Proof of Sun's conjectures on Schröder-like numbers

Ji-Cai Liu[1] and Wilberd van der Kallen[2]

[1]College of Mathematics and Information Science, Wenzhou University, Wenzhou 325035, PR China
jc2051@163.com

[2]Mathematisch Instituut, Universiteit Utrecht, P.O. Box 80010, 3508 TA Utrecht, Netherlands
W.vanderKallen@uu.nl

Abstract. For any non-negative integer n, define R_n and $R_n(x)$ by

$$R_n=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{1}{2k-1}\quad\text{and}\quad R_n(x)=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{x^k}{2k-1},$$

respectively. We mainly prove that for any positive integer n and odd prime p,

$$\frac{3}{n}\sum_{k=0}^{n-1}R_k(x)^2\in\mathbb{Z}[x],$$
$$3\sum_{k=0}^{p-1}R_k^2\equiv(11+4(-1)^{\frac{p+1}{2}})p\pmod{p^2},$$

which were originally conjectured by Z.-W. Sun.

Keywords: congruences; p-adic integer; Zeilberger algorithm

MR Subject Classifications: 11A07; 11B65; 05A10

1 Introduction

In combinatorics, the Schröder numbers are given by

$$S_n=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{1}{k+1},$$

which describes the number of paths from $(0,0)$ to (n,n), using only steps $(1,0),(0,1)$ and $(1,1)$, that do not rise above the line $y=x$. For more information on these numbers, one refer to [7,8]. Some arithmetic properties of the Schröder numbers have been studied by Sun [9,11], Cao and Pan [1], and the first author [5].

Motivated by Schröder numbers, Z.-W. Sun [10] introduced the following interesting numbers (see also http://oeis.org/A245769)

$$R_n=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{1}{2k-1},$$

and obtained many amazing arithmetic properties of these numbers. For example, Sun proved that for any odd prime p,

$$\sum_{k=0}^{p-1} R_k \equiv -p-(-1)^{\frac{p-1}{2}} \pmod{p^2}.$$

Sun also made the following conjecture [10, Conjecture 5.4]:

Conjecture 1.1 *Suppose n is a positive integer and p is an odd prime. Then*

$$\sum_{k=0}^{n-1}(2k+1)R_k^2 \equiv 0 \pmod{n}, \tag{1.1}$$

$$\sum_{k=0}^{p-1}(2k+1)R_k^2 \equiv 4p(-1)^{\frac{p-1}{2}}-p^2 \pmod{p^3}, \tag{1.2}$$

$$3\sum_{k=0}^{n-1} R_k^2 \equiv 0 \pmod{n}, \tag{1.3}$$

$$3\sum_{k=0}^{p-1} R_k^2 \equiv (11+4(-1)^{\frac{p+1}{2}})p \pmod{p^2}. \tag{1.4}$$

Recently, Guo and the first author [3] have successfully proved (1.1) and (1.2) by some combinatorial identities and Zeilberger algorithm.

For any positive integer n, Sun [10, (1.4)] defined the following polynomials:

$$R_n(x)=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{x^k}{2k-1}.$$

The first aim of the paper is to prove Sun's stronger conjecture of (1.3), see the comments of `http://oeis.org/A268136`.

Theorem 1.2 *Suppose n is a positive integer. Then*

$$\frac{3}{n}\sum_{k=0}^{n-1} R_k(x)^2 \in \mathbb{Z}[x]. \tag{1.5}$$

Moreover,

$$\frac{3}{n}\sum_{k=0}^{n-1} R_k^2 \equiv 1 \pmod{2}. \tag{1.6}$$

Guo and the first author [3] introduced the following numbers:

$$W_n=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\frac{3}{2k-3},$$

and some similar arithmetic properties of these numbers have also been studied. The second aim of the paper is to show the following two congruences:

Theorem 1.3 *If p is an odd prime, then* (1.4) *holds and we also have*

$$35\sum_{k=0}^{p-1} W_k^2 \equiv (-77-4(-1)^{\frac{p+1}{2}})p \pmod{p^2}, \quad \text{for } p\geq 5. \tag{1.7}$$

In the next section, we first prove some important lemmas. The proof of Theorem 1.2 and 1.3 will be given in Section 3 and 4, respectively.

