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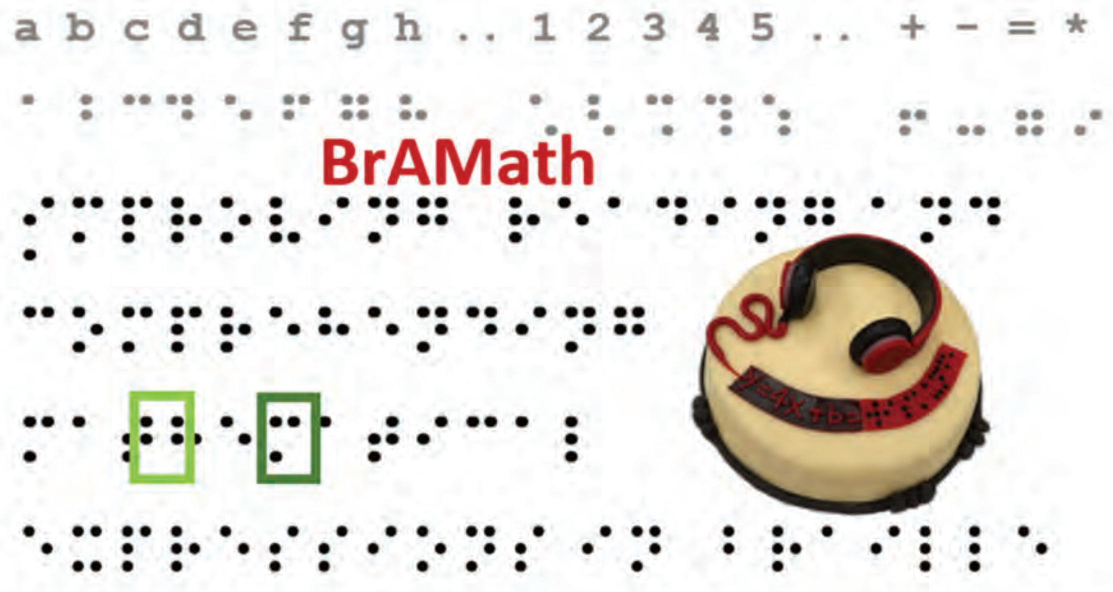

Annemiek van Leendert

Improving reading and comprehending mathematical expressions in braille

Faculteit Bètawetenschappen **FI**

Improving Reading and Comprehending Mathematical Expressions in Braille

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IMPROVING READING AND COMPREHENDING MATHEMATICAL EXPRESSIONS IN BRAILLE

VERBETEREN VAN HET LEZEN EN BEGRIJPEN VAN WISKUNDIGE EXPRESSIES IN BRAILLE

(met een samenvatting in het Nederlands)

Proefschrift

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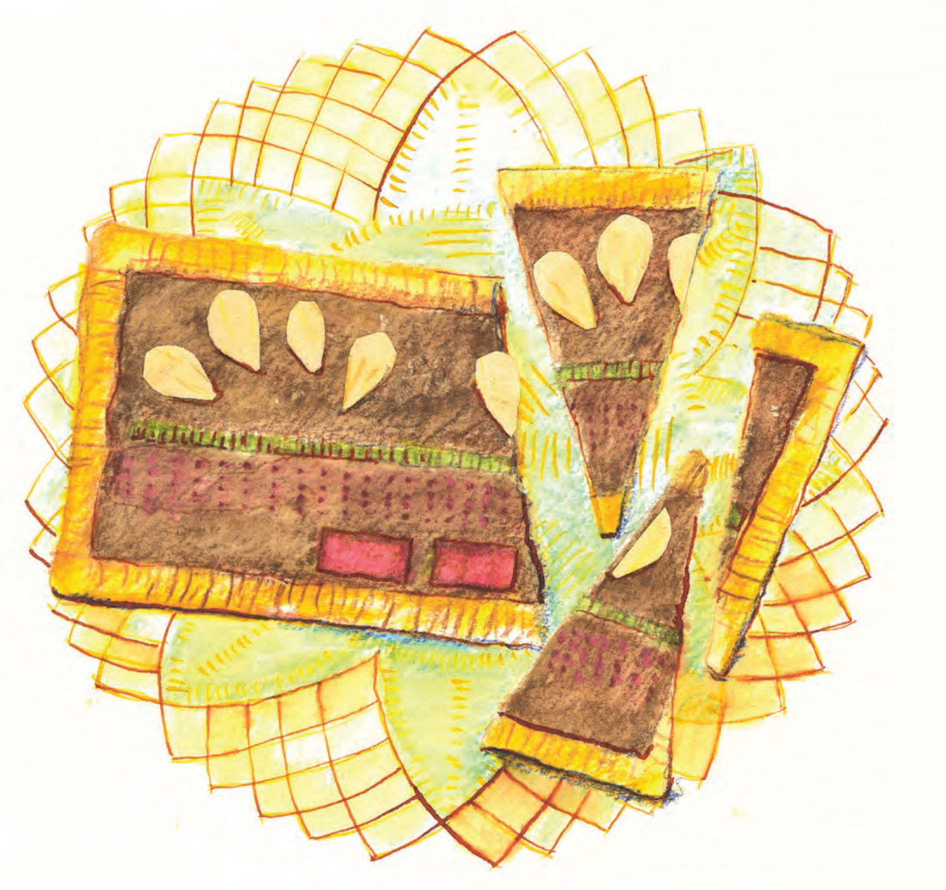
If a child can’t learn the way we teach,
maybe we should teach the way they learn

Ignacio Estrada



Chapter 1

Introduction

1.1 Touching Maths

Mathematics is an important discipline. In mathematics, students learn to combine questioning and reasoning, which is the key to analytical thinking (Robbins, 2011). Mathematics helps to better organize ideas and accurately communicate thoughts (Fatima, 2012). It is all around us and presents itself in various forms: reading diagrams and charts in the newspaper, managing your money, and changing the quantities of ingredients listed in a recipe. We live in a time of extraordinary and increasing change. Consequently, new ways of doing mathematics and communicating mathematically continuously emerge and evolve. Calculators are more common and powerful now than they were just a couple of years ago. Quantitative information that was available to a limited number of people a few years ago is now widely distributed through social media. Mathematics also plays a role in recreational activities, e.g., in building Lego models, in board games and in puzzles. While playing chess, one has to think about a winning strategy by anticipating possible moves during the game, given the conditions under which the different pieces are allowed to move. Even nature embraces mathematics. There are countless examples of symmetry and patterns, such as fractals, in plants and animals.

Mathematical knowledge plays a crucial role in understanding other school subjects such as science, social studies, and even music and art. In our society, most professions require a foundation of mathematical knowledge, and some are mathematically demanding. Therefore, more students are required to follow an educational path that prepares them for work as mathematicians, statisticians, engineers and scientists. Those who understand and can apply mathematics will have significantly more possibilities to shape their future. A society in which few have the mathematical knowledge and skills needed to fulfill crucial economic, political and scientific roles does not match with the values of a fair democratic system or its economic needs (Fatima, 2012). Therefore, all students should be given the opportunity to develop their mathematical abilities.

Although most people would agree that all students should have the opportunity to improve their mathematical abilities, I know from my own experience as a teacher of students whose primary medium for reading is braille (from here on braille readers) that this is not the case. Despite all efforts, different studies show that braille readers are less likely to take advanced mathematics classes (e.g., Klingenberg, Holkesvik, & Augestad, 2019). They underperform in mathematics compared to other subjects (Beal & Shaw, 2008; Freeland, Emerson, Curtis, & Fogarty, 2010). An important reason for this is that in mathematics education, visual modes of perceiving and knowing are frequently favored over other modalities (e.g., Healy & Fernandes, 2011; Nunes, 2004). This works against braille readers. They need to use their touch, whether or not in combination with their hearing, to gather and respond to information. A critical difference between sight and touch is that sight has a simultaneous character whereas touch has a sequential character (Millar, 1994, 1997). This means, for example, that braille readers cannot get an overview of an expression at a glance. The next sections will look into this in more detail. I will start with explaining how people who have typical vision perceive printed text through the making of eye movements. This will serve as a basis to better understand how processing of the same content in braille via tactile reading affects the teaching and learning of braille readers.

1.2 Visual Information Processing

Ballard, Hayhoe and Pelz (1995) did a study in people who had typical vision (from here on print readers) on copying strategies for patterns. The participants, who were adults, were required to manually build a copy of an eight block pattern as quickly and as precise as possible. Eye fixations on the pattern were interpreted as the need to acquire information. If a participant memorizes the model completely, he or she only needs to look at the model once. The more often a participant looks at the model during a task, the less use has been made of an internal representation. Ballard and his colleagues (1995) found that participants did not work at their maximum working memory capacity but instead tried to find ways to minimize its use. Participants reduced their working memory by serializing the tasks with making more eye fixations. These results are consistent with the theory of O’Regan (1992), who postulated that when external information is readily available and accessible, this information will be used instead of relying on an internal representation. Haselen, Van Der Steen, and Frens (2000) investigated whether this behavior is maturation-dependent. They investigated copying strategies by adults and children within the ages of seven to twelve years. Participants had to build an accurate copy of spatial block patterns. They found that both adults and children gathered information per block and therefore mainly used a serializing strategy rather than relying on internal representation. The studies just described show that the participants preferred and were able to reduce their working memory with repeated visual scanning. An interesting question is whether print readers use a similar strategy while reading and comprehending a mathematical expression.

While reading, a person perceives information during a fixation, a period of about 200-300 milliseconds, during which the eyes remain focused on one position (Rayner, 1998). During this period, the perceptual window comprises a foveal area surrounded by a near foveal area. The foveal area is about 6-8 characters, the near foveal area expands to 14 characters (Engbert, Longtin, & Kliegl, 2002). The visual acuity in the foveal area is 100%, in the near foveal area 75% (Hunziker, 2006). People can only discriminate between characters in a word in the foveal and near foveal area (Traxler, 2012). To go from one fixation to another, a person generates a saccade, a rapid eye movement that lasts between 30 and 80 milliseconds. A saccade typical of reading takes about 30 milliseconds (Rayner, 1998). It has been suggested that information extracted from the near foveal area is used to direct a forward or backward saccade (regression) toward the following fixation point (Juhász, White, Liversedge, & Rayner, 2008; Schneider, Maruyama, Dehaene, & Sigman, 2012).

When a person is fixating on a mathematical expression, often not all elements of the expression are in the foveal region. The perception of the elements in the near foveal area can help - although the perception is not very accurate – to gain an overview of the expression. Brackets, for instance, are relatively large and have a shape that stands out. Therefore, even brackets that are in a reader’s near foveal area are easy to spot (Schneider, Maruyama, Dehaene, & Sigman, 2012). The information about the brackets can be used to guide the next saccade to the following fixation (Juhasz, White, Liversedge, & Rayner, 2008; Schneider et al., 2012). This is important, because brackets affect how mathematical expressions are parsed by readers. Several studies (e.g., Schneider et al., 2012) show that, while reading and comprehending an expression, print readers repeatedly scan the expression or parts of the expression rather than relying on an internal representation of the expression. This is in line with the studies of Ballard et al. (1995) and Haselen et al. (2000). Hence, print readers can get an overview of an expression almost immediately while obtaining and using information from the foveal and near foveal area and can reduce their working memory with repeated visual scanning. This is characteristic of visual information processing. The next section shows that tactile information processing is very different.

1.3 Tactile Information Processing

Tactile reading

The medium for tactile reading and writing is braille. Braille uses raised dots to represent the symbols of the visual script. A braille cell consists of a pattern of raised dots arranged in two parallel vertical columns of three (⠿) or four (⣿) dots. Each pattern represents a braille character. For example, the letter *d* is represented as ⠙ . In 6-dot braille, 64 ($= 2^6$) characters, including the braille character for the space, are possible. However, mathematical text needs more characters (Edwards, McCartney, & Fogarolo, 2006). Therefore, 6-dot braille often uses modifier signs. For example, a number sign (⠼) may precede the letter “a, ..., j” to represent the numbers “1, 2, ..., 9, 0”. In that case, the number four is represented as ⠼⠙. In 8-dot braille, 256 ($= 2^8$) characters, including the braille character for the space, are possible. Consequently, many modifier signs can be dropped. An extra dot can be added to discriminate between a letter and a number, e.g., the number four can be represented as ⠹ or ⢙ . Table 1.1 shows how an expression in mathematical notation is transformed into braille. The mathematical braille notation that is used is the 6-dot Dutch mathematical braille notation.

Braille readers have to move their fingertips over the braille characters to gather information (Breidegard, Jönsson, Fellenius, & Strömqvist, 2006). They touch with their fingertips in a sequential pattern, one braille character after the other (Millar, 1997). This is called tactile reading. Bertelson, Mousty and D’Alimonte (1985) recognized two types of finger cooperation in reading: conjoint and disjoint. In conjoint reading, the fingertips of the two index fingers are about one or two braille cells apart. In disjoint reading, the two index fingers explore different parts of the line. This allows a braille reader to connect braille characters that are some distance apart. Perea, Jiménez, Martín-Suėsta and Gómez (2015) used the term linger for when the reading finger is slowing down and staying on one braille cell. This is another tactile reading strategy.

Table 1.1 An expression represented in mathematical notation and in 6-dot Dutch mathematical braille notation

Mathematical notation	Mathematical braille notation
$\frac{x^2}{\sqrt{2}}$	⠭⠬⠼⠃ ⠌⠎⠟⠗⠞⠦⠼⠃ ⠴ or x^#2/sqrt(#2)

Note: # represents the number sign.

Mathematical expressions in braille

In the mathematical notation that print readers use, the differences in the position of symbols provide the necessary information for the correct interpretation of expressions and equations. For instance, the expressions $2x + 1$ and 2^{x+1} have a very different meaning because of the different positions of the symbols. This is not possible in braille, because braille is a linear output modality (Stöger & Miesenberger, 2015), and all braille cells are the same size. Braille readers have to build an overview of an expression while reading one braille character after another (Millar, 1997). This requires a lot from their working memory. Due to the representation in braille and the sequential way of reading, braille readers cannot take advantage from the layout of a mathematical expression that helps print readers to comprehend the mathematical structure at a glance (Karshmer & Bledsoe, 2002). An additional difficulty is that braille characters have low redundancy, which means that braille characters are hard to distinguish (Tobin & Hill, 2015). This makes it challenging to read very precisely. The problem of low redundancy becomes more apparent when reading mathematical text, because this type of text is compact and context-poor. Hence, reading mathematical expressions and equations in braille is challenging because it is difficult to get an overview and read accurately.

1.4 The Dutch Education System

In the Netherlands, children attend primary school for eight years, between the ages of four and twelve. Then they go to one of four different streams for their secondary education, with 7th grade in K-12 school systems the first year of secondary school (see Figure 1.1). Dutch secondary schools are comparable to those in many other countries, with one stream to prepare students for vocational training (vmbo), and another to prepare students for a research university (vwo). The main difference is that, in the Netherlands, there is a third, middle, stream in secondary schools that prepares students to study at universities of applied sciences (havo). Finally, there is a special education stream for students for whom vmbo is too difficult. This stream is called practical education (Praktijk Onderwijs). The streams Praktijk Onderwijs, vmbo, havo and vwo comprise five, four, five and six years, respectively.

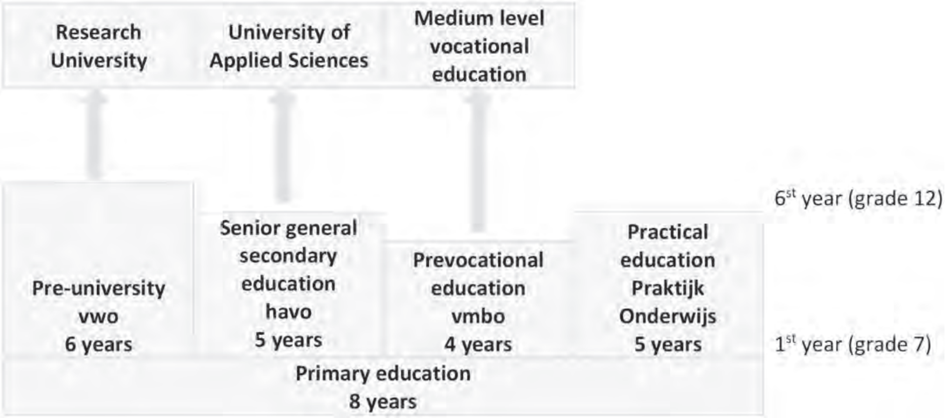


Figure 1.1 Dutch education system

1.5 Focus of this PhD Study

In this study, the focus is on reading and comprehending mathematical expressions and equations in braille. This is critical, because mathematical expressions play an important role in how mathematics is created, stored and communicated (Kohlhase, 2018). The research questions that guided this PhD-study follow from the gaps and the "abuses" that were identified in the previous sections. The main research question of this PhD-study is:

> *How can braille readers improve reading and comprehending of mathematical expressions?*

I have studied this research question from different perspectives. The emphasis was on insight into tactile perception, professional development of mathematics teachers who teach braille readers and on the representation of mathematical expressions in different mathematical braille notations.

1.6 Structure of the PhD Thesis

This section outlines the structure of the thesis and indicates the interrelation between the partial studies—and with that, between the thesis chapters. The thesis includes five peer-reviewed journal articles formatted as chapters. Figure 1.2 shows the structure and provides the theme, title and research question or questions per chapter.

Chapter 1: Introduction		
Tactile reading strategies (*Chapters 2* and *3*)	Theme: Introduction on tactile reading *Chapter 2:* An exploratory study of reading mathematical expressions by braille readers *RQ 2.1:* How do braille and print readers perform on reading and processing mathematical expressions? *RQ 2.2:* Which tactile and visual strategies do braille and print readers, respectively, use while reading mathematical expressions?	Theme: Intervention on tactile reading *Chapter 3:* An exploratory study to improve reading and comprehending mathematical expressions in braille *RQ 3.1:* To what extent does an intervention that gives support on tactile reading with a focus on structure improves the braille readers' performances in reading and comprehending mathematical expressions? *RQ 3.2:* Does the intervention make braille readers more aware and make more use of the mathematical structure of expressions?
Professional Development course for mathematics teachers who taught braille readers in combination with adjustments of the TTS settings of their braille readers' screen reader software (*Chapters 4* and *5*)	Theme: Professional Development course for mathematics teachers *Chapter 4:* Teachers' skills and knowledge in mathematics education for braille readers *RQ 4.1:* How does teachers' TPACK in mathematics for braille readers develop during a four-session professional development course? *RQ 4.2:* Which characteristics of the professional development course were beneficial in helping teachers to develop TPACK knowledge	Theme: Support for braille readers in mathematical practices *Chapter 5:* Supporting braille readers in mathematical practices using the braille display in coordination with the speech synthesizer *RQ 5.1:* What is the effect of a mathematics teachers' professional development course, in combination with adjusted text-to-speech settings, on braille readers' mathematical practices?

	and skills, and which characteristics appeared to be less useful?	
First steps towards a universal mathematical braille notation (*Chapter 6*)	Theme: Study on mathematical braille notations *Chapter 6:* Towards a universal mathematical braille notation *RQ 6.1:* What are similarities and differences in the support that braille notations from different countries offer braille readers in reading and comprehending mathematical expressions?	
Chapter 7: General discussion and conclusions		

Figure 1.2 Structure of the thesis

In the text below, I give a short introduction of each chapter. At first, we wanted to get more insight into how braille readers read and comprehend mathematical expressions on the braille display – a piece of hardware that provides braille output from a computer (see Figure 1.3). Therefore, we set up a study in which three braille readers and five print readers participated. All participants had mastered mathematics at the level of ninth-grade havo. Using finger-tracking technology, the tactile reading strategies of the three braille readers, while reading and comprehending mathematical expressions on the braille display, were recorded and analyzed. We expected that visual reading strategies of print readers would give clues to support braille readers in reading and comprehending mathematical expressions. Since little is known about these visual reading strategies, we also conducted an eye-tracking study with five print readers. They had to read and process the same mathematical expressions as the braille readers. The results of this exploratory study are provided in *Chapter 2*.

Figure 1.3 Reading on the braille display

The results of the first study on tactile reading show that braille readers applied personal tactile reading strategies with little use of the mathematical structure of expressions and equations. This is problematic, because awareness of the structure of an expression or equation is often needed to be able to properly select an operation or a solution strategy (Drijvers, Goddijn, & Kindt, 2010). Additionally, with a better sense of structure, the calculation or solving process is more accurate and faster (Hoch & Dreyfus, 2004). Therefore, an intervention that teaches braille readers to use tactile reading strategies with a focus on structure was developed. The intervention consisted of five individual lessons. Three braille readers, respectively in grade 7, 8 and 11 of havo, participated. Each participant took a pre-, a post- and a retention test. *Chapter 3* reports the results of this study.

During the first two studies, the braille readers were not allowed to use speech synthesis. However, five out of six braille readers that participated in these studies indicated that they usually use braille in coordination with speech synthesis. In addition, all six braille readers indicated that they never had any support in using their assistive devices in mathematics. The results of the first two studies led us to develop an intervention. This intervention consisted of a professional development (PD) course for mathematics teachers who taught braille readers, along with adjustments to the text-to-speech settings of their braille readers' screen reader software. The PD course was given in four three-hour sessions spread over three months. Five mathematics teachers participated in this course. The text-to-speech settings of the braille readers' screen reader software were adapted at the start of the PD course. Nine braille readers, aged 13–18 years, all in either the vmbo or havo stream, underwent the intervention.

The design of the PD course was guided by the so-called TPACK model (Mishra & Koehler, 2016). This model, which we adjusted for mathematics teachers of braille readers, is the result of the recognition that teachers need specific knowledge and skills to successfully integrate technology devices, such as a graphics calculator, in their classroom (Koehler, Mishra, & Yahya, 2007; Mishra, & Koehler, 2006). It has been acknowledged that these devices have the potential to improve mathematics teaching and learning, but many teachers find it challenging to change their teaching practices (Drijvers, Tacoma, Besamusca, Doorman, & Boon, 2013). According to this model, effective integration of technology in the classroom requires knowledge not just of content, technology and pedagogy, but also of their relationship to each other. In the adjusted TPACK model as well as in the PD course, the role of tactile and audible perception was central. *Chapter 4* describes the effect of the intervention on the teachers, while *Chapter 5* describes the effect of the same intervention on the braille readers.

Due to the low number of braille readers, the number of professionals working in the field of mathematics education for braille readers is also small. That is why it is important that professionals also collaborate on an international level. Such cooperation is quite challenging, as the educational context of braille readers differ. For instance, in some countries, braille readers use braille on paper, in others they use a braille display. A significant challenge is that many different mathematical braille notations are used all over the world. That is unfortunate, because a universal notation would greatly facilitate collaboration. To pave the way for such a notation, a study was set up to better understand variations between mathematical braille notations and their characteristics. The results are provided in *Chapter 6*.

Chapter 7 summarizes and connects the findings of the previous chapters. Here, the theoretical and methodological insights as well as the implications for educational practices that this study yielded are elaborated. The chapter is concluded by providing suggestions for further research into mathematics education for braille readers.

At the end of this section, I would like to introduce the research group. Besides myself (the doctoral candidate), this group consisted of two experts in visual information processing and eye-tracking from the vestibular and oculomotor research group of the Department of Neuroscience, Erasmus MC, and two experts in mathematics education from the Freudenthal Institute. The diversity of the members of this research group allowed me to approach the main research question from different perspectives.

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Chapter 2

An Exploratory Study of Reading Mathematical Expressions by Braille Readers

This chapter is based on

Van Leendert, A. J. M., Doorman, L. M. Drijvers, P. H. M. Pel, J., & Van der Steen, J. (2019). An exploratory study of reading mathematical expressions by braille readers. *Journal of Visual Impairment and Blindness, 113*(1)**,** 68–80. https://doi.org/10.1177/0145482X18822024.

Abstract Braille readers encounter difficulties when reading mathematical expressions. In this exploratory study, we created a setting to investigate these difficulties. Using a motion-capturing system, we analyzed the tactile strategies of three braille readers while they read mathematical expressions. To compare tactile and visual reading strategies, we also analyzed the oculomotor performance in five print readers. The analysis showed that the two experienced braille readers – experienced in reading in braille – needed about 3.5 times as much time as print readers to read and solve four items involving mathematical expressions. The other braille reader needed even more time. The braille readers used personal reading strategies for all items with little use of the expression's mathematical structure. In contrast, the reading strategies of print readers showed item-dependent and structure-related characteristics. Within the constraints of tactile reading, the braille readers had difficulties to align their reading strategies to the solution procedures required by the mathematical structure of the items. The practical implication of this study is that mathematics teachers need to become aware of the kind of problems that braille readers face when they try to read and comprehend mathematical expressions.

Keywords*:* braille reader ● eye-tracking ● finger-tracking ● mathematical expression ● reading strategy

2.1 Introduction

Mathematics education is important for professional careers and further study in today's technology-rich society. To access and work with mathematical objects, students need to communicate and manipulate representations such as formulas, graphs, and tables (Duval, 2006). Students who have typical vision (from here on referred to as print readers) have the ability to have an instantaneous overview of a mathematical expression. Back and forth scanning with the eyes allows them to identify the bracketed parts. Students who are visually impaired and use braille as their primary literacy medium (from here on referred to as braille readers) easily lose track when interacting with these representations in braille, in speech synthesis, or in tactile diagrams. This difficulty with serial processing in braille readers may be one reason why they do not have equal opportunities as print readers to fulfill their potential in mathematics (e.g., Fajardo-Flores & Archambault, 2012, 2014). To provide better support for these students, we investigated the performances and strategies of braille readers in mathematics using finger-tracking technology. Since little is known about the visually guided strategies of print readers when they are confronted with mathematical expressions, we also performed a study using eye-tracking in print readers. We expected that visual reading strategies have the potential to provide clues to support braille readers in reading and comprehending mathematical expressions.

2.2 Theory on Tactile and Visual Reading Strategies

Tactile reading strategies

The tactile reading strategies of braille readers can be studied through finger-tracking technology (Breidegard, 2007; Breidegard, Jönsson, Fellenius, & Strömqvist, 2006; Breidegard et al., 2008; Hughes, McClelland, & Henare, 2014; Perea, Jiménez, Martín-Suėsta, & Gómez, 2015). The tactile perceptual window, approximately equal to one finger pad, is much smaller than the visual perceptual window (Loomis, Klatzky, & Lederman, 1991). To identify a word, one needs to perceive each braille character separately and integrate this character with previously stored information (Dimitrova-Radojichikj, 2015). This is time consuming. It has been shown that tactile reading of non-mathematical text is about 3 times slower than visual reading (Mohammed & Omar, 2011; Perea et al., 2015).

Bertelson, Mousty, and D'Alimonte (1985) observed two types of finger cooperation in braille reading of non-mathematical text on paper: conjoint and disjoint. In conjoint reading, the fingertips of the two index fingers are about one or two braille cells apart, often touching each other. In disjoint reading, the two index fingers explore different parts of the line. According to Breidegard, Jönsson, Fellenius, and Strömqvist (2006), conjoint reading is most appropriate for efficient and deep reading and disjoint reading for global scanning of text. Perea, Jiménez, Martin-Suesta, and Gómez (2015) used the term linger when the reading finger is slowing down and lingering on one braille cell. This is another reading strategy. Braille readers only read in the left-to-right direction. Displacements of the finger over the braille display in the right-to-left direction are tactile regressions. These regressions are less intentional than visual regressions because a preview of the braille cell or cells one plans to go to is not available (Perea et al., 2015). The successive positions of each index finger on the braille display while reading a text provide insight into how braille readers perceive information. These consecutive positions form a so-called tactile scan path.

Mathematical text is compact, context-poor, and contains more symbols than non-mathematical text. Therefore, precise reading is critical, which could be difficult for braille readers because characters in braille have a low level of redundancy (Tobin & Hill, 2015). Furthermore, to comprehend the expression's structure, braille readers have to sequentially build an overview of the expression. Consequently, they may be hindered by limitations in working memory or in the process of integrating the tactile information (Loomis et al., 1991).

Visual reading strategies

In general, visual reading strategies are based on oculomotor performance assessed with an eye-tracking system. While reading, a print reader perceives information during a fixation, a period of about 200-300 milliseconds, during which the eyes remain focused on one position (Rayner, 1998). During this period, the perceptual window consists of a foveal area surrounded by a near foveal area. In terms of reading text, the foveal area is about 6-8 characters, the near foveal area extends to 14 characters (Engbert, Longtin, & Kliegl, 2002). People can only differentiate characters in a word in the foveal and near foveal area (Traxler, 2012). Characters in the foveal area can be read very precisely. This is not the case for characters in the near foveal area. To go from one fixation to another, a person generates a saccade, a rapid eye movement that lasts between 30 and 80 milliseconds. It has been proposed that information extracted from the near foveal area is used to direct a forward or backward saccade (regression) toward the following fixation point (Juhász, White, Liversedge, & Rayner, 2008; Schneider, Maruyama, Dehaene, & Sigman, 2012). The position of fixations and the sequence of fixations, saccades, and regressions can be described in a visual scan path (Noton & Stark, 1971).

Methods for exploring reading strategies

In their eye-tracking study, Schneider, Maruyama, Dehaene, and Sigman (2012) investigated reading strategies of print readers. Thirty-five participants processed several expressions: for example, 4 + (3 x 2 - 5) and 4 + (1 - (3 + 2)). The authors found that the spatial sequences of the eye movements, reflected by the visual scan paths, correspond with the expected sequences based on the syntactic structure of the expression. Although brackets do not receive frequent or long fixations, they appear to have a strong influence on how print readers parse mathematical expressions (Schneider et al., 2012).

In the study of Fajardo-Flores and Archambault (2012), braille readers and print readers solved three algebraic expressions and one algebraic equation orally. The instructor read the expression to the participants in a similar way as a speech synthesizer would do. While processing the expressions, the participants were not allowed to take notes and had to rely on their memory and the help of the instructor. The results show that, without the constraints of tactile reading, the intentions of all participants were related to the structure of algebraic procedures. In addition, the result show that both groups of students have difficulties with doing mathematics without an adequate external support.

Perea and his colleagues (2015) examined how letter position was identified in reading sentences that either were intact or involved letter transpositions. They conducted two parallel studies, one with braille readers using finger-tracking and the other with print readers using eye-tracking. Measurements of reading difficulties were number of words per minute, number of lingers and fixations, and percentage of tactile and visual regressions. Their method led to insights that contribute to the knowledge of processing transposed letter words in general. This makes research on tactile processing even more valuable.

Research questions and hypotheses

The aim of the study is twofold. The first aim is to gain insight into the strategies of braille readers while reading mathematical expressions. The second aim is to better understand possibilities and limitations of reading mathematics in braille by comparing tactile and visual scan paths. These aims are divided into two research questions:

> *Research question 2.1: How do braille and print readers perform on reading and processing mathematical expressions?*

Hypothesis 2.1: Based on studies on tactile perception of pictures (Loomis et al., 1991) and of non-mathematical text (Mohammed & Omar, 2011; Perea et al., 2015; Tobin & Hill, 2015), we expect that in our study, braille readers make more reading errors and need more time.

> *Research question 2.2: Which tactile and visual strategies do braille and print readers, respectively, use while reading mathematical expressions?*

Hypothesis 2.2: Braille readers are expected to read the expression from left to right starting at the first braille cell. They have to sequentially build an overview and will therefore have to reread (parts of) the expression more often. Ineffective tactile regressions will augment the amount of rereading further in the process, which will result in (relative) long tactile scan paths. We conjecture that the sequence of the eye movements of print readers corresponds with the expected sequences based on the syntactic structure of the expression (Fajardo-Flores & Archambault, 2012; Schneider et al., 2012). As a result, eye movements will produce (relatively) short visual scan paths. Finally, we expect that print readers use brackets for parsing the expressions but do not explicitly fixate on them (Schneider et al., 2012).

2.3 Method

Design of the study

To investigate the research questions in this exploratory study, two settings were created, one using a finger-tracking system to analyze tactile reading strategies and the other using an eye-tracking system to analyze visual reading strategies. The students were asked to read and solve four items involving mathematical expressions. All students were offered the same items in an identical order. The braille readers read the expressions on a braille display without the support of speech synthesis.

Participants

Due to the limited number of braille readers in secondary education, approximately 80 in the Netherlands, we selected three students who mastered mathematics at the ninth-grade havo level. Sylvia (pseudonym) was in havo 9th grade, and Joris (pseudonym) was a 10th-grade havo student. Michael (pseudonym) was an undergraduate university student. Sylvia and Michael developed blindness at a very young age and started reading braille at the age of 6 years. Joris was sighted and became blind, due to a disease, at the age of 12 years. Soon after, he started to learn braille. All three braille readers had no comorbidities. Sylvia and Joris attended a special secondary school for students with visual impairments. Before the actual test, we checked their reading ability of non-mathematical text on the braille display. For the finger-tracking study, the braille readers visited the Department of Neuroscience at the Erasmus Medical Centre in Rotterdam.

For the eye-tracking study, six havo 10th-grade print readers were recruited from a suburban Dutch secondary school. The mathematics teacher, who was acquainted with the first author, selected the students. The students had typical vision. One of the students was excluded because the eye-tracking data were too inaccurate due to a poor fit of the eye- tracking eyeglasses.

Michael had already passed the national mathematics examination. All other students were still being prepared for the same examination. According to their teachers, Sylvia and Joris were “above average” level and all print readers were at “average” level. We expected all students to have acquired the mathematical abilities needed for reading and solving the selected items.

Items

The students were asked to read four expressions: (3 x 2 - 5) + 4, 4 + (3 x 2 - 5), 4 + (1 - (3 + 2)), and (3 - (2 + 1)) + 4. These items were retrieved from an eye-tracking study by Schneider et al. (2012). We selected these items because the sequence of the order of required operations is not linear, except for the first item, and therefore, students had to make regressions while reading the expression.

For the braille readers, the items were typed in a Word document. The first author, who is an expert in this field, provided the correct representation. A screen reader transformed the expressions in braille. The braille notation used is an 8-dot Dutch braille notation. The first element of each expression was represented on the first, the most left, braille cell on the braille display. For the print readers, the items were created with an equation editor and resized to allow them fit on one line centered on the screen.

