \vdash \phi \supset \psi$.

We shall abbreviate $(\phi \supset \psi) \wedge (\psi \supset \phi)$ by $\phi \leftrightarrow \psi$. Furthermore, we denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$) the conjunction (respectively, the disjunction) of all the formulas in Γ and we let $\neg\mathcal{S} = \{\neg\phi \mid \phi \in \mathcal{S}\}$. In examples based on classical logic (CL), it is assumed that all four connectives are part of the language. In the example instances in Section 4, the properties of possible connectives depend on the underlying deductive base.

Definition 2. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, where $\mathcal{L}$ contains at least the connectives $\neg$ and $\wedge$, and let $\mathcal{T}$ be a set of $\mathcal{L}$-formulas.

- The *closure* of $\mathcal{T}$ is denoted by $\mathsf{CN}(\mathcal{T})$ (thus, $\mathsf{CN}(\mathcal{T}) = \{\phi \mid \Gamma \vdash \phi \text{ for } \Gamma \subseteq \mathcal{T}\}$).
- $\mathcal{T}$ is *consistent* (for $\vdash$), if there are no formulas $\phi_1, \ldots, \phi_n \in \mathcal{T}$ such that $\vdash \neg \bigwedge_{i=1}^{n} \phi_i$.
- A subset $\mathcal{C}$ of $\mathcal{T}$ is a *minimal conflict* of $\mathcal{T}$ (w.r.t. $\vdash$), if $\mathcal{C}$ is inconsistent and for any $c \in \mathcal{C}$, $\mathcal{C} \setminus \{c\}$ is consistent. $\mathsf{Free}(\mathcal{T})$ denotes the set of formulas in $\mathcal{T}$ that are not part of any minimal conflict of $\mathcal{T}$.

2.1 Abstract Argumentation

An *abstract argumentation framework*, as introduced by Dung [36], can be viewed as a directed graph. In this graph nodes represent arguments (which are abstract, i.e., they do not have an internal structure) and the arrows represent attacks between arguments, see Figure 1 for a graphical representation. Formally:

Definition 3. An *(abstract) argumentation framework* is a pair $\mathcal{AF} = \langle \text{Args}, \mathcal{AT} \rangle$, where Args is a set of *arguments* and $\mathcal{AT} \subseteq \text{Args} \times \text{Args}$ is an *attack relation* on these arguments.

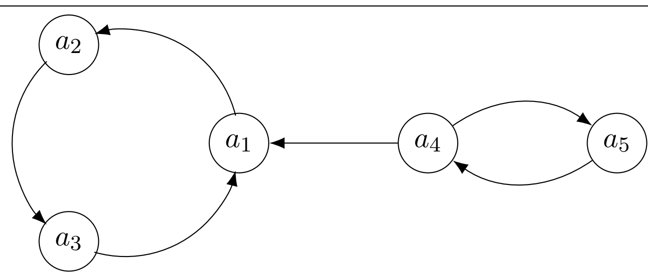

Figure 1: Abstract argumentation framework

Example 1. Consider the abstract argumentation framework from Figure 1. The graph in the figure represents $\mathcal{AF} = \langle \text{Args}, \mathcal{AT} \rangle$ where $\text{Args} = \{a_1, a_2, a_3, a_4, a_5\}$ and $\mathcal{AT} = \{(a_1, a_2), (a_2, a_3), (a_3, a_1), (a_4, a_1), (a_4, a_5), (a_5, a_4)\}$.

Given an argumentation framework $\mathcal{AF}$, Dung-style semantics [36] can be applied to it, to determine what combinations of arguments (called *extensions*) can collectively be accepted from the framework.

Definition 4. Let $\mathcal{AF} = \langle \text{Args}, \mathcal{AT} \rangle$ be an argumentation framework and let $\mathsf{S} \subseteq$ Args be a set of arguments. It is said that:

- S *attacks* an argument a if there is an $a' \in \mathsf{S}$ such that $(a', a) \in \mathcal{AT}$;
- S *defends* an argument a if S attacks every attacker of a;
- S is *conflict-free* if there are no $a_1, a_2 \in \mathsf{S}$ such that $(a_1, a_2) \in \mathcal{AT}$;
- S is *admissible* if it is conflict-free and it defends all of its elements.

An admissible set that contains all the arguments that it defends is a *complete extension* of $\mathcal{AF}$. Below are definitions of some particular complete extensions of $\mathcal{AF}$:

- the *grounded extension* of $\mathcal{AF}$ is the minimal (with respect to $\subseteq$) complete extension of $\mathcal{AF}$;
- a *preferred extension* of $\mathcal{AF}$ is a maximal (with respect to $\subseteq$) complete extension of $\mathcal{AF}$;
- a *stable extension* of $\mathcal{AF}$ is a complete extension of $\mathcal{AF}$ that attacks every argument not in it.

In what follows we shall refer to either complete (cmp), grounded (grd), preferred (prf) or stable (stb) semantics as *completeness-based semantics.* We denote by $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF})$ the set of all the extensions of $\mathcal{AF}$ under the semantics $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{grd}, \mathsf{prf}, \mathsf{stb}\}$. The subscript is omitted when this is clear from the context. As shown in [36], the grounded extension is unique for a given framework, we will therefore sometimes identify $\mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF})$ with its single element.[1]

Throughout the paper we will rely on several properties of the semantics defined above. For example, every stable extension is also a preferred extension, but not vice versa. In fact, the grounded extension always exists and there is always a preferred extension, but there is not necessarily a stable extension. For more details see e.g. [17].

Example 2. Recall the setting from Example 1, for the argumentation framework from Figure 1. Here we have that a_4 and a_5 attack each other and both defend themselves. Examples of conflict-free sets are $\{a_1, a_5\}$ and $\{a_2, a_4\}$.

For the extensions, note that the grounded extensions is $\emptyset$. Furthermore, there are three complete extensions: $\emptyset$, $\{a_5\}$ and $\{a_2, a_4\}$, the last two of these are also preferred extensions and $\{a_2, a_4\}$ is stable.

[1]Other extensions are discussed, e.g., in [16, 17, 18].

It has been argued that abstract argumentation should be instantiated [55], something which Dung already did in his seminal paper [36]. The study of instantiated abstract argumentation frameworks has resulted in several approaches to structured (also called logical or deductive) argumentation [5, 24, 25, 54]. In this paper we consider sequent-based argumentation [5, 10].

2.2 Sequent-Based Argumentation

As usual in logical argumentation (see, e.g., [24, 52, 53, 61]), arguments in this paper will have a specific structure based on the underlying formal language, the so-called *core logic*. In the current setting arguments are represented by the well-known proof-theoretic notion of a *sequent* [39].

Definition 5. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and $\mathcal{S}$ a set of $\mathcal{L}$-formulas.

- An $\mathcal{L}$-*sequent* (*sequent* for short) is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas in $\mathcal{L}$ and $\Rightarrow$ is a symbol that does not appear in $\mathcal{L}$.[2]
- An L-*argument* (*argument* for short) is an $\mathcal{L}$-sequent $\Gamma \Rightarrow \psi$,[3] where $\Gamma \vdash \psi$. Γ is called the *support set* of the argument and ψ its *conclusion*.
- An L-*argument based on* $\mathcal{S}$ is an L-argument $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq \mathcal{S}$. The set of all the L-arguments based on $\mathcal{S}$ will be denoted by $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$.

Given an argument $a = \Gamma \Rightarrow \psi$, we denote $\mathsf{Supp}(a) = \Gamma$ and $\mathsf{Conc}(a) = \psi$.

The formal systems used for the constructions of sequents (and so of arguments) for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, are *sequent calculi* [39], denoted here by C. In what follows it is assumed that C is sound and complete for $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, i.e., $\Gamma \Rightarrow \psi$ is provable in C iff $\Gamma \vdash \psi$. One of the advantages of sequent-based argumentation is that any logic with a corresponding sound and complete sequent calculus can be used as the core logic.[4] The construction of arguments from simpler arguments is done by the *inference rules* of the sequent calculus [39]. See Figure 2 for the sequent calculus LK of classical logic (CL).[5]

[2]Intuitively, in many sequent calculi, a sequent $\Gamma \Rightarrow \Delta$ can be understood as: if all formulas in Γ are true, then at least one formula in Δ is true.

[3]Set signs in arguments are omitted.

[4]See [10] for further discussion and advantages of this approach.

[5]Note that sequents are defined for sets of formulas. This avoids the need for contraction rules in LK. However, the conclusion of arguments (and later on derivations in single conclusioned calculi) contains at most one formula, i.e., $\Gamma \Rightarrow \phi, \psi$ is not allowed.

Axioms: $\phi \Rightarrow \phi$

Logical rules:

$$\frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta}\ [\wedge\Rightarrow] \qquad \frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \wedge \psi, \Delta}\ [\Rightarrow\wedge]$$

$$\frac{\Gamma \Rightarrow \phi, \Delta}{\Gamma, \neg\phi \Rightarrow \Delta}\ [\neg\Rightarrow] \qquad \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\phi, \Delta}\ [\Rightarrow\neg]$$

$$\frac{\Gamma \Rightarrow \phi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \supset \psi \Rightarrow \Delta}\ [\supset\Rightarrow] \qquad \frac{\Gamma, \phi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \phi \supset \psi, \Delta}\ [\Rightarrow\supset]$$

$$\frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta}\ [\vee\Rightarrow] \qquad \frac{\Gamma \Rightarrow \phi, \psi, \Delta}{\Gamma \Rightarrow \phi \vee \psi, \Delta}\ [\Rightarrow\vee]$$

Structural rules:

$$\frac{\Gamma_1 \Rightarrow \phi, \Delta_1 \quad \Gamma_2, \phi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ [\text{Cut}] \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\ [\text{Mon}] \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \phi, \Delta}\ [\text{Mon}]$$

Figure 2: The sequent calculus LK for classical logic.

In addition to arguments, an argumentation system contains attacks between arguments as well. In our case, attacks are represented by *sequent elimination rules*. Such a rule consists of an attacking argument (the first condition of the rule), an attacked argument (the last condition of the rule), conditions for the attack (the conditions in between) and a conclusion (the eliminated attacked sequent). The outcome of an application of such a rule is that the attacked sequent is 'eliminated'. The elimination of a sequent $a = \Gamma \Rightarrow \Delta$ is denoted by $\Gamma \not\Rightarrow \Delta$.

Definition 6. A *sequent elimination rule* (or *attack rule*) is a rule $\mathcal{R}$ of the form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \dots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n} \quad \mathcal{R} \tag{1}$$

Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, C its corresponding sequent calculus and $\mathcal{S}$ a set of $\mathcal{L}$-formulas. It is said that a sequent elimination rule $\mathcal{R}$ is *$Arg_{\mathsf{L}}(\mathcal{S})$-applicable* (with respect to some substitution θ), *applicable* for short, if $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1), \theta(\Gamma_n) \Rightarrow \theta(\Delta_n) \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ and for each $1 < i < n$, $\theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$ is derivable in C. It is then said that $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$ *$\mathcal{R}$-attacks* $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$.

The following example shows some of the possible elimination rules.

Example 3. Suppose $\mathcal{L}$ contains a negation operator $\neg$ and a conjunction operator $\wedge$. See [10, 63] for a definition of many sequent elimination rules. Below are three of them (assuming that $\Gamma_2 \neq \emptyset$):

$$\text{Undercut (Ucut):} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \neg \bigwedge \Gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$$

$$\text{Direct Ucut (DUcut):} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg\gamma \quad \neg\gamma \Rightarrow \psi_1 \quad \gamma, \Gamma_2' \Rightarrow \psi_2}{\gamma, \Gamma_2' \not\Rightarrow \psi_2}$$

$$\text{Consistency Ucut (ConUcut):} \quad \frac{\Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi}$$

A sequent-based framework is now defined as follows:

Definition 7. A *sequent-based argumentation framework* for a set of formulas $\mathcal{S}$ based on the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and a set AR of sequent elimination rules, is a pair $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{AT} \rangle$, where $\mathcal{AT} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \times \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ and $(a_1, a_2) \in \mathcal{AT}$ iff there is an $\mathcal{R} \in \mathsf{AR}$ such that a_1 $\mathcal{R}$-attacks a_2.

In what follows, to simplify notation, the subscripts L and/or AR are omitted when these are clear from the context or arbitrary.

Example 4. Let $\mathcal{AF}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S})$ be an argumentation framework, with classical logic as its core logic, Undercut as the only attack rule and the set $\mathcal{S} = \{p, p \supset q, \neg q\}$. Some of the arguments are:

$$a = p, p \supset q \Rightarrow q \qquad b = \neg q \Rightarrow \neg q \qquad c = p \Rightarrow p$$
$$d = \Rightarrow q \vee \neg q \qquad e = p \supset q, \neg q \Rightarrow \neg p.$$

Note that a attacks b and e since $\Rightarrow q \leftrightarrow \neg\neg q$ is derivable in LK. Similarly, e attacks a and c, since $\Rightarrow \neg p \leftrightarrow \neg p$. The argument d cannot be attacked, since the considered attack rule attacks arguments in their support and d has an empty support set. See Figure 3 for a graphical representation of these arguments and the attacks between them. Note that the figure only shows the five arguments mentioned above. Many other arguments are not shown. However, these five arguments are sufficient to illustrate some of the notions of this section.

A sequent-based argumentation framework $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{AT} \rangle$ can be seen as an instance of a Dung-style argumentation framework $\mathcal{AF} = \langle \mathrm{Args}, \mathcal{AT} \rangle$, where $\mathrm{Args} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ (Definition 3). Therefore, Dung-style semantics (Definition 4) can be applied to it.

From this entailment relations that are induced from a given sequent-based argumentation framework and its semantics can be defined.

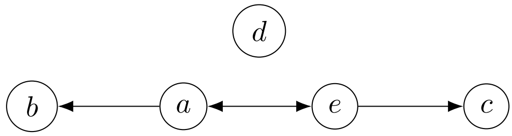

Figure 3: Part of the sequent-based argumentation framework from Example 4 for $\mathcal{S} = \{p, p \supset q, \neg q\}$.

Definition 8. Given a sequent-based argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, the semantics as defined in Definition 4 induce corresponding (nonmonotonic) *entailment relations*:

- $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{L},\mathsf{sem}} \phi$ iff there is an $a \in \bigcap \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, such that $\mathsf{Conc}(a) = \phi$,
- $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{L},\mathsf{sem}} \phi$ iff for some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, there is an argument $\Gamma \Rightarrow \phi \in \mathcal{E}$ where $\Gamma \subseteq \mathcal{S}$,
- $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{sem}} \phi$ iff for every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ there is an $a \in \mathcal{E}$ and $\mathsf{Conc}(a) = \phi$.

Since the grounded extension is unique, $\mid\!\sim^{\cap}_{\mathsf{L},\mathsf{grd}}$, $\mid\!\sim^{\cup}_{\mathsf{L},\mathsf{grd}}$ and $\mid\!\sim^{\Cap}_{\mathsf{L},\mathsf{grd}}$ coincide and will be denoted by $\mid\!\sim_{\mathsf{L},\mathsf{grd}}$.

Example 5. Consider the framework from Example 4, for $\mathcal{S} = \{p, p \supset q, \neg q\}$ and Undercut as the only attack rule. Recall that only a few of the existing arguments were mentioned in the previous example. Since the argument $d = \Rightarrow q \vee \neg q$ is not attacked it holds that $\mathcal{S} \mid\!\sim_{\mathsf{CL},\mathsf{grd}} q \vee \neg q$. It can be shown that there are three preferred extensions: $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S})) = \{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$ where $\mathcal{E}_1 = \mathrm{Arg}_{\mathsf{L}}(\{p, p \supset q\})$, $\mathcal{E}_2 = \mathrm{Arg}_{\mathsf{L}}(\{p, \neg q\})$ and $\mathcal{E}_3 = \mathrm{Arg}_{\mathsf{L}}(\{p \supset q, \neg q\})$. Thus, for $\phi \in \mathcal{S}$ we have that $\mathcal{S} \mid\!\not\sim^{\cap}_{\mathsf{CL},\mathsf{prf}} \phi$ and $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{CL},\mathsf{prf}} \phi$. Now consider the formula $p \vee \neg q$. Although $\mathcal{S} \mid\!\not\sim^{\cap}_{\mathsf{CL},\mathsf{prf}} p \vee \neg q$, $\mathcal{S} \mid\!\not\sim^{\Cap}_{\mathsf{CL},\mathsf{prf}} p$ and $\mathcal{S} \mid\!\not\sim^{\Cap}_{\mathsf{CL},\mathsf{prf}} \neg q$, it holds that $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{CL},\mathsf{prf}} p \vee \neg q$. This follows since in each $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, there is an argument $a_p \in \mathcal{E}$ such that $\mathsf{Conc}(a_p) = p$ and/or there is an argument $a_q \in \mathcal{E}$ such that $\mathsf{Conc}(a_q) = \neg q$. In both cases $p \vee \neg q$ can be derived from the conclusions of $\mathcal{E}$.

3 Assumptive Sequent-Based Argumentation

Sometimes deriving conclusions requires making assumptions, for example, because there is simply not enough information given, or the information provided is conflicting. There are many ways in which assumptions are handled in the literature, e.g., default logic [59], assumption-based argumentation [25], default assumptions [48] and adaptive logics [21]. In this section the sequent-based argumentation framework from Section 2.2, is extended to incorporate assumptions.

In what follows we assume that, instead of one set of formulas, the input consists of two sets of $\mathcal{L}$-formulas: $\mathcal{AS}$, the *defeasible premises*, a set of assumptions, the form of which depends on the application and the logic; and $\mathcal{S}$, the *strict premises*, the formulas of which can intuitively be understood as facts. As before, a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ is assumed to have a corresponding sequent calculus C. This calculus will be adjusted to C', in order to allow for assumptions. Both C and C' are assumed to be sound and complete for L. Furthermore, in the current section, $\mathcal{L}$ will contain at least a negation operator $\neg$ and a conjunction operator $\wedge$, as in Section 2.

Definition 9. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, with a corresponding sound and complete sequent calculus C and the corresponding adjusted calculus C', let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas and $\mathcal{AS}$ a set of assumptions.

- An *assumptive $\mathcal{L}$-sequent* (*(assumptive) sequent* for short) is an expression of the form $A \wr \Gamma \Rightarrow \Delta$.
- An *assumptive* L-*argument* (*(assumptive) argument* for short) is an assumptive sequent $A \wr \Gamma \Rightarrow \psi$, that is provable in C'.[6]
- An *assumptive* L-*argument based on $\mathcal{S}$ and $\mathcal{AS}$* is an assumptive argument $A \wr \Gamma \Rightarrow \psi$ such that $\Gamma \subseteq \mathcal{S}$ and $A \subseteq \mathcal{AS}$. As before, the set of all the assumptive L-arguments based on $\mathcal{S}$ and $\mathcal{AS}$ is denoted by $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$.

Notation 1. Let $a = A \wr \Gamma \Rightarrow \psi$ be an assumptive argument. Then $\mathsf{Ass}(a) = A$ denotes the assumptions of the argument a. As before, $\mathsf{Supp}(a) = \Gamma$ and $\mathsf{Conc}(a) = \psi$. Furthermore, for S a set of arguments, $\mathsf{Concs}(\mathsf{S}) = \{\mathsf{Conc}(a) \mid a \in \mathsf{S}\}$, $\mathsf{Supps}(\mathsf{S}) = \bigcup\{\mathsf{Supp}(a) \mid a \in \mathsf{S}\}$ and $\mathsf{Ass}(\mathsf{S}) = \bigcup\{\mathsf{Ass}(a) \mid a \in \mathsf{S}\}$. In case that $A = \emptyset$, a will sometimes be written as $\Gamma \Rightarrow \psi$.

Because of the additional component (the assumptions) in an argument, rules have to be defined that allow for the movement of assumptions around $\wr$.

Definition 10. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, $\mathcal{S}$ a set of $\mathcal{L}$ formulas and $\mathcal{AS}$ a set of assumptions. The following two rules allow to move assumptions:

$$\frac{A \wr \Gamma, \phi \Rightarrow \psi}{A, \phi \wr \Gamma \Rightarrow \psi}\ \mathrm{AS}^l_{\mathcal{AS}} \qquad \frac{A, \phi \wr \Gamma \Rightarrow \psi}{A \wr \Gamma, \phi \Rightarrow \psi}\ \mathrm{AS}^r_{\mathcal{AS}} \quad \text{where } \phi \in \mathcal{AS}.$$

[6] In this paper, C' will differ from C only in that it is defined in terms of assumptive sequents rather than sequents (as in Definition 5) and that it has rules that allow for assumptions to be moved to and from the left side of $\wr$.

Remark 1. For a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, a set of $\mathcal{L}$-formulas $\mathcal{S}$ and a set of assumptions $\mathcal{AS}$, let $\Gamma \subseteq \mathcal{S}$ and $A \subseteq \mathcal{AS}$, if $\mathrm{AS}^r_{\mathcal{AS}}$ and $\mathrm{AS}^l_{\mathcal{AS}}$ are rules in C' then: $A \cup \Gamma \Rightarrow \phi$ is derivable in C iff $A \wr \Gamma \Rightarrow \phi$ is derivable in C'.

Remark 2. The rules from Definition 10 are necessary to construct assumptive arguments. Note that these rules can only be applied to assumptions (i.e., elements from $\mathcal{AS}$). Thus, although assumptions might occur left and right of $\wr$ in a derivation, assumptive sequents (and therefore the arguments in this paper) are such that assumptions only occur on the left side of $\wr$.

An important rule is [Cut] (see Figure 2). In view of Remark 1, the following two versions are admissible when $\mathrm{AS}^r_{\mathcal{AS}}$ and $\mathrm{AS}^l_{\mathcal{AS}}$ are part of C' and [Cut] is admissible in C:

$$\frac{A_1 \wr \Gamma_1 \Rightarrow \Delta_1, \phi \quad A_2 \wr \Gamma_2, \phi \Rightarrow \Delta_2}{A_1, A_2 \wr \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ [\text{Cut}] \qquad \frac{A_1 \wr \Gamma_1 \Rightarrow \Delta_1, \phi \quad A_2, \phi \wr \Gamma_2 \Rightarrow \Delta_2}{A_1, A_2 \wr \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}\ [\text{Cut}]$$

Figure 4 shows how the sequent calculus for classical logic LK (from Figure 2) can be extended to LK'. IN view of the discussion above, LK' contains only one cut rule.

Example 6. Recall, from Example 4, the set of formulas $\{p, p \supset q, \neg q\}$, where CL is the core logic and LK the corresponding calculus. Let now $\mathcal{S} = \{p\}$ and $\mathcal{AS} = \{p \supset q, \neg q\}$ and take LK' from Figure 4 as the corresponding calculus. The assumptive counterparts of the arguments in Example 4 are then:

$$a_{\mathcal{AS}} = p \supset q \wr p \Rightarrow q \qquad b_{\mathcal{AS}} = \neg q \wr \Rightarrow \neg q \qquad c_{\mathcal{AS}} = p \Rightarrow p$$
$$d_{\mathcal{AS}} = \ \Rightarrow q \vee \neg q \qquad e_{\mathcal{AS}} = p \supset q, \neg q \wr \Rightarrow \neg p.$$

Arguments are attacked in the set of assumptions. When choosing a (set of) attack rule(s), it is important to note that these reflect the interpretation of an assumption. In the rules below, the interpretation of the assumptions is positive: they are assumed to hold. If the interpretation is negative instead, the negation in the condition(s) of the first two rules should be removed. See Section 4.2 on adaptive logics for a setting with negative assumptions.

Example 7. Assume $A_1 \wr \Gamma_1 \Rightarrow \phi_1$; $A_2, \psi \wr \Gamma_2 \Rightarrow \phi_2 \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ and $\Delta \subseteq \mathcal{S}$. Let $a = A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, we continue using $A \wr \Gamma \not\Rightarrow \phi$ to denote that a has been eliminated. Examples of sequent elimination rules for assumptive

Axioms: $\phi \Rightarrow \phi$

Logical rules:

$$\frac{A \wr \Gamma, \phi, \psi \Rightarrow \Delta}{A \wr \Gamma, \phi \wedge \psi \Rightarrow \Delta}\ [\wedge\Rightarrow] \qquad \frac{A \wr \Gamma \Rightarrow \phi, \Delta \quad A \wr \Gamma \Rightarrow \psi, \Delta}{A \wr \Gamma \Rightarrow \phi \wedge \psi, \Delta}\ [\Rightarrow\wedge]$$

$$\frac{A \wr \Gamma \Rightarrow \phi, \Delta}{A \wr \Gamma, \neg\phi \Rightarrow \Delta}\ [\neg\Rightarrow] \qquad \frac{A \wr \Gamma, \phi \Rightarrow \Delta}{A \wr \Gamma \Rightarrow \neg\phi, \Delta}\ [\Rightarrow\neg]$$

$$\frac{A \wr \Gamma \Rightarrow \phi, \Delta \quad A \wr \Gamma, \psi \Rightarrow \Delta}{A \wr \Gamma, \phi \supset \psi \Rightarrow \Delta}\ [\supset\Rightarrow] \qquad \frac{A \wr \Gamma, \phi \Rightarrow \psi, \Delta}{A \wr \Gamma \Rightarrow \phi \supset \psi, \Delta}\ [\Rightarrow\supset]$$

$$\frac{A \wr \Gamma, \phi \Rightarrow \Delta \quad A \wr \Gamma, \psi \Rightarrow \Delta}{A \wr \Gamma, \phi \vee \psi \Rightarrow \Delta}\ [\vee\Rightarrow] \qquad \frac{A \wr \Gamma \Rightarrow \phi, \psi, \Delta}{A \wr \Gamma \Rightarrow \phi \vee \psi, \Delta}\ [\Rightarrow\vee]$$

Structural rules:

$$\frac{A_1 \wr \Gamma_1 \Rightarrow \Pi, \phi \quad A_2 \wr \Gamma_2, \phi \Rightarrow \Delta}{A_1, A_2 \wr \Gamma_1, \Gamma_2 \Rightarrow \Pi, \Delta}\ [\text{Cut}]$$

$$\frac{A \wr \Gamma, \phi \Rightarrow \psi}{A, \phi \wr \Gamma \Rightarrow \psi}\ \text{AS}^{l}_{\mathcal{AS}} \qquad \frac{A, \phi \wr \Gamma \Rightarrow \psi}{A \wr \Gamma, \phi \Rightarrow \psi}\ \text{AS}^{r}_{\mathcal{AS}} \qquad \text{where } \phi \in \mathcal{AS}$$

$$\frac{A \wr \Gamma \Rightarrow \Delta}{A \wr \Gamma, \phi \Rightarrow \Delta}\ [\text{LMon}] \qquad \frac{A \wr \Gamma \Rightarrow \Pi}{A \wr \Gamma \Rightarrow \phi, \Pi}\ [\text{RMon}]$$

Figure 4: The assumptive sequent calculus LK' for classical logic.

sequent-based argumentation are (see Section 4 for other definitions):

$$\frac{A_1 \wr \Gamma_1 \Rightarrow \phi_1 \quad \phi_1 \Rightarrow \neg\psi \quad A_2, \psi \wr \Gamma_2 \Rightarrow \phi_2}{A_2, \psi \wr \Gamma_2 \not\Rightarrow \phi_2}\ \text{AT}^{\Rightarrow}_{\mathcal{AS}}$$

$$\frac{A_1 \wr \Gamma_1 \Rightarrow \phi_1 \quad \phi_1 \Rightarrow \neg\psi \quad \neg\psi \Rightarrow \phi_1 \quad A_2, \psi \wr \Gamma_2 \Rightarrow \phi_2}{A_2, \psi \wr \Gamma_2 \not\Rightarrow \phi_2}\ \text{AT}^{\Leftrightarrow}_{\mathcal{AS}}$$

$$\frac{\Delta \Rightarrow \neg \bigwedge A_1 \quad A_1 \wr \Gamma_1 \Rightarrow \phi_1}{A_1 \wr \Gamma_1 \not\Rightarrow \phi_1}\ \text{AT}^{\mathsf{Con}}_{\mathcal{AS}}$$

When the superscript is clear from the context or arbitrary, it will be omitted. In the remainder we sometimes write that, for the arguments as in the first two rules,

$A_2, \psi \wr \Gamma_2 \Rightarrow \phi_2$ is *attacked in* ψ by $A_1 \wr \Gamma_1 \Rightarrow \phi_1$.