2 Some lemmas

Lemma 2.1 *Suppose m is a non-negative integer. Then*

$$\sum_{i=0}^{m}\sum_{j=0}^{m}\binom{x+j}{j}\binom{x-1}{j}\binom{j}{i}\binom{m}{i}\binom{i}{m-j}\frac{3}{(2i-1)(2m-2i-1)(2j+1)} \tag{2.1}$$

always takes integer values for all $x\in\mathbb{Z}$.

Proof. Let $P_m(x)$ denote the polynomial (2.1). For $m=0,1$, it is easy to check that $P_m(x)$ is integer-valued. Assume $m\geq 2$. Since $\binom{x+j}{j}\binom{x-1}{j}=\binom{-x+j}{j}\binom{-x-1}{j}$, we conclude that $P_m(x)$ is an even polynomial. Let

$$B_k(x)=\binom{x+k}{2k}+\binom{-x+k}{2k}.$$

We can rewrite $P_m(x)$ as

$$P_m(x)=\sum_{k=0}^{m} d(m,k)B_k(x),$$

with $d(m,k)\in\mathbb{Q}$.

Note that

$$(-1)^k\binom{x+k}{k}\binom{x-1}{k}-(-1)^{k-1}\binom{x+k-1}{k-1}\binom{x-1}{k-1}=(-1)^k\binom{2k}{k}B_k(x)/2.$$

Taking the telescoping sum over k gives

$$(-1)^j\binom{x+j}{j}\binom{x-1}{j}=\sum_{k=0}^{j}(-1)^k\binom{2k}{k}B_k(x)/2. \tag{2.2}$$

Substituting (2.2) into (2.1), we conclude that

$$d(m,k)=\sum_{i=0}^{m}\sum_{j=k}^{m}\frac{3(-1)^{k+j}\binom{2k}{k}\binom{j}{i}\binom{m}{i}\binom{i}{m-j}}{2(2i-1)(2j+1)(2m-2i-1)}.$$

It suffices to prove that $d(m,k)\in\mathbb{Z}$ for $m\ge 2$.

We need the following two key results:

$$\begin{aligned}&4(m-1)(m+1)d(m,k)+4(m+2k+2)(m-k+1)d(m+1,k)\\&-(k+1)(2k-m-1)d(m+1,k+1)=0,\end{aligned}\tag{2.3}$$

and

$$(2m+1)d(m,k)/3\in\mathbb{Z}.\tag{2.4}$$

Before proving the key results, let us draw conclusions from them.

Noting that $\binom{2k}{k}/2=\binom{2k-1}{k}$ is an integer and $(2i-1)(2j+1)(2m-2i-1)$ is an odd integer, we immediately get $d(m,k)\in\mathbb{Z}_2$, where $\mathbb{Z}_p$ denotes the set of all p-adic integers for prime p.

If $3\nmid 2m+1$, by (2.4), we have $d(m,k)\in\mathbb{Z}_3$. If $m\equiv 1\pmod 9$ or $m\equiv 7\pmod 9$, then $(2m+1)/3$ is coprime to 3. It follows from (2.4) that $d(m,k)\in\mathbb{Z}_3$. If $m\equiv 4\pmod 9$, then $(m-1)(m+1)/3$ and $2m+3$ are both coprime to 3. From (2.4), we have $d(m+1,k)/3\in\mathbb{Z}_3$ for all k, and so $d(m,k)\in\mathbb{Z}_3$ by (2.3).

Let $p\ge 5$ be a prime. If $p\nmid 2m+1$, by (2.4), $d(m,k)\in\mathbb{Z}_p$. If $p\mid 2m+1$, then $p\nmid 2m+3$, and so $d(m+1,k)\in\mathbb{Z}_p$ for all k by (2.4). Noting that $2m+1=2(m+1)-1$ and $2m+1=2(m-1)+3$, we get $p\nmid(m+1)(m-1)$. It follows from (2.3) that $d(m,k)\in\mathbb{Z}_p$.

