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Pietro Mengoli's 1650 Proof that the Harmonic Series Diverges

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The first published proof that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

exceeds any given quantity was given by Pietro Mengoli in 1650 [9]. The same result had been proved by Nicole Oresme in Question 2 of his *Questiones super geometriam Euclidis* [7, pp. 131–135], dated around 1350. These *Questiones* were copied as a manuscript but were not published until the 20th century. There is no indication that Mengoli knew of this work. Oresme's proof is the one still commonplace today, based on grouping the terms of the series into blocks of 2 terms, 4 terms, 8 terms, etc. Mengoli's proof is also based on the grouping of terms, but in a different manner. He groups the terms into blocks of three and applies the inequality

$$\frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1} > \frac{3}{n}. \tag{1}$$

This inequality follows, as Mengoli says, from the fact that the first term exceeds the middle by more than the middle exceeds the last, i.e., $\frac{1}{n-1} - \frac{1}{n} > \frac{1}{n} - \frac{1}{n+1}$, and therefore, replacing the outer terms by the middle one will diminish the first by more than it will increase the last.

Applying this inequality to the harmonic series gives

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots &= 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \cdots \\ &> 1 + \frac{3}{3} + \frac{3}{6} + \cdots \\ &= 1 + 1 + \frac{1}{2} + \cdots \end{aligned}$$

Note that the harmonic series recurs in the final expression: if we let S denote the sum of the harmonic series, we have just proved that $S > 1 + S$. From this inequality, it follows that S cannot be finite, so this is one way of arriving at the desired result. Indeed, most modern accounts of Mengoli's proof put this exact reasoning in his mouth. In particular, [4, pp. 7–10], [8], and [1, pp. 11–12] all phrase Mengoli's proof in this exact way.

We believe it is highly unfortunate that this has become the standard account of Mengoli's proof, for in fact he does *not* argue in this way, and indeed the fact that

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he does not do so is arguably one of the most interesting and historically illuminating aspects of his proof. Below we give a complete English translation of Mengoli's argument, so that it may be appreciated in his own terms and its persistent misrepresentations eradicated.

In the fourth paragraph of the translation, we see that Mengoli does indeed note and utilise the self-replicating nature of the above estimation procedure. However, he does not treat as an entity the completed series in the manner of the inequality $S > 1 + S$. Instead he notes that (1) tells us that the sum of the first three terms,

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \tag{2}$$

is greater than $\frac{3}{3}$, i.e., greater than 1, and the sum of the next nine terms,

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13}, \tag{3}$$

is greater than

$$\frac{3}{6} + \frac{3}{9} + \frac{3}{12},$$

i.e., greater than the sum (2), and the sum of the next 27 terms is greater than

$$\frac{3}{15} + \frac{3}{18} + \frac{3}{21} + \frac{3}{24} + \frac{3}{27} + \frac{3}{30} + \frac{3}{33} + \frac{3}{36} + \frac{3}{39},$$

i.e., greater than the sum (3), etc. Therefore, when the terms of the original series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

are grouped into blocks of size 3, 9, 27, etc., the sum of each block exceeds 1, since by repeated application of (1) the sum of each such block is greater than the sum of the previous block and thus greater than the sum of the first three terms and hence greater than 1. (Note that Mengoli does not include a leading term 1, as is customary today when talking about the harmonic series.)

Thus a faithful schematic representation of Mengoli's proof is not to conclude from $S > 1 + S$ that S cannot be finite, but rather to apply this inequality repeatedly to yield $S > 1 + S > 2 + S > 3 + S > \cdots$, and a fortiori $S > 1$, $S > 2$, $S > 3$, etc., from which it follows that the series continued sufficiently far can be made to exceed any given quantity. The accounts of Mengoli's proof given by [**11**], [**5**], [**10**, pp. 14–23], and [**2**, pp. 204–205] capture this aspect of the proof much more faithfully than the sources cited above.

But even the very idea of considering the sum of the series as a number or algebraic entity that can be denoted by a single symbol such as S is foreign to Mengoli. And that with good reason. For what grounds do we have for assuming that an infinite series can be considered as a unified algebraic entity and be operated on as such? Mengoli's approach dextrously avoids all the potential pitfalls of dealing with infinities in a careless fashion. Instead of speaking in abstractions such as saying that the harmonic series equals infinity, he remains thoroughly finitistic, saying that the series exceeds 1 if you take 3 terms, 2 if you take $3 + 9$ terms, 3 if you take $3 + 9 + 27$ terms, etc. This captures the infinity of the series in the most concrete and constructive manner possible, in unequivocal terms that are not susceptible to any philosophical qualms about infinities.