Each of the above rules reflects that an assumptive argument can only be attacked in its assumptions. The rule $\mathrm{AT}^{\Leftrightarrow}_{\mathcal{AS}}$ can be seen as the assumptive version of the direct undercut rule from Example 3. The $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$ rule can be understood as the assumptive version of consistency undercut. This rule attacks arguments that have an inconsistent set of assumptions (in which case it could be that $\Delta = \emptyset$) or the set of assumptions is inconsistent with the set of facts. In Example 8 below, if $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$ would be part of the attack rules, the argument a_6 would be $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$-attacked.

Definition 11. An *assumptive sequent-based argumentation framework* for a set of formulas $\mathcal{S}$, set of assumptions $\mathcal{AS}$, based on a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and a set AR of sequent elimination rules (such as those from Example 7), is a pair $\mathcal{AF}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT} \rangle$, where $\mathcal{AT} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) \times \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ and $(a_1, a_2) \in \mathcal{AT}$ iff there is a rule $\mathcal{R} \in \mathsf{AR}$ such that a_1 $\mathcal{R}$-attacks a_2.

Note that, although no restrictions are placed on $\mathcal{S}$ and $\mathcal{AS}$ in the definition above, in Section 3.2 it is shown why $\mathcal{S}$ should be consistent. Such a restriction can not be enforced in general, since there are cases where $\mathcal{S}$ has to be inconsistent, in order for the argumentation process to be interesting. Section 4.2, on adaptive logics, is an example of such a case.

Like before, when these are clear from the context or arbitrary, the subscripts L and/or AR are omitted. The semantics, as defined in Definition 4 can be applied to assumptive sequent-based argumentation frameworks.

Example 8. Let $\mathcal{AF}_{\mathsf{CL},\{\mathrm{AT}^{\Leftrightarrow}_{\mathcal{AS}}\}}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{CL}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT} \rangle$, where $\mathcal{S} = \{p\}$ and $\mathcal{AS} = \{p \supset q, \neg q\}$, as in Example 6. Then some of the arguments in $\mathrm{Arg}_{\mathsf{CL}}(\mathcal{S}, \mathcal{AS})$ are:

$$a_1 = p \Rightarrow p \qquad a_2 = p \supset q \wr \Rightarrow p \supset q \qquad a_3 = \neg q \wr \Rightarrow \neg q$$
$$a_4 = p \supset q \wr p \Rightarrow q \qquad a_5 = \neg q \wr p \Rightarrow p \wedge \neg q \qquad a_6 = p \supset q, \neg q \wr \Rightarrow \neg p$$

As in Example 4, these are only a few of the derivable arguments. However, these arguments are sufficient for the purpose of this example and the other arguments do not change the discussion and evaluation below.

Note that a_4 attacks any argument with $\neg q$ in the assumptions (i.e., a_3, a_5 and a_6), since $\Rightarrow q \leftrightarrow \neg\neg q$ is derivable in LK'. To see why a_5 attacks a_2, a_4 and a_6, take a look at the following derivations:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow p, q}[\text{Mon}] \quad \dfrac{q \Rightarrow q}{p, q \Rightarrow q}[\text{Mon}]}{p, p \supset q \Rightarrow q}[\supset\Rightarrow]}{p, \neg q, p \supset q \Rightarrow}[\neg\Rightarrow]}{p, \neg q \Rightarrow \neg(p \supset q)}[\Rightarrow\neg]}{p \wedge \neg q \Rightarrow \neg(p \supset q)}[\wedge\Rightarrow]}{\Rightarrow (p \wedge \neg q) \supset \neg(p \supset q)}[\Rightarrow\supset]$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \Rightarrow p}{p \Rightarrow p, q}[\text{Mon}] \quad \dfrac{\dfrac{q \Rightarrow q}{p, q \Rightarrow q}[\text{Mon}]}{p \Rightarrow \neg q, q}[\Rightarrow\neg]}{p \Rightarrow p \wedge \neg q, q}[\Rightarrow\wedge]}{\Rightarrow p \wedge \neg q, p \supset q}[\Rightarrow\supset]}{\neg(p \supset q) \Rightarrow p \wedge \neg q}[\neg\Rightarrow]}{\Rightarrow \neg(p \supset q) \supset (p \wedge \neg q)}[\Rightarrow\supset]$$

See Figure 5 for a graphical representation of the given arguments and the attacks between them.

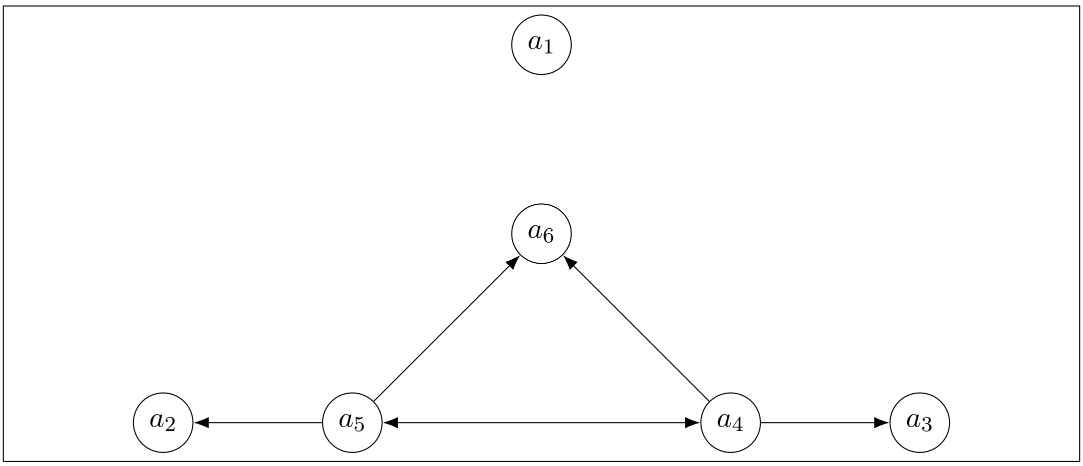

Figure 5: Part of the assumptive sequent-based argumentation framework from Example 8 for $\mathcal{S} = \{p\}$ and $\mathcal{AS} = \{p \supset q, \neg q\}$.

Since $\mathsf{Ass}(a_1) = \emptyset$, the argument a_1 cannot be attacked. It follows that $a_1 \in \bigcap \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{CL},\{\mathrm{AT}^{\Leftrightarrow}_{\mathcal{AS}}\}}(\mathcal{S},\mathcal{AS}))$ and hence $a_1 \in \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{CL},\{\mathrm{AT}^{\Leftrightarrow}_{\mathcal{AS}}\}}(\mathcal{S},\mathcal{AS}))$, where $\mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{CL},\{\mathrm{AT}^{\Leftrightarrow}_{\mathcal{AS}}\}}(\mathcal{S},\mathcal{AS}))$ is identified with its single element. There are five admissible sets in the framework from Figure 5: $\emptyset$, $\{a_1\}$, $\{a_1, a_2, a_4\}$, $\{a_1, a_3, a_5\}$, $\{a_2, a_4\}$ and $\{a_3, a_5\}$. Note that a_6 is not part of any admissible set. To see this, note that both a_4 and a_5 have to be attacked and not defended, yet any attacker of a_4 and a_5 is also an attacker of a_6.

The entailment relations for an assumptive framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S},\mathcal{AS})$ are defined similarly to those in Definition 8 and are denoted by $\mid\!\sim^{\star}_{\mathcal{AS},\mathsf{sem}}$ for $\star \in \{\cap, \cup, \Cap\}$ and where $\mathcal{AS}$ is the set of assumptions.

Example 9. Recall $\mathcal{AF}_{\mathsf{CL},\{\mathrm{AT}^{\Leftrightarrow}_{\mathcal{AS}}\}}(\mathcal{S},\mathcal{AS})$ from Example 8, where $\mathcal{S} = \{p\}$ and $\mathcal{AS} = \{p \supset q, \neg q\}$. In view of the discussion about the extensions in that example, $\mathcal{S} \mid\!\sim_{\mathcal{AS},\mathsf{grd}} p$, since a_1 is not attacked. Moreover, $\mathcal{S} \mid\!\sim^{\cup}_{\mathcal{AS},\mathsf{sem}} \phi$ for $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ and $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{p \supset q, p\} \cup \{\neg q, p\})$, but $\mathcal{S} \not\mid\!\sim^{\cap}_{\mathcal{AS},\mathsf{sem}} \psi$ for $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ and $\psi \in \{p \supset q, \neg q\}$. This follows since for each $\phi \in \{p \supset q, \neg q\}$ there is an argument a with $\mathsf{Conc}(a) = \phi$ and there is some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{CL},\{\mathrm{AT}^{\Leftrightarrow}_{\mathcal{AS}}\}}(\mathcal{S},\mathcal{AS}))$ such that $a \in \mathcal{E}$. However, there is also some $\mathcal{E}' \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{CL},\{\mathrm{AT}^{\Leftrightarrow}_{\mathcal{AS}}\}}(\mathcal{S},\mathcal{AS}))$ such that $a \notin \mathcal{E}'$, for $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$.

In the next sections we study some properties of assumptive sequent-based argumentation frameworks. First, in Section 3.1 priorities among the assumptions are incorporated. Then, in Section 3.2 the rationality postulates from [30] for the resulting prioritized frameworks are shown. In Section 3.3 we discuss how reasoning with maximally consistent subsets, as introduced in [60], can be generalized to the assumptive setting and can be represented by the here introduced framework. In these sections we assume that the rules from Figure 6 are admissible in the sequent calculus C. This way it is not necessary to choose a specific core logic to prove the results and the proofs can be kept relatively simple (i.e., no case distinctions are necessary to cover different kinds of rules). Note that this requirement does not limit the presented assumptive framework, only the calculi for which the results hold.

3.1 Adding Priorities

Another important and often applied way to distinguish between elements of the premises, is by means of priorities. By assigning priorities to some knowledge, or expressing preferences among the knowledge, the derivation process can be adjusted such that as much as possible of the most preferred knowledge is accepted. Within argumentation, for many frameworks prioritized versions have been studied, including sequent-based argumentation [8]. In the assumptive setting, facts always hold, thus these are preferred over any other premise. But among the assumptions, a user might have preferences.

Definition 12. A *priority function* for a language $\mathcal{L}$ is a function $\pi : \mathcal{L} \to \mathbb{N}^+$. Given a set of $\mathcal{L}$-formulas $\mathcal{S}$, we denote $\pi(\mathcal{S}) = \{\pi(\phi) \mid \phi \in \mathcal{S}\}$. Moreover, $\max_\pi(\mathcal{S}) = \{\phi \in \mathcal{S} \mid \nexists \psi \in \mathcal{S}, \pi(\phi) < \pi(\psi)\}$ denotes the set of formulas from $\mathcal{S}$ with maximal π-value. We let $\max_\pi(\emptyset) = 0$.

In what follows, it is assumed that a formula ϕ is preferred over a formula ψ if $\pi(\phi) \leq \pi(\psi)$, ϕ is strictly preferred over ψ if it is preferred over ψ and $\pi(\psi) \not\leq \pi(\phi)$. Thus, intuitively, a lower π-value means a higher preference.

With this priority function, the attack relation induced by $\mathrm{AT}_{\mathcal{AS}}$ can be refined:

Axioms: $\phi \Rightarrow \phi$

Logical rules:

$$\frac{A \wr \Gamma, \phi, \psi \Rightarrow \Delta}{A \wr \Gamma, \phi \wedge \psi \Rightarrow \Delta}\,[\wedge\Rightarrow] \qquad \frac{A \wr \Gamma \Rightarrow \phi, \Pi \quad A \wr \Gamma \Rightarrow \psi, \Pi}{A \wr \Gamma \Rightarrow \phi \wedge \psi, \Pi}\,[\Rightarrow\wedge]$$

$$\frac{A \wr \Gamma \Rightarrow \phi, \Pi}{A \wr \Gamma, \neg\phi \Rightarrow \Pi}\,[\neg\Rightarrow] \qquad \frac{A \wr \Gamma, \phi \Rightarrow \Pi}{A \wr \Gamma \Rightarrow \neg\phi, \Pi}\,[\Rightarrow\neg]$$

Structural rules:

$$\frac{A_1 \wr \Gamma_1 \Rightarrow \Pi, \phi \quad A_2 \wr \Gamma_2, \phi \Rightarrow \Delta}{A_1, A_2 \wr \Gamma_1, \Gamma_2 \Rightarrow \Pi, \Delta}\,[\mathrm{Cut}] \qquad \frac{A_1 \wr \Gamma_1 \Rightarrow \Pi, \phi \quad A_2, \phi \wr \Gamma_2 \Rightarrow \Delta}{A_1, A_2 \wr \Gamma_1, \Gamma_2 \Rightarrow \Pi, \Delta}\,[\mathrm{Cut}]$$

$$\frac{A \wr \Gamma, \phi \Rightarrow \psi}{A, \phi \wr \Gamma \Rightarrow \psi}\,\mathrm{AS}^{l}_{\mathcal{AS}} \qquad \frac{A, \phi \wr \Gamma \Rightarrow \psi}{A \wr \Gamma, \phi \Rightarrow \psi}\,\mathrm{AS}^{r}_{\mathcal{AS}} \quad \text{where } \phi \in \mathcal{AS}$$

$$\frac{A \wr \Gamma \Rightarrow \Delta}{A \wr \Gamma, \phi \Rightarrow \Delta}\,[\mathrm{LMon}] \qquad \frac{A \wr \Gamma \Rightarrow \Pi}{A \wr \Gamma \Rightarrow \phi, \Pi}\,[\mathrm{RMon}]$$

Figure 6: Rules that are assumed to be part of (or admissible in) the calculus C (in the case that C is single-conclusioned Π should be empty and Δ contains at most one formula).

Definition 13. Let $a_1, a_2 \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, it is said that a_1 $AT^{\star,\leq_\pi}_{\mathcal{AS}}$*-attacks* a_2 if and only if a_1 $\mathrm{AT}^{\star}_{\mathcal{AS}}$-attacks a_2 in ψ and $\max_\pi(\mathsf{Ass}(a_1)) \leq \pi(\psi)$, for $\star \in \{\Rightarrow, \Leftrightarrow\}$ or a_1 $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$-attacks a_2.

Remark 3. An $\mathrm{AT}^{\mathsf{Con},\leq_\pi}_{\mathcal{AS}}$-attack is always successful, since the attacker has an empty set of assumptions, the superscript $\leq_\pi$ will therefore often be omitted from the notation.

Example 10. Recall the examples from the previous section, for the assumptive framework $\mathcal{AF}_{\mathsf{CL},\{\mathrm{AT}^{\Leftrightarrow}_{\mathcal{AS}}\}}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{CL}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT} \rangle$, where $\mathcal{S} = \{p\}$, $\mathcal{AS} = \{p \supset q, \neg q\}$. Let $\pi(p \supset q) = 2$ and $\pi(\neg q) = 3$. Then, not all attacks of the flat setting (i.e., the setting without priorities) go through. For example, although a_5 attacks a_4 in the flat setting, this attack goes no longer through given the priority function π. In fact, since a_5 attacks arguments in the assumption $p \supset q$, no argument is attacked by a_5 given this priority function.

Definition 14. A *prioritized assumptive sequent-based argumentation framework* for a set of formulas $\mathcal{S}$, set of assumptions $\mathcal{AS}$, based on a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, π a priority function on $\mathcal{L}$ and AR the set of sequent elimination rules, is a triple $\mathcal{AF}^{\leq_\pi}_{\mathsf{L},\mathsf{AR}}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$, where $\mathcal{AT} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) \times \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ and $(a_1, a_2) \in \mathcal{AT}$ iff there is a rule $\mathcal{R}^{\leq_\pi} \in \mathsf{AR}$ such that a_1 $\mathcal{R}^{\leq_\pi}$-attacks a_2.

Like before, the semantics of Definition 4 can be applied to prioritized assumptive sequent-based argumentation frameworks. The corresponding entailment relations are denoted by $\mid\!\sim^{\leq_\pi,\star}_{\mathcal{AS},\mathsf{sem}}$, where $\star \in \{\cap, \cup, \Cap\}$ and $\mathcal{AS}$ is the set of assumptions.

Example 11. Consider the setting from Example 10, in which CL is the core logic, $\mathrm{AT}^{\Leftrightarrow,\leq_\pi}_{\mathcal{AS}}$ the attack rule, $\mathcal{S} = \{p\}$, $\mathcal{AS} = \{p \supset q, \neg q\}$, the priority function π is such that $\pi(p \supset q) = 2$ and $\pi(\neg q) = 3$. As mentioned, not all attacks as presented in Figure 5 go through. For a graphical representation of this prioritized assumptive framework, see Figure 7. Given the priority function π, a_2 is no longer attacked

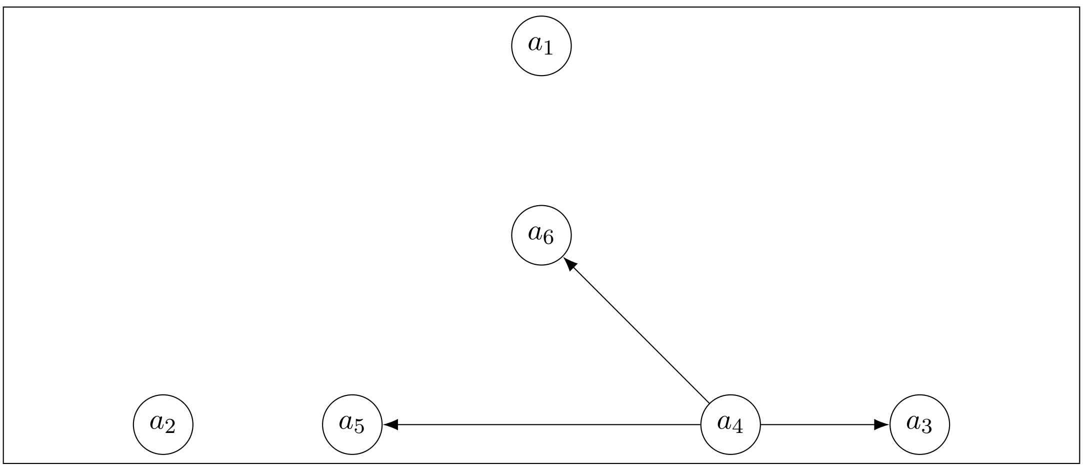

Figure 7: Part of the prioritized assumptive sequent-based argumentation framework from Example 11 for $\mathcal{S} = \{p\}$ and $\mathcal{AS} = \{p \supset q, \neg q\}$, with $\pi(p \supset q) = 2$ and $\pi(\neg q) = 3$.

and a_3 can no longer be defended from the attack by a_4. Thus $\mathcal{S} \mid\!\sim^{\leq_\pi}_{\mathcal{AS},\mathsf{grd}} \phi$, where $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{p, p \supset q\})$. On the other hand $\mathcal{S} \not\mid\!\sim^{\leq_\pi,\cup}_{\mathcal{AS},\mathsf{cmp}} \neg q$.

In the next section some desirable properties of (prioritized) assumptive sequent-based argumentation frameworks are studied, in terms of the rationality postulates from [30].

3.2 Rationality Postulates

There are many structured argumentation frameworks introduced and studied in the literature. It is therefore important to have an objective measure for the usefulness of such frameworks and to make sure that the resulting extensions satisfy some basic desirable properties. To this end, the rationality postulates from [30] are studied. Before introducing the postulates, the notion of a sub-argument will be useful.

Definition 15. Let $\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$ be an argumentation framework and consider two arguments $a, a' \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ such that $a = A \wr \Gamma \Rightarrow \phi$ and $a' = A' \wr \Gamma' \Rightarrow \phi'$. Then a' is a *sub-argument* of a if $\Gamma' \subseteq \Gamma$ and $A' \subseteq A$. The set of sub-arguments of a is denoted by $\mathsf{Sub}(a)$.

Definition 16. Let $\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$ be an assumptive argumentation framework for the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, the set $\mathcal{S}$ of $\mathcal{L}$-formulas, the set $\mathcal{AS}$ of assumptions and some semantics sem. $\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS})$ satisfies:

- *closure of extensions:* iff $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$ for each extension $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}))$;
- *sub-argument closure:* iff $a \in \mathcal{E}$ implies that $\mathsf{Sub}(a) \subseteq \mathcal{E}$ for all extensions $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}))$;
- *consistency:* iff $\mathsf{Concs}(\mathcal{E})$ is consistent for each $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}))$.

Remark 4. In [30], there are two different postulates for inconsistency: direct consistency (the consistency postulate above) and indirect consistency. However, in view of the closure of extensions postulate, indirect consistency follows from the consistency postulate in our setting. This is why the above postulates are discussed in the given order.

Furthermore, sub-argument closure was not defined as a postulate, but is shown as a proposition ([30, Proposition 1]). Note that the framework from [30] is different from the one presented here, thus the notion of a sub-argument is also different. However, the definition of sub-arguments as given here, corresponds to that of e.g., [1, 2, 8].

Remark 5. In the proofs of the rationality postulates below, it will be assumed that $\mathcal{S}$ is consistent. Consider for example $\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$, for CL the core logic, with LK as calculus and where $\mathcal{S} = \{p, \neg p\}$ and $\mathcal{AS} = \{q\}$. Some of the arguments are:

$$a = p \Rightarrow p \qquad b = \neg p \Rightarrow \neg p \qquad c = q \wr \Rightarrow q \qquad d = p, \neg p \Rightarrow \neg q$$

Note that a, b and d cannot be attacked, since $\mathsf{Ass}(a) = \mathsf{Ass}(b) = \mathsf{Ass}(d) = \emptyset$. Thus $a, b, d \in \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{L}}^{\leq\pi}(\mathcal{S}, \mathcal{AS}))$. Moreover, d attacks c and c cannot be defended, though one might argue that the conflict of p should not cause q to be excluded from the conclusions. In Lemma 2, it will be shown that, when $\mathcal{S}$ is inconsistent, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$ is the only extension.

The next lemma introduces some sequent rules that will be used in the proofs of this section.

Lemma 1. *Let* $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ *be a logic with corresponding sequent calculus* C*, in which the rules from Figure 6 are admissible. Then the rules from Figure 8 are admissible as well.*

$$\frac{\Gamma, \neg\neg\phi \Rightarrow \Delta}{\Gamma, \phi \Rightarrow \Delta}\ [\neg\neg{\not\Rightarrow}] \qquad \frac{\Gamma, \phi_1, \ldots, \phi_n \Rightarrow \Pi}{\Gamma \Rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_n), \Pi}\ [\Rightarrow\neg\wedge]$$

$$\frac{\Gamma \Rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_n), \Pi}{\Gamma, \phi_1, \ldots, \phi_n \Rightarrow \Pi}\ [{\not\Rightarrow}\neg\wedge]$$

Figure 8: Admissible rules in the minimal calculus from Figure 6 (in the case that C is single-conclusioned Π should be empty and Δ contains at most one formula).

The proof is by means of derivations in the minimal calculus from Figure 6 and can be found in Appendix A.

The next lemma shows that, when $\mathcal{S}$ is inconsistent, there is exactly one extension that contains only the arguments with an empty set of assumptions. Moreover, together with Remark 5, it provides the motivation to assume that $\mathcal{S}$ is consistent.

Lemma 2. *Let* $\mathcal{AF}_{\mathsf{L}}^{\leq\pi}(\mathcal{S}, \mathcal{AS}) = \langle \mathit{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$ *be an argumentation framework. If* $\mathcal{S}$ *is inconsistent, then* $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq\pi}(\mathcal{S}, \mathcal{AS})) = \{\mathit{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)\}$ *for each* $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$.

Proof. Let $\mathcal{AF}_{\mathsf{L}}^{\leq\pi}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$ be an argumentation framework for the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ with corresponding calculus C, inconsistent set of $\mathcal{L}$-formulas $\mathcal{S}$, set of assumptions $\mathcal{AS}$ and π a priority function. Since an attack is always on formulas in the assumptions of an argument, none of the arguments in $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$ can be attacked, thus $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset) \subseteq \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{L}}^{\leq\pi}(\mathcal{S}, \mathcal{AS}))$.

By assumption $\mathcal{S}$ is inconsistent, thus there are $\phi_1, \ldots, \phi_n \in \mathcal{S}$, such that $\vdash \neg \bigwedge_{i=1}^{n} \phi_i$. Thus, by the completeness of C for L, $\Rightarrow \neg \bigwedge_{i=1}^{n} \phi_i$ and by $[{\not\Rightarrow}\neg\wedge]$ $\phi_1, \ldots, \phi_n \Rightarrow$ are derivable in C. Let $\psi \in \mathcal{AS}$ be arbitrary, by [RMon], $a =$

$\phi_1, \ldots, \phi_n \Rightarrow \neg\psi$ is derivable in C. Note that $a \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$ and a attacks any argument $b \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ for which $\psi \in \mathsf{Ass}(b)$. Since $\psi \in \mathcal{AS}$ was arbitrary, it follows that $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$ attacks any argument with a non-empty set of assumptions. Hence, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)$ attacks any argument not in it. Therefore $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS})) = \{\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \emptyset)\}$ for each $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$. ☐

For the following lemmas let $\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$ be an argumentation framework. The framework is induced by the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ (with corresponding calculus C), the set of $\mathcal{L}$-formulas $\mathcal{S}$, the set of assumptions $\mathcal{AS}$, the priority ordering π on formulas in $\mathcal{L}$ ($\leq_\pi$ is based on π) and the attack rules $\mathrm{AT}_{\mathcal{AS}}^{\star;\leq_\pi}$ and $\mathrm{AT}_{\mathcal{AS}}^{\mathsf{Con}}$, where $\star \in \{\Rightarrow, \Leftrightarrow\}$. Moreover, let $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$. In view of Remark 5 and Lemma 2, suppose that $\mathcal{S}$ is consistent.

Before proving the rationality postulates for assumptive sequent-based argumentation, two helpful lemmas are considered. The first shows that an argument a is only $\mathrm{AT}_{\mathcal{AS}}^{\mathsf{Con}}$-attacked if its set of assumptions is inconsistent with the set of facts. The second lemma shows that the set of assumptions from an extension together with the set of facts is always consistent. Together with the rationality postulates, these are good properties to have: the arguments that are accepted in the end should have no assumptions that are conflicting with the facts.

Lemma 3. *$a = A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ is $\mathrm{AT}_{\mathcal{AS}}^{\mathsf{Con}}$-attacked iff $A \cup \mathcal{S}$ is inconsistent.*

Proof. Let $a = A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ and

- $\Rightarrow$ suppose that a is $\mathrm{AT}_{\mathcal{AS}}^{\mathsf{Con}}$-attacked. Thus there is some $\Delta \subseteq \mathcal{S}$ such that $\Delta \Rightarrow \neg \bigwedge A$ is derivable in C. Hence, by $[\not\Rightarrow\neg\wedge]$ $A, \Delta \Rightarrow$ is derivable, by $[\Rightarrow\neg\wedge]$ it follows that $\Rightarrow \neg \bigwedge(A \cup \Delta)$ is derivable. Thus, by the soundness of C for L, $\vdash \neg \bigwedge(A \cup \Delta)$. Therefore, by Definition 2, $A \cup \mathcal{S}$ is inconsistent.

