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Mathematics

Regular Stars as Critical Points

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ABSTRACT. We show that star-shaped regular planar polygons are non-degenerate critical points of certain natural functions on appropriate configuration spaces and compute their Morse indices. The main new results are concerned with the Coulomb potential of equally charged vertices on the space of *n*-gons with fixed perimeter. *© 2018 Bull. Georg. Natl. Acad. Sci.*

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As is well known convex planar regular n-gons are extremal for several natural functions on n-tuples of points on a circle and on the set of n-gons with fixed perimeter [1]. In case of area and perimeter**,** results of such type are known for a long time [1**].** An analogous result is valid for certain models of interacting particles confined to a planar contour of fixed length [2]. The main aim of the present paper is to complement and generalize these results in the setting of extremal problems on configuration spaces developed in [3, 4].

Namely, we show that, in addition to convex regular polygons, regular star-shaped polygons can also be interpreted as critical points of functions on appropriate configuration spaces. In particular, as has recently been proven in [5, 6], regular n-pointed stars are critical points of oriented area in the space of n-gons with fixed perimeter.

Here and throughout the paper the term *regular n-gon* means as usual a *convex equilateral equiangular planar n-gon.*The term *regular n-pointed star* means a *self-intersecting equilateral equiangular planar n-gon* [1]. For brevity, we often say simply *regular n-star.* It is also convenient to refer to the set of all planar n-gons with fixed perimeter L as *L-isoperimetric n-gons* and denote it by $Per_n(L)$. The group $Iso_+(2)$ of orientation preserving isometries of the plane acts on $Per_n(L)$ in an obvious way. The *configuration space of L-isoperimetric polygons* $M_n(L)$ is defined as the set of orbits of this action of $Iso_+(2)$. The latter set endowed with the canonical topology of orbit space is naturally homeomorhic to the complex projective space CP^{n-2}. So one can consider $M_n(L)$ as a compact differentiable manifold diffeomorphic to CP^{n-2}.

In the sequel we discuss a number of results concerned with the representation of regular n-stars as critical points of various functions defined on $M_n(L)$ (cf. [4], [6]). In particular, we show that regular stars can be interpreted as non-degenerate critical points of Coulomb energy on $M_n(L)$. We also compute the critical values of oriented area and Coulomb energy for isoperimetric n-gons and indicate some corollaries of the main results.

We begin by reproducing some definitions and results on critical points of oriented area given in [6, **7**] (cf. also [8]). For $n \geq 3$, denote by $M_n = M_n(1)$ the moduli space of n-gons with fixed perimeter equal to 1 (cf. [6]). The oriented area defines a differentiable function A on M_n and we may consider its critical points. Due to the skew-symmetry of oriented area, in this context one should distinguish between the two possible orientations of a regular n-star. For even n, we also consider a *degenerate n-gon* with all sides coinciding, which will be called *n-pile*. Obviously, each regular star P is *cyclic* (inscribed in a circle) and its *circumcenter* (center of circumscribed circle) is called the *center* of P. We always assume that the center of P coincides with the origin of the reference coordinate system and the first vertex lies on the x-axis.

The *winding number* $w(P)$ of oriented n-star P is defined as the (signed) number of full turns of the radius-vector of a point p making a full positive turn along P. Notice that non-degenerate regular n-star with given perimeter is completely determined (up to orientation preserving congruence) by its *winding number* $w(P)$. The winding number of pile is set equal to zero. It is easy to show that the spectrum of possible values of $w(P)$ for n-star P is the union of two integer segments $[-[n-1/2], -1]$ and $[1, [n-1/2]]$, where the inner square brackets denote the "entier".

The following three results established in [6], [8] serve as a paradigm for our considerations. We formulate them as propositions in order to distinguish from the new results presented as theorems.

Proposition 1. ([6, 8]) *Critical points of A on M_n include the regular n-stars with both orientations. For even n, n-pile is also a critical point.*

The proof given in [8] is analytical and technically rather involved. A simple geometric proof can be found in [6]. In some cases a regular n-star is a non-degenerate cutical point of A and one can compute its Morse index.

Proposition 2. ([6]) *The Morse index of A at non-degenerate regular n-star P with winding number $w(P)$ equals $2w(P) - 2$ if $w(P)>0$, and $2n - 2w(P) - 2$ if $w(P)<0$.*

We proceed by explicating and complementing these results by giving more detailed information about the shape and size of regular stars with given perimeter L. To this end we need a well known formula presented as lemma for convenience of reference.

Lemma 1. *The length of diagonal joining j-th and i-th vertices of regular n-gon with side a is*

$$d_{ij}(a) = \frac{a \sin \frac{\pi |i-j|}{n}}{\sin \frac{\pi}{n}}. \tag{1}$$

Given a regular n-gon with side a one obtains an w-folded regular n-star by joining its vertices with $|i - j| = w$, where the indices are considered modulo n. The perimeter of such n-star equals $nd_{i(i+w)}(a)$. So the following lemma is a corollary of the above formula.

Lemma 2. *The distance between adjacent vertices of w-fold n-star with perimeter L equals* $a = \dfrac{L}{n}\dfrac{a\sin\dfrac{\pi}{n}}{\sin\dfrac{w\pi}{n}}$.

Denoting by conv P the convex hull of P which is obviously a convex regular n-gon we have the following corollary.