Now we have shown that $d(m,k)\in\mathbb{Z}_p$ for any prime p and $m\ge 2$. This implies that $d(m,k)\in\mathbb{Z}$ for $m\ge 2$. So we still have to prove (2.3) and (2.4).

We first prove that

$$\begin{aligned}&\frac{(2m+1)d(m,k)}{3}\\&=\sum_{i=0}^{m}\binom{2k}{k}\binom{k}{i}\binom{m}{i}\binom{i}{m-k}\left(\frac{m-k}{m(m-1)}-\frac{2m-2k-1}{2(2i-1)(2m-2i-1)}\right).\end{aligned}\tag{2.5}$$

Note that

$$2(2k+1)(-1)^k\binom{2k}{k}+(k+1)(-1)^{k+1}\binom{2k+2}{k+1}=0.$$

Then we have

$$2(2k+1)d(m,k)+(k+1)d(m,k+1)=\sum_{i=0}^{k}\frac{3\binom{2k}{k}\binom{k}{i}\binom{m}{i}\binom{i}{m-k}}{(2i-1)(2m-2i-1)}.\tag{2.6}$$

Define

$$\begin{aligned}S(m,k)=&\,2m(m-1)(2m+1)d(m,k)+(m-2k)(m-2k+1)(2m-2k-1)D(m,k)\\&+16(1-2k)(k-m-1)(k-m)D(m,k-1),\end{aligned}\tag{2.7}$$

where $D(m,k)$ denotes the left-hand side of (2.6). Applying the Zeilberger algorithm [6] to the right-hand side of (2.6), we get the following recurrence for $D(m,k)$:

$$2(2k+3)S(m,k+1)+(k+2)S(m,k+2)=0. \tag{2.8}$$

By Zeilberger algorithm, we find that $d(m,0)=0$ for $m\ge 2$, and so by (2.6) and (2.7) we have $S(m,0)=0$ for $m\ge 2$. It follows from (2.8) and induction that $S(m,k)=0$ for $m\ge 2$ and $k\ge 0$, that is

$$\begin{aligned}&2m(m-1)(2m+1)d(m,k)+(m-2k)(m-2k+1)(2m-2k-1)D(m,k)\\&+16(1-2k)(k-m-1)(k-m)D(m,k-1)=0.\end{aligned} \tag{2.9}$$

Substituting (2.6) into the left-hand side of (2.9) and then noting that

$$\begin{aligned}&\frac{3(m-2k)(m-2k+1)(2m-2k-1)}{(2i-1)(2m-2i-1)}\binom{2k}{k}\binom{k}{i}\binom{m}{i}\binom{i}{m-k}\\&+\frac{48(1-2k)(k-m-1)(k-m)}{(2i-1)(2m-2i-1)}\binom{2k-2}{k-1}\binom{k-1}{i}\binom{m}{i}\binom{i}{m-k+1}\\&=\left(\frac{3m(m-1)(2m-2k-1)}{(2i-1)(2m-2i-1)}+6(k-m)\right)\binom{2k}{k}\binom{k}{i}\binom{m}{i}\binom{i}{m-k},\end{aligned}$$

we conclude the proof of (2.5). Substituting (2.5) into the left-hand side of (2.3) and then applying Zeilberger algorithm again, we can prove (2.3).

In order to prove (2.4), it suffices to prove that every term on the right-hand side of (2.5) is an integer. Note that (see [3, (2.1)])

$$\binom{2i}{i}\binom{2m-2i}{m-i}\Bigg|\binom{2k}{k}\binom{k}{i}\binom{m}{i}\binom{i}{m-k},$$

and $2(2i-1)\mid\binom{2i}{i}$. It follows that

$$\binom{2k}{k}\binom{k}{i}\binom{m}{i}\binom{i}{m-k}\frac{2m-2k-1}{2(2i-1)(2m-2i-1)}$$

is always an integer. We still have to prove

$$\binom{2k}{k}\binom{k}{i}\binom{m}{i}\binom{i}{m-k}\frac{m-k}{m(m-1)}=\binom{2k}{k}\binom{k}{i}\binom{m-1}{i-1}\binom{i-1}{m-k-1}\frac{1}{m-1}$$