- $\Leftarrow$ now suppose that $A \cup \mathcal{S}$ is inconsistent. Then there are $\phi_1, \ldots, \phi_n \in A \cup \mathcal{S}$ such that $\vdash \neg \bigwedge_{i=1}^n \phi_i$. Note that $\{\phi_1, \ldots, \phi_n\} \cap A \neq \emptyset$, since $\mathcal{S}$ is consistent by assumption. Thus, by the completeness of C for L, $\Rightarrow \neg \bigwedge_{i=1}^n \phi_i$. Hence, by $[\not\Rightarrow\neg\wedge]$, $\phi_1, \ldots, \phi_n \Rightarrow$ is derivable in C. Let $\{\phi_1, \ldots, \phi_n\} \cap \mathcal{S} = \Delta$. By [LMon], $\Delta, A \Rightarrow$ is derivable and, by $[\Rightarrow\neg\wedge]$, $\Delta \Rightarrow \neg \bigwedge A$ is derivable. Hence a is $\mathrm{AT}_{\mathcal{AS}}^{\mathsf{Con}}$-attacked. ☐

Lemma 4 (Consistency of the assumptions)**.** *Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}))$, then $\mathsf{Ass}(\mathcal{E}) \cup \mathcal{S}$ is consistent.*

Proof. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}))$ and suppose, towards a contradiction, that $\mathsf{Ass}(\mathcal{E}) \cup \mathcal{S}$ is not consistent. Then there is a minimal set of formulas $\Gamma = \{\phi_1, \ldots,$

$\phi_n\} \subseteq \mathsf{Ass}(\mathcal{E})$ such that there are formulas $\psi_1, \ldots, \psi_m \in \mathcal{S}$ for which $\vdash \neg(\bigwedge_{i=1}^{n} \phi_i \wedge \bigwedge_{j=1}^{m} \psi_j)$. Note that $n \geq 1$, since $\mathcal{S}$ is consistent by assumption.

By the completeness of C for L, it follows that $\Rightarrow \neg(\bigwedge_{i=1}^{n} \phi_i \wedge \bigwedge_{j=1}^{m} \psi_j)$ is derivable in C. By [$\not\Rightarrow\neg\wedge$] $\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi_m \Rightarrow$ is derivable in C. Let $\phi_i \in \{\phi_1, \ldots, \phi_n\}$ be such that $\pi(\phi_i) = \max_\pi(\{\phi_1, \ldots, \phi_n\})$. By [$\Rightarrow\neg$] and $\mathrm{AS}^l_{\mathcal{AS}}$ $a = \phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_n \wr \psi_1, \ldots, \psi_m \Rightarrow \neg\phi_i$. Note that a cannot be $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$-attacked. This follows since $\mathsf{Ass}(a) = \{\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_n\}$ and thus $\mathsf{Ass}(a) \subsetneq \Gamma$, but Γ was assumed to be minimal. Since $\phi_1, \ldots, \phi_n \in \mathsf{Ass}(\mathcal{E})$, any attacker of a is an attacker of some $a' \in \mathcal{E}$. Therefore, because $\mathcal{E}$ is a completeness-based extension, $a \in \mathcal{E}$. Recall that ϕ_i was chosen such that $\max_\pi(\mathsf{Ass}(a)) \leq \pi(\phi_i)$. Since $\phi_i \in \mathsf{Ass}(\mathcal{E})$, there is some $b \in \mathcal{E}$ such that $\phi_i \in \mathsf{Ass}(b)$. Thus a attacks b. A contradiction to the conflict-freeness of $\mathcal{E}$. □

With this the rationality postulates from Definition 16 can be shown.

Lemma 5 (Closure). *$\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ satisfies closure of extensions: for each extension $\mathcal{E} \in \mathsf{Ext_{sem}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ it holds that $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$.*

Proof. ($\subseteq$) This follows immediately by the reflexivity of $\vdash$.

($\supseteq$) Now suppose that $\phi \in \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$. Thus there are $\phi_1, \ldots, \phi_n \in \mathsf{Concs}(\mathcal{E})$ such that $\phi_1, \ldots, \phi_n \vdash \phi$ and $a_i = A_i \wr \Gamma_i \Rightarrow \phi_i \in \mathcal{E}$ for each $i \in \{1, \ldots, n\}$. Since C is complete for L, it follows that $\phi_1, \ldots, \phi_n \Rightarrow \phi$ is derivable in C. Thus, by [Cut], from the a_i's $a = A_1, \ldots, A_n \wr \Gamma_1, \ldots, \Gamma_n \Rightarrow \phi$ is derivable in C'. If a is not attacked (e.g., because $\mathsf{Ass}(a) = \emptyset$) it follows immediately that $a \in \mathcal{E}$ thus that $\phi \in \mathsf{Concs}(\mathcal{E})$. Now suppose that a is attacked by some $b \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$. Note that, by Lemma 3, this is not an $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$-attack, since by Lemma 4, $\mathsf{Ass}(\mathcal{E}) \cup \mathcal{S}$ is consistent and $\mathsf{Ass}(a) \subseteq \mathsf{Ass}(\mathcal{E})$. Thus there is some $\psi \in A_i$, for some $i \in \{1, \ldots, n\}$ such that $\mathsf{Conc}(b) \Rightarrow \neg\psi$ and $\max_\pi(\mathsf{Ass}(b)) \leq \pi(\psi)$. It follows immediately that b attacks a_i as well. Since $a_i \in \mathcal{E}$ and $\mathcal{E}$ is complete, it follows that $a \in \mathcal{E}$ as well. Therefore $\phi \in \mathsf{Concs}(\mathcal{E})$. □

Rather than showing sub-argument closure directly, a stronger property is shown: an argument constructed from assumptions that other arguments in an extension already contain is also part of that extension.

Lemma 6 (Assumption inclusion). *$\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ satisfies assumption inclusion: for $\mathcal{E} \in \mathsf{Ext_{sem}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ and $a \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, if $\mathsf{Ass}(a) \subseteq \mathsf{Ass}(\mathcal{E})$, then $a \in \mathcal{E}$.*

Proof. Let $a = A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ such that $\mathsf{Ass}(a) \subseteq \mathsf{Ass}(\mathcal{E})$. Suppose there is some $b = A' \wr \Delta \Rightarrow \psi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ such that b attacks a (if no such attacker exists it follows immediately that $a \in \mathcal{E}$). By Lemma 3, this is not an $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$-attack, since $\mathsf{Ass}(a) \subseteq \mathsf{Ass}(\mathcal{E})$, hence by Lemma 4, $A \cup \mathcal{S}$ is consistent. Thus there is some $\gamma \in A$ such that $\psi \Rightarrow \neg\gamma$ and $\max_\pi(A') \leq \pi(\gamma)$. Since $\mathsf{Ass}(a) \subseteq \mathsf{Ass}(\mathcal{E})$, there is some $c \in \mathcal{E}$ such that $\gamma \in \mathsf{Ass}(c)$. Thus b attacks c as well. Therefore, since $\mathcal{E}$ is assumed to be complete, $\mathcal{E}$ defends c and thus a from the attack by b. It follows that $a \in \mathcal{E}$. □

From the lemma above it follows immediately that an extension is the set of arguments constructed from $\mathcal{S}$ and some $\mathcal{AS}' \subseteq \mathcal{AS}$.

Corollary 1. *For any $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ there is some $\mathcal{AS}' \subseteq \mathcal{AS}$ such that $\mathcal{E} = Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}')$.*

Proof. First note that, for any $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$, there is always some $\mathcal{AS}' \subseteq \mathcal{AS}$ such that $\mathcal{E} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}')$. In particular, $\mathcal{E} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathsf{Ass}(\mathcal{E}))$. Now let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ and let $\mathcal{AS}' = \mathsf{Ass}(\mathcal{E})$. Consider some $a \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}')$, thus $\mathsf{Ass}(a) \subseteq \mathcal{AS}'$. By Lemma 6 it follows immediately that $a \in \mathcal{E}$. Hence, $\mathcal{E} \supseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}')$ as well. □

The following lemma is a corollary of the above result:

Lemma 7 (Sub-argument closure)**.** *$\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ satisfies sub-argument closure: let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$, then $a \in \mathcal{E}$ implies that $\mathsf{Sub}(a) \subseteq \mathcal{E}$.*

Lemma 8 (Consistency)**.** *$\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ satisfies consistency: for each extension $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ it holds that $\mathsf{Concs}(\mathcal{E})$ is consistent.*

Proof. Suppose, towards a contradiction, that $\mathsf{Concs}(\mathcal{E})$ is not consistent. Then there are $\phi_1, \ldots, \phi_n \in \mathsf{Concs}(\mathcal{E})$ such that $\vdash \neg \bigwedge_{i=1}^n \phi_i$. By the completeness of C for L, $\Rightarrow \neg \bigwedge_{i=1}^n \phi_i$ is derivable in C. Hence, there are arguments $a_1, \ldots, a_n \in \mathcal{E}$ such that $a_i = A_i \wr \Gamma_i \Rightarrow \phi_i$ for $i \in \{1, \ldots, n\}$. Then, by [$\not\Rightarrow\neg\wedge$], $\phi_1, \ldots, \phi_n \Rightarrow$ is derivable and, by [Cut], so is $a = A_1, \ldots, A_n \wr \Gamma_1, \ldots, \Gamma_n \Rightarrow$. By construction $\mathsf{Ass}(a) \subseteq \mathsf{Ass}(\mathcal{E})$. Thus, by Lemma 6, $a \in \mathcal{E}$. However, by Remark 1 and [$\Rightarrow\neg\wedge$], $\Rightarrow \neg \bigwedge_{i=1}^n (A_i \cup \Gamma_i)$ is derivable in C. A contradiction to Lemma 4. Thus $\mathsf{Concs}(\mathcal{E})$ is consistent. □

From these lemmas the next theorem follows.

Theorem 1. *Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with corresponding sound and complete sequent calculus C in which the rules from Figure 6 are admissible, let $\mathcal{S}$ be a consistent set of $\mathcal{L}$-formulas, $\mathcal{AS}$ a set of assumptions and let π be a priority function on the formulas in $\mathcal{L}$. Moreover, let $\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) = \langle Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$ be the corresponding argumentation framework, with $AT^{\star, \leq_\pi}_{\mathcal{AS}}$ and $AT^{\mathsf{Con}}_{\mathcal{AS}}$ as the attack rules, where $\star \in \{\Rightarrow, \Leftrightarrow\}$. Then $\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ satisfies closure of extensions, sub-argument closure and consistency for completeness-based semantics.*

3.3 Maximally Consistent Subsets with Assumptions

In many reasoning contexts, the provided information is inconsistent. A well-known way to maintain consistency when given an inconsistent set of formulas is by means of reasoning with maximally consistent subsets, as introduced in [60]. The representation of reasoning with maximally consistent subsets by means of structured argumentation approaches has been studied in e.g. [2, 32, 40, 41], see [6] for a survey. Moreover, this kind of reasoning has been applied in several areas of artificial intelligence, such as knowledge-based integration systems [15], consistency operators for belief revision [46] and computational linguistics [49]. It is therefore useful to study the representation of reasoning with maximally consistent subsets in assumptive sequent-based argumentation as well. To do so, the notion of a maximally consistent subset has to be adjusted to account for the two sets of premises: facts ($\mathcal{S}$) and assumptions ($\mathcal{AS}$). Following [8], in this section we suppose that both sets are finite. First some basic notions and the entailment relations are recalled.

Notation 2. The set of all maximally consistent subsets of $\mathcal{S}$ for the logic L is denoted by $\mathsf{MCS}_{\mathsf{L}}(\mathcal{S})$. The subscript is omitted when arbitrary or clear from the context.

Definition 17. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and $\mathcal{S}$ a set of $\mathcal{L}$-formulas. Several entailment relations for reasoning with maximally consistent subsets are defined as follows:

- $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{mcs}} \phi$ iff $\phi \in \mathsf{CN}(\bigcap \mathsf{MCS}(\mathcal{S}))$;
- $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{mcs}} \phi$ iff $\phi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}(\mathcal{S})} \mathsf{CN}(\mathcal{T})$;
- $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{mcs}} \phi$ iff $\phi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}(\mathcal{S})} \mathsf{CN}(\mathcal{T})$.

Example 12. Consider the set $\mathcal{S} = \{p, p \supset q, \neg q\}$ and core logic CL, as in Example 4. Then there are three maximally consistent subsets: $\mathsf{MCS}(\mathcal{S}) = \{\{p, p \supset q\}, \{p, \neg q\}, \{p \supset q, \neg q\}\}$. Hence $\bigcap \mathsf{MCS}(\mathcal{S}) = \emptyset$. Moreover, $\mathcal{S} \mid\!\sim^{\cap}_{\mathsf{mcs}} \phi$ if and only if ϕ is a CL-tautology. But $\mathcal{S} \mid\!\sim^{\cup}_{\mathsf{mcs}} \psi$ for $\psi \in \mathcal{S}$ (since for each $\psi \in \mathcal{S}$ there is a $\mathcal{T} \in \mathsf{MCS}(\mathcal{S})$ such that $\psi \in \mathcal{T}$) and $\mathcal{S} \mid\!\sim^{\Cap}_{\mathsf{mcs}} p \vee \neg q$ (since from each $\mathcal{T} \in \mathsf{MCS}(\mathcal{S})$ the formula $p \vee \neg q$ is derivable, thus $p \vee \neg q \in \mathsf{CN}_{\mathsf{CL}}(\mathcal{T})$ for each $\mathcal{T} \in \mathsf{MCS}(\mathcal{S})$).

Recently it was shown that sequent-based argumentation (as recalled in Section 2.2) is a useful platform to incorporate reasoning with maximally consistent subsets [9].

Proposition 1 ([9], Propositions 3.8 and 4.3)**.** *Let $\mathcal{AF}_{\mathsf{CL},\{\mathsf{Ucut}\}}(\mathcal{S}) = \langle Arg_{\mathsf{L}}(\mathcal{S}), \mathcal{AT} \rangle$, take classical logic as core logic, Undercut as attack rule and let $\mathcal{S}$ be a set of formulas:*

- $\mathcal{S} \mathrel{|\!\sim}_{\mathsf{grd}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{prf}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{stb}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{mcs}} \phi$
- $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{prf}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{stb}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{mcs}} \phi$.

For $\mathcal{AF}_{\mathsf{CL},\{\mathsf{DUcut}\}}(\mathcal{S}) = \langle Arg_{\mathsf{L}}(\mathcal{S}), \mathcal{AT}\rangle$, *with Direct Undercut as attack rule, classical logic as core logic and* $\mathcal{S}$ *a set of formulas, it was shown that:*

- $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{prf}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{stb}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{mcs}} \phi$.

Indeed, the results from Examples 5 and 12 coincide.

Following the previous section, it will be assumed that $\mathcal{S}$ is consistent. To allow for assumptions, the set $\mathsf{MCS}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ is defined, which takes an additional set of formulas ($\mathcal{AS}$) as input. Then $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ iff $\mathcal{T} \subseteq \mathcal{AS}$, $\mathcal{T} \cup \mathcal{S}$ is consistent and there is no $\mathcal{T} \subset \mathcal{T}' \subseteq \mathcal{AS}$ such that $\mathcal{T}' \cup \mathcal{S}$ is consistent. Thus, $\mathsf{MCS}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ is the set of all maximally consistent subsets of $\mathcal{AS}$ that are consistent with $\mathcal{S}$. The entailment relations are adjusted as follows:

Definition 18. Let $\mathsf{L} = \langle \mathcal{L}, \vdash\rangle$, $\mathcal{S}$ a consistent set of $\mathcal{L}$-formulas and $\mathcal{AS}$ a set of assumptions.

- $\mathcal{S} \mathrel{|\!\sim}^{\cap,\mathcal{AS}}_{\mathsf{mcs}} \phi$ iff $\phi \in \mathsf{CN}(\bigcap \mathsf{MCS}(\mathcal{S}, \mathcal{AS}) \cup \mathcal{S})$;
- $\mathcal{S} \mathrel{|\!\sim}^{\cup,\mathcal{AS}}_{\mathsf{mcs}} \phi$ iff $\phi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{AS})} \mathsf{CN}(\mathcal{S} \cup \mathcal{T})$;
- $\mathcal{S} \mathrel{|\!\sim}^{\Cap,\mathcal{AS}}_{\mathsf{mcs}} \phi$ iff $\phi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{AS})} \mathsf{CN}(\mathcal{S} \cup \mathcal{T})$.

Example 13. Let CL be the core logic, $\mathcal{S} = \{p\}$ and $\mathcal{AS} = \{p \supset q, \neg q\}$. Recall that in Example 12, where there was no distinction between facts and defeasible assumptions, there where three maximally consistent subsets. Now, given the distinction, there are two: $\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}, \mathcal{AS}) = \{\{p \supset q\}, \{\neg q\}\}$. Therefore, $\mathcal{S} \mathrel{|\!\sim}^{\cap,\mathcal{AS}}_{\mathsf{mcs}} \phi$ iff $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{p\})$, this is the case since p is now a fact and thus should always follow. However, $\mathcal{S} \mathrel{|\!\sim}^{\cup,\mathcal{AS}}_{\mathsf{mcs}} \psi$, for $\psi \in \{p \supset q, \neg q\}$.

In order to generalize reasoning with maximally consistent subsets to the prioritized setting, we define an ordering on sets of $\mathcal{L}$-formulas:

Definition 19. Let $\Gamma, \Delta \subseteq \mathcal{L}$ and let π be a priority function on $\mathcal{L}$. Where $\pi_j(\Gamma) = \{\phi \in \Gamma \mid \pi(\phi) = j\}$, $\Gamma \preceq_\pi \Delta$ if and only if there is some $i \geq 1$, such that $\pi_i(\Gamma) \supsetneq \pi_i(\Delta)$ and for each $j < i$, $\pi_j(\Gamma) = \pi_j(\Delta)$. When $\Delta \not\preceq_\pi \Gamma$, then $\Gamma \prec_\pi \Delta$.

Remark 6. The ordering on sets of formulas from Definition 19 is transitive: if $\mathcal{S}_1 \preceq_\pi \mathcal{S}_2$ and $\mathcal{S}_2 \preceq_\pi \mathcal{S}_3$ then $\mathcal{S}_1 \preceq_\pi \mathcal{S}_3$.

With this, the set of the $\preceq_\pi$-most preferred maximally consistent subsets can be defined:

Definition 20. $\mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S},\mathcal{AS}) = \{\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S},\mathcal{AS}) \mid \nexists \mathcal{T}' \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S},\mathcal{AS})$ such that $\mathcal{T}' \prec_\pi \mathcal{T}\}$.

Example 14. Consider again CL as the core logic, $\mathcal{S} = \{p\}$ and $\mathcal{AS} = \{p \supset q, \neg q\}$. Let π be the priority function from Example 10, where $\pi(p \supset q) = 2$ and $\pi(\neg q) = 3$. Then $\mathsf{MCS}^{\preceq}_{\mathsf{CL}}(\mathcal{S},\mathcal{AS}) = \{\{p \supset q\}\}$.

Now consider $\mathcal{S} = r$ and $\mathcal{AS} = \{p, q, \neg p \vee \neg q\}$. There are three maximally consistent subsets: $\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S},\mathcal{AS}) = \{\{p, q\}, \{p, \neg p \vee \neg q\}, \{q, \neg p \vee \neg q\}\}$. Consider two cases:

- Let $\pi(p) = 1$, $\pi(q) = 2$ and $\pi(\neg p \vee \neg q) = 3$. Then $\{p, q\} \prec_\pi \{p, \neg p \vee \neg q\} \prec_\pi \{q, \neg p \vee \neg q\}$. Thus $\mathsf{MCS}^{\preceq}_{\mathsf{CL}}(\mathcal{S},\mathcal{AS}) = \{\{p, q\}\}$.
- If $\pi(p) = \pi(q) = 2$ and $\pi(\neg p \vee \neg q) = 1$, then $\{p, \neg p \vee \neg q\}$ and $\{q, \neg p \vee \neg q\}$ are incomparable and both are strictly preferred to $\{p, q\}$. Thus $\mathsf{MCS}^{\preceq}_{\mathsf{CL}}(\mathcal{S},\mathcal{AS}) = \{\{p, \neg p \vee \neg q\}, \{q, \neg p \vee \neg q\}\}$.

The prioritized counterparts of the entailment relations from Definition 18 are defined as:

Definition 21. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, $\mathcal{S}$ a consistent set of $\mathcal{L}$-formulas, $\mathcal{AS}$ a set of assumptions and π a priority function on $\mathcal{L}$.

- $\mathcal{S} \mathrel{|\!\sim}^{\cap,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$ iff $\phi \in \mathsf{CN}(\bigcap \mathsf{MCS}^{\preceq}(\mathcal{S},\mathcal{AS}) \cup \mathcal{S})$;
- $\mathcal{S} \mathrel{|\!\sim}^{\cup,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$ iff $\phi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}^{\preceq}(\mathcal{S},\mathcal{AS})} \mathsf{CN}(\mathcal{S} \cup \mathcal{T})$;
- $\mathcal{S} \mathrel{|\!\sim}^{\Cap,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$ iff $\phi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}^{\preceq}(\mathcal{S},\mathcal{AS})} \mathsf{CN}(\mathcal{S} \cup \mathcal{T})$.

Example 15. Recall from Example 14, that for $\mathcal{S} = \{p\}$, $\mathcal{AS} = \{p \supset q, \neg q\}$, π such that $\pi(p \supset q) = 2$ and $\pi(\neg q) = 3$, there is only one assumptive maximally consistent subset: $\mathsf{MCS}^{\preceq}_{\mathsf{CL}}(\mathcal{S},\mathcal{AS}) = \{\{p \supset q\}\}$. It thus follows that $\mathcal{S} \mathrel{|\!\sim}^{\star,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$, where $\star \in \{\cup, \cap, \Cap\}$ iff $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{p, p \supset q\})$.

For the last setting from Example 14, where $\mathcal{S} = r$ and $\mathcal{AS} = \{p, q, \neg p \vee \neg q\}$ such that $\pi(p) = \pi(q) = 2$ and $\pi(\neg p \vee \neg q) = 1$ note that $\mathcal{S} \mathrel{|\!\sim}^{\star,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$ for $\star \in \{\cup, \cap, \Cap\}$ and $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{r, \neg p \vee \neg q\})$. Moreover $\mathcal{S} \mathrel{|\!\sim}^{\cup,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$ for $\phi \in \{p, q\}$.

The next theorem shows that it is no coincidence that the results from the first part of the previous example correspond to that of Example 11. Like in the previous section, in view of Remark 5 and Lemma 2, it will be assumed that $\mathcal{S}$ is consistent.

Theorem 2. *Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be such that the rules from Figure 6 are admissible in its corresponding calculus C, $\mathcal{S}$ a finite and consistent set of $\mathcal{L}$-formulas, $\mathcal{AS}$ a finite set of assumptions and π a priority function on $\mathcal{L}$. For $\mathcal{AF}_{\mathsf{L}}^{\leq_\pi}(\mathcal{S}, \mathcal{AS}) = \langle Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$ an argumentation framework, where $\mathcal{AT}$ is based on the attack rules $AT_{\mathcal{AS}}^{\star,\leq_\pi}$ and $AT_{\mathcal{AS}}^{\mathsf{Con}}$, with $\star \in \{\Rightarrow, \Leftrightarrow\}$:*

1. $\mathcal{S} \mid\!\sim_{\mathsf{mcs},\preceq}^{\cap,\mathcal{AS}} \phi$ *iff* $\mathcal{S} \mid\!\sim_{\mathcal{AS},\mathsf{grd}}^{\leq_\pi} \phi$ *iff* $\mathcal{S} \mid\!\sim_{\mathcal{AS},\mathsf{prf}}^{\leq_\pi,\cap} \phi$ *iff* $\mathcal{S} \mid\!\sim_{\mathcal{AS},\mathsf{stb}}^{\leq_\pi,\cap} \phi$
2. $\mathcal{S} \mid\!\sim_{\mathsf{mcs},\preceq}^{\cup,\mathcal{AS}} \phi$ *iff* $\mathcal{S} \mid\!\sim_{\mathcal{AS},\mathsf{prf}}^{\leq_\pi,\cup} \phi$ *iff* $\mathcal{S} \mid\!\sim_{\mathcal{AS},\mathsf{stb}}^{\leq_\pi,\cup} \phi$
3. $\mathcal{S} \mid\!\sim_{\mathsf{mcs},\preceq}^{\Cap,\mathcal{AS}} \phi$ *iff* $\mathcal{S} \mid\!\sim_{\mathcal{AS},\mathsf{prf}}^{\leq_\pi,\Cap} \phi$ *iff* $\mathcal{S} \mid\!\sim_{\mathcal{AS},\mathsf{stb}}^{\leq_\pi,\Cap} \phi$.

For the next lemmas, needed to prove the above theorem, suppose that the conditions from the theorem statement hold.

The first lemma shows that if there is an attack between two arguments, the union of the assumptions and support sets of these arguments is inconsistent.

Lemma 9. *Let $a_1, a_2 \in Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, if a_1 $AT_{\mathcal{AS}}^{\star,\leq_\pi}$-attacks a_2, then $\mathsf{Ass}(a_1) \cup \mathsf{Ass}(a_2) \cup \mathsf{Supp}(a_1) \cup \mathsf{Supp}(a_2)$ is inconsistent.*

Proof. Let $a_1, a_2 \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ and suppose that $a_1 = A \wr \Gamma \Rightarrow \phi$ $\mathrm{AT}_{\mathcal{AS}}^{\star,\leq_\pi}$-attacks a_2, thus $\phi \Rightarrow \neg\psi$, for some $\psi \in \mathsf{Ass}(a_2)$. Thus, by [Cut] $A \wr \Gamma \Rightarrow \neg\psi$ is derivable in C'. By [$\neg\Rightarrow$] and [$\neg\neg\not\Rightarrow$] it follows that $A \wr \Gamma, \psi \Rightarrow$ is derivable. Then, by Remark 1 and [$\Rightarrow\neg\wedge$] the sequent $\Rightarrow \neg \bigwedge(A \cup \Gamma \cup \{\psi\})$ is derivable in C. Hence, by the soundness of C for L it follows that $\vdash \neg \bigwedge(A \cup \Gamma \cup \{\psi\})$. Therefore $\mathsf{Ass}(a_1) \cup \mathsf{Ass}(a_2) \cup \mathsf{Supp}(a_1) \cup \mathsf{Supp}(a_2)$ is inconsistent. □

The next lemma shows that for any maximally consistent subset of assumptions (i.e, any member of $\mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S}, \mathcal{AS})$), no consistent set of assumptions can be strictly preferred over it.

Lemma 10. *Let $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S}, \mathcal{AS})$, if $\mathcal{AS}' \subseteq \mathcal{AS}$ is such that $\mathcal{AS}' \cup \mathcal{S}$ is consistent, then $\mathcal{AS}' \not\prec_\pi \mathcal{T}$.*

Proof. Let $\mathcal{T} \in \mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S}, \mathcal{AS})$ and $\mathcal{AS}' \subseteq \mathcal{AS}$ such that $\mathcal{AS}' \cup \mathcal{S}$ is consistent. Then there is some $\mathcal{AS} \supseteq \mathcal{AS}^* \supseteq \mathcal{AS}'$ such that $\mathcal{AS}^* \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$. Since $\mathcal{AS}' \subseteq \mathcal{AS}^*$, $\mathcal{AS}' = \mathcal{AS}^*$ or there is an $i \geq 1$ such that $\pi_i(\mathcal{AS}^*) \supsetneq \pi_i(\mathcal{AS}')$ and $\pi_j(\mathcal{AS}^*) = \pi_j(\mathcal{AS}')$ for each $j < i$ and thus $\mathcal{AS}^* \preceq_\pi \mathcal{AS}'$. By definition of $\mathsf{MCS}_{\mathsf{L}}^{\preceq}(\mathcal{S}, \mathcal{AS})$, $\mathcal{AS}^* \not\prec_\pi \mathcal{T}$. Thus in both cases, by Remark 6, $\mathcal{AS}' \not\prec_\pi \mathcal{T}$. □

With help from the above two lemmas, the next three lemmas show how maximally consistent subsets of assumptions are related to grounded (Lemma 11), stable (Lemma 12) and preferred (Lemma 13) extensions.