Corollary 1. *If P is a w-fold n-star with perimeter L then the perimeter of conv P is* $L\dfrac{\sin\dfrac{L}{n}}{\sin\dfrac{w\pi}{n}}$ *and its circumradius is* $\dfrac{L}{2n\sin\dfrac{w\pi}{n}}$.

Any regular n-star P with $|w(P)| > 1$ defines another convex regular n-gon P' called the core. By definition it is the intersection of all triangles $\Delta p_{i-1}p_i p_{i+1}$ constituting P. The sides of P are obtained as extensions of the sides of P' and P is uniquely determined by P'. In order to compute the critical values of oriented area it is useful to know the side of the core. This yields an explicit formula which we only present for 2-fold regular 5-star for the reason of space.

Lemma 3. *The side of the core* P' *of 2-fold regular 5-star P with perimeter equals*

$$\frac{L}{5}\left[1-2\left(1-\frac{1+\sqrt{2-2\cos\dfrac{\pi}{5}}}{1+\cos\dfrac{\pi}{5}}\right)\right].$$

This formula is proved using elementary geometric considerations.

To compute the area of such a 2-fold regular 5-star notice that it equals the area of the core plus the total area of 5 isosceles triangles constituting the complement of core P in P. Since the base of each such triangle is the side of the core and the opposite angle is $\dfrac{\pi}{10}$ we easily find the area of P.

We omit the resulting explicit formula for the area of P. In the same way one can obtain similar results for the critical points and critical values of oriented area in M_n.

Consider now the Coulomb potential E of n equal charges freely sliding along a flexible non-extendible contour of length L as in [2]. It is easy to show that all critical points have polygonal shapes. Thus we can again work in the isoperimetric configuration space $M_n(L)$. Following [2] the configuration with the lowest possible energy is called the ground state. As was shown in [2] the ground state is represented by a regular n-gon with the side L/n.We extend this result in the following way:

Theorem 1. *Non-degenerate regular n-stars of perimeter L are non-degenerate local minima of E on* $M_n(L)$.

One can easily convince himself that regular stars are critical by considering the forces diagram and taking the symmetry into account. An analytical proof can be obtained using the expression for Coulomb energy written in polar coordinates. The fact that each regular star is a non-degenerate minimum of Coulomb potential follows from the formula for the Hessian matrix of Coulomb energy calculated at regular n-gon in [2] by noticing that the Coulomb energy of a regular n-star P is the same as of its convex hull conv

P. The latter observation enables us to compute also the critical values of E on $M_n(L)$ using Lemma 2 and results of [2]. This yields an explicit formula for the critical values of energy on $M_n(L)$.

Theorem 2. *The value of E at a regular n-star with perimeter L and winding number w equals*

$$\frac{nq^2 \sin\frac{w\pi}{n}}{4nL} \sum_{i=1}^{n-1} \sum_{i=1}^{n} \frac{1}{\sqrt{2-2\cos^2\left(\frac{2\pi\,|i-j|}{n}\right)}}. \tag{3}$$

Indeed, according to [2] the Coulomb energy of a regular n-gon with circumradius r can be written as a double sum

$$\frac{q^2}{4r\pi} \sum_{i=1}^{n-1} \sum_{j=1}^{n} \frac{1}{\sqrt{2-2\cos^2\left(\frac{2\pi(i-j)}{n}\right)}}.$$

where r is the radius of its circumcircle. Substituting in the latter expression the value $\frac{L}{2n\sin\frac{w\pi}{n}}$ of the circumradius of convex hull given in Corollary 1 we arrive to formula (3).

Remark 1. In this setting, piles are excluded from consideration since we do not permit coincidences of charges leading to infinite value of Coulomb energy.

Remark 2. We conjecture that there are no other local maxima except the poles caused by coincidence of charges.

Remark 3. It is not true that all critical points of E on Mn with finite energy are non-degenerate regular n-stars. For example, for n=4, we have the "sand clock" configuration which does not have the shape of a regular star but still represents a critical point of E on M_n as can easily be seen by symmetry reasons.

Remark 4. These results suggest that the global minimum of E is attained at a convex regular n-gon with side L/n. This is true for n=5 and tautological for n=4. However in general this remains unclear since according to the preceding remark we do not have description of all critical points of E.

We can now obtain some corollaries and generalizations. First, let us consider n non-equal charges on flexible planar contour of length L. Using our Theorems 1, 2 it is possible to show that in case of nearly equal charges there also exist star-shaped critical points (of course they are in general non-regular). This follows from the non-degeneracy of regular stars for equal charges and stability of non-degenerate critical points.

Another option is to consider critical points of Coulomb energy on a union of concentric circles with non-equal radii sufficiently close to *1*. It follows that all regular stars "survive" for sufficiently close radii and provide equilibrium configurations of equal charges on close concentric circles. Such results may appear useful in some applied problems since Coulomb equilibria on concentric circles are related to the so-called Coulomb control of swarms of small satellites.

One can also obtain similar results for cyclic polygons with the fixed circumcircle. For example, all regular *n*-stars with the fixed circumcircle are critical points of perimeter on the configuration space of cyclic *n*-gons with the fixed circumcircle. Using the results and formulas presented above, one can also show that all regular *n*-stars with the fixed circumcircle are critical points of oriented area and Coulomb potential of equally charged vertices on the same configuration space of cyclic *n*-gons with the fixed circumcircle.

Finally, a whole series of similar results can be obtained for other functions on configuration spaces of n-gons with the fixed perimeter and/or circumcircle. For example, one can consider central forces different form Coulomb, power sums of the sides and like.

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მათემატიკა

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