The settings for finger-tracking and eye-tracking

Finger movement patterns were recorded using a combination of a laptop computer with braille display, an infrared motion-capturing system (Vicon, Oxford, United Kingdom) and a video camera (Figure 2.1). The movements of each index finger and hand that were made within a predefined area in front of a laptop were recorded. A marker of reflective tape was attached on the nail of each index finger. The braille reader wore a fingerless glove on each hand, on which three markers were attached, to assess global hand positions. The infrared cameras detected the position of each marker separately at a sample rate of 200 hertz. The calibration of the cameras was done prior to each measurement series following the standardized calibration procedure prescribed by Vicon. Nexus 1.8.2 software was used to record and analyze the trajectories of each marker separately and to create an animation of the tracking in space. The x-positions of the first and the last braille cell were detected to calibrate the braille display. A video camera recorded the whole session.

The eye movement patterns were recorded using a wearable eye-tracking device (Tobii glasses 2, Tobii Technology, Sweden) in combination with a laptop computer. The eye tracker had a built-in video camera; the software was able to plot the gaze points in the recorded video images simultaneously. The items were displayed on the screen of a laptop because the one-point calibration works best when presented perpendicular to the viewing direction. Every slide in PowerPoint was followed by an empty slide or a slide with a dot on it. The dot was used as an in-between calibration point. A video camera recorded overall performances.

Procedure

Before the start of the experiment, all students were given oral instruction on what to expect and were motivated to do their best. Prior to each measurement, the setup used was calibrated. The students were asked to “think aloud” when reading and solving the expression. The investigator (the first author) started by saying, “Calculate the value of the next expression.” Then the student read the expression and answered orally. At the end of the experiment, all students received a gift voucher to thank them for taking part.

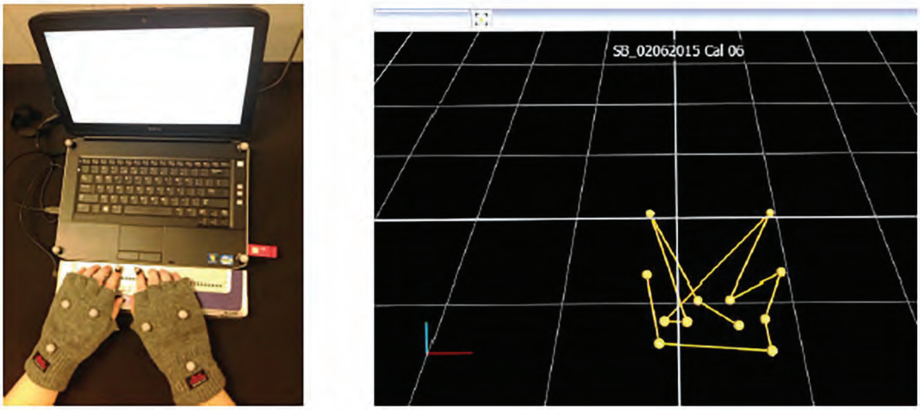

Figure 2.1 Recording of finger movement patterns. The panel on the left shows a laptop with a braille display. The markers of reflective tape are attached to the gloves, to the nails of the index fingers and to the laptop. The panel on the right shows an animation of the tracking in space

Data collection and analysis

For investigating the first research question, data were collected through audio and video recordings. We were not able to separate the reading (including comprehending) from processing because part of comprehending takes place during the processing phase. Therefore, we followed the students from the start of the reading until finding an answer to the task (reading, comprehending, and processing). Data were collected on whether the students could read and process the expression correctly and how much time they needed. The time needed per item for each braille reader and the average time for each group of students were calculated.

To answer the second research question, data were collected from the finger- and eye-tracking systems. The finger-tracking system was used to record positions of both index fingers. Horizontal finger positions were compared with the spatial location of braille cells that represent separate elements in the expressions. The eye-tracking system allowed recording gaze positions. Horizontal fixation positions and their durations were collected for the elements in each expression.

The ratios between the length of the scan path and the length of the expression were calculated as measurements for rereading. For the tactile scan paths, only the movements from left-to-right were taken into account. Tactile and visual time charts, representing the relative amount of time that a student focused on a particular element of the expression, were made to identify tactile, respectively, visual cues and different strategies.

2.4 Results

For each braille reader, the amount of time needed to read and process the expression is shown (see Table 2.1). Sylvia needed 110 seconds, Michael 113 seconds, and Joris 292 seconds to read all items.

For each item and for each condition, the average amount of time used to read the expression was calculated (see Table 2.2). For Item 1, for example, the average time needed by braille readers was 57 seconds (SD = 36.6 seconds); print readers needed 8 seconds (SD = 1.6 seconds). The braille readers needed on average 171 seconds to read all items, and the print readers needed on average 32 seconds.

Table 2.1 Reading and processing time (s.) per item

Item	1 (3 * 2 - 5) + 4	2 4 + (3 * 2 - 5)	3 4 + (1 - (3 + 2))	4 (3 - (2 + 1)) + 4
Sylvia	34	18	35	23
Michael	29	21	36	27
Joris	109	52	85	46

Table 2.2 Average reading and processing time (s.) per item and in total

Item	1 (3 * 2 - 5) + 4	2 4 + (3 * 2 - 5)	3 4 + (1 - (3 + 2))	4 (3 - (2 + 1)) + 4	
	M (SD)	M (SD)	M (SD)	M (SD)	Sum of Means
Visual	8 (1.6)	6 (1.1)	11 (3.1)	7 (1.4)	32
Tactile	57 (36.6)	30 (15.4)	52 (23.3)	32 (10.0)	171

Almost all students read and solved the four items correctly. Joris made a reading error and one print reader, Dennis (pseudonym), made an error while processing an item. The results on Item 3 provided interesting tactile and visual scan paths illustrating the relation between what the students thought and what they did. Figures 2.2 and 2.3 represent the tactile scan paths of Sylvia, Michael, and Joris and the visual scan path of Dennis on this item.

In Figure 2.2, Sylvia's tactile scan path is plotted in the top left panel. It illustrates how she used the second open bracket to start or end a finger movement. She started with a fast movement, from left to right, over the expression on the braille display. At first, she only read with her left and then only with her right index finger. In the first 10 seconds, she read the expression in overlapping parts. In the next 24 seconds, she made two regressions to the first braille cell with her left index finger. Between 24 and 28 seconds, Sylvia ignored the bracket before the number 1 and checked whether it was allowed. While she uttered 4 + 1, her left index finger moved to the left. During this period, her right index finger lingered on the braille cell at the end of the expression. Even after calculating the value of 3 + 2, she frequently moved over the right part of the expression with her index fingers. Sylvia started with a more conjoint and ended with a more disjoint reading strategy. She processed the expression in 35 seconds with a tactile scan path of about 13 times the length of the expression.

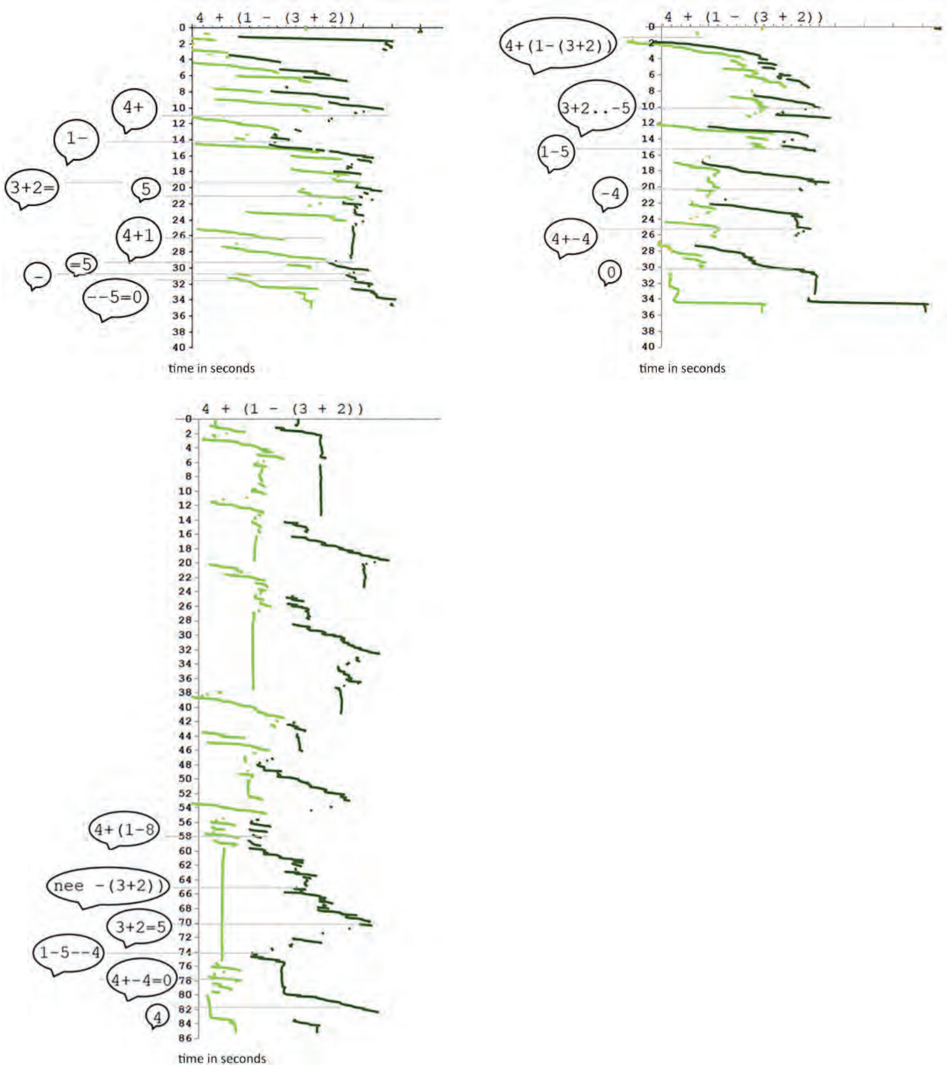

Figure 2.2 The tactile scan paths of Sylvia (top left), Michael (top right) and Joris (bottom). Light green indicates the movements of the left, dark green the movements of the right index finger

Figure 2.2 shows Michael’s tactile scan path in the top right panel. He started with reading the whole expression aloud, moving his index fingers from left to right. Then, after about 10 seconds, he calculated the value of 3 + 2. In the first part of the solving process, his left index finger frequently lingered on the second open bracket and in the second part on the first open bracket. He also started with a more conjoint and finished with a more disjoint reading strategy. He used both pairs of brackets to parse the expression, although the first pair is redundant. He solved the problem correctly in about 36 seconds with a tactile scan path of about 11 times the expression.

Joris’ tactile scan path is shown in the lower left panel of Figure 2.2. He frequently lingered with one or two index fingers on braille cells. Just after 58 seconds, he uttered “four plus one minus eight.” He made an error and corrected himself. He also, like Michael, used both pairs of brackets to parse the expression. Joris’ scan path shows that he did not use a conjoint reading strategy. He solved the problem in about 85 seconds with a tactile scan path of almost 14 times the expression.

The tactile scan graphs of the braille readers were very different. Sylvia often made quite long, almost contiguous movements, from left to right over the braille cells representing the expression. Michael used his left index finger to linger on the open brackets. They both started with a more conjoint and ended with a more disjoint reading strategy. Joris’ scan path looked very different. He lingered much more than the other braille readers and this lingering seemed not to be controlled by the expression’s mathematical structure. He did not use a conjoint reading strategy.

Figure 2.3 shows Dennis’ visual scan path. He started to make utterances after 1.75 seconds. He did not fixate on the closed brackets. He used both pairs of brackets to structure the solving process. He incorrectly calculated the value of 5 - 1, while scanning the expression from right to left. In the graphs of the other print readers, we see an extra kink to the right in this place when the print reader subsequently said 1 - 5. Dennis calculated the value of the expression incorrectly in almost 7 seconds with a visual scan path of about 3 times the length of the expression. The graph shows that his reading strategies were guided by the expression’s structure, a general result for print readers.

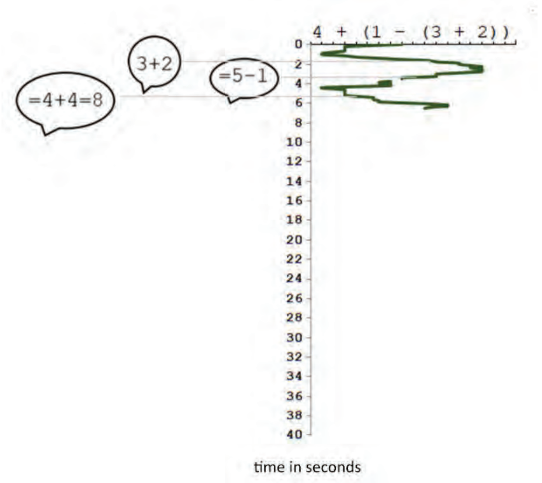

Figure 2.3 Dennis' visual scan path

In addition, we visualized tactile and visual time charts for each of the elements of the mathematical expressions. The results are shown in Figure 2.4. All highest values were on numbers, spaces, or, only for the braille readers, brackets. The tactile time charts show that for all items, the left index finger was the most active finger to the left of Cell 7, the right index finger to the right of Cell 9. The highest or second highest value was always on braille Cell number 6 or 7 when the values of the left and right index finger are added. For print readers, the values on the brackets were very low and, in some cases, even zero.

Another way to analyze tactile and visual reading strategies is to compute the ratio between the length of the scan path and the length of the expression as a measure for rereading (Tables 2.3 and 2.4). The length of the tactile scan path is the total length covered by both index fingers while moving to the right. For each item, the ratio for each braille reader was calculated (Table 2.3). Furthermore, for each item and for each group of students, the ratio of average scan path length and expression length was calculated (Table 2.4). For the first item, the ratio was 13.2 (1,288.743/97.5; SD = 2.41) for the braille readers and 3.3 (416.75/125; SD = 1.41) for the print readers.

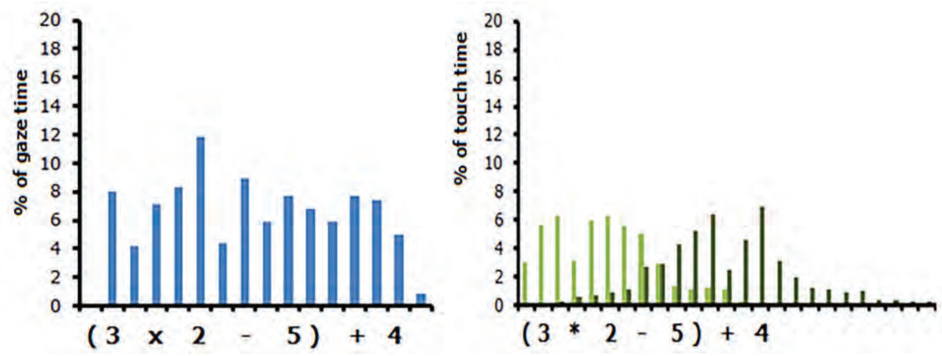

Figure 2.4a Visual and touch time charts for Item 1

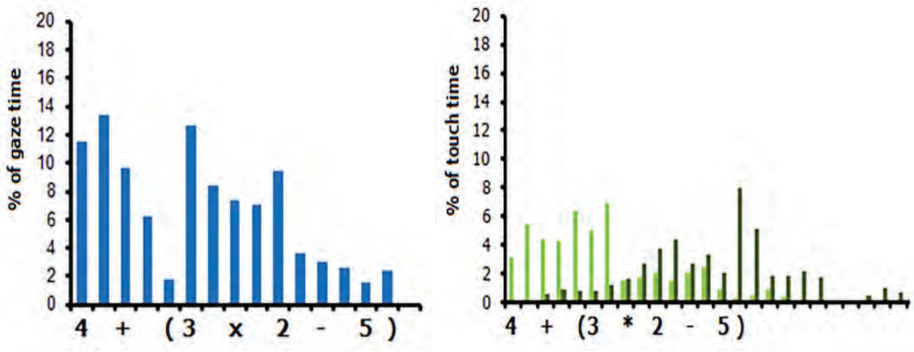

Figure 2.4b Visual and touch time charts for Item 2

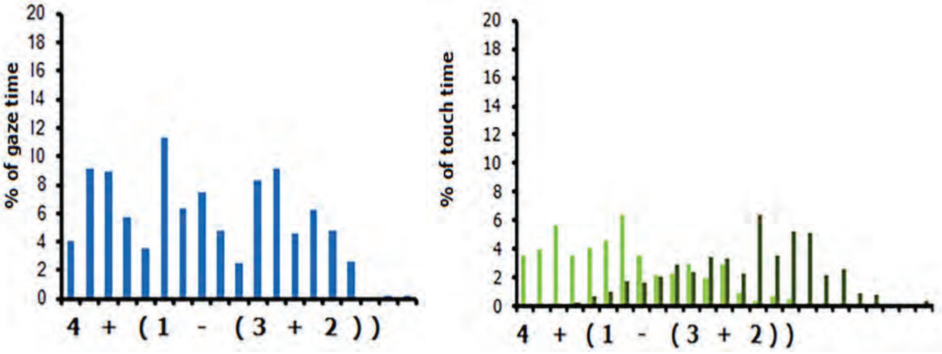

Figure 2.4c Visual and touch time charts for Item 3

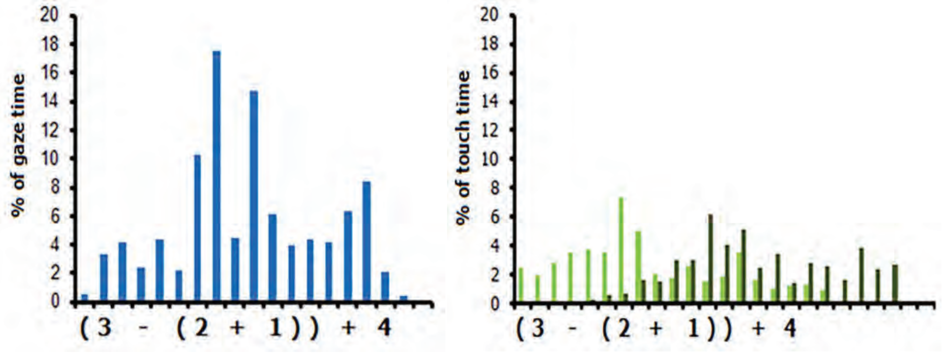

Figure 2.4d Visual and touch time charts for Item 4

Figure 2.4 Visual and touch time charts for items 1-4. In the touch time charts, light green indicates the left index finger, dark green the right index finger. In the left time chart in Figure 2.4a, the element "2" for example, has a value of 12%, meaning that, on average, 12% of the gaze time of the print readers while reading and solving the first item was spent on the element "2". In the right chart in Figure 2.4a, the element " -" has two values: 5.1% for the left index finger and 2.6% for the right index finger, so 7.7% of all the touch time while reading and comprehending Item 1 goes to "-" . The touch time charts (in the right column) show that for all items, the left index finger is the most active finger at the left of Cell 7, the right index finger at the right of Cell 9.

Table 2.3 Ratio of tactile scan path length and expression length

Item	1 (3 * 2 - 5) + 4	2 4 + (3 * 2 - 5)	3 4 + (1 - (3 + 2))	4 (3 - (2 + 1)) + 4
Sylvia	13.0	9.3	12.9	9.8
Michael	10.4	7.9	11.1	9.9
Joris	16.3	8.9	13.6	6.6

Table 2.4 Ratio of average scan path length and expression length

Item	1 (3 * 2 - 5) + 4	2 4 + (3 * 2 - 5)	3 4 + (1 - (3 + 2))	4 (3 - (2 + 1)) + 4
	M (SD)	M (SD)	M (SD)	M (SD)
Visual	3.3 (1.41)	2.7 (.69)	4.2 (.92)	3.6 (.66)
Tactile	13.2 (2.41)	8.7 (.59)	12.5 (1.05)	8.8 (1.53)

2.5 Conclusion and Discussion

The first research question focused on the performance of braille and print readers. Joris needed more than twice as much time to read and process all expressions than the more experienced braille readers. He also made a reading error. On average, Sylvia and Michael needed about 3.5 as much reading and processing time for all items as the print readers. Both groups performed equally well on the items.

The second research question focused on tactile and visual reading strategies. As expected, the braille readers started reading the expression at the first braille cell. They had to reread (parts of) the expressions to precisely decode the braille characters, to sequentially build an overview of the expression, and to compensate for ineffective regressions (Loomis et al., 1991; Perea et al., 2015; Tobin & Hill, 2015). For all items, Sylvia started with a quick movement of the index fingers, from left to right, over the expression on the braille display to locate the end of the expression. Michael lingered, frequently, with his left index finger on an open bracket while reading with his right index finger. Both students started with a more conjoint reading strategy and ended with a more disjoint one. According to Breidegard and his colleagues (2006), this pattern suggests that they started with efficient and profound reading. These basic strategies helped them to read and calculate the values of the expressions. Joris, the less experienced braille reader, did not show such efficient reading strategies. In addition, the touch time charts show that the location of the braille cell is important for the relative amount of touch time. Dennis seemed, as did the other print readers, to have an overview of the expression almost immediately. The visual scan paths and the visual time charts show that brackets did not require much gaze time. This finding is in line with the study by Schneider and her colleagues (2012). In addition, print readers seemed to be tempted to make errors when the gaze direction was not in agreement with the calculation direction, which shows that the interaction between the syntax of the expression and reading directions can be complex (Schneider et al., 2012).

Although we did not investigate this interaction systematically, the think-aloud protocols of braille and print readers revealed very similar steps in processing the information and seemed to be guided by the expressions' mathematical structure, a finding that is in agreement with the study of Fajardo-Flores and Archambault (2012). The braille readers were not able to efficiently align their reading strategies to the required solution procedures within the constraints of tactile reading, which resulted in tactile scan paths that contained many personal choices and required a lot of rereading. As a result, for each item, the graphs of the braille readers were very different, while the graphs of the print readers showed many similarities that could be related to the structure of the item.

This study has some limitations. A first limitation is the small number of students and the particular type of mathematical expressions we used. For these expressions, the representation in braille is very similar to the representation in print. This is because in print, all symbols and operators are on the baseline. Using these kinds of expressions enabled us to focus more on differences in perception and less on differences in representation. A second limitation is that the braille readers, in contrast to what they were accustomed to, were not allowed to use speech synthesis. It is possible that they would use more strategies that are effective if they had been allowed to also use speech synthesis. A third, technical limitation is that, during finger-tracking, the markers of the two index fingers were occasionally too close to distinguish one from the other. In those cases, position data of the visible marker were used to manually repair such gaps.

This study adds to the small number of studies on ways to support braille readers in mathematics. An implication for teachers of braille readers is that they should be aware of the kind of problems that braille readers encounter when reading and comprehending mathematical expressions. We recommend further experiments with similar research methods to investigate strategies of braille and print readers in mathematics. Findings from such studies should enable teachers to better support braille readers in mathematics and in successfully finishing their educational path. Finally, research on how braille readers perceive and process mathematical expressions also leads to insights that contribute to the learning of mathematics in general (Figueiras & Arcavi, 2014).

Authors’ Note

The study presented here was approved by the medical ethical committee of the Erasmus Medical Centre (MEC-2012-097 and MEC-2012-524) and adheres to the tenets of the Declaration of Helsinki (2013) for research involving human subjects. Informed consent was obtained from the subjects.

Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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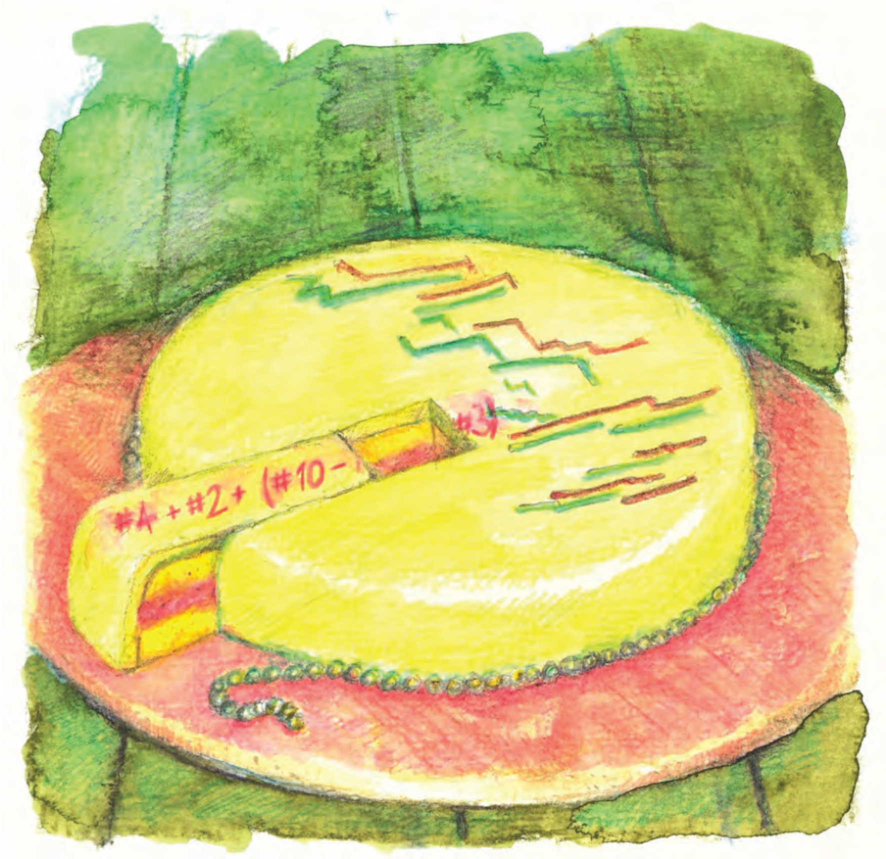

Chapter 3

An Exploratory Study to Improve Reading and Comprehending Mathematical Expressions in Braille

This chapter is based on

Van Leendert, A., Boonstra L. G., Doorman, M., Drijvers, P., van der Steen, J., & Pel. J. (2021). An exploratory study to improve reading and comprehending mathematical expressions in braille. *British Journal of Visual Impairment.* https://doi.org/10.1177/02646196211044972.

And on

Boonstra, L. G. (2017). Improving the mathematical reading skills of students who read braille by scaffolding reading strategies (unpublished master thesis). Utrecht University.

Abstract Braille readers read and comprehend mathematical expressions while moving their fingertips over braille characters. The aim of this exploratory study was to investigate the effect of an intervention that teaches braille readers who use a braille display to use finger movements with a focus on the expression's mathematical structure. The finger movements involved movements where the two index fingers are about one or two braille cells apart and movements where the index fingers explore different parts of the expression. We investigated to what extent the intervention supports an interplay between finger movements and the expression's mathematical structure to make the process of calculating the value of an expression easier and to make braille readers more aware of the expression's structure. Three braille readers, respectively in grades 7, 8 and 11 of havo, received the intervention consisting of five individual lessons. During the pre-, post-, and retention test, the braille readers' finger movements were video recorded, as well as the time needed to read and process the mathematical tasks. Four expressions were selected for further analysis. The results show that during the posttest, each braille reader required at least 29% less time to read and process the expressions. The retention test results were even better. Scan paths indicated that braille readers picked up features of mathematical structures more easily after the intervention. Based on our findings, we recommend that braille readers receive lessons in tactile reading strategies that support the reading and processing of mathematical expressions and equations.

Keywords: braille reader ● mathematical expression ● mathematical structure ● tactile reading

3.1 Introduction

Many high school students struggle with mathematical expressions and equations, both numerical and algebraic (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). Difficulties include the order of operations (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1994), understanding of the equal sign (McNeil & Alibali, 2005), working with brackets (Hoch & Dreyfus, 2004) and working with variables (Knuth et al., 2005). Therefore, it is worthwhile to give students extra support in reading and comprehending mathematical expressions. The current study is about support for students who use braille as their primary reading medium (i.e., braille readers). In a recent study using finger-tracking technology, Van Leendert, Doorman, Drijvers, Pel and Van der Steen (2019) investigated how three braille readers read and comprehend four items involving mathematical expressions on the braille display. One of the expressions was $4 + (1 - (3 + 2))$. Using eye-tracking technology, they also investigated how five students who had typical vision (from here on print readers) read and comprehend the same expressions. They found that the two most experienced braille readers – experienced in reading in braille – needed about 3.5 times as much time as the print readers. The print readers focused directly on typical features of the expression's mathematical structure, such as brackets. In addition, they traced with their eyes the elements of the expression in an order that corresponded to the sequence of steps required to calculate the value of the expression. The braille readers, in contrast, picked up the structure's characteristics either marginally or much later than the print readers did. It was a challenge for them to match their finger movements to the sequence of steps needed to do the calculation. These findings suggest that braille readers may perform better when they use tactile reading strategies with an emphasis on the expression's structure. Therefore, the aim of the current exploratory study was to investigate the effect of an intervention that teaches braille readers who use a braille display finger movements with a focus on the expression's mathematical structure.

3.2 Theoretical Background

Mathematical expressions and equations in braille

The representation of expressions and equations in the mathematical notation, the notation that print readers use, is very different from the representation in braille on a braille display. This difference can be illustrated with the next equation:

$$\frac{(x+2)^2}{\sqrt{3}} = \frac{3(x+2)}{\sqrt{3}}$$

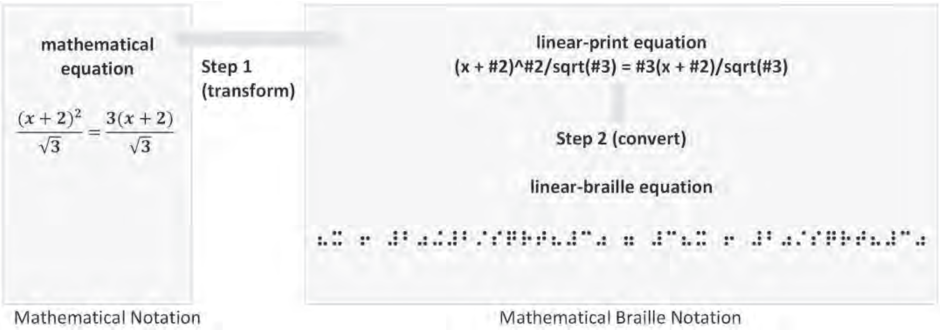

Figure 3.1 Transformation from a mathematical to a linear-braille equation
Note: # represents the number sign

The symbols and numbers are arranged at different heights above, on and below the baseline to convey structure (Stöger & Miesenberger, 2015). Moreover, the symbols, numbers and operators look very different from each other. The specific features of this representation in combination with the use of sight – which has a simultaneous character (Millar, 1994; Millar, 1997), enable print readers to get an overview of this equation in a split second (Schneider, Maruyama, Dehaene, & Sigman, 2012).

The representation in braille on a braille display is very different from the representation in print. Braille is a linear output modality (Stöger & Miesenberger, 2015) and all braille cells have the same size (Karshmer & Bledsoe, 2002). The transformation from an equation in mathematical notation to an equation in braille can be executed in two steps. This is depicted in Figure 3.1. This figure consists of two blocks: Mathematical Notation and Mathematical Braille Notation. *Step 1* is a transformation from an equation into a linear-print equation. This can be done in very different ways depending on the mathematical braille notation used. In the current study, the 6-dot Dutch mathematical braille notation is used. This notation uses number signs (#). The correct linear-print equation is (x + #b)^#b/sqrt(#c) = #c(x + #b)/sqrt(#c). For the sake of convenience, to make it easier to read the equation, #2 and #3 are used instead of #b and #c. *Step 2* is a conversion from the linear-print to the linear-braille equation.

Mathematical structure of expressions and equations

In the current study, the concept of mathematical structure is important. According to Hoch and Dreyfus (2004), a structure can be seen as the way in which a mathematical entity, such as an algebraic or numerical expression or equation, is composed of its parts and how these parts are connected. Awareness of the structure of an expression or equation is important because that is often needed to be able to appropriately select an operation or a solution strategy (Drijvers, Goddijn, & Kindt, 2010). The strength of recognizing the structure can be demonstrated in the following equation:

$$(x-3)(2x+1)=(x-3)(x+2)$$

First, one needs to recognize that the equal sign does not require a calculation (like in “3 + 4 = …”), but requires finding the value of x for which the left and right expression have an equal value. When one recognizes the structure of this equation, which is $\mathrm{A}*\mathrm{B}=\mathrm{A}*\mathrm{C}$, it follows rather easily that $A=0$ or $\mathrm{B}=\mathrm{C}$. Removing brackets may also lead to the correct answer, but takes much more time and calculations. Students who have to make many calculations are more inclined to make errors; that is why using characteristics of the structure is not only faster but also more accurate (Hoch & Dreyfus, 2004).

Interplay between finger movements and comprehending structure

Mathematical structure is more difficult to grasp in braille than in print due to the linear representation in braille (e.g., Stöger & Miesenberger, 2015) and to the small perceptual view of braille readers (e.g., Millar, 1994, 1997). This does not mean that the level of detail is not present in the linear-braille expression or equation. An additional problem is that braille characters have a low redundancy, which means that braille characters are hard to distinguish (Millar, 2003; Tobin & Hill, 2015). This is particularly problematic when reading mathematical text because this type of text is very compact. For instance, in the mathematical notation for print readers, “the square root of 2x” is denoted by “$\sqrt{2x}$”. Reading mathematical text in braille requires several attempts before the purpose and structure of the text becomes clear. What braille reading strategies can be recognized?