is an integer. For the p-adic order of $n!$, there is a known formula

$$\mathrm{ord}_p n! = \sum_{j=1}^{\infty}\left\lfloor\frac{n}{p^j}\right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to a real number x. Applying this method, it suffices to prove that for any positive integer $q \geq 2$

$$\left\lfloor \frac{2k}{q} \right\rfloor - \left\lfloor \frac{k}{q} \right\rfloor - \left\lfloor \frac{k-i}{q} \right\rfloor - \left\lfloor \frac{i}{q} \right\rfloor + \left\lfloor \frac{m-2}{q} \right\rfloor - \left\lfloor \frac{m-i}{q} \right\rfloor - \left\lfloor \frac{m-k-1}{q} \right\rfloor - \left\lfloor \frac{i+k-m}{q} \right\rfloor \geq 0. \qquad (2.10)$$

We distinguish two cases to prove (2.10).

If $\left\lfloor \frac{m-2}{q} \right\rfloor \geq \left\lfloor \frac{m-i}{q} \right\rfloor + \left\lfloor \frac{m-k-1}{q} \right\rfloor + \left\lfloor \frac{i+k-m}{q} \right\rfloor$, then (2.10) is obviously true.

If $\left\lfloor \frac{m-2}{q} \right\rfloor = \left\lfloor \frac{m-i}{q} \right\rfloor + \left\lfloor \frac{m-k-1}{q} \right\rfloor + \left\lfloor \frac{i+k-m}{q} \right\rfloor - 1$, then there exist integers a_1, a_2 and a_3 such that $m - i = a_1 q$, $m - k - 1 = a_2 q$ and $i + k - m = a_3 q$. So we have $k = (a_1 + a_3)q$, $i = (a_2 + a_3)q + 1$ and $k - i = (a_1 - a_2)q - 1$. It follows that

$$\left\lfloor \frac{2k}{q} \right\rfloor - \left\lfloor \frac{k}{q} \right\rfloor - \left\lfloor \frac{k-i}{q} \right\rfloor - \left\lfloor \frac{i}{q} \right\rfloor = 1,$$

which implies that (2.10) is true. □

Lemma 2.2 *Let p be an odd prime and m be an integer such that $0 \leq m \leq 2p-2$. Then*

$$\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (-1)^j \binom{j}{i} \binom{m}{i} \binom{i}{m-j} \frac{3}{(2i-1)(2m-2i-1)(2j+1)}$$
$$\equiv \begin{cases} 3 \pmod p, & \text{if } m = 0, \\ 2 \pmod p, & \text{if } m = 1, \\ 6 \pmod p, & \text{if } m = p, \\ 4(-1)^{\frac{p+1}{2}} \pmod p, & \text{if } m = \frac{3p-1}{2}, \\ 0 \pmod p, & \text{otherwise.} \end{cases} \qquad (2.11)$$

Proof. Let $S_p(m)$ denote the left-hand side of (2.11). We distinguish three cases to prove (2.11).

Case 1. $m = 0$ or 1. It is easy to verify that $S_p(0) = 3$ and $S_p(1) = 2$.

Case 2. $2 \leq m \leq p-1$. In this event, we have

$$\begin{aligned} S_p(m) &= \sum_{i=0}^{m} \binom{m}{i} \frac{3}{(2i-1)(2m-2i-1)} \sum_{j=0}^{m} (-1)^j \binom{j}{i} \binom{i}{m-j} \frac{1}{2j+1} \\ &= (-1)^m \sum_{i=0}^{m} \binom{m}{i}^2 \binom{2m}{2i}^{-1} \frac{3}{(2m+1)(2i-1)(2m-2i-1)} \\ &= 0, \end{aligned}$$

where we have utilized the following two identities:

$$\sum_{j=0}^{m}(-1)^j\binom{j}{i}\binom{i}{m-j}\frac{1}{2j+1}=\frac{(-1)^m}{2m+1}\binom{m}{i}\binom{2m}{2i}^{-1}, \tag{2.12}$$

$$\sum_{i=0}^{m}\binom{m}{i}^2\binom{2m}{2i}^{-1}\frac{1}{(2i-1)(2m-2i-1)}=0, \quad \text{for } m\geq 2,$$

which can be proved by Zeilberger algorithm [4,6].

