# Towards relativistic hydrodynamics with spin currents

Angel Domingo Gallegos Pazos

Promotor: Stefan Vandoren
Copromotor: Umut Gürsoy

The cover depicts the nahuatlahtolli written form of atl-tlachinolli: the mexica concept of duality and war. The topics treated in this thesis started from the study of holographic dualities.


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# Towards relativistic hydrodynamics with spin currents

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(met een samenvatting in het Nederlands)

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**Abstract**

In this thesis, some aspects of hydrodynamics on systems with non-trivial spin degrees of freedom are studied. We start by identifying the relevant spin degrees of freedom and how they can be incorporated in the hydrodynamic expansion. After this has been achieved we modify the constitutive relations to include such degrees of freedom, this is done in a way compatible with the second law of thermodynamics. Conformal symmetry in the presence of spin degrees of freedom is also analyzed and is used to write down an idealized model that resembles the quark gluon plasma generated in heavy ion collisions. From this model the expected polarization of particles after such a collision is computed and compared to experimental data. Finally, we use the fluid gravity correspondence and a five dimensional gravitational toy model to exemplify how the new spin related transport coefficients of a strongly interactive quantum field theory can be computed from a dual gravitational theory.

# Contents

# Publications

The content of the following chapters is based on

- **Chapter 2 & 3** A.D. Gallegos and Umut Gürsoy and Amos Yarom
  *Hydrodynamics of spin currents*,
  *SciPost* **11** (2021) 041, [2101.04759].

- **Chapter 4** A.D. Gallegos and Umut Gürsoy
  *Holographic spin liquids and Lovelock Chern-Simons gravity*,
  *JHEP* **11** (2020) 151, [2004.05148].

Both chapters 2 and 3 have more content than the papers they have been based on. Work which is not part of this dissertation

- A.D. Gallegos and Umut Gürsoy
  *Dynamical gauge fields and anomalous transport at strong coupling*,
  *JHEP* **05** (2019) 001, [1806/07138].

- A.D. Gallegos and Umut Gürsoy and Natale Zinnato
  *Torsional Newton Cartan gravity from non-relativistic strings*,
  *JHEP* **09** (2020) 172, [1906.01607].

- A.D. Gallegos and Umut Gürsoy and Sagar Verma and Natale Zinnato
  *Non-Riemannian gravity actions from double field theory*,
  *JHEP* **06** (2021) 173, [2012.07765].

- Chris Blair and A.D. Gallegos and Natale Zinnato
  *A non-relativistic limit of M-theory and 11-dimensional membrane Newton-Cartan*,
  *JHEP* **10** (2021) 015, [2104.07579].

# Chapter 1

# Introduction

Hydrodynamics is a robust effective theory, valid on distance scales much longer than the typical mean free path, capable of describing a wide variety of phenomena, ranging from stellar evolution through fluid flow in a pipe to heavy ion collisions. The dynamics of fluid flow are captured by the conservation laws of the underlying many body theory at hand, supplemented by constitutive relations which parametrize the conserved currents as local functions of the dynamical variables. For example, energy can be parametrized by a local temperature field and energy conservation serves as a dynamical equation for it.

In a relativistic theory, on Minkowski spacetime, translation invariance implies that the energy momentum tensor, $T^{\mu\nu}$, is conserved. In relativistic hydrodynamics the constitutive relations for the energy momentum tensor are expressed in terms of a temperature field $T$ and a velocity field $u^\mu$ normalized such that $u^\mu u_\mu = -1$. As in any effective theory, the constitutive relations are the most general ones possible, compatible with the physical constraints of the problem. In hydrodynamics these include [1–3] symmetries, unitarity, the second law of hydrodynamics and various Onsager relations. Once the constitutive relations are available (usually in terms of a derivative expansion), the equations of motion for $T$ and $u^\mu$ are given by energy momentum conservation and from the relativistic Navier-Stokes equations. Over the last few years there has been increasing interest in a theory of hydrodynamics in the presence of an independently conserved angular momentum density, $J^{\mu\nu\rho}$, or, alternatively, a non vanishing spin current, $S^{\mu\nu\rho}$;

$$J^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu} - S^{\mu\nu\rho}. \tag{1.1}$$

In flat spacetime, the conservation of energy and angular momentum implies

$$\partial_\mu T^{\mu\nu} = 0\,, \qquad \partial_\lambda S^{\lambda\mu\nu} = 2T^{[\mu\nu]}\,. \tag{1.2}$$

Equations (1.2) will rule the evolution of the flow on a fluid with non-vanishing spin current $S^{\lambda\mu\nu}$. The spin current $S^{\lambda\mu\nu}$ will be present when underlying canonical fields carry nontrivial spin. An instructive example is to look at free Dirac fermions, i.e. the quantum field theory of spinors $\psi$ with mass $m$ that evolve according to the Lagrangian

$$S[\psi] = \int d^d x |e| \bar{\psi} \left(i\gamma^\mu \partial_\mu - m\right) \psi\,, \tag{1.3}$$

with $\gamma_\mu$ the Dirac gamma matrices. From the Noether procedure it can be shown that conservation of angular momentum fixes the spin current to be

$$S^{\lambda\mu\nu} = -\frac{i}{2}\bar{\psi}\left(\gamma^\lambda \gamma_{\mu\nu} + \gamma_{\mu\nu}\gamma^\lambda\right)\psi\,, \tag{1.4}$$

with $\gamma_{\mu\nu} = \gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu$, (1.4) is the simplest example of a theory with non-vanishing spin current as the scalar field will have a vanishing $S^{\lambda\mu\nu}$ as expected.

Relativistic hydrodynamics plays an important role during heavy ion collisions. In these collisions the quark gluon plasma (QGP) is produced along with strong magnetic fields, see figure 1.1.

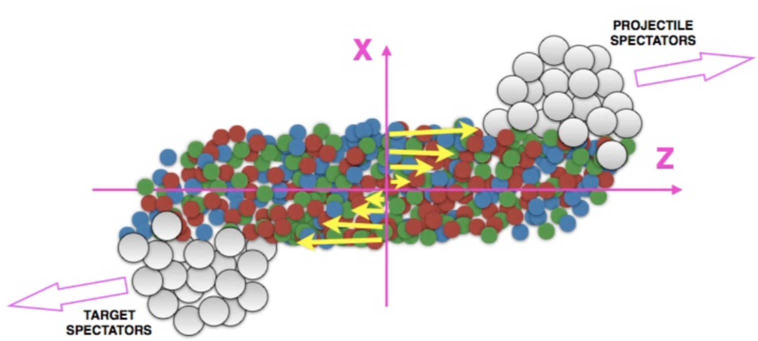


Figure 1.1: Schematic view of the collision with the yellow arrow indicating the direction of the flow.Figure taken from [4].

In off-central collisions the QGP is characterized by a large vorticity, which together with the large magnetic fields that are generated can induce a global polarization of the final hadrons [4]. In particular, there is experimental evidence for correlations between the spin polarization of Λ-hyperons and the angular momentum of the quark gluon plasma in off center collisions [5,6], see figure 1.2.

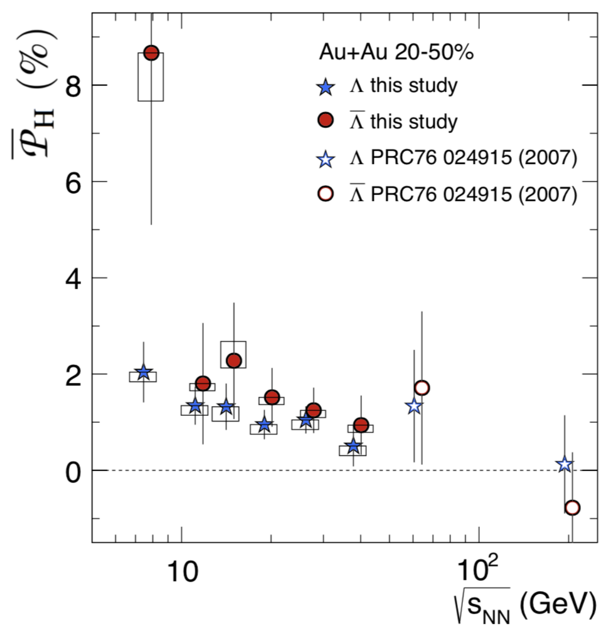


Figure 1.2: Results and figure taken from [5] where the average polarization $\bar{\mathcal{P}}_H$ for Λ-hyperons ($H = \Lambda$) and $\bar{\Lambda}$-hyperions ($H = \bar{\Lambda}$) is plotted against the beam energy per nucleon $\sqrt{s_{NN}}$ for 20-50% central Au+Au collisions. The results of [5] are shown together with the ones from [7] for 62.4 and 200 GeV collisions. Boxes indicate systematic uncertainties.

Fluid-like description of spin dynamics in non-relativistic systems are also crucial in spintronics [8,9] and quantum spin liquids [10]. The experimental realization of spin currents induced by vorticity in liquid metals [11], see figure 1.3, is a recent example. Spin hydrodynamics in (nearly) defect-free crystals might give insights on the spin transport, energy dissipation, and efficiency of the system [12].

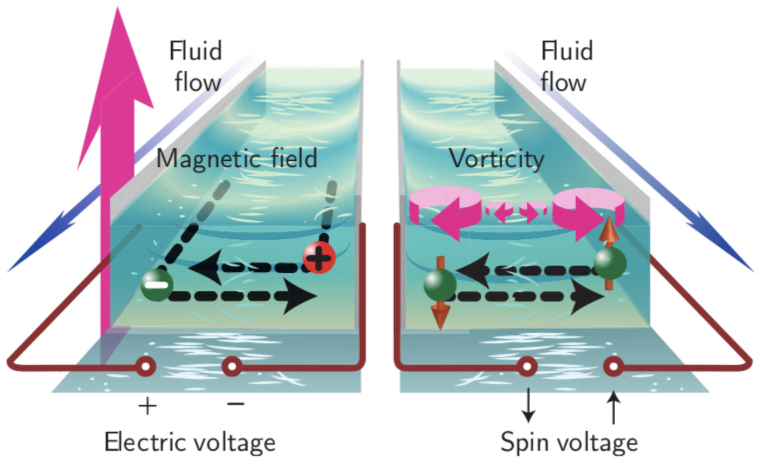


Figure 1.3: Comparison of electron hall effect (left) and an equivalent spin effect in liquid metals due to non-vanishing vorticity (right). Figure taken from [11].

A proper theoretical understanding of these phenomenon necessitates a consistent hydrodynamic theory of spin currents. While there has been a significant amount of work on the matter, see, e.g., [13–33] and [34–38], a fully consistent theory of hydrodynamics of spin currents is lacking. Therefore the main goal of this thesis is to show the steps we have taken into developing an exhaustive and consistent theory of spin hydrodynamics. One of the main issues when trying to develop spin hydrodynamics is the general disregard of the spin current due to the Belinfante-Rosenfeld (BR) symmetry [39, 40], an invariance of (1.2) under the shift

$$T'^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\mathring{\nabla}_\lambda \left(B^{\lambda\mu\nu} - B^{\mu\lambda\nu} - B^{\nu\lambda\mu}\right) , \tag{1.5}$$

$$S'^{\lambda\mu\nu} = S^{\lambda\mu\nu} + B^{\lambda\mu\nu} , \tag{1.6}$$

where $B^{\lambda\mu\nu}$ is antisymmetric on the last two indices and in principle can then be chosen judiciously to completely remove the spin current from the description, leaving the symmetric Belinfante-Rosenfeld energy momentum tensor $T^{\mu\nu}_{BR}$ as the only relevant current

$$T^{\mu\nu}_{BR} = T^{\mu\nu} - \frac{1}{2}\mathring{\nabla}_\lambda \left(S^{\lambda\mu\nu} - S^{\mu\lambda\nu} - S^{\nu\lambda\mu}\right) . \tag{1.7}$$

This conclusion was challenged in [20] (see also [22, 23]) where it was argued that, even though the definition of the spin current operator is subject to the transformation (1.6) this symmetry might be broken by the thermodynamic state of the theory, hence the different choices of $B^{\lambda\mu\nu}$ would lead to different observables. Our approach to the BR symmetry consisted on introducing sources into the description. Once the proper sources of the energy momentum tensor and the spin current were identified the BR symmetry was broken, allowing us to work in an unambiguous way. This identification is the jumping point into exploring the dynamics of the spin current in a consistent way. Although we did not manage to do an exhaustive study of all aspects associated to the spin dynamics we did develop the first steps into a systematic approach to studying relativistic hydrodynamics with spin.

In the rest of this chapter we will discuss how the spin current is consistently sourced, we will review the classical relativistic theory of hydrodynamics and how we can consistently include the spin degrees of freedom into the description. Finally we will mention how spin hydrodynamic transport can be studied using holographic techniques. These topics on themselves are explored on the remaining chapters. In chapter 2 the introduction of spin degrees of freedom into the hydrodynamic description of a theory is performed in a systematic way. Chapter 3 is used to explore conformal symmetry in the presence of spin, we later use this symmetry to simplify the computation of spin polarization in a model containing some of the main features found in the quark gluon plasma generated in heavy ion collisions finding a good fit with collision data. Chapter 4 is relatively self contained and in there we use a holographic five dimensional toy model to show that it is possible to compute hydrodynamic transport via holographic techniques. Finally in chapter 5 we discuss the next steps that should be taken to fully flesh out the hydrodynamic description of systems with spin.

## 1.1 Sourcing the spin current

The dynamics of a finite temperature many body system at a given energy scale are encoded on the partition function $Z[J_{\mathcal{O}}, \Phi_\varepsilon]$, where $J$ are sources coupled to the operators $\mathcal{O}$ of the theory and $\Phi_\varepsilon$ are the microscopic degrees of freedom relevant at such energy scale. Such partition function is obtained after integrating out the high energy degrees of freedom of the microscopic theory up to the chosen scale.

From the partition function the quantum effective action $W$ can be defined as

$$e^{iW[J_{\mathcal{O}},\Phi_\varepsilon]} = Z[J_{\mathcal{O}},\Phi_\varepsilon]\,. \tag{1.8}$$

By taking variations of $W[J_{\mathcal{O},\Phi_\varepsilon}]$ with respect to the sources we can find the correlation functions of the theory for the operators $\mathcal{O}$. In the hydrodynamic limit only conserved currents enter the description, meaning that we only need to know how those can be sourced. As we already discussed, for an underlying theory invariant under translations and rotations there will be two operators associated to the Noether currents of these symmetries, the energy momentum tensor $T^{\mu\nu}$ and the total angular momentum $J^{\lambda\mu\nu}$. In flat spacetime the conservation laws for these currents will be (1.2). In the context of hydrodynamics, which our work will focus on, these two currents will be the only relevant ones for the dynamics while all the higher energy degrees of freedom have been integrated out, namely $Z = Z[J_T, J_S]$. We know that for theories with vanishing spin the energy momentum tensor can be sourced by coupling the underlying QFT to a generic curved background, with the metric $g_{\mu\nu}$ of such background acting as the source of $T^{\mu\nu}$. To illustrate how the spin current can be sourced it is, once again, enough to understand how Dirac fermions couple to gravity. To place fermions in curved spacetime it is necessary to change from a formulation in terms of the background metric $g_{\mu\nu}$ to a first order formulation in terms of the vielbein fields $e^a_\mu$ [41–43], related to the metric by

$$e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}\,. \tag{1.9}$$

The vielbein introduce a flat tangent space at every point in spacetime on which the Clifford algebra can be locally defined. This results in the Dirac Lagrangian

$$S[\psi] = \int d^4x|e|i\bar{\psi}\left[\gamma^\mu\left(\partial_\mu + \frac{i}{2}\omega^{ab}{}_\mu\gamma_{ab}\right) - m\right]\psi + c.c.\,, \tag{1.10}$$

where $|e|$ is the determinant of the vielbein fields, and $\omega^{ab}{}_\mu$ is the spin connection introduced to make the theory invariant under local Lorentz transformation. The connection is minimally coupled to the fermions hence sources the spin current operator $-\frac{i}{2}\bar{\psi}\left(\gamma^\lambda\gamma_{\mu\nu} + \gamma_{\mu\nu}\gamma^\lambda\right)\psi$. Comparing this coupling with (1.4) it becomes clear that the spin connection will act as the source of $S^{\lambda\mu\nu}$.

When the sources in (1.10) are only background fields hence not subject to variations, there is no reason to choose the connection to be Levi-Civita[1]. It can be more general

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) + K^\lambda{}_{\nu\mu} \,, \tag{1.11}$$

where the first term is the Levi-Civita connection $\mathring{\Gamma}^\lambda_{\mu\nu}$and the second term, $K$ is called *contorsion* and its antisymmetric on its first two indices. The metricity requirement (vanishing covariant derivative of the metric) does not fix $K$. Instead, it is determined completely in terms of the *torsion* $T^a{}_{\mu\nu}$, through

$$T^\lambda{}_{\mu\nu} \equiv 2\Gamma^\lambda{}_{[\mu\nu]} = K^\lambda{}_{\nu\mu} - K^\lambda{}_{\mu\nu} \,, \tag{1.12}$$

while torsion itself is related to both the vielbein and spin connection through

$$T^a{}_{\mu\nu} = \partial_{[\mu} e^a_{\nu]} + \omega^a{}_{b[\mu} e^b_{\nu]} \,. \tag{1.13}$$

In the presence of a non-trivial $T^a$, the vielbein and the spin connection stay independent.

We can now extend the discussion from free fermions to a more general QFT with an action $I[e, \omega, \Psi]$ with $\Psi$ a non-trivial representation of the Lorentz algebra, coupled to a first order gravitational background. One then defines the effective action $W[e, \omega]$

$$e^{iW[e,\omega]} = \int D\Psi e^{iI[e,\omega,\Psi]} \,, \tag{1.14}$$

with a variation of the form

$$\delta W = \int d^d x |e| \left[ T^\mu{}_a \delta e^a_\mu + \frac{1}{2} S^\lambda{}_{ab} \delta\omega^{ab}{}_\lambda \right] \,. \tag{1.15}$$

Coupling the underlying theory to a torsionful spacetime will unambiguously define the spin current through (1.15), we will take this relation as the definition of $S^{\lambda\mu\nu}$. At this point we can note that taking the affine connection to be the Christoffel connection,

$$\mathring{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right) \,, \tag{1.16}$$

[1] Choosing the Levi-Civita connection is equivalent to symmetrizing the energy momentum tensor and removing the spin current through a BR transformation.

will make the connection and the vielbein not independent and will recover the usual symmetric Belinfante-Rosenfeld energy momentum tensor [39, 40] and a vanishing spin current. It is then crucial to keep the connection independent throughout any computation to have unambiguous definitions, we are free to set torsion to zero once the relevant macroscopic quantities of interest have been computed.

## 1.2 Relativistic hydrodynamics

Hydrodynamics is a universal low energy effective field theory of many body, finite temperature systems. The equations of motion of hydrodynamics consist of local conservation laws (e.g. energy momentum conservation or charge conservation, we will focus on the set of equations (1.2)). The dynamical variables are given by a temperature field T, a velocity field $u^\mu$ (which we normalize such that $u^\mu u_\mu = -1$ for the relativistic setting we are working in), and chemical potentials associated with other conserved charges present. In the current context angular momentum conservation leads to a (non-)conservation equation for the spin current, which implies the existence of a spin potential $\mu_{ab}$, i.e. the spin analog of electric chemical potential.

To obtain the explicit form of the equations of motion for the hydrodynamic variables one needs a set of constitutive relations whereby the energy momentum tensor and spin current are expressed in terms of T, $u^\mu$, the relevant chemical potentials and their derivatives. Often, such constitutive relations are expressed in terms of a truncated expansion in derivatives of the hydrodynamics variables. In particular we would like to classify all independent and symmetry allowed tensor structures at any given order of the expansion to write down the most general set of constitutive relations. The proportionality constants between the currents and each of the relevant tensor structures are usually called transport coefficients. For example, the constitutive relations for $T^{\mu\nu}$ up to first order in derivatives of an uncharged fluid with trivial spin is known to be of the form

$$T^{\mu\nu} = \varepsilon(T)u^\mu u^\nu + P(T)\Delta^{\mu\nu} - \zeta(T)\theta\Delta^{\mu\nu} - \eta(T)\sigma^{\mu\nu}\,, \tag{1.17}$$

where $\varepsilon$ is the energy density, $P$ is the pressure, the projector $\Delta^{\mu\nu}$ to the space orthogonal to $u^\mu$ is defined as

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu\,, \tag{1.18}$$

the transport coefficients $\{\zeta, \eta\}$, known as the bulk and shear viscosities, appear next to the first order in derivative terms $\{\theta, \sigma^{\mu\nu}\}$ respectively. These first order tensors are related to the gradient of the four velocity through

$$\theta = \mathring{\nabla}_\mu u^\mu \,, \tag{1.19}$$

$$\sigma^{\mu\nu} = \left( \Delta^\mu_{(\alpha} \Delta^\nu_{\beta)} - \frac{\Delta^{\mu\nu} \Delta_{\alpha\beta}}{d-1} \right) \mathring{\nabla}^\alpha u^\beta \,, \tag{1.20}$$

with $d$ the dimension of the spacetime and $\mathring{\nabla}$ is the Christoffel covariant derivative. In general the constitutive relations must satisfy certain criteria which have been shown to be captured by the second law of thermodynamics, at least to leading order in derivative expansion. For the particular case of (1.17) we know these constraints are translated into the positivity of the viscosities $\{\zeta, \eta\}$. One of the main goals of this work was to derive the equivalent of (1.17) for a system with both a non-trivial spin current and spin chemical potential. We will later motivate that the spin potential is naturally associated with terms which are first order in derivatives, this will alter the way we will truncate the theory. We will write down the most general constitutive relations for the energy momentum tensor and spin current of a parity invariant theory on d-dimensional flat spacetime, this will be shown in chapter 2. It is important to mention that this counting will not be the only possible counting on the hydrodynamic derivative expansion, in chapters 2 and 4 we will briefly discuss some alternatives and in [44, 45] some other proposed alternatives can be found.

### 1.2.1 Classification of constitutive relations

The tensor structures appearing on the constitutive relations for the currents can be classified into two categories: equilibrium and non-equililbrium. The non-equilibrium tensors are defined such that they vanish under local thermodynamic equilibrium, while the equilibrium part corresponds to the hydrostatic limit. A fluid which is acted on by time independent external forces will settle down to a hydrostatically equilibrated configuration. Since a hydrostatically equilibrated configuration must be a solution to the hydrodynamic equations of motion, the explicit form of the hydrostatic configuration must be compatible with the constitutive relations of the fluid. Put differently, the existence of hydrostatic equilibrium poses constraints on the constitutive relations of the fluid. Therefore, if we can generate a valid hydrostatically equilibrated configuration we may use it to simplify the construction of the constitutive relations for the fluid.

To obtain a valid hydrostatically equilibrated configuration we use the methods developed in [46, 47]: Consider a hydrostatically equilibrated state. In such state Euclidean correlation functions of generic operators at equal Euclidean time will decay exponentially at large distances (assuming we are not at a critical point or that there are no long range forces at play). That is, there exists a power series expansion of zero frequency correlation functions of generic operators around zero spatial momentum. Let us consider a generating function for such correlation functions valid up to, say, $m$ powers of spatial momentum. In real space such a generating function will contain $m$ derivatives of its arguments, the sources for the operators in question. For instance, the hydrostatic generating function for the energy momentum tensor will be a local function of the background metric and its derivatives. Thinking of hydrodynamics as an expansion around this hydrostatic state justifies the use of a truncated derivative expansion.

There exists an abundant body of literature on the construction of the hydrostatic generating function, and the associated constraints on constitutive relations for a variety of operators and sources. See e.g., [48–77]. For the current work, we are interested in hydrostatics in the presence of a spin current, as defined in (1.15). To this end, we need to construct a Lorentz and coordinate invariant generating function associated with a time independent external vielbein, ${e^a}_\mu$ and spin connection, ${\omega_\mu}^{ab}$ (and their covariant derivatives).

To make time independence of the sources manifest, we denote by $V^\mu$ the Killing vector along which the vielbein and connection are time independent. Then, time independence of ${\omega_\mu}^{ab}$ and ${e^a}_\mu$ amounts to the condition $\mathcal{L}_L(...) = 0$ for any tensor and source of the theory. The existence of $V^\mu$ will allow us to find a solution of the dynamical variables in terms of the sources $\{e^a_\mu, {\omega^{ab}}_\mu\}$. For the ideal fluid, namely a fluid with only energy and pressure taken into account, this solution looks like

$$T = \frac{T_0}{\sqrt{-V^2}}\,, \qquad u^\mu = \frac{V^\mu}{\sqrt{-V^2}}\,, \tag{1.21}$$

with $T_0^{-1}$ the parametric length of a compact Euclidean time direction on which the effective action $W$ has been computed. Through (1.21) the equilibrium/hydrostatic effective action $W$ for the ideal fluid can be written in terms of the dynamical variables as

$$W = \int d^d x |e| \ P(T) \,, \tag{1.22}$$

with $P(T)$ the hydrostatic pressure, this identification will follow once we compute the variation of $W$ using (1.21). In addition to the pressure identification the Gibbs-Duhem relation and first law also follow from this partition function, namely

$$\varepsilon = -P + sT \,, \tag{1.23}$$

$$s = \frac{\partial P}{\partial T} \,, \tag{1.24}$$

$$d\varepsilon = T ds \,, \tag{1.25}$$

where we have identified $\frac{\partial P}{\partial T}$ with the entropy $s$, this identification follows from matching the result of the variation of $W$ with the standard thermodynamic relations expected to be valid at local equilibrium. We can go beyond the ideal case and find the equilibrium corrections to the constitutive relations at any given order of the derivative expansion, this can be done by adding to $W$ all independent equilibrium scalars made out of gradients of the dynamical variables at the desired order. It is important to note that the equilibrium condition, $\mathcal{L}_V(...) = 0$, imposes certain relations between the dynamical variables. For the uncharged fluid the following equilibrium relations can be derived from the background metric Killing equation, $\mathcal{L}_V g_{\mu\nu} = 0$,

$$a_\mu = -\frac{\mathring{\nabla}_\mu T}{T} \,, \tag{1.26}$$

$$\theta = 0 \,, \tag{1.27}$$

$$\sigma^{\mu\nu} = 0 \,, \tag{1.28}$$

with $a$ the acceleration defined as

$$a_\mu = u^\nu \mathring{\nabla}_\nu u_\mu \,. \tag{1.29}$$

If we were to take $\mathring{\nabla}_\mu T$ as the independent equilibrium vector we will be left with the vorticity $\Omega^{\mu\nu}$, defined below, as the only independent equilibrum part of the velocity gradient

$$\Omega_{\mu\nu} = \Delta^\rho_\mu \Delta^\sigma_\nu \mathring{\nabla}_{[\rho} u_{\sigma]} \,. \tag{1.30}$$

In addition to these equilibrium identities we will have $u^\mu \mathring{\nabla}_\mu T = 0$, following from $\mathcal{L}_V T = 0$, allowing us to conclude that there are no non-vanishing first order scalars for the uncharged fluid[2]. We can conclude the first order corrections to the ideal fluid constitutive relations should only be of the non-equilibrium type. There are 2 non-equilibrium scalars $\{\theta, u^\mu \mathring{\nabla}_\mu T\}$, 1 non-equilibrium vector $a_\mu + \frac{\mathring{\nabla}_\mu T}{T}$, and 1 non-equilibrium tensor $\sigma^{\mu\nu}$ resulting in the first order constitutive relation[3]

$$T^{\mu\nu} = \left(\varepsilon + n_1 u^\rho \mathring{\nabla}_\rho T + n_2 \theta\right) u^\mu u^\nu + \left(p + n_3 u^\rho \mathring{\nabla}_\rho T + n_4 \theta\right) \Delta^{\mu\nu} + n_5 \left(a^{(\mu} + \frac{\mathring{\nabla}^{(\mu} T}{T}\right) u^{\nu)} - \eta \sigma^{\mu\nu} , \tag{1.31}$$

with $n_i$ some functions of temperature. We can note (1.31) is not of the form discussed in (1.17), the reason for this is that there are still two simplifications we can make[4]. The first one consists on writing down the constitutive relations on-shell at any given order in the derivative expansion, meaning we can use the equations of motion order by order to write a subset of the non-equilibrium tensors in terms of another independent subset at any given order. For the uncharged fluid we have two projections of the equations of motion at first order in derivatives

$$u_\nu \mathring{\nabla}_\mu T^{\mu\nu} = 0 , \tag{1.32}$$

$$\Delta^\rho_\nu \mathring{\nabla}_\mu T^{\mu\nu} = 0 . \tag{1.33}$$

From (1.32) it follows that $u^\rho \mathring{\nabla}_\rho T$ is related to $\theta$ on shell, allowing to change $(n_3 u^\rho \mathring{\nabla}_\rho T + n_4 \theta) \to -\zeta\theta$ and leaving only the bulk viscosity $\zeta$ as the independent transport coefficient. The second equation (1.33) will set $a^\mu + \frac{\mathring{\nabla}^\mu T}{T}$ to zero. The second simplifications consist on performing a so called frame transformation of the dynamical variables, this transformation is of the form

$$T \to T + \delta T , \tag{1.34}$$

$$u^\mu \to u^\mu + \delta u^\mu , \tag{1.35}$$

[2]This is true for a parity invariant theory. If we allow for parity breaking contributions to W there might be equilibrium contributions at first order.

[3]The uncharged fluid has a symmetric energy momentum tensor.

[4]If this simplifications are not made the constitutive relations will have an ambiguity on them as the transport coefficients will not uniquely characterize the fluid.

with $\{\delta T, \delta u^\mu\}$ first order in derivatives. Such transformation is related to the ambiguity in defining microscopically the dynamical variables and we can note that such shift will not change the equations of motion at the order we are taking into account. We can note that under this transformations the projections of (1.31) transforms, up to $\mathcal{O}(\partial)$, as

$$\delta\left(u_\mu u_\nu T^{\mu\nu}\right) = 0\,, \qquad \delta\left(\Delta_{\mu\nu} T^{\mu\nu}\right) = 0\,, \tag{1.36}$$
$$\delta\left(\Delta_{\mu\rho} u_\sigma T^{\rho\sigma}\right) = -\left(\varepsilon + p\right)\delta u_\mu\,, \tag{1.37}$$
$$\delta\left(\Delta_{\mu\rho}\Delta_{\nu\sigma} T^{\rho\sigma}\right) = 0\,. \tag{1.38}$$

From (1.37) we can choose $\delta u_\mu$ such that $T^{\mu\nu}u_\mu$ is parallel to the four velocity, this choice of frame is known as the Landau frame and will be our choice for the whole work. After fixing $\delta u$ this way we are left with $\delta T$, to fix this transformation we can look closely at (1.36) and note that under the frame transformation the energy and pressure transform as

$$\varepsilon(T) \to \varepsilon(T) + \frac{\partial \varepsilon}{\partial T}\delta T\,, \tag{1.39}$$
$$p(T) \to p(T) + \frac{\partial p}{\partial T}\delta T\,. \tag{1.40}$$

We can then pick $\delta T$ to absorb $\left(n_1 u^\rho \mathring{\nabla}_\rho T + n_2\theta\right)$. We are then left with the constitutive relations on the Landau frame

$$T^{\mu\nu} = \varepsilon(T)u^\mu u^\nu + p(T)\Delta^{\mu\nu} - \zeta(T)\theta\Delta^{\mu\nu} - \eta(T)\sigma^{\mu\nu}\,. \tag{1.41}$$

This whole discussion can be repeated for the case of a fluid with non-trivial spin current and non-vanishing spin potential. After we have chosen the derivative order on which $\mu^{ab}$ is to be considered we can classify all tensor structures on the theory in terms of the equilibrium condition and their order in the expansion. For the equilibrium piece we can write down the effective action $W$ up to a desired order in derivatives by using a solution of the dynamical variables analogous to (1.21), in particular the spin potential will be related to the spin connection via

$$\mu^{ab} = \frac{\omega^{ab}{}_\mu V^\mu}{\sqrt{-V^2}}\,. \tag{1.42}$$

The number of non-equilibrium tensors can then be reduced by using the equations of motion at the desired order in the expansion. This systematic classification of a fluid with spin is one if, if not the, main results of this work.