In reading non-mathematical text, several types of finger cooperation can be recognized: conjoint and disjoint (Bertelson, Mousty, & D’Alimonte, 1985). In conjoint reading, the fingertips of the two index fingers are at a distance of one or two braille cells from each other, often touching each other. In disjoint reading, the two index fingers explore different parts of the line. Another type of finger movement is when a reading finger slows down and rests on a braille cell. This movement is referred to as lingering on a braille cell (Perea, Jiménez, Martín-Suesta, & Gómez, 2015). Conjoint reading is most suitable for efficient and deep reading and disjoint reading for global scanning of text (Breidegard, Jönsson, Fellenius, & Strömqvist, 2006). Our assumption is that braille readers can be supported in comprehending the expression’s mathematical structure and in the calculation or solving process by using specific finger movement strategies. In this study, we only focus on the movements of the index fingers. We assume that the first step in comprehending the structure is to decode every element of the expression or equation accurately. For this, a conjoint reading style seems to be suitable (Breidegard, Jönsson, Fellenius, & Strömqvist, 2006). During this activity, the braille readers are expected to try to get a global overview of the expression. The next step, which can be skipped if the structure is very clear, is to look for relations between different parts of the expression or equation. For this, a disjoint reading style is appropriate, because that enables the braille reader to physically relate parts of the expression or equation to each other and compare these parts almost simultaneously. The following step, inevitable in mathematical activity, is to calculate the value of the expression or solve the equation. For this, one can use different reading styles, depending on the required process and the skills and knowledge of the braille reader. Figure 3.2 gives an illustration of the proposed method. This figure illustrates the finger movements over a braille display while reading and processing $24 + 15 =$. The expression and the time are shown on the horizontal and vertical axis, respectively. The movements of the left and right index finger are shown in light and dark grey, respectively. The diagram shows that the braille reader starts with conjointly reading the entire expression, continues with disjoint reading and ends with conjoint reading (Figure 3.2c). Using a disjoint reading style helps to perform the calculation by first focusing on tens and next on ones in both numbers (20 + 10 = 30 and 4 + 5 = 9).

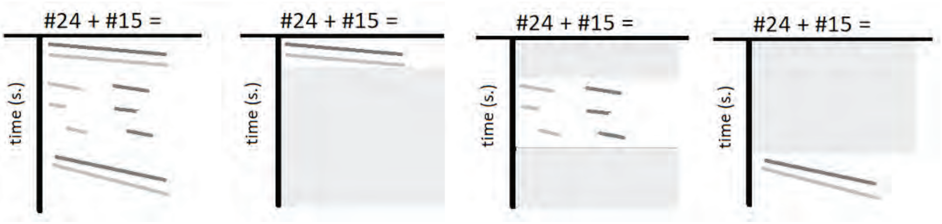

Figure 3.2a

Figure 3.2b
conjoint reading

Figure 3.2c
disjoint reading

Figure 3.2d
conjoint reading

Figure 3.2 Schematic representation of the finger movements while reading the expression $24 + 15 =$ *in braille. Figure 3.2a illustrates the whole reading process. Figure 3.2b, 3.2c and 3.2d show the successive stages of this process.*

Note: Only the movements of the index fingers in the left-to-right direction are depicted. Light grey indicates the movements of the left index finger, dark grey the movements of the right one.

Research questions

The current study investigated whether an intervention on tactile reading supports braille readers to read and comprehend mathematical expressions. In this intervention, we made braille readers aware of their finger movements and reading strategies. Moreover, we suggested and trained ways for improving these finger movements over the braille display to better understand the structure of an expression and to support the calculation process. The following questions were addressed:

> *Research question 3.1: To what extent does an intervention that gives support on tactile reading with a focus on structure improves the braille readers' performances in reading and comprehending mathematical expressions?*
>
> *Research question 3.2: Does the intervention make the braille readers more aware and make more use of the mathematical structure of expressions?*

The first question uses the term performance. In this study, the operationalization of performance is the time needed to read and calculate the value of an expression and the number of correct answers. The attention for structure is expected to lead to better performance in reading and comprehending mathematical expressions (Drijvers et al., 2010; Hoch & Dreyfus, 2004). The second research question addresses awareness and use of the mathematical structure while reading and solving mathematical expressions and tasks. In this study, awareness and use of the structure is operationalized in tactile scan path patterns. It is expected that the intervention changes these patterns towards finger movements that reflect structure or mathematical problem solving. For example, they will use a disjoint reading style when they need to compare two parts of the expression with each other.

3.3 Methods

Design of the study

An intervention consisting of five individual lessons was developed to be able to answer the research questions. The focus was on tactile reading strategies for processing mathematical expressions and equations. A mixed-method approach was taken for investigating the effect of the intervention on the braille readers, using a pre-, post- and retention test. The quantitative data (test results and time needed) and qualitative data (video recordings of finger movements) are used to answer the first and second research question respectively.

This study was approved by the medical ethical committee of the Erasmus Medical Centre (MEC-2012-097 and MEC-2012-524) and adheres to the tenets of the Declaration of Helsinki (2013) for research involving human subjects. Informed consent was obtained from the subjects.

Participants

Three braille readers from a school for visually impaired students participated in the study. The braille readers, called T., R., and S., were in grade 7, 8 and 11 of havo, which is the senior general secondary track, respectively. They all developed blindness at a very young age. They had no comorbidities. The braille readers started at the age of six with reading in braille on paper. In grade 6, they switched gradually to using the braille display in mathematics lessons. They did not learn how to read and explore mathematical text in braille, not on paper and not on the braille display. Consequently, they developed a very personal way of reading and comprehending mathematical expressions and equations. Over time, they had learned to only use their index fingers when reading in braille, regardless of the type of text. Their mathematics teachers classified the braille readers as above average in mathematics.

Design of the intervention

The intervention consisted of five 45-minute lessons. The braille readers were individually taught by an instructor (this paper's second author) who had a good knowledge of mathematics and braille. They did not get exactly the same intervention; adjustments were made according to their grade level. For example, the grade 7 braille reader was given equations with only a variable on one side of the equal sign. The other two braille readers were given equations that are more complicated. Table 3.1 provides an overview of the lessons. In the first lesson, the focus was on the decoding of braille characters and on the mathematical structure of expressions. The braille readers were taught which characteristics of the structure are important – and why – and how to recognize them by touch. For example, they were asked to point out the operator in an expression or the equal sign in an equation. In the second lesson, the emphasis was on using finger movements that support reading and processing mathematical expressions. In the third lesson, the focus was on finger movements that support reading and solving equations. An important strategy was to relate the two sides of an equation in disjoint reading. In the fourth and fifth lessons, the knowledge and skills acquired in the first three lessons were further developed with different expressions and equations. In addition, much attention was paid to mathematical language.

Table 3.1 Design of the intervention

Lesson	Topic of the Lesson
1	Global structure of mathematical expressions
2	Finger movements that support reading and processing mathematical expressions
3	Finger movements that support reading and solving equations
4	Mixed exercises
5	Mixed exercises

Test design

The pre- post- and retention tests consisted of 22 items. Each test took 45 minutes. Examples of the tasks are: point out operators in expressions, read the expression or equation, simplify an expression, calculate the value of an expression, and solve an equation. The tasks were the same for each braille reader, but dependent on the grade level, there were small differences in the expressions and equations involved. For each individual braille reader, the items on the pre-, post- and retention test were identical (see Appendix for the test taken by braille reader R.).

Procedure

The instructor provided the lessons over a period of three weeks. The lessons and tests were scheduled during regular mathematics lessons and were administered in a room separated from other students. The pre-test was administered one week before the start of the intervention, the posttest one week after the end of the intervention and the retention test was conducted six weeks after the posttest. The expressions were typed – represented as linear-print expressions – in Word and converted into braille with a screen reader. The instructor started with explaining the task. Then the braille readers were asked to “think aloud” while reading the expressions and equations on the braille display and to answer orally.

The screen reader software NVDA – NonVisual Desktop Access – and the so-called NL literature table, the 6-dot Dutch braille table, were installed on the laptop used for the lessons and the tests. The braille readers were used to using this braille table.

Data collection and analysis

During the tests, a video camera assessed the movements of the readers’ index fingers over the braille display. The finger movements were analyzed with the help of video analysis software (Kinovea). Every 200 milliseconds, the position of both index fingers was noted in Excel. The positions were matched with the spatial locations of the braille cells that represent separate elements in the expressions. The successive positions of each index finger on the braille display formed a tactile scan path.

To answer the first research question, we first selected the test items whose task was "calculate the value of the expression." From these five test items, we then selected four items for further analysis. These items are depicted in Table 3.2. This table illustrates that every braille reader has their own test. For example, Item *d* involves the expression 3/4 * 2/5 (braille reader T.) and 5/7 * 2/9 (braille readers R. and S.).

These items have been selected because the structure of the expressions provokes braille readers to use lingering or disjoint reading. This enables braille readers to show that that they use a reading strategy with a focus on the structure. In the first two items, the braille readers can use lingering to keep track of the locations of the brackets and solve the sub-expressions between the brackets before solving the whole expression. In the last two items, the braille readers can use a disjoint reading style while comparing the tens, ones and decimals (Item c) or the denominators (Item d).

To answer the first research question, we did a descriptive analysis of the quantitative data. For each test item and for each braille reader, the number of correct answers and the reduction in time needed to give a correct answer were calculated. In addition, for all selected test items in the pre-, post-, and retention test, and for each braille reader the total time necessary to give a correct answer and the reduction in time needed to give a correct answer were calculated.

A qualitative approach was used to answer the second research question. We visualized the scan path patterns of the items *a* and *c* by tracing finger locations on the braille display. These items were selected because the expressions' mathematical structures are very different. We analyzed the resulting scan path patterns and the changes in these patterns for each braille reader and for both items. We tried to recognize the finger movements that braille readers used and investigated whether these can be related to the expression's mathematical structure and the sequence of steps necessary to perform the required calculations.

Table 3.2 Selected test items

	T. grade 7	R. grade 8	S. grade 11
Item a	4 + 2 * (10 - 3)	4 + 2 * (10 - 3)	4 + 2(10 - 3)
Item b	2 + 1 - (2^2 + 1)	2 + 1 - (2^2 + 1)	2 + 1 - (5 - 3)^2
Item c	45.7 + 13.4	45.7 - 13.4	45.7 - 13.4
Item d	3/4 * 2/5	5/7 * 2/9	5/7 * 2/9

Note: In the Netherlands, we use a decimal comma as a decimal separator.

Table 3.3 Time (s.) for reading and processing Item a (Table 3.3a), Item b (Table 3.3b), Item c (Table 3.3c) and Item d (Table 3.3d)

Table 3.3a Time (s.) for reading and processing Item a

Braille reader	Expression	Pre-test	Posttest (reduction %)	Retention test (reduction %)
T. (grade 7)	4 + 2 * (10 - 3)	71	42 (41%)	55 (23%)
R. (grade 8)	4 + 2 * (10 - 3)	87	27 (69%)	15 (83%)
S. (grade 11)	4 + 2(10 - 3)	57	14 (75%)	16 (72%)

Table 3.3b Time (s.) for reading and processing Item b

Braille reader	Expression	Pre-test	Posttest (reduction %)	Retention test (reduction %)
T. (grade 7)	2 + 1 - (2^2 + 1)	47	49 (-4%)	27 (42%)
R. (grade 8)	2 + 1 - (2^2 + 1)	86	33 (62%)	21 (76%)
S. (grade 11)	2 + 1 - (5 - 3)^2	53	35 (34%)	13 (75%)

Table 3.3c Time (s.) for reading and processing Item c

Braille reader	Expression	Pre-test	Posttest (reduction %)	Retention test (reduction %)
T. (grade 7)	45.7 + 13.4	28	10 (64%)	10 (64%)
R. (grade 8)	45.7 - 13.4	25	28 (-12%)	18 (28%)
S. (grade 11)	45.7 - 13.4	32	26 (19%)	30 (6%)

Table 3.3d Time (s.) for reading and processing Item d

Braille reader	Expression	Pre-test	Posttest (reduction %)	Retention test (reduction %)
T. (grade 7)	3/4 * 2/5	8	8 (0%)	12 (-50%)
R. (grade 8)	5/7 * 2/9	22	14 (36%)	24 (-9%)
S. (grade 11)	5/7 * 2/9	30	12 (60%)	8 (73%)

Table 3.4 Total Time (s.) for reading and processing all four Items

Braille reader	Pre-test	Posttest (reduction %)	Retention test (reduction %)
T. (grade 7)	154	109 (29%)	104 (32%)
R. (grade 8)	220	102 (54%)	78 (65%)
S. (grade 11)	172	87 (49%)	67 (61%)

3.4 Results

Research question 3.1: Performances in reading and comprehending expressions

The braille readers calculated the values of the selected expressions correctly. For each braille reader and for each test, the time needed to read and process the selected expression is represented in Table 3.3. The braille readers performed better on the post- and retention test than on the pre-test, except in four cases. T., for example, needed 47 s. in the pre-test and 49 s. in the posttest for processing Item b. However, he performed much better on this item in the retention test, in which he only needed 27 s. For Item c., T. and R. performed worse in the retention test than in the pre-test. In contrast, S. performed much better on this item in the post- and retention test than in the pre-test. The total time needed to read and process the four items is depicted in Table 3.4. In the posttest, the reduction in time was 29%, 54% and 49% for T., R. and S., respectively. The results on the retention tests were even better.

Research question 3.2: Awareness and use of the mathematical structure

To answer the second research question, the scan paths of the expressions that belong to Item a and Item c were analyzed. The scan paths during the pre-test (left), the posttest (middle) and the retention test are shown for Item a (Figure 3.3) and c (Figure 3.4). The scan paths of braille reader T., R. and S. are shown in the top, middle and bottom panel, respectively. Because braille readers can only properly decode braille characters if they move their fingers from left to right on the braille display, those are the only finger movements that are displayed. We start with the analysis of the scan paths for Item a. Overall, for this item, the scan path patterns in the retention test are very similar to the scan path patterns in the post-test. In contrast, the scan path patterns in the pre-test are different and much longer. Let us look at the patterns of each braille readers in more detail. In the pre-test, T. started to read the expression with his left index finger and continued with his right index finger. He stopped just before the open bracket. He spent some time on “#2” and on “*”. After about 30 seconds from the beginning, he continued in a more conjoint reading style. He read the entire expression a few times. The sub-expression between brackets was read more often than the other parts of the expression. In the posttest, he first explored the first part of the expression – using a conjoint and disjoint reading style. Then he read almost 10 seconds only with his left index finger. Finally, he reread the entire expression in conjoint reading. In the retention test, he also used a conjoint and disjoint reading style. After about 10 seconds from the beginning, he spent more than 5 seconds on “(“. R. started in the pre-test with a conjoint reading style. Then she continued with a more disjoint reading style. She also read part of the expression with only one index finger. She read the first elements of the expression more than 10 times. In the posttest, she started with conjointly reading the entire expression. Then she made little jumps while moving her fingers from right to left in the expression. In the retention test, she started with a similar strategy as in the posttest, but did not reread so much in the end. S. started in the pre-test with conjointly reading the entire expression. She conjointly read parts of the expression again and switched, after about 25 seconds, to a more disjoint reading style. In the posttest, she started with conjointly reading the entire expression. Then she switched to a more disjoint reading style. In the retention test, she used similar reading strategies as in the posttest.

Figure 3.4 shows the scan paths for Item c. In the pre-test, T. started with conjoint reading and switched, after about 13 seconds, to disjoint reading. After about 20 seconds from the beginning, he read #45.7 with his left index finger and, almost simultaneously, #13.4 with his right index finger. In the posttest as well as in the retention test, he started with reading the entire expression conjointly and continued in disjoint reading. R. read the entire expression once using different reading styles and continued with disjoint reading. In the post- and retention-test, she started with conjointly reading the entire expression. Then she continued in disjoint reading. For R., the three patterns are not very different from each other. S. started in the pre-test with reading the entire expression. During the whole process of reading, she alternated between different reading strategies: conjoint reading, disjoint reading and reading with just one index finger. In the post- and retention test, she started with conjointly reading the entire expression and continued in disjoint reading. For S., the scan path patterns in the post and retention test are very similar. The pattern in the pre-test, on the other hand, is very different

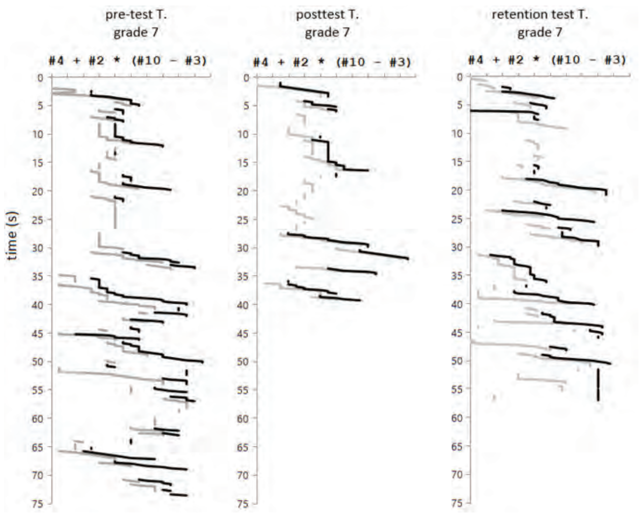

Figure 3.3a The tactile scan paths of braille reader T.

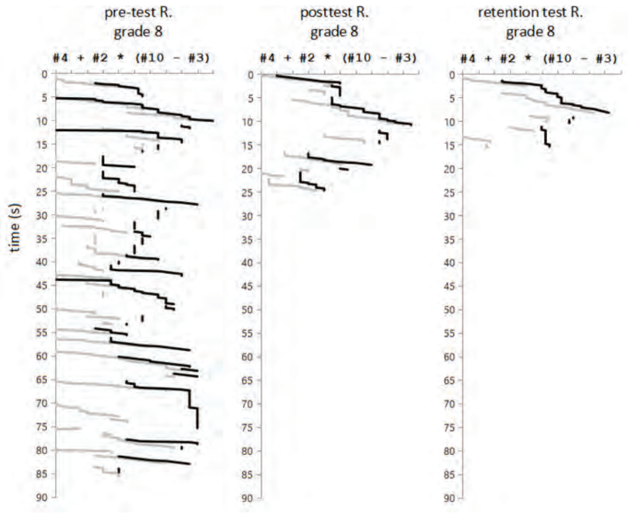

Figure 3.3b The tactile scan paths of braille reader R.

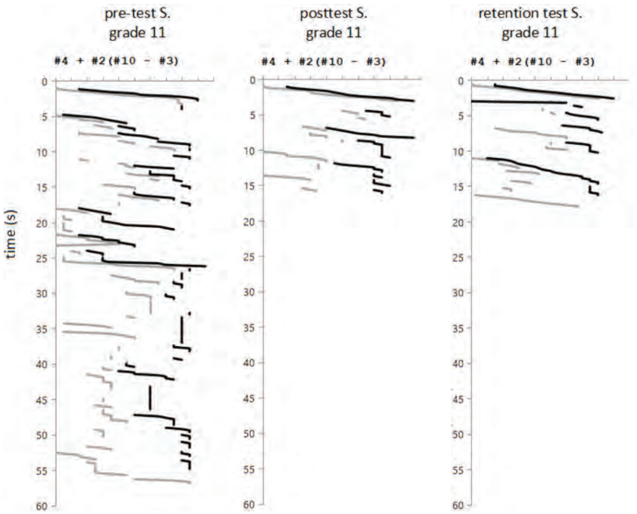

Figure 3.3c The tactile scan paths of braille reader S.

Figure 3.3 The tactile scan paths for Item a during the pre-test (left), the posttest (middle) and the retention test (right). # stands for the number sign. Only the movements of the index fingers in the left-to-right direction are depicted. The movements of the left and right index finger are shown in light and dark grey, respectively

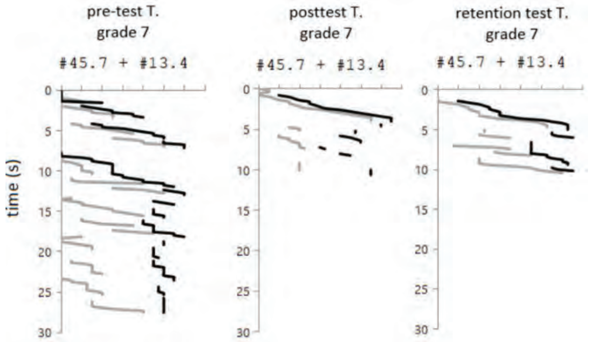

Figure 3.4a The tactile scan paths of braille reader T.

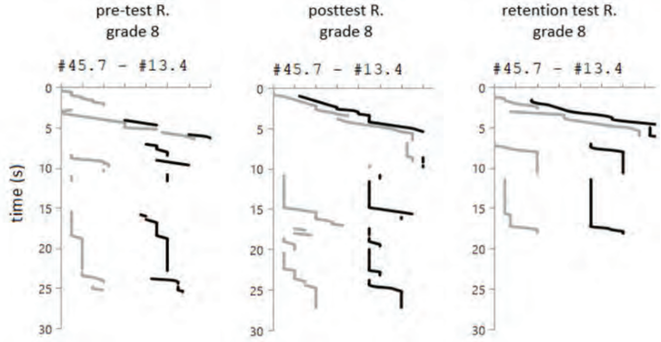

Figure 3.4b The tactile scan paths of braille reader R.

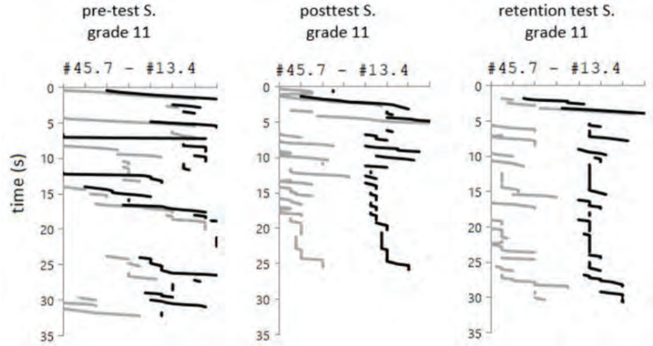

Figure 3.4c The tactile scan paths of braille reader S.

Figure 3.4 The tactile scan paths for Item c during the pre-test (left), the posttest (middle) and the retention test (right). # stands for the number sign. Only the movements of the index fingers in the left-to-right direction are depicted. The movements of the left and right index finger are shown in light and dark grey, respectively

3.5 Conclusion and Discussion

RQ 1: Performances in reading and comprehending expressions

The first research question concerned the performances in reading and comprehending mathematical expressions. In all tests, the braille readers calculated the values of the expressions correctly. The results show that the braille readers improved their performance, because each braille reader needed at least 29% less time to read and process the expressions. A reduction in time was expected, because attention for the mathematical structure should lead to improved results (Drijvers et al., 2010; Hoch & Dreyfus, 2004). The results also show that the performances of the braille readers differ from each other. Overall, it appears that braille reader T. – the youngest braille reader – had benefited less from the intervention than the other braille readers. In some cases, the braille readers performed much better on the retention test than on the posttest. This may indicate that other aspects related to experiences in mathematical learning were also relevant for improving performance. Moreover, for Item *d*, there were large differences between the performance and performance change of the braille readers. This is also an indication that other aspects may be relevant.

RQ 2: Awareness and use of the mathematical structure

The second question concerned whether, after the intervention, the braille readers were more aware of and made more use of the expressions' mathematical structure. The scan paths patterns for Item a show that after the intervention R. used reading strategies that were more in line with the sequence of steps needed to process the expression. After reading the entire expression, she almost immediately started to reread the sub-expression between brackets. To a lesser extent, this was also the case for S. For Item a, the results also show that all braille readers reread less after the intervention. The scan paths for Item c illustrate that T. and R. used a disjoint reading style in all tests. This enabled them to relate 45.7 to 13.4. After the intervention, braille reader S. also used this strategy. The results also show that T. reread less after the intervention. Overall, the results indicate that the braille readers picked up characteristics of the structure more easily. They used finger movements that were more in accordance with the steps needed to do the calculations. This is in line with what we expected.

We recognized in almost all tactile scan paths patterns that are depicted in this article that the changes in scan strategies – that indicated that braille readers were more aware of the expression's structure and use the structure more – improve the braille readers' reading and comprehension time. This is in accordance with what we expected (Drijvers et al., 2010; Hoch & Dreyfus, 2004). However, for braille reader S., the changes in the scan path patterns for Item c were large but did not result in great reductions in the reading and comprehension time. It is also possible that another calculation strategy affected reading and processing time.

This study had some limitations. The first is that the small number of participants offered little basis to generalize findings. However, the braille readers differed in their knowledge of mathematical concepts and in their braille reading skills, but they all benefited from the intervention. Secondly, the braille readers were all classified as good students. It is possible that braille readers who perform on or below average level in mathematics will respond differently on the intervention. Thirdly, in all tests, the braille readers made no errors in calculating the values of the expressions. These results do not make clear whether and to what extent the braille readers have improved their comprehension. The use of more complicated test items may provide insight into the development of the braille readers' comprehension. The fourth limitation concerns threats to the internal validity of the study design, in this case a history and a testing threat. A history threat refers to the possibility that other aspects related to the braille readers' experiences in mathematics learning may have been important for their performances and their awareness and use of the expressions' structure. A testing threat refers to the possibility that taking the pre-test affects how participants do on the posttest. It is possible that there was an interaction between the pre-test and instruction. It is also possible that the braille readers remembered the answers from the pre-test. Adding a comparable control group and using equivalent – and not identical – items, would counter these threats to internal validity.

The current study has provided a new way of supporting braille readers in reading and comprehending mathematical expressions and equations. The results indicate that the presented approach stimulates braille readers to keep improving their tactile reading strategies, since they performed better not only after the intervention but also after a retention period. Follow-up research should investigate whether the change in strategy use is also relevant for braille readers who use more than one or two fingers while reading. This is important, because it is known that some braille readers do (Radojichikj, 2015; Wanja, Murugami, & Bunyasi, 2021).

Based on our findings, we recommend that braille readers receive individual instruction on how to read and comprehend expressions and equations in braille. This instruction should be given by a mathematics teacher who is able to connect finger movements to the mathematical structure of an expression or equation.

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Appendix A: Test for Braille Reader R. (Grade 8, havo)

I Point out the +, - and * on the braille display

9641 + 83153 - 4732

728.3 - 481.2 * 3.1

II Point out the variable on the braille display

x + 1 + x = 5

4x - x + 3 = 6

III Read the expression

45.86 + 11.34 - 3.4

IV Read the equation

3 + sqrt(9) = 3 * 2

V Enter the number two on the dots and calculate the value of the expression

4 * .. + 3

2 * .. + 1 + ..

VI What number should be on the dots?

4 + .. = 6

3 * .. + 1 = 7

VII Simplify the expression

x + 4 + 2x

3x - 2x + 1

VIII Calculate the value of the expression

45.7 - 13.4

5/7 * 2/9

45.8 + 10.4 - 2.4

4 + 2 * (10 - 3)

2 + 1 - (2^2 + 1)

IX Is the equation correct?

4 + 2 * 3 = 10

3 + 1/2 = 2

(5 + 1)/2 = 3

X Solve the equation

4 + x = 3

4x = 1 + 2

Chapter 4

Teachers' Skills and Knowledge in Mathematics Education for Braille Readers

This chapter is based on:

Van Leendert, A., Doorman, M., Drijvers, P., Pel, J., & Van der Steen, J. (2021). Teachers' skills and knowledge in mathematics education for braille readers. *Technology, Knowledge and Learning.* https://doi.org/10.1007/s10758-021-09525-2.

Abstract Braille readers use a braille display and text-to-speech synthesizer while reading and comprehending mathematical expressions and equations. Teachers need to have technological, pedagogical and content (TPACK) knowledge and skills for exploiting the potential of these devices in mathematics education. They have to understand how the use of assistive technology influences the teaching and learning of mathematics. Therefore, the aim of the current study is to support teachers to better understand the dynamic relation between the different TPACK domains. That is why a professional development course was developed in which five mathematics teachers of braille readers in special secondary education participated. The development of the teachers' TPACK knowledge was analyzed, and course characteristics that helped to develop this knowledge were identified. The results show an increased awareness of the importance of assistive devices, but only a small positive effect in TPACK knowledge and skills. Course activities related to the braille display, the mathematical braille notation and mathematical vocabulary helped to develop TPACK knowledge. However, course activities related to the text-to-speech synthesizer and working in heterogeneous groups did not work out so well. A better understanding of what teaching mathematics to braille readers means is expected to improve their learning.

Keywords*:* assistive technology ● braille readers ● mathematics education ● professional development course ● TPACK

4.1 Introduction

The importance of technology in mathematics education can hardly be overestimated. Various studies have shown that the strategic use of technological devices can support both the learning of mathematical procedures and skills and the development of advanced mathematical competences such as problem solving, reasoning and justifying (e.g., Gadanidis & Geiger, 2010; Pierce & Stacey, 2010; Roschelle et al., 2010). Although teachers generally appreciate the benefits of educational technologies, they often find smooth and effective integration of new educational technologies challenging. According to the so-called TPACK model, effective teaching with technology asks for a rich collection of knowledge and skills from teachers varying from technological (TK), pedagogical (PK) and content knowledge (CK) as well as an understanding of the interaction between these components (Koehler & Mishra, 2008; Mishra & Koehler, 2006). This interaction is constantly changing and, therefore, very complex.

The technology boom of the last decades has produced an abundance of devices to assist in learning and teaching, including those useful to teachers of students who are visually impaired (DePountis, Pogrund, Griffin-Shirley, & Lan, 2015). The innovations in assistive technology support the accessibility of higher education and may serve as an equalizer in the lives of people with disabilities (Michaels & McDermott, 2003), but also have an impact on the role and responsibilities of their teachers. Zhou, Parker, Smith and Griffin-Shirley (2011) did a survey of 165 teachers of students with visual impairments to investigate their perceptions of their knowledge of assistive technology. The results showed, in particular, low ratings of the competences related to mathematics and science.

Although the importance of assistive technology for the success of students with visual impairments is no longer subject to debate, there remains a strong concern as to whether these students receive adequate instruction and training in this area (Pogrund & Smith, 2012; Zhou, Parker, Smith, & Griffin-Shirley, 2011). This is particularly the case for visually impaired students whose primary reading medium is braille (from here on braille readers). They need assistive devices, such as the braille display, tactile drawings, and tangible models, to get access to and explore mathematical representations. If we consider the teacher as the key to using assistive technology in mathematics lessons, then knowledge and skills described by the TPACK model should be considered important. The aim of this study, therefore, is to support teachers to better understand the dynamic relation between technological, pedagogical and content knowledge in mathematics education for braille readers. To investigate this, a professional development (PD) course for mathematics teachers of braille readers was developed.

4.2 Theoretical Framework

The TPACK model for teachers' competences

Digital technologies gain interest in many educational contexts around the world (e.g., Mumtaz, 2000). The availability of these technologies challenges teachers' practices (Adler, 2000; Gueudet, Pepin, & Trouche, 2012). The teachers' ability to use the opportunities offered by digital devices is a major determinant of the success of using these devices in mathematics education (Drijvers, Tacoma, Besamusca, Doorman, & Boon, 2013; Powers & Blubaugh, 2005).

Mishra and Koehler (2006) have developed the TPACK model that describes the knowledge and skills that teachers need to successfully integrate technology in their classroom practices (see Figure 4.1). This conceptual model builds on Shulman's (1986) demarcation of the professional knowledge of teachers as pedagogical content knowledge (PCK). This knowledge is unique for teachers and distinguishes, for example, a mathematics teacher from a mathematician. Mishra and Koehler (2006) expanded this model with the addition of technological knowledge (TK). The TPACK model acknowledges the complex interaction between technology (TK), pedagogy (PK) and content (CK) knowledge (Mishra & Koehler, 2006).

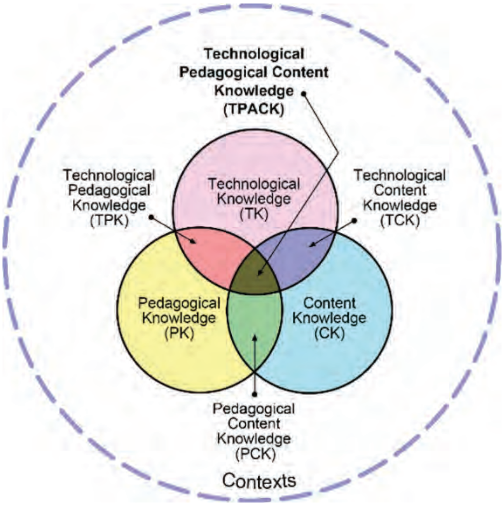


Figure 4.1 The TPACK model. Retrieved from http://www.tpack.org/

Although the TPACK model has been criticized for difficulties in distinguishing between the included constructs (Cox & Graham, 2009; Graham, 2011; Voogt, Fisser, Pareja Roblin, Tondeur, & Van Braak, 2013), the strength of the model is that it questions content-neutral or pedagogical-neutral teaching practices (Harris, Mishra, & Koehler, 2009; Koehler, Mishra, & Cain, 2013). Specific technologies have their own potentials, affordances, and limitations that make them more suitable for certain tasks and pedagogies than others (Koehler & Mishra, 2008). For example, the use of the graphics calculator – in contrast to a calculator that does not display graphs – encourages the integration of geometric and algebraic activities (Drijvers & Doorman, 1996). Understanding how the affordances and limitations of technologies affect what teachers do in their class is not easy and may necessitate rethinking teacher education and professional development. Teachers' knowledge, however, is not the only factor influencing their TPACK construction. Contextual factors, such as the availability of technological support, school culture and teachers' perception of their responsibilities are influential factors as well (Koehler, Mishra, & Cain, 2013; Koh, Chai, & Tay, 2014).

The TPACK's T in mathematics for braille readers. In the current study, the T in TPACK stands for assistive technology that braille readers have to use to get access to mathematical representations and to communicate mathematically. Assistive technology refers to extra materials or time in addition to what "ordinary" students get, and to alternative materials (e.g., text in braille or tactile drawings) (Young & MacCormack, 2018). In this study, the most important assistive devices are braille, the braille display, the text-to-speech synthesizer and the mathematical braille notation.

In the Netherlands, braille readers in secondary education use a laptop with screen reader software. A braille display is connected to the laptop. The screen reader software allows a braille reader to read the text that is displayed on the laptop screen with a braille display or text-to-speech (from here on TTS) synthesizer. The braille display enables a braille reader to read one line of text at a time. Text-to-speech synthesis is a technology that provides a means of transforming text from a descriptive form to spoken language. The TTS synthesizer can read a character, a word, a line or full text screen.