Case 3. $p\leq m\leq 2p-2$. If $m-i\leq j\leq p-1$, then $\binom{m}{i}\equiv 0 \pmod p$. Otherwise $\binom{i}{m-j}=0$. This implies that

$$\binom{j}{i}\binom{m}{i}\binom{i}{m-j}\frac{1}{(2i-1)(2m-2i-1)(2j+1)}\equiv 0 \pmod p,$$

unless $i=\frac{p+1}{2}, m-\frac{p+1}{2}$. It follows that

$$\begin{aligned}
S_p(m) &\equiv \sum_{j=0}^{p-1}(-1)^j\binom{j}{\frac{p+1}{2}}\binom{m}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{m-j}\frac{3}{p(2m-p-2)(2j+1)}\\
&+\sum_{j=0}^{p-1}(-1)^j\binom{j}{m-\frac{p+1}{2}}\binom{m}{\frac{p+1}{2}}\binom{m-\frac{p+1}{2}}{m-j}\frac{3}{p(2m-p-2)(2j+1)}\\
&=2\sum_{j=0}^{p-1}(-1)^j\binom{j}{\frac{p+1}{2}}\binom{m}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{m-j}\frac{3}{p(2m-p-2)(2j+1)}\\
&=\frac{6}{p(2m-p-2)}\binom{m}{\frac{p+1}{2}}\sum_{j=0}^{p-1}(-1)^j\binom{j}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{m-j}\frac{1}{2j+1} \pmod p.
\end{aligned}\tag{2.13}$$

If $\frac{3p+1}{2}\leq m\leq 2p-2$, then $\binom{\frac{p+1}{2}}{m-j}=0$ for $0\leq j\leq p-1$, and so $S_p(m)\equiv 0 \pmod p$.

If $m=\frac{3p-1}{2}$, then $\binom{\frac{p+1}{2}}{m-j}=0$ for $0\leq j\leq p-2$, and so

$$S_p\left(\frac{3p-1}{2}\right)\equiv\frac{2}{p}\binom{\frac{3p-1}{2}}{\frac{p+1}{2}}\binom{p-1}{\frac{p+1}{2}}\equiv 4(-1)^{\frac{p+1}{2}} \pmod p.$$

If $m=p$ or $p+2\leq m\leq\frac{3p-3}{2}$, then $\frac{6}{p(2m-p-2)}\binom{m}{\frac{p+1}{2}}$ is a p-adic integer. Noting that $\binom{j}{\frac{p+1}{2}}\frac{1}{2j+1}\equiv 0 \pmod p$ for $p\leq j\leq m$ and applying (2.12), we obtain

$$\begin{aligned}
&\sum_{j=0}^{p-1}(-1)^j\binom{j}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{m-j}\frac{1}{2j+1}\\
&=\sum_{j=0}^{m}(-1)^j\binom{j}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{m-j}\frac{1}{2j+1}-\sum_{j=p}^{m}(-1)^j\binom{j}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{m-j}\frac{1}{2j+1}\\
&\equiv\frac{(-1)^m}{2m+1}\binom{m}{\frac{p+1}{2}}\binom{2m}{p+1}^{-1} \pmod p.
\end{aligned}\tag{2.14}$$

It is easy to see that

$$\frac{(-1)^m}{2m+1}\binom{m}{\frac{p+1}{2}}\binom{2m}{p+1}^{-1} \equiv \begin{cases}(-1)^{\frac{p+1}{2}} \pmod p, & \text{if } m=p,\\ 0 \pmod p, & \text{if } p+2\le m\le \frac{3p-3}{2}.\end{cases} \tag{2.15}$$

Combining (2.13)-(2.15) and noting that

$$\frac{1}{p}\binom{p}{\frac{p+1}{2}} = \frac{1}{(p+1)/2}\binom{p-1}{\frac{p-1}{2}} \equiv 2(-1)^{\frac{p-1}{2}} \pmod p, \tag{2.16}$$

we obtain

$$S_p(p) \equiv 6 \pmod p,$$
$$S_p(m) \equiv 0 \pmod p, \quad \text{for } p+2\le m\le \frac{3p-3}{2}.$$