### 1.2.2 Constraints on transport coefficients

The coefficients appearing on the constitutive relations (1.41) are generically not unconstrained. They will be restricted by the existence of an entropy current $S^\mu$, namely we will ask for the existance a vector $S^\mu$ such that, on-shell, its divergence is positive definite

$$\mathring{\nabla}_\mu S^\mu \geq 0 \,. \tag{1.43}$$

Equation (1.43) is the hydrodynamic version of the second law of thermodynamics. Following the canonical approach of deriving such current [1] we can look at the following equation of motion

$$u_\nu \mathring{\nabla}_\mu T^{\mu\nu} = -u^\mu \mathring{\nabla}_\mu \varepsilon - \theta \left(\varepsilon + P\right) + \zeta\theta^2 + \eta\sigma^{\mu\nu}\sigma_{\mu\nu} \,, \tag{1.44}$$

using the thermodynamic relations $\mathring{\nabla}_\mu \varepsilon = T\mathring{\nabla}_\mu s$ and $\epsilon + P = sT$ we can rewrite (1.44) as

$$u_\nu \mathring{\nabla}_\mu T^{\mu\nu} = -T\left(u^\mu \mathring{\nabla}_\mu s + s\theta\right) + \xi\theta^2 + \eta\sigma^{\mu\nu}\sigma_{\mu\nu} \,, \tag{1.45}$$

that, on-shell, can be rewritten as

$$\mathring{\nabla}_\mu \left(su^\mu\right) = \frac{\xi\theta^2}{T} + \frac{\eta\sigma^{\mu\nu}\sigma_{\mu\nu}}{T} \,. \tag{1.46}$$

If we are to assume $S^\mu = su^\mu$ then for (1.43) to be satisfied we would need

$$\zeta > 0 \,, \qquad \eta > 0 \,. \tag{1.47}$$

However there could exists a non-canonical part[5] on the entropy current, namely the entropy current can take the form $S^\mu = su^\mu + S^\mu_{non-can}$ with $S^\mu_{non-can}$ a judiciously chosen first order vector such that (1.43) is still satisfied. Although it can be shown that the canonical part is enough to capture

[5] Generically the canonical part of the entropy current is $S_{can} = su^\mu - \frac{u_\nu}{T}\left(T^{\mu\nu} - T^{\mu\nu}_{ideal}\right)$.

the entropy current of an uncharged fluid we know that not taking the non-canonical piece into account can lead to incomplete results, see [78] to see its importance on anomalous transport. When analyzing spin transport an analogous analysis should be done, in this case the canonical part of the spin current will take the form

$$S^{\mu}_{can} = su^{\mu} - \frac{u_\nu}{T}\left(T^{\mu\nu} - T^{\mu\nu}_{ideal}\right) - \frac{\mu_{\rho\sigma}}{2T}\left(S^{\mu\rho\sigma} - S^{\mu\rho\sigma}_{ideal}\right) , \tag{1.48}$$

where $T^{\mu\nu}_{ideal}$ and $S^{\lambda\mu\nu}_{ideal}$ are the ideal pieces of the constitutive relations[6]. To determine the non-canonical piece we need to assume the most general first order vector[7] and impose (1.43).

## 1.3 Holography

Two theories are called dual when you can map the observables from one theory into the observables of the other one. A very successful duality has been the holographic correspondence, that on its original and more general form relates a string theory in an $AdS_5$ x$S_5$ background with $\mathcal{N} = 4$ super Yang-Mills (SYM) theory with $SU(N)$ gauge group living in four spacetime dimensions [80]. In its stronger form the mapping between these two theories is conjectured to be given by [81]

$$Z_{CFT}\left[\mathcal{O}\phi_{(0)}\right] = Z_{String}\big|_{\phi(z,x)_{\text{boundary}}=\phi_0(x)} , \tag{1.49}$$

where $Z_{CFT}$ is the SYM partition function, $Z_{String}$ is the string partition function, $\mathcal{O}$ is some SYM operator and $\phi(x,z)$ is a general field propagating on the AdS bulk with coordinates $(z,x)$, note that $x$ are the coordinates of the 4D spacetime on which SYM is evolving. What (1.49) is telling us is that the boundary value of the bulk field acts as source for the corresponding operator $\mathcal{O}$. On the SYM side there are two relevant couplings: the SYM coupling $g_{YM}$ and the number of colors $N$, while on the string side there is the string length $l_s$ and the string coupling constant $g_s$. The relation between

[6]From the point of view of the hydrostatic partition function $P$ we refer to the ideal part as the one coming from the contributions to the pressure $P$ involving no explicit derivatives.

[7]We are assuming (1.43) is satisfied up to $\mathcal{O}\left(\partial^2\right)$, for an analysis when higher order terms are taken into account look at [79].

these couplings is

$$g_{YM}^2 = 2\pi g_s \,, \qquad 2g_{YM}^2 N = \frac{L^4}{l_s^4} \,, \tag{1.50}$$

with $L$ the AdS radius. On the $g_s \to 0$ and $\frac{l_s^2}{L} \to 0$ limit the string theory can be approximated by classical supergravity, the corresponding limit on the CFT side will correspond to the t'Hooft limit [82] $N \to \infty$ and $\lambda \equiv g_{YM}^2 N$ large. Namely the large N and strongly coupled CFT can be approximated by the on-shell supergravity action $S_{\text{SUGRA}}$, in Euclidean signature, through

$$Z_{CFT}\left[\mathcal{O}\phi_{(0)}\right] = e^{-S_{\text{SUGRA}}}\Big|_{\phi(z,x)_{\text{boundary}}=\phi_0(x)} \,. \tag{1.51}$$

Strong arguments for (1.51) have been made [80] in the particular case of Type IIB supergravity and strongly coupled $\mathcal{N} = 4$ SYM. However, for computational purposes, it is possible to assume that (1.51) is a good approximation to compute observables on any given strongly coupled QFT once an adequate choice of dual gravitational theory has been chosen[8].

Assuming (1.51) works for our theory of interest we can note from comparing (1.51) and (1.8) that $S_{SUGRA}$ takes the role of the effective quantum action. As suggested from (1.15) the boundary value of the supergravity metric will source the energy momentum tensor. In addition to the gravitational fields we would need to include fields for all the relevant operators of the theory, however on the hydrodynamic limit these operators are the ones associated to conserved currents, for us they will be the energy momentum tensor and the spin current. This means the class of gravitational theories that are of interest to us are those of the Einstein-Cartan type [83–85], where the vielbein and the spin connection are the gravitational independent variables

$$S_{SUGRA} = S_{SUGRA}[e_M^A, \omega_M^{AB}] \,, \tag{1.52}$$

where we are using bulk spacetime and tangent space indices $\{M, N, ..\}$ and $\{A, B, ..\}$ respectively. In practice what has to be done is to propose a gravitational action based on the symmetries of the expected dual CFT, and then solve the equations of motion to evaluate the on-shell gravitational action and its fluctuations. From the fluctuations of $S_{SUGRA}$ the correlations functions

[8] This type of holography is known as bottom up holography and the dual gravitational theory is usually chosen based on the symmetries of the particular QFT of interest.

of the theory can be computed. When using holography for computing the constitutive relations it is possible to directly use the hydrodynamic expansion on the gravitational side [86,87]. This procedure consists on solving the gravitational equations of motion order by order in the derivative expansion. This will allow us to compute the correlation functions to any desired order in the gradient expansion. In chapter 4 we will look at a particular gravitational action satisfying this characteristics that will allow us to compute different spin transport coefficients in an analytic and controlled manner.

# Chapter 2

# Hydrodynamics

As discussed in the introduction, the first step in setting up a hydrodynamic theory of spin currents is to construct a well defined spin current. The challenge that arises when working in flat spacetime is the inherent ambiguity in the definition of the stress tensor constructed via Noether's procedure due to improvement terms. These improvement terms modify the structure of the angular momentum density and therefore the structure of the spin current, as defined in (2.6). In what follows we will go over the origin of the ambiguity of the spin current and we will make two observations. The first one is that the ambiguity in the spin current is removed once the theory is placed on a torsionful background geometry. Thus, by coupling the theory to torsion one can obtain a well defined spin current. Our second observation is that the ambiguity in the spin current does not affect the combined dynamics of the stress tensor and spin current. Therefore, to obtain a well defined spin current we can couple our theory to a torsionful background and take the torsionless limit to obtain physical results. Along the way we will introduce some notation relevant for the rest of this work. While the material contained in this section is known, it is central to our construction, so we present it here for completeness.

Given an action $S[\phi;\,\chi]$ with dynamical fields $\phi$ and couplings $\chi$, we define the variation

$$\delta S[\phi;\,\chi] = \int \left[C \cdot \delta\chi + E \cdot \delta\phi\right] d^d x\,, \tag{2.1}$$

such that $E$ denotes the equations of motion. If $\delta S = 0$ and $\delta\chi = 0$ for a particular $\delta\phi$, we say that $\delta\phi$ is an infinitesimal symmetry. If $\delta S = 0$ for a particular (non vanishing) $\delta\chi$ and $\delta\phi$ we say that $\delta\chi$ and $\delta\phi$ constitute a spurionic symmetry. If we parametrize an infinitesimal symmetry via $\delta\phi = \lambda\delta_\lambda\phi$ with $\lambda$ a constant, then for a spacetime dependent $\lambda$ we necessarily have

$$\delta S[\phi;\,\chi] = -\int J^\mu \nabla_\mu \lambda d^d x = \int E \cdot \lambda\delta_\lambda\phi d^d x\,, \tag{2.2}$$

which gives Noether's theorem, $\nabla_\mu J^\mu = 0$, under the equations of motion.

The transformation associated with $\delta\phi = \lambda(x)\delta_\lambda\phi$ is not a symmetry. But, if we manage to couple the action to a source $\chi_\mu$ such that a generic transformation of the action yields

$$\delta S[\phi;\,\chi,\,\chi_\mu] = \int J^\mu \delta\chi_\mu + C \cdot \delta\chi + E \cdot \delta\phi d^d x\,, \tag{2.3}$$

then $\delta\phi = \lambda(x)\delta_\lambda\phi$ and $\delta\chi_\mu = \nabla_\mu\lambda$ will be a spurionic symmetry with the associated conservation law $\nabla_\mu J^\mu = 0$. Conversely, suppose $\delta S = 0$ for some spurionic symmetry $\delta\phi = F(\lambda(x))$, $\delta\chi = \chi(\lambda(x))$ and $\delta\chi_\mu = X_\mu(\lambda(x))$ where $F$, $X$ and $X_\mu$ are linear functions of $\lambda$ with (possibly) a finite number of derivatives acting on them. Then, $J^\mu$ defined in (2.2) will satisfy a certain identity under the equations of motion. Often, this identity is referred to as a conservation equation even though, as it should be clear from the above construction, it will result in $\nabla_\mu J^\mu = 0$ only for $\delta\chi_\mu = \nabla_\mu\lambda$ and $\delta\chi = 0$. In what follows we will refer to identities associated with the above spurionic symmetry as conservation laws if they lead to a conservation equation of the form $\nabla_\mu J^\mu = 0$ and to non conservation laws otherwise. Our entire discussion can be mapped to a quantum field theory (modulo a treatment of anomalies) where the non conservation laws are referred to as Ward identities.

Before applying this somewhat abstract construction to the vielbein and spin connection, let us first discuss it in a more familiar setting. Consider a set of dynamical fields $\phi$ coupled to an external metric $g_{\mu\nu}$ and an external gauge field, $A_\mu$ such that the action is coordinate reparameterization invariant and gauge invariant. In this case, the coordinate reparameterizations and gauge transformations are spurionic symmetries. A general variation of the fields and sources is given by

$$\delta S[\phi;\,g_{\mu\nu},\,A_\mu] = \int \left(\frac{1}{2}T^{\mu\nu}\delta g_{\mu\nu} + J^\mu \delta A_\mu + E\cdot\phi\right)\sqrt{g}d^d x\,. \tag{2.4}$$

Only the gauge field transforms under gauge transformations so $J^\mu$ will satisfy a conservation law, $\nabla_\mu J^\mu = 0$. On the other hand coordinate reparameterizations will modify the metric and the gauge field which leads to the non-conservation law

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu \,, \tag{2.5}$$

with $F^{\mu\nu}$ the field strength associated with the external gauge field $A_\mu$. Moving on to our particular setup, consider a field theory with dynamical fields $\phi$, coupled to a vielbein $e^a{}_\mu$ and spin connection $\omega_\mu{}^{ab}$. The variation of the action associated with this theory is given by

$$\delta W = \int d^d x |e| \left[ T^\mu{}_a \delta e^a_\mu + \frac{1}{2} S^\lambda{}_{ab} \delta \omega^{ab}{}_\lambda \right] . \tag{2.6}$$

Such an action can be made invariant under coordinate reparameterizations, $\xi^\mu$, and local Lorentz transformations, $\theta^a{}_b$, as long as the dynamical fields transform as appropriate tensors, e.g.,

$$\delta Q^a{}_\mu = \pounds_\xi Q^a{}_\mu - \theta^a{}_b Q^b{}_\mu \,, \tag{2.7}$$

for some tensor $Q^a{}_\mu$, and the vielbein and spin connection transform as

$$\begin{aligned} \delta e^a{}_\mu =& \pounds_\xi e^a{}_\mu - \theta^a{}_b e^b{}_\mu \,, \\ \delta \omega^{ab}{}_\mu =& \pounds_\xi \omega^{ab}{}_\mu + \nabla_\mu \theta^a{}_b \,. \end{aligned} \tag{2.8}$$

Here $\pounds_\xi$ is a Lie derivative along $\xi$, and $\nabla_\mu$ denotes a covariant derivative,

$$\nabla_\mu Q^a{}_\nu = \partial_\mu V^a{}_\nu + \omega^a{}_{c\mu} V^c{}_\nu - \Gamma^\alpha{}_{\mu\nu} V^a{}_\alpha \,. \tag{2.9}$$

If $\omega^{ab}{}_\mu$ and $e^a{}_\mu$ are independent fields, the transformations (2.8) will lead to two non conservation laws: one for $T^\mu{}_a$ and one $S^\mu{}_{ab}$. Conversely, if $\omega^{ab}_\mu$ is determined from $e^a{}_\mu$ then the only current is the stress tensor and (2.8) will lead to a non conservation law for it. The improvement term which allows for an ambiguity in the definition of the spin current can be traced to the latter and its absence to the former. Before showing this explicitly, let us introduce some notation. We define the torsion free spin connection $\mathring{\omega}^{ab}{}_\mu$ via

$$\mathring{\omega}^{ab}{}_\mu = e_\nu{}^a \left( \partial_\mu e^{\nu b} + \mathring{\Gamma}^\nu{}_{\sigma\mu} e^{\sigma b} \right) , \tag{2.10a}$$

where

$$\mathring{\Gamma}^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left( \partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma} \right) . \tag{2.10b}$$

The covariant derivative associated with $\mathring{\omega}^{ab}{}_{\mu}$ or $\mathring{\Gamma}^{\alpha}{}_{\beta\gamma}$ will be denoted by $\mathring{\nabla}_{\mu}$. For instance,

$$\mathring{\nabla}_{\mu} Q^{a}{}_{\nu} = \partial_{\mu} V^{a}{}_{\nu} + \mathring{\omega}^{a}{}_{c\mu} V^{c}{}_{\nu} - \mathring{\Gamma}^{\alpha}{}_{\mu\nu} V^{a}{}_{\alpha} \,. \tag{2.11}$$

Likewise, $\mathring{R}^{\alpha}{}_{\beta\gamma\delta}$ and other ringed curvature tensors are associated with ringed connections in contrast to their non-ringed counterparts, e.g., $R^{\alpha}{}_{\beta\gamma\delta}$. The difference between the spin connection and the ringed spin connection is the contorsion tensor

$$\omega^{ab}{}_{\mu} = \mathring{\omega}_{\mu}{}^{ab} + K^{ab}{}_{\mu} \,. \tag{2.12}$$

The contorsion tensor, $K^{ab}{}_{\mu}$, is related to the torsion tensor, $T^{\alpha}{}_{\beta\gamma}$, defined via

$$\Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta} = T^{\alpha}{}_{\beta\gamma} \,, \tag{2.13}$$

through

$$T^{\alpha}{}_{\mu\nu} = K^{\alpha}{}_{\nu\mu} - K^{\alpha}{}_{\mu\nu} \,. \tag{2.14}$$

Going back to (2.8), suppose that $\omega^{ab}{}_{\mu} = \mathring{\omega}^{ab}{}_{\mu}$. In this case (2.8) will reduce to

$$\delta S[\phi,\, e^{a}{}_{\mu},\, \omega^{ab}{}_{\mu}(e)]\Big|_{\omega^{ab}{}_{\mu} = \mathring{\omega}^{ab}_{\mu}} = \int \left( T^{\mu}_{\mathrm{C}\,a} \delta e^{a}{}_{\mu} + E \cdot \phi \right) |e| d^{d}x \,, \tag{2.15}$$

where

$$T^{\mu}_{\mathrm{C}\,a}(x) = T^{\mu}{}_{a}(x) + \frac{1}{|e|} \int \frac{1}{2} S^{\nu}{}_{cb}(y) \frac{\delta \mathring{\omega}^{cb}{}_{\nu}(y)}{\delta e^{a}{}_{\mu}(x)} |e(y)| d^{d}y \,. \tag{2.16}$$

The conservation equations associated with the spurionic symmetries (2.8) read

$$\mathring{\nabla}_{\mu} T^{\mu\nu}_{\mathrm{C}} = 0 \,, \qquad T^{\mu\nu}_{\mathrm{C}} - T^{\nu\mu}_{\mathrm{C}} = 0 \,. \tag{2.17}$$

Note that in our current formulation, the absence of an antisymmetric component of the energy momentum tensor is a dynamical statement. That is, it is satisfied only if the equations of motion, $E = 0$, are satisfied. We will come back to this feature shortly. So far, we have set $\omega_{\mu}{}^{ab} = \mathring{\omega}_{\mu}{}^{ab}$ and then taken the variation (2.6) in order to get the stress tensor. We could have carried out the same procedure in the reverse order, first take the variation (2.6) and then set $\omega^{ab}{}_{\mu} = \mathring{\omega}^{ab}{}_{\mu}$. In this case we would have obtained the non conservation equations

$$\mathring{\nabla}_{\mu} T^{\mu\nu} = \frac{1}{2} \mathring{R}^{\nu}{}_{\alpha\beta\gamma} S^{\alpha\beta\gamma} \,, \qquad T^{\mu\nu} - T^{\nu\mu} = \mathring{\nabla}_{\alpha} S^{\alpha\mu\nu} \,. \tag{2.18}$$

Note that the first equality in (2.18) can be written in the equivalent form,

$$\mathring{\nabla}_\mu T^{\mu\nu} = \frac{1}{2}\mathring{\nabla}_\mu \mathring{\nabla}_\alpha \left(S^{\alpha\mu\nu} - S^{\mu\alpha\nu} - S^{\nu\alpha\mu}\right) . \tag{2.19}$$

The second equality in (2.18) is equivalent to angular momentum conservation, c.f., (1.1).

Using (2.10) and (2.16) it is straightforward to show that $S^{\alpha\mu\nu}$ is related to $T^a_{\mathrm{C}\,\mu}$ via

$$T^{\mu\nu}_{\mathrm{C}} = T^{\mu\nu} - \frac{1}{2}\mathring{\nabla}_\alpha \left(S^{\alpha\mu\nu} - S^{\mu\alpha\nu} - S^{\nu\alpha\mu}\right) . \tag{2.20}$$

Thus, equations (2.17) and (2.18) are, obviously, equivalent, and no information is lost by solving one or the other. In fact, since $\mathring{\omega}^{ab}{}_\mu$ is a function of $e^a{}_\mu$ the dependence of the action, $S$, on the spin connection $\mathring{\omega}^{ab}{}_\mu$ is ambiguous: we may always shuffle a dependence of the action on $\mathring{\omega}^{ab}{}_\mu$ into a dependence on $e^a{}_\mu$ and its derivatives. Thus, in general, we find

$$\mathring{\nabla}_\mu T'^{\mu\nu} = \frac{1}{2}\mathring{R}^\nu{}_{\alpha\beta\gamma} S'^{\,\alpha\beta\gamma} , \qquad T'^{\mu\nu} - T'^{\nu\mu} = \mathring{\nabla}_\alpha S'^{\,\alpha\mu\nu} , \tag{2.21}$$

where

$$T'^{\,\mu\nu} = T^{\mu\nu} + \frac{1}{2}\mathring{\nabla}_\alpha \left(B^{\alpha\mu\nu} - B^{\mu\alpha\nu} - B^{\nu\alpha\mu}\right) . \tag{2.22a}$$

with

$$B^{\alpha\mu\nu} = S'^{\,\alpha\mu\nu} - S^{\alpha\mu\nu} . \tag{2.22b}$$

Equations (2.17) and (2.18) are special cases of (2.21).

Let us emphasize once again that given an action, $S$, equations (2.17), (2.18) and (2.21) are all equivalent and will take the same functional form. The difference between $T^{\mu\nu}$ and $T'^{\,\mu\nu}$ exhibited in equation (2.22a), is often referred to as an improvement term, and amounts to exchanging derivatives of the vielbein in the action between the spin connection and the stress tensor. In equations, it amounts to a decomposition of the stress tensor $T^{\mu\nu}_{\mathrm{C}}$ of the form

$$T^{\mu\nu}_{\mathrm{C}} = T'^{\,\mu\nu} - \frac{1}{2}\mathring{\nabla}_\alpha \left(S'^{\,\alpha\mu\nu} - S'^{\,\mu\alpha\nu} - S'^{\,\nu\alpha\mu}\right) . \tag{2.23}$$

Colloquially, due to the similarity between (2.21) and (2.18), one oftentimes refers to the decomposition in (3.30) as an ambiguity in the spin current. The construction presented in this section makes it clear that the modification of the spin current à la (2.22b) is compensated by a modification of the energy momentum tensor given by (2.22a) so that the equations of motion are unchanged.

Let us now turn our attention to the somewhat simpler situation where ${\omega_\mu}^{ab}$ and ${e^a}_\mu$ are independent parameters. Coordinate invariance and local Lorentz invariance imply the non-conservation equations

$$\mathring{\nabla}_\mu T^{\mu\nu} = \frac{1}{2} R^{\rho\sigma\nu\lambda} S_{\rho\lambda\sigma} - T_{\rho\sigma} K^{\nu ab} {e^\rho}_a {e^\sigma}_b \,, \tag{2.24}$$

$$\mathring{\nabla}_\lambda {S^\lambda}_{\mu\nu} = 2T_{[\mu\nu]} + 2{S^\lambda}_{\rho[\mu} {e_{\nu]}}^a {e_\rho}^b K_{\lambda ab} \,. \tag{2.25}$$

Since ${\omega_\mu}^{ab}$ and ${e^a}_\mu$ are independent sources the freedom leading to (3.30) is absent and the stress tensor and spin current can not be transmuted into one another. It is particularly insightful to rewrite equations (2.24) and (2.25) as[1]

$$\mathring{\nabla}_\mu \left[ T^{\mu\nu}_{\mathrm{BR}} + \frac{1}{2} {S^\mu}_{\rho\sigma} K^{\rho\sigma\nu} \right] = \frac{1}{2} S_{\lambda\rho\sigma} \mathring{\nabla}^\nu K^{\rho\sigma\lambda} \,, \tag{2.26}$$

$$T^{[\mu\nu]}_{\mathrm{BR}} = {S^\lambda}_{\rho[\mu} {K^\rho}_{\nu]\lambda} \,, \tag{2.27}$$

where $T_{\mathrm{BR}}$ is the Belinfant-Rosenfeld energy momentum tensor defined as

$$T^{\mu\nu}_{\mathrm{BR}} = T^{\mu\nu} + \frac{1}{2} \mathring{\nabla}_\lambda \left( S^{\mu\lambda\nu} + S^{\nu\lambda\mu} - S^{\lambda\mu\nu} \right) \,, . \tag{2.28}$$

In the torsionless limit equations these equations reduce to (2.18) (or equivalently, (2.17) or (2.21)) as expected. We will focus on the constitutive relations for $\{T^{\mu\nu}, S^{\lambda\mu\nu}\}$ when there is no background torsion, meaning that most contributions of the spin current will be reabsorbed into the antisymmetric part of the energy momentum tensor. However the spin chemical potential will still be present and $T^{[\mu\nu]}_{\mathrm{BR}} = 0$ will be nothing but the dynamical equation for $\mu^{ab}$. Although we will focus on the zero torsion case we will keep it around for as long as possible both on the constitutive relations and on the equations of motion. The reason for this is that even when ${K^{ab}}_\mu$ is set to zero its fluctuation with respect to both the vielbein and connection is not necessarily vanishing meaning that it will generically contribute, either to the constitutive relations from appearing in $W$ or to higher point functions when appearing on the constitutive relations. We will not look, in detail, beyond the one point functions however we want to leave as much groundwork

[1] To get to this form it is enough to use the definition of the Riemann tensor as the commutator of covariant derivatives.

done as possible for any future work. The main use of torsion, when describing theories without it, is to remove the ambiguity introduced by the shift (2.28), and to allow for a well defined set of hydrodynamic currents through the identification of their respective sources. Before moving on there are a couple of observations we can make around the set of equations (2.24) and (2.25) (or alternatively (2.26) and (2.27)):

1. The derivative expansion: From (2.25) we can note that if the spin current is at least $\mathcal{O}(1)$ in the derivative expansion then the antisymmetric part of the energy momentum tensor will have to be at least $\mathcal{O}(\partial)$.

2. Dynamical degrees of freedom: The dynamical variables for the hydrodynamic equations (2.24) and (2.25) will be the temperature $T$, the four velocity $u^{\mu}$, and the spin potential $\mu^{ab}$. Equation (2.24) compromises $d$ equations which will allow us to solve for $T$ and $u^{\mu}$, while equation (2.25) contains the $\frac{d(d-1)}{2}$ equations necessary to solve for $\mu^{ab}$.

3. Order of $\mu^{ab}$ and $K^{ab}{}_{\mu}$: In principle we are allowed to freely choose the order of the spin potential and external torsion, each choice will correspond to a different physical setting. The main application we have in mind, heavy ion colllisions, sets the external torsion to zero meaning that an adequate counting can be done by assuming $K^{ab}{}_{\mu} \sim \mathcal{O}(\partial)$ before eventually setting it to zero. Taking $K^{ab}{}_{\mu} \sim \mathcal{O}(1)$ should be a valid assumption when other systems, such a condensed matter settings, are taken into account. In equilibrium we will identify the spin potential with the spin connection making $\mu^{ab} \sim \mathcal{O}(\partial)$ a natural choice. We can note that it could be possible for us to consider $\mu^{ab} \sim \mathcal{O}(1)$ once we are far from equilibrium, however we are assuming for the hydrodynamic limit to be valid mostly near equilibrium states.

4. Equation of motion for $\mu^{ab}$: The order on which $\mu^{ab}$ is considered implies its dynamical equation will appear at $\mathcal{O}(\partial^2)$. If we want to write down dynamical equations for all the hydrodynamic degrees of freedom we will need to go up to $\mathcal{O}(\partial^2)$ on the antisymmetric piece of $T^{\mu\nu}$, per the previous observations.

5. We can note that due to the assumed order of $\omega^{ab}{}_{\lambda}$ we will have a decrease in the derivative order when computing variations with respect

to it, namely when computing correlation functions involving the spin connection. This means that an $\mathcal{O}(\partial^2)$ scalar on the effective partition function might contribute to the $\mathcal{O}(\partial)$ spin current constitutive relation.

The program of hydrodynamics aims to find a solution for the dynamical variables order by order in gradients. For this we need to provide the constitutive relations order by order in the expansion as well. Our goal will be to write down the constitutive relations up to $\mathcal{O}(\partial)$ on the symmetric piece of the energy momentum tensor, $\mathcal{O}(\partial)$ on the spin current, and $\mathcal{O}(\partial^2)$ on the antisymmetric piece of the energy momentum tensor, see the aforementioned observations. It is particularly useful to decompose both currents by means of the four velocity $u^\mu$ and its transverse projector $\Delta^{\mu\nu}$ as

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + (q^\mu u^\nu + u^\mu q^\nu) + (h^\mu u^\nu - u^\mu h^\nu)\,\Sigma^{\mu\nu} + \tau^{\mu\nu}\,, \tag{2.29}$$

$$S^{\lambda\mu\nu} = u^\lambda \rho^{\mu\nu} + \left(u^\mu j^{\lambda\nu} - u^\nu j^{\lambda\mu}\right) + l^{\lambda\mu\nu}\,, \tag{2.30}$$

with the projections $\{\mathcal{E}, \mathcal{P}, q, h, \rho, j, l\}$ given by

$$\mathcal{E} = u_\mu u_\nu T^{\mu\nu}\,, \tag{2.31}$$

$$\mathcal{P} = \frac{1}{d-1}\Delta_{\mu\nu}T^{\mu\nu}\,, \tag{2.32}$$

$$q^\mu = -\Delta^\mu_\rho u_\sigma T^{(\rho\sigma)}\,, \tag{2.33}$$

$$\Sigma^{\mu\nu} = \Delta^\mu_\rho \Delta^\nu_\sigma T^{(\rho\sigma)}\,, \tag{2.34}$$

$$h^\mu = -\Delta^\mu_\rho u_\sigma T^{[\rho\sigma]}\,, \tag{2.35}$$

$$\tau^{\mu\nu} = \Delta^\mu_\rho \Delta^\nu_\sigma T^{[\rho\sigma]}\,, \tag{2.36}$$

$$\rho^{\mu\nu} = -u_\lambda S^{\lambda\mu\nu}\,, \tag{2.37}$$

$$j^{\mu\nu} = \Delta^\mu_\rho \Delta^\nu_\sigma u_\alpha S^{\rho\sigma\alpha}\,, \tag{2.38}$$

$$l^{\lambda\mu\nu} = \Delta^\lambda_\kappa \Delta^\mu_\rho \Delta^\nu_\sigma S^{\kappa\rho\sigma}\,. \tag{2.39}$$

What follows is to write down the projections (2.31)-(2.39) in terms of the equilibrium and non-equilibrium tensor structures at the relevant hydrodynamic order. To write the lowest order dynamical equation for the hydrodynamic degrees of freedom we would need the following:

- All independent equilibrium scalars up to $\mathcal{O}(\partial^2)$ so that we can write down the hydrostatic partition function $W$ by means of the equilibrium ansatz. Once again we can recall the reason for wanting the partition function up to this order is twofold, first we need the antisymmetric parts of the energy momentum up to $\mathcal{O}(\partial^2)$ and second we can recall that the variations with respect to the spin connection lower the order in the derivative expansion.

- $\{\mathcal{E}, \mathcal{P}, q^\mu, \Sigma^{\mu\nu}\}$ up to $\mathcal{O}(\partial)$. This means we will need all independent non-equilibrium scalars up to $\mathcal{O}(\partial)$, all independent non-equilibrium transverse vectors up to $\mathcal{O}(\partial)$, and all independent non-equilibrium symmetric traceless and transverse rank 2 tensors up to $\mathcal{O}(\partial)$.

- $\{h^\mu, \tau^{\mu\nu}\}$ up to $\mathcal{O}(\partial^2)$. This means we will need all independent non-equilibrium transverse vectors up to $\mathcal{O}(\partial^2)$, and all independent non-equilibrium rank 2 transverse antisymmetric tensors up to $\mathcal{O}(\partial^2)$.

- $\{\rho^{\mu\nu}, j^{\mu\nu}, l^{\lambda\mu\nu}\}$ up to $\mathcal{O}(\partial)$. The density $\rho^{\mu\nu}$ can still be further decomposed into a transverse vector and a transverse antisymmetric tensor. The reason for not doing this decomposition explicitly is because $\rho^{\mu\nu}$ will take the role of a thermodynamic spin density and is more intuitive to leave it as a whole. After considering this further splitting we note we will need all non-equilibrium transverse vectors up to $\mathcal{O}(\partial)$, all non-equilibrium transverse rank-2 tensors up to $\mathcal{O}(\partial)$, and all non-equilibrium transverse rank-3 tensors with a pair of antisymmetric indices up to $\mathcal{O}(\partial)$.

In the following sections we will do a recap of the equilibrium ansatz follow up by the classification for all relevant aforementioned tensor structures.