Lemma 11. *If* $A \subseteq \bigcap \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ *and* $a = A \wr \Gamma \Rightarrow \phi \in Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$*, for some* $\Gamma \subseteq \mathcal{S}$*, then* $a \in \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$*.*

Proof. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ and $A \subseteq \bigcap \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ and suppose that $a = A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ for some $\Gamma \subseteq \mathcal{S}$. Suppose that there is some $b \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, such that b attacks a. Note that, since $A \subseteq \bigcap \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, $A \cup \mathcal{S}$ is consistent. Thus, by Lemma 3, b cannot $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$-attack a. By Lemma 9, it follows that $A \cup \Gamma \cup \mathsf{Ass}(b) \cup \mathsf{Supp}(b)$ is inconsistent. Also since $A \subseteq \bigcap \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, $\Gamma \cup \mathsf{Ass}(b) \cup \mathsf{Supp}(b)$ is inconsistent. Therefore, there are $\psi_1, \ldots, \psi_n \in \Gamma \cup \mathsf{Ass}(b) \cup \mathsf{Supp}(b)$, such that $\vdash \neg(\psi_1 \wedge \ldots \wedge \psi_n)$. Note that, since $\Gamma, \mathsf{Supp}(b) \subseteq \mathcal{S}$ and $\mathcal{S}$ is consistent by assumption $\mathsf{Ass}(b) \cap \{\psi_1, \ldots, \psi_n\} \neq \emptyset$.

Suppose, wlog., that $\Delta = \{\psi_1, \ldots, \psi_l\} \subseteq \mathcal{S}$ and $\{\psi_{l+1}, \ldots, \psi_n\} \subseteq \mathsf{Ass}(b)$. Then, by the completeness of C for L and $[\not\Rightarrow\neg\wedge]$, $\Delta, \psi_{l+1}, \ldots, \psi_n \Rightarrow$ is derivable in C. By [LMon], and $[\Rightarrow\neg\wedge]$ $c = \Delta \Rightarrow \neg \bigwedge \mathsf{Ass}(b)$ is derivable in C. Since $\mathsf{Ass}(c) = \emptyset$ and $\Delta \subseteq \mathcal{S}$ it follows that $c \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$. This also means that c cannot be attacked, therefore $c \in \mathcal{E}$. From this it follows that b is $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$-attacked by $\mathcal{E}$. Thus $\mathcal{E}$ defends a from any attacker. Moreover, since $\mathcal{E}$ was arbitrarily chosen, a is part of any complete extension. Recall that the grounded extension is the $\subseteq$-minimal complete extension. Therefore $a \in \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$. □

The proofs of the following two lemmas are based on proofs in [45].

Lemma 12. *If* $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$*, then* $Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$*.*

Proof. Let $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ and let $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$. In what follows we show that $\mathcal{E} \in \mathsf{Ext}_{\mathsf{stb}}\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, by showing that $\mathcal{E}$ is conflict-free and stable.[7]

$\mathcal{E}$ **is conflict-free.** Suppose towards a contradiction, that $\mathcal{E}$ is not conflict-free. Then there are $a_1, a_2 \in \mathcal{E}$ such that $a_1 = A_1 \wr \Gamma_1 \Rightarrow \phi_1$; $a_2 = A_2 \wr \Gamma_2 \Rightarrow \phi_2$ and a_1 $\mathrm{AT}^{\star;\leq_\pi}_{\mathcal{AS}}$-attacks a_2, for $\star \in \{\Rightarrow, \Leftrightarrow, \mathsf{Con}\}$. Since $A_2 \subseteq \mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, $A_2 \cup \mathcal{S}$ is consistent. Thus, by Lemma 3, this is not an $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$ attack. However then, by Lemma 9, $\mathsf{Ass}(a_1) \cup \mathsf{Ass}(a_2) \cup \mathsf{Supp}(a_1) \cup \mathsf{Supp}(a_2)$ is inconsistent, a contradiction to the assumption that $\mathsf{Ass}(a_1), \mathsf{Ass}(a_2) \subseteq \mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$. Thus $\mathcal{E}$ is conflict-free.

$\mathcal{E}$ **is stable.** Now suppose that there is some $b = A' \wr \Gamma' \Rightarrow \phi' \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) \setminus \mathcal{E}$ and $\mathcal{E}$ does not attack b. Thus, since $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ and $b \notin \mathcal{E}$, there is some $\phi \in \mathsf{Ass}(b)$ such that $\phi \notin \mathcal{T}$. Suppose first that $A' \cup \mathcal{S}$ is inconsistent. Then b is $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$-attacked by an argument that has an empty assumptions set and thus cannot be attacked itself. It follows immediately that $\mathcal{E}$ attacks b, a contradiction. Now

[7]The statements "$\mathcal{E}$ conflict-free and stable" (i.e., $\mathcal{E}$ is conflict-free and attacks all arguments not in it) and "$\mathcal{E}$ is complete and stable" are equivalent [17, Proposition 3.39].

suppose that $A' \cup \mathcal{S}$ is consistent. Since $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ (i.e., $\mathcal{T}$ is maximally consistent w.r.t. $\mathcal{S}$) and $\phi \notin \mathcal{T}$, $\mathcal{T} \cup \mathcal{S} \cup \{\phi\}$ is inconsistent. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots \subseteq \mathcal{T}$ be all the minimal subsets of $\mathcal{T}$ such that $\mathcal{C}_i \cup \mathcal{S} \cup \{\phi\}$ is inconsistent. Thus, for each i, there are $\psi^i_1, \ldots, \psi^i_{n_i} \in \mathcal{C}_i \cup \mathcal{S}$ such that $\vdash \neg(\psi^i_1 \wedge \ldots \wedge \psi^i_{n_i} \wedge \phi)$. By the completeness of C for L, $\Rightarrow \neg(\psi^i_1 \wedge \ldots \wedge \psi^i_{n_i} \wedge \phi)$ is derivable in C. By $[\not\Rightarrow\neg\wedge]$ and $[\Rightarrow\neg]$, $\psi^i_1, \ldots, \psi^i_{n_i} \Rightarrow \neg\phi$ is derivable in C. Let $A_i = \{\psi^i_1, \ldots, \psi^i_{n_i}\} \cap \mathcal{AS}$ and $\Gamma_i = \{\psi^i_1, \ldots, \psi^i_{n_i}\} \cap \mathcal{S}$. Note that, by assumption, $\mathcal{AS} \cap \mathcal{S} = \emptyset$ and $\{\psi^i_1, \ldots, \psi^i_{n_i}\} \subseteq \mathcal{AS} \cup \mathcal{S}$, hence $A_i \cup \Gamma_i = \{\psi^i_1, \ldots, \psi^i_{n_i}\}$. Thus, by $\mathrm{AS}^l_{\mathcal{AS}}$, $a_i = A_i \between \Gamma_i \Rightarrow \neg\phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$. However, since a_i does not attack b, $\max_\pi(A_i) \not\leq \pi(\phi)$, for all i.

Let $\Pi \subseteq \bigcup_{i \geq 1} \max_\pi(\mathcal{C}_i)$ be such that it contains at least one member of each $\{\psi \in \mathcal{C}_i \mid \pi(\psi) = \max_\pi(\mathcal{C}_i)\}$ and let $\Theta = (\mathcal{T} \setminus \Pi) \cup \{\phi\}$. Note that, since $\max_\pi(\mathcal{C}_i) > \pi(\phi)$ for each i, $\min_\pi(\Pi) > \pi(\phi)$, thus $\Theta \prec_\pi \mathcal{T}$. Suppose first that $\Theta \cup \mathcal{S}$ is not consistent. Then there are $\Theta' \subseteq \Theta$ and $\Delta \subseteq \mathcal{S}$ such that $\vdash \neg \bigwedge(\Theta' \cup \Delta)$. Note that $\phi \in \Theta'$, since $\Theta' \setminus \{\phi\} \subseteq \mathcal{T}$. Therefore, there is some i such that $\Theta' \setminus \{\phi\} \supseteq \mathcal{C}_i$. However, by construction, there is some $\psi \in \mathcal{C}_i$ such that $\psi \notin \Theta$. A contradiction. Therefore, $\Theta \cup \mathcal{S}$ is consistent. Hence, by Lemma 10 $\Theta \not\prec_\pi \mathcal{T}$. Also a contradiction. Therefore, $\mathcal{E}$ attacks b, from which it follows that $\mathcal{E}$ is stable. ☐

Lemma 13. *Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$, then there is some $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ such that $\mathcal{E} = Arg_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$.*

Proof. Suppose, for a contradiction, that there is some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ such that there is no $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ for which $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$. Note that, by Corollary 1, there is some $\mathcal{AS}' \subseteq \mathcal{AS}$ such that $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}')$. If $\mathcal{AS}' \cup \mathcal{S}$ would be inconsistent we have an immediate contradiction with Lemma 4. Hence, there is some $\mathcal{T}' \in \mathsf{MCS}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ such that $\mathcal{AS}' \subseteq \mathcal{T}'$. If $\mathcal{T}' \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, then by Lemma 12 $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}') \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\leq\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ and therefore (by [36, Lemma 15] any stable extension is a preferred extension) $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}') \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$, a contradiction with the assumption that no such set exists. Therefore there is some $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ for which $\mathcal{T} \prec_\pi \mathcal{T}'$. It follows that there is some i, such that $\pi_i(\mathcal{T}) \supsetneq \pi_i(\mathcal{T}')$ and for each $j < i$, $\pi_j(\mathcal{T}) = \pi_j(\mathcal{T}')$. Let $\Delta = \pi_i(\mathcal{T}) \setminus \pi_i(\mathcal{T}')$ and let $\mathsf{S} = \{a \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}' \cup \Delta) \mid \Gamma \Rightarrow \neg \bigwedge \mathsf{Ass}(a) \text{ is derivable for some } \Gamma \subseteq \mathcal{S}\}$.

Since $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}') = \mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$, by Lemma 3 none of the arguments in $\mathcal{E}$ are $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$-attacked thus $\mathcal{E} \cap \mathsf{S} = \emptyset$. Note that, since $\pi_j(\mathcal{AS}' \cup \Delta) = \pi_j(\mathcal{T})$ for $j \leq i$ and there is some $\psi \in \pi_i(\mathcal{AS}' \cup \Delta) \setminus \pi_i(\mathcal{AS}')$, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}' \cup \Delta) \setminus \mathsf{S} \neq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}')$ thus $\psi \between \Rightarrow \psi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}' \cup \Delta)$. We show that $\mathcal{E}' = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}' \cup \Delta) \setminus \mathsf{S}$ is admissible.

$\mathcal{E}'$ is conflict-free. To see this, note first that $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ and $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \Delta) \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\leq\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ are conflict-free. Suppose, for some arguments $a_1, a_2 \in \mathcal{E}'$, that a_1 attacks a_2. By the definition of $\mathcal{E}'$ this is not an $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$ attack. Thus $\mathsf{Conc}(a_1) \Rightarrow \neg\psi$ for $\psi \in \mathsf{Ass}(a_2)$ and $\max_\pi(\mathsf{Ass}(a_1)) \leq \pi(\psi)$. Suppose first that $\psi \in \Delta$. Since $\max_\pi(\mathsf{Ass}(a_1)) \leq \pi(\psi) = i$, it follows that $\mathsf{Ass}(a_1) \subseteq \mathcal{T}$. But then $a_1 \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$, a contradiction, since a_1 attacks any argument with ψ in the set of assumptions and $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ is a stable extension and thus conflict-free. Let now $\psi \in \mathcal{AS}'$. Then there is some $a_3 \in \mathcal{E}$ such that a_1 attacks a_3 as well. Thus $a_1 \notin \mathcal{E}$, since $\mathcal{E}$ is conflict-free. Since $a_3 \in \mathcal{E}$, there is an $a_4 \in \mathcal{E}$, such that a_4 attacks a_1 in some formula $\psi' \in \mathsf{Ass}(a_1) \cap \Delta$. Hence $\max_\pi(\mathsf{Ass}(a_4)) \leq \pi(\psi') = i$. But then $a_4 \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$, again a contradiction. Thus $\mathcal{E}'$ is conflict-free.

$\mathcal{E}'$ **is admissible**. Note that, since $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$, any attack in a formula in $\mathcal{AS}'$ is defended by $\mathcal{E}$. Let $a = A \wr \Gamma \Rightarrow \phi \in \mathcal{E}'$ be such that it is attacked by some $b = A' \wr \Gamma' \Rightarrow \phi' \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ in $\gamma \in A \cap \Delta$. Thus $\max_\pi(A') \leq \pi(\gamma)$. By Lemma 12, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$. Moreover, since $\gamma \in \mathcal{T}$, there is some $a' \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$, such that $\gamma \in \mathsf{Ass}(a')$. Hence there is some $c \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ such that c attacks b in some $\gamma' \in A'$. Thus $\max_\pi(\mathsf{Ass}(c)) \leq \pi(\gamma')$ and since $\gamma' \in A'$, $\max_\pi(\mathsf{Ass}(c)) \leq \pi(\gamma) = i$. Therefore $c \in \mathcal{E}'$ as well. It follows that $\mathcal{E}'$ defends itself against all attackers. Hence $\mathcal{E}'$ is admissible. Since $\mathcal{E}' \supsetneq \mathcal{E}$ this is a contradiction to the assumption that $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$.[8]

Therefore $\mathcal{E} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$, for some $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$. By Lemma 12 it follows that $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$, thus $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$. □

With the above lemmas Theorem 2 can be proven:

Proof. Let $\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT}, \leq_\pi \rangle$ be an assumptive sequent-based argumentation framework for the logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, such that the rules from Figure 6 are admissible in the corresponding calculus C. Let $\mathcal{S}$ be a finite and consistent set of $\mathcal{L}$-formulas, $\mathcal{AS}$ a finite set of assumptions, let ϕ be an $\mathcal{L}$-formula and suppose that π is a priority function on $\mathcal{L}$. Furthermore, let $\mathcal{AT}$ be based on the attack rules $\mathrm{AT}^{\star;\leq_\pi}_{\mathcal{AS}}$ and $\mathrm{AT}^{\mathsf{Con}}_{\mathcal{AS}}$, where $\star \in \{\Rightarrow, \Leftrightarrow\}$.

- ($\Rightarrow$) Note that $\mathcal{S} \mid\!\sim^{\leq_\pi}_{\mathcal{AS},\mathsf{grd}} \phi$ implies $\mathcal{S} \mid\!\sim^{\leq_\pi,\cap}_{\mathcal{AS},\mathsf{prf}} \phi$ implies $\mathcal{S} \mid\!\sim^{\leq_\pi,\cap}_{\mathcal{AS},\mathsf{stb}} \phi$. Suppose that $\mathcal{S} \mid\!\sim^{\cap,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$, thus there are $A \subseteq \bigcap \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ and $\Gamma \subseteq \mathcal{S}$, such that $A \cup \Gamma \vdash \phi$. By the completeness of C for L and Remark 1, $A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$. By Lemma 11, it follows that $A \wr \Gamma \Rightarrow \phi \in \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$. Therefore $\mathcal{S} \mid\!\sim^{\leq_\pi}_{\mathcal{AS},\mathsf{grd}} \phi$ and thus $\mathcal{S} \mid\!\sim^{\leq_\pi,\cap}_{\mathcal{AS},\mathsf{prf}} \phi$ and $\mathcal{S} \mid\!\sim^{\leq_\pi,\cap}_{\mathcal{AS},\mathsf{stb}} \phi$ as well.

[8] The statements "$\mathcal{E}$ is a $\subseteq$-maximal admissible set of $\mathcal{AF}$" and "$\mathcal{E}$ is a $\subseteq$-maximal complete extension of $\mathcal{AF}$" (the definition of preferred extensions in Definition 4) are equivalent [17, Proposition 3.35].

($\Leftarrow$) Suppose that $\mathcal{S} \mathrel{|\!\!\sim}^{\leq_\pi,\cap}_{\mathcal{AS},\mathsf{stb}} \phi$. Then there is an $a \in \bigcap \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ such that $\mathsf{Ass}(a) \subseteq \mathcal{AS}$, $\mathsf{Supp}(a) \subseteq \mathcal{S}$ and $\mathsf{Conc}(a) = \phi$. By Lemma 12, for each $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, $a \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$. Thus $\mathsf{Ass}(a) \subseteq \bigcap \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$. From a, by Remark 1 and the soundness of C for L, $\mathsf{Ass}(a) \cup \mathsf{Supp}(a) \vdash \phi$. Therefore, $\mathcal{S} \mathrel{|\!\!\sim}^{\cap,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$.

- ($\Rightarrow$) Note that $\mathcal{S} \mathrel{|\!\!\sim}^{\leq_\pi,\cup}_{\mathcal{AS},\mathsf{stb}} \phi$ implies $\mathcal{S} \mathrel{|\!\!\sim}^{\leq_\pi,\cup}_{\mathcal{AS},\mathsf{prf}} \phi$. Suppose that $\mathcal{S} \mathrel{|\!\!\sim}^{\cup,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$. Then, there is some $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ such that $\mathcal{T} \cup \mathcal{S} \vdash \phi$. By the completeness of C for L and Remark 1, there are $A \subseteq \mathcal{T}$ and $\Gamma \subseteq \mathcal{S}$ such that $A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$. By Lemma 12, $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$. Thus there is some stable extension $\mathcal{E}$, such that $\phi \in \mathsf{Concs}(\mathcal{E})$. Therefore $\mathcal{S} \mathrel{|\!\!\sim}^{\leq_\pi,\cup}_{\mathcal{AS},\mathsf{stb}} \phi$ and thus $\mathcal{S} \mathrel{|\!\!\sim}^{\leq_\pi,\cup}_{\mathcal{AS},\mathsf{prf}} \phi$.

  ($\Leftarrow$) Suppose that $\mathcal{S} \mathrel{|\!\!\sim}^{\leq_\pi,\cup}_{\mathcal{AS},\mathsf{prf}} \phi$. Then there is some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ such that there is some $A \wr \Gamma \Rightarrow \phi$ where $A \subseteq \mathcal{AS}$ and $\Gamma \subseteq \Gamma$. Hence, by Remark 1 and the soundness of C for L, $A \cup \Gamma \vdash \phi$. By Lemma 13, there is some $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ such that $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$. Thus $A \subseteq \mathcal{T}$. Hence, $\mathcal{S} \mathrel{|\!\!\sim}^{\cup,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$.

- ($\Rightarrow$) Note that $\mathcal{S} \mathrel{|\!\!\sim}^{\leq_\pi,\Cap}_{\mathcal{AS},\mathsf{prf}} \phi$ implies $\mathcal{S} \mathrel{|\!\!\sim}^{\leq_\pi,\Cap}_{\mathcal{AS},\mathsf{stb}} \phi$. Suppose that $\mathcal{S} \not\mathrel{|\!\!\sim}^{\leq_\pi,\Cap}_{\mathcal{AS},\mathsf{prf}} \phi$, then there is some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$ such that there is no $a \in \mathcal{E}$ with $\mathsf{Conc}(a) = \phi$. By Lemma 13, there is some $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$ such that $\mathcal{E} = \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$. Hence, there is no $A \subseteq \mathcal{T}$ and $\Gamma \subseteq \mathcal{S}$, such that $A \cup \Gamma \vdash \phi$. Therefore $\phi \notin \mathsf{CN}_{\mathsf{L}}(\mathcal{S} \cup \mathcal{T})$. Thus $\mathcal{S} \not\mathrel{|\!\!\sim}^{\Cap,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$.

  ($\Leftarrow$) Now suppose that $\mathcal{S} \not\mathrel{|\!\!\sim}^{\Cap,\mathcal{AS}}_{\mathsf{mcs},\preceq} \phi$. Thus there is some $\mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})$, such that there are no $A \subseteq \mathcal{T}$, $\Gamma \subseteq \mathcal{S}$, for which $A \cup \Gamma \Rightarrow \phi$. Hence there is no $a \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T})$ such that $\mathsf{Conc}(a) = \phi$. By Lemma 12, it follows that $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$. Therefore $\mathcal{S} \not\mathrel{|\!\!\sim}^{\leq_\pi,\Cap}_{\mathcal{AS},\mathsf{stb}} \phi$ and thus also $\mathcal{S} \not\mathrel{|\!\!\sim}^{\leq_\pi,\Cap}_{\mathcal{AS},\mathsf{prf}} \phi$. □

Remark 7. As can be seen from the results above, the preferred and stable extensions coincide, when the rules from Figure 6 are admissible in the calculus of the core logic. In fact, by Lemmas 12 and 13 $\mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})) = \{\mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{T}) \mid \mathcal{T} \in \mathsf{MCS}^{\preceq}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})\} = \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\leq_\pi}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}))$. Although it is possible that no stable extension exists in abstract argumentation (see [36]), assumptive sequent-based argumentation is not the only approach to logical argumentation in which the preferred and stable extensions coincide. For example, this is the case for instances of ASPIC^+ (see [51]), simple contrapositive assumption based argumentation (see [41]) and sequent-based argumentation (see [9]). For an overview see [6].

When stable and preferred extensions do not coincide in abstract argumentation, this is because of odd cycles in the argumentation framework. In, for example, ASPIC$^+$, such cycles may also exist, since the contrariness function might be one-sided. However, given the assumptions made to prove the results (i.e., because [$\Rightarrow\neg$], [$\neg\Rightarrow$] and [Cut] are admissible), such cycles do not cause these problems in the setting of the theorem. For example, a possible odd cycle might exist when $p \wedge \neg p \in \mathcal{AS}$, since then $p \wedge \neg p \wr \Rightarrow \neg(p \wedge \neg p)$ would be derivable with the rules from Figure 6. However, this cycle is attacked by $\Rightarrow \neg(p \wedge \neg p)$, which cannot be attacked.

In the next section the general framework defined here will be applied to several well-known approaches to nonmonotonic reasoning with assumptions.

4 Some Assumptive Approaches and Their Properties

We will consider three well-known frameworks for nonmonotonic reasoning with assumptions. Assumption-based argumentation in Section 4.1, adaptive logics in Section 4.2 and default assumptions in Section 4.3. For each of these approaches the representation by the introduced assumptive sequent-based approach, maximally consistent subsets, as well as the rationality postulates from [30] are discussed.

In this paper only the flat approaches are considered. On the one hand, because the objective of this paper is just to show that the presented assumptive frameworks are expressive enough to represent other approaches to reasoning with assumptions and, on the other hand, because there are often several possibilities to introduce priorities, for assumption-based argumentation see e.g., [35, 42] and for adaptive logics see e.g., [57, 58].

4.1 Assumption-Based Argumentation

Assumption-based argumentation (ABA) was introduced in [25], see [37, 64] for an introduction and an overview. In contrast to the other two examples that will be discussed, ABA is already defined in terms of argumentation frameworks. It takes as input a formal deductive system, a set of assumptions and a contrariness mapping for each assumption. There are only few requirements placed on each of these, keeping the framework semi-abstract on the one hand, while the arguments have a formal structure and the attacks are based on the latter. We first consider some of the most important definitions of the ABA-framework, from [25]:

Definition 22. A *deductive system* is a pair $\langle \mathcal{L}, \mathcal{R} \rangle$, where:

- $\mathcal{L}$ is a formal language;

- $\mathcal{R}$ is a set of rules of the form $\phi_1, \ldots, \phi_n \to \phi$, for $\phi_1, \ldots, \phi_n, \phi \in \mathcal{L}$ and $n \geq 0$.

Definition 23. A *deduction* from a theory Γ is a sequence $\psi_1, \ldots, \psi_m$, where $m > 0$, such that for all $i = 1, \ldots, m$, $\psi_i \in \Gamma$, or there is a rule $\phi_1, \ldots, \phi_n \to \psi_i \in \mathcal{R}$ with $\phi_1, \ldots, \phi_n \in \{\psi_1, \ldots, \psi_{i-1}\}$. A deduction from Γ using rules in $\mathcal{R}$ is denoted by $\Gamma \vdash^{\mathcal{R}} \psi_m$.

Clearly, a deductive system is not necessarily based on a logic in the sense of Section 2, thus the possible connectives do not necessarily have the properties they were assumed to have in the previous sections. However, in this section, the examples will be based on classical logic, in which the connectives have the properties as discussed after Definition 1.

Example 16. An example of a deductive system is classical logic, such that $\phi_1, \ldots, \phi_n \to \phi \in \mathcal{R}_{\mathsf{CL}}$ if and only if $\phi_1, \ldots, \phi_n \vdash_{\mathsf{CL}} \phi$. Thus, $\Gamma \vdash^{\mathcal{R}} \psi$ if and only if $\Gamma \vdash_{\mathsf{CL}} \psi$.

From this ABA argumentation frameworks can be defined:[9]

Definition 24. An *ABA-framework* is a tuple $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \bar{\cdot} \rangle$ where:

- $\langle \mathcal{L}, \mathcal{R} \rangle$ is a deductive system;
- $\mathcal{S} \subseteq \mathcal{L}$ a set of formulas, that satisfies non-triviality (there is some $\mathcal{L}$-formula ϕ such that $\mathcal{S} \nvdash^{\mathcal{R}} \phi$);
- $\mathcal{A} \subseteq \mathcal{L}$ a non-empty set of *assumptions* for which $\mathcal{S} \cap \mathcal{A} = \emptyset$; and
- $\bar{\cdot}$ a mapping from $\mathcal{A}$ into a set of $\mathcal{L}$-formulas, where $\overline{\phi}$ is the set of the *contrary* formulas of ϕ.

In the remainder, if a set of formulas $\mathcal{S}$ satisfies non-triviality, it is said that $\mathcal{S}$ is non-trivializing.

A simple way of defining contrariness in the context of classical logic is by $\overline{\phi} = \{\neg\phi\}$. In what follows, by $A', \Gamma \vdash^{\mathcal{R}} \overline{\phi}$ it is meant that there is some $\psi \in \overline{\phi}$ such that $A', \Gamma \vdash^{\mathcal{R}} \psi$. Moreover, to avoid clutter, the superscript $\mathcal{R}$ in $\vdash^{\mathcal{R}}$ is sometimes omitted.

The consistency notions from Definition 2 can be defined in terms of a contrariness function as well, in order to avoid confusion with the previously defined notion, we will refer to (maximally) contrary-consistent sets of assumptions:

Definition 25. Given an ABA-framework $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})$, a set $A \subseteq \mathcal{A}$ is:

[9] Note that not in all the literature on ABA the set of facts (in the notation of this paper $\mathcal{S}$) is part of the framework. Rather, these are special rules (so-called domain oriented rules), denoted by $\to \phi$ for $\phi \in \mathcal{S}$. Thus, one could understand $\mathcal{S}$ such that $\phi \in \mathcal{S}$ in our setting iff $\to \phi \in \mathcal{R}$ if a set of facts is not part of the framework.

- *contrary-consistent* if and only if there is no $\phi \in A$ such that $A', \Gamma \vdash^{\mathcal{R}} \overline{\phi}$ for some $A' \subseteq A$ and some $\Gamma \subseteq \mathcal{S}$;
- *maximally contrary-consistent*, denoted by $A \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$, if and only if A is contrary-consistent and there is no contrary-consistent A' such that $A \subset A' \subseteq \mathcal{A}$.

The *closure* of $\mathcal{T} \subseteq \mathcal{L}$ is defined as $\mathsf{CN}(\mathcal{T}) = \{\phi \mid \Gamma \vdash^{\mathcal{R}} \phi \text{ for } \Gamma \subseteq \mathcal{T}\}$.

ABA-arguments are defined in terms of deductions and an attack is on the assumptions of the attacked argument. Following [37], arguments are not required to be contrary-consistent.

Definition 26. Let $\mathcal{AF}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \bar{\cdot} \rangle$. An *ABA-argument* for $\phi \in \mathcal{L}$ is a deduction $A \cup \Gamma \vdash^{\mathcal{R}} \phi$, where $A \subseteq \mathcal{A}$ and $\Gamma \subseteq \mathcal{S}$. The set $\mathrm{Arg}^{\mathsf{ABA}}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S}, \mathcal{A})$ denotes the set of all ABA-arguments for $\mathcal{S}$ and $\mathcal{A}$.

Definition 27. Let $\mathcal{AF}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \bar{\cdot} \rangle$. An argument $A \cup \mathcal{S} \vdash^{\mathcal{R}} \phi$ *attacks* an argument $A' \cup \mathcal{S} \vdash^{\mathcal{R}} \phi'$ iff $\phi \in \overline{\psi}$ for some $\psi \in A'$.