Mengoli's cautious approach to the infinite is very much in keeping with the way the infinite was treated in classical Greek mathematics. In particular, the Greek "method of exhaustion" was a fundamental technique for avoiding appeal to the infinite. Greek mathematicians used this technique to determine many areas by, in effect, limiting processes of polygonal approximations. But to avoid explicit use of the infinite and instead phrase their results in safe, finitistic terms, they showed, by a double reduction ad absurdum, that all other possible values for the area, except the one claimed in the theorem at hand, would be impossible. To rule out any given value for the area other than the correct one, only a finite number of steps in the polygonal approximation would be needed, whence the proof avoids assuming the completion of an infinite number of operations or making any explicit reliance on the infinite. In other words, the method of exhaustion deals only with the *potential* infinity (as Aristotle called it) of the procedure having the potential of being extended indefinitely, as opposed to the *actual* infinity of considering the approximation process or series as having been carried through to its completion.

Indeed, Mengoli cites Archimedes' *Quadrature of the Parabola* as having occasioned his work on infinite series. This is a prime example of the method of exhaustion. In it, Archimedes shows that the area of a segment of a parabola can be approximated by inscribed triangles in such a way that the triangles added at each step of the approximation have one quarter the area of those at the preceding step. Thus the total area of the parabolic segment is

$$A + \frac{1}{4}A + \frac{1}{4^2}A + \frac{1}{4^3}A + \cdots = \frac{4}{3}A,$$

where A is the area of the initial inscribed triangle (which itself is straightforward to determine). Again, Archimedes does not speak of the sum of an infinite series but rather shows that, by bringing the approximation far enough, any other possible value for the area is ruled out.

Mengoli was surely very sensitive to this context. In fact, as Eneström [3] points out, this even explains the title of Mengoli's work, which is called "New arithmetical quadratures" even though it contains no actual quadratures (i.e., area determinations). Thus Mengoli evidently associated the theory of series very closely with the classical method of exhaustion, so it is not surprising that he remains committed to its finitistic paradigm.

The nature of Mengoli's proof makes it a perfect showcase for the great importance attached to these considerations at the time. For if there ever were a time to employ a form of reasoning (such as that $S > 1 + S$ implies $S = \infty$) that considers a series as a single, completed algebraic entity, then this was it. Indeed, as we have pointed out, the temptation to reason in this way in this case is so strong that even several writers on the history of mathematics have succumbed to it when describing Mengoli's proof. Thus the fact that Mengoli does *not* do so is an especially telling testament to his dedication to the ancient manner of dealing with infinities in strictly finitistic terms. It is a great pity, therefore, that this crucial aspect of his proof is misrepresented in the standard modern accounts of it.

The translation

Mengoli's proof of the divergence of the harmonic series occurs in the first five paragraphs of the Preface of his 1650 *Novae quadraturae arithmeticae* [9]. We now give a complete English translation of this passage. The translation will be followed by some explanatory notes.

Meditating often on Archimedes' quadrature of the parabola, in which infinitely many triangles, being in continued quadruple proportion, do not exceed certain bounds, the universal quadrature came to mind, demonstrated by geometers using the same proof, in which infinitely many magnitudes in some continued proportion of greater inequality are gathered into determined homogeneous quantities. This admirable theorem! In contemplating it, I was led to the question, whether magnitudes arranged under such a rule, whatever it may be, such that some can be taken smaller than any given quantity, or that decreasing terms vanish *in infinitum*, when composed infinitely can exceed each given quantity.

Having gone about to try arithmetical fractions for the purpose of such an experiment, I set them out thus, so that all the unities are denominated by all the numbers after unity,

$$\frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}\ \frac{1}{5}\ \frac{1}{6}\ \frac{1}{7}\ \frac{1}{8}\ \frac{1}{9}\ \frac{1}{10}\ \frac{1}{11}\ \frac{1}{12}\ \frac{1}{13}\ \frac{1}{14}.$$

In this arrangement, the magnitude can be taken less than any given amount, and therefore these magnitudes decreasing in quantity according to the increase of the rank, disappear into infinity.

Propounding the question in the terms of the assumed arrangement, I was therefore searching for an argument to decide whether the unities denominated by every number starting with unity, laid out to infinity, taken together would make up some infinite or finite extent. It seemed that the answer would have to be in favor of a finite extent, since the powers of numbers and of fractions are opposed: that of numbers in multiplication, by which quantities progress towards infinity, but that of fractions in division, by which a thing is reduced downright to indivisibles: now the numbers taken together exceed any given quantity; so by the opposite reasoning it seems that the fractions cannot exceed any given quantity. This sophism was the reason of my expectation, held for almost an entire month, that I would be able to decide in favor of this geometrical view about the matter; but when I now examine the procedure of proof, my judgment changes to the other view.