If $m=p+1$, then

$$\begin{aligned}
&S_p(p+1)\\
&= \frac{6}{p^2}\binom{p+1}{\frac{p+1}{2}}\sum_{j=0}^{p-1}(-1)^j\binom{j}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{p+1-j}\frac{1}{2j+1}\\
&= \frac{6}{p^2}\binom{p+1}{\frac{p+1}{2}}\left(\sum_{j=0}^{p+1}(-1)^j\binom{j}{\frac{p+1}{2}}\binom{\frac{p+1}{2}}{p+1-j}\frac{1}{2j+1}+\binom{p}{\frac{p+1}{2}}\frac{p+1}{2(2p+1)}-\binom{p+1}{\frac{p+1}{2}}\frac{1}{2p+3}\right)\\
&= \frac{6}{p^2}\binom{p+1}{\frac{p+1}{2}}\left(\frac{1}{2p+3}\binom{p+1}{\frac{p+1}{2}}\binom{2p+2}{p+1}^{-1}+\binom{p}{\frac{p+1}{2}}\frac{p+1}{2(2p+1)}-\binom{p+1}{\frac{p+1}{2}}\frac{1}{2p+3}\right)\\
&\equiv \frac{1}{p}\binom{p+1}{\frac{p+1}{2}}\left(\frac{3}{p}\binom{p}{\frac{p+1}{2}}-\frac{3}{2p}\binom{p+1}{\frac{p+1}{2}}\right) \pmod p.
\end{aligned}$$

Noting that

$$\frac{1}{p}\binom{p+1}{\frac{p+1}{2}} = \frac{2}{p}\binom{p}{\frac{p+1}{2}} \equiv 4(-1)^{\frac{p-1}{2}} \pmod p$$

with the help of (2.16), we get $S_p(p+1)\equiv 0 \pmod p$. This completes the proof. □

Lemma 2.3 *Let $p \geq 11$ be a prime and m be an integer such that $0 \leq m \leq 2p-2$. Then*

$$35\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}(-1)^j\binom{j}{i}\binom{m}{i}\binom{i}{m-j}\frac{9}{(2i-3)(2j+1)(2m-2i-3)}$$

$$\equiv \begin{cases} 35, -70, 64 \pmod p, & \text{if } m=0,1,3, \text{ respectively,} \\ 70, -140, -36 \pmod p, & \text{if } m=p, p+1, p+3, \text{ respectively,} \\ 80(-1)^{\frac{p+1}{2}} \pmod p, & \text{if } m=\frac{3p-1}{2}, \\ 84(-1)^{\frac{p-1}{2}} \pmod p, & \text{if } m=\frac{3p+1}{2}, \\ 0 \pmod p, & \text{otherwise.} \end{cases} \tag{2.17}$$

Proof. Let $T_p(m)$ denote the left-hand side of (2.17). If $m=0,1,2,3$, we can check the values of $T_p(m)$ directly. If $4 \leq m \leq p-1$, we use the following identity

$$\sum_{i=0}^{m}\binom{m}{i}^2\binom{2m}{2i}^{-1}\frac{1}{(2i-3)(2m-2i-3)}=0, \quad \text{for } m \geq 4.$$

If $p \leq m \leq 2p-2$, then

$$\begin{aligned} T_p(m) \equiv& 35\sum_{j=0}^{p-1}(-1)^j\binom{j}{\frac{p+3}{2}}\binom{m}{\frac{p+3}{2}}\binom{\frac{p+3}{2}}{m-j}\frac{9}{p(2m-p-6)(2j+1)} \\ &+35\sum_{j=0}^{p-1}(-1)^j\binom{j}{m-\frac{p+3}{2}}\binom{m}{\frac{p+3}{2}}\binom{m-\frac{p+3}{2}}{m-j}\frac{9}{p(2m-p-6)(2j+1)} \\ &=70\sum_{j=0}^{p-1}(-1)^j\binom{j}{\frac{p+3}{2}}\binom{m}{\frac{p+3}{2}}\binom{\frac{p+3}{2}}{m-j}\frac{9}{p(2m-p-6)(2j+1)} \pmod p. \end{aligned}$$