Example 17. Recall the deductive system $\mathcal{R}_{\mathsf{CL}}$ for classical logic, described in Example 16 and let $\overline{\phi} = \{\neg\phi\}$. Consider the sets $\mathcal{S} = \{s\}$ and $\mathcal{A} = \{p, q, \neg p \vee \neg q, \neg p \vee r, \neg q \vee r\}$. Some of the arguments of $\mathcal{AF}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S}, \mathcal{A})$ are:

$$a = s \vdash s \qquad b = p, \neg p \vee \neg q \vdash \neg q$$
$$c = q, \neg p \vee \neg q \vdash \neg p \qquad d = p, q, \neg p \vee r, \neg q \vee r \vdash r.$$

Note that a cannot be attacked, since the set of assumptions of a is empty. For the other arguments, b attacks c and d, and c attacks b and d.

Semantics are defined as usual, see Definition 4. From this the corresponding entailment relations can be defined:

Definition 28. Let $\mathcal{AF}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \bar{\cdot} \rangle$ and $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$. Then:

- $\mathcal{A} \cup \mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{ABA,sem}} \phi$ iff for some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S}, \mathcal{A}))$ there is an argument $A \cup \Gamma \vdash^{\mathcal{R}} \phi \in \mathcal{E}$.
- $\mathcal{A} \cup \mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{ABA,sem}} \phi$ iff there is an $a \in \bigcap \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S}, \mathcal{A}))$, where $a = A \cup \Gamma \vdash^{\mathcal{R}} \phi$.
- $\mathcal{A} \cup \mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{ABA,sem}} \phi$ iff for every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S}, \mathcal{A}))$ there is an $a \in \mathcal{E}$ with $\mathsf{Conc}(a) = \phi$.

Example 18. Recall the setting from Example 17, where the deductive system was $\mathcal{R}_{\mathsf{CL}}$ from Example 16, $\mathcal{S} = \{s\}$ and $\mathcal{A} = \{p, q, \neg p \vee \neg q, \neg p \vee r, \neg q \vee r\}$. It can be shown that $\mathcal{A} \cup \mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathsf{ABA,sem}} s$, for $\star \in \{\cap, \cup, \Cap\}$, $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$, this follows since s is a fact. Furthermore, $\mathcal{A} \cup \mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{ABA,sem}} \phi$, but $\mathcal{A} \cup \mathcal{S} \not\mathrel{|\!\sim}^{\cap}_{\mathsf{ABA,sem}} \phi$ and $\mathcal{A} \cup \mathcal{S} \not\mathrel{|\!\sim}^{\Cap}_{\mathsf{ABA,sem}} \phi$ for $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ and $\phi \in \{p, q, \neg p \vee \neg q\}$, to see this, note that for each formula $\phi \in \{p, q, \neg p \vee \neg q\}$ there is an extension from which ϕ can be derived, but there is also an extension from which ϕ cannot be derived.

Based on the above notions from assumption-based argumentation, a corresponding sequent-based ABA-framework can be defined:

Definition 29. Let $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \bar{\cdot} \rangle$ be an ABA-framework. The corresponding *sequent-based ABA-framework* is defined as a pair$\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}) = \left\langle \mathrm{Arg}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}), \mathcal{AT} \right\rangle$, where:

- $\mathcal{R}_\Rightarrow$ is defined as follows:
  - if $\langle \mathcal{L}, \mathcal{R} \rangle$ is a logic in the sense of Definition 1 with corresponding sound and complete sequent calculus C in which [Cut] is admissible, $\mathcal{R}_\Rightarrow = \mathsf{C} \cup \{\mathrm{AS}_{\mathsf{ABA}}\}$ such that:

$$\frac{A \wr \Gamma, \phi \Rightarrow \psi}{A, \phi \wr \Gamma \Rightarrow \psi}\ \mathrm{AS}_{\mathsf{ABA}} \qquad \frac{A, \phi \wr \Gamma \Rightarrow \psi}{A \wr \Gamma, \phi \Rightarrow \psi}\ \mathrm{AS}_{\mathsf{ABA}} \qquad \text{where } \phi \in \mathcal{A}.$$

  - otherwise $\mathcal{R}_\Rightarrow = \{\mu(r) \mid r \in \mathcal{R}\} \cup \{\mathrm{AS}_{\mathsf{ABA}}, [\mathrm{Cut}], [\mathsf{id}]\}$ where, for each $r = \phi_1, \ldots, \phi_n \to \phi \in \mathcal{R}$

$$\frac{}{\phi_1, \ldots, \phi_n \Rightarrow \phi}\ \mu(r) \qquad \text{and} \qquad \frac{}{\phi \Rightarrow \phi}\ [\mathsf{id}]$$

- $a = A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$ for $A \subseteq \mathcal{A}$, $\Gamma \subseteq \mathcal{S}$ iff there is a derivation of a using rules in $\mathcal{R}_\Rightarrow$.
- $(a_1, a_2) \in \mathcal{AT}$ iff a_1 $\mathcal{R}$-attacks a_2 as defined in Definition 7, for $\mathsf{AR} = \{\mathrm{AT}_{\mathsf{ABA}}\}$ and:

$$\frac{A_1 \wr \Gamma_1 \Rightarrow \overline{\phi} \quad A_2, \phi \wr \Gamma_2 \Rightarrow \psi}{A_2, \phi \wr \Gamma_2 \not\Rightarrow \psi}\ \mathrm{AT}_{\mathsf{ABA}} \tag{2}$$

Remark 8. Similar to Remark 1, since the rules $\mathrm{AS}_{\mathsf{ABA}}$ are part of the calculus of any sequent-based ABA-framework: $A \cup \Gamma \Rightarrow \phi$ is derivable iff $A \wr \Gamma \Rightarrow \phi$ is derivable.

In the next example we show how classical logic, with corresponding sequent calculus LK can be taken as underlying deductive system.

Example 19. Let $\mathsf{CL} = \langle \mathcal{L}, \vdash \rangle$, where $\overline{\phi} = \{\neg\phi\}$ and $\mathcal{R}_{\Rightarrow} = \mathsf{LK}$. According to Definition 9 $A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{A})$ iff $\Gamma \cup A \Rightarrow \phi$ is derivable in LK, for some finite $A \subseteq \mathcal{A}$ and $\Gamma \subseteq \mathcal{S}$. Since $\mathcal{R}_{\Rightarrow} = \mathsf{LK} \cup \{\mathrm{AS}_{\mathsf{ABA}}\}$ it follows immediately that $A \cup \Gamma \Rightarrow \phi$ is derivable in $\mathcal{R}_{\Rightarrow}$ iff it is derivable in LK.

The next proposition formalizes the representation of ABA in assumptive sequent-based argumentation, via the above described translation.

Proposition 2. *Let $\langle \mathcal{L}, \mathcal{R} \rangle$ be a deductive system, $\mathcal{S} \subseteq \mathcal{L}$ a non-trivializing set of formulas and $\mathcal{A} \subseteq \mathcal{L}$ a set of assumptions, such that $\Gamma \subseteq \mathcal{S}$ and $A \subseteq \mathcal{A}$ are finite and $\mathcal{A} \cap \mathcal{S} = \emptyset$. Let $\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}) = \left\langle Arg^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}), \mathcal{AT} \right\rangle$ be a sequent-based ABA-framework that corresponds to the ABA-framework $\mathcal{AF}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A}) = \langle \mathcal{L}, \mathcal{R}, \mathcal{S}, \mathcal{A}, \bar{\cdot} \rangle$. $\mathcal{A} \cup \mathcal{S} \mid\!\sim^{\star}_{\mathsf{ABA},\mathsf{sem}} \phi$ iff $\mathcal{S} \mid\!\sim^{\star}_{\mathcal{A},\mathsf{sem}} \phi$ for $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ and $\star \in \{\cup, \cap, \Cap\}$.*

The above proposition is a corollary of the following two lemmas. Suppose that the conditions from the proposition statement hold:

Lemma 14. *$A \cup \Gamma \vdash^{\mathcal{R}} \phi \in Arg^{\mathsf{ABA}}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})$ iff $A \wr \Gamma \Rightarrow \phi \in Arg^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$.*

Proof. If the deductive system is a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, with corresponding sound and complete sequent calculus C, it follows that $A \cup \Gamma \vdash^{\mathcal{R}} \phi \in \mathrm{Arg}^{\mathsf{ABA}}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})$ iff $A \cup \Gamma \vdash_{\mathsf{L}} \phi$. By the soundness and completeness of C for L we have that $A \cup \Gamma \Rightarrow \phi$ is derivable iff $A \cup \Gamma \vdash_{\mathsf{L}} \phi$. And by Remark 8 it follows that, since $A \subseteq \mathcal{A}$ and $\Gamma \subseteq \mathcal{S}$, $A \cup \Gamma \Rightarrow \phi$ is derivable in C iff $A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$.

For other types of deductive systems, consider both directions:

- ⇒ Assume that $A \cup \Gamma \vdash^{\mathcal{R}} \phi \in \mathrm{Arg}^{\mathsf{ABA}}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})$. Then there is a deduction from the theory $A \cup \Gamma$ for the formula ϕ. By Definition 23, there is a sequence $\psi_1, \ldots, \psi_m$ ($\psi_m = \phi$), such that for each $i = 1, \ldots, m$, $\psi_i \in A \cup \Gamma$ or there is a rule $\phi_1, \ldots, \phi_n \to \psi_i = r \in R$ and $\phi_1, \ldots, \phi_n \in \{\psi_1, \ldots, \psi_{i-1}\}$. We proceed by induction on m, showing that for each ψ_i, there is a sequent $s_i = A_i \cup \Gamma_i \Rightarrow \psi_i$:

  m=1 Then either $\psi_1 \in A \cup \Gamma$ and thus $\psi_1 \Rightarrow \psi_1$ is derivable in $\mathcal{R}_{\Rightarrow}$, by [id]. Or there is a rule $\to \psi_1 \in R$. Hence $\Rightarrow \psi_1 \in \mathcal{R}_{\Rightarrow}$ for $A \cup \Gamma = \emptyset$. Since $\psi_1 = \psi_m = \phi$, $A \cup \Gamma \Rightarrow \phi$ is derivable.

  m=k+1 Assume that for sequences up to $k \geq 1$, for each ψ_i there is a sequent $s_i = A_i \cup \Gamma_i \Rightarrow \psi_i$. Now consider ψ_{k+1}. Then $\psi_{k+1} \in A \cup \Gamma$, from which it follows immediately that $A \cup \Gamma \Rightarrow \psi_{k+1}$ is derivable in $\mathcal{R}_{\Rightarrow}$, or there is a rule $\phi_1, \ldots, \phi_n \to \psi_{k+1} = r \in \mathcal{R}$ and $\phi_1, \ldots, \phi_n \in \{\psi_1, \ldots, \psi_k\}$. By Definition 29, $\overline{\phi_1, \ldots, \phi_n \Rightarrow \psi_{k+1}} \in \mathcal{R}_\Rightarrow$. Furthermore, by induction hypothesis, for each $\psi_i \in \{\psi_1, \ldots, \psi_k\}$, there is a sequent $s_i = A_i \cup \Gamma_i \Rightarrow \psi_i$. Hence, $\phi_1, \ldots, \phi_n \in \{\mathsf{Conc}(s_1), \ldots, \mathsf{Conc}(s_k)\}$. By applying [Cut] a sequent $s_{k+1} = A_{k+1} \cup \Gamma_{k+1} \Rightarrow \psi_{k+1}$ is obtained.

Hence, there is a sequence of sequents $s_1, \ldots, s_m$, such that s_i is derived from $s_1, \ldots, s_{i-1}$ by applying rules in $\mathcal{R}_\Rightarrow$ and $s_m = A \cup \Gamma \Rightarrow \phi$. That $A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$ follows by Remark 8.

$\Leftarrow$ Now suppose that $a = A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$. By Remark 8, $A \cup \Gamma \Rightarrow \phi$ is derivable in $\mathcal{R}_\Rightarrow$ as well. Then there is a derivation via a sequence of sequents $s_1, \ldots, s_m$, where $s_i = A_i \cup \Gamma_i \Rightarrow \psi_i$ for each $i \in \{1, \ldots, m\}$ is the result of applying rules from $\mathcal{R}_\Rightarrow$ to sequents in $\{s_1, \ldots, s_{i-1}\}$ and $s_m = A \cup \Gamma \Rightarrow \phi$. Again by induction on the length of the derivation m, for each s_i, there is a deduction $\mathsf{Ass}(s_i) \cup \mathsf{Supp}(s_i) \vdash^{\mathcal{R}} \mathsf{Conc}(s_i)$ via the sequence $\Phi_i = \psi^i_1, \ldots, \psi^i_{m_i}$:

m=1 Then $\phi \in A \cup \Gamma$ in which case $s_m = \phi \Rightarrow \phi$ or there is a $\mu(r) \in \mathcal{R}_\Rightarrow$ such that $\mu(r) = \Rightarrow \phi$ and thus, by Definition 29, $r = \to \phi \in \mathcal{R}$. Hence $A \cup \Gamma \vdash^{\mathcal{R}} \phi$.

m=k+1 Now assume that for derivations up to length $k \geq 1$, for each s_i, there is a deduction from $\mathsf{Ass}(s_i) \cup \mathsf{Supp}(s_i)$ for $\mathsf{Conc}(s_i)$ via the sequence Φ_i. That s_m is derivable implies that $\mathsf{Conc}(s_m) \in \mathsf{Ass}(s_m) \cup \mathsf{Supp}(s_m)$, in which case $s_m = \mathsf{Conc}(s_m) \Rightarrow \mathsf{Conc}(s_m)$, from which it follows immediately that there is a deduction $\mathsf{Ass}(s_m) \cup \mathsf{Supp}(s_m) \vdash^{\mathcal{R}} \mathsf{Conc}(s_m)$ or s_m is the result of applying a rule to sequents in $\{s_1, \ldots, s_k\}$:

* suppose that [Cut] was applied to $s_{j_1}, s_{j_2} \in \{s_1, \ldots, s_k\}$. By induction hypothesis, there are deductions $\mathsf{Ass}(s_{j_1}) \cup \mathsf{Supp}(s_{j_1}) \vdash^{\mathcal{R}} \mathsf{Conc}(s_{j_1})$ and $\mathsf{Ass}(s_{j_2}) \cup \mathsf{Supp}(s_{j_2}) \vdash^{\mathcal{R}} \mathsf{Conc}(s_{j_2})$ via the sequence Φ_{j_1} respectively Φ_{j_2}. Then $\mathsf{Ass}(s_m) \cup \mathsf{Supp}(s_m) \vdash^{\mathcal{R}} \mathsf{Conc}(s_m)$ is obtained via the sequence $\Phi_m = \Phi_{j_1} \circ_{\mathsf{Conc}(s_{j_1})} \Phi_{j_2}$, where $\Phi^1 \circ_\psi \Phi^2$ denotes the concatenation of Φ^1 with Φ^2 such that all occurrences of ψ in Φ^2 are taken out.
* suppose that s_m is the result of applying $\overline{\phi_1, \ldots, \phi_n \Rightarrow \phi}\ \mu(r) \in \mathcal{R}_\Rightarrow$. By construction, $\phi_1, \ldots, \phi_n \to \phi = r \in \mathcal{R}$ such that $\phi_j \in \{\psi_1, \ldots, \psi_k\}$ is obtained via a sequence Φ'_j, for each $j \in \{1, \ldots, n\}$. Therefore, $\mathsf{Ass}(s_m) \cup \mathsf{Supp}(s_m) \vdash^{\mathcal{R}} \mathsf{Conc}(s_m)$.

Thus, for the derivation of a, of any length m, via the sequence of sequents, $s_1, \ldots, s_m$, there is a deduction from $A \cup \Gamma$ via the sequence Φ_m, for ϕ. Hence $A \cup \Gamma \vdash^{\mathcal{R}} \phi \in \text{Arg}^{\mathsf{ABA}}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A})$.

□

Lemma 15. *Let $a, b \in Arg^{\mathsf{ABA}}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A})$ and a', b' their corresponding ABA-sequent arguments, thus $a', b' \in Arg^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A})$.*[10] *Then a attacks b in $\mathcal{AF}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A})$ iff a' attacks b' in $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A})$.*

Proof. Consider the $\Rightarrow$-direction, the $\Leftarrow$-direction is similar and left to the reader.

Let $a, b \in \text{Arg}^{\mathsf{ABA}}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A})$ and assume $a = A \cup \Gamma \vdash^{\mathcal{R}} \phi$ attacks $b = A' \cup \Gamma' \vdash^{\mathcal{R}} \phi'$. Then, by Definition 27, $\phi \in \overline{\psi}$ for $\psi \in A'$. By Lemma 14, $a' = A \wr \Gamma \Rightarrow \phi$ and $b' = A' \wr \Gamma' \Rightarrow \phi'$ are arguments in $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A})$ ($a', b' \in \text{Arg}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A})$). Since $\phi \in \overline{\psi}$ for $\psi \in A'$, it follows that a' AT_{ABA}-attacks b'. □

With this Proposition 2 can be shown:

Proof. Let $\langle \mathcal{L},\mathcal{R}\rangle$ be a deductive system, $\mathcal{S} \subseteq \mathcal{L}$ a non-trivializing set of formulas and $\mathcal{A} \subseteq \mathcal{L}$ a set of assumptions, such that $\Gamma \subseteq \mathcal{S}$ and $A \subseteq \mathcal{A}$ are finite and $\mathcal{A} \cap \mathcal{S} = \emptyset$. Let $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}) = \left\langle \text{Arg}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}), \mathcal{AT} \right\rangle$ be a sequent-based ABA-framework that corresponds to the ABA-framework $\mathcal{AF}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A}) = \langle \mathcal{L},\mathcal{R},\mathcal{S},\mathcal{A},\bar{\cdot}\rangle$. We show only some cases, leaving the others to the reader. First note that:

1. if $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A}))$ then $\{a' \mid a \in \mathcal{E}$ and a' corresponds to a as in Lemma 14$\} = \mathcal{E}' \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}))$;
2. if $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}))$ then $\{a' \mid a \in \mathcal{E}$ and a' corresponds to a as in Lemma 14$\} = \mathcal{E}' \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A}))$.

We show only the first item, leaving the second item to the reader. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A}))$ and let $\mathcal{E}' = \{a' \mid a \in \mathcal{E}$ where a' corresponds to a as in Lemma 14$\}$. To show that $\mathcal{E}'$ is complete.

$\mathcal{E}'$ **is conflict-free**. Since $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A}))$ it follows immediately that $\mathcal{E}$ is conflict-free. By the construction of $\mathcal{E}'$ and Lemma 15 it follows that $\mathcal{E}'$ is conflict-free as well.

$\mathcal{E}'$ **defends itself**. Suppose $a' \in \mathcal{E}'$ is attacked by some $b' \in \text{Arg}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A})$. By the construction of $\mathcal{E}'$ and Lemma 14 there exist $a, b \in \text{Arg}^{\mathsf{ABA}}_{\langle \mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A})$ corresponding

[10] That a' and b' exist follows from Lemma 14.

to a' and b' respectively, such that $a \in \mathcal{E}$. Moreover, by Lemma 15, b attacks a. Since $a \in \mathcal{E}$, there is some $c \in \mathcal{E}$ such that c defends a. By the construction of $\mathcal{E}'$ it follows that $c' \in \mathcal{E}'$ and by Lemma 15 it defends a' against the attack from b'. Thus $\mathcal{E}'$ defends a'.

$\mathcal{E}'$ **contains the arguments it defends**. Suppose that $a' \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S},\mathcal{A})$ is defended by $\mathcal{E}'$. Then there is some $b' \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S},\mathcal{A})$ such that b' attacks a' and there is some $c' \in \mathcal{E}'$ such that c' attacks b'. By the construction of $\mathcal{E}'$ and Lemma 14, there are corresponding arguments $a, b, c \in \mathrm{Arg}^{\mathsf{ABA}}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A})$ such that a is attacked by b, b is attacked by c and $c \in \mathcal{E}$. Thus c defends a against the attack by b. Since $\mathcal{E}$ is complete $a \in \mathcal{E}$. Hence, by the construction of $\mathcal{E}'$ it follows that $a' \in \mathcal{E}'$.

Therefore we have that $\{a' \mid a \in \mathcal{E}$ and a' corresponds to a as in Lemma 14$\} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S},\mathcal{A}))$. It remains to show that $\mathcal{A}\cup\mathcal{S} \mid\!\sim^{\star}_{\mathsf{ABA},\mathsf{sem}} \phi$ iff $\mathcal{S} \mid\!\sim^{\star}_{\mathcal{A},\mathsf{sem}} \phi$ for $\star \in \{\cup, \cap, \Cap\}$ and completeness-based semantics sem. We show the case for $\star = \cup$ and $\mathsf{sem} = \mathsf{cmp}$.

- $\Rightarrow$ Let $\mathcal{A} \cup \mathcal{S} \mid\!\sim^{\cup}_{\mathsf{ABA},\mathsf{cmp}} \phi$. Then there is some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A}))$ such that there is some $a \in \mathcal{E}$ where $a = A \cup \Gamma \vdash^{\mathcal{R}} \phi$ for some $A \subseteq \mathcal{A}$ and $\Gamma \subseteq \mathcal{S}$. By Lemma 14 $a' = A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S},\mathcal{A})$. Moreover, by the first item above it follows that there is some $\mathcal{E}' \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S},\mathcal{A}))$ and $a' \in \mathcal{E}'$. Therefore $\mathcal{S} \mid\!\sim^{\cup}_{\mathcal{A},\mathsf{cmp}} \phi$.

- $\Leftarrow$ Let $\mathcal{S} \mid\!\sim^{\cup}_{\mathcal{A},\mathsf{cmp}} \phi$. Then there is some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A}))$ such that there is some $a \in \mathcal{E}$ where $a = A \wr \Gamma \Rightarrow \phi$ for some $A \subseteq \mathcal{A}$ and $\Gamma \subseteq \mathcal{S}$. By the second item above it follows that there is some $\mathcal{E}' \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S},\mathcal{A}))$ such that $a' \in \mathcal{E}'$ where a' corresponds to a as in Lemma 14, thus $a' = A \cup \Gamma \vdash^{\mathcal{R}} \phi$. Hence $\mathcal{A} \cup \mathcal{S} \mid\!\sim^{\cup}_{\mathsf{ABA},\mathsf{cmp}} \phi$. □

Example 20. Recall the setting from Example 18, in which $\mathcal{S} = \{s\}$, $\mathcal{A} = \{p, q, \neg p \vee \neg q, \neg p \vee r, \neg q \vee r\}$ and classical logic is the core logic. Let $\mathcal{R}_\Rightarrow = \mathsf{LK}$. Some of the arguments of the sequent-based ABA-framework $\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S},\mathcal{A}) = \left\langle \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S},\mathcal{A}), \mathcal{AT} \right\rangle$ are:

$$a = s \Rightarrow s \qquad b = p, \neg p \vee \neg q \wr \Rightarrow \neg q$$
$$c = q, \neg p \vee \neg q \wr \Rightarrow \neg p \qquad d = p, q, \neg p \vee r, \neg q \vee r \wr \Rightarrow r.$$

Note that a cannot be attacked, since $\mathsf{Ass}(a) = \emptyset$. Thus $\mathcal{S} \mid\!\sim^{\star}_{\mathcal{A},\mathsf{sem}} s$ for $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ and $\star \in \{\cup, \cap, \Cap\}$. However, the argument d is attacked by both b and c. Moreover b attacks c and c attacks b. It can be shown that, for $\phi \in \{p, q, \neg p \vee \neg q\}$, $\mathcal{S} \not\mid\!\sim^{\star}_{\mathcal{A},\mathsf{sem}} \phi$ for $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ and $\star \in \{\cap, \Cap\}$ but also $\mathcal{S} \mid\!\sim^{\cup}_{\mathcal{A},\mathsf{sem}'} \phi$ for $\mathsf{sem}' \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$.

We will now turn to the representation of reasoning with maximally consistent subsets in the here presented framework.

Remark 9. In this section maximally consistent subsets are defined as in Definition 25. The corresponding entailment relations are defined in the same way as those in Definition 18, now with respect to the definition of contrary-consistent sets. We continue using the notation $\mid\!\sim^{\star,\mathcal{AS}}_{\mathsf{mcs}}$ for $\star \in \{\cap, \cup, \Cap\}$.

The relations between ABA and reasoning with maximally consistent subsets and between sequent-based argumentation and maximally consistent subsets have been studied before, see [7, 9, 41]. In addition to the two entailment relations in [41] (in the notation of this paper $\mid\!\sim^{\Cap,\mathcal{AS}}_{\mathsf{mcs}}$ and $\mid\!\sim^{\cup,\mathcal{AS}}_{\mathsf{mcs}}$), we will also consider the entailment relation $\mid\!\sim^{\cap,\mathcal{AS}}_{\mathsf{mcs}}$. For the proof of these relations, like in [41], it is assumed that $\vdash^{\mathcal{R}}$ is contrapositive:

Definition 30. $\vdash^{\mathcal{R}}$ is said to be *contrapositive for assumptions* if for any $\phi \in A$ and any $\psi \in \mathcal{A}$ it holds that $A \cup \Gamma \vdash^{\mathcal{R}} \overline{\psi}$ if and only if $(A \setminus \{\phi\}) \cup \{\psi\} \cup \Gamma \vdash^{\mathcal{R}} \overline{\phi}$.

Similar to the assumption made in the previous section, that the rules from Figure 6 are admissible in the sequent calculus of the core logic, requiring that $\vdash^{\mathcal{R}}$ is contrapositive restricts the generality of the result, not the above introduced representation.

The proofs of Proposition 3 and the lemmas necessary for it are partially based on proofs in [9]. For similar reasons as those in the previous section we will assume that $\mathcal{S}$ is non-trivializing.

Proposition 3. *Let $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S},\mathcal{A})$ be a sequent-based ABA-framework for a deductive system $\langle\mathcal{L},\mathcal{R}\rangle$, $\mathcal{S} \subseteq \mathcal{L}$ a non-trivializing set of formulas and $\mathcal{A}$ a set of assumptions. Suppose that $\vdash^{\mathcal{R}}$ is contrapositive for assumptions. Then:*

1. *$\mathcal{S} \mid\!\sim^{\cap}_{\mathcal{A},\mathsf{prf}} \phi$ iff $\mathcal{S} \mid\!\sim^{\cap}_{\mathcal{A},\mathsf{stb}} \phi$ iff $\mathcal{S} \mid\!\sim^{\cap,\mathcal{A}}_{\mathsf{mcs}} \phi$;*
2. *$\mathcal{S} \mid\!\sim^{\cup}_{\mathcal{A},\mathsf{prf}} \phi$ iff $\mathcal{S} \mid\!\sim^{\cup}_{\mathcal{A},\mathsf{stb}} \phi$ iff $\mathcal{S} \mid\!\sim^{\cup,\mathcal{A}}_{\mathsf{mcs}} \phi$;*
3. *$\mathcal{S} \mid\!\sim^{\Cap}_{\mathcal{A},\mathsf{prf}} \phi$ iff $\mathcal{S} \mid\!\sim^{\Cap}_{\mathcal{A},\mathsf{stb}} \phi$ iff $\mathcal{S} \mid\!\sim^{\Cap,\mathcal{A}}_{\mathsf{mcs}} \phi$.*

As in Section 3.3 we first consider two lemmas that will be useful in the proofs of the above proposition. The first shows that for any maximally consistent subset of assumptions $\mathcal{T}$, if some assumption ϕ is not part of $\mathcal{T}$, there is some argument a such that the conclusion of a is a contrary of ϕ. The second shows that the set of assumptions from which the arguments in a complete extension is constructed, are contrary-consistent.

For the next proofs, suppose that the conditions of the statement of the proposition hold.

Lemma 16. *For each set $\mathcal{T} \subseteq \mathcal{A}$: if $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ then for each $\phi \in \mathcal{A} \setminus \mathcal{T}$, there is some finite $A \subseteq \mathcal{T}$ and some finite $\Gamma \subseteq \mathcal{S}$ such that $A \wr \Gamma \Rightarrow \overline{\phi} \in Arg^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$.*

Proof. Assume that $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ and consider some $\phi \in \mathcal{A} \setminus \mathcal{T}$. By Definition 25, there is some $A' \subseteq \mathcal{T} \cup \{\phi\}$ and some $\Gamma \subseteq \mathcal{S}$ such that $A' \cup \Gamma \vdash^{\mathcal{R}} \overline{\psi}$ for some $\psi \in \mathcal{T} \cup \{\phi\}$. Consider two cases:

- $\psi \in \mathcal{T}$. By contraposition, $(A' \setminus \{\phi\}) \cup \{\psi\} \cup \Gamma \vdash^{\mathcal{R}} \overline{\phi}$.
- $\psi = \phi$. Then $A' \subseteq \mathcal{T}$.