## 2.1 Tensorial Classification

We can recall that the equilibrium condition implies the existence of a time-like Killing vector $V^\mu$, this Killing vector then allow us to write down an ansatz for the dynamical variables in term of the sources

$$T = \frac{T_0}{\sqrt{-V^2}}\,, \qquad u^\mu = \frac{V^\mu}{\sqrt{-V^2}}\,, \qquad \mu^{ab} = \frac{\omega^{ab}{}_\lambda V^\lambda}{\sqrt{-V^2}}\,, \tag{2.40}$$

with $T_0$ the inverse Euclidean time length. From the Killing condition on the background metric $\mathcal{L}_V g_{\mu\nu} = 0$ we find the equilibrium conditions

$$a_\mu = -\frac{\mathring{\nabla}_\mu T}{T}\,, \tag{2.41}$$

$$\theta = 0\,, \tag{2.42}$$

$$\sigma^{\mu\nu} = 0\,, \tag{2.43}$$

with $\{a, \theta, \sigma\}$ the acceleration, compressibility, and shear tensor as defined from the velocity gradient decomposition

$$\mathring{\nabla}_\mu u_\nu = \frac{1}{d-1}\theta\Delta_{\mu\nu} - u_\mu a_\nu + \sigma_{\mu\nu} + \Omega^{\mu\nu}\,, \tag{2.44}$$

$$\theta = \mathring{\nabla}_\mu u^\mu\,, \tag{2.45}$$

$$a^\nu = u^\mu \mathring{\nabla}_\mu u^\nu\,, \tag{2.46}$$

$$\Omega_{\mu\nu} = \Delta^\rho_\mu \Delta^\sigma_\nu \mathring{\nabla}_{[\rho} u_{\sigma]}\,, \tag{2.47}$$

$$\sigma^{\mu\nu} = \Delta^{\mu\nu\rho\sigma}\mathring{\nabla}_\rho u_\sigma\,, \tag{2.48}$$

where we defined $\Delta^{\mu\nu\rho\sigma}$ as

$$\Delta^{\mu\nu\rho\sigma} = \Delta^{\mu(\rho}\Delta^{\sigma)\nu} - \frac{1}{d-1}\Delta^{\mu\nu}\Delta^{\rho\sigma}\,. \tag{2.49}$$

In addition to the Killing equation we also have the Killing condition of the vielbein $e^a_\mu$, resulting in

$$K^{ab}{}_\mu u^\mu = \mu^{ab} + e^a{}_\mu e^b{}_\nu \left(\Omega^{\mu\nu} - 2u^{[\mu}a^{\nu]}\right) = \mu^{ab} + Te^{a\rho}e^{b\sigma}\mathring{\nabla}_{[\rho}\left(\frac{u_{\sigma]}}{T}\right)\,, \tag{2.50}$$

with $\Omega^{\mu\nu}$ the previously defined vorticity. The second term on the right hand side of (2.50) can be rewritten as the thermal vorticity, defined as $\frac{T}{2}\left[\mathring{\nabla}_\rho\left(\frac{u_\sigma}{T}\right) - \mathring{\nabla}_\sigma\left(\frac{u_\rho}{T}\right)\right]$ allowing us to identify (2.50) with the known relation between thermal vorticity and spin potential [88], while generalizing it in the presence of torsion. The last equilibrium relation that can be found follows from the Killing condition on the spin connection and results in

$$Te^\rho_a e^\sigma_b \nabla_\lambda \frac{\mu^{ab}}{T} = R^{\rho\sigma}{}_{\lambda\alpha}u^\alpha\,, \tag{2.51}$$

with $\nabla$ and $R^\lambda{}_{\rho\mu\nu}$ the covariant derivative and Riemann tensor respectively associated to the torsionful affine connection. The spin potential itself can also be decomposed as

$$\mu^{ab} = u^a m^b - u^b m^a + M^{ab}\,, \tag{2.52}$$
$$m^a = \mu^{ab} u_b\,, \tag{2.53}$$
$$M^{ab} = \Delta^a_c \Delta^b_d \mu^{cd}\,, \tag{2.54}$$

Before moving on, it is convenient to split the contorsion into its projections along and transverse to the four velocity

$$\begin{aligned} K^{ab}{}_{\mu} &= -u_\mu \left(u^a k^b - u^b k^a + K^{ab}\right) + 2u^{[a}\left[\frac{\kappa \Delta^{b]}_\mu}{d-1} + (\kappa_S)^{b]}{}_{\mu} + (\kappa_A)^{b]}{}_{\mu}\right] \\ &+ \frac{1}{d-2}\Delta^{[a}_\mu (\mathcal{K}_V)^{b]} + (\mathcal{K}_T)^{ab}{}_{\mu} + (\mathcal{K}_A)^{ab}{}_{\mu}\,, \end{aligned} \tag{2.55}$$

with the properties

$$(\mathcal{K}_T)^{\mu\sigma}{}_{\mu} = 0\,, \qquad (\mathcal{K}_T)^{[\rho\sigma\lambda]} = 0\,, \qquad (\mathcal{K}_A)^{\rho\sigma\lambda} = (\mathcal{K}_A)^{[\rho\sigma\lambda]}\,,$$

with the inverse projections

$$k^a = u^\mu u_b K^{ab}{}_{\mu}\,, \tag{2.56}$$
$$K^{ab} = \Delta^a_c \Delta^b_d u^\mu K^{cd}{}_{\mu}\,, \tag{2.57}$$
$$\kappa = e^\mu_c u_d K^{cd}{}_{\mu}\,, \tag{2.58}$$
$$(\kappa_s)^{\mu\nu} = \Delta^{\mu\nu\rho}{}_c u_d K^{cd}{}_{\rho}\,, \tag{2.59}$$
$$(\kappa_a)^{\mu\nu} = \left(\frac{\Delta^\mu_c \Delta^{\rho\nu} - \Delta^\nu_c \Delta^{\mu\rho}}{2}\right) u_d K^{cd}{}_{\rho}\,, \tag{2.60}$$
$$(\mathcal{K}_V)^\mu = 2\Delta^\mu_d \Delta^\rho_c K^{cd}{}_{\rho}\,, \tag{2.61}$$
$$(\mathcal{K}_A)^{\rho\sigma\lambda} = \Delta^{[\rho}_c \Delta^\sigma_d \Delta^{\lambda]\alpha} K^{cd}{}_{\alpha}\,, \tag{2.62}$$
$$(\mathcal{K}_T)^{\alpha\beta}{}_{\mu} = \Delta^\alpha_c \Delta^d_\beta \Delta^\nu_\mu K^{cd}{}_{\nu} - \frac{1}{d-2}\Delta^{[\alpha}_\mu (\mathcal{K}_a)^{\beta]} - (\mathcal{K}_A)^{\alpha\beta}{}_{\mu}\,. \tag{2.63}$$

In summary, the contorsion tensor can be decomposed into a scalar $\kappa$, two transverse vectors $\{k^a, (\mathcal{K}_V)^\mu\}$, two transverse antisymmetric rank-2 tensors $\{K^{ab}, (\kappa_a)^{\mu\nu}\}$, a symmetric traceless transverse rank-2 tensor $(\kappa_s)^{\mu\nu}$, a traceless transverse rank-3 tensor $(\mathcal{K}_T)^{\alpha\beta}{}_{\mu}$, and a totally antisymmetric transverse rank-3 tensor $(\mathcal{K}_A)^{\rho\sigma\mu}$. It is convenient to use this decomposition to rewrite

(2.50) as

$$k^{\mu} = m^{\mu} - a^{\mu}\,, \tag{2.64}$$

$$K^{\mu\nu} = M^{\mu\nu} + \Omega^{\mu\nu}\,. \tag{2.65}$$

We will now look at tensor structures vanishing/non-vanishing under equilibrium conditions (2.44)-(2.48), (2.51), (2.64), and (2.65).

### 2.1.1 Equilibrium Terms

For the equilibrium piece we only need to use (2.40) to write down the equilibrium partition function $W$, for this we in principle need all independent equilibrium scalars up to $\mathcal{O}\left(\partial^2\right)$. We list these, parity invariant, scalars on the following table

| | | | |
|---|---|---|---|
| $\mathcal{S}_1^{(0)}$ | | | |
| $T$ | | | |
| $\mathcal{S}_1^{(1)}$ | | | |
| $\kappa$ | | | |
| $\mathcal{S}_1^{(2)}$ | $\mathcal{S}_2^{(2)}$ | $\mathcal{S}_3^{(2)}$ | $\mathcal{S}_4^{(2)}$ |
| $\left(\kappa_a\right)^{\mu\nu} M_{\mu\nu}$ | $K^{\mu\nu} M_{\mu\nu}$ | $m_{\mu} k^{\mu}$ | $m_{\mu}\left(\mathcal{K}_V\right)^{\mu}$ |
| $\mathcal{S}_5^{(2)}$ | $\mathcal{S}_6^{(2)}$ | $\mathcal{S}_7^{(2)}$ | $\mathcal{S}_8^{(2)}$ |
| $\kappa^2$ | $k_{\mu} k^{\mu}$ | $K^{\mu\nu} K_{\mu\nu}$ | $k_{\mu}\left(\mathcal{K}_V\right)^{\mu}$ |
| $\mathcal{S}_9^{(2)}$ | $\mathcal{S}_{10}^{(2)}$ | $\mathcal{S}_{11}^{(2)}$ | $\mathcal{S}_{12}^{(2)}$ |
| $\left(\mathcal{K}_V\right)^{\mu}\left(\mathcal{K}_V\right)_{\mu}$ | $\left(\kappa_A\right)^{\mu\nu} K_{\mu\nu}$ | $\left(\kappa_A\right)^{\mu\nu}\left(\kappa_A\right)_{\mu\nu}$ | $\left(\kappa_S\right)^{\mu\nu}\left(\kappa_S\right)_{\mu\nu}$ |
| $\mathcal{S}_{13}^{(2)}$ | $\mathcal{S}_{14}^{(2)}$ | | |
| $\left(\mathcal{K}_A\right)^{\mu\nu\rho}\left(\mathcal{K}_A\right)_{\mu\nu\rho}$ | $\left(\mathcal{K}_T\right)^{\mu\nu\rho}\left(\mathcal{K}_T\right)_{\mu\nu\rho}$ | | |

Table 2.1: Summary of zeroth, first and second order independent equilibrium scalars.

We have denoted the nth scalar of order i as $\mathcal{S}_n^{(i)}$. Not all second order scalars will contribute to the currents once torsion is set to zero, however all of them will contribute to the two point functions of the theory.

### 2.1.2 Non-equilibrium terms

The classification for the non-equilibrium tensors will be shown in tables 2.1.2-2.1.2 and will be organized slightly different than how the equilibrium scalars were previously presented. First we have to note that there are no zeroth order non-equilibrium terms while the first and second order structures are listed on their own tables, two for each order. The first table, for each order, shows all independent tensors for each relevant tensor structure that are not composed of products of other tensor structures, we will then show the equations of motion and constraints that will allow us to reduce the number of terms, we will then list the remaining independent structures and finally we will mention which of these remaining independent tensors are non-equilibrium. The second table, for each order, will list the independent tensor composed of products of other independent tensors. There is one last caveat that should be mentioned before moving on, we are using the equation of motion $L^{\mu\nu}$ as if it appeared first at second order in derivatives. The reasoning for this will be discussed at the end of the chapter.

| Type | Before eqm | Eqm | Ind. Data | Ind. Non-eq |
|---|---|---|---|---|
| Scalar | $u^\mu \mathring{\nabla}_\mu T, \theta, \kappa$ | $u_\nu D^\nu$ | $\theta, \kappa$ | $\theta$ |
| Vector | $\Delta^{\mu\nu}\mathring{\nabla}_\nu T + T a^\mu, \hat{m}^\mu,$ $m^\mu, k^\mu, (\mathcal{K}_V)^\mu$ | $\Delta^\mu_\nu D^\nu$ | $\hat{m}^\mu,$ $k^\mu, (\mathcal{K}_V)^\mu$ | $\hat{m}^\mu$ |
| S. Traceless | $\sigma^{\mu\nu}, (\kappa_S)^{\mu\nu}$ | | $\sigma^{\mu\nu}, (\kappa_S)^{\mu\nu}$ | $\sigma^{\mu\nu}$ |
| Antisymmetric | $M^{\mu\nu}, \hat{M}^{\mu\nu}$ $K^{\mu\nu}, (\kappa_A)^{\mu\nu}$ | | $M^{\mu\nu}, \hat{M}^{\mu\nu}$ $K^{\mu\nu}, (\kappa_A)^{\mu\nu}$ | $\hat{M}^{\mu\nu}$ |
| Spin Symmetry | $(\mathcal{K}_A)^{\mu\nu\rho}, (\mathcal{K}_T)^{\mu\nu\rho}$ | | $(\mathcal{K}_A)^{\mu\nu\rho}$ $(\mathcal{K}_T)^{\mu\nu\rho}$ | |

Table 2.2: Relevant non-composite first order tensor structures.

| Type | Composite data |
|---|---|
| Spin Symmetry | $\Delta^{\mu[\rho}\hat{m}^{\sigma]}$ |

Table 2.3: Relevant composite non-equilibrium first order tensor structures.

where we defined $\{\hat{m}, \hat{M}\}$ as

$$\hat{m}^\mu = m^\mu - k^\mu - a^\mu \,, \tag{2.66}$$

$$\hat{M}^{\mu\nu} = M^{\mu\nu} - K^{\mu\nu} + \Omega^{\mu\nu} \,, \tag{2.67}$$

and where we are denoting the equations of motion as

$$D^\mu \equiv \mathring{\nabla}_\mu T^{\mu\nu} - \ldots \,, \tag{2.68}$$

$$L^{\mu\nu} \equiv \mathring{\nabla}_\lambda S^{\lambda\mu\nu} - \ldots \,. \tag{2.69}$$

At second order we simply need the vectors and tensors

| | | | | | |
|---|---|---|---|---|---|
| $\mathfrak{V}_1^{(2)\mu}$ | $\mathfrak{V}_2^{(2)\mu}$ | $\mathfrak{V}_3^{(2)\mu}$ | $\mathfrak{V}_4^{(2)\mu}$ | $\mathfrak{V}_5^{(2)\mu}$ | $\mathfrak{V}_6^{(2)\mu}$ |
| $\sigma^{\mu\rho}\hat{m}_\rho$ | $\hat{M}^{\mu\rho}\hat{m}_\rho$ | $M^{\mu\rho}\hat{m}_\rho$ | $\Theta\hat{m}^\mu$ | $\kappa\hat{m}^\mu$ | $\sigma^{\mu\rho}m_\rho$ |
| $\mathfrak{V}_7^{(2)\mu}$ | $\mathfrak{V}_8^{(2)\mu}$ | $\mathfrak{V}_9^{(2)\mu}$ | $\mathfrak{V}_{10}^{(2)\mu}$ | $\mathfrak{V}_{11}^{(2)\mu}$ | $\mathfrak{V}_{12}^{(2)\mu}$ |
| $\sigma^{\mu\rho}k_\rho$ | $K^{\mu\rho}\hat{m}_\rho$ | $\hat{M}^{\mu\rho}m_\rho$ | $\hat{M}^{\mu\rho}k_\rho$ | $\sigma^{\mu\rho}\mathcal{K}_\rho$ | $\left(\kappa_S\right)^{\mu\rho}\hat{m}_\rho$ |
| $\mathfrak{V}_{13}^{(2)\mu}$ | $\mathfrak{V}_{14}^{(2)\mu}$ | $\mathfrak{V}_{15}^{(2)\mu}$ | $\mathfrak{V}_{16}^{(2)\mu}$ | $\mathfrak{V}_{17}^{(2)\mu}$ | $\mathfrak{V}_{18}^{(2)\mu}$ |
| $\hat{M}^{\mu\rho}\mathcal{K}_\rho$ | $\left(\kappa_A\right)^{\mu\rho}\hat{m}_\rho$ | $\left(\mathcal{K}_T\right)^{\mu\rho\sigma}\sigma_{\rho\sigma}$ | $\theta m^\mu$ | $\theta k^\mu$ | $\Theta\mathcal{K}^\mu$ |
| $\mathfrak{V}_{19}^{(2)\mu}$ | $\mathfrak{V}_{20}^{(2)\mu}$ | $\mathfrak{V}_{21}^{(2)\mu}$ | | | |
| $\left(\mathcal{K}_A\right)^{\mu\rho\sigma}\hat{M}_{\rho\sigma}$ | $\left(\mathcal{K}_T\right)^{\rho\sigma\mu}\hat{M}_{\rho\sigma}$ | $\left(\mathcal{K}_T\right)^{\mu\rho\sigma}\hat{M}_{\rho\sigma}$ | | | |

Table 2.4: Relevant composite non-equilibrium second order vectors.

| | | | | |
|---|---|---|---|---|
| $\mathfrak{A}_1^{(2)\mu}$ | $\mathfrak{A}_2^{(2)\mu}$ | $\mathfrak{A}_3^{(2)\mu}$ | $\mathfrak{A}_4^{(2)\mu}$ | $\mathfrak{A}_5^{(2)\mu}$ |
| $\sigma^{[\mu}{}_\rho\hat{M}^{\nu]\rho}$ | $\sigma^{[\mu}{}_\rho M^{\nu]\rho}$ | $\hat{M}^{[\mu}{}_\rho M^{\nu]\rho}$ | $\kappa\hat{M}^{\mu\nu}$ | $\theta\hat{M}^{\rho\sigma}$ |
| $\mathfrak{A}_6^{(2)\mu}$ | $\mathfrak{A}_7^{(2)\mu}$ | $\mathfrak{A}_8^{(2)\mu}$ | $\mathfrak{A}_9^{(2)\mu}$ | $\mathfrak{A}_{10}^{(2)\mu}$ |
| $\theta M^{\mu\nu}$ | $\theta K^{\mu\nu}$ | $\theta\left(\kappa_A\right)^{\mu\nu}$ | $\hat{m}^{[\mu}m^{\nu]}$ | $\hat{m}^{[\mu}k^{\nu]}$ |
| $\mathfrak{A}_{11}^{(2)\mu}$ | $\mathfrak{A}_{12}^{(2)\mu}$ | $\mathfrak{A}_{13}^{(2)\mu}$ | $\mathfrak{A}_{14}^{(2)\mu}$ | $\mathfrak{A}_{15}^{(2)\mu}$ |
| $\hat{m}^{[\mu}\mathcal{K}^{\nu]}$ | $\left(\mathcal{K}_A\right)^{\mu\nu\rho}\hat{m}_\rho$ | $\left(\mathcal{K}_T\right)^{\mu\nu\rho}\hat{m}_\rho$ | $\hat{m}_\rho\left(\mathcal{K}_T\right)^{\rho[\mu\nu]}$ | $\sigma^{[\mu}{}_\rho\left(\kappa_S\right)^{\nu]\rho}$ |
| $\mathfrak{A}_{16}^{(2)\mu}$ | $\mathfrak{A}_{17}^{(2)\mu}$ | $\mathfrak{A}_{18}^{(2)\mu}$ | $\mathfrak{A}_{19}^{(2)\mu}$ | $\mathfrak{A}_{20}^{(2)\mu}$ |
| $\sigma^{[\mu}{}_\rho K^{\nu]\rho}$ | $\sigma^{[\mu}{}_\rho\left(\kappa_A\right)^{\nu]\rho}$ | $\left(\kappa_S\right)^{[\mu}{}_\rho\hat{M}^{\nu]\rho}$ | $K^{[\mu}{}_\rho\hat{M}^{\nu]\rho}$ | $\left(\kappa_A\right)^{[\mu}{}_\rho\hat{M}^{\nu]\rho}$ |

Table 2.5: Relevant composite non-equilibrium second order tensors.

| Before eqm | Eqm | Ind. Data | Ind. Non-Eq |
|---|---|---|---|
| $\Delta^{\mu\nu}u^{\rho}\mathring{\nabla}_{\nu}\mathring{\nabla}_{\rho}T$ | $\Delta^{\mu}_{\nu}u^{\rho}\mathring{\nabla}_{\rho}D^{\nu}$ | $\Delta^{\mu\nu}\mathring{\nabla}_{\nu}\theta$ | $\Delta^{\mu\nu}\mathring{\nabla}_{\nu}\theta$ |
| $\Delta^{\mu\nu}\mathring{\nabla}_{\nu}\theta,$ | $u_{\rho}\Delta^{\mu\nu}\mathring{\nabla}_{\nu}D^{\rho}$ | $\Delta^{\mu\nu}\mathring{\nabla}_{\nu}\kappa$ | $\Delta^{\mu}_{\nu}\mathcal{L}_{\frac{u}{T}}k^{\nu}$ |
| $\Delta^{\mu\nu}\mathring{\nabla}_{\nu}\kappa$ | $\Delta^{\mu}_{\rho}u_{\sigma}L^{\rho\sigma}$ | $\Delta^{\mu}_{\nu}\mathcal{L}_{\frac{u}{T}}m^{\nu},$ | $\Delta^{\mu}_{\nu}\mathcal{L}_{\frac{u}{T}}k^{\nu}$ |
| $\Delta^{\mu}_{\nu}\mathcal{L}_{\frac{u}{T}}m^{\nu},$ | | $\Delta^{\mu}_{\nu}\mathcal{L}_{\frac{u}{T}}\left(\mathcal{K}_V\right)^{\nu}$ | $\Delta^{\mu}_{\nu}\mathcal{L}_{\frac{u}{T}}\left(\mathcal{K}_V\right)^{\nu}$ |
| $\Delta^{\mu}_{\nu}\mathcal{L}_{\frac{u}{T}}a^{\nu}$ | | $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\sigma^{\alpha\beta}$ | $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\sigma^{\alpha\beta}$ |
| $\Delta^{\mu}_{\nu}\mathcal{L}_{\frac{u}{T}}k^{\nu},$ | | $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\left(\kappa_S\right)^{\alpha\beta}$ | $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\hat{M}^{\alpha\beta}$ |
| $\Delta^{\mu}_{\nu}\mathcal{L}_{\frac{u}{T}}\left(\mathcal{K}_V\right)^{\nu}$ | | $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}M^{\alpha\beta}$ | |
| $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\sigma^{\alpha\beta}$ | | $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\hat{M}^{\alpha\beta}$ | |
| $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\left(\kappa_S\right)^{\alpha\beta}$ | | $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\left(\kappa_A\right)^{\alpha\beta}$ | |
| $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}M^{\alpha\beta}$ | | | |
| $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\hat{M}^{\alpha\beta}$ | | | |
| $\Delta^{\mu}_{\alpha}\mathring{\nabla}_{\beta}\left(\kappa_A\right)^{\alpha\beta}$ | | | |
| $\mathring{R}_{\alpha\nu}\Delta^{\mu\alpha}u^{\beta}$ | | | |

Table 2.6: Relevant non-composite second order vectors.

| | | | |
|---|---|---|---|
| $\Delta^{\mu}_{\rho}\Delta^{\nu}_{\sigma}\mathcal{L}_{\frac{u}{T}}M^{\rho\sigma}$ | $\Delta^{\mu}_{[\rho}\Delta^{\nu}_{\sigma]}\mathring{\nabla}_{\rho}D_{\sigma}$ | $\Delta^{\mu}_{\rho}\Delta^{\nu}_{\sigma}\mathcal{L}_{\frac{u}{T}}K^{\rho\sigma}$ | $\Delta^{\mu}_{\rho}\Delta^{\nu}_{\sigma}\mathcal{L}_{\frac{u}{T}}K^{\rho\sigma}$ |
| $\Delta^{\mu}_{\rho}\Delta^{\nu}_{\sigma}u^{\beta}\mathring{\nabla}_{\beta}\hat{M}^{\rho\sigma}$ | $\Delta^{\mu}_{\rho}\Delta^{\nu}_{\sigma}L^{\rho\sigma}$ | $\Delta^{\mu}_{\rho}\Delta^{\nu}_{\sigma}\mathcal{L}_{\frac{u}{T}}\left(\kappa_A\right)^{\rho\sigma}$ | $\Delta^{\mu}_{\rho}\Delta^{\nu}_{\sigma}\mathcal{L}_{\frac{u}{T}}\left(\kappa_A\right)^{\rho\sigma}$ |
| $\Delta^{\mu}_{\rho}\Delta^{\nu}_{\sigma}\mathcal{L}_{\frac{u}{T}}K^{\rho\sigma}$ | | $\Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_{\rho}\hat{m}_{\sigma}$ | $\Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_{\rho}\hat{m}_{\sigma}$ |
| $\Delta^{\mu}_{\rho}\Delta^{\nu}_{\sigma}\mathcal{L}_{\frac{u}{T}}\left(\kappa_A\right)^{\rho\sigma}$ | | $\Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_{\rho}m_{\sigma}$ | |
| $\Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_{\rho}\hat{m}_{\sigma}$ | | $\Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_{\rho}k_{\sigma}$ | |
| $\Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_{\rho}m_{\sigma}$ | | $\Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_{\rho}\left(\mathcal{K}_V\right)_{\sigma}$ | |
| $\Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_{\rho}k_{\sigma}$ | | $\mathring{\nabla}_{\alpha}\left(\mathcal{K}_A\right)^{\alpha\mu\nu}$ | |
| $\Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_{\rho}\left(\mathcal{K}_V\right)_{\sigma}$ | | $\mathring{\nabla}_{\alpha}\left(\mathcal{K}_T\right)^{\mu\nu\alpha}$ | |
| $\mathring{\nabla}_{\alpha}\left(\mathcal{K}_A\right)^{\alpha\mu\nu}$ | | $\Delta^{[\mu}_{\rho}\Delta^{\nu]}_{\sigma}\mathring{\nabla}_{\alpha}\left(\mathcal{K}_T\right)^{\alpha\rho\sigma}$ | |
| $\mathring{\nabla}_{\alpha}\left(\mathcal{K}_T\right)^{\mu\nu\alpha}$ | | | |
| $\Delta^{[\mu}_{\rho}\Delta^{\nu]}_{\sigma}\mathring{\nabla}_{\alpha}\left(\mathcal{K}_T\right)^{\alpha\rho\sigma}$ | | | |

Table 2.7: Relevant non-composite second order rank 2 antisymmetric tensors.

## 2.2 Hydrostatic Partition Function

The most general hydrostatic partition function contributing to the desired order can be written by using the independent scalars summarized in table 2.1. The resulting partition function can be split as[2].

$$W = W_{\text{ideal}} + W_{\text{mixed}} + W_{\text{torsion}} \,, \tag{2.70}$$

where the ideal, mixed and torsion pieces are defined as

$$W_{\text{ideal}} = \int d^dx|e|p(T, m^2, M^2)\,, \tag{2.71}$$

$$W_{\text{mixed}} = \int d^dx|e|\left[\chi_1^{(2)}\left(\kappa_A\right)^{\mu\nu}M_{\mu\nu} + \chi_2^{(2)}K^{\mu\nu}M_{\mu\nu} + \chi_3^{(2)}m_\mu k^\mu + \chi_4^{(2)}m_\mu\left(\mathcal{K}_V\right)^\mu\right]\,, \tag{2.72}$$

$$\begin{aligned} W_{\text{torsion}} = \int d^dx|e| \Big[ & \chi^{(1)}\kappa + \Upsilon_0^{(2)}\kappa^2 + \Upsilon_1^{(2)}k_\mu k^\mu + \Upsilon_2^{(2)}K^{\mu\nu}K_{\mu\nu} \\ & + \Upsilon_3^{(2)}k_\mu\left(\mathcal{K}_V\right)^\mu + \Upsilon_4^{(2)}\left(\mathcal{K}_V\right)^\mu\left(\mathcal{K}_V\right)_\mu \\ & + \Upsilon_5^{(2)}\left(\kappa_A\right)^{\mu\nu}K_{\mu\nu} + \Upsilon_6^{(2)}\left(\kappa_A\right)^{\mu\nu}\left(\kappa_A\right)_{\mu\nu} \\ & + \Upsilon_7^{(2)}\left(\kappa_S\right)^{\mu\nu}\left(\kappa_S\right)_{\mu\nu} + \Upsilon_8^{(2)}\left(\mathcal{K}_A\right)^{\mu\nu\rho}\left(\mathcal{K}_A\right)_{\mu\nu\rho} \\ & + \Upsilon_9^{(2)}\left(\mathcal{K}_T\right)^{\mu\nu\rho}\left(\mathcal{K}_T\right)_{\mu\nu\rho}\Big]\,. \end{aligned} \tag{2.73}$$

Where $\chi_i$ and $\Upsilon_i$ are all functions of temperature. We call $W_{\text{ideal}}$ the ideal piece as it only depends on the dynamical variables with no explicit derivative dependence. The $W_{\text{mixed}}$ is linear in torsion and will contribute to the constitutive relations even in the zero torsion limit. The terms in $W_{\text{torsion}}$ with coefficients $\Upsilon_i$ are quadratic in torsion and will only contribute to the constitutive relations whenever torsion is kept, however it will contribute to the two point functions even in the absence of torsion.

[2] The ideal term can take a more general form once parity invariance is broken or we allow for higher order contributions. The ideal term valid at all orders in derivatives is shown in appendix A.

### 2.2.1 Ideal constitutive relations

From the fluctuation of $W_{\text{ideal}}$ we find the ideal constitutive relations

$$(T_{\text{ideal}})^{\mu\nu} = \varepsilon u^\mu u^\nu + p\Delta^{\mu\nu} - 2\left(\frac{\partial p}{\partial m^2} + 4\frac{\partial p}{\partial M^2}\right) u^\alpha M^{\beta\gamma} m_\gamma\,, \tag{2.74}$$

$$(S_{\text{ideal}})^{\lambda\mu\nu} = u^\lambda \rho^{\mu\nu}\,, \tag{2.75}$$

with the energy density $\varepsilon$ and spin density $\rho$ defined as

$$\varepsilon = -p + \frac{\partial p}{\partial T}T + \frac{1}{2}\rho_{\alpha\beta}\mu^{\alpha\beta}\,, \tag{2.76}$$

$$\rho^{\alpha\beta} = 8\frac{\partial p}{\partial M^2}M^{\alpha\beta} + 2\frac{\partial p}{\partial m^2}\left(u^\alpha m^\beta - m^\alpha u^\beta\right)\,. \tag{2.77}$$

From (2.76) and (2.77) together with the analytic structure of the pressure we find the following differential relations

$$dp = sdT + \frac{1}{2}\rho_{cd}d\mu^{cd} + \left[2\left(\frac{\partial p}{\partial m^2} + 4\frac{\partial p}{\partial M^2}\right) m^d M_{dc}\right] du^c\,, \tag{2.78}$$

$$d\varepsilon = Tds + \frac{1}{2}\mu_{cd}d\rho^{cd} - \left[2\left(\frac{\partial p}{\partial m^2} + 4\frac{\partial p}{\partial M^2}\right) m^d M_{dc}\right] du^c\,, \tag{2.79}$$

where we defined the entropy density $s = \frac{\partial p}{\partial T}$. We can then identify (2.79) with the first Law of thermodynamics and (2.78) with the Gibbs-Duhem relation. A couple of observations are in order

- Even in the ideal case there will be an antisymmetric piece on the energy momentum tensor as long as both projections of the chemical potential are turned on.
- The spin density $\rho^{cd}$ and spin potential $\mu^{cd}$ satisfy the expected thermodynamic relation (2.76). However the first law is modified as seen in (2.79). This modification will be crucial when asking for the second law of thermodynamics to hold.

### 2.2.2 Contorsion variation and Belinfante completion

The zero torsion contributions to the constitutive relations and two point functions from $\{W_{\text{mixed}}, W_{\text{torsion}}\}$ will necessarily contain a variation of the contorsion tensor, in particular the two point function contributions from $\Upsilon$ coefficients will come only from contorsion fluctuations as all other contributions will vanish on the torsionless limit. It is then convenient to note the following identity

$$\int d^dx|e|B^{\mu}{}_{ab}\delta K^{ab}{}_{\mu} = \int d^dx|e|\left[\mathring{\nabla}_{\lambda}\left(B^{\lambda\mu\nu} - B^{\mu\lambda\nu} - B^{\nu\lambda\mu}\right)e_{a\nu}\delta e^{a}_{\mu}\right] + \int d^dx|e|\left[\left(B^{\mu}{}_{ab}\right)\delta\omega^{ab}{}_{\mu}\right] , \tag{2.80}$$

That follows from the fluctuation identity

$$\delta K^{ab}{}_{\mu} = \delta\omega^{ab}{}_{\mu} - \left[\mathring{\mathcal{D}}_{\mu}\left(e^{\sigma[a}\delta e^{b]}_{\sigma}\right) - \mathring{\mathcal{D}}_{\sigma}\left(e^{\sigma[a}\delta e^{b]}_{\mu}\right) - \mathring{\mathcal{D}}_{\sigma}\left(e^{\sigma[a}e^{\rho b]}e_{c\mu}\delta e^{c}_{\rho}\right)\right] .$$

We note that as expected there is a contribution to the spin current following from the coupling with contorsion, however it is more notable that the contribution to the energy momentum tensor is nothing but the Belinfante improvement associated to the spin current contribution, see (2.28)

$$(T_B)^{\mu\nu} = \mathring{\nabla}_{\lambda}\left(B^{\lambda\mu\nu} - B^{\mu\lambda\nu} - B^{\nu\lambda\mu}\right) , \tag{2.81}$$

$$(S_B)^{\lambda\mu\nu} = 2B^{\lambda\mu\nu} , \tag{2.82}$$

We will call (2.81) the Belinfante completion of the spin contribution $B^{\lambda\mu\nu}$. Note that this completion includes not only equilibrium terms but also non-equilibrium contributions. Even though terms containing variations of this form will be derived from the hydrostatic partition function we will keep the associated non-equilibrium pieces coming from (2.81). The main reason for this is two-fold: Using (2.81) without specifying the equilibrium condition allows for a compact notation and second, we will show that the non-equilibrium terms encoded in (2.81) will be necessary for the second law of thermodynamics to hold[3].