Braille readers and students who have typical vision (hereafter called print readers) make different use of their senses to perceive mathematical expressions. Because braille readers lack visual input, they have to rely on assistive devices to access mathematical representations. They read mathematical expressions and equations in braille, either in coordination with speech synthesis or not. This is challenging for several reasons. First, perception based on touch or hearing is more serial than parallel perception based on vision (Millar, 1994; Millar 1997). With serial processing, it is more difficult to get an overview of expressions or equations (Van Leendert, Doorman, Drijvers, Pel, & Van der Steen, 2019). Second, braille characters have a low redundancy because they only differ from each other in the presence or absence of a dot in one of the six (or eight) possible locations (Millar, 1997; Tobin & Hill, 2015). This low redundancy makes it challenging to discriminate between the characters. Compare, for instance, the braille characters ⠍ (*m*) and ⠭ (*x*). A third factor is the lack of contextual clues, like those experienced in non-mathematical text that may include familiar words and phrases. In summary, it is difficult to read mathematical expressions accurately in braille, because of serial perception, low redundancy and lack of contextual clues.

The (formal) mathematical notation that print readers use contains many unfamiliar symbols and makes use of techniques such as arranging characters at different heights above or below the base line to convey structure (Stöger & Miesenberger, 2015). Examples are the square root sign and the fraction notation (e.g., √4 and $\frac{1}{4}$). These visual cues help print readers to understand the structure of an expression. Braille and speech, however, are both linear output modalities. Therefore, an expression in the mathematical notation has to be transformed into an expression in a linear notation. Consider the expression $(x+2)^2$. When following the rules of an 8-dot Dutch braille notation, this expression needs to be transformed into the expression (x + 2)^2. We call this a linear-print expression, which stands between the representation in the formal mathematical notation and the representation in braille. The screen reader software transforms this linear-print expression into a linear-braille expression (⠦⠭ ⠖ ⠣⠴⡮⠣) or into speech (“open bracket x plus two closed bracket squared”) (Figure 4.2). There is a one-to-one correspondence between the ASCII characters in the linear-print expression and the braille characters in the linear-braille expression. How the linear-print expression is pronounced depends on the settings of the screen reader software.

It is challenging to perceive the linear-braille expressions through touch. It is very different from scanning mathematical expressions with the eyes. When one has typical vision, one can get an overview of these expressions and, for instance, distinguish the difference between an equation and a formula within a split second. Braille readers use their fingertips while reading in braille. This reading is serial, because the fingertips must necessarily pass over all sequential braille characters on a line (Hughes, 2011; Millar, 1994, 1997; Pring, 1984).

Hence, braille readers need to decode the braille characters and at the same time build an overview of the expression. When braille readers use a speech synthesizer, they must also build an overview. Speech synthesis provides less structural information than braille, because braille offers spatial information about the expression and allows for more control while scanning. An advantage of speech synthesis over braille is that it allows for a faster pace of reading.

In summary, the representation of expressions in braille or through speech synthesis is different from the representation in the mathematical notation that print readers use. In addition, tactile or audible perception is different from visual perception. Therefore, the use of braille or speech synthesis influences the way in which mathematical content is presented and can be manipulated, and they influence the teaching and learning of braille readers.

The TPACK's P in mathematics for braille readers. The classes at schools for visually impaired students are often very heterogeneous (Baril, 2008). The students differ in visual ability, in academic ability and in their use of assistive devices. Teachers have to use assistive technology, such as the mathematical braille notation, tactile drawings and 3D models, to connect their teaching to the personal needs and preferences of their braille readers in order to enhance individual skills.

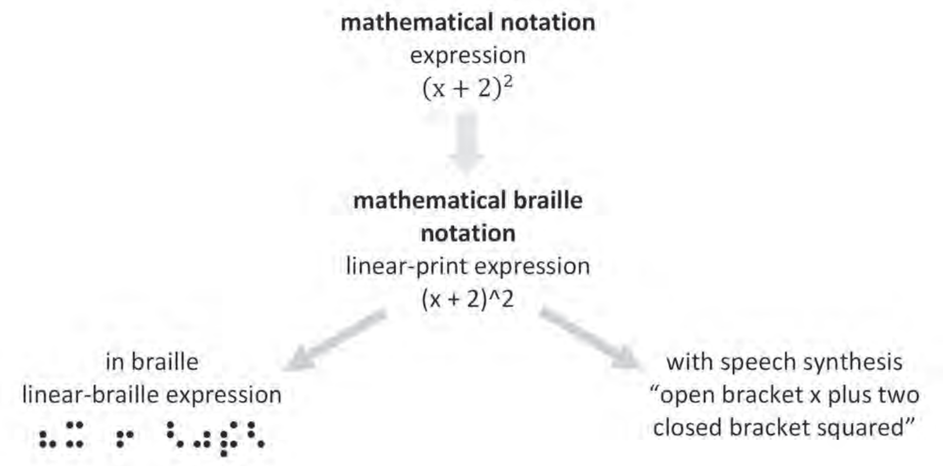


Figure 4.2 Transformation of an expression into braille or speech synthesis

Note: The mathematical braille notation used is an 8-dot Dutch mathematical braille notation.

Another, more inclusive, approach is cooperative learning. In this approach, the emphasis is on creating opportunities for students to learn to collaborate and communicate mathematically, which is important for the overall development of their mathematical proficiency needed for participating in future work and society (Riccomini, Smith, Hughes & Fries, 2015). A special form of cooperative learning in mathematics is inquiry based learning (IBL). This approach includes student-centered teaching strategies with a focus on processes of inquiry that include planning, collaborating and communicating. Mathematics teachers need to support these processes of inquiry with carefully designed tasks, structured lessons to ensure progress in the learning of knowledge and skills (Artigue & Blomhøj, 2013). IBL is driven by open questions and multiple solution strategies, and teachers need pedagogical skills to have students' approaches formulated, compared, evaluated and discussed (Maass & Doorman, 2013).

In summary, the P in TPACK includes skills to value and connect to individual students' preferences in reading and to communicate mathematically, as well as skills to organize inquiry in their classrooms.

The TPACK model for mathematics teachers of braille readers

In this paper, TPACK is introduced as a conceptual model for the knowledge base that teachers need to effectively teach mathematics to braille readers. We interpret TK as knowledge about assistive devices such as the braille display and TTS synthesizer. Teachers need to be able to identify useful devices for braille readers and to be able to continually adapt to technical innovations. Technological Content Knowledge (TCK) is an understanding of how assistive devices and content influence and constrain each other, for example, understanding that it is difficult to get an overview of a mathematical expression while reading in braille. Technological Pedagogical Knowledge (TPK) is an understanding of how teaching and learning change when assistive devices are used. For instance, reading speed probably will decrease when a braille reader switches from reading with speech synthesis to reading in braille. Finally, Technological Pedagogical Content Knowledge (TPACK) is an understanding of how to represent and explain mathematical concepts to braille readers and the ability to use and select the best assistive devices for specific mathematical practices. For instance, the ability to use and select an audible graph when the braille reader needs to get a global overview of the relation between two variables and a tactile graph (or a table in braille) when more detailed information is needed. According to TPACK, the use of assistive devices is an integral tool for teaching mathematics to braille readers (Figure 4.3).

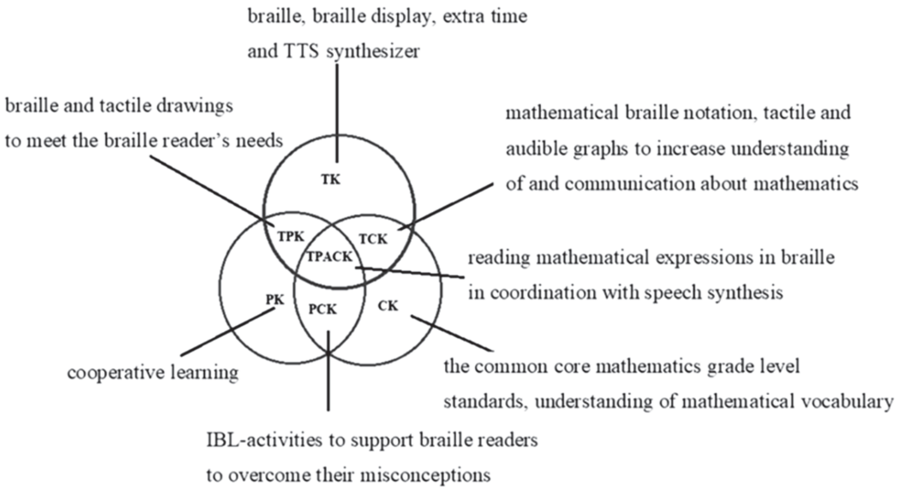


Figure 4.3 A TPACK model for mathematics teachers of braille readers

We developed a TPACK-based course for mathematics teachers of braille readers with the aim to support them to better understand the dynamic relation between technological, pedagogical and content knowledge in mathematics education for braille readers. The design of the professional development (from here on PD) course was guided by different principles. First, the activities have a focus on content, pedagogy and technology closely linked to the practices of the teachers (Koehler, Mishra, & Cain, 2013; Koh, Chai, & Tay, 2014). Second, the course activities are inquiry-based (Van Veen, Zwart, & Meirink, 2010). An example of such activities are the IBL-activities (Swan, Pead, Doorman, & Mooldijk, 2013). Third, in some of the activities, the instructor serves as a role model to illustrate expected teaching practice (Lunenberg, Korthagen, & Swennen, 2007). Fourth, the teachers actively construct knowledge and learn together during the course through small group activities (Van Veen, Zwart, & Meirink, 2010). Fifth, the teachers get opportunities to try out in their practice (Guskey, 2002) and reflect on and discuss their experiences at follow up meetings.

Research questions and hypotheses

The following research questions were addressed:

> *Research question 4.1: How does teachers' TPACK in mathematics for braille readers develop during a four-session PD course?*

Although this study has an explorative character, based on previous research (Koehler, Mishra, & Cain, 2013; Koh, Chai, & Tay, 2014) we expect that the teachers will develop their knowledge and skills from believing that the use of assistive devices is the responsibility of the braille readers to understanding that they have a role in the use of these devices. This is one of the results of their better understanding of the intertwinement between technology, mathematical content and pedagogy.

> *Research question 4.2: Which characteristics of the PD course were beneficial in helping teachers to develop TPACK knowledge and skills, and which characteristics appeared to be less useful?*

The course was guided by design principles that characterized the content and the way of working in the PD course. By connecting activities in the course to individual and collective mathematical practices, we expected the teachers to feel the need for a better understanding of how to work and communicate mathematically with assistive devices and how to support braille readers in using these devices. The alternation of modelling new teaching methods with activities during which the teachers can construct or adapt these methods was intended to give them the opportunity to make the new knowledge and skills usable for their practice. Finally, we expected that try-outs and reflective discussions would help them to share positive experiences and to make explicit what works and how to overcome challenges in daily practice.

4.3 Methods

Design of the study

To answer the research questions, a PD course was developed. The design of this course was guided by the TPACK model for mathematics teachers of braille readers. A mixed-method approach was taken for investigating the effect of the course on the teachers, using a pre- and a post-interview, an evaluation questionnaire and a logbook to record relevant observations and to contrast these observations with hypothesized learning goals. Results from the pre- and post-interviews and the evaluation questionnaires were used to answer the first research question. To answer the second research question, we analyzed the learning and reported feedback of two case teachers in more detail. In the case reports, we included contextual factors that influenced the TPACK development of the teachers (Koehler, Mishra, & Cain, 2013; Koh, Chai, & Tay, 2014).

Participants and context

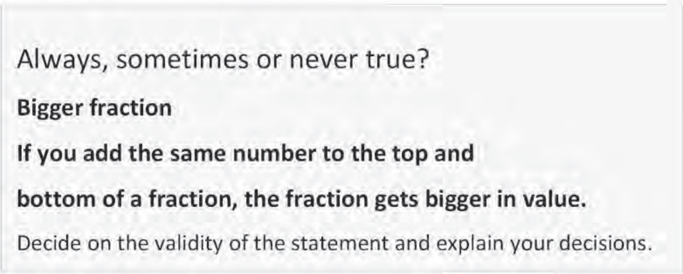
In the Netherlands, there are four secondary schools for students with a visual impairment. More than half of all braille readers in secondary education, from 12 to 18 years, attend one of these schools. Braille readers who, in addition to the visual impairment, have serious psychological or physical problems sometimes go to another special school. The other braille readers are in regular education. The schools for students with a visual impairment offer education on Praktijk Onderwijs, vmbo and havo level. Students of different levels, but of more or less the same age, are often placed in one class. Students who have a visual impairment and want to do vwo must attend a regular school. Five mathematics teachers, from two of the schools for students with a visual impairment, volunteered to participate in the PD course. Three of them had less than five years of experience teaching students with visual impairments, while two teachers had more than five years of experience. The teachers did not receive any specific training in teaching mathematics to students with a visual impairment. They were all qualified teachers, but only one was qualified to teach mathematics. About 30% of their students were braille readers. The other students were visually impaired but did not use braille. The teachers taught in small groups of two to eight students. All students used the same mathematics books, in print or in braille, as students with typical vision (in regular education). A braille teacher had taught the braille readers how to use the braille display and TTS synthesizer, but not how to use these devices in mathematics.

Design of the professional development course

The design of the PD course was guided by the TPACK model. The intended and hypothesized learning goals were expressed in TPACK goals (see Table 4.1). The topics in the first two sessions were related to supporting braille readers in reading and comprehending mathematical expressions in braille and speech synthesis. The affordances and constraints of the braille display and the TTS synthesizer were discussed and applied to tasks involving mathematical expressions. We expected that this would support teachers to develop TK, TPK, TCK and TPACK knowledge and skills.

The third session was about working with heterogeneous groups with an emphasis on cooperative learning. The teachers participated in and reflected on IBL-activities. We expected that this would improve their PCK and TPACK knowledge and skills. Figure 4.4 shows one of the IBL-activities in which the teachers took part. This activity was derived from the PRIMAS (Promoting Inquiry in Mathematics and Science Education across Europe) Project. The teachers had to decide and explain, in small groups, whether the statement about fractions was always, sometimes or never true.

The last (fourth) session was oriented on sustaining the acquired skills in daily practice. New perspectives were also discussed, such as the 3D printer to make tangible models. As recommended by Bleicher (2011), we used the last session to stimulate reflection and evaluation of the course. Table 4.1 provides an overview of the course.



Always, sometimes or never true?

Bigger fraction

If you add the same number to the top and bottom of a fraction, the fraction gets bigger in value.

Decide on the validity of the statement and explain your decisions.

Figure 4.4 Example of an IBL-activity

Table 4.1 Design of the PD course

Learning goals	Corresponding tasks and activities
Session 1: Supporting braille readers to read and comprehend mathematical expressions on the braille display	
1 Knowledge of the problems braille readers encounter when reading mathematical expressions and equations on the braille display.	Braille-experience. Instructor provides vignettes on students struggling with mathematics and lets the participants actively connect to their practices and pose possible explanations.
2 Knowledge about the braille notation.	Participants make analyses of errors committed by braille readers when simplifying an expression or solving an equation. *Homework assignment*: Develop a plan for supporting a braille reader, in your own practice, to understand the structure of an expression. Make a short report or video clip of the interaction.
Session 2: Supporting braille readers to read and comprehend mathematical expressions on the braille display and with the TTS synthesizer	
3 Knowledge about mathematical vocabulary.	Instructor helps teachers to use mathematical vocabulary when talking about expressions and equations and explains why this is important.
4 Knowledge of and skills in reading mathematical expressions and equations with speech synthesis.	Instructor models how to adapt the verbosity settings of the Screen Reader Software.
5 Understanding of how the use of braille and speech synthesis complement and reinforce each other while doing mathematics.	Discussion, related to the participants' experiences, on the advantages, limitations and challenges of reading in braille or with speech synthesis in mathematics lessons. Instructor models how to use braille in

	coordination with speech synthesis while reading and comprehending mathematical expressions. *Homework assignment*: Record your findings with speech synthesis during the next three weeks. Discuss your findings with a braille reader. Write a report of the discussion.
Session 3: Working with heterogeneous groups	
6 Knowledge about cooperative learning.	Information on the importance of inquiry and communication while learning mathematics. Teachers participate in and reflect on IBL-activities. *Homework assignment:* Support a braille reader with exploring a tactile graph. Write a report of the intervention.
Session 4: Sustaining daily practice	
7 Knowledge about new perspectives.	E.g., 3D printer to create tangible models, different software to create audible graphs.
8 Ideas for sustaining daily practices	E.g., possibilities to share practice and experiences in follow-up meetings, Microsoft Teams for online collaboration.
9 Evaluation of the PD course	Reflection on and evaluation of the PD course

Procedure

The research followed the order of pre-interviews, PD course, evaluation questionnaires and post-interviews, all within four months. The pre- and post-interviews contained the same questions. We investigated the responses and the differences between those given in the pre- and post-interviews. The course was provided by the first author in four three-hour sessions spread over three months. The pre- and post-interviews were conducted by the course instructor at the participants' schools: once at the start and once at the end of the course. The interviews were scheduled for 40 minutes, but sometimes took almost an hour. The interviews were recorded and transcribed verbatim. At the end of the course, each teacher completed the evaluation questionnaire. They could take the time to answer the questions. No one needed more than 20 minutes to complete the evaluation questionnaire.

Data collection instruments: Interviews and evaluation questionnaires

The data from the interviews and the evaluation questionnaires were used to trace the development of the teachers' understanding of TPACK in mathematics for braille readers. The interviews (see Appendix A) were semi-structured with open-ended questions (Whiting, 2008). These questions allow the participants to fully express their viewpoints and opinions (Turner III, 2010). The first question of the interview concerned the pace of reading mathematical expressions by braille readers. The second question was about errors made by braille readers when processing expressions in braille. The teachers were required to explain the errors. To introduce the third and last question, the teachers were shown a model of three overlapping circles: technology, mathematical content and pedagogy. No other information was given. The teachers were asked to report their perception of confidence in the different domains of this model. The evaluation questionnaire (see Appendix B) consisted of five open-ended questions. The questionnaire was developed so that the answers could provide information about the teachers' background, their evaluation of the course, and reported changes in their (TPACK) knowledge and in their attitude. Finally, the course provider's logbook recorded observations and deviations from the original plans for the course. Most notes were made immediately after a PD session.

Development of the coding instrument

The interviews and evaluation questionnaires were coded. The coding is used to segment and reassemble the data and is therefore the most important aid in conducting an analysis (Boeije, 2009). Prior to the actual coding, we began with an "initial list" of potential codes based on the PCK (Shulman, 1986) and TPACK model (Koehler, Mishra & Cain, 2013; Mishra & Koehler, 2006, 2008). This was the start of the codebook. The codes were TK, PK, CK, TCK, PCK, TPK and TPACK. Next, the definitions of these codes were adapted for this particular study. After attempting to code the data utilizing the definitions, we recognized that we could give more explicit definitions. Once we agreed on the definitions, we selected example quotes within the data that best illustrated each code (for codebook, see Appendix C).

Data analysis

In preparation of the coding of the interviews and evaluation questionnaires, we segmented the answers into units of analysis that retained focus on one topic. For example, "They sometimes wear headphones. And then they only use one earpiece, so they can still hear me. At the end of the lesson, those students are exhausted." These units ranged in length from a few words to a couple of lines. The units were coded with one of the TPACK components when they showed proficiency in the related type of knowledge or skill. For example, the unit of analysis just described was coded as PK.

To address the first research question, the interviews and evaluation questionnaires were coded using the codebook. Two interviews and one evaluation questionnaire were also coded by a second coder, resulting in 20% disagreements. For example, sometimes a teacher indicated very clearly which TPACK knowledge he or she was missing. In that case, one coder wanted to assign a code and the other coder did not. These cases were discussed until consensus was reached. This resulted in keeping the original set of codes, but with improved and better discriminating definitions. For each question, the assigned codes from the pre- and post-interviews were compared on frequency. In addition, we described the nature of the change in codes.

To address the second research question, two cases studies were conducted. B. and W. were selected for these studies because they seem to be at opposite ends of the participating teachers' skills and knowledge spectrum. For the case reports, different sources of data were collected: teachers' interviews, teachers' evaluation questionnaires and the course provider's logbook. For each case study, the data from the case as well as the data from the other teachers were used. The logbook contained information on the conditions of the actual activities (e.g., time available, involvement of participants) and provided context for the responses on the interviews and evaluation questionnaires. We checked consistency by triangulating data from interviews, evaluation questionnaires and observations during the course (see Figure 4.5). The case reports were described chronologically, which helped to find causal relationships between the characteristics of the course and the development of the teachers' TPACK knowledge and skills. We analyzed whether the intended learning goals of the course (Table 4.1) were achieved and which characteristics of the PD course helped to achieve these goals.

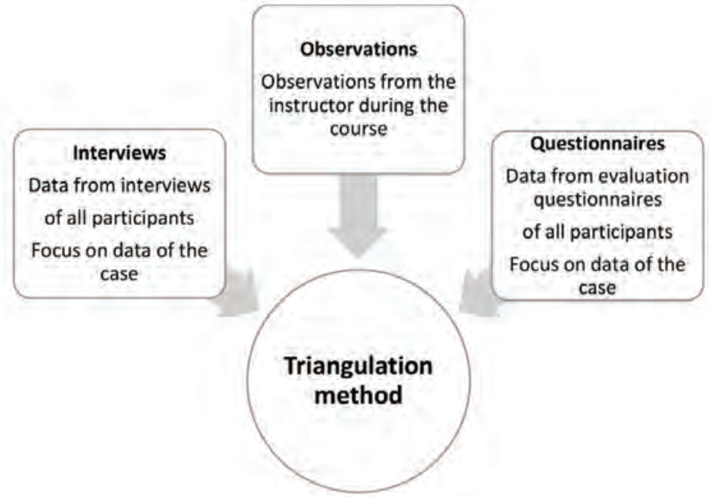


Figure 4.5 Triangulation method for the case study

4.4 Results

Research question 1: Teachers' TPACK development

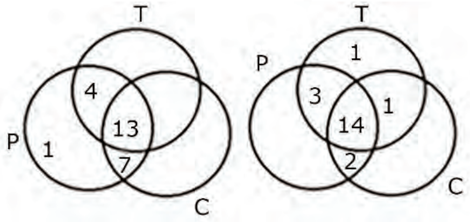
Reading mathematical expressions by braille readers. The first research question addresses the teachers' development with respect to their technological, pedagogical and mathematical content knowledge and skills. The first question of the interview, consisting of two sub-questions, was about reading mathematical expressions. In question 1a, teachers had to explain why braille readers need more time to read and comprehend mathematical expressions and equations than print readers. Teacher W. answered in the post-interview, "They have to build an overview of an expression." This response was assigned the code TPACK.

In question 1b, the teachers were asked whether they take into account in their classes that braille readers need more time to read and comprehend mathematical expressions and equations. The answers to question 1b were almost the same for the pre-interview as for the post-interview. All teachers said that braille readers get all the time they need to complete the mathematical tasks. Teacher T. gave individual instruction. This allowed the braille readers to work at their own pace. Teacher B. gave the braille reader an extra half hour of support every week. All teachers allowed the braille readers to skip assignments if they could show that they had mastered the material sufficiently.

Figure 4.6a shows the total TPACK scores of the teachers for the first question. In the pre-interview, 25 codes were assigned to the responses to question 1, and in the post-interview 21 codes. In the pre-interview, the most frequent category was TPACK (13), followed by PCK (7), TCK (4) and PK (1). In the post-interview, the most frequent category was TPACK (14), followed by TCK (3), PCK (2), TCK and TK (both 1). The result shows a very small shift from only P-related categories to more T-related categories.

question 1

Reading mathematical expressions

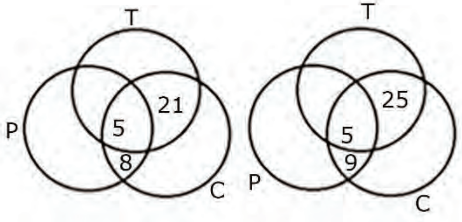


pre-interview post-interview

Figure 4.6a

question 2

Errors in processing expressions

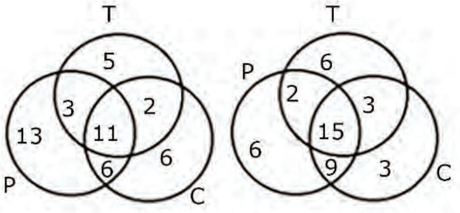


pre-interview post-interview

Figure 4.6b

question 3

Perception of confidence in TPACK



pre-interview post-interview

Figure 4.6c

Errors in processing expressions. The second question of the interview involved errors committed by braille readers while processing expressions. An illustrative TPACK response on question 2a (“Give a possible explanation for 5 - –7 = -2”) was recognizing that the representation of the expression is not according to the rules and unfamiliar for braille readers: “The braille reader doesn’t recognize this as a minus sign on the braille display. Therefore, the braille reader processed five minus seven”. (What is meant here is that the screen reader does not recognize the symbol – as a minus sign). In the pre-interviews, all teachers, except for teacher B., said that they shared their own difficulties with the braille notation with their braille readers. One teacher said that it would be fine if the braille readers knew that their teacher also made errors. That would strengthen the bond between teacher and student. Similar comments were made in the post-interviews, but to a much lesser extent.

The results show that 34 codes were assigned to the responses in the pre- interviews, and 39 in the post-interview (Figure 4.6b). In the pre-interview, the most frequent category was TCK (21), followed by PCK (8) and TPACK (5). In the post-interview, the most frequent category was TCK (25), followed by PCK (9) and TPACK (5).

Perception of confidence in TPACK. The third question of the interview addresses the teachers’ perceptions of confidence in the different domains of TPACK. In the pre- and post-interviews, all teachers mentioned that they felt confident on the P-domain (in isolation). In the post-interviews, all teachers mentioned that they increasingly checked whether the braille readers could read the expressions in braille and corrected them when necessary. In the post-interviews, they also mentioned that they wanted to integrate more assistive devices, such as tactile drawings and tangible models, in their lessons, but they still felt insecure in the area of assistive technology. In the pre-interview, B. said that her mathematical knowledge was “good enough”. This response was assigned the code CK. In the post-interview, she said that her mathematical knowledge was not good enough to properly adapt the assignments in the mathematical textbook for the braille readers. Although awareness of one’s own limitation of knowledge is relevant for the teacher’s learning process, this answer was not assigned a code since no proficiency was shown. Figure 4.6c shows that 46 codes were assigned to the responses in the pre-interview, and 44 in the post-interview. The most frequent category in the pre-interview was PK (13) followed by TPACK (11), PCK and CK (both 6), TK (5) and TPK (3). The most frequent category in the post-interview was TPACK (15), followed by PCK (9), PK and TK (both 6), CK and TCK (both 3) and TPK (2).

Self-reported changes in working practice. Finally, we give the results of the responses on the evaluation questionnaires. These results are not depicted in Figure 4.6. Only the responses to the questions that relate to TPACK knowledge and skills were coded (question 3 and 4, see Appendix B). All teachers mentioned that they were more conscious of the problems that braille readers encounter while doing mathematics. However, they felt that they still had a lot to learn in this area and they wanted more support and direction. For example, the teachers wanted more support from the ICT department. The responses to the evaluation questionnaire were assigned the codes TPACK (13), TPK (4), TCK (4) and TK (2).

Research question 2: Characteristics of the PD course that support the TPACK development of the teachers

The second research question refers to which characteristics of the course helped to develop TPACK knowledge and skills. To answer this question, two case studies were conducted. The cases were teacher B. and teacher W. Teacher B. had worked at the school for visually impaired students for more than 5 years. She was a qualified primary school teacher. Her class, at the time of the study, consisted of four 7th grade students at vmbo and havo level: one braille reader and three students who were visually impaired but did not use braille. She used a more teacher-centered than student-centered learning approach. She gave the braille reader extra mathematics lessons for half an hour every week. In these lessons, she supported the braille reader, for example, to explore tactile graphs. Thanks to these extra lessons, the braille reader could keep up with his classmates.

B. was responsible for braille education in her school. She could read non-mathematical braille text very well, but had difficulties reading mathematical text in braille. She had little knowledge about the braille display and the TTS synthesizer. B. mastered the mathematical content she had to teach, but not at a level that exceeded the level of teaching.

Teacher W. has been working at the school for visually impaired students for less than 5 years and taught students from grade 9 to 11 at vmbo and havo level. He was educated as an engineering teacher. He taught at a different school than B. His classes, at the time of the study, consisted of two to five students. There were at most two braille readers in each class. In each mathematics lesson, W. gave instruction to one or two different students at a time. In the meantime, the other students worked independently.

The first session of the PD course was about how to support braille readers with reading and comprehending mathematical expressions in braille.

After this session, B. practiced reading mathematical expressions in braille on her own. She used to read expressions aloud to the braille reader to support him to make pace. She realized, due to discussions during the first session of the course, that this would not help him to improve his braille reading. Therefore, over the time of the course she increasingly asked the braille reader to read expressions aloud and corrected him when needed. She was very surprised to find that the braille reader had so many difficulties reading mathematical expressions accurately.

W. knew just a little about braille, the braille display and the TTS synthesizer. In the Dutch mathematical braille notations – as well as in many other mathematical braille notations – more brackets are used as grouping symbols than in the (formal) mathematical notation. This is because all characters need to be represented on the baseline. Compare, for example, $\frac{4+3}{9-7}$ with (4 + 3)/(9 - 7). W. found it difficult to understand the structure of a linear-print expression when it contains many brackets. Thanks to the activities in the course, he was able to overcome these problems. Over the time of the course, W. increasingly asked braille readers to read expressions aloud and corrected them if necessary. In the post-interview, he said that he would like to support braille readers with using the braille display in mathematics but not at this stage of his career. “First, I have to put the other things on track.”

In the second session of the PD course, the teachers were taught how to teach braille readers to use speech synthesis while reading and comprehending mathematical expressions. The idea was that braille readers start with using speech synthesis to get an overview of an expression and continue in braille. B. and W. both struggled to understand which symbols in an expression are important to get an overview. In addition, B. feared that teaching braille readers to use braille in coordination with speech synthesis would be detrimental to their motivation to (learn to) read mathematical expressions in braille. About the use of the TTS synthesizer W. said, “Braille readers use the TTS synthesizer themselves. They don't need our support.” Furthermore, in this session, the teachers learned to talk about expressions in mathematical vocabulary.

The third session was about “working with heterogeneous groups.” The focus was on cooperative learning. This approach was very new to B. and W., and they felt it was not a very effective approach. Moreover, they had difficulties with understanding the underlying mathematics in the IBL activities. In the evaluation questionnaire, they indicated that they wanted more information about teaching heterogeneous groups.

The last session was about supporting daily practice. The teachers discussed setting up a community for mathematics teachers of braille readers. B. and W. were enthusiastic about this idea. At the end of this session, the teachers reflected on the main ideas and theories of the course.

In the evaluation questionnaire, all teachers wrote that they were more aware of the difficulties and challenges that braille readers encounter in mathematics. They all wanted to enroll in the next PD course to learn more about the mathematics education of braille readers. W. wrote, "I realize that I lack knowledge about the mathematics education of braille readers."

4.5 Conclusion and Discussion

RQ1: Teachers' TPACK development

Reading mathematical expressions by braille readers. In this paper, we set out two questions, the first of which was: How does teachers' TPACK in mathematics for braille readers develop during a four-session PD course? The data from the interviews and evaluation questionnaires were used to answer this question. For interview question 1, that was related to reading mathematical expressions in braille or with speech synthesis, the results show a very small shift from only P-related categories to more T-related categories. The teachers did not change their daily practice in dealing with the fact that braille readers need extra time to read and comprehend mathematical expressions. None of the teachers indicated that they encouraged or helped the braille readers to speed up the pace when reading mathematical expressions.

Errors in processing expressions. The results with respect to interview question 2 show that the participating teachers improved their knowledge of the braille notation (which is operationalized in an increment of TCK). More specifically, they gained a better understanding of linear-print expressions. However, they did not improve their ability to relate braille readers' errors in processing expressions to problems with braille reading (which would increment the TPACK-category).

Perception of confidence in TPACK. The results with respect to interview question 3 show minimal changes in TPACK codes. In the post-interviews, the teachers focused more on the interaction between the TPACK components, and less on the TK, CK and PK in isolation. However, this did not always result in more assigned codes on the overlapping areas (TPK, TCK, PCK or TPACK). Teacher B. said, for example, that she needs more mathematical knowledge to properly adapt the commands in the braille readers' books. While this was a meaningful comment, no code was assigned to this answer as no proficiency was shown.

Self-reported changes in working practice. The responses for question three and four of the evaluation questionnaire both refer to self-reported changes in the teachers' working practice. The results show a domination of technology-oriented responses.

Overall, the results show an increased awareness that the use of assistive devices influences the teaching and learning of mathematics. The results also illustrate that the teachers feel more responsible for the use of the assistive devices. We assumed, based on research by Koehler et al. (2013) and Koh et al. (2014), a positive relation between teachers' TPACK knowledge and their perception of responsibilities. The results are in line with what we expected, although we hoped for a better result.

RQ2: Characteristics of the PD course that support the TPACK development of the teachers

As a second research question, we wondered which characteristics of the PD course were beneficial in helping teachers to develop TPACK knowledge and skills, and which characteristics appeared to be less useful. The results show that the teachers did improve their understanding of how the braille display and the mathematical braille notation influence the teaching and learning of mathematics (TCK and TPACK knowledge). They also improved their mathematical vocabulary (CK knowledge). The course activities were closely linked to their practice (design principle 1) (Koehler, Mishra, & Cain, 2013; Koh, Chai, & Tay, 2014). That supported teachers to connect with what they already know and to feel the need to support braille readers. The instructor often acted as a role model (design principle 3) to support teachers in putting what they learned into practice (Lunenberg, Korthagen, & Swennen, 2007). The teachers were given ample opportunities to discuss with each other (design principle 4) (Van Veen, Zwart, & Meirink, 2010) and reflected on their experiences in follow up meetings (design principle 5) which supported them to share positive experiences and make explicit what works and how to overcome challenges in daily practice. In summary, the activities that were characterized by design principles 1, 3, 4 and 5 were beneficial for improving TPACK knowledge and skills related to the braille display, mathematical braille notation and mathematical vocabulary. This indicates that the learning goals 1, 2 and 3 (see Table 4.1) were achieved.