The procedure is the following. Because in the given fractions equal magnitudes are denominated by numbers in arithmetic proportion, and thus three consecutive terms, say A, B, and C, are in harmonic proportion, for example,

$$\begin{array}{ccc} A & B & C \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{array}$$

and A has the same proportion to C as the excess of A to B has to the excess of B to C, and moreover, A is greater than C, therefore the excess of A to B is greater than the excess of B to C. The total of A and C is greater than twice B, and the total of the three A, B, C is greater than thrice the middle term B. So by this argument, the fractions in this arrangement taken three at a time from the first,

$$\frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}\qquad \frac{1}{5}\ \frac{1}{6}\ \frac{1}{7}\qquad \frac{1}{8}\ \frac{1}{9}\ \frac{1}{10}\qquad \frac{1}{11}\ \frac{1}{12}\ \frac{1}{13}\qquad \frac{1}{14}\ \frac{1}{15}\ \frac{1}{16},$$

are greater than thrice the middle terms: and the middle terms are unities denominated by numbers multiplied by three, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{9}$, $\frac{1}{12}$, and thrice these are 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, the same ones which in the above argument taken three at a time are greater than thrice the middle terms. Therefore the given fractions of the arrangement, taken according to numbers in subtriple proportion, 3, 9, 27, 81, all exceed unities. For any given number, one can take equally many numbers in continued subtriple proportion starting from three, and then the fractions of the proposed arrangement taken according to the sum of the numbers in continued proportion will exceed the given number. Therefore the proposed fractions, arranged up to infinity and taken together, are capable of filling an infinite extent.

For example, let 4 be the assigned number, and starting from three take four numbers in continued subtriple proportion, 3, 9, 27, 81, whose sum is 120: then 120 of the given fractions exceed the assigned number 4. For, the first three exceed thrice $\frac{1}{3}$, namely unity; the next nine exceed thrice the sum of $\frac{1}{6}$, $\frac{1}{9}$, $\frac{1}{12}$, namely the sum of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, but as I have shown the sum of those exceeds unity, and so these nine exceed unity; and by the same demonstration the following 27 and 81 exceed unities.

Notes on the text

Mengoli opens his discussion with a reference to Archimedes' *Quadrature of the Parabola*. This shows the context in which he was led to the study of series, namely using them—in particular geometric series—for the determination of areas according to the method of exhaustion. This explains why he uses the term "quadrature" to mean, in effect, the sum of a series, even though this term normally means finding an area. Indeed, as noted above, the "new quadratures" promised in the title of Mengoli's work are quadratures only in the sense of summing series. Mengoli's interest is not in the geometrical application of series. In the above translated paragraphs, he instead poses a more abstract question which arises from reflecting on such series, namely the question of whether the sum of an infinite series can exceed any quantity even though its terms become smaller and smaller and approach zero.

In the terminology of the first paragraph, then, "universal quadrature" does not mean finding the area of any figure, but rather finding the sum of any sequence of magnitudes in a "continued proportion of greater inequality," this being "universal" in that it generalizes the sequence of areas Archimedes used in which each was one-quarter the previous. To say that homogeneous magnitudes $a_1, a_2, a_3, \ldots$ are in *continued proportion* means that $a_1 : a_2 = a_2 : a_3$, $a_2 : a_3 = a_3 : a_4$, etc. In other words, a sequence of homogeneous magnitudes $a_1, a_2, a_3, \ldots$ is in continued proportion when there is some dimensionless quantity r such that $a_1 = ra_2$, $a_2 = ra_3$, $a_3 = ra_4$, etc. "Quadruple proportion" means $r = 4$. To say that a ratio $a : b$ is in *greater inequality* means that $a > b$. Thus, the "universal quadrature" means summing any geometric series with strictly decreasing terms. To speak about the magnitudes "composed infinitely" means summing the magnitudes. Heath [6, p. 85] may be consulted for a summary of the notions of arithmetic, geometric, and harmonic proportions in classical Greek mathematics.

In the second paragraph, by "arithmetical fractions" Mengoli means that the denominators are in *arithmetic proportion* rather than geometric proportion, i.e., the terms are of the form $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \frac{1}{a_4}, \ldots$ where $a_1, a_2, a_3, a_4, \ldots$ satisfy $a_1 - a_2 = a_2 - a_3$, $a_2 - a_3 = a_3 - a_4$, etc. "Decreasing in quantity according to the increase of the rank" means that the greater the index of a term, the smaller the magnitude of the term.