### 2.2.3 Mixed Constitutive relations

The variation of the term $W_{\text{mixed}}$ will contribute to the first order part of the spin current and to the second order part of the energy momentum tensor, of

[3] We will actually only show this for $\chi^{(1)}$ as all other contributions will be subleading on the entropy production.

which we will only be concerned with the antisymmetric piece. The resulting constitutive relations take the form

$$\left(T_{\text{mixed}}\right)^{\mu\nu} = \left(T^{B}_{\text{mixed}}\right)^{\mu\nu} + \left(T^{EOM}_{\text{mixed}}\right) , \tag{2.83}$$

$$\left(S_{\text{mixed}}\right)^{\lambda\mu\nu} = 2\left(B_{\text{mixed}}\right)^{\lambda\mu\nu} + \left(S^{EOM}_{\text{mixed}}\right)^{\lambda\mu\nu} , \tag{2.84}$$

Where we defined

$$\left(T^{B}_{\text{mixed}}\right)^{\mu\nu} = \mathring{\nabla}_{\lambda}\left[\left(B_{\text{mixed}}\right)^{\lambda\mu\nu} - \left(B_{\text{mixed}}\right)^{\mu\lambda\nu} - \left(B_{\text{mixed}}\right)^{\nu\lambda\mu}\right] , \tag{2.85}$$

$$\begin{aligned}\left(B_{\text{mixed}}\right)^{\lambda\mu\nu} = {} & u^{\lambda}\left[\chi_2^{(2)} M^{\mu\nu} - \chi_3^{(2)} u^{[\mu} m^{\nu]}\right] - \chi_1^{(2)} M^{\lambda[\mu} u^{\nu]} \\ & + 2\chi_4^{(2)} \Delta^{\lambda[\mu} m^{\nu]} ,\end{aligned} \tag{2.86}$$

$$\begin{aligned}\left(T^{\text{EOM}}_{\text{mixed}}\right)^{\mu\nu} = {} & -\chi_1^{(2)} u^{[\mu}\left[\left(\mathcal{K}_T\right)^{\nu]\alpha\beta} + \left(\mathcal{K}_A\right)^{\nu]\alpha\beta}\right] M_{\alpha\beta} + \frac{\chi_1^{(2)} \kappa M^{\mu\nu}}{d-1} \\ & + \frac{\chi_1^{(2)}}{2(d-2)} u^{[\mu} M^{\nu]\beta}\left(\mathcal{K}_V\right)_{\beta} - 2\chi_1^{(2)} u^{[\mu}\left(\kappa_A\right)^{\nu]\beta} m_{\beta} \\ & + \chi_1^{(2)} M^{\alpha[\mu}\left[\left(\kappa_A\right)^{\nu]}{}_{\alpha} - \left(\kappa_S\right)^{[\nu}{}_{\alpha}\right] - 2\chi_2^{(2)} u^{[\mu}\left(M^{\nu]\alpha} k_{\alpha} + K^{\nu]\alpha} m_{\alpha}\right) \\ & - \chi_3^{(2)} u^{[\mu}\left(M^{\nu]\alpha} k_{\alpha} + K^{\nu]\alpha} m_{\alpha}\right) + 2\chi_4^{(2)}\left(\mathcal{K}_A\right)^{\mu\nu\alpha} m_{\alpha} \\ & + 2\chi_4^{(2)}\left(\mathcal{K}_T\right)^{\alpha[\mu\nu]} m_{\alpha} + \frac{\chi_4^{(2)}}{(d-2)}\left(\mathcal{K}_V\right)^{[\mu} m^{\nu]} - \chi_4^{(2)} u^{[\mu} M^{\nu]\alpha}\left(\mathcal{K}_V\right)_{\alpha} \\ & + 2\chi_4^{(2)} u^{[\mu}\left[\left(\kappa_A\right)^{\nu]\alpha} - \left(\kappa_S\right)^{\nu]\alpha}\right] m_{\alpha} + \frac{2\chi_4^{(2)}(d-2)}{(d-1)} \kappa u^{[\mu} m^{\nu]} + \ldots ,\end{aligned} \tag{2.87}$$

$$\begin{aligned}\left(S^{\text{EOM}}_{\text{mixed}}\right)^{\lambda\mu\nu} = {} & 2u^{\lambda}\left[\chi_1^{(2)}\left(\kappa_A\right)^{\mu\nu} + \chi_2^{(2)} K^{\mu\nu} - \chi_3^{(2)} u^{[\mu} k^{\nu]}\right] \\ & - 2\chi_4^{(2)} u^{\lambda} u^{[\mu}\left(\mathcal{K}_V\right)^{\nu]} ,\end{aligned} \tag{2.88}$$

where +... represents terms irrelevant at the order in derivatives taken into account for either the one point functions or the two point functions, this means $\mathcal{O}\left(\partial^2\right)$ in the antisymmetric piece of $T^{\mu\nu}$ and keeping contorsion up to its linear terms. In the absence of torsion only the Belinfante-Rosenfel (BR) like terms (2.85) and (2.86) will be left, however due to their BR nature they will not contribute to the equations of motion in this limit.

### 2.2.4 Torsion constitutive relations

There are two types of contributions coming from $W_{\text{torsion}}$. The first one is the single contribution with the $\chi^{(1)}$ coefficient, this contribution will be of zeroth order to the spin current and of first order to the energy momentum tensor. The second type of contribution comes from the terms with $\Upsilon$ coefficients. These terms are quadratic in contortion and as such we will only keep their $\mathcal{O}(\partial)$ contribution to the spin current and their $\mathcal{O}(\partial^2)$ linear in contorsion contribution to the antisymmetric piece of the energy momentum tensor, namely all relevant constitutive relations will come from the fluctuation of contortion. With all these considerations taken into account we will decompose the constitutive relations as

$$(T_{\text{torsion}})^{\mu\nu} = \left(T^{\text{B}-\chi}_{\text{torsion}}\right)^{\mu\nu} + \left(T^{\text{B}-\Upsilon}_{\text{torsion}}\right)^{\mu\nu} + \left(T^{\text{EOM}}_{\text{torsion}}\right)^{\mu\nu}, \tag{2.89}$$

$$(S_{\text{torsion}})^{\lambda\mu\nu} = 2\left(B^{\chi}_{\text{torsion}}\right)^{\lambda\mu\nu} + 2\left(B^{\Upsilon}_{\text{torsion}}\right)^{\lambda\mu\nu}, \tag{2.90}$$

Where we defined

$$\begin{aligned}
\left(T^{\text{B}-\chi}_{\text{torsion}}\right)^{\mu\nu} &= -\theta\chi^{(1)}u^{\mu}u^{\nu} + \left(\frac{d-2}{d-1}\theta\chi^{(1)} + u^{\lambda}\mathring{\nabla}_{\lambda}\chi^{(1)}\right)\Delta^{\mu\nu} && (2.91)\\
&\quad - u^{\mu}\Delta^{\nu\lambda}\mathring{\nabla}_{\lambda}\chi^{(1)} + \chi^{(1)}u^{\nu}a^{\mu} - \chi^{(1)}\sigma^{\mu\nu} + \chi^{(1)}\Omega^{\mu\nu}, \\
\left(T^{\text{B}-\Upsilon}_{\text{torsion}}\right)^{\mu\nu} &= \mathring{\nabla}_{\lambda}\left[\left(B^{\Upsilon}_{\text{torsion}}\right)^{\lambda\mu\nu} - \left(B^{\Upsilon}_{\text{torsion}}\right)^{\mu\lambda\nu} - \left(B^{\Upsilon}_{\text{torsion}}\right)^{\nu\lambda\mu}\right], && (2.92)\\
\left(B^{\chi}_{\text{torsion}}\right)^{\lambda\mu\nu} &= \chi^{(1)}\Delta^{\lambda[\mu}u^{\nu]}, && (2.93)\\
\left(B^{\Upsilon}_{\text{torsion}}\right)^{\lambda\mu\nu} &= 2\Upsilon_0\kappa\Delta^{\lambda[\mu}u^{\nu]} + u^{\lambda}\left[2\Upsilon_1 k^{[\mu}u^{\nu]} + 2\Upsilon_2 K^{\mu\nu} + \Upsilon_5\left(\kappa_A\right)^{\mu\nu}\right] && (2.94)\\
&\quad + u^{\lambda}\left[\Upsilon_3\left(\mathcal{K}_V\right)^{[\mu}u^{\nu]}\right] + 2\Delta^{\lambda[\mu}\left[\Upsilon_3 k^{\nu]} + 2\Upsilon_4\left(\mathcal{K}_V\right)^{\nu]}\right] \\
&\quad - \Upsilon_5 K^{\lambda[\mu}u^{\nu]} - 2\Upsilon_6\left(\kappa_A\right)^{\lambda[\mu}u^{\nu]} + 2\Upsilon_7\left(\kappa_S\right)^{\lambda[\mu}u^{\nu]} \\
&\quad + 2\Upsilon_8\left(\mathcal{K}_A\right)^{\lambda\mu\nu} + 2\Upsilon_9\left(\mathcal{K}_T\right)^{\mu\nu\lambda}, \\
\left(T^{\text{EOM}}_{\text{torsion}}\right)^{\mu\nu} &= \left[\left(T\frac{\partial\chi^{(1)}}{\partial T} - \chi^{(1)}\right)\kappa\right]u^{\mu}u^{\nu} + \left[\chi^{(1)}\frac{d-2}{d-1}\kappa\right]\Delta^{\mu\nu} && (2.95)\\
&\quad + 2\chi^{(1)}u^{(\mu}k^{\nu)} + \frac{1}{2}\chi^{(1)}u^{\mu}\left(\mathcal{K}_V\right)^{\nu} - \chi^{(1)}\left(\kappa_S\right)^{\mu\nu} - \chi^{(1)}\left(\kappa_A\right)^{\mu\nu}.
\end{aligned}$$

In the absence of torsion only the $\chi^{(1)}$ contribution is left, although it will have no contribution to the equations of motion as it becomes a BR like term on this limit.

## 2.3 Non-equilibrium Constitutive Relations

We have already taken care of writing down the equilibrium part of the constitutive relations and we are left to write the non-equilibrium piece. To write these terms we will use the relevant independent non-equilibrium structures and write the most general constitutive relations that we can build from them. The relations are decomposed as

$$\left(T_{\text{non-eq}}\right)^{\mu\nu} = \left(T^{B}_{\text{non-eq}}\right)^{\mu\nu} + \left(T^{(1)}_{\text{non-eq}}\right)^{\mu\nu} + \left(T^{(2)}_{\text{non-eq}}\right)^{\mu\nu} , \tag{2.96}$$

$$\left(T^{B}_{\text{non-eq}}\right)^{\mu\nu} = \frac{1}{2}\mathring{\nabla}_{\lambda}\left[\left(S_{\text{non-eq}}\right)^{\lambda\mu\nu} - \left(S_{\text{non-eq}}\right)^{\mu\lambda\nu} - \left(S_{\text{non-eq}}\right)^{\nu\lambda\mu}\right] , \tag{2.97}$$

$$\left(T^{(1)}_{\text{non-eq}}\right)^{\mu\nu} = -\xi\theta\Delta^{\mu\nu} - \eta\sigma^{\mu\nu} + \sigma_7 u^{[\mu}\hat{m}^{\nu]} - \sigma_8 \hat{M}^{\mu\nu} , \tag{2.98}$$

$$\begin{aligned}\left(T^{(2)}_{\text{non-eq}}\right)^{\mu\nu} = {}& \Delta^{[\mu}_{\alpha}u^{\nu]}\left[\lambda_1\mathring{\nabla}_{\beta}\sigma^{\alpha\beta} + \lambda_2\mathring{\nabla}_{\beta}\hat{M}^{\alpha\beta} + \lambda_3\mathcal{L}_{\frac{u}{T}}k^{\alpha} + \lambda_4\mathcal{L}_{\frac{u}{T}}\left(\mathcal{K}_V\right)^{\alpha}\right] \\ & + \lambda_5\Delta^{\alpha[\mu}u^{\nu]}\mathring{\nabla}_{\alpha}\theta + \Delta^{[\mu}_{\alpha}\Delta^{\nu]}_{\beta}\left[\lambda_6\mathcal{L}_{\frac{u}{T}}K^{\alpha\beta} + \lambda_7\mathcal{L}_{\frac{u}{T}}\left(\kappa_A\right)^{\alpha\beta}\right] \\ & + \lambda_8\Delta^{\alpha[\mu}\Delta^{\nu]\beta}\mathring{\nabla}_{\alpha}\hat{m}_{\beta} + \sum_{i=9}^{29}\lambda_i u^{[\mu}\mathfrak{V}^{(2)\nu]}_{i-8} + \sum_{i=30}^{49}\lambda_i\mathfrak{A}^{(2)\mu\nu}_{i-29} ,\end{aligned} \tag{2.99}$$

$$\begin{aligned}\left(S_{\text{non-eq}}\right)^{\lambda\mu\nu} = {}& u^{\lambda}\left(2\sigma_1 u^{[\mu}\hat{m}^{\nu]} + 2\sigma_2\hat{M}^{\mu\nu}\right) + 2\sigma_3\theta\Delta^{\lambda[\mu}u^{\nu]} + 2\sigma_4\sigma^{\lambda[\mu}u^{\nu]} \\ & + 2\sigma_5\hat{M}^{\lambda[\mu}u^{\nu]} + 2\sigma_6\Delta^{\lambda[\mu}\hat{m}^{\nu]} .\end{aligned} \tag{2.100}$$

This encompasses 53 non-equilibrium transport coefficients on the energy momentum tensor and 8 non-equilibrium transport coefficients on the spin current, note that according to our decomposition of the currents the spin current transport coefficients will only contribute to the equations of motion when torsion is non-vanishing. We have also chosen a frame where $\left(T_{\text{non-eq}}\right)^{(\mu\nu)}u_{\nu} = 0$.

## 2.4 Summary of Constitutive Relations

To avoid confusion we summarize the form of the constitutive relations

$$T^{\mu\nu} = \left(T_{\text{ideal}}\right)^{\mu\nu} + \left(T_{\text{mixed}}\right)^{\mu\nu} + \left(T_{\text{torsion}}\right)^{\mu\nu} + \left(T_{\text{non-eq}}\right)^{\mu\nu} , \tag{2.101}$$

$$S^{\lambda\mu\nu} = \left(S_{\text{ideal}}\right)^{\lambda\mu\nu} + \left(S_{\text{mixed}}\right)^{\lambda\mu\nu} + \left(S_{\text{torsion}}\right)^{\lambda\mu\nu} + \left(S_{\text{non-eq}}\right)^{\lambda\mu\nu} . \tag{2.102}$$

The explicit form can be found after using (2.74), (2.75), (2.83),(2.84), (2.89), (2.90), (2.96), and (2.100).

## 2.5 First Order Constraint

When the equations of motion are being solved order by order in the derivative expansion we expect for the dynamical equation of motion for the spin dynamical degrees of freedom to appear as second order equations. The first order spin current equation should either be identically satisfied or should impose constraints on transport. From (2.74), (2.75), (2.83), (2.84), (2.89), (2.90), (2.96), and (2.100) the first order spin current equation of motion becomes

$$\sigma_7 u^{[\mu}\hat{m}^{\nu]} + \sigma_8 \hat{M}^{\mu\nu} = 0\,. \tag{2.103}$$

There are two possibilities for this equation to be satisfied. Either the first order transport coefficients $\{\sigma_7, \sigma_8\}$ are zero or the non-equilibrium spin chemical potentials are zero at this order in derivatives, meaning the spin potential is equal to thermal vorticity up to $\mathcal{O}\left(\partial^2\right)$ terms. Choosing the former option keeps the constitutive relations just as were shown in this chapter, choosing the latter allow us to remove the $\{\lambda_{2,8-14,16,17-22,24-26,30,32-34,38-43,47-49}, \sigma_{1-2,5-6}\}$ contributions[4]. For chapter 3 we will assume $\{\sigma_7, \sigma_8\}$ to see the implications of having a dynamically independent variable on a heavy ion collision setting.

Before moving on we have to note that there is an additional possibility to avoid (2.103) showing up, this involves modifying the derivative expansion by the inclusion of an additional expansion parameter [44, 45] that parametrizes the scaling of the transport coefficients $\{\sigma_7, \sigma_8\}$. The hierarchy of this new scaling can be chosen such that $\mathcal{O}\left(\partial\right)$ and $\mathcal{O}\left(\partial^2\right)$ equations, under an ordinary derivative counting, mixes allowing for $\hat{m}$ and $\hat{M}$ to become dynamically independent variables, namely they are not forced into their equilibrium value at leading order. This scaling is known as the spin hydrodynamic limit [44, 45]. Under this novel scaling the constitutive relations that we wrote down will need to be revised.

## 2.6 Entropy Current

We are interested on computing an entropy current $S^\mu$ such that

$$\mathring{\nabla} S^\mu \geq 0 + \mathcal{O}\left(\partial^3\right)\,. \tag{2.104}$$

[4] Note that when $\{\sigma_7, \sigma_8\}$ are not vanishing we can in principle perform a frame transformation on the spin potentials such that the whole non-equilibrium second order terms can be absorbed into it.

To achieve this we need to analyze the canonical and non-canonical pieces of the entropy current. We will start by analyzing only the ideal piece and noting that it satisfies

$$
\begin{aligned}
u_\nu \mathring{\nabla}_\mu T^{\mu\nu}_{ideal} &= -u^\mu \mathring{\nabla}_\mu \varepsilon - \theta\left(p+\varepsilon\right) + 2\left(\frac{\partial p}{\partial m^2} + 4\frac{\partial p}{\partial M^2}\right) a_\mu m_\nu M^{\mu\nu} \\
&= -T u^\mu \mathring{\nabla}_\mu s - \frac{1}{2}\mu_{\alpha\beta}\mathring{\nabla}_\mu \rho^{\alpha\beta} - \theta\left(sT + \frac{1}{2}\rho_{\alpha\beta}\mu^{\alpha\beta}\right) \\
&= -T\mathring{\nabla}_\mu\left(s u^\mu\right) - \frac{1}{2}\mu_{\alpha\beta}\mathring{\nabla}_\mu\left(u^\mu \rho^{\alpha\beta}\right) \\
&= -T\mathring{\nabla}_\mu\left(s u^\mu\right) - \frac{1}{2}\mu_{\alpha\beta}\mathring{\nabla}_\mu\left(S^{\mu\alpha\beta}_{ideal}\right) ,
\end{aligned}
\tag{2.105}
$$

where we have used the first law (2.79) and the thermodynamic relation (2.76). Note that, given (2.74) and (2.75), (2.105) is true to all orders in derivatives[5]. From the definition of the canonical entropy current (1.48) and (2.105) it follows that

$$
\begin{aligned}
\mathring{\nabla}_\mu S^\mu_{can} = &-\frac{u_\nu}{T}\mathring{\nabla}_\mu T^{\mu\nu} - \frac{1}{2}\frac{\mu_{\rho\sigma}}{T}\mathring{\nabla}_\lambda S^{\lambda\mu\mu} - \left(T^{\mu\nu} - T^{\mu\nu}_{ideal}\right)\mathring{\nabla}_\mu\left(\frac{u_\nu}{T}\right) \\
&- \frac{1}{2}\left(S^{\lambda\rho\sigma} - S^{\lambda\rho\sigma}_{ideal}\right)\mathring{\nabla}_\lambda\left(\frac{\mu_{\rho\sigma}}{T}\right) \\
= &-\frac{u_\nu}{T}\left(\frac{1}{2}R^{\rho\sigma\nu\lambda}S_{\lambda\rho\sigma} - T_{\rho\sigma}K^{\rho\sigma\nu}\right) - \frac{\mu_{\rho\sigma}}{T}\left(T^{\rho\sigma} - S^{\lambda\alpha\rho}K_\alpha{}^\sigma{}_\lambda\right) \\
&- \left(T^{\mu\nu} - T^{\mu\nu}_{ideal}\right)\mathring{\nabla}_\mu\left(\frac{u_\nu}{T}\right) - \frac{1}{2}\left(S^{\lambda\rho\sigma} - S^{\lambda\rho\sigma}_{ideal}\right)\mathring{\nabla}_\lambda\left(\frac{\mu_{\rho\sigma}}{T}\right) ,
\end{aligned}
\tag{2.106}
$$

where we have used the equations of motion (2.24) and (2.25). At this point

[5] The ideal terms discussed in this chapter are valid up to $\mathcal{O}\left(\partial^2\right)$ in the derivative expansion. However it is possible to write down an ideal term valid to all order in derivatives, this is done in appendix A where an analogous relation to (2.105) is also derived. From such relation it is also shown, to all order in derivatives, that an ideal fluid with spin has an entropy current with a vanishing divergence.

we will only keep terms of $\mathcal{O}\left(\partial^2\right)$ in[6] (2.106), this reduces (2.106) to

$$\begin{aligned}
\mathring{\nabla}_\mu S^\mu_{can} =& -\frac{1}{T}\left(u_\nu R^{\rho\sigma\nu\lambda} - 2K^\rho{}_\beta{}^\lambda \mu^{\sigma\beta}\right)\left(B^\chi_{\text{torsion}}\right)_{\lambda\rho\sigma} \\
&+ \frac{\left(K^{\rho\sigma} - \mu^{\rho\sigma}\right)}{T}\left[\left(T^{B-\chi}_{\text{torsion}}\right)_{\rho\sigma} + \left(T^{EOM}_{\text{torsion}}\right)_{\rho\sigma} + \left(T^{(1)}_{\text{non-eq}}\right)_{\rho\sigma}\right] \\
&- \left[\left(T^{B-\chi}_{\text{torsion}}\right)^{\rho\sigma} + \left(T^{EOM}_{\text{torsion}}\right)^{\rho\sigma} + \left(T^{(1)}_{\text{non-eq}}\right)^{\rho\sigma}\right]\mathring{\nabla}_\rho\left(\frac{u_\sigma}{T}\right) \\
&- \left(B^\chi_{\text{torsion}}\right)^{\lambda\rho\sigma}\mathring{\nabla}_\lambda\left(\frac{\mu_{\rho\sigma}}{T}\right) + \mathcal{O}\left(\partial^3\right), \\
=& \frac{\zeta\theta^2}{T} + \frac{\eta\sigma^{\rho\sigma}\sigma_{\rho\sigma}}{T} + \frac{\sigma_7\hat{m}^\alpha\hat{m}_\alpha}{T} + \frac{\sigma_8\hat{M}^{\rho\sigma}\hat{M}_{\rho\sigma}}{T} - \frac{\chi^{(1)}\mathring{\nabla}_\alpha\hat{m}^\alpha}{T} \\
&- \frac{\chi^{(1)}u^\rho\mathring{\nabla}_\rho\left(\theta+\kappa\right)}{T} - \frac{\chi\theta\left(\theta+\kappa\right)}{T} + a_\rho\hat{m}^\rho\left(\frac{\partial\chi^{(1)}}{\partial T} - \frac{\chi^{(1)}}{T}\right) \\
&- \frac{sT}{\frac{\partial\varepsilon}{\partial T}}\frac{\theta\left(\theta+\kappa\right)\left(T\frac{\partial\chi^{(1)}}{\partial T} - \chi^{(1)}\right)}{T^2} + \mathcal{O}\left(\partial^3\right).
\end{aligned} \tag{2.107}$$

Where we have used the constitutive relations (2.89) and (2.96) as well as the first order equation $u_\mu D^{(1)\mu} = 0$. The next step is to write down the non-canonical part of the entropy current, this is done by writing the most general independent first order vector. This will restrict the form of the non-canonical piece to

$$S^\lambda_{non-can} = \left(n_1\Theta + n_2\kappa\right)u^\lambda + n_3\hat{m}^\lambda + n_4 a^\lambda + n_5 k^\lambda + n_6\left(\mathcal{K}_V\right)^\lambda, \tag{2.108}$$

where $n_i$ are functions of the temperature to be fixed by imposing (2.104). By taking the choice

$$\begin{aligned}
n_1 &= \frac{\chi^{(1)}}{T}, & n_4 &= 0, \\
n_2 &= \frac{\chi^{(1)}}{T}, & n_5 &= 0, \\
n_3 &= \frac{\chi^{(1)}}{T}, & n_6 &= 0,
\end{aligned} \tag{2.109}$$

we find the quadratic form

$$\mathring{\nabla}_\lambda S^\lambda = \frac{\zeta\theta^2}{T} + \frac{\eta\sigma^{\rho\sigma}\sigma_{\rho\sigma}}{T} + \frac{\sigma_7\hat{m}^\alpha\hat{m}_\alpha}{T} + \frac{\sigma_8\hat{M}^{\rho\sigma}\hat{M}_{\rho\sigma}}{T}, \tag{2.110}$$

[6] When the fluid is an ideal fluid it is not necessary to truncate the theory as the divergence will vanish to all orders, as expected.

with the entropy current given by

$$S^\lambda = S^\lambda_{can} + S^\lambda_{non-can} = \left[ s + \left( \frac{\partial \chi^{(1)}}{\partial T} \right) \kappa \right] u^\lambda - \frac{\sigma_7 \hat{m}^\lambda}{2T} \,. \tag{2.111}$$

From (2.110) and (2.104) we can conclude that the viscosities $\{\zeta, \eta\}$ should be greater than zero. The same condition seems to be true for $\{\sigma_7, \sigma_8\}$, however if these viscosities are non-vanishing then we can use the first order spin equation to set $\hat{m}$ and $\hat{M}$ to zero at this order on the derivative expansion, keeping the sign of $\{\sigma_7, \sigma_8\}$ unfixed.

# Chapter 3

# Conformal Symmetry in the presence of spin

We will now analyze conformal symmetry in the presence of a spin current. We can recall that conformal symmetry is the invariance of the theory under scaling of the background metric. In an uncharged fluid this can be translated into the vanishing of

$$\delta S = \int \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} = 0 \,, \tag{3.1}$$

whenever $\delta g_{\mu\nu} = 2\phi g_{\mu\nu}$ with $\phi$ an arbitrary function of the coordinates. This condition will imply the energy momentum is traceless. The generalization to a theory with a non-trivial spin current is given by the vanishing of

$$\delta S = \int |e| \left[ T^{\mu}{}_{a} \delta e^{a}_{\mu} + \frac{1}{2} S^{\lambda}{}_{ab} \delta \omega^{ab}{}_{\lambda} \right] = 0 \,, \tag{3.2}$$

under the Weyl transformations [89]

$$\delta e^{a}_{\mu} = \phi e^{a}_{\mu} \,, \tag{3.3}$$

$$\delta \omega^{ab}{}_{\mu} = (1-q) \left( e^{a}_{\mu} e^{b\nu} - e^{b}_{\nu} e^{a\nu} \right) \partial_{\nu} \phi \,, \tag{3.4}$$

where $q$ is a parameter that shifts between the spin connection transforming as the Christoffel connection at $q = 0$, and the spin connection being invariant when we set $q = 1$. Note that torsion can only be set to zero consistently when $q = 0$, namely when torsion is invariant under Weyl transformations, whenever $q \neq 0$ torsion gets reintroduced through a Weyl transformation.

Condition (3.2) under (3.3) and (3.4) implies the following relation

$$T^{\mu}{}_{\mu} = (1-q)\mathring{\nabla}_{\mu}\left(S_{\lambda}{}^{\lambda\mu}\right) . \tag{3.5}$$

Invariance of the equations of motion under (3.3) and (3.4) imply the currents should transform as

$$\delta T^{\mu\nu} = -(d+2)\phi T^{\mu\nu} + (q-1)S^{\mu\nu\rho}\partial_{\rho}\phi - (q-1)S_{\lambda}{}^{\lambda\mu}\partial^{\nu}\phi\,, \tag{3.6}$$
$$\delta S^{\lambda\mu\nu} = -(d+2)\phi S^{\lambda\mu\nu}\,, \tag{3.7}$$

The transformation of the hydrodynamic variables is

$$\delta T = -\phi T\,, \tag{3.8}$$
$$\delta u^{\mu} = -\phi u^{\mu}\,, \tag{3.9}$$
$$\delta m^{\rho} = (1-q)\Delta^{\beta}_{\alpha}\partial_{\beta}\phi\,, \tag{3.10}$$
$$\delta M^{\alpha\beta} = -3\phi M^{\alpha\beta}\,, \tag{3.11}$$

where the transformation rules of the spin potentials were inferred from their relation with the spin connection in hydrostatic equilibrium.

## 3.1 Weyl connection

Let's assume there exists a one form $\mathcal{A}_{\mu}$ satisfying the transformation properties

$$\delta\mathcal{A}_{\mu} = \partial_{\mu}\phi\,, \tag{3.12}$$

We can then note that the transformation rule for the Christoffel connection and the contorsion can be written as

$$\delta\mathring{\Gamma}^{\lambda}_{\mu\nu} = -\delta\mathcal{A}^{\lambda}_{\mu\nu}\,, \tag{3.13}$$
$$\delta K^{\lambda}{}_{\mu\nu} = \delta\left[q\left(g_{\mu\nu}g^{\lambda\sigma} - \delta^{\lambda}_{\mu}\delta^{\sigma}_{\nu}\right)\mathcal{A}_{\sigma}\right]\,, \tag{3.14}$$

where we defined

$$\mathcal{A}^{\lambda}_{\mu\nu} = g_{\mu\nu}\mathcal{A}^{\lambda} - \delta^{\lambda}_{\mu}\mathcal{A}_{\nu} - \delta^{\lambda}_{\nu}\mathcal{A}_{\mu}\,, \tag{3.15}$$

Given any field $Q^{\alpha_1}_{\beta_1}$ that transforms uniformly under Weyl transformation with weight $\omega$, namely

$$\delta Q^{\alpha_1\cdots}_{\beta_1\cdots} = \omega Q^{\alpha_1\cdots}_{\beta_1\cdots}\,, \tag{3.16}$$

we can look for a Weyl covariant derivative [90] $\mathcal{D}$ such that

$$\delta\left(\mathcal{D}_\mu Q^{\alpha_1\cdots}_{\beta_1\cdots}\right) = \omega \mathcal{D}_\mu Q^{\alpha_1\cdots}_{\beta_1\cdots}\,, \tag{3.17}$$

This can be achieved by defining $\mathcal{D}$ as

$$\mathcal{D}_\mu Q^{\alpha_1\cdots}_{\beta_1\cdots} = \mathring{\nabla}_\mu Q^{\alpha_1\cdots}_{\beta_1\cdots} - \omega \mathcal{A}_\mu Q^{\alpha_1\cdots}_{\beta_1\cdots} - \mathcal{A}^{\rho_1}_{\mu\beta_1} Q^{\alpha_1\cdots}_{\rho_1\cdots} + \ldots + \mathcal{A}^{\alpha_1}_{\mu\sigma_1} Q^{\sigma_1\cdots}_{\beta_1\cdots} + \ldots\,. \tag{3.18}$$

Making use of this connection we can define the field strength $\mathcal{F}$ associated to $\mathcal{A}$, Weyl invariant contorsion $(K_W)^{\mu\nu}$, its Weyl invariant field strength $(G_W)^\lambda{}_{\alpha\rho\sigma}$, a Weyl invariant Riemann tensor $(R_W)^\lambda{}_{\alpha\mu\nu}$, and a Weyl invariant chemical potential $(\mu_W)^{ab}$ as

$$\mathcal{F}_{\mu\nu} = \mathring{\nabla}_\mu \mathcal{A}_\nu - \mathring{\nabla}_\nu \mathcal{A}_\mu\,, \tag{3.19}$$

$$(K_W)^\lambda{}_{\nu\mu} = K^\lambda{}_{\mu\nu} - q\left(g_{\mu\nu} g^{\lambda\sigma} - \delta^\lambda_\mu \delta^\sigma_\nu\right)\mathcal{A}_\sigma\,, \tag{3.20}$$

$$\begin{aligned}(G_W)^\lambda{}_{\alpha\rho\sigma} = {} & \mathcal{D}_\rho (K_W)^\lambda{}_{\alpha\sigma} - \mathcal{D}_\sigma (K_W)^\lambda{}_{\alpha\rho} + (K_W)^\lambda{}_{\beta\rho} (K_W)^\beta{}_{\alpha\sigma} \\ & - (K_W)^\lambda{}_{\beta\sigma} (K_W)^\beta{}_{\alpha\rho}\,,\end{aligned} \tag{3.21}$$

$$\begin{aligned}(R_W)^\alpha{}_{\lambda\mu\nu} = {} & \mathring{R}^\alpha{}_{\lambda\mu\nu} + \mathring{\nabla}_\mu \mathcal{A}^\alpha_{\lambda\nu} - \mathring{\nabla}_\nu \mathcal{A}^\alpha_{\lambda\mu} + \mathcal{A}^\beta_{\lambda\nu}\mathcal{A}^\alpha_{\beta\mu} - \mathcal{A}^\beta_{\lambda\mu}\mathcal{A}^\alpha_{\beta\nu} \\ & + (G_W)^\alpha{}_{\lambda\mu\nu}\,,\end{aligned} \tag{3.22}$$

$$(\mu_W)^{ab} = \mu^{ab} - (1-q)\left(u^a \mathcal{A}^b - u^b \mathcal{A}^a\right)\,, \tag{3.23}$$

The Weyl commutator of a tensor will be related to the field strength $\mathcal{F}$, and the Weyl invariant curvature $R_W$ through

$$\left[\mathcal{D}_\mu, \mathcal{D}_\nu\right] V_\lambda = -\omega \mathcal{F}_{\mu\nu} V_\lambda - (R_W - G_W)^\alpha{}_{\lambda\mu\nu} V_\alpha\,, \tag{3.24}$$

From the transformation rules (3.8)-(3.11) we can note there are multiple choices for connections depending on the value of $q$, see table 3.1. As we want to eventually set torsion to zero we will mainly work with $q = 0$, and we will pick $\mathcal{A}_1$ as our connection of choice. From (3.6) we can see that the energy momentum tensor does not transform covariantly under Weyl rescaling, unless $q = 1$. However we can define a modified energy momentum tensor $(T_W)^{\mu\nu}$ that does, through

$$(T_W)^{\mu\nu} = T^{\mu\nu} - (q-1) S^{\mu\mu\rho}\mathcal{A}_\rho + (q-1) S_\lambda{}^{\lambda\mu}\mathcal{A}^\nu\,, \tag{3.25}$$

$$\delta\, (T_W)^{\mu\nu} = -(d+2)\phi\, (T_W)^{\mu\nu}\,, \tag{3.26}$$

| $\mathcal{A}_i$ | $q=0$ | $q=1$ | $q\neq\{0,1\}$ |
|---|---|---|---|
| $(\mathcal{A}_1)_\mu = a_\mu - \frac{\Theta}{d-1}u_\mu$ | ✓ | ✓ | ✓ |
| $(\mathcal{A}_2)_\mu = -\frac{1}{q-1}m_\mu - \frac{\Theta}{d-1}u_\mu$ | ✓ | × | ✓ |
| $(\mathcal{A}_3)_\mu = -\frac{1}{2q(d-2)}(\mathcal{K}_a)_\mu - \frac{\Theta}{d-1}u_\mu$ | × | ✓ | ✓ |
| $(\mathcal{A}_4)_\mu = a_\mu + \frac{\kappa}{q(d-1)}u_\mu$ | × | ✓ | ✓ |
| $(\mathcal{A}_5)_\mu = -\frac{1}{q-1}m_\mu + \frac{\kappa}{q(d-1)}u_\mu$ | × | × | ✓ |
| $(\mathcal{A}_6)_\mu = -\frac{1}{2q(d-2)}(\mathcal{K}_a)_\mu + \frac{\kappa}{q(d-1)}u_\mu$ | × | ✓ | ✓ |

Table 3.1: Different connection options given the value of q.