The rest of the proof is similar to that of (2.11), and we omit the details. □

3 Proof of Theorem 1.2

Proof of (1.5). By [2, (2.5)], we have

$$\begin{aligned} \binom{k}{i}\binom{k+i}{i}\binom{k}{j}\binom{k+j}{j} &= \sum_{r=0}^{i}\binom{i+j}{i}\binom{j}{i-r}\binom{j+r}{r}\binom{k}{j+r}\binom{k+j+r}{j+r} \\ &= \sum_{s=j}^{i+j}\binom{i+j}{i}\binom{j}{s-i}\binom{s}{j}\binom{k}{s}\binom{k+s}{s}. \end{aligned} \tag{3.1}$$

Using (3.1) and the fact that $\binom{k+i}{2i}\binom{2i}{i} = \binom{k}{i}\binom{k+i}{i}$, we get

$$\begin{aligned}
&3\sum_{k=0}^{n-1} R_k(x)^2 \\
&= \sum_{k=0}^{n-1}\sum_{i=0}^{k}\sum_{j=0}^{k}\binom{k+i}{2i}\binom{2i}{i}\binom{k+j}{2j}\binom{2j}{j}\frac{3x^{i+j}}{(2i-1)(2j-1)} \\
&= \sum_{k=0}^{n-1}\sum_{i=0}^{k}\sum_{j=0}^{k}\sum_{s=j}^{i+j}\binom{i+j}{i}\binom{j}{s-i}\binom{s}{j}\binom{k}{s}\binom{k+s}{s}\frac{3x^{i+j}}{(2i-1)(2j-1)} \\
&= \sum_{k=0}^{n-1}\sum_{m=0}^{2k}\sum_{s=0}^{m}\sum_{i=0}^{s}\binom{m}{i}\binom{m-i}{m-s}\binom{s}{m-i}\binom{k}{s}\binom{k+s}{s}\frac{3x^{m}}{(2i-1)(2m-2i-1)} \\
&= \sum_{m=0}^{2n-2}\sum_{s=0}^{m}\sum_{i=0}^{s}\sum_{k=0}^{n-1}\binom{m}{i}\binom{m-i}{m-s}\binom{s}{m-i}\binom{k}{s}\binom{k+s}{s}\frac{3x^{m}}{(2i-1)(2m-2i-1)},
\end{aligned}$$

where $m = i + j$. Applying $\binom{m-i}{m-s}\binom{s}{m-i} = \binom{s}{i}\binom{i}{m-s}$ and the following identity

$$\sum_{k=0}^{n-1}\binom{k}{s}\binom{k+s}{s} = \binom{n+s}{s}\binom{n-1}{s}\frac{n}{2s+1},$$

which can be easily proved by induction on n, we obtain

$$\begin{aligned}
&3\sum_{k=0}^{n-1} R_k(x)^2 \\
&= n\sum_{m=0}^{2n-2}\sum_{s=0}^{m}\sum_{i=0}^{m}\binom{n+s}{s}\binom{n-1}{s}\binom{m}{i}\binom{s}{i}\binom{i}{m-s}\frac{3x^{m}}{(2s+1)(2i-1)(2m-2i-1)}.
\end{aligned} \tag{3.2}$$

Then the proof of (1.5) directly follows from Lemma 2.1 and (3.2). □

Proof of (1.6). Letting $x = 1$ in (3.2), we immediately get

$$\frac{3}{n}\sum_{k=0}^{n-1} R_k^2 \equiv \sum_{m=0}^{2n-2}\sum_{s=0}^{m}\sum_{i=0}^{m}\binom{n+s}{s}\binom{n-1}{s}\binom{m}{i}\binom{s}{i}\binom{i}{m-s} \pmod 2. \tag{3.3}$$

Noting that

$$\binom{s}{i}\binom{i}{m-s} = \binom{s}{m-s}\binom{2s-m}{s-i},$$

and then applying the Chu-Vandermonde identity to (3.3) yields

$$\frac{3}{n}\sum_{k=0}^{n-1} R_k^2 \equiv \sum_{m=0}^{2n-2}\sum_{s=0}^{m}\binom{n+s}{s}\binom{n-1}{s}\binom{s}{m-s}\binom{2s}{s} \pmod 2. \tag{3.4}$$