The results also show that activities related to the TTS synthesizer and working with heterogeneous groups did not work out well. These activities were guided by the same design principles as the activities just described. However, the teachers did not improve their understanding of how to use speech synthesis while reading and comprehending mathematical expressions (TCK knowledge). Nor did they understand how IBL-activities might resolve misconceptions in mathematics (PCK knowledge). This may be because teachers had preconceptions that were different from the views of learning and teaching that we wished to develop. For example, teacher W. remarked "Braille readers use the TTS synthesizer themselves. They don't need our support." In addition, the teachers felt that cooperative learning is an ineffective approach. Another difficulty was that the teachers did not fully understand the mathematics underlying the IBL-activities. As a result, intended learning goals and underlying activities with respect to the TTS synthesizer and working with heterogeneous groups (see Table 4.1, learning goal 4, 5 and 6) did not emerge.

This study has some limitations. The first is the small number of participants. This offers little basis to generalize the findings. However, we provided detailed descriptions that allow readers to make connections between elements of this study and their own experience. The second limitation concerns the teachers' knowledge of mathematical content and assistive technology. This knowledge was less than we expected. Therefore, the course material was sometimes too difficult for the teachers. Further research into how teachers with lack of mathematical content knowledge and assistive technology can develop their TPACK knowledge and skills – in mathematics education for braille readers – is needed. The third limitation is related to the role and responsibilities of the teachers. A basic idea behind TPACK is that the teacher is the key to using assistive technology in the classroom. This was something we had to discuss very often on the course, as teachers did not always agree. We did not expect this to be the case to that extent. In future studies, we recommend research to investigate how to support mathematics teachers of braille readers in taking responsibility for the use and selection of assistive devices.

This study shows how the TPACK model can be adapted for mathematics teachers of braille readers. This adaptation guided our choices for the topics in the PD course. The role of tactile and auditory perception in mathematics learning were central in our TPACK model and in the PD course. For example, we addressed the complex interaction between mathematical, technological and pedagogical content knowledge needed to support students in reading a mathematical expression with the braille display. This included knowledge about the selection of assistive devices, their strengths and their shortcomings for learning mathematics (Koehler & Mishra, 2008). This study also illustrates how the TPACK model can become instrumental in educational contexts that require specific technological devices or sensorimotor coordination.

In addition to theoretical implications, there are also more practical ones. The TPACK model was useful for the design of the PD course. It helped to identify topics on the domains that overlap, e.g., the mathematical braille notation in TCK. The TPACK model also guided us during the analysis, when we mapped the needs and skills of the teachers. For example, it helped in recognizing that teachers did develop their knowledge of the mathematical braille notation. However, the teachers did not link the errors in reading and processing expressions to the complexity of reading in braille. That would have resulted in an improvement in the TPACK domain. The TPACK model shows that assistive devices all have their own potential, features and limitations that make them more appropriate for certain tasks and pedagogies than for others (Koehler & Mishra, 2008). We created an awareness over the choice of assistive devices and suggested teachers to use braille as well as speech synthesis while reading and comprehending an expression. This study illustrates that the task of the teachers is complex because of all the overlapping areas in TPACK. The PD course provides clues for its significance for practice and for supporting teachers. These suggestions can be used in training both new teachers and in-service teachers.

Finally, a better understanding of what teaching mathematics to braille readers comprises is expected to improve their learning and to afford successful school careers for these students. Our study shows that teaching mathematics to braille readers is not limited to translating instruction materials into braille. It affects all domains of the TPACK model.

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Appendix A: Interview Structure

Question 1: Reading mathematical expressions by braille readers

a) Braille readers need more time to read and comprehend mathematical expressions and equations than students who can see. Can you explain why?

b) Do you take this into account during your lessons? How?

Question 2: Errors in processing expressions

The next questions were typed in Word. The braille reader read the questions on the braille display and calculated the value of the expressions. Can you explain the errors he made?

a) 5 - –7 = -2
b) 3/5 * 2 = 3/10
c) 2^2 + 3^2 = 10
d) 2^2 + 3^2 = 26
e) 1 - sqrt((1 + 3)/(2 + 7)) = 7/9

Question 3: Perception of confidence in the different domains

Can you say something about your perception of confidence in the different domains of this model?

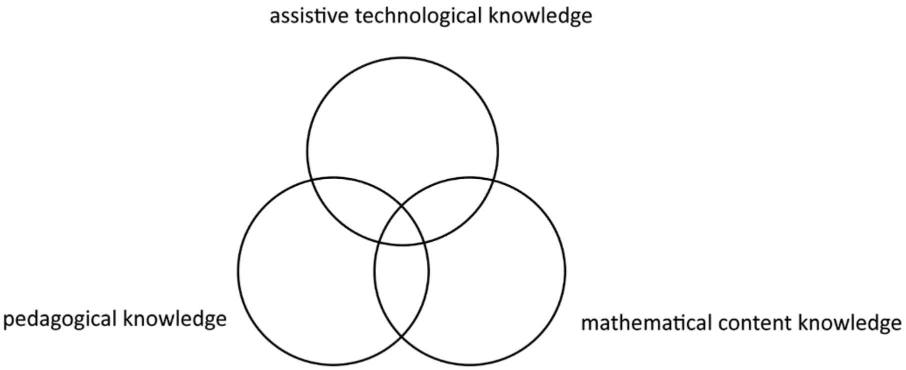

Figure A1 The interview structure

Appendix B: Evaluation Questionnaire

1. What are your education and work experiences?
2. What kind of work do you do at the school for visually impaired students?
3. Which assignments and / or information from the Professional Development course did you use in your daily practice? Explain.
4. On which topics do you want to deepen your knowledge after the course? Explain.
5. What did you think of the course?

Appendix C: Codebook

Table C1 Codebook of the TPACK model of mathematics for braille readers

Code	Description of codes for mathematics for braille readers	Examples
TK	Knowledge about assistive devices for braille readers, being able to identify useful assistive devices and to continually adapt to changes in these devices.	“For braille readers, the most important assistive device is the braille display.”
PK	Knowledge about the processes and practices of learning and teaching braille readers.	“I know how to give braille readers individual attention.”
CK	Knowledge about the actual subject matter that is to be learned and taught to braille readers. This includes knowledge about mathematical content that is not accessible (yet) for braille readers.	“I have sufficient knowledge about mathematics.”
TCK	Knowledge about the manner that assistive devices and mathematical content are reciprocally related.	“In AllerCalc* the number of brackets in an expression can be extremely high.” *AllerCalc is a calculator for braille readers.
PCK	Knowledge about how mathematical instruction and mathematical instruction materials need to be adapted for the learning and the teaching of braille readers.	“Braille readers have the talent but not the pace to do mathematics on a higher level. The pace is the problem.”
TPK	Knowledge of the affordances and constraints of the assistive devices that may be used by braille readers and how the use of these devices influence the teaching and learning of braille readers.	“I can’t really support braille readers when they use the speech synthesizer because I don’t hear what is being said.”
TPACK	Technological pedagogical content knowledge (TPACK) requires an understanding of how to represent and explain mathematical concepts to braille readers using assistive devices. The teacher knows how to select the best assistive devices and how to support braille readers to use these devices optimally while doing mathematics.	“Braille readers need more time to get an overview of an expression because they have to build such an overview while reading in braille.”

Note: This codebook is inspired by Koehler, Mishra and Cain (2013), Mishra and Koehler (2006, 2008) and Shulman (1986).

Chapter 5

Supporting Braille Readers in Mathematical Practices Using the Braille Display in Coordination with the Speech Synthesizer

This chapter is based on:

Van Leendert, A. J. M., Doorman, L. M., Drijvers, P. H. M., Pel, J., & Steen, J. V. D. (2021). Supporting braille readers in mathematical practices using the braille display in coordination with the speech synthesizer. *Journal of the International Association of Special Education, 20*(1), 20–31.

Abstract It is difficult to read mathematical expressions in braille, whether or not in coordination with text-to-speech (TTS) synthesis. In this study, we investigated the effect of an intervention on the braille readers' performance in reading and comprehending mathematical expressions as well as the braille readers' ability to make well-founded choices for the use of the braille display or the TTS synthesizer. The intervention consisted of a professional development course for mathematics teachers who taught to braille readers, in combination with adjustments of the TTS settings of their braille readers' screen reader software. Nine experienced braille readers, aged 13–18 years, whose teachers participated in the course, underwent the intervention. A pre- and posttest and a pre- and post-interview were administered. For mathematical braille reading skills, 6 (22%) and 11 (41%) of a total of 27 answers were correct in the pre- and posttest, respectively. For mathematical speech comprehension, 15 (56%) and 16 (59%) of a total of 27 answers were correct in the pre- and posttest, respectively. Although the study had a small scale, the results of the interviews indicated an improvement in making well-founded choices for the use of braille and speech synthesis. The results suggest that improving mathematical braille reading skills is the key to better mathematical performance.

Keywords*:* mathematical expression ● braille ● braille display ● braille reader ● speech synthesizer

5.1 Supporting Braille Readers in Mathematical Practices Using the Braille Display in Coordination with the Speech Synthesizer

In this study, braille readers are students who use braille as their primary reading medium. These students encounter difficulties while reading and understanding mathematical expressions and equations in braille or braille in coordination with speech synthesis (Van Leendert, Doorman, Drijvers, Pel, & Van der Steen, 2019). There are several reasons for this. First, for braille readers, mathematical expressions and equations have to be represented in a linear notation, because braille and speech are both linear output modalities (Stöger & Miesenberger, 2015). Second, braille readers read with their fingertips or listen to a voice in a sequential pattern (Millar, 1994, 1997). This means that they have a limited perceptual view. While reading in braille, for instance, they only perceive one or two braille characters at a time. Consequently, braille readers cannot benefit from the layout of a mathematical expression that helps sighted students to understand the expression's structure at a glance (Karshmer & Bledsoe, 2002).

Braille readers read braille on paper or on a braille display. A braille display is a device that is typically attached to a computer keyboard. A screen reader allows braille readers to read the text that appears on the screen with a braille display or with a text-to-speech (TTS) synthesizer. It is possible to adjust the TTS software to the needs and wishes of the braille reader, e.g., the speech dictionary can be expanded with a mathematical vocabulary.

The current study focused on reading and comprehending mathematical expressions on the braille display, either on its own or in coordination with the TTS synthesizer. Coordinating these assistive technologies is not straightforward. The issue at stake is how braille readers can be supported in improving their mathematical reading skills. Therefore, in this study, we investigated the effect of adjusted TTS settings—adjusted to mathematical text—of the screen reader software in combination with a teachers' professional development (PD) course on braille readers' mathematical practices. In this study, these practices involved reading and comprehending mathematical expressions and equations in braille or in braille in coordination with speech synthesis.

5.2 Theoretical Background

What is braille?

Braille is a tactile reading and writing system. Braille uses raised dots to represent the characters of the visual script. The braille characters are formed within units of spaces known as braille cells. A full braille cell consists of a pattern of raised dots in a 2 * 3 configuration (⠿). The dots are numbered 1, 2, 3 (left column) and 4, 5, 6 (right column). In 6-dot braille, 64 (= 2^6) combinations, including the braille character for the space, are possible. Hence, not all letters, numbers and symbols from the visual script can be displayed with six dots. This is why announcement signs such as number, uppercase or letter signs are used. For example, the number two may be represented—depending on the mathematical braille notation used—as ⠼⠃ (dot 3456 12). The braille character ⠼ (dot 3456) represents the number sign and ⠃ (dot 12) represents the letter b.

With the arrival of computer technology, there was a desire to drop announcement signs and allow more combinations. Thus, 8-dot braille was created. In 8-dot braille, a full braille cell consists of a pattern of raised dots in a 2 * 4 configuration (⣿). The dots are numbered 1, 2, 3, 7 (left column) and 4, 5, 6, 8 (right column). The number 2, for example, may be represented in 8-dot braille as ⢃ (dot 128) or ⠣ (dot 126). Again, the representation depends on the braille notation used. 8-dot braille is mostly used in combination with a braille display.

Braille characters differ from each other in the presence or absence of a raised dot in one of the possible locations. This implies that braille characters miss the redundancy of print characters which can be recognized by the manifest characteristics that different spatial combinations of straight lines and curves provide (Millar, 2003; Tobin & Hill, 2015). This low redundancy may lead to errors in decoding braille characters. Most braille readers can overcome these problems through experience and adequate training (Radojichikj, 2015).

Reading mathematical expressions with the braille display

Reading and comprehending a mathematical expression requires accuracy and insight into the structure of the expression. Accuracy is critical in reading and comprehending mathematical expressions, because these expressions are context-poor and condensed. As mentioned above, it is difficult to read accurately in braille, because braille characters have low redundancy (Millar, 2003; Tobin & Hill, 2015). It is also challenging to comprehend the structure of an expression while using braille. This is due to the linear representation of expressions and the limited perceptual view that braille readers have. Every braille character must be read individually and sequentially, which makes braille reading exhaustive rather than selective (Hughes, 2011; Hughes, McClelland, & Henare, 2014). It helps that braille is, to a certain extent, static (Archambault, Stöger, Batusic, Fahrengruber, & Miesenberger, 2007). The braille reader can inspect the expression in braille for as long as he or she wants, which at least provides a limited spatial overview. A typical approach to create an overview in braille is to repeatedly read an expression in full, retain and then integrate the information. Specific finger movements, such as lingering with the left index finger on an open bracket while reading and understanding a nested expression, have been shown to support this complicated and time-consuming task (Van Leendert et al., 2019).

As well as these behavioral aspects, the use of the braille display involves some technical issues. When braille readers use a braille display, expressions in the mathematical notation need to be entered into the computer in linear notation. This can be done in different ways dependent on the mathematical braille notation used. For example, when using the 8-dot Dutch mathematical braille notation the expression $\frac{2x^2+1}{3}$ need to be entered as (2x^2 + 1)/3. The screen reader software converts this expression into ⠦⠣⠭⡮⠣ ⠖ ⠡⠴⠌⠩ (dot 236 126 1346 23467 126 space 235 space 16 356 34 146). There is a one-to-one correspondence between the ASCII characters in the linear expression and the expression in braille. Hence, these expressions have the same mathematical structure.

Reading mathematical expressions with TTS synthesizer

Many braille readers also listen to mathematical expressions and equations spoken aloud through speech synthesis. This is a specific way of “reading” mathematical expressions. An advantage of speech over braille is the pace of reading, because spoken language can be comprehended much quicker than braille characters (Jackson & Presley, 2012). A problem when listening to moderately complex information spoken aloud is an overload of the hearing memory (Freitas & Kouroupetroglou, 2008). It has therefore been suggested that to support learning, a student should not try to remember everything, but select information that is important (Kellman, Massey, & Son, 2010). The last two studies mentioned were not related to mathematics. However, these studies are important for the current study, because mathematical expressions and equations are often considered complex.

Speech synthesis can be used in two ways: reading or spelling. When an expression is read aloud, a speech dictionary determines the spoken text. This dictionary can be expanded with a mathematical vocabulary. When an expression (or part of an expression) is not included in the speech dictionary, the punctuation settings of the screen reader software determine which elements of the expression are read aloud and how. The punctuation setting has four levels of verbosity: none, some, most, and all. A higher level of verbosity means that more punctuation marks and symbols are read aloud. When reading mathematical text, the braille reader must select a high verbosity level, because one missing character changes the meaning of the expression. The punctuations settings and the speech dictionary can be adapted so that that all expressions (or almost all expressions) can be spoken aloud in mathematical vocabulary. This supports the braille reader in learning the mathematical vocabulary, which is vital for the development of mathematical skills (Riccomini, Smith, Hughes, & Fries, 2015).

When an expression is spelled, it is spoken aloud element-by-element. How an element is spelled depends on the punctuation settings. For example, “^” can be spelled as “caret” or as “circumflex,” depending on the settings. The spelling is independent of the verbosity level the braille reader uses and independent of the content of the speech dictionary. Hence, uncertainties in braille can always be verified or checked by spelling aloud.

Research question and hypothesis

In this study, we investigated the effect of an intervention consisting of adjusted TTS settings of the screen reader software in combination with a PD course for mathematics teachers. The PD course focused on the teachers’ knowledge and skills to integrate the braille display and the TTS synthesizer into mathematics lessons. We addressed the following research question:

> *Research question 5.1: What is the effect of a mathematics teachers’ PD course, in combination with adjusted TTS settings, on braille readers’ mathematical practices?*

We expected the adjusted TTS settings to help students better understand mathematical expressions and equations, because all expressions are spoken aloud in mathematical vocabulary. Furthermore, we hypothesized that the PD course improves the mathematics teachers’ knowledge and skills regarding the use of braille, the braille display, and the braille display in coordination with the TTS synthesizer. We expected that this knowledge would help the braille readers on two levels: (1) to make well-founded choices for the use of the braille display or the TTS synthesizer, and (2) to read and comprehend mathematical expressions in braille, whether or not in coordination with speech synthesis.

5.3 Methods

Design of the study

To answer the research question, nine braille readers received an intervention consisting of adjusted settings of the screen reader software in combination with a PD course for their mathematics teachers. The intervention was partly indirect, because the mathematics teachers, and not the braille readers themselves, participated in the PD course. A pre- and posttest and semi-structured interviews were administered.

Context and participants

In the Netherlands, more than 50% of all braille readers in secondary education, aged 12–18, go to schools for students with a visual impairment. Five mathematics teachers from two of these schools participated in the PD course. They did not receive training in teaching mathematics to students with a visual impairment prior to this course. About 30% of their students were braille readers. Some braille readers used braille in almost all classes except in mathematics. They had some residual vision and preferred to read mathematical text in print. These braille readers were excluded from this study. The braille teachers taught the braille readers how to use the braille display and TTS synthesizer, but not how to use these assistive technologies in mathematics.

The braille readers who used braille in mathematics lessons and whose teachers enrolled in the PD course were recruited for this study. A condition was that the braille readers were, according to the braille teachers at the schools, proficient in reading non-mathematical text in braille. One braille reader was retroactively excluded because her screen reader software was not adjusted immediately after the pre-test. Finally, nine braille readers participated in the study. None of them had learning or cognitive limitations in addition to their visual impairment. All participating braille readers used JAWS (Job Access With Speech) as screen reader software. Table 5.1 shows the participants' demographics.

Table 5.1 Participants' Demographics

ID	Sex	Age (Years)	Grade	Visual Abilities	Age of Onset of Visual Impairment (Years)	Age of Starting Braille Reading (Years)
1	M	13	7 [1]	none	0	6
2	M	14	8 [2]	some form, light	0	6
3	F	14	8 [2]	some form, light	0	11
4	F	15	9 [2]	none	0	6
5	M	16	10 [1]	none	0	6
6	F	15	8 [1]	light	0	6
7	F	17	10 [1]	some form, light	0	6
8	M	16	9 [1]	none	8	8
9	M	18	9 [1]	colors, light	0	16

Note: [1] stands for pre-vocational secondary education (vmbo) and [2] for senior general secondary education (havo).

Intervention

Adjustments of settings of the screen reader software. The intervention consisted of a combination of adjustments of the settings of the screen reader software of braille readers and a PD course for their mathematics teachers. Hence, the intervention has a direct and an indirect component. The adjustments were made by the course conductor. First, the content of the verbosity levels was adjusted. Possible verbosity levels are *none*, *some*, *most* and *all*. A braille reader can at any time select the verbosity level at which he or she wants to read. Table 5.2 shows the punctuation marks and symbols that braille readers in the Netherlands use when reading and writing mathematics on the laptop, e.g., in Word. When reading at a specific verbosity level, only the symbols and punctuations marks that are depicted at that level or at a lower level are spoken aloud. In the default settings, for example, when reading at verbosity level *some*, “2(x + 3)” will be spoken aloud as “two x plus three”. In this case, the meaning is lost because the brackets are missing. In the current study, the verbosity settings were adjusted so that all symbols and punctuation marks used in mathematics were spoken aloud while reading at verbosity level *most*. Second, the speech dictionary was expanded with a mathematical vocabulary. Because of the adapted settings, the braille readers read all expressions and equations without missing elements when reading at verbosity level *most*. Moreover, most expressions and equations can be spoken aloud in mathematical vocabulary.

Table 5.2 Content of the verbosity levels in the default and adjusted settings

Verbosity Level	Default Settings	Adjusted Settings
None		
Some	# $ % * + / = ^ { } &	$ % &
Most	“ ( ) - : ; < > \ | ~	! “ ‘ ( ) * + , - . / : ; < > = ? [ ] \ ^ { } | ~
All	! ‘ , . ?	#

Note: This table shows the content of the verbosity levels for the default and adjusted settings for punctuation marks that are used in the mathematical braille notations in the Netherlands. When reading at a specific level of verbosity, only the symbols and punctuation marks that are displayed at that level or at a lower level are spoken aloud.

A PD course for mathematics teachers of braille readers The intervention also included a PD course for the braille readers’ mathematics teachers. The PD course was provided in four three-hour sessions spread over three months. The course focused on integrating the braille display and TTS synthesizer in mathematics lessons. Several studies have investigated the knowledge and skills teachers need to successfully integrate technology into their lessons (e.g., Graham et al., 2009; Mishra & Koehler, 2006). These studies do not explicitly relate to teaching mathematics to braille readers, but we realized we could use their findings to design the PD course. The course's topics were: "Supporting braille readers to read and comprehend mathematical expressions on the braille display" (session 1), "Supporting braille readers to read and comprehend mathematical expressions on the braille display and with the TTS synthesizer" (session 2), "Working with heterogeneous groups" (session 3), and "Sustaining daily practice" (session 4). The content of the course is further described in Appendix A (Table A1).

Instruments

Pre- and posttest about reading and comprehending mathematical expressions. The pre- and posttests investigated the development of the braille readers' performances in reading and comprehending mathematical expressions. All pre- and posttests were audio- and video-recorded and transcribed verbatim. The video camera pointed to the braille display to record the finger movements of the braille readers when reading a mathematical expression or equation. The pre- and posttest contained the same items. We did not expect a testing effect, because the time interval between the two tests was more than four months. The tests included items on mathematical speech comprehension and on mathematical braille reading skills. The tasks were not too complex, from a mathematical point of view, but required careful reading skills due to the use of various operations and brackets.

The purpose of mathematical speech comprehension was to investigate whether braille readers were able to select and understand information from an expression or equation spoken aloud. We did not want the braille readers to be distracted by technology or technological problems. Therefore, the instructor spoke the expression or equation aloud with the same mathematical vocabulary that the speech synthesizer would have used (in the adjusted settings). The instructor first spoke the expression aloud at a slow pace—slower than the braille readers were used to when reading with speech synthesis. The expressions and equations could be spoken aloud several times, at any pace, at the request of the braille reader. The items were:

a) $1/7 + 7 - 2/5 + 8 = ..$ What are the fractions in this expression?
b) $4 - (5 + -(6 + 3)) = ..$ Where do you start calculating the value of the expression?
c) $4 * (.. - 5) = 8$ Solve this equation.

The items are not in symbol font because they are represented in an 8-dot Dutch braille notation (in ASCII characters). Data were collected and analyzed on the number of correct answers and on how much time on average the braille readers needed to give the correct answers.

The purpose of mathematical braille reading skills was to investigate whether braille readers were able to read the mathematical text accurately on the braille display, to recognize the structure of the expression, and to verbalize the expression in mathematical vocabulary. With Item b, we intended to investigate whether the braille readers were supported by context ("volume"). The items were:

a) y = 2 1/2 * 3
b) The volume is 12 m^3
c) y = sqrt(2/(x + 3)^2)

Note: In the first item, the term 2 1/2 is a mixed fraction, a combination of a whole number and a proper fraction.

For mathematical braille reading skills, data were collected on the number of correct answers and on the time needed to give a correct answer. An answer was correct when the braille reader could read each element of the expression without errors and verbalize the expression accurately using mathematical vocabulary. The average time needed to give a correct answer was calculated. Errors were divided into three categories: E(1), E(2) and E(3). E(1) is an error due to malfunctioning of a technology device. E(2) is an error due to difficulties with decoding braille characters, e.g., with identifying the position of raised dots, or problems in connecting the pattern of raised dots to characters in print. For example, when decoding the character ⠭ (dot 1346), the braille readers must *feel*, with their fingertips, that the dots in the first and third rows are up and they must *know* that this pattern matches the letter x. Finally, E(3) is an error due to difficulties in recognizing the structure of an expression or in verbalizing the expression in mathematical vocabulary. Only one category can be assigned to an error. If an E(1) error is assigned, no E(2) or E(3) error can be assigned. If an E(2) error is assigned, no E(3) error can be assigned.

Pre- and post-interview about the use of braille display and TTS synthesizer. The braille readers were interviewed about their visual impairment, the assistive technologies they use, and the support they receive in mathematics lessons. The interviews were used to interpret the results of the pre- and posttests and to trace the development of braille readers in making substantiated choices for the braille display or TTS synthesizer when reading and comprehending mathematical expressions. The interviews (Appendix B) were semi-structured and open-ended (Whiting, 2008). The interviews were recorded and transcribed verbatim. In some cases, the order of the questions was changed to avoid interrupting the flow of the discussion.

To investigate the braille readers' development in making solid choices for the use of the braille display or TTS synthesizer, a codebook was created. The codes were "WHY", "WHEN", and "HOW." The code WHY was assigned when the braille reader *explained* for what purpose or reason he or she is *using* or *selecting* (or not using or selecting) the braille display or TTS synthesizer. The code WHEN was assigned when the braille reader *described* in which case he or she *selected* the braille display or TTS synthesizer. The code HOW was given when the braille reader *described* in what way or by what method he or she *used* the braille display or TTS synthesizer. We selected examples of quotes within the data that illustrate each code (Table 5.3). With these examples, we expected to be able to communicate the data clearly and concisely and ensure that the data is correctly understood and interpreted. To code the text properly, the researchers concentrated on text about the use and/or selection of the braille display or TTS synthesizer in mathematics. This text was segmented into units of analysis that focused on one explanation or description. These units ranged in length from a few words to a couple of lines and were coded with one of the three codes WHY, WHEN, and HOW if that type of explanation or description was involved. Two interviews—a pre- and post-interview—were also coded by a second coder. We had 10% cases of disagreement. These cases were discussed until consensus was reached. This did not result in adaptations in the codebook. Finally, the codes were compared on frequency.

Table 5.3 Codebook

Code	Description of the code	Examples
WHY	An explanation for what purpose or reason the braille reader uses or selects (or does not use or select) the braille display or TTS synthesizer.	"I use braille because then you can read short pieces of an expression a little faster."
WHEN	A description of a situation in which the braille reader selects the braille display or TTS synthesizer.	"I read non-mathematical text with speech."
HOW	A description of in what way or by what method the braille reader uses the braille display or TTS synthesizer.	"I first read the expression thoroughly on the braille display before I start to solve the expression."

Procedure

The study's procedure included a pre-interview, pre-test, intervention, post-interview, and posttest, all within five months. The interviews and the pre- and posttests took place at the braille readers' schools. All sessions, consisting of an interview and a pre- or posttest, were scheduled for 30 minutes but sometimes lasted longer due to technical problems with the braille readers' laptops. Within two weeks of completing the pre-tests, the course conductor adjusted the settings of the screen reader software, and the PD course for the mathematics teachers started. The PD course consisted of four three-hour sessions spread over three months. Within a month after the end of the PD course, the braille readers were interviewed for a second time and took the posttest.

5.4 Results

Reading and comprehending mathematical expressions

The research question addresses the effect of the intervention on braille readers' mathematical practices. Therefore, the results of the pre- and posttests for mathematical speech comprehension and mathematical braille reading skills are presented. For mathematical speech comprehension, the test results show that 15 (56%) and 16 (59%) of a total of 27 answers were correct in the pre- and posttest, respectively. The average time needed to give the correct answers decreased from 48 seconds (SD = 28.5) to 34 seconds (SD = 16.5). During the mathematical speech comprehension test, braille readers often asked to read the expression again at a slower pace because the initial pace was too fast to select and understand information from the expression spoken aloud.

For mathematical braille reading skills, 6 (22%) and 11 (41%) of a total of 27 answers were correct in the pre- and posttest, respectively. The average time needed to give the correct answers decreased from 18 seconds (SD = 26.8) to 15 seconds (SD = 8.7). Almost all errors, in the pre-test as well as in the posttest, were E(2) errors (Table 5.4). E(2) errors occurred because braille readers had difficulty with identifying the position of the raised dots or with connecting the braille characters to ASCII characters.

Table 5.4 Categorization of errors made in mathematical braille reading skills

	Pre-test				Posttest			
Mathematical braille	E(1)	E(2)	E(3)	C	E(1)	E(2)	E(3)	C
reading skills	2	16	3	6	0	14	2	11

Note: E(1) is an error due to malfunctioning of a technology device. E(2) is an error due to difficulties with decoding the braille characters. E(3) is an error due to difficulties in recognizing the structure of an expression or in verbalizing the expression in mathematical vocabulary. C stands for "correct answers."

Table 5.5 Reading "The volume is 12 m^3" on the braille display

Time (s)	Utterances of braille reader K. when reading "The volume is 12 m^3"	Remarks on the braille reader's utterances
00 – 04	The volume is	He needs 4 seconds to read this part of the sentence.
04 –17	Two thousand three hundred ninety-three. Wait two thousand. What should this be?	He missed the number "1" and the space after "12". He read "3" instead of "m" and "9" instead of "^".
22 – 40	The volume is... one, two, three; no wait, two, m, s, three, I do not know if that is an s. It looks a bit strange. {stop}	He could not decode the "^" properly and stopped.

The reading practice of braille reader K. is an illustrative example of how braille readers tried to read and comprehend "The volume is 12 m^3" on the braille display (Table 5.5). K. was a senior general secondary education student (ID 2, Table 5.1). He used a 6-dot braille notation at home and an 8-dot braille notation at school. The representation of 12 m^3 in the 8-dot braille notation that K. used is ⠡⠣ ⠍⡮⠩ (dot 16 126 space 134 23467 146).

K., as well as other braille readers, spent a lot of time reading "12 m^3" compared to "The volume is." He made many errors when reading "12 m^3". He missed the number "1" and read "3" (⠩ dot 146) instead of "m" (⠍ dot 134). He also read "9" (⠪ dot 246) instead of "^" (⡮ dot 23467). He could not decode the braille character for "^"and stopped. It was not clear whether he could not identify the position of raised dots or whether he was unable to connect this pattern to the "^"- symbol.

Use of braille display and TTS synthesizer

The research question also addresses the effect of the intervention on making solid choices for the use of the braille display or TTS synthesizer. Therefore, we present the results of the coding of the interviews. Table 5.6 shows that the total number of codes assigned to the answers in the pre-interview was 47, and 109 in the post-interview. Hence, the number of codes assigned had more than doubled. Most of the codes assigned were related to the braille display. The absolute number of assigned codes increased in all categories. The HOW category was the only one in which the percentage of assigned codes also increased, both for the braille display and for the TTS synthesizer.

Table 5.6 Braille readers' absolute and relative scores in the pre- and post-interviews

	Braille display				TTS synthesizer				
		Codes				Codes			Overall
	WHY	WHEN	HOW	Total	WHY	WHEN	HOW	Total	Total
Pre-interview	13 (28%)	8 (17%)	6 (13%)	27 (57%)	8 (17%)	9 (19%)	3 (6%)	20 (43%)	47 (100%)
Post-interview	24 (22%)	14 (13%)	20 (18%)	58 (53%)	16 (15%)	15 (14%)	20 (18%)	51 (47%)	109 (100%)

In the pre- and post-interviews, the braille readers said they did not get help from their teachers on how to use and select the braille display or TTS synthesizer. They also said they did not feel that the teachers changed their teaching practice in the period between the pre- and posttest. In the pre- and post-interviews, the braille readers mentioned that they need to use braille when an expression is complicated. However, they also remarked that they often avoided using braille because reading with speech synthesis is faster and less tiring. All braille readers provided more detailed explanations and descriptions for the selection and use of the braille display or TTS synthesizer. After the adjustments of the screen reader software, two braille readers used a new strategy. Depending on the type of text they had to read, they used a different level of verbosity. They read on a lower verbosity level in, for example, history lessons and on a high verbosity level in mathematics lessons. Five braille readers mentioned that they liked the expanded speech dictionary because the mathematical vocabulary helped them to understand mathematics better.

Some technical issues related to the braille display also emerged during the interviews. In each interview, the instructor asked the braille reader to type a few symbols and numbers in Microsoft Word. Based on the representation on the braille display, the instructor determined the braille notation used. It appeared that five out of nine braille readers changed braille notation in the period between the pre- and posttest. Two braille readers had accidently changed the braille notation, probably by hitting key combinations that triggered this change. One braille reader used 8-dot braille in the pre-test and 6-dot braille in the posttest. He explained that he switched to 6-dot braille because the number signs in 6-dot braille helped him to distinguish between numbers and letters. In the other cases, a teacher advised the braille readers to switch from the 6-dot to an 8-dot braille notation, probably a consequence of what the teacher learned during the PD course. According to the braille readers, this switch was advised but not guided by the teachers.

5.5 Conclusion and Discussion

Reading and comprehending mathematical expressions

In this study, we addressed the effect of the adjusted settings of the screen reader software in combination with a PD course for mathematics teachers on braille readers' mathematical practices. We first discuss the effect of the intervention on the braille readers' performance in reading and comprehending mathematical expressions. For mathematical speech comprehension, the test results show that 15 (56%) and 16 (59%) of a total of 27 answers were correct in the pre- and posttest, respectively. The intervention had a positive effect on the average time needed to give a correct answer. The standard deviation is high for all items, which means that there were large individual differences in the time needed to give correct answers. An unexpected positive result was that the braille readers felt it was easier to select and understand information from an expression or equation spoken aloud when the rate of speech was slow—much slower than they were used to. Five braille readers said they liked the expanded speech dictionary because the mathematical vocabulary helped them to better understand mathematics.

For mathematical braille reading skills, the initial level was low. Twenty-one of a total of 27 items could not be read correctly in braille. We did not expect this because seven of the nine participating braille readers had more than six years of experience with braille reading. We also expected a greater effect of the intervention. Still, even after being supported, sixteen of a total of 27 items could not be read correctly in braille. The number of E(2) errors was high, indicating that the braille readers had difficulties decoding the braille characters. This means that they had difficulties with identifying the position of raised dots or with connecting the pattern of raised dots to ASCII characters. Five of the nine braille readers changed braille notation during the intervention. This is critical because numbers and symbols can look very different in another braille notation. According to the braille readers, the switch from one braille notation to another was not guided by the teachers. This may help explain the large number of E(2) errors.