We can try to rewrite the equations of motion in an explicit Weyl covariant way by writing them in terms of the covariant tensors $\{S^{\lambda\mu\nu}, (T_W)^{\mu\nu}, \mathcal{F}, (R_W)^{\alpha}{}_{\lambda\mu\nu}, (K_W)^{\lambda}{}_{\nu\mu}\}$ and their Weyl covariant derivatives, finding

$$\mathcal{D}_\mu (T_W)^{\mu\nu} = \frac{(R_W)^{\rho\sigma\nu\lambda} S_{\lambda\rho\sigma}}{2} - (T_W)_{\rho\sigma} (K_W)^{\rho\sigma\nu} + 4(q-1)S^{\lambda\mu\nu}\mathcal{F}_{\mu\nu}\,, \tag{3.27}$$

$$\mathcal{D}_\lambda S^{\lambda\mu\nu} = (T_W)^{[\mu\nu]} + 2S^{\lambda\rho[\mu} (K_W)^{\nu]}{}_{\rho\lambda}\,, \tag{3.28}$$

where we used the equation of motion for the spin current, the trace condition (3.5), as well as the weights for the currents with all upper indices equal to $\omega_T = \omega_S = -(d+2)$. We can note that the form of the equations is almost the same as simply changing the fields for their Weyl versions, however for $q \neq 1$ there is an additional coupling between the spin current and the Weyl connection field strength.

## 3.2 Constitutive Relations

The constitutive relations will be of the generic form (2.101) and (2.102), however conformal symmetry will restrict the transport coefficients and will fix their temperature dependence. We are interested on studying polarization on heavy ion collisions when the spin potential becomes an independent degree of freedom. This means we will focus on 3+1-dimensions and set $q = 0$, torsion to zero and $\{\sigma_7, \sigma_8\}$ to zero[1]. Under this assumptions the

[1] It is not standard to set transport coefficients to zero in arbitrary way, however the theory on this limit is quite distinct from the one when they do not vanish. We want to explore the consequences on polarization on the limit when spin matters the most.

constitutive relations take the form

$$\begin{aligned}
T^{(\mu\nu)} =& \epsilon_0 T^4 u^\mu u^\nu + \frac{1}{3}\epsilon_0 T^4 \Delta^{\mu\nu} - 2\eta_0 T^3 \sigma^{\mu\nu} + T^{(\mu\nu)}_{BR} + \mathcal{O}(\partial^2)\,, \\
T^{-2}T^{[\mu\nu]} =& \Delta_\beta{}^{[\mu} u^{\nu]}\left(\ell_1 \mathcal{D}_\alpha \sigma^{\alpha\beta} + \ell_2 \mathcal{D}_\alpha \hat{M}^{\alpha\beta}\right) + \ell_3 \Delta^{\rho[\mu}\Delta^{\nu]\sigma}\mathring{\nabla}_\rho \hat{m}_\sigma \\
& + \ell_4 u^{[\mu}\sigma^{\nu]\rho}\hat{m}_\rho + \ell_5 u^{[\mu}\hat{M}^{\nu]\rho}\hat{m}_\rho + \ell_6 u^{[\mu}M^{\nu]\rho}\hat{m}_\rho \\
& + \ell_7 \sigma^{[\mu}{}_\rho \hat{M}^{\nu]\rho} + \ell_8 \sigma^{[\mu}{}_\rho M^{\nu]\rho} + \ell_9 \hat{M}^{[\mu}{}_\rho M^{\nu]\rho} \\
& - T^{-2} S^{[\mu\nu]\rho}\left(a_\rho - \frac{1}{3}\Theta u_\rho\right) \\
& + T^{-2}\left(S^\rho{}_\rho{}^{[\mu}a^{\nu]} - \frac{1}{3}\Theta S^\rho{}_\rho{}^{[\mu}u^{\nu]}\right) + T_B^{[\mu\nu]}\,, \\
T^{-2}S^\mu{}_{\nu\rho} =& \, 8\rho_0 u^\lambda M_{\nu\rho} + 2s_1 u^\lambda u_{[\nu}\hat{m}_{\rho]} + 2s_2 u^\lambda \hat{M}_{\nu\rho} + S_B{}^\lambda{}_{\nu\rho}\,,
\end{aligned} \tag{3.29}$$

with

$$\begin{aligned}
T_B^{\mu\nu} =& \frac{1}{2}\mathring{\nabla}_\lambda\left(S_B{}^{\mu\nu\lambda} + S_B{}^{\nu\mu\lambda} - S_B{}^{\lambda\nu\mu}\right)\,, \\
S_B{}^\lambda{}_{\mu\nu} =& 2T^3\chi_1 \Delta^\lambda{}_{[\mu}u_{\nu]} + 2T^2\chi_2 M^\lambda{}_{[\mu}u_{\nu]} + 2\sigma_1 T^2 \sigma^\lambda{}_{[\mu}u_{\nu]} \\
& + 2\sigma_2 T^2 \hat{M}^\lambda{}_{[\mu}u_{\nu]} + 2\sigma_3 T^2 \Delta^\lambda{}_\mu \hat{m}_{\nu]}\,,
\end{aligned} \tag{3.30}$$

where $\{\epsilon_0, \rho_0, \eta_0, \ell_{1-9}, s_{1-2}, \chi_{1-2}, \sigma_{1-3}\}$ are constants and the Weyl derivatives are explicitly given by

$$\begin{aligned}
\mathcal{D}_\alpha \sigma^{\alpha\beta} &= \mathring{\nabla}_\alpha \sigma^{\alpha\beta} - 3a_\alpha \sigma^{\alpha\beta}\,, \\
\mathcal{D}_\alpha \hat{M}^{\alpha\beta} &= \mathring{\nabla}_\alpha \hat{M}^{\alpha\beta} - a_\alpha \hat{M}^{\alpha\beta}\,,
\end{aligned} \tag{3.31}$$

## 3.3 An application to heavy ion collisions

We do not presume to carry out a full fledged analysis of heavy ion collision experiments with possible spin currents manifesting during the short collision period. Instead, we consider a perturbed solution to the hydrodynamic equations of motion in the presence of spin, in the limit of dynamical spin, with an underlying Bjorken $(SO(1,1) \times ISO(2) \times Z_2)$ symmetry [91]. We then attempt to relate the dependence of the spin potential on the initial temperature to the dependence of the average hyperon polarization vector on the beam energy. Of course, a complete analysis, which we do not carry out, should include a proper treatment of initial conditions, a full hydrodynamic simulation, and a comprehensive treatment of hadronization of the quark gluon plasma before reaching the detector.

Consider a collision of two gold ions of radii $R$ initially moving with a relativistic velocity directed along a Cartesian '$z$' coordinate. Let's assume that the fluid formed after the collision has Bjorken symmetry, that is, it is invariant under boosts along the beam direction, translations and rotations along the '$x$' and '$y$' directions, and under $z/t \to -z/t$. Going to a Milne coordinate system, $ds^2 = -d\tau^2 + \tau^2 d\eta^2 + dx^2 + dy^2$ where $\tau = \sqrt{t^2 - z^2}$ and $\eta = \mathrm{arctanh}(z/t)$ are proper time and pseudo-rapidity respectively, we find that

$$u^\tau = 1, \qquad T = T_0 \left(\frac{\tau_0}{\tau}\right)^{\frac{1}{3}} - \frac{\eta_0}{2\epsilon_0 \tau}\,, \tag{3.32}$$

with all other components of $u^\mu$ and $\mu_{ab}$ vanishing, solve the equations of motion. Here $T_0$ is the temperature at the initial time $\tau_0$ when the fluid description is a viable one. Note that the shear viscosity to entropy ratio, $\eta/s$, satisfies $\eta/s = 3\eta_0/4\epsilon_0$.

Since the spin potential vanishes on account of Bjorken symmetry, let's consider linear perturbations of Bjorken flow which break transverse translations and axial rotation, $T \to T + \int d^2q \delta T e^{i(q_x x + q_y y)}$, $u^\mu \to u^\mu + \int d^2q \delta u^\mu e^{i(q_x x + q_y y)}$, and $\mu_{ab} \to \mu_{ab} + \int d^2q \delta\mu_{ab} e^{i(q_x x + q_y y)}$. To mimic the experiment, we consider a peripheral collision with impact parameter $b$ along the '$x$' axis. Glancing beams are expected to create a non-trivial velocity gradient in the $x$ direction at initial proper time $\tau_0$ at which we assume hydrodynamics becomes applicable. To this end, we consider an initial velocity profile where $\delta u^\eta(\tau_0) \propto b q_x$, and other components of the perturbations to the velocity vanish. As a result, we find that $\delta m^\eta$, $\delta M^{\eta x} = \delta M^\eta i q_x$ and $\delta M^{\eta y} = \delta M^\eta i q_y$ are non zero while the temperature perturbations and all other components of the spin potential vanish.

To solve the equations of motion we will use the Floerchinger-Wiedemann (FW) approximation [92], where $\frac{3\eta_0}{4\epsilon_0}\frac{1}{T\tau}$ is perturbatively small but $q^2\tau^2\frac{3\eta_0}{4\epsilon_0}\frac{1}{T\tau}$ (with $q^2 = q_x^2 + q_y^2$) is finite. In this approximation, only the leading term for the temperature in (3.32) becomes relevant, the velocity field perturbations take the form

$$\delta u^\eta = i u_0\, b\, q_x\, \tau^{-\frac{5}{3}} e^{-\frac{9q^2\eta_0\tau_0}{16T_0\epsilon_0}\left(\frac{\tau}{\tau_0}\right)^{\frac{4}{3}}}\,, \tag{3.33}$$

and $\delta M^\eta$ and $\delta m^\eta$ are determined algebraically from $\delta u^\eta$ and its derivatives.

Presumably, the stress tensor and spin current will evolve according to hydrodynamic theory from an initial Bjorken time $\tau_0$ to a final time $\tau_f$ when matter hadronizes, $T(\tau_f) = T_f \simeq 150MeV$. The hadrons yield is then collected by the detector which measures its properties. Converting a hydrodynamics spin current and energy momentum tensor to a Hadron distribution is fraught with difficulty. One often used prescription for doing so works under the assumption that the particle distribution after hadronization follows a thermal distribution with temperature, velocity and chemical potential of the hydrodynamic configuration leading to it [93]. Within this framework the polarization vector reads

$$\Pi_\alpha(p) = -\frac{1}{4}\epsilon_{\alpha\rho\sigma\beta}\frac{p^\beta}{m}\frac{\int d\Sigma_\lambda p^\lambda B\mu^{\rho\sigma}}{2\int d\Sigma_\lambda p^\lambda n_F}\,, \tag{3.34}$$

where $\int d\Sigma_\mu$ is an integral over the hadronization surface, $d\Sigma_\mu = \tau\delta^\tau_\mu d\eta dx dy$ in Bjorken coordinates, $p^\mu$ is the particle momentum, $m$ its mass, $n_F$ is the Fermi Dirac distribution and $B$ is an additional distributional quantity that depends on $u^\mu$ and $T$. See, e.g., [17, 94] for details.

It is tempting to use our solution to evaluate (3.34) and compare to data. However, one should keep in mind that our hydrodynamic solution is rather simple minded, involving a linear perturbation on Bjorken symmetry on top of which we used the FW approximation. This perturbation should presumably capture a non vanishing impact parameter. Realistic collisions at mid centrality have an impact parameter of order of the nucleus size and are unlikely to resemble Bjorken flow. They should generate a large enough vorticity for a non trivial spin current to be generated which makes the validity of our linearized approximation somewhat suspect. Still, we have at our disposal an analytic solution to the hydrodynamic equations of motion with spin and it is hard to resist the temptation to compare it with the experimental results using (3.34). Hence, throwing caution to the wind, and inserting the perturbed Bjorken solution into (3.34), we find

$$\Pi_\mu(p) = \frac{16 b e^{-\frac{4T_f^4\epsilon_0 x_0^2}{9T_0^3\eta_0\tau_0}} u_0 \pi^{\frac{3}{2}} T_f^8 x_0 \epsilon_0^{\frac{3}{2}}(\ell_1+\ell_2)\mathrm{Erf}\left(\sqrt{\frac{4T_f^4\epsilon_0 y_0^2}{9T_0^3\eta_0\tau_0}}\right)}{27 m T_0^{\frac{13}{2}}\eta_0^{\frac{3}{2}}\ell_2\tau_0^{\frac{13}{6}}} \times \begin{pmatrix} -p^y I^{(1)} \\ 0 \\ 0 \\ I^{(2)} \end{pmatrix}, \tag{3.35}$$

where now $x_0 = R - \frac{b}{2}$, $y_0 = \sqrt{R^2 - \frac{b^2}{4}}$ and we have integrated over the range $-x_0 < x < x_0$ and $-y_0 < y < y_0$ which approximates the area of overlap of the two colliding nuclei. The expressions for $I^{(n)}$ are given by

$$I^{(n)}(\tau_f) = \frac{\int d\eta\ B\ (p^\tau)^n}{2x_0 \times 2y_0 \times \int d\eta n_F p^\tau}\Bigg|_{\tau=\tau_f}, \tag{3.36}$$

and Erf denotes the error function.

We are particularly interested in the dependence of the spin polarization on the initial temperature, related to the beam energy. Making the reckless approximation that energy and nucleons are distributed uniformly in the nucleus and that the relation between energy density and temperature is of the form $\epsilon = \epsilon_0 T^4$ as dictated by conformal invariance, we find

$$T_0 = \left(\frac{2N}{\pi R^2 \epsilon_0 \tau_0}\right)^{\frac{1}{4}} s_{NN}^{\frac{1}{8}}, \tag{3.37}$$

where $N$ is the number of nucleons, $\sqrt{s_{NN}}$ is the beam energy per nucleon, and we approximated the volume of the nucleus as $\pi R^2 \tau_0$. It is clear that each of these approximations may be improved and upgraded, but as a preliminary order of magnitude estimate relating our hydrodynamic solution to the polarization vector, they are good enough.

Using, $\epsilon_0 = 12$, $T_f = 150 MeV$, $\eta/s = 1/4\pi$, $\tau_0 = 1fm$, $R = 7fm$ and $b = 10fm$ (see [92, 95–97]), we find

$$\frac{4T_f^4 \epsilon_0 x_0^2}{9T_0^3 \eta_0 \tau_0} \simeq \frac{5.1}{\left(\frac{s_{NN}}{\mathrm{GeV}^2}\right)^{\frac{3}{8}}} \qquad \sqrt{\frac{4T_f^4 \epsilon_0 y_0^2}{9T_0^3 \eta_0 \tau_0}} \simeq \frac{5.5}{\left(\frac{s_{NN}}{\mathrm{GeV}^2}\right)^{\frac{3}{16}}}. \tag{3.38}$$

The overall scaling of $\Pi$ in terms of the energy per nucleon is of the form

$$\Pi = \alpha \frac{\exp\left(-\frac{5.1}{\left(\frac{s_{NN}}{\mathrm{GeV}^2}\right)^{\frac{3}{8}}}\right) \mathrm{Erf}\left(\frac{5.5}{\left(\frac{s_{NN}}{\mathrm{GeV}^2}\right)^{\frac{3}{16}}}\right)}{\left(\frac{s_{NN}}{\mathrm{GeV}^2}\right)^{\frac{13}{16}}}, \tag{3.39}$$

where we have replaced the overall constant in (3.35), which depends on the undetermined initial value for the velocity field perturbations $u_0$, and on the

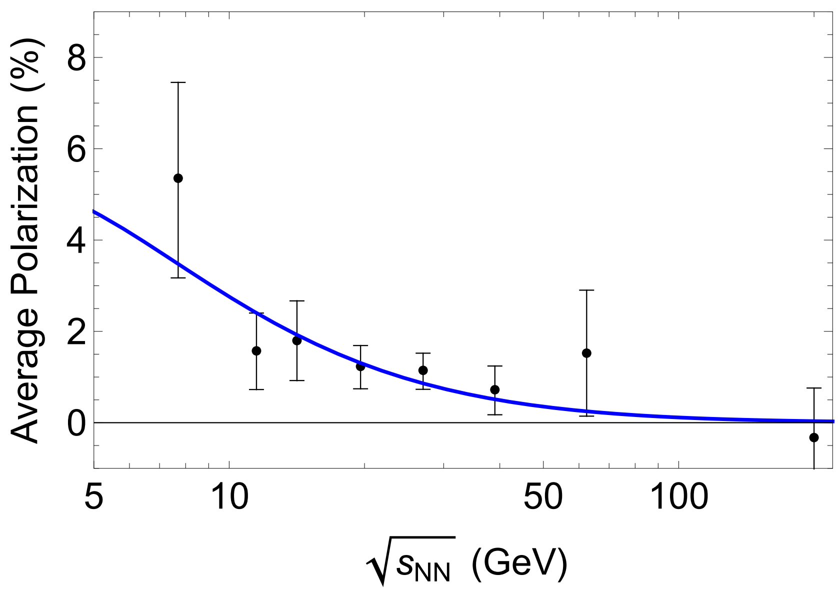


Figure 3.1: A comparison of our estimate (3.39) of the average hyperon polarization (blue) to the STAR measurement [5]. Since a magnetic field was not incorporated in our setting, we have compared our estimate to the average value of the polarization of $\Lambda$ and $\overline{\Lambda}$.

coefficients $\ell_1$ and $\ell_2$, with $\alpha$. To compare to experimental data we need to work out $\Pi_\mu$ in the center of mass frame of the hyperon. Such a Lorentz transformation will not affect the dependence of $\Pi_\mu$ on $s_{NN}$. Therefore, we can attempt to fit (3.39) to experiment by fitting to a single parameter, $\alpha$. Using the data from [5], we find a surprisingly good fit to $\alpha = 286 \pm 52$. See figure 3.1. We emphasize that our phenomenological analysis crucially depend on the new transport coefficients $\ell_1$ $\ell_2$ added to the constitutive relations.

# Chapter 4

# Holography

In this section we will look at a particular example where we use the fluid gravity correspondence [86,87] to derive the constitutive relations for a fluid with a dynamical spin current. We can recall that the holographic correspondence amounts to reading the currents from the relation

$$\delta S_{grav} = \int d^d x |e| \left[ T^{\mu}{}_{a} \delta e^{a}_{\mu} + \frac{1}{2} S^{\lambda}{}_{ab} \delta \omega^{ab}{}_{\lambda} \right] , \tag{4.1}$$

where $S_{grav}$ is a dual gravitational action with both a dynamical and independent vielbein $\delta e^a_\mu$ and spin connection $\delta\omega^{ab}{}_\lambda$, and $\{T^{\mu\nu}, S^{\lambda\mu\nu}\}$ are the conserved currents of the dual quantum field theory (QFT). The fluid gravity treatment will amount to finding gravitational solutions order by order in a gradient expansion, and using the holographic prescription (4.1) to compute the currents. We will focus on 3+1 QFT's, namely we will work on a dual 5D gravitational theory.

For this purpose we focus on the 5 dimensional Lovelock-Chern-Simons (LCS) gravity without matter [85,98,99]. We pick this theory for two reasons: unlike Einstein gravity (1) the spin connection and vielbein are independent, and (2)the equations of motion can be reduced to algebraic equations. The first point is crucial for obtaining a non-trivial spin current as discussed above. The second is practical. We will first review the Chern-Simons model of [85] and the holographic prescription for obtaining the currents is derived as well. We later present an ansatz for the holographic model that is suitable for the fluid-gravity correspondence. We then solve the dynamical equations of motion to all orders in the hydrodynamic derivative expansion by reducing them to a set of constraint equations.

After this generic treatment we then consider zeroth order hydrodynamics, namely constant sources, and present two zeroth order gravitational blackhole solutions. These are the only solutions with independent and unconstrained spin sources. The energy momentum tensor, spin current, and thermodynamic potentials for the solutions are derived. We then promote all the sources to slowly varying functions of the boundary coordinates and obtain the corresponding first order solutions and constitutive relations.

## 4.1 5D Lovelock Chern-Simons Gravity

The holographic backgrounds that we consider in this work as dual to quantum field theories with a non-trivial spin current, are 5-dimensional backgrounds that solve the Chern-Simons action.

$$S = \int \langle \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A} - \frac{1}{2} \mathcal{F} \wedge A \wedge A \wedge A + \frac{1}{10} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \rangle \,, \tag{4.2}$$

where $\langle\ \rangle$ indicates a group trace over the SO(4,2) algebra with generators $\mathcal{J}_{\bar{A}}$ on which the connection one form $\mathcal{A}$ is valued, and, where the field strength is defined as usual $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. It was shown in [85] that, for this five dimensional gravitational Chern-Simons (CS) theory a certain class of gauge allows for a finite Fefferman-Graham (FG) expansion. Therefore a well defined holographic recipe can be established for this theory. Their work was generalized in [100] to any odd dimension gravitational CS theory where an analysis of the boundary gauge symmetries was also performed. Here, we first outline the necessary details of the theory. Our discussion closely follows the presentation in [85]. The SO(4,2) indices $\bar{A}$ can be written as a pair of antisymmetric indices via $\bar{A} = AB, A6$ with the indices $A = 0, 1, 2, 3, 5$. Using this index decomposition the following identification can be done $\{\mathcal{J}_{A6} = P_A, \mathcal{J}_{AB} = J_{AB}\}$ with $P_A$ and $J_{AB}$ the generators of translations and Lorentz transformations in five dimensions satisfying the SO(4,2) algebra

$$[P_A, P_B] = J_{AB} \,, \tag{4.3}$$

$$[P_A, J_{BC}] = \eta_{AB} P_C - \eta_{AC} P_B \,, \tag{4.4}$$

$$[J_{AB}, J_{CE}] = \eta_{BC} J_{AE} + \eta_{AE} J_{BC} - \eta_{AC} J_{BE} - \eta_{BE} J_{AB} \,. \tag{4.5}$$

The action (4.2) is invariant under infinitesimal gauge transformations

$$\delta\mathcal{A} = d\tau + [\mathcal{A}, \tau] \ , \tag{4.6}$$

$$\tau = \eta^A P_A + \frac{1}{2}\lambda^{AB} J_{AB} \, , \tag{4.7}$$

with gauge parameters $\{\eta^A, \lambda^{AB}\}$ representing local translations and Lorentz transformations, up to a boundary term

$$\delta S = -\frac{1}{2}\int_{\partial\mathcal{M}_5} \langle d\tau \wedge (\mathcal{A}\wedge\mathcal{A} + d\mathcal{A}\wedge\mathcal{A} + \mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})\rangle \, . \tag{4.8}$$

To connect to the five dimensional gravity the 5D vielbein[1] and the spin connection $\hat{\omega}^{AB}$ are identified as components of the gauge connection $\mathcal{A}$ via [101]

$$\mathcal{A} \equiv \hat{e}^A P_A + \frac{1}{2}\hat{\omega}^{AB} J_{AB} \, , \tag{4.9}$$

$$\mathcal{F} = \hat{T}^A P_A + \frac{1}{2}\left(\hat{R}^{AB} + \hat{e}^A \wedge e^B\right) J_{AB} \, , \tag{4.10}$$

where $\hat{T}^A$ and $\hat{R}^{AB}$ are the five dimensional torsion and curvature two forms defined as

$$\hat{T}^A = d\hat{e}^A + \hat{\omega}^A{}_B \wedge \hat{e}^B \, , \tag{4.11}$$

$$\hat{R}^{AB} = d\hat{\omega}^{AB} + \hat{\omega}^A{}_C \wedge \hat{\omega}^{CB} \, . \tag{4.12}$$

Using the identification (4.9) the CS action (4.2) can be written in the more familiar form

$$S_{LCS} = \kappa_{CS}\int_{\mathcal{M}_5} \epsilon_{ABCDE}\left[\hat{R}^{AB}\hat{R}^{CD}\hat{e}^E + \frac{2}{3}\hat{R}^{AB}\hat{e}^C\hat{e}^D\hat{e}^E + \frac{1}{5}\hat{e}^A\hat{e}^B\hat{e}^C\hat{e}^D\hat{e}^E\right] \, , \tag{4.13}$$

where for notational simplicity we omit the wedge product symbol. Here $\kappa_{CS}$ is the CS parameter that arises from the non-vanishing group trace

$$\langle \mathcal{J}_{AB}\mathcal{J}_{CD}\mathcal{J}_{E6}\rangle = \frac{\kappa_{CS}}{2}\epsilon_{ABCDE} \, . \tag{4.14}$$

[1] Below we denote the 5D quantities with a hat symbol to distinguish them from the corresponding quantities in 4D.

The equations of motion for action (4.13) are

$$\epsilon_{ABCDE}\left(\hat{R}^{AB}+\hat{e}^{A}\hat{e}^{B}\right)\left(\hat{R}^{AB}+\hat{e}^{A}\hat{e}^{B}\right)=0\,, \tag{4.15}$$

$$\epsilon_{ABCDE}\left(\hat{R}^{AB}+\hat{e}^{A}\hat{e}^{B}\right)\hat{T}^{E}=0\,, \tag{4.16}$$

and can be compactly written in the CS form

$$g_{\bar{A}\bar{B}\bar{C}}\mathcal{F}^{\bar{B}}\mathcal{F}^{\bar{C}}=0\,, \tag{4.17}$$

where $g_{\bar{A}\bar{B}\bar{C}}=\langle\mathcal{J}_{\bar{A}}\mathcal{J}_{\bar{B}}\mathcal{J}_{\bar{C}}\rangle$ is the trilinear symmetric invariant trace whose only non-vanishing component is given by (4.14). There exist non-trivial solutions to (4.15) and (4.16) with non-vanishing torsion $\hat{T}^A\neq 0$, see [102, 103]. We will indeed find such novel solutions later in this section.

## 4.2 Gauge Fixing and Fefferman-Graham expansion

We look for asymptotically AdS solutions to (4.15) and (4.16) to find holographic duals to fluids with spin degrees of freedom. We work with a radial foliation of the spacetime coordinates $x^M=(x^\mu,r)$ and similarly of the tangent space indices $A=(a,5)$ where the asymptotic boundary of the manifold is located at $r=r_0$ which wee send to infinity $r_0\to\infty$ after holographic renormalization. For metric formulations of gravity Fefferman-Graham (FG) theorem tell us that the metric near this boundary takes the form [104]

$$ds^2=\frac{d\rho^2}{4\rho^2}+\frac{1}{\rho^2}g_{\mu\nu}\left(x^\mu,\rho\right)dx^\mu dx^\nu\,, \tag{4.18}$$

where we have introduced the FG coordinate $\rho=\frac{1}{r^2}$. On even $d$-dimensional boundary spacetimes, $g_{\mu\nu}$ admits the expansion

$$g_{\mu\nu}(x^\mu,\rho)=g^{0}_{\mu\nu}\left(x^\mu\right)+...+\rho^{\frac{d}{2}}g^{d}_{\mu\nu}\left(x^\mu\right)\,. \tag{4.19}$$

To derive the equivalent of (4.18) in the first order formulation, it is convenient to use the CS form (4.17) of the equations of motion [85] and note that under the radial foliation they split as

$$g_{\bar{A}\bar{B}\bar{C}}\epsilon^{\mu\nu\alpha\beta}\mathcal{F}^{\bar{B}}{}_{\mu\nu}\mathcal{F}^{\bar{C}}{}_{\alpha\beta}=0\,, \tag{4.20}$$

$$g_{\bar{A}\bar{B}\bar{C}}\epsilon^{\mu\nu\alpha\beta}\mathcal{F}^{\bar{B}}{}_{\mu\nu}\mathcal{F}^{\bar{C}}{}_{\alpha r}=0\,. \tag{4.21}$$

Equation (4.20) contains no radial derivative and can be regarded as a constraint that holds at every value of $r$. It was shown in [85] that the Ward identities of the dual CFT are contained in this set of constraints. The bulk dynamics of the theory are contained in (4.21) and it is from this expression that the FG structure arises. Equations (4.20) and (4.21) are not independent and for a generic solution of (4.20) it follows that (4.21) implies [105]

$$\mathcal{F}^{\bar{B}}{}_{r\mu} = \mathcal{F}^{\bar{B}}{}_{\mu\nu} N^{\nu}\,, \tag{4.22}$$

with $N^{\nu}$ arbitrary functions. Equation (4.22) can be rewritten as

$$\partial_r \mathcal{A}^{\bar{B}}{}_{\mu} = D_{\mu} \mathcal{A}^{\bar{B}}{}_{r} + \mathcal{F}^{\bar{B}}{}_{\mu\nu} N^{\nu}\,, \tag{4.23}$$

where $D_\mu$ denotes a covariant derivative. We note that the right hand side of (4.23) corresponds to a gauge transformation parametrized by $\mathcal{A}_r$ and a diffeomorphism[2] with parameter $N^\nu$. This means $A_\mu(r+\delta r)$ is determined from $A_\mu(r)$ by means of a gauge transformation and a diffeomorphism, allowing us to choose the functions $\{\mathcal{A}_r, N^\nu\}$ at will. By fixing diffeomorphisms on the transverse direction $x^\mu$ we can set $N^\mu = 0$ leaving only $\mathcal{A}_r$ to be fixed by a gauge transformation. One crucial remark here is that different gauge choices for $\mathcal{A}_r$ give rise to non-equivalent boundary theories [100] as the CS action is only gauge invariant up to boundary terms. We consider the following parametrization of the gauge choice $\mathcal{A}_r$

$$\mathcal{A}_r = H(r,x)P_5 + H^a_+(r,x)J^+_a + H^a_-(r,x)J^-_a + \frac{1}{2}H^{ab}(r,x)J_{ab}\,, \tag{4.24}$$

with $J^\pm_a \equiv P_a \pm J_{a5}$ and where $\{H, H^a_\pm, H^{ab}\}$ are arbitrary functions of the holographic coordinate and of the boundary coordinates. Setting $\{H^a_\pm, H^{ab}\}$ to zero and $H = H(r)$ corresponds to the gauge choice[3] in [85, 100]. Here we will work with a slight generalization by allowing $H = H(r,x)$. In particular we assume $H(r,x)$ to assume the following form

$$H = \frac{1}{rg(r,x)}\,, \tag{4.25}$$

[2]We are using the gauge invariant form for diffeomorphisms [106] that differs from a Lie derivative by a local gauge transformation with parameter $\tau = \mathcal{A}_\mu \xi^\mu$.