In both cases there is an $A \subseteq \mathcal{T}$ and a $\Gamma \subseteq \mathcal{S}$ such that $A \cup \Gamma \vdash^{\mathcal{R}} \overline{\phi} \in \mathrm{Arg}^{\mathsf{ABA}}_{\langle \mathcal{L}, \mathcal{R} \rangle}(\mathcal{S}, \mathcal{A})$. Hence, by Lemma 14, $A \wr \Gamma \Rightarrow \overline{\psi} \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$. □

Lemma 17. *The set $\mathsf{Ass}(\mathcal{E})$, for any $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}))$ is contrary-consistent.*

Proof. Assume, towards a contradiction, that $\mathsf{Ass}(\mathcal{E}) = \{\phi_1, \ldots, \phi_n\}$ is not contrary-consistent. Then, by Definition 25 there are $A \subseteq \mathsf{Ass}(\mathcal{E})$ and $\Gamma \subseteq \mathcal{S}$ such that $A, \Gamma \vdash^{\mathcal{R}} \overline{\phi_i}$ for some $\phi_i \in \mathsf{Ass}(\mathcal{E})$. By Lemma 14, $a = A \wr \Gamma \Rightarrow \overline{\phi_i}$ is derivable. Note that, if a is not attacked, $a \in \mathcal{E}$. Suppose that a is attacked by an argument $b = A' \wr \Gamma' \Rightarrow \psi \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$. Then $\psi \in \overline{\psi'}$ for some $\psi' \in A$. Hence $\psi' \in \mathsf{Ass}(\mathcal{E})$. Thus b attacks some argument $a' \in \mathcal{E}$ as well. Since $a' \in \mathcal{E}$, there is an argument $c \in \mathcal{E}$ which defends a' and thus a from the attack by b. Since $\mathcal{E}$ is complete, $a \in \mathcal{E}$. Thus whether a is attacked or not, $a \in \mathcal{E}$. However, a attacks each $a_j \in \mathcal{E}$ with $\phi_i \in \mathsf{Ass}(a_j)$. A contradiction with the conflict-freeness of the complete extension $\mathcal{E}$. □

The next two lemmas show how maximally consistent subsets relate to stable (Lemma 18) and preferred (Lemma 19) extensions.

Lemma 18. *If $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ then $Arg^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}))$.*

Proof. Assume that $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ and let $\mathcal{E} = \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{T})$. We show that $\mathcal{E}$ is conflict-free and stable.

$\mathcal{E}$ is conflict-free. Suppose, towards a contradiction, that $\mathcal{E}$ is not conflict-free. Then there are arguments $a_1 = A_1 \wr \Gamma_1 \Rightarrow \phi_1$ and $a_2 = A_2 \wr \Gamma_2 \Rightarrow \phi_2$, such that

$a_1, a_2 \in \mathcal{E}$ and a_1 attacks a_2. Thus $\phi_1 \in \overline{\psi}$ for some $\psi \in A_2$. However, by assumption $A_1 \cup A_2 \subseteq \mathcal{T}$. A contradiction with the assumption that $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$.

$\mathcal{E}$ **is stable.** Now suppose that $b = A' \wr \Gamma' \Rightarrow \phi' \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A}) \setminus \mathcal{E}$ for some $\Gamma' \subseteq \mathcal{S}$ and $A' \subseteq \mathcal{A}$. Since $b \notin \mathcal{E} = \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$, there is some $\phi \in A'$ such that $\phi \notin \mathcal{T}$. Since, by supposition $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$, from Lemma 16, there are finite $A \subseteq \mathcal{T}$, $\Gamma \subseteq \mathcal{S}$ such that $A \wr \Gamma \Rightarrow \overline{\phi} \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A})$. Because $A \subseteq \mathcal{T}$ it follows that $A \wr \Gamma \Rightarrow \overline{\phi} \in \mathcal{E}$. Hence b is attacked by $\mathcal{E}$. Therefore $\mathcal{E}$ attacks every argument in $\mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A}) \setminus \mathcal{E}$ and thus $\mathcal{E} \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A}))$. □

Lemma 19. *If $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A}))$ then there is some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ such that $\mathcal{E} = \mathit{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$.*

Proof. Assume, towards a contradiction, that for some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A}))$ there is no $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ such that $\mathcal{E} = \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$. Consider first the case that there is some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ such that $\mathcal{E} = \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T}')$ for $\mathcal{T}' \subsetneq \mathcal{T}$. Thus $\mathcal{E} \subsetneq \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$. By Lemma 18, it follows that $\mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A}))$. A contradiction to the assumption that $\mathcal{E}$ is preferred and thus maximal. Thus if $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ does not exist such that $\mathcal{E} = \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$, a $\mathcal{T}' \subsetneq \mathcal{T}$ for which $\mathcal{E} = \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T}')$ does not exist either. Thus, since there is no $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ such that $\mathcal{E} = \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$, there is no $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ such that $\mathsf{Ass}(\mathcal{E}) \subseteq \mathcal{T}$ and hence, $\mathsf{Ass}(\mathcal{E})$ is contrary-inconsistent. A contradiction with Lemma 17 and the assumption that $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A}))$. Thus, $\mathcal{E} \subseteq \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$ for some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$. By Lemma 18, $\mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$ is stable (and therefore preferred) and thus $\mathcal{E} = \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$. □

We now turn to the proof of Proposition 3:

Proof. Let $\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A})$ be a sequent-based ABA-framework, where $\langle\mathcal{L}, \mathcal{R}\rangle$ is a deductive system, $\mathcal{S}$ is a non-trivializing set of $\mathcal{L}$-formulas and $\mathcal{A}$ is a set of assumptions. Consider each item in both directions:

1. ($\Rightarrow$) Note that $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathcal{A},\mathsf{prf}} \phi$ implies $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathcal{A},\mathsf{stb}} \phi$. Suppose $\mathcal{S} \not\mathrel{|\!\sim}^{\cap,\mathcal{A}}_{\mathsf{mcs}} \phi$, but that there is some finite $A \subseteq \mathcal{A}$ and some $\Gamma \subseteq \mathcal{S}$ such that $A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{A})$. Now, by assumption, $A \not\subseteq \bigcap \mathsf{MCS}(\mathcal{S}, \mathcal{A})$. Hence, there is some $\phi' \in A \setminus \bigcap \mathsf{MCS}(\mathcal{S}, \mathcal{A})$. From which it follows that there is some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S}, \mathcal{A})$ such that $\phi' \notin \mathcal{T}$. Therefore $A \wr \Gamma \Rightarrow \phi \notin \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle\mathcal{L},\mathcal{R}_\Rightarrow\rangle}(\mathcal{S}, \mathcal{T})$.

By Lemma 18, $\mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}))$, thus $\mathcal{S} \not\mathrel{|\!\sim}^{\cap}_{\mathcal{A},\mathsf{stb}} \phi$ (and thus $\mathcal{S} \not\mathrel{|\!\sim}^{\cap}_{\mathcal{A},\mathsf{prf}} \phi$) as well.

($\Leftarrow$) Suppose that $\mathcal{S} \mathrel{|\!\sim}^{\cap,\mathcal{A}}_{\mathsf{mcs}} \phi$. Thus, there are finite $A \subseteq \bigcap \mathsf{MCS}(\mathcal{S},\mathcal{A})$ and $\Gamma \subseteq \mathcal{S}$ such that $A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A})$ is derivable. By Lemma 19 $\mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\Gamma, A) \subseteq \bigcap \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}))$. Hence we have that $A \wr \Gamma \Rightarrow \phi \in \bigcap \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}))$. From which it follows that $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathcal{A},\mathsf{prf}} \phi$ and thus $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathcal{A},\mathsf{stb}} \phi$.

2. ($\Rightarrow$) Note that $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathcal{A},\mathsf{stb}} \phi$ implies $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathcal{A},\mathsf{prf}} \phi$. Suppose that $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathcal{A},\mathsf{prf}} \phi$. Then there is some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}))$ such that $A \wr \Gamma \Rightarrow \phi \in \mathcal{E}$, for $A \subseteq \mathcal{A}$ and $\Gamma \subseteq \mathcal{S}$. From Lemma 19 it follows that there is some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{A})$ such that $\mathcal{E} = \mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{T})$ (thus $A \subseteq \mathcal{T}$). Hence, by Definition 25 and Lemma 14, $\phi \in \mathsf{CN}(\mathcal{T} \cup \mathcal{S})$ it follows that $\mathcal{S} \mathrel{|\!\sim}^{\cup,\mathcal{A}}_{\mathsf{mcs}} \phi$.

   ($\Leftarrow$) Assume that $\mathcal{S} \mathrel{|\!\sim}^{\cup,\mathcal{A}}_{\mathsf{mcs}} \phi$. Then there is some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{A})$ such that $\phi \in \mathsf{CN}(\mathcal{T} \cup \mathcal{S})$. Therefore, there is a deduction from $A \cup \Gamma \subseteq \mathcal{T} \cup \mathcal{S}$ for ϕ ($A \cup \Gamma \vdash^{\mathcal{R}} \phi \in \mathrm{Arg}^{\mathsf{ABA}}_{\langle\mathcal{L},\mathcal{R}\rangle}(\mathcal{S},\mathcal{A})$) and thus, by Lemma 14 $A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A})$. From Lemma 18 it follows that $\mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}))$. Thus $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathcal{A},\mathsf{stb}} \phi$ as well.

3. $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathcal{A},\mathsf{stb}} \phi$ implies $\mathcal{S} \mathrel{|\!\sim}^{\Cap,\mathcal{A}}_{\mathsf{mcs}} \phi$: suppose that $\mathcal{S} \not\mathrel{|\!\sim}^{\Cap,\mathcal{A}}_{\mathsf{mcs}} \phi$, then there is some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{A})$ for which $\phi \notin \mathsf{CN}(\mathcal{S} \cup \mathcal{T})$. Hence, there are no $A \subseteq \mathcal{T}$ and $\Gamma \subseteq \mathcal{S}$ with $A \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{T})$. By Lemma 18 it follows that $\mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{T}) \in \mathsf{Ext}_{\mathsf{stb}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}))$, thus $\mathcal{S} \not\mathrel{|\!\sim}^{\Cap}_{\mathcal{A},\mathsf{stb}} \phi$.

   $\mathcal{S} \mathrel{|\!\sim}^{\Cap,\mathcal{A}}_{\mathsf{mcs}} \phi$ implies $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathcal{A},\mathsf{prf}} \phi$: suppose that $\mathcal{S} \not\mathrel{|\!\sim}^{\Cap}_{\mathcal{A},\mathsf{prf}} \phi$. Then there is some preferred extension $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{A}))$ such that there is no $A \wr \Gamma \Rightarrow \phi \in \mathcal{E}$ for $A \subseteq \mathcal{A}$ and $\Gamma \subseteq \mathcal{S}$. From Lemma 19 it follows that there is some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S},\mathcal{A})$ such that $\mathrm{Arg}^{\mathsf{ABA}_{\Rightarrow}}_{\langle\mathcal{L},\mathcal{R}_{\Rightarrow}\rangle}(\mathcal{S},\mathcal{T}) = \mathcal{E}$ and $\phi \notin \mathsf{CN}(\mathcal{S} \cup \mathcal{T})$. Thus $\mathcal{S} \not\mathrel{|\!\sim}^{\Cap,\mathcal{A}}_{\mathsf{mcs}} \phi$.

   $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathcal{A},\mathsf{prf}} \phi$ implies $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathcal{A},\mathsf{stb}} \phi$: this follows immediately since any stable extension is a preferred extension [36, Lemma 15]. □

Example 21. Recall from Example 18 the sets $\mathcal{S} = \{s\}$ and $\mathcal{A} = \{p, q, \neg p \vee \neg q, \neg p \vee r, \neg q \vee r\}$. Then $\mathsf{MCS}(\mathcal{S},\mathcal{A}) = \{\{p, q, \neg p \vee r, \neg q \vee r\}, \{p, \neg p \vee \neg q, \neg p \vee r, \neg q \vee r\}, \{q, \neg p \vee \neg q, \neg p \vee r, \neg q \vee r\}\}$. Hence $\bigcap \mathsf{MCS}(\mathcal{S},\mathcal{A}) = \{\neg p \vee r, \neg q \vee r\}$. Therefore, $\mathcal{S} \mathrel{|\!\sim}^{\cap,\mathcal{A}}_{\mathsf{mcs}} \phi$ and $\mathcal{S} \mathrel{|\!\sim}^{\Cap,\mathcal{A}}_{\mathsf{mcs}} \phi$ for $\phi \in \mathsf{CN}(\{s, \neg p \vee r, \neg q \vee r\})$ and $\mathcal{S} \mathrel{|\!\sim}^{\cup,\mathcal{A}}_{\mathsf{mcs}} \phi$ for $\phi \in \{p, q, \neg p \vee \neg q\}$. Moreover, by the results from Proposition 3 it follows that $\mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathcal{A},\mathsf{sem}} \phi$ for $\star \in \{\cap, \cup, \Cap\}$ and $\phi \in \{s, \neg p \vee r, \neg q \vee r\}$ and $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathcal{A},\mathsf{sem}} \phi$ for $\phi \in \{p, q, \neg p \vee \neg q\}$, which corresponds indeed to the results from Example 20.

The results presented above are summarized in the following theorem.

Theorem 3. *Let $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$ be a sequent-based ABA-framework, for $\langle \mathcal{L}, \mathcal{R} \rangle$ a deductive system, $\mathcal{S}$ a non-trivializing set of $\mathcal{L}$-formulas and $\mathcal{A}$ a set of assumptions such that $\mathcal{S} \cap \mathcal{A} = \emptyset$, then:*

1. *$\mathcal{A} \cup \mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathsf{ABA,sem}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\star}_{\mathcal{A},\mathsf{sem}} \phi$ for $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$ and $\star \in \{\cup, \cap, \Cap\}$ (Proposition 2).*

   For the following, let $\vdash^{\mathcal{R}}$ be contrapositive for assumptions:

2. *$\mathcal{A} \cup \mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{ABA,sem}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathcal{A},\mathsf{sem}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cap, A}_{\mathsf{mcs}} \phi$, for $\mathsf{sem} \in \{\mathsf{prf}, \mathsf{stb}\}$ (Propositions 2 and 3.1).*

3. *$\mathcal{A} \cup \mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{ABA,sem}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathcal{A},\mathsf{sem}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cup, A}_{\mathsf{mcs}} \phi$ for $\mathsf{sem} \in \{\mathsf{prf}, \mathsf{stb}\}$ (Propositions 2 and 3.2).*

4. *$\mathcal{A} \cup \mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathsf{ABA,sem}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathcal{A},\mathsf{sem}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\Cap, A}_{\mathsf{mcs}} \phi$, for $\mathsf{sem} \in \{\mathsf{prf}, \mathsf{stb}\}$ (Propositions 2 and 3.3).*

We will now turn to the rationality postulates from [30], see also Section 3.2. For these proofs consider the sequent-based ABA-framework $\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}) = \left\langle \mathrm{Arg}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}), \mathcal{AT} \right\rangle$ for some deductive system $\langle \mathcal{L}, \mathcal{R} \rangle$, let $\mathcal{S}$ be a non-trivializing set of $\mathcal{L}$-formulas and $\mathcal{A}$ a set of assumptions, where $\Gamma \subseteq \mathcal{S}$, $A \subseteq \mathcal{A}$ are finite and $\mathcal{S} \cap \mathcal{A} = \emptyset$. Let $\mathsf{sem} \in \{\mathsf{grd}, \mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$. Note that, due to the definition of contrary-consistency from Definition 25, the consistency postulate, defined in Definition 16, has to be adjusted:

- $\mathsf{Concs}(\mathcal{E})$ is consistent if and only if there is no $\phi \in \mathcal{A}$ such that $\phi, \overline{\phi} \in \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$.

Lemma 20 (Sub-argument closure)**.** *$\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$ satisfies sub-argument closure: let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A}))$, then for all $a \in \mathcal{E}$, $\mathsf{Sub}(a) \subseteq \mathcal{E}$.*

Proof. Let $a = A \wr \Gamma \Rightarrow \phi \in \mathcal{E}$, $a' = A' \wr \Gamma' \Rightarrow \phi' \in \mathsf{Sub}(a)$ and assume $b \in \mathrm{Arg}^{\mathsf{ABA}\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_{\Rightarrow} \rangle}(\mathcal{S}, \mathcal{A})$ attacks a'. Then $\mathsf{Conc}(b) = \overline{\psi}$ for some $\psi \in A'$. By definition of a sub-argument $A' \subseteq A$, hence b attacks a as well. Since $\mathcal{E}$ is complete and $a \in \mathcal{E}$, it follows that there is a $c \in \mathcal{E}$ which defends a and thus a' from the attack by b. Therefore $a' \in \mathcal{E}$ and hence $\mathsf{Sub}(a) \subseteq \mathcal{E}$. ∎

Lemma 21 (Closure). $\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$ *satisfies closure of extensions, for each extension* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}))$ *it holds that* $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$.

Proof. By Definition 25, it follows immediately that $\mathsf{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$. Suppose $\phi \in \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$. Then there are arguments $a_1, \ldots, a_n \in \mathcal{E}$, with $\mathsf{Supp}(a_i) = \Gamma_i$, $\mathsf{Conc}(a_i) = \phi_i$, $\mathsf{Ass}(a_i) = A_i$ for $1 \leq i \leq n$ and $\phi_1, \ldots, \phi_n \Rightarrow \phi$ is derivable, using rules in $\mathcal{R}_\Rightarrow$. By [Cut] $a = A_1, \ldots, A_n \wr \Gamma_1, \ldots, \Gamma_n \Rightarrow \phi$.

Note that, if a is not attacked, $a \in \mathcal{E}$, thus $\phi \in \mathsf{Concs}(\mathcal{E})$. Now suppose $b \in \mathrm{Arg}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$ attacks a. Then $\mathsf{Conc}(b) = \overline{\psi}$ for some $\psi \in A_1 \cup \ldots \cup A_n$. Without loss of generality assume $\psi \in A_i$. Then b attacks a_i as well. Since $a_i \in \mathcal{E}$ it follows that $\mathcal{E}$ defends against the attack from b. Therefore $a \in \mathcal{E}$ as well. □

Lemma 22 (Consistency). $\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$ *satisfies consistency: for each extension* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}))$*,there is no* $\phi \in \mathcal{A}$ *such that* $\phi, \overline{\phi} \in \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$.

Proof. Assume, towards a contradiction, that $\mathsf{Concs}(\mathcal{E})$ is not consistent. Then there are arguments $a, b \in \mathcal{E}$, such that $\overline{\mathsf{Conc}(a)} = \mathsf{Conc}(b)$ (since $\mathsf{CN}(\mathsf{Concs}(\mathcal{E})) = \mathsf{Concs}(\mathcal{E})$). However, by Lemma 17, $\mathsf{Ass}(\mathcal{E})$ is consistent. Hence, by Definition 25, no such arguments exist. □

Theorem 4. *Let* $\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}) = \left\langle Arg^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A}), \mathcal{AT} \right\rangle$ *be an sequent-based ABA-framework, for the deductive system* $\langle \mathcal{L}, \mathcal{R} \rangle$*,* $\mathcal{S}$ *a non-trivializing set of* $\mathcal{L}$*-formulas,* $\mathcal{A}$ *a set of assumptions and* AT_{ABA} *the only attack rule. Then the framework* $\mathcal{AF}^{\mathsf{ABA}_\Rightarrow}_{\langle \mathcal{L}, \mathcal{R}_\Rightarrow \rangle}(\mathcal{S}, \mathcal{A})$ *satisfies sub-argument closure, closure under strict rules and consistency.*

4.2 Adaptive Logics

Adaptive logics, originally introduced by Batens (see e.g., [21, 62] for an overview), are a logical framework that offer contributions to the research on formalizations of defeasible reasoning forms. It was developed to interpret (possibly) inconsistent theories as consistently as possible. From the perspective of epistemology, the introduction of adaptive logics has been motivated by the lack of a proof-theoretic account that captures the dynamic and defeasible aspects of human reasoning [20]. Adaptive logics have been frequently applied to reasoning forms typical for scientific reasoning (such as handling inconsistencies, inductive generalizations and abductive inferences). From the perspective of nonmonotonic logics, adaptive logics are a subclass of the preferential models known from [47]. Adaptive logics differ from other approaches based on preferential models in that they offer an adequate dynamic proof theory for the resulting nonmonotonic consequence relations. Nowadays adaptive logics cover many application contexts, such as inconsistent knowledge bases, default reasoning and circumscription, abstract argumentation, abduction, fuzzy logic, induction and deontic conflict. The idea is to interpret the premises *as normally as possible*. What this means depends on the logic and the application. The most common form for adaptive logics is the so-called standard format:

Definition 31. *Adaptive logics in the standard format* consist of three elements:

- the lower limit logic (**LLL**), the logic that is strengthened by the adaptive logic, with:
  - a Tarskian consequence relation (see Definition 1); and
  - a characteristic semantics.
- a set of abnormalities Ω, the form of the abnormalities depends on the lower limit logic and the application; and
- an adaptive strategy, either the reliability strategy which is a more cautious reasoning form or minimal abnormality strategy, which is a more credulous form of reasoning.

$\mathsf{AL}^x_{\mathbf{LLL}}$, where $x \in \{r, m\}$ is the adaptive logic with lower limit logic **LLL** and strategy x, which can be the reliability strategy (r) or the minimal abnormality strategy (m). When the strategy and/or lower limit logic are arbitrary or clear from the context, the superscript and/or subscript are omitted.

A third strategy, that is not part of the standard format, is the *normal selections strategy* (n), which is even more credulous than the minimal abnormality strategy. In this section we will also consider this third strategy and will therefore also discuss the adaptive logic $\mathsf{AL}^n_{\mathbf{LLL}}$.

In the literature there are many logics that are used as lower limit logic. For example da Costa's C_i systems [22] and classical modal logics [23, 50] for which interpreting the premises as normally as possible means as non-conflicting as possible. Another example is the logic CLuN, introduced by Batens [19] under the name PI. It is obtained by adding the axioms $\phi \vee {\sim}\phi$ to full positive classical logic, as such, it is a very weak paraconsistent logic. For CLuN interpreting the premises as normally as possible means as consistent as possible.

The set of abnormalities, denoted by Ω, contains all the formulas of a logical form that depends on the lower limit logic of the adaptive logic and its application. Elements of Ω will be denoted by $!\phi$, where ϕ is the abnormal formula. In terms of abnormalities, interpreting the premises as normally as possible means that the premises are interpreted in a way that as few abnormalities as possible are validated.

Example 22. Consider the paraconsistent logic CLuN. Let Ω be the set of formulas of the form $\sim\phi\wedge\phi$, where ϕ is a CLuN-formula. Then $\mathsf{AL}^r_{\mathsf{CLuN}} = \langle \mathsf{CLuN}, \Omega, \text{reliability}\rangle$ is the adaptive logic with lower limit logic CLuN and the reliability strategy.

The following notation will be useful in the definition of the consequence relations and proofs:

Notation 3. Let $\text{Dab}(\Pi)$ denote the classical disjunction of members in Π, where Π is a finite subset of Ω, then:

- the *minimal Dab consequences* for a premise set Γ are all the $\text{Dab}(\Pi)$ such that $\Gamma \vdash_{\mathbf{LLL}} \text{Dab}(\Pi)$ and there is no $\Pi' \subset \Pi$ such that $\Gamma \vdash_{\mathbf{LLL}} \text{Dab}(\Pi')$;
- if $\text{Dab}(\Pi_1), \text{Dab}(\Pi_2), \ldots$ are the minimal Dab consequences for Γ, then $U(\Gamma) = \Pi_1 \cup \Pi_2 \cup \ldots$ is called the set of *unreliable abnormalities*; and
- let $\Sigma(\Gamma) = \{\Pi_1, \Pi_2, \ldots\}$, then $\Phi(\Gamma)$ denotes the set of all minimal choice sets of $\Sigma(\Gamma)$.[11]

Example 23. Let $\mathcal{S} = \{p, q, \sim p \vee \sim q, \sim p \vee r, \sim q \vee r\}$ and suppose that CLuN is the lower limit logic. Then $(p \wedge \sim p) \vee (q \wedge \sim q)$ is a minimal Dab-consequence of $\mathcal{S}$. When reasoning skeptically, both p and q are considered unreliable, thus intuitively r should not follow. However, when reasoning more credulously, r can follow. To see this, suppose that p is unreliable (it is abnormal), then q could be normal, thus from q and $\sim q \vee r$, r follows.

In this paper we define the entailment relations of an adaptive logic semantically, based on [62]. For the dynamic proof theory of adaptive logics see [62, Chapter 2]. In what follows let $\mathcal{M}_{\mathbf{LLL}}(\Gamma)$ denote the set of all $\mathbf{LLL}$-models for the set of formulas Γ.

Definition 32. Let M be an $\mathbf{LLL}$-model, the *abnormal part of* M is then $\text{Ab}(M) = \{\phi \in \Omega \mid M \models \phi\}$.

From the abnormal part of a model a (strict) partial order can be defined on the models of a given premise set Γ:

- $M \sqsubset^{\Gamma}_{\text{Ab}} M'$ iff $\text{Ab}(M) \subset \text{Ab}(M')$;
- $M \sqsubseteq^{\Gamma}_{\text{Ab}} M'$ iff $\text{Ab}(M) \subseteq \text{Ab}(M')$.

Definition 33. A model $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ is a *reliable model of* Γ if $\text{Ab}(M) \subseteq U(\Gamma)$. The set of all reliable models of Γ is denoted by $\mathcal{M}_{\mathsf{AL}^r}(\Gamma)$.

[11] A *choice set* of $\Sigma(\Gamma)$ is a set of formulas Δ, such that $\Delta \cap \Pi_i \neq \emptyset$ for each $\Pi_i \in \Sigma(\Gamma)$. Δ is minimal when there is no choice set Δ' of $\Sigma(\Gamma)$ such that $\Delta' \subset \Delta$.

Definition 34. A model $M \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ is a *minimally abnormal model of* Γ when for all other models $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ of Γ, $\mathrm{Ab}(M') \not\subset \mathrm{Ab}(M)$. The set of all minimally abnormal models of Γ is denoted by $\mathcal{M}_{\mathsf{AL}^m}(\Gamma)$.

Thus, the minimally abnormal models are the minimal elements of the partial order $\sqsubset^{\Gamma}_{\mathrm{Ab}}$.

Definition 35. The entailment relations for the three strategies are then defined by:

- $\Gamma \mid\!\sim^{r,\Omega}_{\mathbf{LLL}} \phi$ if and only if for each $M \in \mathcal{M}_{\mathsf{AL}^r}(\Gamma)$, $M \vDash \phi$.
- $\Gamma \mid\!\sim^{m,\Omega}_{\mathbf{LLL}} \phi$ if and only if for each $M \in \mathcal{M}_{\mathsf{AL}^m}(\Gamma)$, $M \vDash \phi$.
- $\Gamma \mid\!\sim^{n,\Omega}_{\mathbf{LLL}} \phi$ if and only if there is a model $M \in \mathcal{M}_{\mathsf{AL}^m}(\Gamma)$ such that for all $M' \in \mathcal{M}_{\mathbf{LLL}}(\Gamma)$ for which $\mathrm{Ab}(M) = \mathrm{Ab}(M')$, $M' \vDash \phi$.