The reading practice of braille reader K. demonstrated that he encountered many difficulties. For example, he mixed up the letter *m* (⠍ dot 134) and the number *3* (⠩ dot 146). This could be due to these characters being symmetrical in the braille notation used (Tobin & Hill, 2015). Additionally, he was not able to decode the braille character for "^". This may be due to him using a different braille notation at home than at school. Like the other braille readers except one, he was not supported by the context ("The volume is") when reading "12 m^3".

Use of braille display and TTS synthesizer

The interviews investigated the effect of the intervention on the braille readers' substantiated choices for using the braille display or the TTS synthesizer. In the post-interviews, as expected, the braille readers could better explain and describe their use and selection of the braille display and TTS synthesizer. We also expected that more insight into this would result in better performances in reading and comprehending mathematical expressions. This was the case, but we expected a greater effect.

The responses in the pre- and post-interviews showed that the braille readers did not feel that the teachers supported them in the use and selection of the braille display or TTS synthesizer. They also did not feel that the teachers changed their daily practice during the intervention. In the pre- and post-interviews, all braille readers mentioned that they should use braille when they need to read and comprehend a complex expression. However, the test results show that they had difficulties with reading and comprehending mathematical expressions in braille. This may indicate that the braille readers needed and wanted more support from their teachers.

The results suggest that improving mathematical braille reading skills is the key to better mathematical performance. Both assistive technologies, the braille display as well as the TTS synthesizer, have the potential to support braille readers in mathematics. To use these technologies to their full potential, the TTS settings of the screen reader software must be adjusted, and braille readers must be taught how to use the braille display in coordination with the TTS synthesizer when reading and understanding mathematical expressions. An adequate strategy seems to be, based on the previously discussed literature (e.g., Archambault et al., 2007; Tobin & Hill, 2015; Freitas & Kouroupetroglou, 2008; Kellman et al., 2010), that braille readers use speech synthesis at the beginning of the reading process to get a global structure of the expression within a few seconds. They then switch to braille to be able to read and understand the expression (or parts of it) more accurately. Uncertainties in braille can be checked or verified with spelling aloud. To use this strategy successfully, braille readers need to know what information to select with speech synthesis and they need to have a proper level of reading mathematical expressions in braille. Another strategy is to read the expression in braille without using speech synthesis. In that case, the braille reader probably needs more time to read and process the expression.

This study has some limitations. The first is that the small number of participants offers little basis to generalize findings or to consider statistical significance. However, the results are transferable to similar situations because we gave a detailed description of the research setting. A second limitation is that we did not distinguish between E(2) errors due to difficulties with identifying the position of raised dots or with connecting the pattern of raised dots to ASCII characters. This would give more insight into the difficulties that braille readers encounter when reading mathematical expressions in braille.

Overall, this study adds to the small number of studies into ways to support braille readers in mathematics (e.g., Figueiras & Arcavi, 2014; Healy & Fernandes, 2011; Van Leendert et al., 2019). Findings from this type of research should enable mathematics teachers to better support braille readers in improving their mathematical practices and successfully finishing their educational path.

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Appendix A: PD Course Design

Table A1 Design of the PD course

Learning goals	Corresponding tasks and activities
Session 1: Supporting braille readers to read and comprehend mathematical expressions on the braille display	
1 Knowledge of the problems braille readers encounter when reading mathematical expressions and equations on the braille display.	Braille-experience. Instructor provides vignettes on students struggling with mathematics and lets the participants actively connect to their practices and pose possible explanations.
2 Knowledge about the braille notation.	Participants make analyses of errors committed by braille readers when simplifying an expression or solving an equation. *Homework assignment*: Develop a plan for supporting a braille reader, in your own practice, to understand the structure of an expression. Make a short report or video clip of the interaction.
Session 2: Supporting braille readers to read and comprehend mathematical expressions on the braille display and with the TTS synthesizer	
3 Knowledge about mathematical vocabulary.	Instructor helps teachers to use mathematical vocabulary when talking about expressions and equations and explains why this is important.
4 Knowledge of and skills in reading mathematical expressions and equations with speech synthesis.	Instructor models how to adapt the verbosity settings of the Screen Reader Software.
5 Understanding of how the use of braille and speech synthesis complement and reinforce each other while doing mathematics.	Discussion, related to the participants' experiences, on the advantages, limitations and challenges of reading in braille or with speech synthesis in mathematics lessons. Instructor models how to use braille in coordination with speech synthesis while reading and comprehending mathematical expressions. *Homework assignment*: Record your findings with speech synthesis during the next three

Learning goals	Corresponding tasks and activities
	weeks. Discuss your findings with a braille reader. Write a report of the discussion.
Session 3: Working with heterogeneous groups	
6 Knowledge about cooperative learning.	Information on the importance of inquiry and communication while learning mathematics. Teachers participate in and reflect on IBL-activities. *Homework assignment:* Support a braille reader with exploring a tactile graph. Write a report of the intervention.
Session 4: Sustaining daily practice	
7 Knowledge about new perspectives.	E.g., 3D printer to create tangible models, different software to create audible graphs.
8 Ideas for sustaining daily practices	E.g., possibilities to share practice and experiences in follow-up meetings, Microsoft Teams for online collaboration.
9 Evaluation of the PD course	Reflection on and evaluation of the PD course

Appendix B: Interview Form

Personal details

- Number
- Name
- Sex
- Date of birth
- Class
- Achieved results in mathematics
- School career (special, regular education)
- Visual abilities
- Age of onset of visual impairment
- Start braille reading
- Comorbidities

Questions

- Which screen reader do you use at school/home?
- Which assistive devices do you use in mathematics lessons? Do you use the same devices in other lessons? Explain your answer.
- Do you get any help from your mathematics teacher on how to use braille or speech synthesis in mathematics? Explain your answer.
- Do you use a specific strategy when reading and comprehending mathematical expressions? Explain your answer.
- What are the advantages and disadvantages of the use of the braille display or TTS synthesizer in mathematics? Explain your answer.

Additional question for the post-interview:

- Did you feel that your mathematics teacher had changed his or her teaching practice in the last few weeks? Explain your answer.

Chapter 6

Towards a Universal Mathematical Braille Notation

This chapter is based on

Van Leendert, A. J. M., Doorman, L. M., Drijvers, P. H. M., Pel, J., & Van der Steen, J. (in press). Towards a universal mathematical braille notation. *Journal of Visual Impairment and Blindness.*

Abstract Across the world, mathematical expressions are represented very differently in braille. The aim of this study was 1) to gain an overall insight in mathematical braille notations and 2) to investigate how mathematical braille notations support braille readers in reading and comprehending mathematical expressions. Twenty teachers from sixteen countries (thirteen EU, three non-EU) were asked to transform 21 mathematical expressions and equations into the mathematical braille notation currently used by their braille readers. Three mathematical expressions were selected, and the transformed expressions in the different braille notations were qualitatively compared at braille and mathematical structure level. The results illustrated that most mathematical braille notations use mathematical structures that either support braille readers in getting an overview of an expression – for example, by announcing the start and end of a fraction – or facilitate communication between braille readers and people with typical vision. In our study, there is only one example of a notation that uses a structure that both helps in obtaining an overview and supports communication between braille and print readers. The method of comparing transformed expressions at structure level can be extended to other types of mathematical expressions and other mathematical braille notations. Agreement on the structure of different mathematical expressions can be a first step towards a universal mathematical braille notation. The practical implications of this study are that mathematics teachers should be aware of and use the strengths of the mathematical braille notation and try to compensate for weaknesses of the notation in the support of braille readers.

Keywords*:* braille reader ● mathematical notation ● mathematical braille notation

6.1 Towards a Universal Mathematical Braille Notation

It is generally recognized that language plays an important role in teaching and learning mathematics (e.g., Morgan, Craig, Schütte, & Wagner, 2014). Developments in the study of language in mathematics education are closely related to other developments. The shift in thinking about learning as an individual activity to a socially organized one stimulates language-oriented studies to contribute to the understanding of mathematics education (Kress & Selander, 2012). To access mathematics, students must communicate through different languages (Riccomini, Smith, Hughes, & Fries, 2015; Schleppegrell, 2007; Van Eerde, 2009). For example, students need to distinguish the meaning of the word “function” in daily life, school language and formal mathematical language. Moreover, a grasp of symbols is needed to act and communicate mathematically. A mathematical notation like $\frac{3}{4}$ is supposed to evoke images such as  and actions such as dividing 3 among 4. These practices are part of a mathematical culture that has developed over centuries and resulted in a shared symbolism (Nasir & Cobb, 2002). This symbolism, however, has barriers to entry for students who cannot hear or who cannot see (e.g., O’Neill, Cameron, Quinn, O’Neill, & McLean, 2015; Schermer, 2003). These students use rather recently developed alternatives to our spoken and written language that have not developed universal mathematical notations. In the current study, we investigated how mathematical braille notations (from here on also referred to as braille notations) in different countries support braille readers in reading and comprehending mathematical expressions. This can be a first step towards a uniform mathematical braille notation.

6.2 Mathematical Notation

The formal mathematical notation (from here on mathematical notation) uses two-dimensional arrangements of symbols to convey information. The symbols are arranged according to specific rules. For instance, 2^x has a different meaning than 2x. In mathematical notations, Latin and Greek letters, e.g., e, π and ∑, are used as well as specific forms, e.g., $\sqrt{\quad}$. In general, symbols are used to save time and space. For instance, “the square root of x to the power of three” is denoted by “$\sqrt{x^3}$ ”. This example shows that the mathematical notation can be very compact and offers little redundancy. It is in many instances impossible to guess the identity of a symbol based on context.

Braille

A braille cell consists of a pattern of raised dots arranged in a 2 * 3 (⠿) or 2 * 4 (⣿) configuration. Each pattern represents a braille character. The mathematical braille notations that braille readers use have rules for transforming expressions into braille. Worldwide, different mathematical braille notations are used. Every mathematical braille notation has its own rules. In this article, we use the 6-dot or 8-dot Dutch mathematical braille notation. If another braille notation is used, this is explicitly stated.

In 6-dot braille, 64 braille characters – including the character for space – are possible. Mathematical text, however, needs more characters (Edwards, McCartney, & Fogarolo, 2006). For instance, extra characters are needed to distinguish between ⠼⠙⠃ (dot 3456 145 12) (42) and ⠼⠙ ⠠⠃ (dot 3456 145 6 12) (4b) or between ⠼⠃ ⠭ (dot 3456 12 1346) (2x) and ⠼⠃ ⠬⠭ (dot 3456 12 346 1346) (2^x). In 8-dot braille, 256 – including the character for space – braille characters are possible. As a consequence, many modifier signs can be removed. For example, 42 and 4b are transformed into ⠹⠣ (dot 1456 126) and ⠹⠃ (dot 1456 12) , 2x and 2^x into ⠣⠭ (dot 126 1346) and ⠣⡮⠭ (dot 126 23467 1346) .

Braille can be read on paper or on a braille display linked to a computer. Typically, braille readers read or write on paper in 6-dot braille. There are, however, new developments that make it possible to use 8-dot braille on paper (e.g., Four Line 8-Dot Braille Slate, MakerBot Industries, LLC). When braille readers write on paper, they use a slate and stylus, or a braillewriter (Dixon, 2009) (Figure 6.1). With a slate and stylus, you write from right to left, one dot at a time, and reverse the dots because they are embossed on the other side of the paper. This device is still widely used in developing countries. In western countries, most braille readers use a braillewriter or a one-line braille display (Figure 6.1). A braillewriter is a typewriter with a key corresponding to each of the six dots, a space key, a backspace key and a line space key. With a braillewriter, you write – in contrast to a slate and stylus – one braille character at a time. The one-line braille display allows braille readers to read the content on a computer screen one text line at a time in the form of a line of braille characters. When using this device, it is difficult to get an overview of a few lines of text. This is less of a problem when using a multi-line braille display – which is, as far as we know, not yet widely used – or braille on paper.

slate and stylus braillewriter braille display

Figure 6.1 Assistive devices for reading and writing in braille

Challenges in reading mathematical expressions in braille

It is challenging to read and comprehend mathematical expressions in braille (e.g., Van Leendert, Doorman, Drijvers, Pel, & Van der Steen, 2019). These challenges are related to accurate reading, getting an overview of an expression, and mathematical communication. Accurate reading is important, because an error in decoding the braille characters of an expression can change the meaning. Accurate reading is difficult, because braille characters have low redundancy, which means that characters are difficult to distinguish (Millar, 1997; Tobin & Hill, 2015). Getting an overview is challenging, because braille is a linear output modality (Stöger & Miesenberger, 2015). Some braille notations also allow for spatially arranged structures such as matrices and grade school level arithmetic sum, multiplication and division problems. That does not completely solve the challenge of getting an overview, because braille readers still need to build an overview by touching one braille character after the other (Millar, 1997; Van Leendert et al., 2019). Therefore, braille readers cannot take advantage of the layout of a mathematical expression that helps people who have typical vision (from here on print readers) to understand the structure of an expression at a glance (Karshmer & Bledsoe, 2002). Finally, mathematical communication between braille and print readers is difficult due to the differences in perception and notation. This is critical, as communicating mathematically is essential for the overall development of mathematical abilities (Riccomini et al., 2015).

The transformation from a mathematical expression to an expression in braille

The transformation from a mathematical expression to an expression in braille can be considered as a two-step process, see Figure 6.2. Step 1 is the transformation from a mathematical to a linear-print expression. This may result in a change in the mathematical structure of the expression. Step 2 is the conversion from the linear-print to the linear-braille expression. This transformation is called a conversion because of the one-to-one correspondence between the ASCII characters and the braille characters. This conversion depends on the braille table used. It does not change the mathematical structure of the expression. If no distinction between the linear-print and the corresponding linear-braille expression is necessary or desired, we use the term transformed expression. In some cases, when no confusion is possible, we use the term expressions instead of transformed, linear-print or linear-braille expressions.

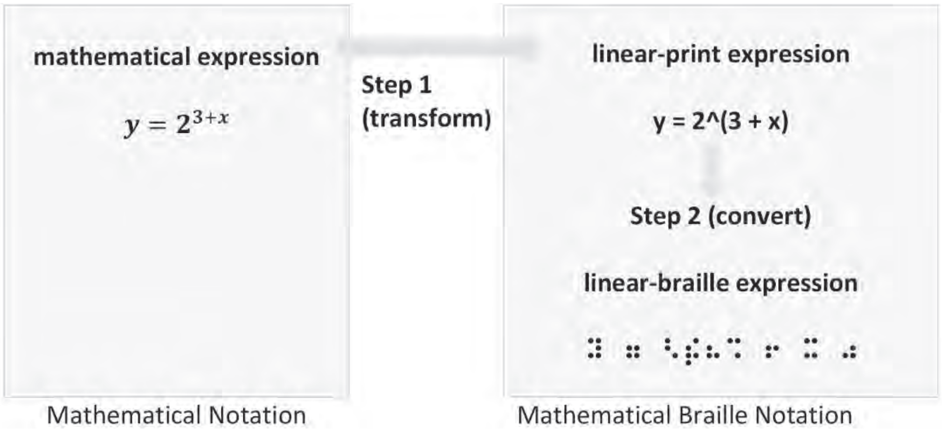

Figure 6.2 Transformation from a mathematical to a linear-braille expression

Support in reading mathematical expressions in braille

Braille notations differ from each other in how they transform mathematical expressions into linear-print expressions and/or in the braille table that they use. In this section, we will describe how notations can support braille readers in reading accurately, in getting an overview and in communication. Accurate reading is supported by using braille tables that are unambiguous and use good mnemonics (Martos, Kouroupetroglou, & Argyropoulos, 2015; Nemeth, 2001). An example of good mnemonics is using symmetric braille characters for the "(" and ")" signs. Getting an overview is supported by transformed expressions that are 1) compact, use 2) structure announcement and/or use 3) context awareness. We will explain this in more detail. An expression that is compact helps to get an overview because such an expression does not include unnecessary characters. An expression that uses structure announcement also supports getting an overview. This will be illustrated with the fraction $\frac{x+1}{x-1}$. We start with two non-examples of structure announcement. In the Dutch braille notation, this fraction is transformed into (x+1)/(x-1). This expression introduces many brackets that are not present in mathematical notation and are therefore called phantom brackets. The French braille notation uses blocks to avoid the use of phantom brackets. This results in bbx+1eb/bbx–1eb. The abbreviation bb (⠰ dot 56) stands for begin and eb (⠆ dot 23) for end block. The problem with the previously mentioned transformed expressions is that braille readers only know that they are reading a fraction when they come across the symbol "/" (Karshmer, Gupta, & Pontelli, 2007). Therefore, some braille notations provide a variety of grouping symbols to announce the start and end of the structure of an expression or sub-expression. For instance, the Nemeth Code, a notation that is mainly used in the United States, transforms the above fraction as ?x+1/x-1#, where ? (⠹ dot 1456) stands for start and # (⠼ dot 3456) for end fraction. This is called structure announcement. Another important feature that also helps to get an overview is keeping the braille reader aware of the context he/she is in at all times (Karshmer, Gupta, & Pontelli, 2007). This is because braille readers are focused on one braille character and the part to the right of the finger is not known to them at all, while the text to the left is in the braille readers' memory. For example, the expression $y^{x^{a}+3}$ is transformed into the Dutch notation as y^(x^a + 3). This does not work so well, as the braille reader needs to remember the exponent level as he/she moves left to right in the expression and a significant number of backtracking with the finger will be required to comprehend it. Therefore, some notations use context awareness. The Nemeth Code, for example, transforms $x^{a}+3$ and $y^{x^{a}+3}$ into x^a”+3 and y^x^^a^+3 respectively. The superscripted expression y^x^^a^+3 is terminated by the space. The symbol ^ indicates superscript and the symbol ” indicates a shift to the baseline. The sub-expression $x^{a}+3$ (in $y^{x^{a}+3}$) is transformed into x^^a^+3 rather than x^a”+3 because of the context.

Finally, a transformed expression can support mathematical communication between braille and print readers. This is the case for an expression that is true to the print (Nemeth, 2001). This means that a transformed expression is very similar to the expression in the mathematical notation, apart from spacing and format. For example, the previously mentioned expression y^x^^a^+3 is true to the print. Expressions that use Excel or LaTeX conventions also support communication between braille and print readers even when these expressions are not true to the print. This is because many print readers are familiar with these notations. An example is y^(x^a+3). This expression is not true to the print, but most print readers will recognize and comprehend the structure of this transformed expression. Table 6.1 summarizes the features of expressions that support braille readers when reading and comprehending mathematical expressions.

Table 6.1 Support in reading and comprehending mathematical expressions in braille

	Feature of expression	Support
Braille level	unambiguity good mnemonics	accurate reading
	compactness	getting an overview
Structure level	compactness structure announcement context awareness	getting an overview
	Excel or LaTeX conventions true to the print	communication

Research question

The challenges that braille readers face when reading and understanding mathematical expressions relate to accurate reading, getting an overview of an expression and communication with print readers. In this study, we want to investigate whether and how braille notations from different countries support braille readers with reading and comprehending mathematical expressions. This is why we addressed the following research question:

> *Research question 6.1: What are similarities and differences in the support that braille notations from different countries offer braille readers in reading and comprehending mathematical expressions?*

We assume that most braille notations use good mnemonics to support accurate reading. This seems a very natural thing to do. In addition, we expect that braille notations differ in how they support braille readers. They have to make choices between the features described in the last section.

6.3 Methods

Design of the study

An English-language questionnaire was made (see Appendix). In the first part, the participants were required to give demographic information. In the second part, the participants had to transform mathematical expressions into the mathematical braille notation used by their braille reader or braille readers.

Participants

In 2019, a conference on mathematics for braille readers took place in France. There were 22 teachers – in addition to other professionals – from fourteen European countries, who worked in mathematics education for braille readers. After the conference, we approached them to participate in the current study.

Table 6.2 Participants' demographics

Number of participants	Experience in Mathematics Education of Braille readers (Years)	Braille (B) or Print (P) Reader	European (E) or non-European (N-E)
1	<1	P	E
4	5-10	P	E
10	>10	P	E
1	>10	B	E
4	>10	P	N-E

None of them used the Nemeth Code or UEB (Unified English Braille) notation, which are common in English-speaking countries. That is why we used our personal contacts and approached two teachers who used the Nemeth Code (both from USA) and two teachers who used the UEB notation (one from Ireland and one from New Zealand). In addition, we approached a teacher from Mexico, who is a friend of one of the authors. After agreeing to participate in the current study, each teacher received an e-mail and was asked to complete the questionnaire. Finally, twenty teachers from thirteen European and three non-European countries participated. Table 6.2 shows the participants' demographics.

Procedure

Each teacher was requested to complete the questionnaire within six weeks. 50% of the teachers responded within this period. After a reminder, all remaining teachers responded within three months after the first contact.

Pilot study

We conducted a pilot and asked four teachers, two from the Czech Republic and two from Flanders (northern Belgium), to complete the questionnaire. They identified some issues in readability, understanding and phrasing. We discussed these issues and adapted the text accordingly. They mentioned that the selected expressions contribute directly to the factors being evaluated for comparison.

Data collection and analysis

The questionnaire consisted of 21 items involving expressions and equations. The teachers transformed these expressions and equations into the braille notation that their braille readers use. To address the research question, we first analyzed the representations of numbers and the "+" and "-" symbols in braille for the presence of mnemonics and compactness. Three mathematical expressions have been selected for further analysis:

$$\frac{1}{4} \tag{1}$$

$$\frac{2a+3b}{n} \tag{2}$$

$$y^{x^{a}+b} \tag{3}$$

Expression (1) was selected to investigate the extent to which braille notations differ from each other. Expression (2) and (3) were selected to investigate how different braille notations support braille readers in reading and comprehending expressions. For expression (2), we investigated whether braille notations use structure announcement or other ways to group symbols. For expression (3), we investigated whether braille notations use context awareness or other ways to transform the mathematical expression.

6.4 Results

The response rate was 80%. Twenty teachers from sixteen countries completed the questionnaire. We checked each completed questionnaire for inconsistencies in the transformed expressions and equations. In five cases, we discovered some inconsistencies and these teachers corrected their answers. The results show that most countries have their own braille notation. In some countries, 6-dot braille is not – or hardly – used in secondary education. In that case, we only gave the representation in 8-dot braille. In some cases, we referred to a braille notation only using the country's name. In other cases, we needed to give some additional information. This is necessary when a country uses different braille notations or when a braille notation is used in different countries. Czech Republic, for example, uses three different notations: Czech Republic (6-dot), Czech Republic (BlindMoose) and Czech Republic (Lambda). BlindMoose is a Microsoft add-in and provides access through braille and visual display (Wiazowski, 2018). Lambda is a mathematical editor that provides access through braille, speech synthesis, and visual display (Edwards, McCartney, & Fogarolo, 2006). Flanders uses two braille notations: Flanders Mathematical Notation (FMN) and Spermalie. A plug-in for MS Word enables on-the-fly conversion between expressions in the mathematical notation and FMN. Both Ireland and New Zealand use the Unified English Braille (UEB) notation. This is referred to as UEB (Ireland & New Zealand). In the United States, UEB as well as the Nemeth Code are used. Our teachers from the United States used the Nemeth Code. This notation is named USA (Nemeth Code). Finally, we refer to the Swedish 8-dot notation as Sweden (AsciiMath). This notation is very similar to AsciiMath, a well-known notation for mathematics teachers.

Table 6.3 summarizes how numbers are transformed in different braille notations. This table shows that most 6-dot braille notations use number signs. The French notation uses “letter a, ..., j + dot 6”. However, the number zero is transformed into ⠼ (dot 3456) instead of ⠺ (dot 2456) to avoid a conflict with the letter w (⠺) (dot 2456). The Nemeth Code uses “dropped” letters. This notation requires that ⠼ (dot 3456), the numeric indicator, is used before numbers that would otherwise be preceded by a space. That helps determine the braille character alignment. In this study, the 8-dot notations use “letter a, ..., j + dot 6”, except for the number zero, or “letter a, ..., j + dot 8”. The number zero is transformed into ⠬ (dot 346) to avoid a conflict with the letter w (⠺) (dot 2456). Table 6.4 shows how the “+” sign is transformed in braille. The “-” sign is transformed into ⠤ (dot 36) in all notations, except for the UEB notation that uses ⠐⠤ (dot 5 36).

Table 6.3 Transformation of numbers in braille

Table	Numbers in braille	Country
6-dot	number sign + letter a, ..., j example number 3: ⠼⠉	Czech Republic (6-dot), Estonia (6-dot), Ireland & New Zealand (UEB), Latvia, Lithuania, Mexico, the Netherlands (6-dot), Poland, Slovenia (6-dot), Sweden (6-dot)
	letter a, ..., j + dot 6 exception number 0: ⠼ dot 3456 example number 3: ⠩ dot 146	France
	“dropped” letter a, ..., j example number 3: ⠒ dot 25	USA (Nemeth Code)
8-dot	letter a, ..., j + dot 6 exception number 0: ⠬ dot 346 example number 3: ⠩ dot 146	Flanders (FMN), Germany (pseudo-LaTeX, LaTeX), the Netherlands (8-dot)
	letter a, ..., j + dot 8 example number 3: ⢉ dot 148	Czech Republic (BlindMoose, Lambda), Estonia (8-dot), Flanders (Spermalie), Norway, Slovenia (8-dot), Sweden (AsciiMath)

Table 6.4 Transformation of the plus sign in braille

Table	Plus sign	Braille Notation
6-dot	⠲ dot 256	Czech Republic (6-dot), Sweden (6-dot)
	⠖ dot 235	Estonia (6-dot), France, Latvia, Lithuania, Mexico, the Netherlands (6-dot), Poland
	⠐⠖ dot 5 235	Ireland & New Zealand (UEB)
	⠳ dot 1256	Slovenia (6-dot)
	⠬ dot 346	USA (Nemeth Code)
8-dot	⠲ dot 256	Czech Republic (BlindMoose, Lambda), Sweden (AsciiMath)
	⠖ dot 235	Estonia (8-dot), Flanders (Spermalie), Germany (pseudo-LaTex, LaTeX), the Netherlands (8-dot)
	⡖ dot 2357	Norway,
	⢖ dot 2358	Flanders (FMN), Slovenia (8-dot)

Table 6.5 shows how different braille notations transform $\frac{1}{4}$. The braille notations are divided into four categories based on the structure of the transformed expressions. In the first category, the notations use "dropped" characters for the numerator or denominator. For example, a dropped character of the letter d (⠙) (dot 145) is the character (⠲) (dot 256). This results in a very compact expression. Estonian (6-dot) uses two different structures: numerator - denominator (category 1) and numerator - fraction line - denominator (category 2).

The notations for expression (2) are divided into five categories, based on how grouping symbols are used (Table 6.6). The Polish notation transforms the expression in two different ways. One transformed expression uses structure announcement, the other is true to the print. Usually, in Polish notation, a space is placed before the plus sign. However, in the representation in category 2, this space is replaced by the braille character ⠈ (dot 4). This character is needed to group the numerator "2a plus 3b". If you were to write 2a +3b/n, the numerator would be 3b. The French notation uses blocks. This notation is also true to the print. Sweden (6-dot), which is typically read on paper, uses structure announcement. In contrast, Sweden (AsciiMath), which is read on the braille display, uses Excel conventions.

Table 6.5 Different ways to represent $\frac{1}{4}$ *in braille*

Category 1 Mathematical structure: *numerator - denominator*
Feature: compactness

table	linear-print expression	linear-braille expression	Braille notation
6-dot	#a.	⠼⠁⠲	Estonia (6-dot), Latvia, Lithuania
	#a/	⠼⠁⠲	Poland
	#,d	⠼⠂⠙	Mexico

Category 2 Mathematical structure: start fraction - *numerator* - fraction line - *denominator* - end fraction
Feature: structure announcement

table	linear-print expression	linear-braille expression	Braille notation
6-dot	;#a/#d[caps lock] [1]	⠆⠼⠁⠻⠼⠙⠰	Czech Republic (6-dot)
	?1/4#	⠹⠂⠌⠲⠼	USA (Nemeth Code)
	;#a:#d[letter prefix] [2]	⠆⠼⠁⠒⠼⠙⠰	Slovenia (6-dot)
8-dot	;1/4¨	⠆⢁⠻⢙⠰	Czech Republic (BlindMoose)
	//1ϕ4\\	⡄⢁⢶⢙⢠	Czech Republic (Lambda)

Category 3 Mathematical structure: *numerator* - fraction line - *denominator*
Feature: Excel conventions

table	linear-print expression	linear-braille expression	Braille notation
6-dot	#a/#d	⠼⠁⠌⠼⠙	Estonia (6-dot), the Netherlands (6-dot)
	1/4	⠡⠌⠹	France
	#aü#d	⠼⠁⠳⠼⠙	Sweden (6-dot)
	#a/d	⠼⠁⠌⠙	Ireland & New Zealand (UEB)
8-dot	1/4	⢁⠌⢙	Estonia (8-dot), Sweden (AsciiMath)
	1/4	⠡⠌⠹	Flanders (FMN), the Netherlands (8-dot)
	1/4	⠡⠲⠹	Germany (pseudo-LaTeX)
	1/4	⢁⡐⢙	Norway, Flanders (Spermalie)

Category 4 Mathematical structure: backslash - fraction - opening accolade - *numerator*- closing accolade - opening accolade - *denominator* - closing accolade
Feature: LaTeX conventions

table	linear-print expression	linear-braille expression	Country
8-dot	\ul{1}{4}	⡳⠥⠇⠣⢁⠜⠣⢙⠜	Slovenia (8-dot)
	\frac{1}{4}	⡌⠋⠗⠁⠉⠷⠡⠾⠷⠹ ⠾	Germany (LaTeX)

Note: [1] The braille character ⠰ (dot 56) represents Caps Lock. [2] The braille character ⠰ (dot 56) represents letter prefix.

Table 6.6 Different ways to support reading $\frac{2a+3b}{n}$ *in braille*

Mathematical Structure	Table	Braille notation
Category 1 start fraction - *numerator*- fraction line - *denominator*- end fraction Example: ?2a+3b/n# (Nemeth Code) Feature: structure announcement	6-dot	Czech Republic (6-dot), Estonia (6-dot), Lithuania, USA (Nemeth Code), Ireland & New Zealand (UEB), Poland, Sweden (6-dot)
	8-dot	Czech Republic (BlindMoose, Lambda), Estonia (8-dot), Flanders (FMN), Norway
Category 2 *numerator* - fraction line - *denominator* Example: 2a⠈+3b/n [1] Feature: true to the print	6-dot	Poland
Category 3 begin block - *numerator* - end block - fraction line - *denominator* Example: ⠰2a+3b⠰/n [2] Feature: true to the print	6-dot	France
Category 4 opening bracket - *numerator*- closing bracket - fraction line - *denominator* Example: (2a+3b)/n Feature: Excel conventions	6-dot	Latvia, Mexico, the Netherlands (6-dot)
	8-dot	Flanders (Spermalie), Germany (pseudo-LaTeX), the Netherlands (8-dot), Sweden (AsciiMath)
Category 5 backslash - “fraction” - opening accolade - *numerator* - closing	8-dot	Germany (LaTeX), Slovenia (8-dot)

accolade - opening accolade - *denominator*- closing accolade Example: \frac{2a+3b}{n} Feature: LaTeX conventions		

Note: [1] The braille character ⠈ (dot 4) is used as a grouping symbol. [2] The braille characters ⠰ (dot 56) and ⠆ (dot 23) denote begin and end block.

The braille notations for expression (3) are divided into six categories (Table 6.7). In the first category, the notations announce “that the exponent of y is an exponential expression.” This is an example of structure announcement. The Nemeth Code, in the second category, uses context awareness and is true to the print. The notations in the third and fourth category are also true to the print. In the third category, the notations give the location of each exponent relative to the neighbor. The notations in category four use blocks – in the UEB notation named braille grouping symbols – to avoid the use of phantom brackets. Most notations use Excel or LaTeX conventions (category five and six).

Table 6.7 Different ways to support reading y^{x^a+b} *in braille*

Mathematical Structure	Table	Braille notation
Category 1 y - shift up for exponential expression - x - shift up for expression - a - plus b (eventually: - shift down) [1] Example: y⠘⠬x⠬a+bû (Sweden 6-dot) [2] Feature: structure announcement	6-dot	Poland, Sweden (6-dot)
	8-dot	Czech Republic (Lambda)
Category 2 y - shift up - x - shift up shift up (two times) - a - shift up - plus b - space Example: y^x^^a^+b Feature: context awareness, true to the print	6-dot	USA (Nemeth Code)
Category 3 y - shift up - x - shift up - a - shift down - plus b - shift down Example: yíxíaš +bš (Czech Republic BlindMoose) Feature: true to the print	6-dot	Czech Republic (6-dot), Latvia, Lithuania
	8-dot	Czech Republic (BlindMoose), Flanders (FMN)

Category 4 y - shift up - begin block - x - shift up - a - plus b - end block Example: y^ ⠰x^a+b⠆ [3] Feature: true to the print	6-dot	France, Ireland & New Zealand (UEB)
Category 5 y - shift up - open bracket - x - shift up - a - plus b - closed bracket Example: y^(x^a + b) (the Netherlands) Feature: Excel conventions	6-dot	Estonia (6-dot), Mexico, the Netherlands (6-dot)
	8-dot	Estonia (8-dot), Flanders (Spermalie), Germany (pseudo-LaTeX), the Netherlands (8-dot), Norway, Sweden (AsciiMath)
Category 6 y - shift up - opening accolade - x - shift up - opening accolade - a - closing accolade - plus b - closing accolade Example: y^{x^{a}+b} Feature: LaTeX conventions	8-dot	Germany (LaTeX), Slovenia (8-dot)

Note: [1] Shift up means shift to a higher level. Shift down means shift to a lower level. [2] The combination of braille characters ⠘⠬ (dot 45 346) denotes shift up for exponential expression, the braille character ⠬ (dot 346) denotes shift up for expression. [3] The braille character ⠰ (dot 56) denotes begin block and the braille character ⠆ (dot 23) denotes end block.