[3]The simplest gauge sets $\{H, H^a, H^{ab}\}$ to zero, however this will result in a degenerate metric.

with $g(r,x)$ playing the role of a blackening factor which in principle can be fixed by making use of the remaining radial diffeomorphism. In particular, in [85] it is set to 1 with the corresponding solution being global AdS. However when $g(r,x)$ has poles, as in a blackhole, the radial diffeomorphism required to set $g$ to unity would be singular. These give rise to a class of blackhole solutions distinct from the global AdS solutions considered in [85, 100]. We therefore consider $g(r,x)$ with simple poles. A generic near boundary asymptotic is then given by

$$g(r,x) = 1 + \sum \frac{c_i(x)}{r^{i+1}}\,, \tag{4.26}$$

with $c_i(x)$ some real functions. The solution to (4.23) in this gauge becomes

$$\mathcal{A}(\rho,x) = e^{-PP_5\int h(r,x)d\rho}\mathcal{A}(0,x)e^{P_5\int H(r,x)d\rho} + P_5\left[dx^\mu\partial_\mu\int H(r,x)\right]\,. \tag{4.27}$$

For convenience we switched back to the FG coordinates and where $\mathcal{A}(0,x)$ is the boundary condition for the gauge connection at the AdS boundary. One finds, using (4.25) and (4.26), that the function $H$ when integrated over the radial coordinate satisfies the following asymptotic expansion

$$\int H d\rho = -\frac{\ln\rho}{2} + c_0\rho^{\frac{1}{2}} + \frac{c_1 - c_0^2}{2}\rho + \frac{c_0^3 - 2c_0c_1 + c_2}{3}\rho^{\frac{3}{2}} \tag{4.28}$$

$$+\frac{c_3 - c_0^4 + 3c_0^2c_1 - c_1^2 - 2c_0c_2}{4}\rho^2 + \ldots\,. \tag{4.29}$$

On the other hand, using the gauge choice discussed above, the boundary condition $A(0,x)$ can be parametrized as

$$A(0,x) = e^a(x)J_a^+ + k^a(x)J_a^- + \frac{1}{2}\omega^{ab}(x)J_{ab}\,, \tag{4.30}$$

Combining (4.27), (4.28), and (4.30) the near boundary expansion of $\hat{e}^M$ and $\hat{\omega}^{MN}$ is

$$\hat{e}^5 = -\frac{d\rho}{2\rho}\,,\quad \hat{e}^a = \frac{1}{\sqrt{\rho}}\left[\tilde{e}^a + \rho\tilde{k}^a\right]\,,\quad \hat{\omega}^{a5} = \frac{1}{\sqrt{\rho}}\left[\tilde{e}^a - \rho\tilde{k}^a\right]\,,\quad \hat{\omega}^{ab} = \omega^{ab}\,, \tag{4.31}$$

with $\tilde{e}^a$ and $\tilde{k}^a$ a shorthand notation for

$$\tilde{e}^a = \left[1 + c_0\rho^{\frac{1}{2}} + \frac{c_1}{2}\rho + \frac{2c_2 - c_0c_1}{5}\rho^{\frac{3}{2}} + \frac{6c_3 - 4c_0c_2 - 3c_1^2 + 2c_0^2c_1}{24}\rho^2 + ...\right] e^a \,, \tag{4.32}$$

$$\tilde{k}^a = \left[1 - c_0\rho^{\frac{1}{2}} + \frac{2c_0^2 - c_1}{2}\rho + ...\right] k^a \,. \tag{4.33}$$

Here we only show the terms that are relevant for the thermodynamics of the solutions. The near boundary expansion (4.31) is the equivalent of the FG expansion (4.18) in the first order formulation of gravity. The asymptotic expansion (4.31) should still satisfy the constraint equation (4.20). As it was shown in [85,100], the constraint implies the hydrodynamic equations of motion for the energy momentum tensor and spin current, leaving the fields $e^a$ and $\omega^{ab}$ unconstrained, which will then be identified with the sources for the currents.

## 4.3 Holographic Counterterm Action

An important aspect of the holographic theory is renormalization. In order to identify the on-shell gravity action with the effective action of the field theory, we should first construct a finite action which at the same time preserve the boundary symmetries

$$I_{ren} = \lim_{\epsilon\to 0}\left[S_{\text{on-shell}}(\epsilon) - V(\epsilon)\right] \,, \tag{4.34}$$

where $V(\epsilon)$ is the counterterm action and $\epsilon$ a cutoff for the FG coordinate. The renormalized action (4.34) should have a well defined variational problem, so that, upon the holographic identification of $I_{ren} = S_{grav}$ in (4.1), it becomes the generating function in the dual field theory. Following a procedure analogous to the one in [85], we obtain the following counterterm action

$$V = 4\kappa_{CS}\epsilon_{abcd}\int_{\mathcal{M}_4}\int\left[\left(R^{ab} + e^ak^b\right)k^ce^d + \frac{\tilde{e}^a\tilde{e}^b\tilde{e}^c\tilde{e}^d}{6\epsilon^2} - \frac{\tilde{e}^a\tilde{e}^b\left(R^{cd} + \frac{4}{3}\tilde{e}^c\tilde{k}^d\right)}{2\epsilon}\right] \,, \tag{4.35}$$

with $R^{cd} = d\omega^{cd} + \omega^c{}_d\omega^{bd}$ the boundary field strength of the source $\omega^{ab}$. The first term of (4.35) is a Gibbons-Hawking term, while the last two terms cancel the divergences in $S_{\text{on-shell}}$. By varying $\delta I_{ren}$ and identifying it with the variation of the boundary generating function the energy momentum and spin curren three forms are found as[4]

$$\tau_a = -8\kappa_{CS}\epsilon_{abcd}\left(R^{bc} + 2e^b k^c\right) k^d\,, \tag{4.36}$$

$$\sigma_{ab} = 16\kappa_{CS}\epsilon_{abcd}T^c k^d\,, \tag{4.37}$$

with $T^c$ the torsion coming from the boundary sources $e^a$ and $\omega^{ab}$, and with the three forms $\tau_a$ and $\sigma_{ab}$ related to the usual currents through

$$T^{\mu\nu} = \frac{\epsilon^{\rho\sigma\lambda\nu}}{|e|} e^{b\mu}\tau_{b,\rho\sigma\lambda}\,, \qquad S^{\lambda}{}_{\mu\nu} = \frac{\epsilon^{\lambda\rho\sigma\tau}}{|e|} e^a_\mu e^b_\nu \sigma_{ab,\rho\sigma\tau}\,. \tag{4.38}$$

Following [85] it follows that the conservation equations for currents are the expected ones. The chosen gauge leaves not only these symmetries as residual boundary symmetries but also Weyl symmetry and a particular non-abelian symmetry are present [100]. These last two symmetries become anomalous and it has been suggested in [85] that the non-abelian anomaly could be related to a chiral anomaly through the antisymmetric part of the spin current.

## 4.4 Thermodynamic properties of the black-hole solutions

In equilibrium we can Wick rotate the theory to Euclidian signature, allowing us to identify the (grand-)canonical free energy $F_{free}$ as

$$\beta F_{free} \equiv \beta \int_{\mathcal{M}_3} \mathcal{F}_{free} = I[e,\omega]_{on-shell}\,, \tag{4.39}$$

with $\beta$ the inverse temperature which equals the length of the thermal cycle and $\mathcal{F}_{\text{free}}$ the free energy density integrated over the spatial boundary $\mathcal{M}_3$. We are also interested in the thermal entropy $S_{thermal}$. For blackhole solutions the entropy can be computed as the Noether charge associated with

[4] We note that the form of these currents correspond to a particular choice of finite counterterms given in (4.34).

diffeomorphisms[5] and is found to be

$$S_{thermal} \equiv \int_{\mathcal{M}_3^h} \mathcal{S}_{thermal} \quad = 4\pi \int_{\mathcal{M}_3^h} \epsilon_{ABCDF} \left( \hat{R}^{CD} \hat{e}^F + \frac{1}{3} \hat{e}^C \hat{e}^D \hat{e}^F \right) n^{AB} , \tag{4.40}$$

where $M_3^h$ denotes the horizon manifold, $n^{AB} = D^{[A}\xi^{B]}$ is the binormal at the horizon, and $\xi^B$ the Killing vector generating the horizon. We denote the entropy density by $\mathcal{S}_{thermal}$. Expression (4.40) is obtained from the general entropy formula (B.19) which is valid for a generic Lovelock gravity[6]. This formula is the analogue of Wald's entropy formula [108] for the first order formulation of gravity and its derivation is shown in appendix B. Our derivation closely follows [109, 110] where a similar formula was derived for torsionless theories[7]. The free energy and the entropy satisfy the Smarr relation and the first law of thermodynamics

$$\mathcal{F}_{free} = M - T\mathcal{S}_{thermal} - \mu_I Q^I , \tag{4.41}$$

$$d\mathcal{F}_{free} = -\mathcal{S}_{thermal} dT - Q^I d\mu_I , \tag{4.42}$$

with $M \equiv M_0 - \mu_I Q^I$ the mass of the blackhole, $\mu_I$ represent all independent components of torsion, and $Q_I = \frac{\partial \mathcal{F}_{free}}{\partial \mu_I}$. This mass can be independently computed from the energy tensor and the spin current as

$$M = \int_{\mathcal{M}_3} d^3x \left[ n_\mu \xi_\nu T^{\mu\nu} + \frac{1}{2} n_\lambda S^\lambda{}_{cd} \omega^{cd} \xi^\alpha \right] , \tag{4.43}$$

with $n_\mu$ the normal to the timelike direction.

## 4.5 Generic Holographic Background

In this section we introduce a hydrodynamic ansatz for $\{\hat{e}^M, \hat{\omega}^{MN}\}$ suitable for a blackhole solution. We will introduce this ansatz in a form appropriate

[5] To be precise, here we consider the gauge invariant diffeomorphisms discussed below equation (4.23).

[6] To see another concrete analysis of blackhole thermodynamics in a higher derivative theory see [107] for the thermodynamics of Ads and dS blackholes in Gauss-Bonnet gravity.

[7] There exist some subtleties for deriving a first law in the context of the first order formalism.

to perform the hydrodynamic expansion in (boundary) derivatives, allowing us to interpret the currents of the solutions as those of a thermal conformal fluid. Just as in the derivation of the holographic currents above, we consider the following gauge

$$\hat{e}^5 = \frac{dr}{rg} + \partial_\mu \left( \int \frac{dr}{rg} \right) dx^\mu \,, \tag{4.44}$$

$$\hat{e}^a_r = \hat{\omega}^{a5}{}_r = \hat{\omega}^{ab}{}_r = \hat{T}^A{}_{r\mu} = F^{ab}{}_{r\mu} = 0 \,. \tag{4.45}$$

where $F^{AB} \equiv R^{AB} + e^A e^B$ is the Chern-Simons field strength. We know that solving for $F^{AB}{}_{r\mu} = 0$ and $\hat{T}^A_{r\mu} = 0$ is sufficient to determine the radial dependence of the functions $\{\hat{e}^a_\mu, \hat{\omega}^{a5}{}_\mu, \hat{\omega}^{ab}{}_\mu\}$ leaving a set of constraint equations for the integration constants and sources. The dynamical equations we need to solve are

$$\partial_r \hat{\omega}^{ab}{}_\mu = 0 \,, \tag{4.46}$$

$$\hat{\omega}^{a5}{}_\mu = rg \partial_r \hat{e}^a_\mu \,, \tag{4.47}$$

$$\partial_r \left( rg \partial_r \hat{e}^a_\mu \right) = \frac{1}{rg} \hat{e}^a_\mu \,. \tag{4.48}$$

This system of equations has a simple solution which was already presented. We will rewrite such solution in an alternative form by using the hydrodynamic ansatz presented in the rest of this section. This manipulation will allow us to solve the remaining constraint equations more easily in terms of the hydrodynamic derivative expansion. This, in particular, entails a decomposition of the background ansatz in terms of the fluid velocity $u^\mu$ and the projection operator $\Delta^\mu_\nu$.

### 4.5.1 Ansatz for the vielbien

The one form $e^a$ can be decomposed as

$$\hat{e}^a = \theta^a_\mu \left[ F(x,r) u^\mu u_\nu + F_\sigma(x,r) u^\mu \Delta^\sigma_\nu + \tilde{F}_\rho(x,r) \Delta^{\rho\mu} u_\nu + F_{\rho\sigma}(x,r) \Delta^{\rho\mu} \Delta^\sigma_\nu \right] dx^\nu \,, \tag{4.49}$$

where $\{F, F_\sigma, \tilde{F}_\rho, F_{\rho\sigma}$ are functions of both the boundary and the holographic coordinate and $\theta^a$ us a boundary vielbein that is related to the vielbein $e^a$ in the previous section as

$$e^a = \theta^a_\mu \left( u^\mu u_\nu + \Delta^\mu_\nu \right) dx^\nu \,. \tag{4.50}$$

The corresponding bulk metric reads

$$ds^2 = \frac{dr^2}{r^2 g^2} + \Big[ -\left(F^2 - \tilde{F}_\alpha \tilde{F}_\beta \Delta^{\alpha\beta}\right) u_\mu u_\nu + 2\left(\tilde{F}_\alpha F_{\beta\sigma}\Delta^{\alpha\beta} - F F_\sigma\right) u_\mu \Delta^\sigma_\nu \\ + \left(F_{\alpha\rho}F_{\beta\sigma}\Delta^{\alpha\beta} - F_\rho F_\sigma\right)\Delta^\rho_\mu \Delta^\sigma_\nu \Big] dx^\mu dx^\nu \,. \tag{4.51}$$

The AdS boundary conditions for the functions $\{g, F, F_\sigma, \tilde{F}_\sigma, F_{\rho\sigma}\}$ are such that

$$\lim_{r\to\infty} ds^2 \sim \frac{dr^2}{r^2} + r^2 \gamma_{\mu\nu} dx^\mu dx^\nu \,, \tag{4.52}$$

where $\gamma_{\mu\nu}$ is the boundary metric coupled to the dual CFT. For simplicity we take $\gamma_{\mu\nu} = \eta_{\mu\nu}$ in the rest of the paper. It is now possible to write down an ansatz for the functions $\{g, F, F_\sigma, \tilde{F}_\sigma, F_{\rho\sigma}\}$ as an expansion in derivatives

$$F(x,r) = -r f(r) + \sum_{m=1}^{\infty}\sum_{i_m=1}^{I^s_m} \epsilon^m r f^{[m]}_{i_m}(r,x) s^{[m]}_{i_m}(x) \,, \tag{4.53}$$

$$F_\sigma(x,r)\Delta^{\sigma\mu} = \sum_{m=1}^{\infty}\sum_{i_m=1}^{I^v_m} \epsilon^m r l^{[m]}_{i_m}(r,x) v^{[m]\mu}_{i_m}(x) \,, \tag{4.54}$$

$$\tilde{F}_\rho(x,r)\Delta^{\rho\mu} = \sum_{m=1}^{\infty}\sum_{i_m=1}^{I^v_m} \epsilon^m r \tilde{l}^{[m]}_{i_m}(r,x) v^{[m]\mu}_{i_m}(x) \,, \tag{4.55}$$

$$F_{\rho\sigma}(x,r)\Delta^{\rho\mu}\Delta^{\sigma\nu} = r h(r,x)\Delta^{\mu\nu} + \sum_{m=1}^{\infty}\sum_{i_m=1}^{I^t_m} \epsilon^m r h^{[m]}_{i_m}(r,x) t^{[m]\mu\nu}_{i_m}(x) \,, \tag{4.56}$$

where $f$ is a function of the radial coordinate[8], $\{h, f^{[m]}_{i_m}, l^{[m]}_{i_m}, \tilde{l}^{[m]}_{i_m}, h^{[m]}_{i_m}\}$ functions of both the holographic and the boundary coordinates[9], index $[m]$ indicates the order of appearance in the derivative expansion, $\epsilon$ is a book keeping parameter explicitly counting the number of derivatives. Here $s^{[m]}_{i_m}$, $v^{[m]\mu}_{i_m}$, and $t^{[m]\mu\nu}_{i_m}$ are, respectively, a scalar, a vector, and a tensor form the corresponding $I^s_m$, $I^v_m$, and $I^t_m$ independent quantities constructed from all available sources

[8]Using radial diffeomorphisms f can be chosen to depend only on the holographic coordinate.

[9]Dependence on the boundary coordinates in these functions arises from their temperature dependence.

with $m$ number of derivatives. Although the ansatz in (4.53)-(4.56) is formal it is possible to find solutions for all the functions in terms of the blackening factor $f$ as follows[10]

$$g = \frac{\sqrt{h_0 + (rf)^2}}{r(rf)'} \,, \tag{4.57}$$

$$h = \frac{(h_0 + h_1)(rf) + (h_0 - h_1)\sqrt{h_0 + (rf)^2}}{2rh_0} \,, \tag{4.58}$$

$$f^{[m]}_{i_m} = a^{[m]}_{i_m}\left(\frac{rf - \sqrt{h_0 + (rf)^2}}{2rh_0}\right) \,, \tag{4.59}$$

$$l^{[m]}_{i_m} = b^{[m]}_{i_m}\left(\frac{rf - \sqrt{h_0 + (rf)^2}}{2rh_0}\right) \,, \tag{4.60}$$

$$\tilde{l}^{[m]}_{i_m} = c^{[m]}_{i_m}\left(\frac{rf - \sqrt{h_0 + (rf)^2}}{2rh_0}\right) \,, \tag{4.61}$$

$$h^{[m]}_{i_m} = d^{[m]}_{i_m}\left(\frac{rf - \sqrt{h_0 + (rf)^2}}{2rh_0}\right) \,, \tag{4.62}$$

with prime denoting a radial derivative, $\{h_0, a^{[m]}_{i_m}, b^{[m]}_{i_m}, c^{[m]}_{i_m}, d^{[m]}_{i_m}\}$ integration constants that a priori are functions of the boundary coordinates, and where the asymptotic AdS boundary conditions have already been implemented. As seen from (4.57)-(4.62) we have rewritten the solution for the functions $\{g, h, f^{[m]}_{i_m}, l^{[m]}_{i_m}, \tilde{l}^{[m]}_{i_m}, h^{[m]}_{i_m}\}$ as algebraic functions of $f$. This function $f$ is arbitrary and can be fixed via the remaining diffeomorphisms of the holographic coordinate.

It is instructive to look at the zeroth order metric

$$ds^2 = \frac{dr^2}{r^2 g^2} + r^2\left(-f^2 u_\mu u_\nu + h^2 \Delta_{\mu\nu}\right) dx^\mu dx^\nu \,. \tag{4.63}$$

This corresponds to a non-extremal blackhole with a horizon at $r_h$ if the function $f$ satisfies $f^2(r_h) = 0$ and $(f')^2(r_h) \neq 0$. Then the blackhole temperature $T(x)$ can be related to $h_0$ by

$$h_0 = (2\pi T)^2 \,. \tag{4.64}$$

[10]This choice of ansatz with the corresponding AdS asymptotic behavior identifies the boundary source as in (4.50).

### 4.5.2 Ansatz for the connection

The connection $\omega^{a5}$ can be decomposed in a similar fashion

$$\hat{\omega}^{a5} = \theta^a_\mu \Big[ K(x,r) u^\mu u_\nu + K_\sigma(x,r) u^\mu \Delta^\sigma_\nu + \tilde{K}_\rho(x,r) \Delta^{\rho\mu} u_\nu + K_{\rho\sigma}(x,r) \Delta^{\rho\mu} \Delta^\sigma_\nu \Big] dx^\mu \,, \tag{4.65}$$

where $\{K, K_\sigma, \tilde{K}_\rho, K_{\rho\sigma}\}$ are functions of the holographic and the boundary coordinates. From (4.46)-(4.48) we see that these functions are solved in terms of $\{F, F_\sigma, \tilde{F}_\rho, F_{\rho\sigma}\}$ as

$$K = (rf) F' \,, \tag{4.66}$$

$$K_\sigma = (rg) F'_\sigma \,, \tag{4.67}$$

$$\tilde{K}_\sigma = (rg) \tilde{F}'_\sigma \,, \tag{4.68}$$

$$K_{\rho\sigma} = (rg) F'_{\rho\sigma} \,, \tag{4.69}$$

where prime denotes a derivative with respect to $r$. From (4.46)-(4.48) it also follows that the connection $\tilde{\omega}^{ab}$ should be independent of the holographic coordinate and can only depend on the boundary coordinates. These, then, corresponds to the external spin sources $\omega^{ab}$ in the dual field theory.

## 4.6 Solutions dual to zeroth order hydrodynamics

We first consider the zeroth order in the derivative expansion where all sources are taken to be constant and of $\mathcal{O}(1)$. Using (4.57)-(4.62) we reduce the equations of motion to a set of algebraic constraints. Before attempting to solve this set of constraints, we will first limit the space of solutions by analyzing the behavior of the Ricci scalar $\mathcal{R}$ near the horizon and demanding that it stays finite. At the horizon the Ricci scalar behaves as

$$R\,(r_h + \epsilon) = -\frac{1}{\epsilon} \Big[ (\mathcal{K}_V)^\alpha k_\alpha + 2\,(\kappa_a)^{\alpha\beta} K_{\alpha\beta} + 3\,(h_0 - h_1) \Big] + \mathcal{O}\left(\epsilon^0\right) \,. \tag{4.70}$$

A particular solution reads

$$h_1 = -h_0 \,, \tag{4.71}$$

$$(\kappa_A)^{\alpha\beta} K_{\alpha\beta} = -\frac{1}{2} (\mathcal{K}_V)^\alpha k_a \,. \tag{4.72}$$

Clearly (4.71)-(4.72) is not the most general regular solution, but it is easy to show that this condition automatically implies the first law (4.42) of blackhole thermodynamics is satisfied. We found two classes of solutions compatible with (4.71)-(4.72)

- Vanishing $\{(\kappa_s)^{\mu\nu}, \kappa, k^\mu, (\kappa_A)^{\mu\nu}\}$ and independent sources $\{(\mathcal{K}_T)^{\lambda\mu\nu}, \mathcal{K}_V^\mu, \left((\mathcal{K}_A)^{\lambda\mu\nu}, K^{\mu\nu}\right\}$. We will refer to this solution as the Scalar-Vector-Tensor solution. The reason for this is that in 3+1 dimensions the sources $(\mathcal{K}_A)^{\lambda\mu\nu}$, $(\mathcal{K}_T)^{\lambda\mu\nu}$ and $K^{\mu\nu}$ can be dualized into a scalar, a symmetric traceless rank 2 tensor, and an axial vector respectively.

- Vanishing $\{(\kappa_S)^{\mu\nu}, (\mathcal{K}_T)^{\lambda\mu\nu}, \kappa, k^\mu, (\mathcal{K}_V)^\mu, (\mathcal{K}_A)^{\lambda\mu\nu}, K^{\mu\nu}\}$ and the single independent source $(\kappa_A)^{\mu\nu}$. We will refer to this solution as the Axial solution. Once again, the reason for this name is that $(\kappa_A)^{\mu\nu}$ can be dualized into an axial vector.

Below we analyze these two solutions by computing their zeroth order constitutive relations and determining their thermodynamic behavior.

### 4.6.1 Scalar-Vector-Tensor solution

For sake of clarity we present the explicit form of the metric and the torsion with non-vanishing scalar-vector-tensor in a convenient gauge[11] $f^2 = 1 - \frac{r_h^2}{r^2}$.

$$
\begin{aligned}
ds^2 &= -\frac{dr^2}{r^2 g^2} + r^2\left(-f^2 u_\mu u_\nu + h^2 \Delta_{\mu\nu}\right) dx^\mu dx^\nu\,,\\
f^2 &= 1 - \frac{r_h^2}{r^2}\,,\\
g^2 &= \left(1 - \frac{r_h^2}{r^2}\right)\left(1 - \frac{r_h^2 - 4\pi^2 T^2}{r^2}\right)\,,\\
h^2 &= 1 - \frac{r_h^2 - 4\pi^2 T^2}{r^2}\,,\\
T^a &= r h \theta^a_\mu \left[K^\mu{}_\rho u_\sigma - \frac{(\mathcal{K}_V)_\rho}{4}\Delta^\mu_\sigma - (\mathcal{K}_A)_{\rho\sigma}{}^\mu - \frac{(\mathcal{K}_T)_{\rho\sigma}{}^\mu}{2}\right] dx^\rho \wedge dx^\sigma\,,\\
T^5 &= 0\,.
\end{aligned}
\tag{4.73}
$$

[11] This gauge puts the blackhole solution in the familiar form in higher derivative gravity, e.g. the Gauss-Bonnet gravity with Gauss-Bonnet coupling set to $\lambda_{GB} = 1/4$, see for example [107, 111, 112].

The energy momentum tensor of the holographic dual fluid follows from (4.36) as

$$
\begin{aligned}
T^{\mu\nu} = 32\pi^4\kappa_{CS}T^4 \Bigg[ & \left(1 + \frac{3\left(\mathcal{K}_T\right)_{\alpha\rho\beta}\left(\mathcal{K}_T\right)^{\alpha\beta\rho} - \left(\mathcal{K}_A\right)_{\alpha\beta\rho}\left(\mathcal{K}_A\right)^{\alpha\beta\rho}}{12\pi^2T^2}\right)\left(4u^\mu u^\nu + \gamma^{\mu\nu}\right) \\
& + \frac{u^\mu\left(\left(\mathcal{K}_T\right)^{\nu\alpha\beta} + \left(\mathcal{K}_A\right)^{\nu\alpha\beta}\right)K_{\alpha\beta}}{2\pi^2T^2} + \frac{\left(\mathcal{K}_A\right)^{\mu}{}_{\alpha\beta}\left(\mathcal{K}_T\right)^{\nu\alpha\beta} - \left(\mathcal{K}_T\right)_{\alpha\beta}{}^{\nu}\left(\mathcal{K}_T\right)^{\mu\alpha\beta}}{2\pi^2T^2} \\
& - \frac{u^\mu u^\nu\left(16\left(\mathcal{K}_T\right)_{\alpha\rho\beta}\left(\mathcal{K}_T\right)^{\alpha\beta\rho} + \left(\mathcal{K}_V\right)^{\alpha}\left(\mathcal{K}_V\right)_{\alpha}\right) + \left(\mathcal{K}_V\right)^{\mu}\left(\mathcal{K}_V\right)^{\nu}}{32\pi^2T^2} \\
& - \frac{\left(\mathcal{K}_V\right)^{\alpha}\left(\left(\mathcal{K}_A\right)^{\mu\nu}{}_{\alpha} + \left(\mathcal{K}_T\right)^{\mu}{}_{\alpha}{}^{\nu} - \left(\mathcal{K}_T\right)^{\nu}{}_{\alpha}{}^{\mu} - u^\mu K^\nu{}_\alpha\right)}{8\pi^2T^2}\Bigg] ,
\end{aligned} \tag{4.74}
$$

The energy momentum is traceless and satisfies the usual equation of state $p = -\frac{\varepsilon}{3}$ for a conformal fluid. There exists a non-vanishing shear (traceless symmetric) component when $\left(\mathcal{K}_T\right)^{\lambda\mu\nu}$ and $\left(\mathcal{K}_V\right)^{\mu}$ are non vanishing. Also we observe that the energy momentum tensor is not symmetric unless $\{K^{\mu\nu}, \left(\mathcal{K}_V\right)^{\mu}\}$ or $\{K^{\mu\nu}, \left(\mathcal{K}_T\right)^{\lambda\mu\nu}, \left(\mathcal{K}_A\right)^{\lambda\mu\nu}\}$ vanish. Finally, we observe several novel transport coefficients associated to the spin sources appear in (4.74). The spin current in this holographic fluid follows from (4.37) as

$$
\begin{aligned}
S^{\lambda\mu\nu} = 8\pi^2\kappa_{CS}T^2 \Big[ & 4\left(\mathcal{K}_A\right)^{\lambda\mu\nu} - 2\left(\mathcal{K}_T\right)^{\mu\nu\lambda} - 2u^\lambda\left(\mathcal{K}_V\right)^{[\mu}u^{\nu]} \\
& + 4K^{\lambda[\mu}u^{\nu]} + \Delta^{\lambda[\mu}\left(\mathcal{K}_V\right)^{\nu]}\Big] .
\end{aligned} \tag{4.75}
$$

An interesting observation here is that, there exists a non-trivial spin current, due to the presence of non-trivial spin sources, even when the energy momentum tensor is symmetric.

Finally the thermodynamic potentials which correspond to the mass $M_0$, the axial charge $\mathcal{Q}_A^{\lambda\mu\nu}$, the vector charge $\mathcal{Q}_V^{\mu}$, and the tensor charge $\mathcal{Q}_T^{\lambda\mu\nu}$, the free energy density, and the entropy density of the holographic fluid are given by

$$
M_0 = 96\pi^4\kappa_{CS}T^4\left(1 - \frac{3\mu_{\text{eff}}^2}{2\pi^2T^2}\right) , \tag{4.76}
$$

$$
\mathcal{F}_{free} = -32\pi^4\kappa_{CS}T^4\left(1 - \frac{3\mu_{eff}^2}{2\pi^2T^2}\right) , \tag{4.77}
$$

$$
\mathcal{S}_{thermal} = 128\pi^4\kappa_{CS}T^3\left(1 - \frac{3\mu_{eff}^2}{4\pi^2T^2}\right) , \tag{4.78}
$$

$$
\begin{aligned}
(Q_A)^{\lambda\mu\nu} &= -16\pi^2\kappa_{CS}T^2\,(\mathcal{K}_A)^{\lambda\mu\nu}\ , && (4.79)\\
(Q_V)^{\mu} &= -2\pi^2\kappa_{CS}T^2\,(\mathcal{K}_V)^{\mu}\ , && (4.80)\\
(Q_T)^{\lambda\mu\nu} &= 16\pi^2\,(\mathcal{K}_T)^{\lambda\nu\mu}\ , && (4.81)
\end{aligned}
$$

where we defined an effective potential $\mu_{eff}$ through

$$
\mu_{eff}^2 \equiv \frac{1}{6}\,(\mathcal{K}_A)_{\lambda\mu\nu}\,(\mathcal{K}_A)^{\lambda\mu\nu} - \frac{1}{6}\,(\mathcal{K}_T)_{\lambda\mu\nu}\,(\mathcal{K}_T)^{\lambda\nu\mu} + \frac{1}{48}\,(\mathcal{K}_V)_{\alpha}\,(\mathcal{K}_V)^{\alpha}\ , \quad (4.82)
$$

which enters into the free energy and the entropy density. We note that the energy of the fluid, and the total mass of the blackhole agrees with $\varepsilon \equiv u_\mu u_\nu T^{\mu\nu} = M_0 - \mu_I Q^I$. Positivity of the thermal entropy requires $\mu_{eff} < \frac{4\pi^2T^2}{3}$ for $\kappa_{CS} > 0$ and $\mu_{eff} > \frac{4\pi^2T^2}{3}$ for $\kappa_{CS} < 0$. It can be easily verified that the thermodynamic potentials satisfy the first law of thermodynamics

$$
\begin{aligned}
d\mathcal{F}_{free} = &-\mathcal{S}_{thermal}dT - (Q_A)_{\lambda\mu\nu}\,d\,(\mathcal{K}_A)^{\lambda\mu\nu} - (Q_V)_{\mu}\,d\,(\mathcal{K}_V)^{\mu} \\
&- (Q_T)_{\lambda\mu\nu}\,d\,(\mathcal{K}_T)^{\lambda\mu\nu}\ .
\end{aligned} \quad (4.83)
$$

The thermodynamic stability of the solution requires positivity of the specific heat $c$, and the chemical susceptibilities $\chi$,

$$
\begin{aligned}
C_V &\equiv -T\frac{\partial^2\mathcal{F}_{free}}{\partial T^2} = 384\pi^4\kappa_{CS}T^3\left(1-\frac{\mu_{eff}^2}{4\pi^2T^2}\right) > 0\,, && (4.84)\\
\chi_A^{\lambda\rho\sigma\kappa\mu\nu} &\equiv \frac{\partial Q_A^{\lambda\rho\sigma}}{\partial\,(\mathcal{K}_A)^{\kappa\mu\nu}} = -96\pi^2\kappa_{CS}T^2\Delta^{\lambda\kappa}\Delta^{\rho\mu}\Delta^{\sigma\nu} > 0\,, && (4.85)\\
\chi_V^{\mu\nu} &\equiv \frac{\partial Q_V^{\mu}}{\partial\,(\mathcal{K}_V)^{\nu}} = -32\pi^2\kappa_{CS}T^2\Delta^{\mu\nu} > 0\,, && (4.86)\\
\chi_T^{\lambda\rho\sigma\kappa\mu\nu} &\equiv \frac{\partial Q_T^{\lambda\rho\sigma}}{\partial\,(\mathcal{K}_T)^{\kappa\mu\nu}} = 16\pi^2\kappa_{CS}T^2\Delta^{\lambda\kappa}\Delta^{\rho\mu}\Delta^{\sigma\nu} > 0\,. && (4.87)
\end{aligned}
$$

To analyze (4.84)-(4.87) we need to distinguish the two cases: $\kappa > 0$ and $\kappa < 0$. For $\kappa > 0$ we first observe that the positivity of the susceptibilities requires setting $(\mathcal{K}_A)^{\lambda\mu\nu} = (\mathcal{K}_V)^{\mu} = 0$. Then, from (4.82) we find $\mu_{eff}^2 > 0$ hence there are no further conditions that arise from positivity of the specific heat or positivity of the entropy. We conclude that, for $\kappa > 0$ the black hole with a tensor source is thermodynamically stable. For $\kappa < 0$, on the other hand, positivity susceptibilities require the vanishing of the tensor source. From positivity of the specific heat then we obtain an upper bound on the temperature $T < \mu_{eff}/2\pi$. Dynamical stability of the solutions is another question which can be settled by considering perturbations around the background and calculating the quasi-normal modes. We will no further investigate the dynamical stability of the solutions.