Example 24. Recall the set $\mathcal{S} = \{p, q, {\sim}q \vee {\sim}p, {\sim}q \vee r, {\sim}p \vee r\}$ from Example 23, where CLuN is the lower limit logic. Three types of models can be considered, they differ in their abnormal parts: M_1 for which $\mathrm{Ab}(M_1) = \{p \wedge {\sim}p\}$, M_2 for which $\mathrm{Ab}(M_2) = \{q \wedge {\sim}q\}$ and M_3 for which $\mathrm{Ab}(M_3) = \{p \wedge {\sim}p, q \wedge {\sim}q\}$. As mentioned in Example 23, intuitively it is expected that r follows when reasoning credulously, but not when reasoning skeptically. Indeed, $\mathcal{S} \not\mid\!\sim^{r}_{\mathsf{CLuN}} r$, while $\mathcal{S} \mid\!\sim^{m}_{\mathsf{CLuN}} r$ and $\mathcal{S} \mid\!\sim^{n}_{\mathsf{CLuN}} r$.

In assumptive sequent-based argumentation with a lower limit logic **LLL** as core logic, an inference rule (RC) is added to the sequent calculus C of **LLL**. The idea is similar to the rules $\mathrm{AS}_{\mathcal{AS}}$ introduced in Definition 10. Let ϕ be a formula in the language of **LLL** and let $!\phi$ denote the abnormality for the formula ϕ. We consider two variations and will refer in both cases to the *RC*-rule:

$$\frac{\Pi \wr \Gamma \Rightarrow \Delta, \psi \vee !\phi}{\Pi, !\phi \wr \Gamma \Rightarrow \Delta, \psi}\ \mathrm{RC} \qquad\qquad \frac{\Pi \wr \Gamma \Rightarrow \Delta, !\phi}{\Pi, !\phi \wr \Gamma \Rightarrow \Delta}\ \mathrm{RC} \tag{3}$$

For a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, with corresponding sequent calculus C, let $\mathsf{C}' = \mathsf{C} \cup \{\mathrm{RC}\}$. AL-sequent arguments are then defined as follows:

Definition 36. Let **LLL** be a lower limit logic, with corresponding sound and complete sequent calculus C, let $\mathcal{S}$ be a set of **LLL**-formulas and Ω a set of abnormalities. An *assumptive* **LLL**-*argument based on* $\mathcal{S}$ *and* Ω (*AL-(sequent)argument* for short) is an assumptive sequent $\Pi \wr \Gamma \Rightarrow \psi$, provable in C', where $\Pi \subseteq \Omega$ and $\Gamma \subseteq \mathcal{S}$. $\mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ denotes the set of all AL-arguments based on $\mathcal{S}$ and Ω.

Definition 37. The sequent elimination rule for assumptive sequent-based argumentation with adaptive logics is defined as, where $\Pi \wr \Gamma \Rightarrow \phi,\ \Theta, \phi \wr \Delta \Rightarrow \psi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$:

$$\frac{\Pi \wr \Gamma \Rightarrow \phi \quad \Theta, \phi \wr \Delta \Rightarrow \psi}{\Theta, \phi \wr \Delta \not\Rightarrow \psi}\ \mathrm{AT}_{\mathsf{AL}} \tag{4}$$

An assumptive sequent-based argumentation framework for adaptive logics is now defined as:

Definition 38. An *adaptive logic sequent-based argumentation framework* (*(sequent-based) AL-framework* for short) for the lower limit logic $\mathbf{LLL} = \langle \mathcal{L}, \vdash \rangle$, with corresponding sequent calculus C, set of abnormalities Ω, set of formulas $\mathcal{S}$ and $\mathrm{AT}_{\mathsf{AL}}$ as sequent elimination rule, is a pair $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \left\langle \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \right\rangle$. Where $\mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ is the set of AL-arguments based on $\mathcal{S}$ and Ω, $\mathcal{AT} \subseteq \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}) \times \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ and $(a_1, a_2) \in \mathcal{AT}$ iff a_1 $\mathrm{AT}_{\mathsf{AL}}$-attacks a_2.

Example 25. Consider again the set $\mathcal{S} = \{p, q, \sim q \vee \sim p, \sim q \vee r, \sim p \vee r\}$ and let $\mathcal{AF}_{\mathsf{CLuN},\Omega}(\mathcal{S}) = \left\langle \mathrm{Arg}_{\mathsf{CLuN},\Omega}(\mathcal{S}), \mathcal{AT} \right\rangle$, where $\mathcal{AT}$ is based on $\mathrm{AT}_{\mathsf{AL}}$. Note that $!\psi \in \Omega$ if and only if ψ is a CLuN-formula and $!\psi = \psi \wedge \sim\psi$. Some of the arguments in $\mathrm{Arg}_{\mathsf{CLuN},\Omega}(\mathcal{S})$ are:

$$a = p \Rightarrow p \qquad b = q \Rightarrow q \qquad c = \sim q \vee \sim p \Rightarrow \sim q \vee \sim p$$
$$d = p, \sim p \vee \sim q \Rightarrow \sim q \vee !p \qquad e = q, \sim p \vee \sim q \Rightarrow \sim p \vee !q$$
$$f = !p \wr \mathcal{S} \Rightarrow !q \qquad g = !q \wr \mathcal{S} \Rightarrow !p \qquad h = !p \wr \mathcal{S} \Rightarrow r \qquad k = !q \wr \mathcal{S} \Rightarrow r.$$

As in previous sections, these are only a subset of the available arguments. See Figure 9 for a graphical representation.

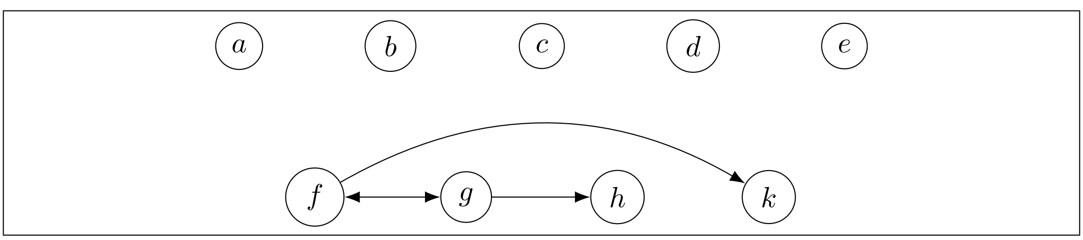

Figure 9: Part of the AL-framework of Example 25 for $\mathcal{S} = \{p, q, \sim q \vee \sim p, \sim q \vee r, \sim p \vee r\}$.

The consequence relation corresponding to an adaptive logic sequent-based argumentation framework $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S})$ is denoted by $\mid\!\sim^{\star}_{\Omega,\mathsf{sem}}$ for each semantics and

$\star \in \{\cap, \cup, \Cap\}$. Similar to Proposition 2 the following representational theorem can be shown:

Theorem 5. *Let* $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \left\langle Arg_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT}\right\rangle$ *be a sequent-based argumentation framework for the lower limit logic* $\mathbf{LLL} = \langle \mathcal{L}, \vdash \rangle$, *with corresponding sequent calculus* C, *set of abnormalities* Ω *and set of* $\mathcal{L}$*-formulas* $\mathcal{S}$.

1. $\mathcal{S} \mid\!\sim^{m,\Omega}_{\mathbf{LLL}} \phi$ *if and only if* $\mathcal{S} \mid\!\sim^{\Cap}_{\Omega,\mathsf{prf}} \phi$.
2. $\mathcal{S} \mid\!\sim^{r,\Omega}_{\mathbf{LLL}} \phi$ *if and only if* $\mathcal{S} \mid\!\sim^{\cap}_{\Omega,\mathsf{prf}} \phi$.
3. $\mathcal{S} \mid\!\sim^{n,\Omega}_{\mathbf{LLL}} \phi$ *if and only if* $\mathcal{S} \mid\!\sim^{\cup}_{\Omega,\mathsf{prf}} \phi$.

Due to the requirement of further notation and many technical details, the proof of the above theorem is placed in Appendix B.

For adaptive logic sequent-based argumentation frameworks, the representation of reasoning with maximally consistent subsets (recall Section 3.3) follows from the results in [56], in which it was shown that the consequence relations of adaptive logics are directly related to those of default assumptions, discussed in the next section. We therefore refer to Corollary 2 on page 53.

Example 26. Recall the setting from Example 25, for the sequent-based AL-framework $\mathcal{AF}_{\mathsf{CLuN},\Omega}(\mathcal{S}) = \left\langle \mathrm{Arg}_{\mathsf{CLuN},\Omega}(\mathcal{S}), \mathcal{AT}\right\rangle$, $\mathcal{S} = \{p, q, {\sim}q \vee {\sim}p, {\sim}q \vee r, {\sim}p \vee r\}$ and nine arguments were introduced. Two preferred extensions can be considered: $\mathcal{E}_1 \supseteq \{a, b, c, d, e, f, h\}$ and $\mathcal{E}_2 \supseteq \{a, b, c, d, e, g, k\}$. Hence $\mathcal{S} \mid\!\sim^{\Cap}_{\Omega,\mathsf{prf}} r$ but $\mathcal{S} \not\mid\!\sim^{\cap}_{\Omega,\mathsf{prf}} r$.

The above example shows that the consistency postulate (Definition 16) does not hold in sequent-based AL-frameworks. This is the case since $\mathcal{S}$ is not necessarily consistent. In fact, applying argumentation to a set of formulas $\mathcal{S}$ is only interesting when it is inconsistent, since otherwise the consequences would be the same as the conclusions that are already derivable with the lower limit logic. However, we will show below that the other two postulates (i.e., closure and sub-argument closure) can be shown for adaptive logic sequent-based argumentation.

In what follows let $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \left\langle \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT}\right\rangle$ be a sequent-based AL-framework for $\mathcal{S}$ a set of formulas, Ω a set of assumptions and $\mathrm{AT}_{\mathsf{AL}}$ the attack rule.

Lemma 23 (Sub-argument closure)**.** *Let* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$, *if* $a \in \mathcal{E}$ *then* $\mathsf{Sub}(a) \subseteq \mathcal{E}$.

Proof. Assume $a \in \mathcal{E}$. Let $a' \in \mathsf{Sub}(a)$ and assume $b \in \text{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ attacks a'. Thus $\mathsf{Conc}(b) \in \mathsf{Ass}(a')$. Since, by definition, $\mathsf{Ass}(a') \subseteq \mathsf{Ass}(a)$, it follows that b attacks a as well. Therefore, there is some $c \in \mathcal{E}$, which defends a, and thus a' from the attack by b. Hence, $a' \in \mathcal{E}$. $\square$

Lemma 24 (Closure)**.** *Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$, then $\mathsf{Concs}(\mathcal{E})$ is closed under strict rules.*

Proof. To show $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$. Note that $\mathsf{Concs}(\mathcal{E}) \subseteq \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$ by the reflexivity of $\vdash$, it remains to show that $\mathsf{Concs}(\mathcal{E}) \supseteq \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$. Suppose $\phi \in \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$. Then there are arguments $a_1, \ldots, a_n \in \mathcal{E}$ such that $\mathsf{Conc}(a_i) = \phi_i$, $\mathsf{Supp}(a_i) = \Gamma_i$ and $\mathsf{Ass}(a_i) = \Pi_i$ for $1 \leq i \leq n$ and $\phi_1, \ldots, \phi_n \vdash \phi$. By the completeness of C and applying [Cut] it follows that $a = \mathsf{Ass}(a_1), \ldots, \mathsf{Ass}(a_n) \wr \Gamma_1, \ldots, \Gamma_n \Rightarrow \phi$ is derivable. Note that any attacker of a is an attacker of one of the arguments $a_1, \ldots, a_n$. Since $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$, it follows that $a \in \mathcal{E}$ as well. Therefore $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}(\mathsf{Concs}(\mathcal{E}))$. $\square$

As noted after Example 26, the consistency postulate does not hold for sequent-based AL-frameworks since $\mathcal{S}$ can be inconsistent.

Theorem 6. *Let $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \left\langle Arg_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \right\rangle$ for $\mathcal{S}$ a set of formulas, Ω a set of abnormalities and AT_{AL} the attack rule. $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S})$ satisfies sub-argument closure and closure under strict rules under completeness-based semantics. But it does not satisfy consistency.*

4.3 Default Assumptions

In [48], Makinson presents three ways of turning a classical consequence relation nonmonotonic. The first of which uses additional background assumptions, called *default assumptions*. The resulting nonmonotonic consequence relation is directly related to the assumptive maximally consistent subset consequence relations from Definition 18, as well as to the adaptive consequence relation for minimal abnormality $\mid\!\sim^{m,\Omega}_{\mathbf{LLL}}$ from Definition 35, see [56]. Because of the relations between the different approaches, default assumptions are used in this section to show how adaptive sequent-based argumentation as introduced in the previous section is related to reasoning with (assumptive) maximally consistent subsets.

In addition to the default assumption consequence relation introduced in [48] ($\mid\!\sim^{\Cap,\mathcal{AS}}_{\mathsf{mcs}}$ in Section 3.3), the two other relations from Definition 18 (i.e., $\mid\!\sim^{\cap,\mathcal{AS}}_{\mathsf{mcs}}$ and $\mid\!\sim^{\cup,\mathcal{AS}}_{\mathsf{mcs}}$) will be considered as well. For the remainder of this section, it is assumed that $\mathcal{L}$ contains at least a negation operator $\neg$ as introduced in Section 2.

Example 27. Let CL be the core logic and, as in Example 18, $\mathcal{S} = \{s\}$ and $\mathcal{AS} = \{p, q, \neg p \vee \neg q, \neg p \vee r, \neg q \vee r\}$. Then $\mathsf{MCS}(\mathcal{S}, \mathcal{AS}) = \{\{p, \neg p \vee \neg q, \neg p \vee r, \neg q \vee r\}, \{q, \neg p \vee \neg q, \neg p \vee r, \neg q \vee r\}, \{p, q, \neg p \vee r, \neg q \vee r\}\}$. Clearly $\mathcal{S} \mathrel{|\!\sim}^{\star,\mathcal{AS}}_{\mathsf{mcs}} s$, additionally $\mathcal{S} \mathrel{|\!\sim}^{\star,\mathcal{AS}}_{\mathsf{mcs}} \neg p \vee r$ and $\mathcal{S} \mathrel{|\!\sim}^{\star,\mathcal{AS}}_{\mathsf{mcs}} \neg q \vee r$ for $\star \in \{\Cap, \cap\}$. Furthermore, $\mathcal{S} \mathrel{|\!\sim}^{\cup,\mathcal{AS}}_{\mathsf{mcs}} \phi$ for $\phi \in \mathcal{S} \cup \mathcal{AS}$.

Recall the entailment relations $\mathrel{|\!\sim}^{r,\Omega}_{\mathbf{LLL}}$ and $\mathrel{|\!\sim}^{m,\Omega}_{\mathbf{LLL}}$ from Definition 35. For $\mathcal{S}$ a set of formulas, **LLL** a monotonic logic, Ω a set of abnormalities and $\mathcal{AS}$ a set of default assumptions, in [56] it is shown that, where the maximally consistent subsets are taken with respect to the core logic **LLL**:

- $\mathcal{S} \mathrel{|\!\sim}^{m,\Omega}_{\mathbf{LLL}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\Cap,\neg\Omega}_{\mathsf{mcs}} \phi$ and similarly $\mathcal{S} \mathrel{|\!\sim}^{\Cap,\mathcal{AS}}_{\mathsf{mcs}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{m,\neg\mathcal{AS}}_{\mathbf{LLL}} \phi$
- $\mathcal{S} \mathrel{|\!\sim}^{r,\Omega}_{\mathbf{LLL}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\cap,\neg\Omega}_{\mathsf{mcs}} \phi$ and similarly $\mathcal{S} \mathrel{|\!\sim}^{\cap,\mathcal{AS}}_{\mathsf{mcs}} \phi$ iff $\mathcal{S} \mathrel{|\!\sim}^{r,\neg\mathcal{AS}}_{\mathbf{LLL}} \phi$.

Let $\mathrel{|\!\sim}^{\star,\mathrm{AL}}_{\Omega,\mathsf{prf}}$ for $\star \in \{\cap, \Cap, \cup\}$ denote the consequence relation corresponding to an adaptive logic sequent-based argumentation framework, as defined in the previous section. The following corollary is obtained from the results in [56], Theorem 5 and Proposition 2.

Corollary 2. *Let $\mathcal{AF}_{\mathsf{L},\mathcal{K}}(\mathcal{S}) = \left\langle Arg_{\mathsf{L},\mathcal{K}}(\mathcal{S}), \mathcal{AT} \right\rangle$, where $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ is a monotonic logic with corresponding sequent calculus C, $\mathcal{S}$ is a set of formulas and $\mathcal{K}$ is a set of default assumptions. Then:*

1. $\mathcal{S} \mathrel{|\!\sim}^{\Cap,\mathcal{AS}}_{\mathsf{mcs}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\mathcal{AS},\mathsf{prf}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{m,\neg\mathcal{AS}}_{\mathsf{L}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\Cap,AL}_{\neg\mathcal{AS},\mathsf{prf}} \phi$.
2. $\mathcal{S} \mathrel{|\!\sim}^{\cap,\mathcal{AS}}_{\mathsf{mcs}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathcal{AS},\mathsf{prf}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{r,\neg\mathcal{AS}}_{\mathsf{L}} \phi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\cap,AL}_{\neg\mathcal{AS},\mathsf{prf}} \phi$.

5 Related Literature

That one framework can be expressed by another (and vice versa), is nothing new. Relations between different formal approaches to nonmonotonic reasoning have been studied in the literature. As mentioned in the introduction, default logic is an instance of ABA [25]. The results in [56] were used in Section 4.3, to relate reasoning with maximally consistent subsets and the presented adaptive logic setting. In [43], ABA in relation to adaptive logics and vice versa, and ASPIC$^+$ to ABA were studied. Furthermore, reasoning with maximally consistent subsets and the related consequence relations are studied for other structured argumentation frameworks [2, 7, 9, 32, 41, 65], see [6] for a survey. By introducing assumptive sequent-based argumentation, a first step was made into the study of how sequent-based argumentation fits within this group of nonmonotonic reasoning systems.

Although different approaches to formal argumentation can be expressed by one another, one way of making a distinction between them is by their level of abstraction. Abstract argumentation (see Dung [36] and recall Section 2.1) is the most abstract and, as mentioned, it has been argued that it should be instantiated [55]. When looking at some approaches to logical argumentation (i.e., ABA, (assumptive) sequent-based argumentation and ASPIC^+ mentioned below), we can distinguish different levels of abstraction. ASPIC^+ [51, 54] is the most fine-grained perspective, where arguments come with a full proof structure. On the other hand, ABA is the most abstract of the three, since the semantics are applied to sets of sets of assumptions and the derivation of a conclusion is completely abstract. (Assumptive) sequent-based argumentation lies between these two approaches, it is less abstract than ABA, since an argument consists of a support set and a conclusion (and in the case of assumptive sequent-based argumentation, it is clear which strict and defeasible assumptions were used in the construction of an argument), but the exact derivation of the argument is not part of the argument itself.

In Section 4 we have only studied three of the well-known approaches to reasoning with defeasible assumptions. Two other well-known approaches were not mentioned here: ASPIC^+ [51, 54] and default logic [4, 59]. The first, like ABA, is an approach to structured argumentation, in which a distinction is made between axioms (the strict premises in the setting of this paper) and ordinary premises (the assumptions in this paper) and there are two types of rules: strict and defeasible ones. Moreover, an extensive study into the use of preferences was done in [51]. The result is an expressive structured argumentation system.

Research on ASPIC^+ has focused on applications and on the enrichment of the expressive power of the underlying language (such as the addition of preferences and having strict and defeasible rules) to be able to model different aspects of human reasoning. Research on sequent-based argumentation, which was introduced in the tradition of instantiating abstract argumentation with Tarskian logics (see also [24, 40]), has focused on studying logical properties of the resulting entailment relations and semantic extensions of an argumentation framework. As pointed out in, e.g., [29], defining argumentation frameworks with a robust meta-theory (e.g., satisfying the rationality postulates from [30, 31]), is not only interesting from a theoretical point of view, but is also beneficial for practical purposes. However, because of the many components from which an argumentation framework is constructed, this has been challenging for ASPIC^+ in case the set of strict rules is sufficiently rich (e.g., when these are based on a Tarskian logic) [29]. The only instantiation that satisfies all standard rationality postulates is $\text{ASPIC}^\ominus$, see [44]. In contrast, sequent-based argumentation has been studied with these challenges in mind. Classes of frameworks, instantiated with Tarskian logics, have been identified that satisfy all rationality postulates and other logical properties. Moreover, dynamic derivations ([11, 12]), introduced for sequent-based argumentation, provide a proof-theoretic approach to formal argumentation with which Gentzen-type sequent calculi can be applied to study the reasoning process of argumentation. Thus, while ASPIC$^+$ frameworks are very expressive and many possible applications have been studied, sequent-based argumentation has mainly been investigated to obtain a clear view of its meta-theoretic properties. How ASPIC$^+$ and (assumptive) sequent-based argumentation relate remains a question for future work. A good starting point for this investigation are the results in [28], where it is shown that both, in a setting without priorities, can be translated in a very simple argumentation setting.

The second approach, default logic, was already shortly mentioned in the introduction as one of the best-known approaches to reasoning with defeasible rules. There are however several specific additional problems one faces when representing default logic in sequent-based argumentation, besides the handling of default assumptions. One is that, although default logic has CL as underlying deductive system, classical connectives are not handled in a standard way when they occur in default rules. For example, disjunction does not allow for reasoning by cases and negation does not allow for contraposition. This is shown in the following example.

Example 28. Recall from the introduction that a default rule is of the form $\phi : \phi_1, \ldots, \phi_n/\psi$, which represents that ψ can be derived, if ϕ is given and no inconsistencies arise when $\phi_1, \ldots, \phi_n$ hold. Intuitively, one could expect that such a default rule can be translated into an assumptive sequent: $\phi_1, \ldots, \phi_n \wr \phi \Rightarrow \psi$. Suppose that $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS}), \mathcal{AT} \rangle$, where CL is the core logic, the sequent calculus is LK' with in addition the sequents obtained by translating the rules from $\mathcal{D}$ and $\mathcal{AS}$ contains the assumptions from the rules in $\mathcal{D}$ (i.e., $\phi_1, \ldots, \phi_n \in \mathcal{AS}$ if the rule above is part of $\mathcal{D}$).

- Let $\mathcal{S} = \{\neg q\}$ and let $\mathcal{D} = \left\{\frac{\emptyset:p}{q}\right\}$. This rule would be translated into $p \wr \Rightarrow q$. However, then by $\mathrm{AS}^r_{\mathcal{AS}}$, [$\Rightarrow\neg$] and [$\neg\Rightarrow$] the sequent $\neg q \Rightarrow \neg p$ is derivable. Moreover, since $\neg q \in \mathcal{S}$, $\neg q \Rightarrow \neg p$ is an argument that cannot be attacked: its set of assumptions is empty. Therefore $\neg p \in \mathsf{Concs}(\bigcap \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})))$, for any of the considered semantics. Yet $\neg p$ is not a default conclusion.

- Now suppose that $\mathcal{S} = \emptyset$ and let $\mathcal{D} = \left\{\frac{\emptyset:p}{q \vee t}, \frac{\emptyset:q}{v}\right\}$. These default rules are translated into the sequents $p \wr \Rightarrow q \vee t$ and $q \wr \Rightarrow v$. From these, by applying [$\Rightarrow\vee$], [$\vee\Rightarrow$], [Cut] and weakening, $p \wr \Rightarrow v \vee t$ can be derived. Since there are no attackers, $v \vee t \in \mathsf{Concs}(\bigcap \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}, \mathcal{AS})))$. However, in deault logic, $v \vee t$ is not a consequence.

In the first case, the problem arises because of the application of the sequent rules for negation. Similarly, in the second case, the rules for disjunction make it possible to derive $v \vee t$.

Because of examples such as the ones above, there is an asymmetry when reasoning classically with the consequences of applications of defaults where all connectives have their standard meaning, and reasoning with the defaults themselves. This asymmetry poses an additional challenge for a representation of default logic within the presented framework. As a solution for this, in the representation of default logic in ABA (see [25, §2.3]), classical logic cannot be applied to the assumptions in the default rule. However, one of the advantages of (assumptive) sequent-based argumentation, is the modularity of the approach (any logic with corresponding sequent calculus can be taken as the deductive base) and the availability of dynamic proofs [12], which allow for the automatic derivation of arguments. In light of this, the representation of default logic in assumptive sequent-based argumentation without such restrictions is left for future work.

6 Conclusion

In order to incorporate defeasible assumptions, sequent-based argumentation was extended to assumptive sequent-based argumentation. An additional component was added to each sequent, to contain the defeasible assumptions. As in sequent-based argumentation, any logic with a corresponding sound and complete sequent calculus can be taken as the core logic. It was shown how the assumptive framework can be generalized to a prioritized setting and several desirable properties were investigated. Furthermore, three well-known and much researched approaches to reasoning with assumptions were investigated in the context of assumptive sequent-based argumentation. It was shown that assumption-based argumentation (ABA), adaptive logics and default assumptions can be embedded in the here introduced framework.

Due to its generic and modular setting (only few requirements are placed on the logic and its corresponding calculus) assumptive sequent-based argumentation is a very general approach to reasoning with assumptions. In addition, the presented proofs do not rely on specific properties of the logic and only a few rules are assumed to be admissible in the calculus. This paper therefore paves the way to equip many well-known logics (e.g., intuitionistic logic and many modal logics) with defeasible assumptions. Moreover, although we required the logic to be Tarskian in this paper (recall Definition 1), this is not strictly necessary for the general definitions of assumptive sequent-based argumentation. It would therefore be possible to take a substructural logic, often characterized in terms of sequent calculi, as the core logic of an assumptive sequent-based argumentation framework. This would, for example, allow to incorporate a non-transitive system such as ST, which has been applied to study paradoxes [33, 34]. Note that such a system cannot be represented by a deductive system underlying ABA, since these are assumed to be transitive.

Though relations to other forms of reasoning with defeasible assumptions have been discussed in detail, it was not the objective of this paper to show how various approaches relate to each other, but instead to introduce a general logical argumentation framework, that allows for reasoning with assumptions in different settings. For example, situations in which assumptions are supposed to hold (such as in ABA) or supposed not to be satisfied (such as in adaptive logics), different core logics, such that different settings can be modeled, allowing for a priority function as additional input and with different mechanisms (Dung-style semantics and maximally consistent subsets). For the three approaches that were taken as example in Section 4, it was shown that the resulting sequent-based framework satisfies the rationality postulates from [30] (except for consistency in the case of adaptive logic). Therefore, assumptive sequent-based argumentation is a very general and flexible framework (it allows for many instances and can easily be adjusted to the requirements of different situations), that is also well-behaved (it satisfies some desirable properties).

The presented assumptive sequent-based argumentation framework can be extended to include other research on sequent-based argumentation. For example, the notion of a sequent can be generalized to a hypersequent, as in [27]. This way further core logics and calculi can be taken as the deductive base and the results of the extensive studies on sequent calculi in proof theory can be benefited from in formal argumentation. Furthermore, the dynamic proof theory from [12] can be adjusted to the assumptive setting presented here, thus extending this proof-theoretic approach to formal argumentation to account for defeasible assumptions. The availability of first-order sequent calculi opens up the possibility to investigate nonmonotonic systems such as circumscription. Though these extensions are left for future work, they will further strengthen the benefits of the assumptive sequent-based approach to formal argumentation. In addition, it would be interesting to know if assumptive sequent-based argumentation is more expressive than ABA, adaptive logics and/or default assumptions, or if they are equivalent. Therefore, in future work, we will investigate instances of the example frameworks, to see if these can express (assumptive) sequent-based argumentation.

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A Admissible Rules in the Minimal Calculus

In this appendix we show that the rules from Figure 8 are indeed derivable in any (single conclusioned) sequent calculus in which the rules from Figure 6 are admissible. We show this by sequent derivations in the minimal calculus from Figure 6.

Lemma 1. *Let* $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ *be a logic with corresponding sequent calculus* C*, in which the rules from Figure 6 are admissible. Then the rules from Figure 8 are admissible as well.*

Proof. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with corresponding sequent calculus C in which the rules from Figure 6 are admissible. We show that the rules from Figure 8 are admissible. Recall that Π is empty if C is a single conclusioned calculus and Δ contains at most one formula. We consider each of the axioms and rules in turn, note that each of the derivations can also be done in a single conclusioned calculus.