In the third paragraph, Mengoli states that he at first believed that the harmonic series must have a finite value since when one makes a number bigger and bigger, the corresponding fraction becomes smaller and smaller, and thus eventually "indivisible", so that it would contribute nothing to a sum. Mengoli's point that he was misled by this "sophism" (i.e., a confusing or deceptive argument) "for almost an entire month" serves to make the reader appreciate the counterintuitive nature of his result.

The fourth paragraph contains the proof that the harmonic series diverges. In symbols, Mengoli says that for A, B, and C to be in harmonic proportion means that

$$\frac{A}{C} = \frac{A-B}{B-C}.$$

If a, b, and c are in arithmetic progression, i.e., $a - b = c - b$, then their reciprocals $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in harmonic proportion:

$$\frac{\frac{1}{a}-\frac{1}{b}}{\frac{1}{b}-\frac{1}{c}} = \frac{\frac{b-a}{ab}}{\frac{c-b}{bc}} = \frac{b-a}{c-b}\frac{c}{a} = \frac{c}{a} = \frac{\frac{1}{a}}{\frac{1}{c}}.$$

If A, B, and C are in harmonic proportion and $A > C$, then $A - B > B - C$, so $A + C > 2B$ and hence $A + B + C > 3B$. Using this fact that applies to triples in harmonic proportion, we chunk the sequence $\frac{1}{2}, \frac{1}{3}, \ldots$ into triples of the form $\frac{1}{3n-1}, \frac{1}{3n}, \frac{1}{3n+1}$, and the sum of each triple is greater than thrice the middle term, i.e., greater than $\frac{1}{n}$.

Thus, first, the sum of the three terms

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$$

is greater than $\frac{3}{3} = 1$. Second, the sum of $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ is greater than $\frac{3}{6} = \frac{1}{2}$, the sum of $\frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ is greater than $\frac{3}{9} = \frac{1}{3}$, and the sum of $\frac{1}{11}, \frac{1}{12}, \frac{1}{13}$ is greater than $\frac{3}{12} = \frac{1}{4}$, so the sum of the nine terms

$$\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}, \frac{1}{11}, \frac{1}{12}, \frac{1}{13} \tag{4}$$

is greater than the sum of

$$\frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}.$$

But we know that the latter sum is itself greater than 1, so the sum of (4) is greater than 1. Third, the sum of $\frac{1}{14}, \frac{1}{15}, \frac{1}{16}$ is greater than $\frac{3}{15} = \frac{1}{5}$, the sum of $\frac{1}{17}, \frac{1}{18}, \frac{1}{19}$ is greater than $\frac{3}{18} = \frac{1}{6}$, etc., and the sum of $\frac{1}{38}, \frac{1}{39}, \frac{1}{40}$ is greater than $\frac{3}{39} = \frac{1}{13}$, so the sum of the 27 terms

$$\frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}, \frac{1}{18}, \frac{1}{19}, \ldots, \frac{1}{38}, \frac{1}{39}, \frac{1}{40} \tag{5}$$

is greater than the sum of

$$\frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{13}.$$

But we know that the latter sum is itself greater than 1, so the sum of (5) is greater than 1. Thus, Mengoli chunks the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \tag{6}$$

into blocks with 3, 9, 27, etc. terms, and the sum of the terms in each block is greater than 1. There are infinitely many blocks, and therefore the sum of (6) is greater than $1 + 1 + 1 + \cdots$, namely it fills an "infinite extent."

The ratio 1 : 3 is a "subtriple proportion," and saying that the numbers 3, 9, 27, 81 are in subtriple proportion means that the consecutive ratios 3 : 9, 9 : 27, 27 : 81 are subtriple proportions.

In the fifth paragraph Mengoli spells out an explicit recipe for how many terms will suffice for the series to exceed any given number. Generally, for a positive integer n, the sum of the first $\sum_{k=1}^{n} 3^k = \frac{3^{n+1}-3}{2}$ terms of the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is greater than n. Mengoli gives as an example $n = 4$, for which $\frac{3^{4+1}-3}{2} = 120$, and $\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{121} > 4$. (In fact, ones computes that this sum is equal to $5.368\ldots$.) Again, this shows very clearly Mengoli's commitment to a finitistic or constructive notion of what it means for a sum to be infinite.

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Summary. Pietro Mengoli proved that harmonic series diverges in his 1650 *Novae quadraturae arithmeticae* – the first published proof of this result. His proof is discussed in a number of places in the secondary literature, but is often misrepresented in a crucial manner. We give a full English translation of the proof with explanatory notes and argue for a less anachronistic interpretation of it.

JORDAN BELL (MR Author ID: 776262) received his M.Sc. in mathematics at the University of Toronto. His earlier work in the history of mathematics includes a translation (published in Stephen Hawking's *God Created the Integers*) of the 1735 paper in which Euler finds $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots = \frac{\pi^2}{6}$.

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