### 4.6.2 Single axial solution

Now we consider the second class of solutions to the constraint equations, outlined in the beginning of this section: the blackholes with a single axial charge. The explicit forms of the metric and the torsion, choosing once again the gauge $f^2 = 1 - \frac{r_h^2}{r^2}$, reads

$$\begin{aligned}
ds^2 &= -\frac{dr^2}{r^2 g^2} + r^2\left(-f^2 u_\mu u_\nu + h^2 \Delta_{\mu\nu}\right) dx^\mu dx^\nu \,, \\
f^2 &= 1 - \frac{r_h^2}{r^2}\,, \\
g^2 &= \left(1 - \frac{r_h^2}{r^2}\right)\left(1 - \frac{r_h^2 - 4\pi^2 T^2}{r^2}\right)\,, \\
h^2 &= 1 - \frac{r_h^2 - 4\pi^2 T^2}{r^2}\,, \\
T^a &= r\theta^a_\mu \left(\kappa_A\right)^\lambda{}_\rho \left[(f-2h) u_\lambda u^\mu u_\sigma - f\gamma^\mu_\lambda u_\sigma + h u^\mu \gamma_{\lambda\rho}\gamma_{\kappa\sigma}\right] dx^\rho \wedge dx^\sigma\,, \\
T^5 &= 0\,.
\end{aligned} \tag{4.88}$$

We find the following energy-momentum tensor

$$T^{\mu\nu} = 32\pi^4 \kappa_{CS} T^4 \left[4u^\mu u^\nu + \gamma^{\mu\nu} + \frac{\left(\kappa_A\right)^{\rho\sigma}\left(\kappa_A\right)_{\rho\sigma}\left(u^\mu u^\nu + \Delta^{\mu\nu}\right) - 2\left(\kappa_A\right)^{\mu\alpha}\left(\kappa_A\right)^\nu{}_\alpha}{4\pi^2 T^2}\right]\,, \tag{4.89}$$

We observe that the energy momentum tensor for the single axial fluid is traceless and symmetric with a shear component. The spin current follows from (4.37) as

$$S^{\lambda\mu\nu} = -32\pi^2 \kappa_{CS} T^2 \left[u^\lambda \left(\kappa_A\right)^{\mu\nu} + \left(\kappa_A\right)^{\lambda[\mu} u^{\nu]}\right]\,. \tag{4.90}$$

As in the previous example we note that there exists a non-trivial spin current sourced by the torsion even though the energy-momentum tensor is symmetric. As for the thermodynamics, there is an axial charge $(Q_A)^{\mu\nu}$ associated to $(\kappa_A)^{\mu\nu}$ as shown in the following thermodynamic potentials

$$M_0 = 96\pi^4 \kappa_{CS} T^4 \left(1 + \frac{(\kappa_A)_{\mu\nu}(\kappa_A)^{\mu\nu}}{4\pi^2 T^2}\right), \tag{4.91}$$

$$(Q_A)^{\mu\nu} = 16\pi^2 \kappa_{CS} T^2 (\kappa_A)^{\mu\nu}, \tag{4.92}$$

$$\mathcal{F}_{free} = -32\pi^4 \kappa_{CS} T^4 \left(1 + \frac{(\kappa_A)_{\mu\nu}(\kappa_A)^{\mu\nu}}{4\pi^2 T^2}\right), \tag{4.93}$$

$$\mathcal{S}_{thermal} = 128\pi^4 \kappa_{CS} T^3 \left(1 + \frac{(\kappa_A)_{\mu\nu}(\kappa_A)^{\mu\nu}}{8\pi^2 T^2}\right). \tag{4.94}$$

We again observe that the energy $\varepsilon = u_\mu u_\nu T^{\mu\nu}$ agrees with the total mass $M_0$. We also note that positivity of thermal entropy automatically discards the case $\kappa < 0$. The aforementioned thermodynamic potentials satisfy the first law in the form

$$d\mathcal{F}_{\text{free}} = -\mathcal{S}_{thermal} dT - (\kappa_A)_{\mu\nu} d(\kappa_A)^{\mu\nu}. \tag{4.95}$$

The stability of the solution is determined from the condition on the specific heat and the susceptibilities

$$C_V \equiv -T\frac{\partial^2 \mathcal{F}_{free}}{\partial T^2} = 384\pi^4 \kappa T^3 \left(1 + + \frac{(\kappa_A)_{\mu\nu}(\kappa_A)^{\mu\nu}}{12\pi^2 T^2}\right) > 0, \tag{4.96}$$

$$(\chi_A)^{\mu\nu\rho\sigma} = \frac{\partial (Q_A)^{\mu\nu}}{\partial (\kappa_A)^{\rho\sigma}} = 16\pi^2 \kappa T^2 \Delta^{\mu[\rho}\Delta^{\sigma]\nu} > 0, \tag{4.97}$$

We observe that the thermodynamic stability is guaranteed for $\kappa > 0$.

## 4.7 Solutions Dual to First Order Hydrodynamics

We now promote the hydrodynamic variables $u^\mu$ and $T$ as well as the spin sources ${\omega^{ab}}_\mu$ to slowly varying functions of the boundary coordinates to study hydrodynamic expansion at first order in derivatives. For this purpose we consider the generic solution in (4.57)-(4.62) to solve the constraint equations (4.20) up to first order in the derivative expansion. For simplicity we explicitly treat only two cases: non vanishing $(\mathcal{K}_A)^{\lambda\mu\nu}$ and non vanishing $(\mathcal{K}_V)^{\mu}$. This is sufficiently rich to explore the spin dependent transport in our holographic model. For all cases we can choose the gauge $f^2 = 1 - \frac{r_h^2}{r^2}$ as before. In this gauge the generic solution for the metric at first order can be written as

$$\begin{aligned}
ds^2 &= \frac{dr^2}{r^2 g^2} + r^2\left[-f^2 u_\mu u_\nu + h^2 \Delta_{\mu\nu} + j\left(2hc_I u_\mu v^I_\nu - 2fb_I u_\mu v^I_\nu + hd_I t^I_{\mu\nu}\right)\right] dx^\mu dx^\nu\,,\\
f^2 &= 1 - \frac{r_h^2}{r^2}\,,\\
g^2 &= \left(1 - \frac{r_h^2}{r^2}\right)\left(1 - \frac{r_h^2 - 4\pi^2 T^2}{r^2}\right),\\
h^2 &= 1 - \frac{r_h^2 - 4\pi^2 T^2}{r^2}\,,\\
j &= \frac{1}{2h_0}\left[\sqrt{1 - \frac{r_h^2}{r^2}} - \sqrt{1 - \frac{r_h^2 - 4\pi^2 T^2}{r^2}}\right],
\end{aligned} \tag{4.98}$$

where $\{c_I v^I_\mu, b_I v^I_\mu\}$ denotes two distinct combinations of linearly independent vectors with a single derivative of any of the hydrodynamic $(u^\mu, T)$ or spin sources $\omega^{ab}{}_\mu$, and $\{d_I t^I_{\mu\nu}\}$ the same for linearly independent tensors. To obtain the solution we used the regularity of the metric determinant to set $F = -rf$ in the metric ansatz (4.51) and (4.53). The corresponding torsion two form at first order for non-vanishing $\{(\mathcal{K}_A)^{\lambda\mu\nu}, (\mathcal{K}_V)^{\mu}, (\mathcal{K}_T)^{\lambda\mu\nu}, (\kappa_A)^{\mu\nu}\}$ is given by

$$\begin{aligned}
T^a = \theta^a_\alpha\Bigg[& r\left(2u^{[\alpha}(\kappa_A)^{\beta]}{}_\mu - \frac{1}{2}(\mathcal{K}_V)^{[\alpha}\Delta^{\beta]}_\mu - (\mathcal{K}_A)^{\alpha\beta}{}_\mu - (\mathcal{K}_T)^{\alpha\beta}{}_\mu\right) \\
&\left(-fu_\beta u_\nu + h\Delta_{\beta\nu} + ju_\beta b_I v^I_\nu + ju_\nu c_I v^I_\beta + jd_I t^I_{\beta\mu}\right) + \frac{\Delta^\alpha_\nu \partial_\mu h_0}{2rh} + 4h_0 rju_{(\mu}\partial_{\nu}u_{\alpha)} \\
&- \frac{hu_\alpha u_\mu - f\Delta_{\alpha\mu} + j\left(u_\mu c_I v^I_\alpha + u_\alpha b_I v^I_\mu\right) + d_I t^I_{\mu\alpha}}{2rh(f+h)}\partial_\nu h_0\Bigg] dx^\mu \wedge dx^\nu\,,
\end{aligned} \tag{4.99}$$

$$T^5 = \frac{1}{2}\left[d_I t^I_{\mu\nu} + u_\mu(b_I - d_I)v^I_\nu\right] dx^\mu \wedge dx^\nu\,. \tag{4.100}$$

The regularity of the Ricci scalar at the horizon can be written as

$$d_I t^{I\mu}_\mu = -a_\mu (\mathcal{K}_V)^\mu + 2\Omega_{\mu\nu}(\kappa_A)^{\mu\nu}\,. \tag{4.101}$$

Below we derive the constraint equations to be satisfied by the sources and the integration constants $\{b_I v^I_\mu, c_I v^I_\mu, d_I t^I_{\mu\nu}\}$ for the particular solutions.

### 4.7.1 Non-vanishing $(\mathcal{K}_A)^{\lambda\mu\nu}$

The solution at first order in derivatives is found to be

$$b_I = c_I\,, \tag{4.102}$$

$$c_I v^I_\nu = \frac{4\pi^2 T^2}{\epsilon_{\alpha\beta\mu\nu} u^\alpha \left(\mathcal{K}_A\right)^{\beta\mu\nu}} \tilde{\Omega}_\nu\,, \tag{4.103}$$

$$d_I t^I_{\mu\nu} = 0\,, \tag{4.104}$$

$$\theta = 0\,, \tag{4.105}$$

$$a^\mu = 0\,, \tag{4.106}$$

$$\partial_\mu T = 0\,, \tag{4.107}$$

$$u^\alpha \partial_\alpha \left(\mathcal{K}_A\right)^{\lambda\mu\nu} = 0\,. \tag{4.108}$$

The energy momentum tensor and spin current are found to be

$$\begin{aligned} T^{\mu\nu} = 32\pi^4 \kappa_{CS} T^4 \Bigg[ & \left(1 - \frac{\left(\mathcal{K}_A\right)^{\alpha\beta\lambda} \left(\mathcal{K}_A\right)_{\alpha\beta\lambda}}{12\pi^2 T^2}\right) \left(4u^\mu u^\nu + \gamma^{\mu\nu}\right) \\ & - \frac{4u^{(\mu}\tilde{\Omega}^{\nu)}}{\epsilon_{\alpha\beta\lambda\kappa} u^\alpha \left(\mathcal{K}_A\right)^{\beta\lambda\kappa}} + \frac{2u^\nu \left(\mathcal{K}_A\right)^\mu{}_{\alpha\beta}\Omega^{\alpha\beta} + \partial_\alpha \left(\mathcal{K}_A\right)^{\mu\nu\alpha}}{2\pi T^2} \Bigg]\,, \end{aligned} \tag{4.109}$$

$$S^{\lambda\mu\nu} = 32\pi^2 T^2 \left[ \left(\mathcal{K}_A\right)^{\lambda\mu\nu} + 2u^\lambda \Omega^{\mu\nu} + 4u^{[\mu}\Omega^{\nu]\lambda} \right]\,. \tag{4.110}$$

We note the energy-momentum tensor is generically non-symmetric and traceless. The spin current (4.110) is totally antisymmetric so it can be dualized into an axial current $J_A$ given

$$J^\lambda_A = -32\pi^2 \kappa_{CS} T^2 \left[ \epsilon^{\rho\sigma\mu\nu} u_\sigma \left(\mathcal{K}_A\right)_{\rho\mu\nu} u^\lambda + \tilde{\Omega}^\lambda \right]\,. \tag{4.111}$$

This axial current comprises a charge density $\rho_A = 32\pi^2\kappa_{CS} T^2 \epsilon^{\rho\sigma\mu\nu} u_\sigma \left(\mathcal{K}_A\right)_{\rho\mu\nu}$ and a linear response proportional to the vorticity with coefficient $32\pi^2\kappa_{CS}T^2$. This last contribution has the same form as the known chiral separation vortical effect (CVSE) [113,114] that is typically associated to anomalous transport in chiral fluids. We note however it resembles more the chiral torsional effect [115] where an axial current is generated due to the presence of defects. The appearance of vorticity as a source of the spin current in (4.110) is interesting, implying magnetization by rotation, akin to the Barnett effect [116].

### 4.7.2 Non vanishing $(\mathcal{K}_V)^\lambda$

Next, we work out an example with a non-vanishing vector-like spin source. The solution to (4.20) for this choice of sources is

$$b_I = c_I \,, \tag{4.112}$$

$$c_I v^I_\nu = \frac{16\pi^2 T^2 \left(\theta \left(\mathcal{K}_V\right)_\nu - 2\Omega_{\nu\rho} \left(\mathcal{K}_V\right)^\rho\right)}{\left(\mathcal{K}_V\right)_\alpha \left(\mathcal{K}_V\right)^\alpha} \,, \tag{4.113}$$

$$t^I_{\mu\nu} = 0 \,, \tag{4.114}$$

together with the flow solution

$$\theta = \frac{u^\alpha \partial_\alpha \left(\mathcal{K}_V\right)^2}{64\pi^2 T^2 - \left(\mathcal{K}_V\right)^2} \,, \tag{4.115}$$

$$\sigma^{\mu\nu} = \frac{\Theta}{6}\left(\Delta^{\mu\nu} - \frac{3\left(\mathcal{K}_V\right)^\mu \left(\mathcal{K}_V\right)^\nu}{\left(\mathcal{K}_V\right)^2}\right) \,, \tag{4.116}$$

$$a^\mu = 0 \,, \tag{4.117}$$

$$\partial_\alpha T = 0 \,. \tag{4.118}$$

We note that whenever the vector source $(\mathcal{K}_V)^\mu$ is time independent, both the compressibility and the shear tensor vanish. Using (4.36) we arrive at the following energy momentum tensor

$$\begin{aligned} T^{\mu\nu} = 32\pi^4 \kappa_{CS} T^4 \Bigg[ & 4u^\mu u^\nu + \gamma^{\mu\nu} - \frac{\left(\mathcal{K}_V\right)^\mu \left(\mathcal{K}_V\right)^\nu + u^\mu u^\nu \left(\mathcal{K}_V\right)^2}{32\pi^2 T^2} \\ & + \frac{u^\mu \Omega^{\nu\kappa} \left(\mathcal{K}_V\right)_\kappa}{8\pi^2 T^2} + \frac{8u^{(\mu}\Omega^{\nu)\kappa} \left(\mathcal{K}_V\right)_\kappa}{\left(\mathcal{K}_V\right)^2} + \left(\frac{\Delta^{\mu[\alpha}\Delta^{\nu]}_\beta + 2u^\mu u^{[\alpha}\Delta^{\nu]}_\beta}{4\pi^2 T^2}\right) \partial_\alpha \left(\mathcal{K}_V\right)^\beta \\ & + \left(\frac{\left(\mathcal{K}_V\right)^\mu u^\nu + 2u^\mu \left(\mathcal{K}_V\right)^\nu}{8\pi^2 T^2} - \frac{16 u^{(\mu} \left(\mathcal{K}_V\right)^{\nu)}}{\left(\mathcal{K}_V\right)^2}\right) \theta \Bigg] \,. \end{aligned} \tag{4.119}$$

We observe that the energy momentum tensor is not symmetric but it remains traceless. Using (4.37) the spin current is found to be

$$S^{\lambda\mu\nu} = 32\pi^2\kappa_{CS}T^2 \left[ \frac{\Delta^{\lambda[\mu}\left(\mathcal{K}_V\right)^{\nu]} + 2u^\lambda u^{[\mu}\left(\mathcal{K}_V\right)^{\nu]}}{4} - \Delta^{\lambda[\mu}u^{\nu]}\theta \right. \tag{4.120}$$
$$\left. + \frac{\left(\mathcal{K}_V\right)^\lambda \left(\mathcal{K}_V\right)^{[\mu} u^{\nu]}\theta}{\left(\mathcal{K}_V\right)^2} + \frac{2\left(\mathcal{K}_V\right)^\kappa \left(u^\lambda \Omega^{[\mu}{}_\kappa \left(\mathcal{K}_V\right)^{\nu]} + u^{[\mu}\left(\mathcal{K}_V\right)^{\nu]}\Omega^\lambda{}_\kappa\right)}{\left(\mathcal{K}_V\right)^2} \right],$$

A tensor-axial current is contained within the spin current from the projection

$$\begin{aligned} \bar{J}^\mu_A &= \frac{1}{2}\left(\epsilon_{\beta\lambda\nu\sigma}u_\alpha - \epsilon_{\alpha\beta\nu\sigma}u_\lambda\right) u^\sigma \gamma^{\mu\nu} S^{\lambda\alpha\beta} \\ &= -16\pi^2\kappa_{CS}T^2 \left(\gamma^{\mu\nu} - \frac{\left(\mathcal{K}_V\right)^\mu \left(\mathcal{K}_V\right)^\nu}{\left(\mathcal{K}_V\right)^2}\right)\tilde{\Omega}_\nu \,. \end{aligned} \tag{4.121}$$

Perhaps the most remarkable finding in this analysis is this last equation: a novel type of torsional anomalous transport in the direction proportional to the projection of vorticity transverse to the spin source vector.

# Chapter 5

# Discussion and Outlook

In this thesis we took some first steps towards a consistent and exhaustive treatment of relativistic hydrodynamics with spin degrees of freedom. The first step was to identify the relevant current associated to spin, the spin current $S^{\lambda\mu\nu}$, as well as providing a consistent definition in terms of its coupling to sources, background torsion. Just as with any current there will exist an associated canonical charge, which we denoted as the spin potential $\mu^{ab}$. This associated charge will be the dynamical degree of freedom whose equation of motion will correspond to the (non-)conservation equation of the spin current. Together with the temperature $T$ and the four velocity $u^{\mu}$ it will form a closed system of equations that describe the evolution of spin, temperature, and flow of the fluid of interest.

We will summarize the advances presented in this work

- *Belinfante-Rosenfeld (BR) symmetry*: The ambiguity when defining the spin current was resolved by the inclusion of sources. Adding torsion to the background breaks the BR symmetry and allows for an unambiguous treatment of the spin current. Torsion can be set to zero at the end of any physical computation but this way of computing the quantities will have observable effects in the expectation values of these observables.

- *Order of $\mu^{ab}$ and torsion on the derivative expansion*: It was argued that the most natural order in which both of these quantities should be treated is $\mathcal{O}(\partial)$. The possibility of having $\mathcal{O}(1)$ torsion is not discarded, but such theory of torsion hydrodynamics should be treated independently.

- *Hydrostatic limit*: The most general parity invariant hydrostatic partition function and hydrostatic constitutive relation, contributing up to $\mathcal{O}\left(\partial^2\right)$ to the equations of motion was derived, see (2.70). From here 15 equilibrium transport coefficients associated to the spin potential and the background torsion were identified, in addition to the hydrostatic ideal pressure $p$.

- *Non-equilibrium transport*: In addition to the bulk $\zeta$ and shear viscosities $\eta$ there are two additional $\mathcal{O}\left(\partial\right)$ non-equilibrium viscosities $\{\sigma_7, \sigma_8\}$ associated to the antisymmetric part of the energy momentum tensor, and six $\mathcal{O}\left(\partial\right)$ transport coefficients $\{\sigma_{1-6}\}$ associated to the spin current. The way we structured the constitutive relations (2.101) and (2.102) implies $\{\sigma_{1-6}\}$ will only contribute to the equations of motion in the presence of torsion. There are in principle additional $\mathcal{O}\left(\partial^2\right)$ transport coefficients associated to the antisymmetric piece of $T^{\mu\nu}$ that will contribute to the second order equations of motion. However their inclusion requires some extra term

  1. Independent spin degrees of freedom: When $\{\sigma_6, \sigma_7\}$ are non-vanishing the equations of motion imply that, up to $\mathcal{O}\left(\partial^2\right)$, the spin potentials take their equilibrium value and are completely determined by the vorticity and acceleration of the fluid. This implies the spin degrees of freedom are not independent. To have independent spin degrees of freedom it is needed that $\{\sigma_6, \sigma_7\}$ vanish or an alternative expansion counting is taken into account, see [44, 45] for an example of a novel counting proposal.
  2. $\mathcal{O}\left(\partial^2\right)$ transport coefficients: When $\{\sigma_6, \sigma_7\}$ are non-vanishing, and no additional scaling is assumed on the coefficients, it is possible to absorb all higher order transport coefficients associated to the antisymmetric part of the energy momentum tensor into the spin potential projections $\{m^a, M^{ab}\}$, namely we can remove all related higher order transport coefficients through a frame transformation. However on the independent spin limit, either from vanishing $\{\sigma_6, \sigma_7\}$ or when some exotic scaling is assumed, the second order transport coefficients cannot be removed and we are left with 49 spin related transport coefficients. It is on this independent spin limit where we analyzed a toy model of polarization on the quark gluon plasma, see chapter 3.

3. Entropy current analysis: Asking for the second law to hold up to $\mathcal{O}(\partial^2)$ constraints the shear and bulk viscosities to be positive. All other transport coefficients will contribute at a higher order. We can note that $\{\sigma_6, \sigma_7\}$ appear in a quadratic form that upon using the equations of motion contributes at $\mathcal{O}(\partial^4)$ to the divergence of the entropy current. This seems to suggest that $\{\sigma_6, \sigma_7\}$ should also be subject to a positivity constraint, however in principle a higher order analysis analogous to the one of [79] should be performed to verify this. There might be a simpler argument, that so far has escaped us, to impose positivity on those coefficients without having to do such higher order analysis.

- *Conformal symmetry*: We identified the conformal transformations for the energy momentum tensor and spin current by asking invariance of the conservation equations under such transformations. A consequence of the found transformations was that the energy momentum tensor does not transform uniformly under conformal transformations whenever the spin current is non-trivial.

- *Holography*: We showed with a particular example that the fluid gravity correspondence can be successfully used to compute the currents, and consequently the transport coefficients, of a theory with non-trivial spin degrees of freedom.

There are still many things to be done and questions to be answered regarding the hydrodynamic description of spin, to conclude this work we will list some of the future direction we want to explore

- Kubo formulas: From the constitutive relations presented in (2.101) and (2.102) it is straightforward, albeit involved, to compute the variation of the current on a flat background. Allowing us to compute the two point functions and ideally resulting in Kubo formulas for all transport coefficients involved.

- Holographic computation of viscosities: We already showed that a holographic computation of the spin current in holography is possible. A natural next step is to go beyond the CS toy model and perform a fluid gravity analysis to compute the spin transport coefficients on a more realistic dual theory. Alternatively, once the Kubo formulas are derived a holographic two point function analysis can be performed to compute these transport coefficients as well.

- Polarization in quark gluon plasma: We have shown, on a particular spin limit, that taking into account spin transport and solving, in a naive way, the hydrodynamic equations give a good match with data (see figure 3.1). A natural step from this is to go beyond the Bjorken approximation of the background flow and do a more robust computation of hyperion polarization.

- Torsion hydrodynamics and condensed matter: The constitutive relations we provided are valid even in the presence of torsion. Although in heavy ion collisions torsion is not present, we know that in some condensed matter systems it can appear as lattice defects [117–119]. The hydrodynamic equations in the presence of torsion might give further insights into this type of materials.

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# Samenvatting

In dit proefschrift worden aspecten van hydrodynamica op systemen met nontriviale spinvrijheidsgraden bestudeerd. We beginnen met het identificeren van de relevante spinvrijheidsgraden en hoe deze in de hydrodynamische expansie kunnen worden opgenomen. Nadat dit is bereikt passen we de constitutieve relaties aan zodat ze deze vrijheidsgraden meenemen, wat gedaan wordt op een manier die verenigbaar is met de tweede wet van de thermodynamica. Conforme symmetrie in de aanwezigheid van spinvrijheidsgraden is ook geanalyseerd en wordt gebruikt om een geïdealiseerd model op te schrijven dat op het quark-gluon plasma lijkt wat gegenereerd wordt in zware ionenbotsingen. Met dit model wordt de verwachte polarisatie van deeltjes na een dergelijke botsing uitgerekend en vergeleken met experimentele data. Tenslotte gebruiken we de vloeistof-zwaartekrachtcorrespondentie en een vijfdimensionaal gravitationeel versimpeld model om te illustreren hoe de nieuwe spingerelateerde transportcoëfficiënten van een sterk interagerende kwantumveldentheorie uitgerekend kan worden middels een duale gravitationele theorie.

# Appendix A

# Ideal fluid with spin

The generating function whose variation will give us constitutive relations with no explicit derivatives is given by

$$\mathcal{W} = P(T,\, u^\mu,\, \mu^{ab},\, e^a{}_\mu) + \mathcal{O}\left(\mathring{\nabla},\, K_\mu{}^{ab}\right)\,. \tag{A.1}$$

A fluid whose constitutive relations are obtained by varying the leading term on the right hand side of (A.1) is often referred to as an ideal fluid. Ideal fluids are fluids whose constitutive relations contain no explicit derivatives of the hydrodynamic variables. Note that we have phrased the previous sentence rather carefully. In most instances an equivalent definition of an ideal fluid is one whose constitutive relations are zeroth order in derivatives. We will see shortly that in our current scheme, these two definitions of an ideal fluid are not interchangeable.

From the projections $\{m^a, M^{ab}\}$ of the spin potential we find that the available scalars in a generic number of spacetime dimensions $d$ are given by contractions of chains of $M^{ab}$,

$$\mathcal{M}_{(n)}{}^a{}_b = M^a{}_{c_2} M^{c_2}{}_{c_2} \ldots M^{c_n}{}_b \qquad n \geq 1\,, \tag{A.2}$$

with a pair of $m_a$'s or with themselves:

$$\begin{aligned} m_{(n)} &= m^c \mathcal{M}_{(2n)}{}^{cd} m_d\,, \\ M_{(n)} &= \mathcal{M}_{(2n)}{}^c{}_c\,. \end{aligned} \tag{A.3}$$

For notational convenience it is useful to define

$$\mathcal{M}_{(0)}{}^a{}_b = \delta^a{}_b\,, \tag{A.4}$$

so that $m_{(0)} = m_c m^c$ and $M_{(0)} = d$ are well defined. The number of possible independent scalars is bound by the dimensionality of $M^{ab}$ and $m^a$. In a generic number, $d$, of spacetime dimensions the independent scalars are given by $m_{(0)}, \ldots, m_{(\lfloor \frac{d-2}{2} \rfloor)}$, and $M_{(0)}, \ldots, M_{(\lfloor \frac{d-1}{2} \rfloor)}$. Of course, in addition to (A.3) one may construct various dimension dependent pseudo scalars using the Levi Civita tensor. For example, in $3+1$ dimensions one may use,

$$\widetilde{M} = \epsilon^{abcd} u_a m_b M_{cd} \,. \tag{A.5}$$

Since $\widetilde{M}^2 = 4m_{(1)} - 2M_{(1)} m_{(0)}$ we should replace $m_{(1)}$ with $\widetilde{M}$. Unless stated otherwise, in what follows we will consider contributions to the constitutive relations for a generic $d$ dimensional spacetime theory.

The dependence of the ideal stress tensor and current on the external fields can now be computed by varying $P(m_{(n)}, M_{(n)}, T^2)$ with respect to the vielbein or spin connection respectively. Inserting (A.1) into (2.6) and denoting the resulting stress tensor and spin current by $(T_{\text{ideal}})^{\mu\nu}$ and $(S_{\text{ideal}})^{\mu\nu\rho}$ we find

$$\begin{aligned} (T_{\text{ideal}})^{\mu\nu} &= \epsilon u^\mu u^\nu + P\Delta^{\mu\nu} + u^\mu \Delta^{\nu\alpha} P_\alpha \,, \\ (S_{\text{ideal}})^{\lambda}{}_{\mu\nu} &= u^\lambda \rho_{\mu\nu} \,, \end{aligned} \tag{A.6}$$

where $\epsilon$ is given by

$$\epsilon = -P + sT + \frac{1}{2}\rho_{\alpha\beta}\mu^{\alpha\beta} \,, \tag{A.7}$$

and $s$, $\rho_{ab}$ and $P_a$ are given by the variation of the pressure with respect to the variables $T$, $\mu^{ab}$ and $u^a$,

$$s = \frac{\partial P}{\partial T} \,, \qquad \frac{1}{2}\rho_{ab} = \frac{\partial P}{\partial \mu^{ab}} \,, \qquad P_a = \frac{\partial P}{\partial u^a} \,. \tag{A.8}$$

Their explicit dependence on derivatives of the pressure with respect to our basis of scalars is given by

$$\begin{aligned} \rho_{ab} = {}& 4 \sum_{n=0}^{\lfloor \frac{d-2}{2} \rfloor} \frac{\partial P}{\partial m_{(n)}} \left( m_c \mathcal{M}_{(2n)}{}^c{}_{[a} u_{b]} + \sum_{k=0}^{\lfloor \frac{2n-1}{2} \rfloor} m_c \mathcal{M}_{(k)}{}^c{}_{[a} \mathcal{M}_{(2n-1-k)b]}{}^d m_d \right) \\ & - 4 \sum_{n=1}^{\lfloor \frac{d-1}{2} \rfloor} n \frac{\partial P}{\partial M_{(n)}} \mathcal{M}_{(2n-1)ab} \,, \end{aligned} \tag{A.9}$$

$$P_a = -2 \sum_{n=0}^{\lfloor \frac{d-2}{2} \rfloor} \frac{\partial P}{\partial m_{(n)}} \left( m_{(n)} u_a - m_b \mathcal{M}_{(2n+1)}{}^b{}_a + \sum_{k=0}^{\lfloor \frac{2n-1}{2} \rfloor} m_b \mathcal{M}_{(2n-1-2k)}{}^b{}_a m_{(k)} \right)$$
$$- 4 \sum_{n=1}^{\lfloor \frac{d-1}{2} \rfloor} n \frac{\partial P}{\partial M_{(n)}} m_b \mathcal{M}_{(2n-1)}{}^b{}_a \,, , \tag{A.10}$$

where square brackets represent an antisymmetric combination, $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$. Note that the stress tensor is not symmetric due to the term proportional to (the transverse part of) $P^\mu$.

A few comments are in order. The relations given in (A.6) describe the dependence of the energy momentum tensor and spin current on the external sources which have been conveniently packaged into the variables, $T$, $\mu^{ab}$ and $u^a$ given in (2.40). Since these are hydrostatically equilibrated configurations the resulting stress tensor and current should also be expressible in terms of solutions to the hydrodynamic equations. This interpretation allows us to identify $T$ as the temperature, $u^\mu$ as the velocity field and $\mu^{ab}$ as the spin chemical potential. From this point of view, the relations (A.6) are the constitutive relations for an ideal fluid and (2.40) give the dependence of the hydrodynamic values on the external sources once hydrostatic equilibrium is achieved.