[$\Rightarrow\wedge\wedge$] First a useful derivation, that shows that $\phi_1, \ldots, \phi_n \Rightarrow \phi_1 \wedge \ldots \wedge \phi_n$ is derivable. We show the case for $n = 3$, the cases for other values of n are similar.

$$\dfrac{\dfrac{\dfrac{\dfrac{\phi_1 \Rightarrow \phi_1}{\phi_1, \phi_2 \Rightarrow \phi_1}\,[\text{LMon}]}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1}\,[\text{LMon}] \quad \dfrac{\dfrac{\phi_2 \Rightarrow \phi_2}{\phi_2, \phi_3 \Rightarrow \phi_2}\,[\text{LMon}]}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_2}\,[\text{LMon}]}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \wedge \phi_2}\,[\Rightarrow\wedge] \quad \dfrac{\dfrac{\phi_3 \Rightarrow \phi_3}{\phi_2, \phi_3 \Rightarrow \phi_3}\,[\text{LMon}]}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_3}\,[\text{LMon}]}{\phi_1, \phi_2, \phi_3 \Rightarrow \phi_1 \wedge \phi_2 \wedge \phi_3}\,[\Rightarrow\wedge]$$

[$\neg\neg\not\Rightarrow$]

$$\dfrac{\Gamma, \neg\neg\phi \Rightarrow \Delta \quad \dfrac{\dfrac{\phi \Rightarrow \phi}{\phi, \neg\phi \Rightarrow}\,[\neg\Rightarrow]}{\phi \Rightarrow \neg\neg\phi}\,[\Rightarrow\neg]}{\Gamma, \phi \Rightarrow \Delta}\,[\text{Cut}]$$

$$[\Rightarrow\neg\wedge] \qquad \cfrac{\cfrac{\cfrac{\Gamma, \phi_1, \ldots, \phi_n \Rightarrow \Pi}{\vdots}\ [\wedge\Rightarrow]}{\Gamma, \phi_1 \wedge \ldots \wedge \phi_n \Rightarrow \Pi}\ [\wedge\Rightarrow]}{\Gamma \Rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_n), \Pi}\ [\Rightarrow\neg]$$

$$[\not\Rightarrow\neg\wedge] \qquad \cfrac{\Gamma \Rightarrow \neg(\phi_1 \wedge \ldots \wedge \phi_n), \Pi \qquad \cfrac{\cfrac{}{\phi_1, \ldots, \phi_n \Rightarrow \phi_1 \wedge \ldots \wedge \phi_n}\ [\Rightarrow\wedge\wedge]}{\phi_1, \ldots, \phi_n, \neg(\phi_1 \wedge \ldots \wedge \phi_n) \Rightarrow}\ [\neg\Rightarrow]}{\Gamma, \phi_1, \ldots, \phi_n \Rightarrow \Pi}\ [\text{Cut}]$$

□

B Representation Adaptive Logics

In this appendix we turn to the proof of the following theorem:

Theorem 5. *Let* $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \left\langle Arg_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \right\rangle$ *be a sequent-based argumentation framework for the lower limit logic* $\mathbf{LLL} = \langle \mathcal{L}, \vdash \rangle$*, with corresponding sequent calculus* C*, set of abnormalities* Ω *and set of* $\mathcal{L}$*-formulas* $\mathcal{S}$*.*

1. $\mathcal{S} \mathrel{|\!\sim}^{m,\Omega}_{\mathbf{LLL}} \phi$ *if and only if* $\mathcal{S} \mathrel{|\!\sim}^{\Cap}_{\Omega,\mathsf{prf}} \phi$*.*
2. $\mathcal{S} \mathrel{|\!\sim}^{r,\Omega}_{\mathbf{LLL}} \phi$ *if and only if* $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\Omega,\mathsf{prf}} \phi$*.*
3. $\mathcal{S} \mathrel{|\!\sim}^{n,\Omega}_{\mathbf{LLL}} \phi$ *if and only if* $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\Omega,\mathsf{prf}} \phi$*.*

In order to prove the above theorem, some further notation, facts and lemmas are necessary. Let $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \left\langle \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \right\rangle$ be a sequent-based argumentation framework as defined in Definition 38, with as the core logic the lower limit logic $\mathbf{LLL}$, the corresponding sequent calculus C, where Ω is a set of abnormalities and $\mathcal{S}$ is a set of formulas.

Notation 4. Let $\Psi \in \Phi(\mathcal{S})$, define $\mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S}) =_{\mathrm{df}} \{\Pi \wr \Gamma \Rightarrow \psi \mid \Pi \subseteq \Omega \setminus \Psi \text{ and } \Gamma \subseteq \mathcal{S}\}$.

The following result from [62, Lemma 5.5.1] will be useful in the proof of Theorem 5.

Lemma 25. *Let* Ξ *be a set of finite subsets of* $\mathcal{S}$*, and let* $\mathsf{CS}(\cdot)$ *denote the function that returns the set of all the choice sets of a set of sets. Let* $\Psi = \{\phi_i \mid i \in \mathbb{N}^+\} \in \mathsf{CS}(\Xi)$ *and define* $\widehat{\Psi} = \bigcap_{i \geq 0} \Psi_i$ *where* $\Psi_0 = \Psi$ *and (where* $i + 1 \leq n$*):*

$$\Psi_{i+1} = \begin{cases} \Psi_i & \text{if there is a } \Delta \in \Xi \text{ such that } \Psi_i \cap \Delta = \{\phi_{i+1}\} \\ \Psi_i \setminus \{\phi_{i+1}\} & \text{else} \end{cases}$$

we have: $\widehat{\Psi} \in \min_{\subset}(\mathsf{CS}(\Xi))$.

Corollary 3.

1. *For each choice set* Ψ *there is a minimal choice set* Ψ' *such that* $\Psi' \subseteq \Psi$.
2. *Let* $\Psi \in \Phi(\mathcal{S})$, *then for each* $\phi \in \Psi$ *there is a* $\Pi \in \Sigma(\mathcal{S})$ *such that* $\Psi \cap \Pi = \{\phi\}$.

Proof. Consider both items:

1. This follows immediately from Lemma 25.
2. Let $\Psi \in \Phi(\mathcal{S})$ and suppose that there is some $\phi \in \Psi$, such that there is no $\Pi \in \Sigma(\mathcal{S})$ for which $\Psi \cap \Pi = \{\phi\}$. Since Ψ is a choice set of $\Sigma(\mathcal{S})$, there must be some $\Pi \in \Sigma(\mathcal{S})$ such that $\phi \in \Pi$. Therefore for each $\Pi \in \Sigma(\mathcal{S})$ such that $\phi \in \Pi$, $\Psi \cap \Pi \supsetneq \{\phi\}$. However, then $\Psi \setminus \{\phi\}$ would also be a choice set of $\Sigma(\mathcal{S})$. A contradiction to the minimality of Ψ. □

Fact 1. *Let* $\Gamma \subseteq \mathcal{S}$ *and* $\Pi \subseteq \Omega$ *be finite. Moreover, let* $\Psi \in \Phi(\mathcal{S})$. *Then:*

1. *For each* $\phi \in \Psi$, $\Pi \setminus \{\phi\} \wr \Gamma \Rightarrow \phi \in Arg_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$.
2. $\mathsf{Concs}(Arg_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})) \supseteq \Psi$.
3. *Let* $\Pi \wr \Gamma \Rightarrow \phi \in Arg_{\mathbf{LLL},\Omega}(\mathcal{S})$ *and* $\mathsf{S} \subseteq Arg_{\mathbf{LLL},\Omega}(\mathcal{S})$. *If* $\Pi \cap \mathsf{Concs}(\mathsf{S}) \neq \emptyset$ *then* S *attacks* $\Pi \wr \Gamma \Rightarrow \phi$.
4. *Let* $\Pi \wr \Gamma \Rightarrow \phi \in Arg_{\mathbf{LLL},\Omega}(\mathcal{S})$. *If* $\Pi \cap \Psi \neq \emptyset$ *then* $Arg_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ *attacks the argument* $\Pi \wr \Gamma \Rightarrow \phi$.
5. *Let* $\phi \in \Omega$, *then for any* $\Pi \wr \Gamma \Rightarrow \phi \in Arg_{\mathbf{LLL},\Omega}(\mathcal{S})$, *there is some* $\Pi' \subseteq \Pi \cup \{\phi\}$ *such that* $\Pi' \in \Sigma(\mathcal{S})$.

Proof. Let $\Gamma \subseteq \mathcal{S}$, $\Pi \subseteq \Omega$ and $\Psi \in \Phi(\mathcal{S})$. Consider each of the items in turn.

1. Let $\phi \in \Psi$, then, by Corollary 3.2 there is some $\Pi \in \Sigma(\mathcal{S})$, such that $\phi \in \Pi$ and $\Psi \cap \Pi = \{\phi\}$. Where $\mathrm{Dab}(\Pi)$ is a minimal Dab consequence of $\mathcal{S}$. Thus $\mathcal{S} \vdash_{\mathbf{LLL}} \mathrm{Dab}(\Pi)$. Hence, by the completeness of C for **LLL** for some $\Gamma \subseteq \mathcal{S}$, $\Gamma \Rightarrow \bigvee \Pi$ is derivable. And thus, by applying RC (several times) $\Pi \setminus \{\phi\} \wr \Gamma \Rightarrow \phi$ is derivable in C'. Since $\Gamma \subseteq \mathcal{S}$, $\Pi \setminus \{\phi\} \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$. Moreover, since $\Psi \cap \Pi = \{\phi\}$, $(\Pi \setminus \{\phi\}) \subseteq \Omega \setminus \Psi$. Therefore $\Pi \setminus \{\phi\} \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$.
2. Suppose that $\phi \in \Psi$. Then, by the previous item $\phi \in \mathsf{Concs}(\mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S}))$.

3. Let $\Pi \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ and $\mathsf{S} \subseteq \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ and suppose that $\Pi \cap \mathsf{Concs}(\mathsf{S}) \neq \emptyset$. Then there is some $a \in \mathsf{S}$, such that $\mathsf{Conc}(a) \in \Pi$. By definition of the attack rule $\mathrm{AT}_{\mathsf{AL}}$, it follows that a attacks $\Pi \wr \Gamma \Rightarrow \phi$.

4. Let $\Pi \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ and suppose that $\Pi \cap \Psi \neq \emptyset$. Thus there is some $\psi \in \Pi$, such that $\psi \in \Psi$. By Item 2, $\psi \in \mathsf{Concs}(\mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S}))$. Thus, by the previous item, $\mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ attacks $\Pi \wr \Gamma \Rightarrow \phi$.

5. Let $\phi \in \Omega$ and $a = \Pi \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$, since a is derivable in C', $a' = \Gamma \Rightarrow \phi \vee \bigvee \Pi$ is derivable in C as well. Thus, by the soundness and monotonicity of C for **LLL** $\mathcal{S} \vdash \phi \vee \bigvee \Pi$. Hence, by the definition of minimal Dab consequences (Notation 3), there is some $\Pi' \subseteq \Pi \cup \{\phi\}$ such that $\mathrm{Dab}(\Pi')$ is a minimal Dab consequence for Γ and thus $\Pi' \in \Sigma(\Gamma)$. □

The following facts can be found in [21, 62]:

Fact 2.

1. *$\mathcal{S} \mid\!\sim^{r,\Omega}_{\mathbf{LLL}} \phi$ iff there is a (finite) set of abnormalities $\Pi \subseteq \Omega \setminus U(\mathcal{S})$ such that $\mathcal{S} \vdash_{\mathbf{LLL}} \phi \vee Dab(\Pi)$ [21, Theorem 7].*

2. *$\mathcal{S} \mid\!\sim^{m,\Omega}_{\mathbf{LLL}} \phi$ iff for all $\Psi \in \Phi(\mathcal{S})$ there is a $\Pi \subseteq \Omega \setminus \Psi$ such that $\mathcal{S} \vdash_{\mathbf{LLL}} \phi \vee Dab(\Pi)$ [21, Theorem 8].*

3. *$\mathcal{S} \mid\!\sim^{n,\Omega}_{\mathbf{LLL}} \phi$ iff there is a $\Pi \subseteq \Omega$ such that $\mathcal{S} \vdash_{\mathbf{LLL}} \phi \vee Dab(\Pi)$ and for some $\Psi \in \Phi(\mathcal{S})$, $\Psi \cap \Pi = \emptyset$ [62, Theorem 2.8.3].*

4. *$U(\mathcal{S}) = \bigcup \Phi(\mathcal{S})$ [21, Theorem 11.5].*

Fact 3. *If $\Gamma \vdash_{\mathbf{LLL}} \phi \vee Dab(\Pi)$, where $\Gamma \subseteq \mathcal{S}$ there is some $\Gamma' \subseteq \Gamma$ such that $\Pi \wr \Gamma' \Rightarrow \phi \in Arg_{\mathbf{LLL},\Omega}(\mathcal{S})$.*

Proof. Suppose that $\Gamma \vdash_{\mathbf{LLL}} \phi \vee \mathrm{Dab}(\Pi)$ and that $\Gamma \subseteq \mathcal{S}$. By the completeness of C for **LLL**, $\Gamma' \Rightarrow \phi \vee \mathrm{Dab}(\Pi)$ is derivable in C for some $\Gamma' \subseteq \Gamma$. Thus, by applying RC (several times), $\Pi \wr \Gamma' \Rightarrow \phi$ is derivable in C. Since $\Gamma \subseteq \mathcal{S}$, $\Pi \wr \Gamma' \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$. □

Before proving the theorem, we first show how preferred extensions relate to minimal Dab consequences. In particular, we show that preferred extensions are closely related to the above defined set of arguments $\mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$.

Lemma 26. *Let $\Psi \in \Phi(\mathcal{S})$, then $Arg_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S}) \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$.*

Proof. Let $\Psi \in \Phi(\mathcal{S})$ and let $\mathcal{E} = \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$. We show that $\mathcal{E}$ is admissible and maximal.

$\mathcal{E} = \mathbf{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ **is admissible.** Let $a = \Theta \wr \Delta \Rightarrow \psi \in \mathcal{E}$, and assume $b = \Pi \wr \Lambda \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ attacks a. By Definition 37, it follows that $\phi \in \Theta$. Note that, since $\phi \in \Theta$, $\phi \in \Omega$. By Fact 1.5, there is some $\Pi' \subseteq \Pi \cup \{\phi\}$ such that $\Pi' \in \Sigma(\mathcal{S})$. Since $\phi \notin \Psi$ (by assumption $a \in \mathcal{E}$) and $\Psi \cap \Pi' \neq \emptyset$ (by Corollary 3.2, recall that $\Pi' \in \Sigma(\mathcal{S})$), also $\Psi \cap \Pi \neq \emptyset$. Therefore, $b \notin \mathcal{E}$. From which it follows that $\mathcal{E}$ is conflict-free and since $\Psi \cap \Pi \neq \emptyset$, it follows by Fact 1.4 that $\mathcal{E}$ is admissible.

$\mathcal{E}$ **is maximally admissible**. Assume that there is an argument $\Pi \wr \Lambda \Rightarrow \gamma \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}) \setminus \mathcal{E}$ such that $\mathcal{E} \cup \{\Pi \wr \Lambda \Rightarrow \gamma\}$ is admissible. By Fact 1.4 it follows that $\Pi \cap \Psi = \emptyset$. Hence $\Pi \wr \Lambda \Rightarrow \gamma \in \mathcal{E}$: $\mathcal{E}$ is maximally admissible. □

Lemma 27. *Let* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$ *and* $\Pi \in \Sigma(\mathcal{S})$*, then* $\mathsf{Concs}(\mathcal{E}) \cap \Pi \neq \emptyset$*.*

Proof. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$. Let $\Sigma_{\mathcal{E}}$ denote all sets Π' in $\Sigma(\mathcal{S})$ for which $\mathsf{Concs}(\mathcal{E}) \cap \Pi' = \emptyset$. Assume towards a contradiction that $\Sigma_{\mathcal{E}} \neq \emptyset$. Let Ψ be a minimal choice set over $\Sigma_{\mathcal{E}}$. That Ψ exists follows from Corollary 3.1. From Corollary 3.2 it is known that for each $\phi \in \Psi$ there is a $\Pi_\phi \in \Sigma_{\mathcal{E}}$ such that $\Psi \cap \Pi_\phi = \{\phi\}$. Since $\Pi_\phi \in \Sigma_{\mathcal{E}} \subseteq \Sigma(\mathcal{S})$, there is some $\Lambda \subseteq \mathcal{S}$ such that $\Lambda \vdash_{\mathbf{LLL}} \mathrm{Dab}(\Pi_\phi)$. By the completeness of C for L, $\Lambda \Rightarrow \bigvee \Pi_\phi$ is derivable and thus, by (several) application(s) of RC, so is $\Pi_\phi \setminus \{\phi\} \wr \Lambda \Rightarrow \phi$. Let $\mathcal{E}' = \mathcal{E} \cup \{\Pi_\phi \setminus \{\phi\} \wr \Lambda \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}) \mid \phi \in \Sigma_{\mathcal{E}}\}$. It can be shown that $\mathcal{E}'$ is admissible:

$\mathcal{E}'$ **is conflict-free.** Suppose $a = \Pi_\phi \setminus \{\phi\} \wr \Lambda' \Rightarrow \phi$ attacks $\mathcal{E}$. By assumption $\mathcal{E}$ is admissible, hence there is an argument $a' \in \mathcal{E}$ such that a' attacks a. From this it follows that $\mathsf{Concs}(\mathcal{E}) \cap (\Pi_\phi \setminus \{\phi\}) \neq \emptyset$, which is a contradiction with the assumptions that $\Pi_\phi \in \Sigma_{\mathcal{E}}$ and $\mathsf{Concs}(\mathcal{E}) \cap \Pi = \emptyset$ for each $\Pi \in \Sigma_{\mathcal{E}}$. For the same reason, no argument $b \in \mathcal{E}$ attacks $\Pi_\phi \setminus \{\phi\} \wr \Lambda^* \Rightarrow \phi$, for any $\Lambda^* \subseteq \mathcal{S}$. Now suppose that $\Pi_\phi \setminus \{\phi\} \wr \Lambda \Rightarrow \phi$ attacks $\Pi_\psi \setminus \{\psi\} \wr \Lambda' \Rightarrow \psi$. By definition $\phi \in \Pi_\psi$, which is a contradiction with the assumption that $\phi \in \Psi$ and $\Psi \cap \Pi_\psi = \{\psi\}$. Hence $\mathcal{E}'$ is conflict-free.

$\mathcal{E}'$ **defends its arguments.** Suppose, for some argument $b = \Theta \wr \Delta \Rightarrow \psi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}) \setminus \mathcal{E}'$, that b attacks $\Pi_\phi \setminus \{\phi\} \wr \Lambda \Rightarrow \phi$ and $\mathcal{E}$ does not attack b. Since $\mathcal{E}'$ is conflict-free it follows that $\mathsf{Concs}(\mathcal{E}) \cap (\{\psi\} \cup \Theta) = \emptyset$. Note that, by Definition 37 $\psi \in \Pi_\phi \setminus \{\phi\} \subseteq \Omega$, thus by Fact 1.5, there is a $\Pi \in \Sigma(\mathcal{S})$ such that $\Pi \subseteq \{\psi\} \cup \Theta$. Hence $\Pi \in \Sigma_{\mathcal{E}}$. By the construction of $\mathcal{E}'$, for each $\gamma \in \Pi$ and any $\Delta' \subseteq \mathcal{S}$ such that $c = \Pi_\gamma \setminus \{\gamma\} \wr \Delta' \Rightarrow \gamma$ is derivable in C', $c \in \mathcal{E}'$. Note that $\gamma \neq \psi$, since it was shown above that $\mathcal{E}'$ is conflict-free and otherwise $\mathcal{E}'$ would attack $\Pi_\phi \setminus \{\phi\} \wr \Lambda \Rightarrow \phi$. Thus $\gamma \in \Theta$. Therefore $\Pi_\gamma \setminus \{\gamma\} \wr \Delta' \Rightarrow \gamma$ attacks b, and thus $\mathcal{E}'$ is admissible.

However, since $\mathcal{E}'$ attacks b and, by assumption, $\mathcal{E}$ does not, $\mathcal{E} \subsetneq \mathcal{E}'$. This is a contradiction with $\mathcal{E}$ being a preferred extension. □

Lemma 28. *If* $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$*, then there is a* $\Psi \in \Phi(\mathcal{S})$ *such that* $\mathcal{E} = \mathit{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$.

Proof. Suppose $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$. By Lemma 27 and Corollary 3.1 it follows that $\mathsf{Concs}(\mathcal{E}) \supseteq \Psi$ for some $\Psi \in \Phi(\mathcal{S})$. By Fact 1.3, for all arguments $\Pi \wr \Lambda \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$, with $\Pi \cap \Psi \neq \emptyset$, $\Pi \wr \Lambda \Rightarrow \phi \notin \mathcal{E}$. Hence $\mathcal{E} \subseteq \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$, with Lemma 26 it thus follows that $\mathcal{E} = \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$. □

From Lemmas 26 and 28 it follows that:

Corollary 4. $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$ *iff* $\mathcal{E} = \mathit{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ *for some* $\Psi \in \Phi(\mathcal{S})$.

With this Theorem 5 can be proven:

Proof. Let $\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}) = \left\langle \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S}), \mathcal{AT} \right\rangle$ be a sequent-based AL-framework for the lower limit logic $\mathbf{LLL} = \langle \mathcal{L}, \vdash \rangle$, with corresponding set of abnormalities Ω and $\mathcal{S}$ a set of $\mathcal{L}$-formulas. Consider each strategy, in both directions.

1. Start with minimal abnormality.

   ($\Rightarrow$) Suppose that $\mathcal{S} \mid\!\sim^{m,\Omega}_{\mathbf{LLL}} \phi$. By Fact 2.2 and Fact 3, for all $\Psi \in \Phi(\mathcal{S})$ there is a $\Pi \subseteq \Omega \setminus \Psi$ such that $\Pi \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$, for some $\Gamma \subseteq \mathcal{S}$. By Corollary 4, for each preferred extension $\mathcal{E}$ there is a $\Psi \in \Phi(\mathcal{S})$ such that $\mathcal{E} = \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$. From this it follows that for each preferred extension $\mathcal{E}$ there is an argument $\Pi' \wr \Gamma' \Rightarrow \phi \in \mathcal{E}$ for some $\Gamma' \subseteq \mathcal{S}$ and $\Pi' \subseteq \Omega \setminus \Psi$. Therefore $\phi \in \mathsf{Concs}(\mathcal{E})$ for each $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$. Hence $\mathcal{S} \mid\!\sim^{\Cap}_{\Omega,\mathsf{prf}} \phi$.

   ($\Leftarrow$) Now suppose that $\mathcal{S} \mid\!\sim^{\Cap}_{\Omega,\mathsf{prf}} \phi$. Let $\Psi \in \Phi(\mathcal{S})$ be arbitrary. Then, by Corollary 4, there is an $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$, such that $\mathcal{E} = \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$. Hence, there is an argument $\Pi \wr \Gamma \Rightarrow \phi \in \mathcal{E}$, for some $\Gamma \subseteq \mathcal{S}$, from which it follows that $\Pi \subseteq \Omega \setminus \Psi$. Thus, by Definition 36 and the definition of the sequent RC-rule, $\Gamma \Rightarrow \phi \vee \mathrm{Dab}(\Pi)$ is derivable in C'. Hence, by soundness of C and monotonicity of $\mathbf{LLL}$, $\mathcal{S} \vdash_{\mathbf{LLL}} \phi \vee \mathrm{Dab}(\Pi)$. Since $\Psi \in \Phi(\mathcal{S})$ is arbitrary, for each such Ψ, such a Π exists. Therefore, by Fact 2.2 it follows that $\mathcal{S} \mid\!\sim^{m,\Omega}_{\mathbf{LLL}} \phi$.

2. The reliability strategy.

   ($\Rightarrow$) Suppose that $\mathcal{S} \mid\!\sim^{r,\Omega}_{\mathbf{LLL}} \phi$. By Fact 2.1 and Fact 2.4, there is a set $\Pi \subseteq \Omega \setminus \bigcup \Phi(\mathcal{S})$ of abnormalities, such that $\mathcal{S} \vdash_{\mathbf{LLL}} \phi \vee \mathrm{Dab}(\Pi)$. By Fact 3 for some $\Gamma \subseteq \mathcal{S}$ it follows that $\Pi \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$. Furthermore, by the construction of $\mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$ and Corollary 4, $\Pi \wr \Gamma \Rightarrow \phi \in \mathcal{E}$, for every $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$. Hence $\mathcal{S} \mid\!\sim^{\cap}_{\Omega,\mathsf{prf}} \phi$.

($\Leftarrow$) Now suppose that $\mathcal{S} \mathrel{|\!\!\sim}^{\cap}_{\Omega,\mathsf{prf}} \phi$. By assumption there is an argument $a = \Pi \wr \Gamma \Rightarrow \phi$ for some $\Gamma \subseteq \mathcal{S}$ such that for all $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$, $a \in \mathcal{E}$. By Corollary 4 and the construction of $\mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$, it follows that $\Pi \cap \Psi = \emptyset$, for every $\Psi \in \Phi(\mathcal{S})$. Hence, $\Pi \subseteq \Omega \setminus \bigcup \Phi(\mathcal{S})$. By the soundness of C and the RC-rule that is available in C', for some $\Gamma \subseteq \mathcal{S}$, we have that $\Gamma \vdash_{\mathbf{LLL}} \phi \vee \mathrm{Dab}(\Pi)$. Hence, by Fact 2.1 and the monotonicity and soundness of **LLL** $\mathcal{S} \mathrel{|\!\!\sim}^{r,\Omega}_{\mathbf{LLL}} \phi$.

3. The normal selections strategy.

   ($\Rightarrow$) Suppose that $\mathcal{S} \mathrel{|\!\!\sim}^{n,\Omega}_{\mathbf{LLL}} \phi$. By Fact 2.3, there is a $\Pi \subseteq \Omega$ such that (a) $\mathcal{S} \vdash_{\mathbf{LLL}} \phi \vee \mathrm{Dab}(\Pi)$ and (b) for some $\Psi \in \Phi(\mathcal{S})$, $\Psi \cap \Pi = \emptyset$. From (a) and Fact 3, $a = \Pi \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$ for some $\Gamma \subseteq \mathcal{S}$. By construction of $\mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$, since by (b) $\Psi \cap \Pi = \emptyset$, $a \in \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$. Thus, by Corollary 4, $a \in \mathcal{E}$, for some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$. Therefore $\mathcal{S} \mathrel{|\!\!\sim}^{\cup}_{\Omega,\mathsf{prf}} \phi$.

   ($\Leftarrow$) Now assume that $\mathcal{S} \mathrel{|\!\!\sim}^{\cup}_{\Omega,\mathsf{prf}} \phi$. Then there is an $a = \Pi \wr \Gamma \Rightarrow \phi \in \mathrm{Arg}_{\mathbf{LLL},\Omega}(\mathcal{S})$, with $\Gamma \subseteq \mathcal{S}$, such that $a \in \mathcal{E} \in \mathsf{Ext}_{\mathsf{prf}}(\mathcal{AF}_{\mathbf{LLL},\Omega}(\mathcal{S}))$. By Corollary 4, there is a $\Psi \in \Phi(\mathcal{S})$, such that $\mathcal{E} = \mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$. Hence, by construction of $\mathrm{Arg}_{\mathbf{LLL},\overline{\Psi}}(\mathcal{S})$, $\Psi \cap \Pi = \emptyset$. Moreover, by adjusting the derivation of a, such that RC is never applied, the sequent $a' = \Gamma \Rightarrow \phi \vee \bigvee \Pi$ is derived. By soundness and monotonicity of C it follows that $\mathcal{S} \vdash_{\mathbf{LLL}} \phi \vee \mathrm{Dab}(\Pi)$. Thus, by Fact 2.3 $\mathcal{S} \mathrel{|\!\!\sim}^{n,\Omega}_{\mathbf{LLL}} \phi$. $\square$

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