Since $\binom{2s}{s} = 2\binom{2s-1}{s-1}$ for $s \geq 1$, we conclude that every term on the right-hand side of (3.4) is even except for $m = s = 0$. It follows that

$$\frac{3}{n}\sum_{k=0}^{n-1} R_k^2 \equiv 1 \pmod{2}.$$

This completes the proof of (1.6). □

4 Proof of Theorem 1.3

Proof of (1.4). Letting $n = p$ and $x = 1$ in (3.2), and then noting that for $0 \leq s \leq p-1$

$$\binom{p+s}{s}\binom{p-1}{s} \equiv (-1)^s \pmod{p^2},$$

and for $0 \leq s, i \leq p-1$ and $0 \leq m \leq 2p-2$

$$\binom{m}{i}\binom{s}{i}\binom{i}{m-s}\frac{3p}{(2s+1)(2i-1)(2m-2i-1)} \in \mathbb{Z}_p,$$

we obtain

$$\begin{aligned} &3\sum_{k=0}^{p-1} R_k^2 \\ &\equiv p\sum_{m=0}^{2p-2}\sum_{s=0}^{p-1}\sum_{i=0}^{p-1}(-1)^s\binom{m}{i}\binom{s}{i}\binom{i}{m-s}\frac{3}{(2s+1)(2i-1)(2m-2i-1)} \pmod{p^2}. \end{aligned} \tag{4.1}$$

Combining (2.11) and (4.1), we have

$$\begin{aligned} 3\sum_{k=0}^{p-1} R_k^2 &\equiv (3+2+6+4(-1)^{\frac{p+1}{2}})p \\ &\equiv (11+4(-1)^{\frac{p+1}{2}})p \pmod{p^2}. \end{aligned}$$

This completes the proof of (1.4). □

Proof of (1.7). For $p = 5, 7$, it is easy to verify that (1.7) holds. For $p \geq 11$, we apply (2.17) and then obtain

$$35\sum_{k=0}^{p-1} W_k^2 \equiv (-77-4(-1)^{\frac{p+1}{2}})p \pmod{p^2}.$$

The proof runs analogously, and we omit the details. □

Acknowledgments. The authors would like to thank the anonymous referee for valuable comments which improved the presentation of the paper.

References

[1] H.-Q. Cao and H. Pan, A Stern-type congruence for the Schröder numbers, preprint, 2015, arXiv:1512.06310v1.

[2] V.J.W. Guo, Some congruences involving powers of Legendre polynomials, Integral Transforms Spec. Funct. 26 (2015), 660–666.

[3] V.J.W. Guo and J.-C. Liu, Proof of a conjecture of Z.-W. Sun on the divisibility of a triple sum, J. Number Theory 156 (2015), 154–160.

[4] W. Koepf, Hypergeometric Summation, an Algorithmic Approach to Summation and Special Function Identities, Friedr. Vieweg & Sohn, Braunschweig, 1998.

[5] J.-C. Liu, A supercongruence involving Delannoy numbers and Schröder numbers, J. Number Theory 168 (2016), 117–127.

[6] M. Petkovšek, H. S. Wilf and D. Zeilberger, $A = B$, A K Peters, Ltd., Wellesley, MA, 1996.

[7] R. P. Stanley, Hipparchus, Plutarch, Schröder, and Hough, Amer. Math. Monthly 104 (1997), 344–350.

[8] R. P. Stanley, Enumerative Combinatorics, Vol. 2, Cambridge Univ. Press, Cambridge, 1999.

[9] Z.-W. Sun, On Delannoy numbers and Schröder numbers, J. Number Theory 131 (2011), 2387–2397.

[10] Z.-W. Sun, Two new kinds of numbers and related divisibility results, preprint, 2014, arXiv:1408.5381v8.

[11] Z.-W. Sun, Arithmetic properties of Delannoy numbers and Schröder numbers, preprint, 2016, arXiv:1602.00574v3.