Once the hydrodynamic variables $T$, $u^\mu$ and $\mu^{ab}$ have been ascertained, we may identify $P$ with the pressure, $\epsilon$ with the energy density, $s$ with the entropy density, $\rho_{\alpha\beta}$ with the spin charge density and $P_a$ with what we refer to as a momentum density. With the above interpretation of thermodynamic variables, equation (A.7) can be interpreted as the Gibbs Duhem Relation relating energy density and pressure. This expression together with (A.8) gives us the first law of thermodynamics

$$d\epsilon = T ds + \frac{1}{2}\mu^{ab} d\rho_{ab} - P_a du^a \,. \tag{A.11}$$

From (A.11) (or (A.8)) we observe that the momentum density is the variable conjugate to the velocity field, $u^a$. Such a quantity is absent in relativistic thermodynamics as long as the velocity field is the only tensor which breaks Lorentz invariance. Thus, normal, charged and uncharged fluids, do not support such variable. The two component model of superfluids includes a superfluid velocity field which further breaks Lorentz invariance but since the superfluid flow is a gradient flow thermodynamic quantities are packaged in a somewhat different way than here. The instance we are aware of where the momentum density plays a thermodynamic role similar to the one in (A.11) is in non relativistic thermodynamics without boost invariance. See [120].

# Appendix B

# Thermal Entropy in Riemann-Cartan Spacetimes

An entropy formula and the corresponding first law for blackhole solutions in arbitrary theories of gravity carrying a metric and a Levi-Civita connection was derived in [108] by Lee and Wald. An analogous formula for gravitational theories described in vielbein formalism, albeit with vanishing torsion, was not addressed until much later[1] in [121–123]. In this appendix we derive the blackhole entropy formula for gravitational theories in the first order formalism of gravity, with independent vielbein and connection, by extending the approach in [123]. We first present a quick review of the necessary covariant phase space formalism — for a more complete review see [124–126] — and then we apply it to gravitational theories that we are interested in.

## B.1 Covariant phase space formalism

To start the discussion let us consider a local Lagrangian $\mathcal{L} = \mathcal{L}(\phi)$ depending on canonical fields $\{\phi\}$. Through the variation of the Lagrangian one obtains not only the equations of motion $E_\phi$ but also the so called symplectic potential $\theta(\delta\phi)$

$$\delta\mathcal{L} = E_\phi \delta\phi + d\theta(\delta\phi)\,. \tag{B.1}$$

[1] The complication arises from the internal degrees of freedom of the vielbein associated to the Lorentz symmetry of the tangent space. See the discussion in [121] for a summary.

It is important to note that $\theta$ is not uniquely defined as any exact form $d\alpha$ could be added to it. We instead consider a class of symplectic potentials $\Theta$ defined by

$$\Theta = \theta + d\alpha \,, \tag{B.2}$$

where the form $\alpha$ will be used to fix the gauge invariance of the symplectic potential. The anti-symmetrized field variation of $\Theta$ defines a symplectic form $\Omega$ as

$$\Omega(\delta_1, \delta_2) = \delta_1\Theta - \delta_2\Theta \,. \tag{B.3}$$

For a symmetry parametrized by $\omega$, the symplectic structure allow us to define a Hamiltonian flow by

$$\delta H_\omega = \int_\Sigma \Omega\left(\delta, \delta_\omega\right) \,, \tag{B.4}$$

where $\Sigma$ denotes the spacetime manifold. For a diffeomorphism $\xi$ we expect all linear variations $\delta_\xi$ to vanish, as this is a symmetry of the theory, implying that the Hamiltonian flow should also vanish, namely $\delta H_\xi = 0$. In Riemann-Cartan spacetimes there is, in addition to diffeomorphisms, local Lorentz symmetry parametrized by $\lambda^{AB}$, so we will equivalently ask for absence of an associated charge by demanding $\delta H_\lambda = 0$. This allows us to fix $\alpha$.

It is also possible to use the symplectic potential to define a Noether current associated to diffeomorphisms $\xi$. This can be done by noting that, if we define this current $J_\xi$ as

$$J_\xi = \Theta(\delta_\xi) - \mathcal{L} \cdot \xi \,, \tag{B.5}$$

then $dJ = 0$ on shell using (B.1). This implies we should be able to write the current as an exact form

$$J_\xi = dQ_\xi \,, \tag{B.6}$$

where $Q_\xi$ will be the Noether charge. The Noether charge and the Hamiltonian flow for a diffeomorphism can easily be related by noticing that

$$\Omega(\delta, \delta_\xi) = d\left(\delta Q_\xi - \Theta(\delta) \cdot \xi\right) \,, \tag{B.7}$$

which implies the Hamiltonian flow is given by a boundary term

$$\delta H_\xi = \int_{\partial\Sigma} \left[\delta Q_\xi - \Theta(\delta)\cdot\xi\right] , \tag{B.8}$$

whenever $\Theta\cdot\xi = \delta B$ with $B$ being some differential form. We then say the theory is integrable. We are concerned with theories where $\partial\Sigma$ is formed by a Killing horizon, i.e. $\xi = 0$ on this surface, and with an asymptotic boundary at $\infty$. Taking this into account together with the vanishing of the Hamiltonian flow we are left with

$$\int_{\partial\Sigma_h} \delta Q^h_\xi = \int_{\partial\Sigma_\infty} \left[\delta Q^\infty_\xi - \Theta^\infty(\delta)\cdot\xi\right] . \tag{B.9}$$

This equation is the first law of thermodynamics when the left hand side identified with $T_0\delta S_{\text{thermal}}$. We now proceed to calculate the relevant Noether charge.

## B.2 Gauge Invariant symplectic potential

To compute $Q_\xi$ it is necessary to find a form $\alpha$ such that $\delta H_\lambda = 0$. As in [123] we rather proceed via a simpler road by requiring for the symplectic potential itself to be invariant, namely $\Theta(\delta_\lambda) = 0$. For completeness, we show below the relevant local Lorentz transformations for the coframes and connection [2]

$$\begin{aligned} \delta_\lambda e^A &= -\lambda^A{}_F \hat{e}^F , \\ \delta_\lambda \omega^{AB} &= D\lambda^{AB} , \end{aligned} \tag{B.10}$$

We now consider gravitational Lagrangians that are functions of both the coframes $e^A$ and the connection $\omega^{AB}$. In particular we expect the Lagrangian to depend on the local Lorentz covariant forms $\{e^A, T^A, R^{AB}\}$, i.e. $\mathcal{L} = \mathcal{L}\,(e, T, R)$.

The corresponding variation of the Lagrangian can then be written as

$$\delta\mathcal{L} = \delta e^A E_A + \delta\omega^{AB} E_{AB} + d\Theta(\delta e, \delta\omega) , \tag{B.11}$$

[2] Note that at this point we are working in arbitrary dimensions and $A$ represent D-dimensional indices, not necessarily the 5D ones.

with $E_A$, $E_{AB}$ the equations of motion and $\Theta$ the symplectic potential given by

$$\begin{aligned} E_A &= \frac{\partial \mathcal{L}}{\partial e^A} + D\left(\frac{\partial \mathcal{L}}{\partial T^A}\right) , \\ E_{AB} &= e_B \frac{\partial \mathcal{L}}{\partial T^A} + D\left(\frac{\partial \mathcal{L}}{\partial R^{AB}}\right) , \\ \Theta(\delta e, \delta\omega) &= \delta e^A \frac{\partial \mathcal{L}}{\partial T^A} + \delta\omega^{AB} \frac{\partial \mathcal{L}}{\partial R^{AB}} + d\alpha \, . \end{aligned} \tag{B.12}$$

We obtain

$$\begin{aligned} \int_\Sigma \Theta\left(\delta_\lambda\right) &= \int_\Sigma \left[-\lambda^A{}_F e^F \frac{\partial \mathcal{L}}{\partial T^A} + D\lambda^{AB}\frac{\partial \mathcal{L}}{\partial R^{AB}} + d\alpha(\delta_\lambda)\right] \\ &= -\int_\Sigma \lambda^{AB}\left[e_B \frac{\partial \mathcal{L}}{\partial T^A} + D\left(\frac{\partial \mathcal{L}}{\partial R^{AB}}\right)\right] + \int_{\partial\Sigma}\left[\lambda^{AB}\frac{\partial \mathcal{L}}{\partial R^{AB}} + \alpha(\delta_\lambda)\right] \\ &= -\int_\sigma \lambda^{AB} E_{AB} + \int_{\partial\Sigma}\left[\lambda^{AB}\frac{\partial \mathcal{L}}{\partial R^{AB}} + \alpha(\delta_\lambda)\right] . \end{aligned} \tag{B.13}$$

Requiring on-shell gauge invariance of the potential fixes $\alpha$ as

$$\alpha(\delta) = -\left(e^{a\mu}\delta e^b_\mu\right)\frac{\partial \mathcal{L}}{\partial R^{ab}} \, . \tag{B.14}$$

We note that the form of $\alpha$ is valid for any theory in a Riemann-Cartan spacetime. To calculate the Noether charge we will need to specify a particular Lagrangian.

## B.3 Noether Charge in 5D Lovelock Gravity

We consider a (non necessarily Chern-Simons) 5D Lovelock Lagrangian characterized by the action

$$S = \int \epsilon_{ABCDF}\left[c_1 R^{AB}R^{CD}e^F + \frac{c_2}{3}R^{AB}e^C e^D e^F + \frac{c_3}{5}e^A e^B e^C e^D e^F\right] , \tag{B.15}$$

with free parameters $\{c_1, c_2, c_3\}$. The equations of motion for this action read

$$\begin{aligned} E_A &= \epsilon_{ABCDF}\left[c_1 R^{BC}R^{DF} + c_2 R^{BC}e^D e^F + c_3 e^B e^C e^D e^F\right] \\ E_{AB} &= \epsilon_{ABCDF}\left[2c_1 R^{CD} + c_2 e^C e^D\right] T^F , \end{aligned} \tag{B.16}$$

with the symplectic potential $\Theta$

$$\Theta = \epsilon_{ABCDF}\left(2c_1 R^{AB}e^C + \frac{c_2}{3}e^A e^B e^C\right)\delta\omega^{DF}\,, \tag{B.17}$$

and the symplectic current

$$\begin{aligned} J = d&\left[\epsilon_{ABCDF}\left(2c_1 R^{AB} + \frac{c_2}{3}e^A e^B e^C\right)\left(e^{D\mu}T^F_{\mu\nu}\xi^\nu + D^D\xi^F\right)\right] \\ &+ E^{AB}\left(\omega_{AB}\cdot\xi\right) + E^A\left(e_A\cdot\xi\right)\,. \end{aligned} \tag{B.18}$$

We can now read off the Noether charge and evaluate it at the horizon, by noting that $D^{[F}\xi^{F]} \to n^{DF}$. This yields the following expression for the entropy

$$S = 2\pi\int_{M_3}\left[n^{AB}\frac{\partial\mathcal{L}}{\partial R^{AB}}\right] = 2\pi\int_{M_3}\epsilon_{ABCDF}n^{AB}\left[2c_1 R^{AB} + \frac{c_2}{3}e^A e^B e^C\right]\,. \tag{B.19}$$

with the $2\pi$ a normalization factor. Setting $c_1 = 1$ and $c_2 = 2$ we find the result of (4.40).

# Bibliography


[1] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics, Second Edition: Volume 6 (Course of Theoretical Physics)*, Course of theoretical physics / by L. D. Landau and E. M. Lifshitz, Vol. 6. Butterworth-Heinemann, 2 ed., Jan., 1987.

[2] L. Onsager, *Reciprocal relations in irreversible processes. i.*, *Phys. Rev.* **37** (1931) 405.

[3] L. Onsager, *Reciprocal relations in irreversible processes. ii.*, *Phys. Rev.* **38** (1931) 2265.

[4] F. Becattini, I. Karpenko, M. A. Lisa, I. Upsal and S. A. Voloshin, *Global hyperon polarization at local thermodynamic equilibrium with vorticity, magnetic field, and feed-down*, *Phys. Rev. C* **95** (2017) 054902.

[5] STAR collaboration, *Global $\Lambda$ hyperon polarization in nuclear collisions: evidence for the most vortical fluid*, *Nature* **548** (2017) 62 [1701.06657].

[6] STAR collaboration, *Global polarization of $\Lambda$ hyperons in Au+Au collisions at $\sqrt{s_{NN}}$ = 200 GeV*, *Phys. Rev. C* **98** (2018) 014910 [1805.04400].

[7] STAR Collaboration collaboration, *Global polarization measurement in au+au collisions*, *Phys. Rev. C* **76** (2007) 024915.

[8] D. A. Bandurin, I. Torre, R. K. Kumar, M. Ben Shalom, A. Tomadin, A. Principi et al., *Negative local resistance caused by viscous electron backflow in graphene*, *Science* **351** (2016) 1055 [https://science.sciencemag.org/content/351/6277/1055.full.pdf].

[9] J. Crossno, J. K. Shi, K. Wang, X. Liu, A. Harzheim, A. Lucas et al., *Observation of the dirac fluid and the breakdown of the wiedemann-franz law in graphene*, Science **351** (2016) 1058 [https://science.sciencemag.org/content/351/6277/1058.full.pdf].

[10] L. Savary and L. Balents, *Quantum spin liquids: a review*, Rept. Prog. Phys. **80** (2017) 016502 [1601.03742].

[11] R. Takahashi et al., *Spin hydrodynamic generation*, Nature **12** (2016) 52.

[12] R. J. Doornenbal, M. Polini and R. A. Duine, *Spin–vorticity coupling in viscous electron fluids*, Journal of Physics: Materials **2** (2019) 015006.

[13] S. A. Voloshin, *Polarized secondary particles in unpolarized high energy hadron-hadron collisions?*, nucl-th/0410089.

[14] N. Armesto, N. Borghini, S. Jeon, U. A. Wiedemann, S. Abreu, S. V. Akkelin et al., *Heavy-ion collisions at the LHC—last call for predictions*, Journal of Physics G: Nuclear and Particle Physics **35** (2008) 054001.

[15] F. Becattini, F. Piccinini and J. Rizzo, *Angular momentum conservation in heavy ion collisions at very high energy*, Phys. Rev. C **77** (2008) 024906 [0711.1253].

[16] B. Betz, M. Gyulassy and G. Torrieri, *Polarization probes of vorticity in heavy ion collisions*, Phys. Rev. C **76** (2007) 044901 [0708.0035].

[17] F. Becattini, V. Chandra, L. Del Zanna and E. Grossi, *Relativistic distribution function for particles with spin at local thermodynamical equilibrium*, Annals Phys. **338** (2013) 32 [1303.3431].

[18] W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski and E. Speranza, *Spin-dependent distribution functions for relativistic hydrodynamics of spin-1/2 particles*, Phys. Rev. D **97** (2018) 116017 [1712.07676].

[19] W. Florkowski, B. Friman, A. Jaiswal and E. Speranza, *Relativistic fluid dynamics with spin*, Phys. Rev. C **97** (2018) 041901 [1705.00587].

[20] F. Becattini, W. Florkowski and E. Speranza, *Spin tensor and its role in non-equilibrium thermodynamics*, Phys. Lett. B **789** (2019) 419 [1807.10994].

[21] F. Becattini and M. A. Lisa, *Polarization and Vorticity in the Quark–Gluon Plasma*, Ann. Rev. Nucl. Part. Sci. **70** (2020) 395 [2003.03640].

[22] F. Becattini, *Polarization in relativistic fluids: A quantum field theoretical derivation*, in *Strongly Interacting Matter under Rotation*, F. Becattini, J. Liao and M. Lisa, eds., (Cham), pp. 15–52, Springer International Publishing, 2021, DOI.

[23] F. Becattini, *Does the spin tensor play any role in non-gravitational physics?*, Nucl. Phys. A **1005** (2021) 121833 [2003.01406].

[24] R. Singh, G. Sophys and R. Ryblewski, *Spin polarization dynamics in the Gubser-expanding background*, Phys. Rev. D **103** (2021) 074024 [2011.14907].

[25] S. Bhadury, J. Bhatt, A. Jaiswal and A. Kumar, *New developments in relativistic fluid dynamics with spin*, Eur. Phys. J. ST **230** (2021) 655 [2101.11964].

[26] W. Florkowski and R. Ryblewski, *On the interpretation of $\Lambda$ spin polarization measurements*, 2102.02890.

[27] S. Floerchinger and E. Grossi, *Conserved and non-conserved Noether currents from the quantum effective action*, 2102.11098.

[28] R. Singh, M. Shokri and R. Ryblewski, *Spin polarization dynamics in the Bjorken-expanding resistive MHD background*, Phys. Rev. D **103** (2021) 094034 [2103.02592].

[29] A. Das, W. Florkowski, R. Ryblewski and R. Singh, *Pseudogauge dependence of quantum fluctuations of the energy in a hot relativistic gas of fermions*, Phys. Rev. D **103** (2021) L091502 [2103.01013].

[30] N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang and D. H. Rischke, *Derivation of the nonlocal collision term in the relativistic Boltzmann equation for massive spin-1/2 particles from quantum field theory*, Phys. Rev. D **104** (2021) 016022 [2103.04896].

[31] F. Becattini, M. Buzzegoli, A. Palermo, G. Inghirami and I. Karpenko, *Local polarization and isothermal local equilibrium in relativistic heavy ion collisions*, `2103.14621`.

[32] E. Speranza, F. S. Bemfica, M. M. Disconzi and J. Noronha, *Challenges in Solving Chiral Hydrodynamics*, `2104.02110`.

[33] K. J. Gonçalves and G. Torrieri, *Vector resonances spin alignement as a probe of spin hydrodynamics and freeze-out*, `2104.12941`.

[34] E. Sonin, *Spin currents and spin superfluidity*, *Advances in Physics* **59** (2010) 181 [`https://doi.org/10.1080/00018731003739943`].

[35] J. König, M. C. Bønsager and A. H. MacDonald, *Dissipationless spin transport in thin film ferromagnets*, *Phys. Rev. Lett.* **87** (2001) 187202.

[36] W. Chen and M. Sigrist, *Dissipationless multiferroic magnonics*, *Phys. Rev. Lett.* **114** (2015) 157203.

[37] S. Takei and Y. Tserkovnyak, *Superfluid spin transport through easy-plane ferromagnetic insulators*, *Phys. Rev. Lett.* **112** (2014) 227201.

[38] S. Takei, B. I. Halperin, A. Yacoby and Y. Tserkovnyak, *Superfluid spin transport through antiferromagnetic insulators*, *Phys. Rev. B* **90** (2014) 094408.

[39] F. Belinfante, *On the current and the density of the electric charge, the energy, the linear momentum and the angular momentum of arbitrary fields*, *Physica* **7** (1940) 449.

[40] L. Rosenfeld, *Sur le tenseur D'Impulsion-Energie*, *Acad. Roy. Belg. Memoirs de Classes de Science* **18** (1940) 063504.

[41] T. W. B. Kibble, *Lorentz invariance and the gravitational field*, *Journal of Mathematical Physics* **2** (1961) 212 [`https://doi.org/10.1063/1.1703702`].

[42] D. W. SCIAMA, *The physical structure of general relativity*, *Rev. Mod. Phys.* **36** (1964) 463.

[43] Y. N. Obukhov and V. A. Korotky, *The weyssenhoff fluid in einstein-cartan theory*, *Classical and Quantum Gravity* **4** (1987) 1633.

[44] S. Li, M. A. Stephanov and H.-U. Yee, *Non-dissipative second-order transport, spin, and pseudo-gauge transformations in hydrodynamics*, 2011.12318.

[45] M. Hongo, X.-G. Huang, M. Kaminski, M. Stephanov and H.-U. Yee, *Relativistic spin hydrodynamics with torsion and linear response theory for spin relaxation*, 2107.14231.

[46] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, *Towards hydrodynamics without an entropy current*, *Phys. Rev. Lett.* **109** (2012) 101601 [1203.3556].

[47] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, *Constraints on Fluid Dynamics from Equilibrium Partition Functions*, *JHEP* **09** (2012) 046 [1203.3544].

[48] S. Jain and T. Sharma, *Anomalous charged fluids in 1+1d from equilibrium partition function*, *JHEP* **01** (2013) 039 [1203.5308].

[49] K. Jensen, *Triangle Anomalies, Thermodynamics, and Hydrodynamics*, *Phys. Rev. D* **85** (2012) 125017 [1203.3599].

[50] M. Valle, *Hydrodynamics in 1+1 dimensions with gravitational anomalies*, *JHEP* **08** (2012) 113 [1206.1538].

[51] S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, *Constraints on Superfluid Hydrodynamics from Equilibrium Partition Functions*, *JHEP* **01** (2013) 040 [1206.6106].

[52] N. Banerjee, S. Dutta, S. Jain, R. Loganayagam and T. Sharma, *Constraints on Anomalous Fluid in Arbitrary Dimensions*, *JHEP* **03** (2013) 048 [1206.6499].

[53] K. Jensen, R. Loganayagam and A. Yarom, *Thermodynamics, gravitational anomalies and cones*, *JHEP* **02** (2013) 088 [1207.5824].

[54] J. Bhattacharya, S. Bhattacharyya and M. Rangamani, *Non-dissipative hydrodynamics: Effective actions versus entropy current*, *JHEP* **02** (2013) 153 [1211.1020].

[55] C. Eling, Y. Oz, S. Theisen and S. Yankielowicz, *Conformal Anomalies in Hydrodynamics*, *JHEP* **05** (2013) 037 [1301.3170].

[56] J. Armas, *How Fluids Bend: the Elastic Expansion for Higher-Dimensional Black Holes*, *JHEP* **09** (2013) 073 [1304.7773].

[57] K. Jensen, P. Kovtun and A. Ritz, *Chiral conductivities and effective field theory*, *JHEP* **10** (2013) 186 [1307.3234].

[58] S. Chapman, C. Hoyos and Y. Oz, *Superfluid Kubo Formulas from Partition Function*, *JHEP* **04** (2014) 186 [1310.2247].

[59] J. L. Mañes and M. Valle, *Parity odd equilibrium partition function in 2 + 1 dimensions*, *JHEP* **11** (2013) 178 [1310.2113].

[60] K. Jensen, R. Loganayagam and A. Yarom, *Anomaly inflow and thermal equilibrium*, *JHEP* **05** (2014) 134 [1310.7024].

[61] K. Jensen, R. Loganayagam and A. Yarom, *Chern-Simons terms from thermal circles and anomalies*, *JHEP* **05** (2014) 110 [1311.2935].

[62] S. Bhattacharyya, *Entropy current and equilibrium partition function in fluid dynamics*, *JHEP* **08** (2014) 165 [1312.0220].

[63] J. Armas, *(Non)-Dissipative Hydrodynamics on Embedded Surfaces*, *JHEP* **09** (2014) 047 [1312.0597].

[64] S. Bhattacharyya, *Entropy Current from Partition Function: One Example*, *JHEP* **07** (2014) 139 [1403.7639].

[65] E. Megias and M. Valle, *Second-order partition function of a non-interacting chiral fluid in 3+1 dimensions*, *JHEP* **11** (2014) 005 [1408.0165].

[66] K. Jensen, *Aspects of hot Galilean field theory*, *JHEP* **04** (2015) 123 [1411.7024].

[67] L. Di Pietro and Z. Komargodski, *Cardy formulae for SUSY theories in $d = 4$ and $d = 6$*, *JHEP* **12** (2014) 031 [1407.6061].

[68] M. Harder, P. Kovtun and A. Ritz, *On thermal fluctuations and the generating functional in relativistic hydrodynamics*, *JHEP* **07** (2015) 025 [1502.03076].

[69] M. Valle, *Torsional response of relativistic fermions in* $2+1$ *dimensions*, *JHEP* **07** (2015) 006 [1503.04020].

[70] N. Banerjee, S. Dutta and A. Jain, *Equilibrium partition function for nonrelativistic fluids*, *Phys. Rev. D* **92** (2015) 081701 [1505.05677].

[71] J. Armas, J. Bhattacharya, A. Jain and N. Kundu, *On the surface of superfluids*, *JHEP* **06** (2017) 090 [1612.08088].

[72] J. Hernandez and P. Kovtun, *Relativistic magnetohydrodynamics*, *JHEP* **05** (2017) 001 [1703.08757].

[73] J. Armas and A. Jain, *One-form superfluids & magnetohydrodynamics*, *JHEP* **01** (2020) 041 [1811.04913].

[74] P. Kovtun and A. Shukla, *Einstein's equations in matter*, *Phys. Rev. D* **101** (2020) 104051 [1907.04976].

[75] P. Kovtun, *First-order relativistic hydrodynamics is stable*, *JHEP* **10** (2019) 034 [1907.08191].

[76] J. Armas, J. Hartong, E. Have, B. F. Nielsen and N. A. Obers, *Newton-Cartan Submanifolds and Fluid Membranes*, *Phys. Rev. E* **101** (2020) 062803 [1912.01613].

[77] J. Armas and A. Jain, *Effective field theory for hydrodynamics without boosts*, 2010.15782.

[78] D. Son and P. Sur, *Hydrodynamics with triangle anomalies*, *Physical review letters* **103** (2009) 191601.

[79] S. Bhattacharyya, *Constraints on the second order transport coefficients of an uncharged fluid*, *JHEP* **07** (2012) 104 [1201.4654].

[80] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *AIP Conf. Proc.* **484** (1999) 51.

[81] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[82] G. t Hooft, *A planar diagram theory for strong interactions*, *Nuclear Physics B* **72** (1974) 461.

[83] E. Cartan, *Sur une généralisation de la notion de courbure de Riemann et les espaces à torsion.*, *C. R. Acad. Sci., Paris* **174** (1922) 593.

[84] E. Cartan, *Sur les variétés à connexion affine et la théorie de la relativité généralisée (première partie)*, *Annales scientifiques de l'École Normale Supérieure* **3e série, 40** (1923) 325.

[85] M. Banados, O. Miskovic and S. Theisen, *Holographic currents in first order gravity and finite Fefferman-Graham expansions*, *JHEP* **06** (2006) 025 [hep-th/0604148].

[86] V. E. Hubeny, S. Minwalla and M. Rangamani, *The Fluid/Gravity Correspondence*, 1107.5780.

[87] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, *Fluid dynamics of R-charged black holes*, *JHEP* **01** (2009) 055.

[88] F. Becattini, *Covariant statistical mechanics and the stress-energy tensor*, *Phys. Rev. Lett.* **108** (2012) 244502.

[89] I. Shapiro, *Physical aspects of the spacetime torsion*, *Physics Reports* **357** (2002) 113.

[90] R. Loganayagam, *Entropy Current in Conformal Hydrodynamics*, *JHEP* **05** (2008) 087 [0801.3701].

[91] J. D. Bjorken, *Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region*, *Phys. Rev. D* **27** (1983) 140.

[92] S. Floerchinger and U. A. Wiedemann, *Fluctuations around Bjorken Flow and the onset of turbulent phenomena*, *JHEP* **11** (2011) 100 [1108.5535].

[93] F. Cooper and G. Frye, *Comment on the Single Particle Distribution in the Hydrodynamic and Statistical Thermodynamic Models of Multiparticle Production*, *Phys. Rev. D* **10** (1974) 186.

[94] W. Florkowski, A. Kumar, R. Ryblewski and R. Singh, *Spin polarization evolution in a boost invariant hydrodynamical background*, *Phys. Rev. C* **99** (2019) 044910 [1901.09655].

[95] F. Karsch, E. Laermann and A. Peikert, *The Pressure in two flavor, (2+1)-flavor and three flavor QCD*, *Phys. Lett. B* **478** (2000) 447 [hep-lat/0002003].

[96] P. Kovtun, D. T. Son and A. O. Starinets, *Viscosity in strongly interacting quantum field theories from black hole physics*, *Phys. Rev. Lett.* **94** (2005) 111601 [hep-th/0405231].

[97] I. Karpenko and F. Becattini, *Study of $\Lambda$ polarization in relativistic nuclear collisions at $\sqrt{s_{\rm NN}} = 7.7$ –200 GeV*, *Eur. Phys. J. C* **77** (2017) 213 [1610.04717].

[98] A. H. Chamseddine, *Topological Gauge Theory of Gravity in Five-dimensions and All Odd Dimensions*, *Phys. Lett. B* **233** (1989) 291.

[99] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008)*, in *Geometric and topological methods for quantum field theory. Proceedings, Summer School, Villa de Leyva, Colombia, July 9-27, 2001*, 2, 2005, hep-th/0502193.

[100] B. Cvetković, O. Miskovic and D. Simić, *Holography in lovelock chern-simons ads gravity*, *Phys. Rev. D* **96** (2017) 044027.

[101] A. Chamseddine, *Topological gauge theory of gravity in five and all odd dimensions*, *Physics Letters B* **233** (1989) 291.

[102] M. Banados, L. J. Garay and M. Henneaux, *The Dynamical structure of higher dimensional Chern-Simons theory*, *Nucl. Phys. B* **476** (1996) 611 [hep-th/9605159].

[103] M. Banados, L. J. Garay and M. Henneaux, *The Local degrees of freedom of higher dimensional pure Chern-Simons theories*, *Phys. Rev. D* **53** (1996) 593 [hep-th/9506187].

[104] C. Fefferman and C. R. Graham, *Conformal invariants*, in *Élie Cartan et les mathématiques d'aujourd'hui - Lyon, 25-29 juin 1984*, no. S131 in Astérisque, pp. 95–116, Société mathématique de France, (1985), http://www.numdam.org/item/AST$_1$985$_{S131_{95_0}}$.

[105] M. P. Bañados, L. J. Garay and M. Henneaux, *The local degrees of freedom of higher dimensional pure chern-simons theories*, 1995.

[106] R. Jackiw, *Gauge-covariant conformal transformations*, *Phys. Rev. Lett.* **41** (1978) 1635.

[107] M. Cvetič, S. Nojiri and S. Odintsov, *Black hole thermodynamics and negative entropy in de sitter and anti-de sitter einstein–gauss–bonnet gravity*, *Nuclear Physics B* **628** (2002) 295.

[108] R. M. Wald, *Black hole entropy is the noether charge*, *Phys. Rev. D* **48** (1993) R3427.

[109] T. Jacobson and A. Mohd, *Black hole entropy and lorentz-diffeomorphism noether charge*, *Phys. Rev. D* **92** (2015) 124010.

[110] E. De Paoli and S. Speziale, *A gauge-invariant symplectic potential for tetrad general relativity*, *Journal of High Energy Physics* **2018** (2018) 40.

[111] T. Clunan, S. F. Ross and D. J. Smith, *On Gauss-Bonnet black hole entropy*, *Class. Quant. Grav.* **21** (2004) 3447 [gr-qc/0402044].

[112] H. Liu and P. Glorioso, *Lectures on non-equilibrium effective field theories and fluctuating hydrodynamics*, p. 008, 07, 2018, DOI.

[113] D. T. Son and P. Surowka, *Hydrodynamics with Triangle Anomalies*, *Phys. Rev. Lett.* **103** (2009) 191601 [0906.5044].

[114] K. Landsteiner, *Notes on Anomaly Induced Transport*, *Acta Phys. Polon. B* **47** (2016) 2617 [1610.04413].

[115] S. Imaki and A. Yamamoto, *Lattice field theory with torsion*, *Phys. Rev. D* **100** (2019) 054509 [1906.02406].

[116] S. J. Barnett, *Magnetization by rotation*, *Phys. Rev.* **6** (1915) 239.

[117] T. L. Hughes, R. G. Leigh and O. Parrikar, *Torsional anomalies, hall viscosity, and bulk-boundary correspondence in topological states*, *Phys. Rev. D* **88** (2013) 025040.

[118] M. Geracie, S. Golkar and M. M. Roberts, *Hall viscosity, spin density, and torsion*, 1410.2574.

[119] A. Randono and T. L. Hughes, *Torsional monopoles and torqued geometries in gravity and condensed matter*, *Phys. Rev. Lett.* **106** (2011) 161102.

[120] J. de Boer, J. Hartong, N. A. Obers, W. Sybesma and S. Vandoren, *Perfect Fluids*, *SciPost Phys.* **5** (2018) 003 [1710.04708].

[121] K. Prabhu, *The first law of black hole mechanics for fields with internal gauge freedom*, *Classical and Quantum Gravity* **34** (2017) 035011.

[122] T. Jacobson and A. Mohd, *Black hole entropy and Lorentz-diffeomorphism Noether charge*, *Phys. Rev. D* **92** (2015) 124010 [1507.01054].

[123] E. De Paoli and S. Speziale, *A gauge-invariant symplectic potential for tetrad general relativity*, *JHEP* **07** (2018) 040 [1804.09685].

[124] C. Crnkovic and E. Witten, *Covariant description of canonical formalism in geometrical theories.*, pp. 676–684. 1987.

[125] C. Crnkovic, *Symplectic geometry of the convariant phase space*, *Classical and Quantum Gravity* **5** (1988) 1557.

[126] J. Lee and R. M. Wald, *Local symmetries and constraints*, *J. Math. Phys.* **31** (1990) 725.