6.5 Conclusions and Discussion

We investigated how braille notations of different countries support braille readers while reading and comprehending mathematical expressions. The results of the transformations of numbers and the "+"and "-" signs show that braille notations differ in compactness. All notations, except the UEB notation transform the "-" sign into ⠤ (dot 36) which is very similar to the representation in print. UEB uses two braille characters ⠐⠤ (dot 5 36). In the Czech Republic, Estonia, the Netherlands and Sweden, the braille characters for the "+" sign in 6-dot and 8-dot braille are the same. These are all examples of good mnemonics.

For expression (1), the transformed expressions were compared at structure and braille level. This resulted in eighteen different linear-braille expressions. These expressions were grouped into four categories based on mathematical structure. For expression (2) and (3), the transformed expressions were only compared at structure level. For expression (2), the notation in the first category supports getting an overview of an expression. The notations in the other categories support communication between print and braille readers. As to expression (3), the notations in the first category support getting an overview. The

Nemeth Code (category two) supports getting an overview and communication. That is because this structure uses context awareness and is true to the print. The notations in the last four categories support communication. These results are in line with what we expected. Most notations, except the Nemeth Code for expression (3), do not support both getting an overview of the expression and communication between braille and print readers. Other findings are that the categories are not stable. For example, Latvia and Mexico fall in the same category for expression (1) and (2), but in different categories for expression (3). Another finding is, related to the one we just mentioned, that a notation can support getting an overview for one transformed expression and support communication for another transformed expression.

A limitation of this study is the low number of mathematical expressions, as well as the low number of mathematical braille notations. However, the method of comparing expressions at structure level can be easily scaled up to other types of mathematical expressions and other mathematical braille notations. A second limitation is that we investigated the notations in isolation. We did not take into account the context of the braille reader and/or teacher. For example, the assistive devices that braille readers use and how they use them also play a role in reading and comprehending mathematical expressions (e.g., Van Leendert et al., 2019). Future studies should investigate the notations in relation to different contexts.

Our study sheds light on how braille notations support braille readers in reading and comprehending mathematical expressions. For expressions (2) and (3), we compared the transformed expressions only at structure level. That resulted in manageable differences and similarities. Therefore, we suggest that mathematics teachers of braille readers from different countries come together and try to agree on (features of) the structure of different kind of expressions and equations. That could be a first step towards a universal mathematical braille notation. As a next step, we might opt for a more comprehensive universal mathematical approach to supporting braille readers in doing mathematics. Such an approach should be developed in close collaboration with braille readers. A universal mathematical braille notation can be part of it. Speech synthesis can also play an important role and may compensate for the weaknesses of the mathematical braille notation.

The practical implications are that mathematics teachers of braille readers should get opportunities to study the mathematical braille notation that their braille reader(s) use at braille and structure level. They should use the strengths of the braille notation and compensate for its weaknesses.

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Appendix A: Demographic Data

1. Country

   In which country do you live?

   …

2. Experience

How many years of experience do you have with teaching mathematics to braille readers?

- ☐ less than 1 year
- ☐ 1-3 years
- ☐ 3-5 years
- ☐ 5-10 years
- ☐ more than 10 years

3. What is your primary medium for reading and writing?

- ☐ braille
- ☐ print
- ☐ other (specify) …

Appendix B: Transformation of Expressions and Equations

Please transform the expressions and equations in Table 2 and Table 3 in the braille notation that your braille reader(s) use. Write the transformed expressions and equations in ASCII characters, in dots or in braille (see Table 1). If an expression or equation cannot be transcribed into braille, e.g., because the braille notation is not suitable for advanced mathematics, put “can’t be transcribed into braille”.

Table B1 Example of the transformation of an expression in the 8-dot Dutch mathematical braille notation

Mathematical Notation	Mathematical Braille Notation
$2 \times 3 =$	ASCII characters: 2 * 3 = In dots: 126 space 35 space 146 space 2356 In braille: ⠣ ⠔ ⠩ ⠶

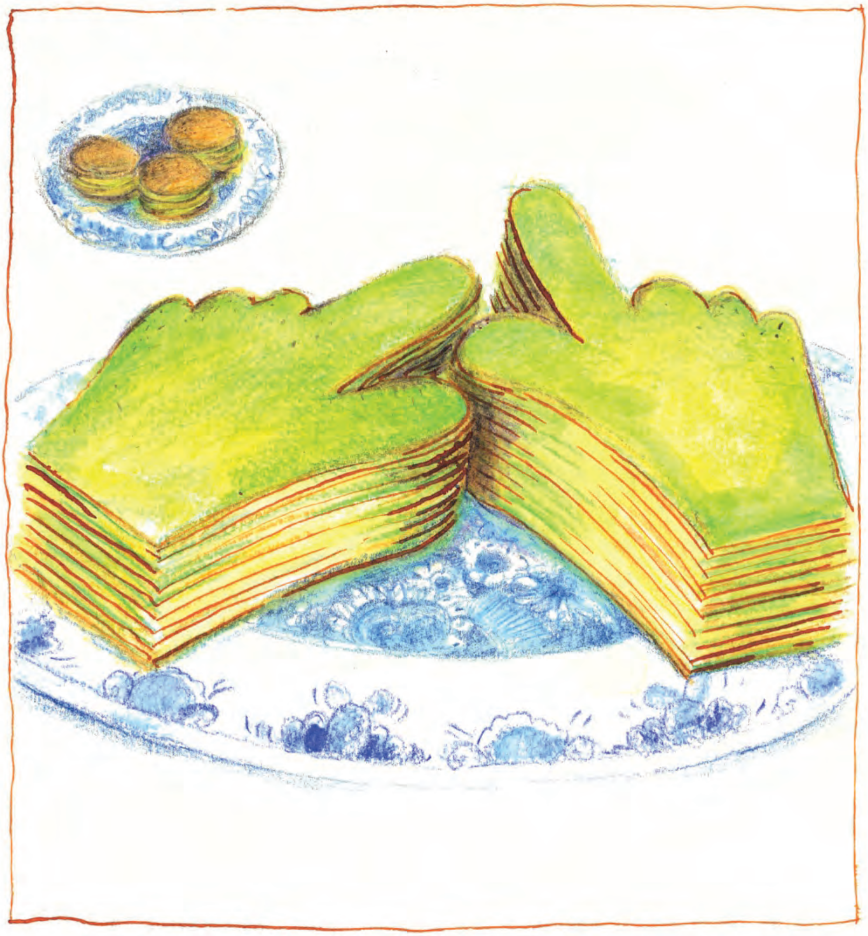
Table B2

Number	Mathematical Notation	Mathematical Braille Notation
1	$2 \times 3 =$	
2	$\frac{1}{4}$	
3	$-3 - 9 =$	
4	$ax^2 + bx + c$	
5	$x^a + b$	
6	y^{x^a+b}	
7	$sin\alpha = \cos\,(\pi - a)$	
8	$3\frac{1}{2}$	
9	$\frac{3}{7a}$	
10	$\frac{3}{7}a$	
11	$4 + (1 - (3 + 2))$	
12	$\frac{2a + 3b}{n}$	
13	$p = \frac{2}{\frac{1}{3+4}}$	

14	$2 > \sqrt{a}$	
15	$\sqrt{ab} \cdot \sqrt[3]{c}$	
16	$\sqrt{\frac{x^2}{2}} = \sqrt{\frac{3+x}{2}}$	

Table B3

Number	Mathematical Notation	Mathematical Braille Notation
1	$X_{n+2} = X_{n+1} + X_n$	
2	$\int_1^3 x^2 dx =$	
3	$\lim_{h \to 0} \frac{\Delta l(x)}{h}$	
4	$M = \begin{pmatrix} 1 & 6 \\ 2 & 7 \\ 4 & 5 \end{pmatrix}$	
5	$\sum_{i=0}^{n} a_i$	



Chapter 7

Conclusion and Discussion

7.1 Summary and Conclusion

This thesis reports the results of five partial studies. The main research question guiding this thesis is: How can braille readers improve reading and comprehending of mathematical expressions? I have studied that question from different perspectives. The first two studies (*Chapters 2* and *3*) concern tactile perception. These studies investigated how braille readers read and comprehend mathematical expressions on the braille display and how they can improve their tactile reading strategies. The following two studies (*Chapters 4* and *5*) relate to teachers' professional development. These studies investigated the effect of an intervention consisting of a professional development course for mathematics teachers who taught braille readers, in combination with adjustments of their braille readers' screen reader software. Finally, the last study (*Chapter 6*) examined how expressions are represented in different mathematical braille notations and how these representations support braille readers.

Chapter 2: An exploratory study of reading mathematical expressions by braille readers

Chapter 2 reports on a study that aimed to gain a better understanding of how braille readers read and comprehend mathematical expressions in braille. Three braille readers and five students who had typical vision (hereafter print readers) took part in this study. All participants mastered mathematics at grade 9 havo level. Using finger-tracking technology, the braille readers' tactile reading strategies in reading and comprehending mathematical expressions were recorded. The assumption was that knowledge of the visual reading strategies of print readers would provide indications for improvements in the reading strategies of braille readers. Therefore, with the help of eye-tracking technology, the reading strategies of print readers while reading and comprehending the same expressions, have also been recorded. Subsequently, the scan paths of the eye and finger movements were analyzed to gain more insight into the students' reading strategies. These scan paths illustrate the order in which the elements of the expression are read and how much time the students need to read and process the expression or parts of the expression. The results show that the two experienced braille readers – experienced in reading in braille – took 3.5 times as long as the print readers to read and process four items with mathematical expressions. The less experienced braille reader needed even more time. The results also show that braille readers reread more than print readers did. As a measure of the rereading, we used the ratio between the length of the scan path and the length of the expression. For the first item $(3 \times 2 - 5) + 4$, for example, the ratio was 13.2 for braille readers and 3.3 for print readers. The analyzes of the scan paths show that the braille readers applied very personal reading strategies for all items without making much use of the mathematical structure of the expression. The print readers, on the other hand, used structure-related strategies. In summary, compared to the print readers, the braille readers required more time, reread more and used reading strategies that were less structure related.

The results of this first study show that braille readers take longer than print readers to read and process mathematical expressions. This can be especially a problem when braille readers are in a regular school. If the pace of the class is too fast for the braille reader, he / she will not have enough time to properly process the subject matter. Working with classmates will therefore be difficult. In addition, a braille reader who works slowly will have to spend a lot of time on his / her homework. This also means that the braille reader has little time for relaxation. We have therefore designed an intervention that should help braille readers to read and understand expressions more efficiently.

Chapter 3: An exploratory study to improve reading and comprehending mathematical expressions in braille

Chapter 3 describes an intervention that teaches braille readers to use tactile reading strategies with attention to the mathematical structure of expressions and equations. The intervention consisted of five individual lessons. Three braille readers, in grade 7, 8 and 11 of havo, took part in the intervention. During the intervention, much attention was paid to the use of a disjoint reading style. In this reading style, both index fingers move at some distance from each other on the braille display. This reading style allows the braille reader to compare and (mentally) connect different parts of an expression or equation – even when these parts are relatively far apart. During a pre-, post- and retention test, the movements of the index fingers over the braille display and the time it took to read and process the expressions were video recorded. The braille readers made no errors in the tests. Four test items have been selected for further investigation. The results show that, in the posttest, each braille reader took at least 29% less time to read and process the expressions. The results of the retention test were even better. Analyses of the scan paths show that, after the intervention, the braille readers picked up characteristics of the mathematical structure more easily.

Five of the six braille readers who participated in the first two studies reported that they commonly use braille in coordination with speech synthesis. I think this is certainly advantageous if all elements of the expression can be pronounced using speech synthesis and the speech dictionary is expanded with mathematical vocabulary. I know from experience that this is often not the case, and if so, the screen reader software should be adapted. The results of the first two studies indicate that braille readers need more help reading and comprehending mathematical expressions and equations in braille. I think that the mathematics teacher is the key figure in providing this support. That is why an intervention has been developed that consists of a professional development course (from now on PD-course) for mathematics teachers who teach braille readers in combination with adjustments of their braille readers' screen reader software. *Chapter 4* describes the intervention from the perspective of the teachers and *Chapter 5* describes the same intervention from the perspective of the braille readers.

Chapter 4: Teachers' skills and knowledge in mathematics education for braille readers

Braille readers use a braille display and text-to-speech synthesizer while reading and processing mathematical expressions and equations. Mathematics teachers need to have adequate technological, pedagogical and content (TPACK) knowledge and skills for utilizing the potential of these devices. They need to understand how the use of these devices affects the teaching and learning of braille readers. Therefore, the aim of the study described in *Chapter 4* was to help mathematics teachers who taught braille readers to better understand the dynamic relationship between technology, pedagogy and mathematics. First, we adapted the TPACK model for mathematics teachers teaching braille readers. Then we developed a PD course the design of which was guided by the (adapted) TPACK model. Tactile and auditory perception played a crucial role in the (adjusted) TPACK model and in the PD-course. For example, much attention was paid to the complex interaction between assistive devices, pedagogy and mathematics in supporting the reading and processing of mathematical expressions in braille. Five mathematics teachers who taught braille readers in special secondary education participated in the course.

The effect of the PD- course on the teachers was investigated using a mixed-method approach. The instruments used were pre- and post-interviews and evaluation questionnaires. The development of the TPACK knowledge and skills of the teachers has been analyzed. The characteristics of the PD-course that help to develop this knowledge and skills have also been identified. The results show that teachers were more aware of the importance of assistive devices, and show a positive effect on the teachers' TPACK knowledge and skills. The activities of the PD course related to the braille display, mathematical braille notation and mathematical vocabulary contributed positively to the development of TPACK knowledge and skills. The activities related to the text-to-speech synthesizer and working in heterogeneous groups were less successful. This study shows that teaching mathematics to braille readers is not limited to translating instructional material into braille. It affects all domains of the TPACK model.

Chapter 5: Supporting braille readers in mathematical practices using the braille display in coordination with the speech synthesizer

Chapter 5 reports on the same intervention as described in *Chapter 4*, but now from the perspective of the braille readers. Nine braille readers aged 13 to 18, who were at vmbo or havo level, took part in the intervention. We examined the effect of the intervention on the braille readers' performance in reading and comprehending mathematical expressions and on their ability to make well-founded choices - when reading and comprehending mathematical expressions - between the braille display and the text-to-text speech synthesizer. A pre- and posttest and a pre- and post-interview were conducted with each braille reader. The tests include items related to "mathematical speech comprehension" and "mathematical braille reading skills". The purpose of "mathematical speech comprehension" was to investigate whether braille readers were able to select and comprehend information from an expression that is spoken aloud. The purpose of "mathematical braille reading skills" was to investigate whether braille readers can read the mathematical expression without errors, recognize the structure of the expression, and verbalize the expression in mathematical vocabulary. We used the results of the pre- and post-interviews to interpret the results of the pre- and posttests. In addition, we used these results to investigate the development in making well-founded choices between the braille display and text-to-speech synthesizer in reading and comprehending mathematical expressions. For this, we have created a codebook, based on codes resulting from the TPACK model. Immediately after the pre-test, we expanded the speech dictionary of the braille readers’ screen reader software with a mathematical vocabulary and adjusted the settings of the screen reader software. As a result, while using the speech synthesizer, all expressions can be spoken aloud without skipping expressions’ elements. Most expressions can be spoken aloud in mathematical vocabulary.

For mathematical braille reading skills, 6 (22%) and 11 (41%) of a total of 27 answers were correct in the pre- and posttest, respectively. In both tests, the braille readers made many errors in decoding the braille characters. This indicates that they already struggled at the beginning of the reading and solving process. For mathematical speech comprehension, 15 (56%) and 16 (59%) of a total of 27 answers were correct in the pre- and posttest, respectively. An unintended positive result was that the braille readers found that it is easier to select and comprehend information from an expression spoken aloud when the speech rate is low – much lower than they were used to using speech synthesis. During the post-interviews, five braille readers reported that using the mathematical speech dictionary – the expanded speech dictionary – helped them understand mathematics better.

The responses from the pre- and post-interviews show that the braille readers did not feel that they were supported by their mathematics teachers in reading and comprehending mathematical expressions using the braille display and / or the text-to-speech synthesizer. Some braille readers noted that they did not always feel like using braille. One said that reading in braille takes more time and effort than reading with speech synthesis. However, all braille readers indicated that they cannot read and comprehend a complex expression if they do not use braille.

Although the study had a small scale, the results of the pre- and post-interviews indicated that, after the intervention, the braille readers were better able to explain how they use the braille display or text-to-speech synthesizer. They could also better explain why and when they prefer to use the braille display over the text-to-speech synthesizer and vice versa. We expected this to result in better performances in reading and comprehending mathematical expressions. This was the case, but we expected a greater effect.

The results of the intervention, described in *Chapters 4* and *5*, indicate that improving the braille reading skills of braille readers is crucial for improving their mathematical performance. Mathematics teachers must be aware that they have a responsibility in this and trust that they can support the braille readers. They can support them if they have adequate TPACK knowledge and skills.

Chapter 6: Towards a universal mathematical braille notation

All over the world, mathematical expressions are represented in different ways in braille. *Chapter 6* describes a study whose aim was to get an idea of the different mathematical braille notations and to investigate how these notations help braille readers to read and comprehend mathematical expressions. Twenty teachers from 13 European and three non-European countries participated in this study. They had to transform 21 mathematical expressions and equations into expressions and equations in the mathematical braille notation currently used by their braille readers. We selected three mathematical expressions for further analysis: $\frac{1}{4}$, $\frac{2a+3b}{n}$ en y^{x^a+b} . Subsequently, the transformed expressions were examined at braille and at mathematical structure level. The variation at braille level was great. For the fraction $\frac{1}{4}$ we found eighteen different transformed expressions. Based on the mathematical structure, these transformed expressions are classified into four categories. For example, one category uses the structure "start of fraction–numerator–fraction line–denominator–end of fraction". An example of a transformed expression that has this structure is ⠆ ⠡⠛⠹ ⠰ (; 1/4¨). The braille characters ⠆ (;) and ⠰ (¨) represent the start and end of a fraction. The findings show that most mathematical structures either help in obtaining an overview of an expression – for example, by announcing the start and end of a fraction – or help facilitate communication between braille and print readers. In our study, there is only one example of a notation that uses a mathematical structure that both helps to obtain an overview and supports the communication between braille and print readers. The method of comparing expressions at structure level can be extended to other mathematical expressions and other mathematical braille notations. If professionals could agree on which type of support – for example, overview or communication – has the highest priority, it could be a very first step towards a universal mathematical braille notation. The practical implications of this study are that mathematics teachers of braille readers should get opportunities to study the mathematical braille notation that their braille readers use at braille and structure level. They can then use the strengths of the mathematical braille notation and compensate for weaknesses by providing targeted verbal support.

The main question guiding this thesis was: How can braille readers improve reading and comprehending mathematical expressions? To answer this question five partial studies were conducted. Figure 7.1 gives the results at a glance. The results show that braille readers can improve reading and comprehending mathematical expressions when they improve their tactile reading skills (*Chapter*s *2*, *3* and *5*). To achieve this, they need support from their mathematics teachers. Their teachers need to have adequate TPACK knowledge and skills to provide this support (*Chapter 4*). The most critical is that the teachers understand how the assistive devices, such as the braille display and the text-to-speech synthesizer, affect the teaching and learning of braille readers. This also includes an understanding of how the mathematical braille notation that their braille readers use affect the teaching and learning of braille readers (*Chapter 6*).

7.2 Findings of the Study

Theoretical insights

Metacognitive control: use of tactile reading strategies with a focus on the mathematical structure. This study has provided theoretical as well as methodological insights. The theoretical insights are related to metacognition and to TPACK. Metacognition refers to "an individual's awareness of his own thinking processes and his ability to control these processes" (Özsoy & Ataman, 2009). Note that these are two different aspects of metacognition: awareness of thought processes on the one hand, and being able to control them on the other. Relating the two aspects of metacognition to the reading and comprehension of expressions in braille: metacognitive awareness can be seen as awareness and recognition of the mathematical structure of expressions and metacognitive control as the ability to make use of these structures during the reading and comprehension process. We illustrate this with the next item: "Calculate the value of the expression 1/4 + 3/5 = ". Braille readers who have metacognitive awareness recognize that this is an addition of two fractions and know that they first have to calculate the value of a common denominator. Braille readers who also have metacognitive control know what tactile reading strategies support the reading and calculation process. The intervention described in *Chapter* 3 involved metacognitive instruction. It concentrates on metacognitive awareness and control. The results show that after the intervention each braille reader needed at least 29% less time to read and process the expressions. In addition, the results suggest that braille readers recognized characteristics of the structure easier. These findings are in line with different studies in print readers. For example, Özsoy and Ataman (2009) found students who received metacognitive strategy training performed better in mathematical problem solving than students who did not. Cardelle-Elawar (1992) found similar results for students with low mathematical ability. Onu, Eskay, Igbo, Obiyo and Agbo (2012) found that a training in mathematics metacognition on fractional mathematics among primary school pupils, improves pupils' achievement in fractional mathematics.

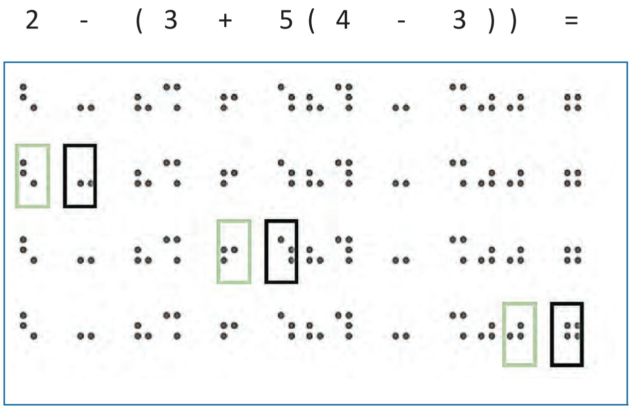
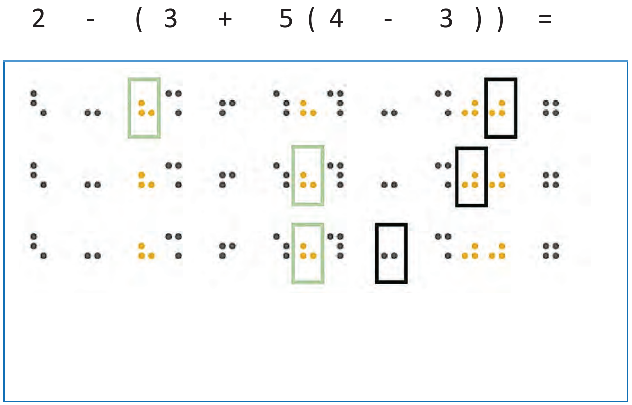
Tactile reading strategies (*Chapters 2* and *3*)	Introduction on tactile reading *Chapter 2* *Results:* Braille readers made little use, in contrast to print readers, of the mathematical structure of expressions when scanning, exploring and solving mathematical tasks.	Intervention on tactile reading *Chapter 3* *Results*: Training of structure-based finger movements for scanning expressions can support the reading and processing of mathematical expressions; each braille reader needed at least 29% less time to read and comprehend the expressions after a short training. They seemed to pick up features of the structure faster.
Professional Development course for mathematics teachers who taught to braille readers in combination with adjustments of the TTS settings of their braille readers' screen reader software (*Chapters 4* and *5*)	PD course for mathematics teachers *Chapter 4* *Results*: The results show an increased awareness of the importance of assistive devices, but only a small positive effect in TPACK knowledge and skills. We expected a better result.	Support for braille readers in mathematical practices *Chapter 5* *Results*: For mathematical braille reading skills, 6 (22%) and 11 (41%) of a total of 27 answers were correct in the pre- and posttest, respectively. Braille readers still made many errors in decoding. For mathematical speech comprehension, 15 (56%) and 16 (59%) of a total of 27 answers were correct in the pre- and posttest, respectively.
	Conclusions of the intervention described in *Chapters 4* and *5* Improving the reading skills of braille readers is the key to better mathematical performances. An important point of attention is the interplay between finger movements, text-to-speech devices, and the mathematical structure of expressions and equations. Mathematics teachers can only properly support braille readers if they have sufficient TPACK knowledge and skills.	

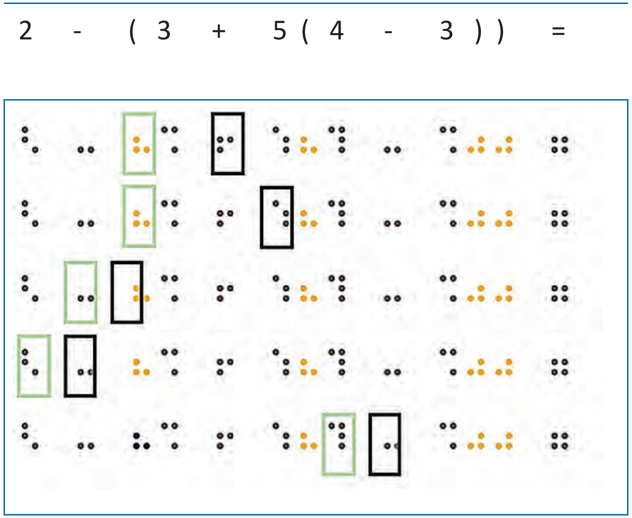
First steps towards a universal mathematical braille notation (*Chapter 6*)	Study on mathematical braille notations *Chapter 6* *Results:* Braille notations across the world use structures that either help braille readers to get an overview of an expression, or facilitate communication between braille and print readers. In our study, there is only one example of a notation that uses a structure that both helps in obtaining an overview and supports communication between braille and print readers.

Figure 7.1 Results at a glance

Metacognitive tools: cues in tactile reading. Mathematical expressions — in the notation that print readers use — consist of different types of visual cues to convey their semantic meaning (e.g., Stöger, & Miesenberger, 2015). Examples are superscripts, subscripts, brackets and fraction lines. The way these visual cues are constructed can make a difference in the ease and effectiveness of cognition and metacognition. Effective visual cues can help students to process an expression. Brackets, for example, are strong visual cues that do not receive frequent or long fixations and can be perceived in peripheral vision (e.g., Schneider, Maruyama, Dehaene, & Sigman, 2012). They also have a strong influence on the way mathematical expressions are parsed by the reader. Hence, these visual cues can serve as metacognitive tools. It would be very helpful, in particular for novices, if braille characters representing brackets would also be very easy to recognize. Then these characters can also act as metacognitive tools. This is challenging because the braille notation is linear (Stöger & Miesenberger, 2015) and all braille characters have the same size (Karshmer & Bledsoe, 2002). Therefore, no braille character really stands out among others. Imagine — it is not possible yet — that braille characters can be represented with higher raised dots than normal. Then these characters can act as metacognitive tools. This will be demonstrated in Figure 7.2. This figure illustrates the reading and comprehending process of an expression that represents brackets in higher dots than normal. The expression is 2 - (3 + 5(4 - 3)) = or ⠣ ⠤ ⠦⠩ ⠖ ⠱⠦⠹ ⠤ ⠩⠴⠴ ⠶ The expression is represented in a Dutch mathematical braille notation. In the left column, the expression is depicted in ASCII characters and in braille. The small rectangles image the location of the left (green) and right (black) index finger. These rectangles are the perceptual windows of the braille reader. The text in the right column describes the reading and comprehension process or, more precisely, the interplay between the finger movements and the mathematical structure. From line five, the dots of the braille characters representing brackets are slightly higher than normal. One can recognize these dots by their orange color. As a result of the adjustments in the height of the dots, the braille readers are cued to the braille characters that represent brackets. This helps to process the expression.

Another way to direct braille readers to specific elements of an expression is to represent only these elements in braille and leave out the other elements. Figure 7.3 illustrates the reading and comprehending process of an expression that uses this method. The expression is 4/5 + 3/4 = or ⠹⠌⠱ ⠖ ⠩⠌⠹ ⠶ . This expression is also represented in a Dutch mathematical braille notation. The expression is an addition of two fractions. To support the calculation of the common denominator, only the denominators of the two fractions are represented in braille (see line 4, 5 and 6 in Figure 7.3). Hence, the braille characters that represent the denominators serve as tactile cues. They act as metacognitive tools because they influence how braille readers perceive and comprehend the expression. As far as is known, no study has yet investigated the use of these tools for the learning of mathematics of braille readers. It may be valuable to get more insight into this because the intervention described in *Chapter 3* illustrates that attention for metacognition improves the performances of braille readers.

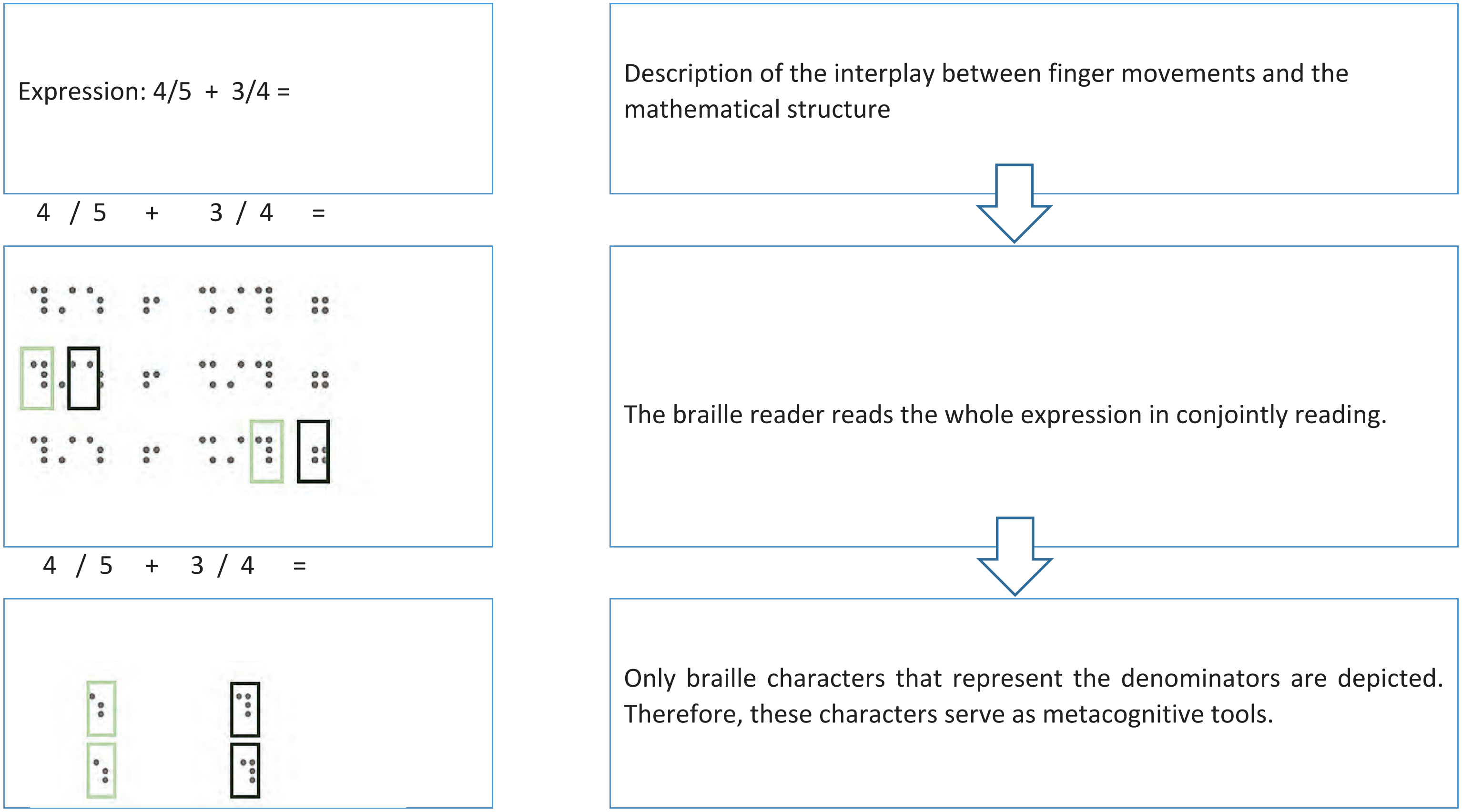
Expression: 2 - (3 + 5(4 - 3))	Description of the interplay between finger movements and mathematical structure
	The braille reader reads the whole expression in conjoint reading.
	The dots of the braille characters representing brackets are slightly higher and serve as metacognitive tools. The braille reader first moves his index fingers to the outer and then to the inner pair of brackets. In the last line, the braille reader lingers with his left index finger on the second open bracket while reading the sub-expression 4 - 3 with his right index finger.



After calculating the value of 4 - 3 the braille reader moves his left index finger to the first open bracket. He lingers with his left index finger on this bracket and continues reading with his right index finger. He calculates the value of 3 + 5 * 1. Then he starts again at the beginning of the expression and calculates the value of 2 - 8 Finally, he reads the whole expression again, in conjointly reading, to check his answer.

Figure 7.2 Brackets as tactile cues

Note: The small rectangles image the location of the left (green) and right (black) index finger. The braille characters that have dots that are colored orange are higher than normal. They serve as metacognitive tools. In this example, these braille characters represent brackets. The expression is represented in a Dutch mathematical braille notation.

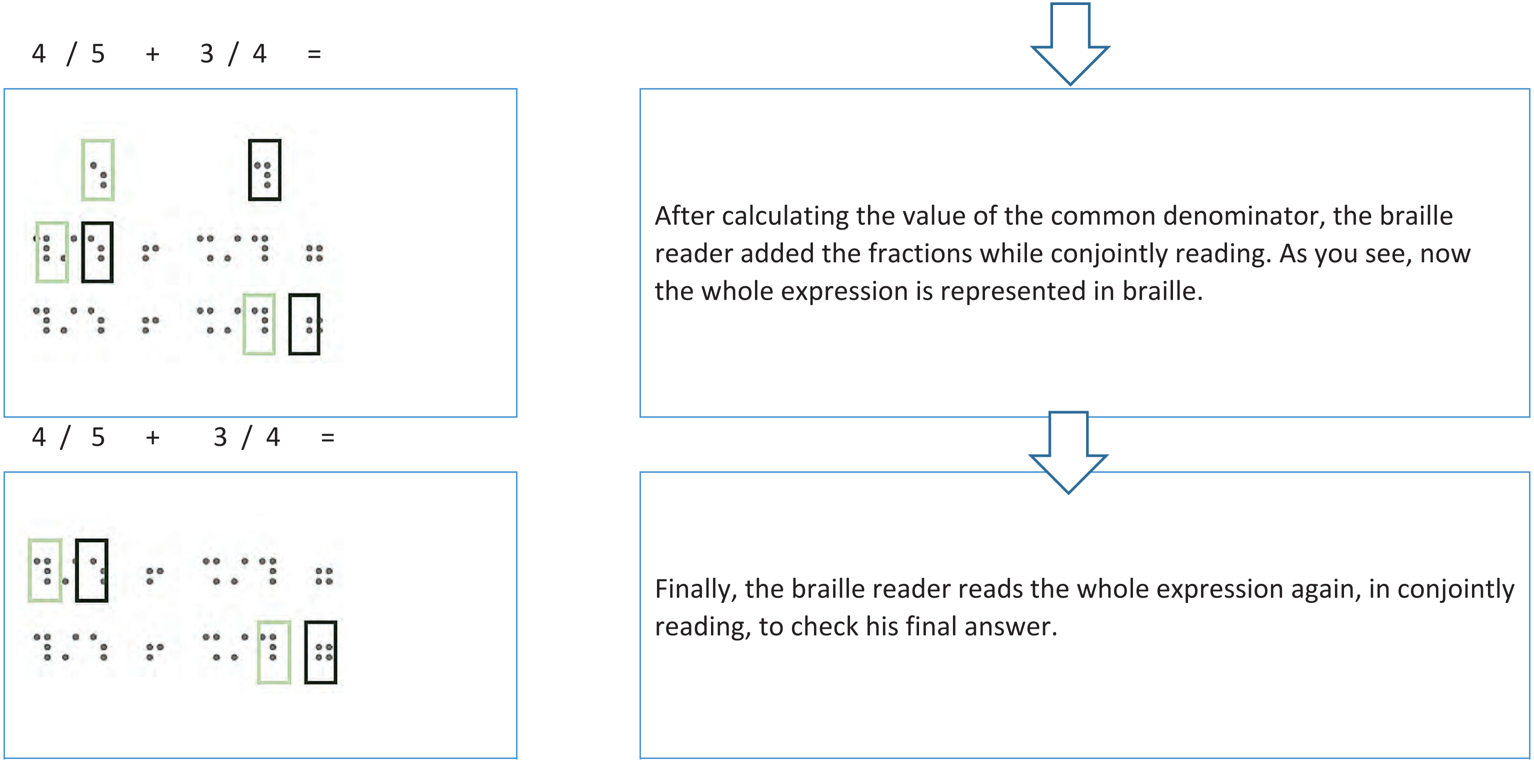


Description of the interplay between finger movements and the mathematical structure

The braille reader reads the whole expression in conjointly reading.

Only braille characters that represent the denominators are depicted. Therefore, these characters serve as metacognitive tools.

Expression: 4/5 + 3/4 =



After calculating the value of the common denominator, the braille reader added the fractions while conjointly reading. As you see, now the whole expression is represented in braille.

Finally, the braille reader reads the whole expression again, in conjointly reading, to check his final answer.

Figure 7.3 Denominators as metacognitive tools

Note: The small rectangles image the location of the left (green) and right (black) index finger. In line 4, 5 and 6, the braille readers are cued to the braille characters that represent the denominators, because the other characters are left out. The expression is represented in a Dutch mathematical braille notation.

Using TPACK to improve the teaching and learning of braille readers. ***Chapters 4*** and *5* present the results of an intervention that included a PD course for mathematics teachers who taught braille readers. The design of this PD course was guided by the TPACK model that was adapted for mathematics teachers who taught braille readers. In the (adapted) TPACK model and in the PD course, the role of auditory and tactile perception in mathematics teaching and learning was important. This involved knowledge of the strengths and weaknesses of the assistive devices that the braille readers used. TPACK acknowledges that these devices are not neutral (Koehler & Mishra, 2008). This insight is important for the education of braille readers. The teaching and learning of braille readers is not just a matter of translating instruction materials into braille or speech synthesis. It affects all the domains of the TPACK model. The study described in *Chapter 4* also illustrates how the TPACK model can become useful in educational practices requiring specific technological devices or sensorimotor coordination.

Methodological insights

Finger-tracking. In the first study about tactile reading (*Chapter 2*) the movements of the index fingers of braille readers while reading and comprehending mathematical expressions were recorded and represented in tactile scan paths. Analyses of these scan paths give insight into how braille readers perceive and comprehend mathematical expressions. These results provided cues for improvement and guided the design of the intervention on tactile reading (*Chapter 3*). We also used finger-tracking to determine the effectiveness of this intervention. Finger-tracking is, as far as is known, a new technology in research in mathematics education for braille readers. The results of our studies on tactile reading (*Chapters 2* and *3*) illustrate that this is a promising method.

Towards a universal mathematical braille notation. Two mathematical braille notations differ from each other when there is a difference in the mathematical structure of the transformed expressions and / or in the braille table used. In the study of mathematical braille notations (*Chapter 6*) the focus is on the mathematical structure. We were particularly interested in the support that structures provide and the way they do so. For example, a structure can support getting an overview of a transformed expression when the mathematical structure is compact, uses structure announcement or uses context awareness. In our study, we found that most structures either support braille readers in getting an overview or facilitate communication between braille and print readers. There was only one example of a structure that both supports getting an overview and facilitates communication (Table 7.1). This structure uses context awareness and is true to the print.

Table 7.1. Support in reading the expression y^{x^a+b} in braille

Expression	Transformed expression	Mathematical structure	Feature	Support
y^{x^a+b}	y^x^^a^+b	y - shift up - x - shift up shift up (two times) - a - shift up - plus b - space	context awareness true to the print	getting an overview communication

Note: This is an example of a transformed expression that supports getting an overview and communication. The mathematical braille notation used is the Nemeth Code.

In our study, when examining the mathematical braille notations, the emphasis was on the support provided by the structures. This method can be easily extended to other expressions, equations and mathematical braille notations. When this is realized, it is possible to think about developing a (more) universal mathematical braille notation. A first step is that professionals agree with each other which form of support should have the highest priority, for example support in gaining an overview or support with communication. The next step is to agree on how to provide that support, for example with structure announcement or context awareness.

Implications and recommendations for educational practice

The results of the current study show that improving braille reading skills is crucial for better performance in mathematics. Braille readers need to be able to read very accurately and use tactile reading strategies with a focus on the mathematical structure of expressions and equations. They can speed up their reading process by using speech synthesis to get a first impression of the mathematical structure of an expression. Then they should switch to braille using the knowledge of the structure of the expression obtained with speech synthesis.

Of course, braille readers need to be taught how to read mathematical expressions and equations in braille or in braille in coordination with speech synthesis. In my opinion, the mathematics teachers – and not the braille teachers – should be responsible for this. The mathematics teachers have to teach the braille readers how to use these devices in mathematics. For example, they have to teach the braille readers when and how to use a disjoint reading style while processing a mathematical expression in braille. This requires that they have, at least, some TPACK knowledge and skills. When teachers view themselves as potential learners in their interaction with braille readers, they can extend their teaching practices (Healy, & Fernandez, 2011) and, hence, improve their TPACK knowledge and skills. In addition, they must be given sufficient opportunities to participate in a PD course where the design is guided by the TPACK model. The role of tactile and auditory perception in mathematics learning need to be crucial in the PD course. This course can include online and face-to-face elements. It will have to take place annually because teachers come and go, assistive devices change and insights about teaching and learning of braille readers develop. Additionally, teachers can acquire and exchange knowledge and experience through international cooperation. This is important because the number of people that teach mathematics to braille readers is very small. For international cooperation, it would be very useful if professionals know each other's mathematical braille notation or, even better, use the same notation.

In the Netherlands, mathematics teachers in special secondary education must have a qualification for mathematics or a general one for primary education. The requirements for mathematics teachers in regular secondary education are stricter. They need to have a qualification for mathematics. To my opinion, braille readers in special secondary education are also entitled to well-trained mathematics teachers. Therefore, I strongly recommend that the same high standards be set for teachers in special secondary education. This can be a challenge because in the Netherlands, as in many other countries, there is a shortage of (qualified) mathematics teachers. Nevertheless, I think we should strive for that. Maybe we should even strive for “the best” mathematics teachers, as the results of *Chapter 5* illustrate that the teaching of braille readers is very challenging and affects all domains of the TPACK model.

7.4 Suggestions for Further Research

Reflecting upon the results and process of this PhD study brings the realization that there is a need for further study. I will do four suggestions for follow-up studies.

Support for print readers who are severally visually impaired reading mathematical expressions

In our study, we recruited students who were visually impaired and used braille. However, there is a larger group of students, all over the world, who are severely visually impaired but do not use braille. This group is rather diverse, since there are different types of visual impairment, e.g., loss of central vision, loss of peripheral vision or blurred vision (www.coavision.org). Therefore, these students use different assistive devices. Some students need, for instance, devices that magnify while other students need devices that increase contrast. The use of these devices affects the teaching and learning of the students. For example, a student who uses a magnification device may have problems with getting an overview of an expression because he/ she only perceives a few characters at once. Further studies are needed to explore how these students can be supported and at what level their mathematics teachers should be trained in the use of the assistive devices. I suggest, based on our findings described in *Chapter 5*, to use the TPACK model in these studies.

Mathematical structures that support doing mathematics

In the study about mathematical braille notations (*Chapter 6*), we investigated whether and how the mathematical structure of a transformed expression helps at the beginning of the reading and calculation process. This is important because it helps the braille reader to start solving the problem. The following step, inevitable in mathematical activity, is to calculate the value of the expression. At this stage, it is often very helpful to write down steps in between. We do not know whether it is helpful to use the same structures for this writing process. It is possible that, at this stage, using a compact structure is most preferable. Follow-up studies are needed to get more insight into structures or combinations of structures that help braille readers to do mathematics.

Tactile scan paths as a metacognitive tool for braille readers

In *Chapters 2* and *3*, the researchers analyzed the tactile scan paths of braille readers. That helped them to get more insight into the braille readers' finger movements over the braille display while reading and comprehending an expression. To date, no representations of tactile scan paths have been used in mathematics education to give the braille readers themselves more insight into the tactile reading strategies they use. In that case, the representation of the scan path – which needs to be a tactile representation – could be used as a metacognitive tool that supports braille readers to get insight into how their perception influence their cognitive processes. For print readers, the counterpart of the tactile scan path for braille readers is the visual scan path. Therefore, it is interesting that different studies have investigated the effect of confronting print readers with gaze data. For example, Sommer, Hinojosa and Polman (2016) used gaze data in the context of Science Technology Engineering Mathematics (STEM) literacy. Their study revealed that confronting print readers with their own gaze data helped them to gain insight into their perceptual and interpretive processes. Van den Broek (2018) described an intervention in which a similar approach was used in the context of developing structure sense in algebra. In her study, the print readers did not reflect on their own gaze data but on that of other print readers. These gaze data were collected while print readers solved mathematical equations. In the intervention, the teacher used the gaze data to stimulate both metacognitive awareness and control. The results of that study show that print readers did develop some structure sense during the intervention. Hence, it seems valuable to investigate whether such an approach also has a positive impact on braille readers. Therefore, I recommend future studies in which the braille readers' reflections on their own tactile scan paths are used as a metacognitive tool to improve reading and comprehending mathematical expressions.

Integration of tactile and audible graphs in mathematics lessons

The current study is about mathematical expressions in braille. Other representations, such as diagrams, graphs and physical models, also play a central role in mathematics. The use of various representations (Cai & Lester, 2005) and the translation activities from one representation to another representation are very important in mathematics (Duval, 2006). For functions, for example, one may use an algebraic or a graphical representation, e.g., $y = x^2$ or the image of a parabola. Braille readers do not have access to visual graphs. They need to use tactile and/or audible graphs (see Figure 7.4). Both representations have their own benefits and limitations. For example, while using a tactile graph a braille reader can easily compare different parts of the graph. This seems more difficult while using an audible graph. On the other hand, an audible graph can give information about the relation between two variables in a split second. This shows that it is probably very useful to use both representations, depending on the task that needs to be solved. The translation activity from a tactile to an audible graph and vice versa, is also important. As far as is known, no studies have investigated the use of both tactile and audible graphs in mathematics. Therefore, I recommend future studies to investigate how both representations can be integrated in mathematics education for braille readers.

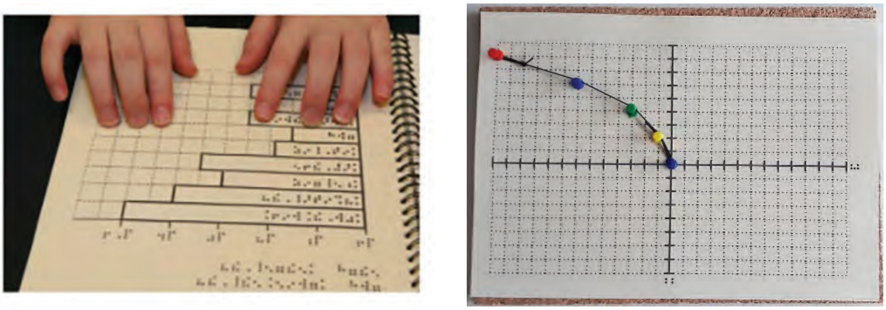


Figure 7.4 Examples of a tactile graph

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Samenvatting

Dit proefschrift doet verslag van de resultaten van vijf deelstudies. De centrale onderzoeksvraag van dit proefschrift is: Hoe kunnen braillelezers het lezen en begrijpen van wiskundige expressies verbeteren? Ik heb die vraag vanuit verschillende invalshoeken benaderd. De eerste twee studies (*Hoofdstukken 2* en *3*) hebben betrekking op tactiele waarneming. In deze studies is onderzocht hoe braillelezers wiskundige expressies op de brailleleesregel lezen en begrijpen en hoe ze hun tactiele leesstrategieën kunnen verbeteren. De volgende twee studies (*Hoofdstukken 4* en *5*) hebben betrekking op de professionele ontwikkeling van leraren. In deze studies is onderzocht wat het effect is van een interventie bestaande uit een professionaliseringscursus voor wiskundedocenten die lesgeven aan braillelezers, in combinatie met aanpassingen van de schermleessoftware van hun braillelezers. Ten slotte is in de laatste studie (*Hoofdstuk 6)* onderzocht hoe expressies worden weergegeven in verschillende wiskundige braillenotaties en hoe deze representaties braillelezers ondersteunen bij het lezen en begrijpen van deze expressies.

Hoofdstuk 2: Een verkennend onderzoek naar het lezen van wiskundige expressies in braille

Hoofdstuk 2 doet verslag van een studie die als doel had om een beter inzicht te krijgen in hoe braillelezers wiskundige expressies in braille lezen en begrijpen. Aan deze studie namen drie braillelezers en vijf studenten met een normaal gezichtsvermogen (hierna zwartdruklezers) deel. Alle deelnemers beheersten de wiskunde op klas 3 havo niveau. Met behulp van finger-tracking technologie zijn de tactiele leesstrategieën van de braillelezers bij het lezen en begrijpen van wiskundige expressies vastgelegd. De veronderstelling was dat kennis van de visuele leesstrategieën van zwartdruklezers aanwijzingen kunnen geven voor verbeteringen in de tactiele leesstrategieën van braillelezers. Daarom zijn met behulp van eye-tracking technologie ook de leesstrategieën van zwartdruklezers tijdens het lezen en begrijpen van dezelfde expressies vastgelegd. Vervolgens hebben we de scanpaden van de oog- en vingerbewegingen geanalyseerd om meer inzicht te krijgen in de leesstrategieën van de studenten. Deze scanpaden laten zien in welke volgorde de elementen van de expressie worden gelezen en hoeveel tijd de studenten nodig hebben om de expressie of delen van de expressie te lezen of uit te rekenen. De resultaten laten zien dat de twee ervaren braillelezers – ervaren in het lezen in braille – 3,5 keer zoveel tijd nodig hadden als de zwartdruklezers om vier items met wiskundige expressies te lezen en uit te rekenen. De minder ervaren braillelezer had zelfs nog meer tijd nodig. De resultaten laten ook zien dat braillelezers meer herlezen dan zwartdruklezers. Als maat voor het herlezen gebruikten we de verhouding tussen de lengte van het scanpad en de lengte van de expressie. Voor het eerste item $(3 \times 2 - 5) + 4$, bijvoorbeeld, was de verhouding 13,2 voor braillelezers en 3,3 voor zwartdruklezers. Uit de analyses van de scanpaden blijkt dat de braillelezers voor alle items heel persoonlijke leesstrategieën toepasten zonder veel gebruik te maken van de wiskundige structuur van de expressies. De zwartdruklezers daarentegen gebruikten strategieën die een relatie hadden met de wiskundige structuur. Samengevat, in vergelijking tot de zwartdruklezers hadden de braillelezers meer tijd nodig, herlazen ze meer en gebruikten ze leesstrategieën die minder gericht waren op de wiskundige structuur van expressies.

De resultaten van de eerste studie laten zien dat braillelezers meer tijd nodig hebben dan zwartdruklezers om wiskundige expressies te lezen en te begrijpen. Dit kan vooral een probleem zijn wanneer braillelezers op een reguliere school zitten. Als het tempo van de klas te hoog is voor de braillelezer, zal hij of zij onvoldoende tijd hebben om de leerstof goed te kunnen verwerken Het samenwerken met klasgenoten zal dan ook lastig zijn. Bovendien zal een braillelezer die langzaam werkt veel tijd moeten besteden aan huiswerk. Dat betekent ook dat de braillelezer weinig tijd heeft voor ontspanning. We hebben daarom een interventie ontworpen die braillelezers moet helpen om expressies efficiënter te lezen en te begrijpen.

Hoofdstuk 3: Een verkennend onderzoek naar het verbeteren van het lezen en begrijpen van wiskundige expressies in braille

Hoofdstuk 3 beschrijft een interventie die braillelezers leert tactiele leesstrategieën met aandacht voor de wiskundige structuur van expressies en vergelijkingen te gebruiken. De interventie bestond uit vijf individuele lessen. Drie braillelezers, uit 1, 2 en 5 havo, namen deel aan de interventie. Gedurende de interventie was er veel aandacht voor het gebruik van een zogenaamde disjuncte leesstijl. Bij deze leesstijl bewegen de beide wijsvingers zich op enige afstand van elkaar over de brailleleesregel. Dat maakt het mogelijk om verschillende delen van een expressie of vergelijking – ook wanneer deze delen relatief ver uit elkaar staan – met elkaar te vergelijken en (in gedachten met elkaar) te verbinden. Tijdens een pre-, post- en retentietest zijn de bewegingen van de wijsvingers over de brailleleesregel en de tijd die nodig was om de expressies te lezen en te begrijpen vastgelegd met behulp van een videocamera. De braillelezers maakten geen enkele fout in de testen. Vier testitems zijn geselecteerd voor nader onderzoek. De resultaten laten zien dat elke braillelezer in de posttest tenminste 29% minder tijd nodig had om de expressies te lezen en te berekenen. De resultaten van de retentietest waren zelfs nog beter. De analyses van de scanpaden laten zien dat de braillelezers na de interventie kenmerken van de wiskundige structuur gemakkelijker oppikten.

Vijf van de zes braillelezers die deelnamen aan de eerste twee studies gaven aan dat ze gewoonlijk braille in coördinatie met spraaksynthese gebruiken. Dit heeft, naar mijn mening, zeker voordelen als alle elementen van de expressie kunnen worden uitgesproken met behulp van spraaksynthese en het spraakwoordenboek is uitgebreid met wiskundige vocabulaire. Ik weet uit ervaring dat dit vaak niet zo is. In dat geval zou de schermlezerssoftware aangepast moeten worden. De resultaten van de eerste twee studies geven aan dat braillelezers meer hulp nodig hebben bij het lezen en begrijpen van wiskundige expressies en vergelijkingen in braille. Ik denk dat de wiskundedocent dé sleutelfiguur is voor het geven van deze ondersteuning. Daarom is een interventie ontwikkeld die bestaat uit een professionaliseringscursus (vanaf nu P-cursus) voor wiskundedocenten die lesgeven aan braillelezers in combinatie met aanpassingen van de schermlezerssoftware van de braillelezers. *Hoofdstuk 4* beschrijft de interventie vanuit het perspectief van de docenten en *Hoofdstuk 5* beschrijft dezelfde interventie vanuit het perspectief van de braillelezers.

Hoofdstuk 4: Vaardigheden en kennis van wiskundedocenten van braillelezers

Braillelezers gebruiken een brailleleesregel en tekst-naar-spraak-synthesizer bij het lezen en verwerken van wiskundige expressies en vergelijkingen. De wiskundedocenten moeten voldoende technologische, pedagogische en inhoudelijke (TPACK) kennis en vaardigheden hebben om de mogelijkheden van deze hulpmiddelen te kunnen benutten. Ze moeten begrijpen hoe het gebruik van deze hulpmiddelen van invloed is op het lesgeven aan en leren van braillelezers. Het doel van de studie die beschreven wordt in *Hoofdstuk 4* was om wiskundedocenten die lesgaven aan braillelezers te helpen de dynamische relatie tussen technologie, pedagogiek en wiskunde beter te begrijpen. We hebben daarom eerst het TPACK-model aangepast voor wiskundedocenten die lesgeven aan braillelezers. Vervolgens hebben we een P-cursus ontwikkeld. Voor het ontwerp van deze cursus was het aangepaste TPACK-model leidend. Tactiele en auditieve waarneming speelden een cruciale rol in het (aangepaste) TPACK-model en in de P-cursus. Er was bijvoorbeeld veel aandacht voor de complexe interactie tussen hulpmiddelen, pedagogiek en wiskunde bij het ondersteunen van het lezen van wiskundige expressies in braille. Vijf wiskundedocenten die in het speciaal voortgezet onderwijs lesgaven aan braillelezers namen deel aan de cursus.

Met behulp van een mixed-method benadering is het effect van de P-cursus op de docenten onderzocht. De gebruikte instrumenten waren pre- en post-interviews en evaluatievragenlijsten. We hebben de ontwikkeling van de

TPACK-kennis en vaardigheden van de docenten onderzocht. Ook zijn kenmerken van de P-cursus die helpen bij het ontwikkelen van deze kennis en vaardigheden geïdentificeerd. De resultaten wijzen uit dat de docenten zich na de interventie meer bewust zijn van het belang van hulpmiddelen. Ook laten de resultaten zien dat de interventie een positief effect heeft op de TPACK-kennis en -vaardigheden van de docenten. De activiteiten van de P-cursus die betrekking hadden op de brailleleesregel, de wiskundebraillenotatie en de wiskundige woordenschat leverden een positieve bijdrage aan het ontwikkelen van TPACK-kennis en -vaardigheden. De activiteiten die betrekking hadden op de tekst-naar-spraak synthesizer en het werken in heterogene groepen waren minder succesvol. Deze deelstudie laat zien dat het onderwijzen van wiskunde aan braillelezers niet beperkt is tot het vertalen van instructiemateriaal in braille, maar invloed heeft op alle domeinen van het TPACK-model.

Hoofdstuk 5: Braillelezers ondersteunen bij wiskunde met behulp van de braille leesregel in coördinatie met de spraaksynthesizer

Hoofdstuk 5 doet verslag over dezelfde interventie als beschreven in *Hoofdstuk 4*, maar nu vanuit het perspectief van de braillelezers. Negen braillelezers van 13 tot 18 jaar, allemaal vmbo- of havoleerlingen, namen deel aan de interventie. We onderzochten het effect van de interventie op de prestaties van de braillelezers bij het lezen en begrijpen van wiskundige expressies en op hun vaardigheid om goed onderbouwde keuzes te maken – bij het lezen en begrijpen van wiskundige expressies – tussen de brailleleesregel en de tekst-naar-spraak synthesizer. Elke braillelezer deed een pre- en posttest en bij elke braillelezer werd een pre- en post-interview afgenomen. De testen bevatten items die betrekking hebben op ‘begrip van wiskundige spraak’ en op ‘wiskundige brailleleesvaardigheid’. Het doel van ‘begrip van wiskundige spraak’ was om te onderzoeken of de braillelezers in staat waren om informatie uit een expressie die hardop wordt voorgelezen te selecteren en te begrijpen. Het doel van ‘wiskundige brailleleesvaardigheid’ was om te onderzoeken of de braillelezers de wiskundige expressie foutloos kunnen lezen, de structuur van de expressie herkennen en in staat zijn om in wiskundetaal over de expressie te praten. We gebruikten de resultaten van de pre- and postinterviews voor het duiden van de resultaten van de pre- en posttests. Bovendien hebben we deze resultaten gebruikt om de ontwikkeling in het maken van goed onderbouwde keuzes tussen de brailleleesregel en tekst-naar-spraak synthesizer bij het lezen en begrijpen van wiskundige expressies te onderzoeken. Daarvoor hebben we een codeboek gemaakt, gebaseerd op codes die voortkomen uit het TPACK-model. Direct na de pretest hebben we het spraakwoordenboek van de schermlezerssoftware van de braillelezers uitgebreid met een wiskundig vocabulaire en zijn ook andere instellingen van de schermlezerssoftware aangepast. Hierdoor kan, met behulp van de tekst-naar-spraak synthesizer, elke expressie worden uitgesproken zonder elementen van de expressie over te slaan. De meeste expressies kunnen in wiskundetaal worden uitgesproken.

Voor wiskundige brailleleesvaardigheid waren 6 (22%) en 11 (41%) van de in totaal 27 antwoorden correct in respectievelijk de pre-test en de posttest. De braillelezers maakten in beide testen veel fouten bij het decoderen van de brailletekens. Dit geeft aan dat ze al problemen hadden bij de start van het lees- en oplossingsproces. Voor begrip van wiskundige spraak waren respectievelijk 15 (56%) en 16 (59%) van de in totaal 27 antwoorden correct in respectievelijk de pre-test en posttest. Een onbedoeld positief resultaat was dat de braillelezers ontdekten dat het gemakkelijker is om informatie uit een expressie die voorgelezen wordt te selecteren en te begrijpen wanneer de spreeksnelheid laag is – veel lager dan ze gewend waren bij gebruik van spraaksynthese. Tijdens de post-interviews vertelden vijf braillelezers dat het gebruik van het wiskunde spraakwoordenboek – het aangepaste spraakwoordenboek- hen hielp wiskunde beter te begrijpen.

Uit de antwoorden van de pre- en post-interviews blijkt dat de braillelezers niet het gevoel hadden dat ze door hun wiskundedocenten werden ondersteund bij het lezen en begrijpen van wiskundige expressies met behulp van de brailleleesregel en/of de tekst-naar-spraak synthesizer. Sommige braillelezers merkten op dat ze niet altijd zin hadden om braille te gebruiken. Een van hen zei dat het lezen in braille meer tijd en moeite kost dan het lezen met spraaksynthese. Alle braillelezers gaven echter wel aan dat ze een complexe expressie niet kunnen lezen en begrijpen als ze geen gebruik maken van braille.

Hoewel dit een kleinschalig onderzoek was, laten de resultaten van de pre- en post-interviews zien dat de braillelezers na de interventie beter in staat waren om toe te lichten hoe ze de brailleleesregel of tekst-naar-spraak synthesizer gebruiken. Ze konden ook beter uitleggen waarom en wanneer ze een voorkeur hebben voor de brailleleesregel boven de tekst-naar-spraak synthesizer en andersom. We verwachtten dat dit zou resulteren in betere prestaties bij het lezen en begrijpen van wiskundige expressies. Dit was het geval, maar we hadden een groter effect verwacht.

De resultaten van de interventie beschreven in de *Hoofdstukken 4* en *5* geven aan dat het verbeteren van de brailleleesvaardigheid cruciaal is voor het verbeteren van de wiskundeprestaties van braillelezers. De wiskundedocenten moeten zich ervan bewust zijn dat ze hier een verantwoordelijkheid in hebben en er ook op vertrouwen dat zij de braillelezers kunnen ondersteunen. Dat kunnen ze als ze over adequate TPACK-kennis en -vaardigheden beschikken.

Hoofdstuk 6: Op weg naar een universele wiskundige braillenotatie

Over de hele wereld worden wiskundige expressies op verschillende manieren in braille weergegeven. *Hoofdstuk 6* beschrijft een studie waarvan het doel was om een beeld te krijgen van de verschillende wiskundebraillenotaties en om te onderzoeken hoe deze notaties braillelezers helpen bij het lezen en begrijpen van wiskundige expressies. Twintig docenten uit dertien Europese en drie niet-Europese landen namen deel aan deze studie. Ze moesten 21 wiskundige expressies en vergelijkingen transformeren naar expressies en vergelijkingen in de wiskundebraillenotatie die op dat moment door hun braillelezers werd gebruikt. We selecteerden drie wiskundige expressies voor verdere analyse: $\frac{1}{4}$, $\frac{2a+3b}{n}$ en y^{x^a+b}. Vervolgens zijn de getransformeerde expressies op braille-niveau en op het niveau van de wiskundige structuur geanalyseerd. De variatie op braille-niveau was groot. Voor de breuk $\frac{1}{4}$ vonden we achttien verschillende getransformeerde expressies. Op basis van de wiskundige structuur zijn deze getransformeerde expressies ingedeeld in vier categorieën. Er is bijvoorbeeld een categorie die de structuur ‘begin breuk–teller–breukstreep–noemer–einde breuk’ gebruikt. Een voorbeeld van een getransformeerde expressie die deze structuur heeft is ⠆⠡⠛⠹⠆ (;1/4¨). De braillekarakters ⠆ (;) en ⠆ (¨) geven het begin en einde van de breuk aan. De bevindingen laten zien dat de meeste wiskundige structuren helpen óf om overzicht te krijgen over een expressie – bijvoorbeeld door het begin en einde van een breuk te vermelden – óf de communicatie tussen braillelezers en zwartdruklezers vergemakkelijken. Er is in onze studie slechts één voorbeeld van een wiskundige structuur die én helpt om overzicht te verkrijgen én de communicatie tussen braillelezers en zwartdruklezers ondersteunt. De methode voor het vergelijken van expressies op structuurniveau kan worden uitgebreid naar andere wiskundige expressies en andere wiskundige braillenotaties. Als professionals het eens zouden kunnen worden over welke type ondersteuning – bijvoorbeeld overzicht of communicatie – de hoogste prioriteit heeft, zou dat een allereerste stap kunnen zijn naar een universele wiskundebraillenotatie. De praktische implicaties van deze deelstudie zijn dat wiskundeleraren van braillelezers de mogelijkheid moeten krijgen om de wiskundige braillenotatie op braille- en structuurniveau te bestuderen. Ze kunnen dan de sterke punten van de wiskundebraillenotatie gebruiken en de zwakke punten compenseren door gerichte verbale ondersteuning te geven.

De hoofdvraag in dit proefschrift was: Hoe kunnen braillelezers het lezen en begrijpen van wiskundige expressies verbeteren? Er zijn vijf deelonderzoeken uitgevoerd om deze vraag te beantwoorden. Figuur 8.1 geeft een overzicht van de resultaten. De resultaten laten zien dat braillelezers het lezen en begrijpen van wiskundige expressies kunnen verbeteren als ze hun tactiele leesvaardigheid verbeteren (*Hoofdstukken 2*, *3* en *5*). Om dit te bereiken, hebben ze de hulp van hun wiskundedocenten nodig. De wiskundedocenten moeten over voldoende TPACK-kennis en -vaardigheden beschikken om deze hulp te kunnen geven (*Hoofdstuk 4*). Het belangrijkste is dat ze begrijpen hoe de hulpmiddelen, zoals de brailleleesregel en de tekst-naar-spraak synthesizer, het lesgeven aan en het leren van braillelezers beïnvloeden. Dit omvat ook kennis van hoe de wiskundige braillenotatie het onderwijzen en het leren van de braillelezers beïnvloedt (*Hoofdstuk 6*).

Tactiele lees-strategieën (*Hoofdstukken 2* en *3*)	Inleiding op tactiel lezen *Hoofdstuk 2* *Resultaten*: Braillelezers maakten, in tegenstelling tot zwartdruklezers, weinig gebruik van de wiskundige structuur van expressies bij het scannen, onderzoeken en oplossen van wiskundige taken.	Interventie op tactiel lezen *Hoofdstuk 3* *Resultaten:* Training van op structuur gebaseerde vingerbewegingen voor het scannen van expressies kan het lezen en verwerken van wiskundige expressies ondersteunen; elke braillelezer had na een korte training tenminste 29% minder tijd nodig om de expressies te lezen en te begrijpen. Ze leken ook de kenmerken van de structuur gemakkelijker op te pakken.
Professionaliserings-cursus voor wiskundedocenten die lesgeven aan braillelezers in combinatie met aanpassingen van de tekst-naar-spraak instellingen van de schermlezerssoftware van hun braillelezers (*Hoofdstukken 4* en *5*)	Professionaliseringscursus voor wiskundedocenten *Hoofdstuk 4* *Resultaten:* De docenten zijn zich meer bewust van het belang van hulpmiddelen. Er is een klein positief effect op TPACK-kennis en -vaardigheden. We verwachtten een beter resultaat.	Het ondersteunen van braillelezers bij wiskunde *Hoofdstuk 5* *Resultaten:* Voor wiskundige brailleleesvaardigheid waren respectievelijk 6 (22%) en 11 (41%) van de in totaal 27 antwoorden correct in de pre- en posttest. De braillelezers maakten nog steeds veel fouten bij het decoderen. Voor begrip van wiskundige spraak waren respectievelijk 15 (56%) en 16 (59%) van de in totaal 27 antwoorden correct in de pre- en posttest.

	Conclusies van de interventie beschreven in *Hoofdstukken 4* en *5* Het verbeteren van de leesvaardigheid van braillelezers is de sleutel tot betere prestaties in de wiskunde. Een belangrijk aandachtspunt is het samenspel tussen vingerbewegingen, het gebruik van spraaksynthese en de wiskundige structuur van expressies en vergelijkingen. Wiskundedocenten kunnen braillelezers alleen ondersteunen als ze over adequate TPACK-kennis en -vaardigheden beschikken.
Eerste stappen naar een universele wiskundebraillenotatie (*Hoofdstuk 6*)	Studie van wiskundebraillenotaties *Hoofdstuk 6* *Resultaten:* Braillenotaties uit vele delen van de wereld gebruiken structuren die braillelezers helpen een overzicht van een expressie te krijgen, of de communicatie tussen braillelezers en zwartdruklezers vergemakkelijken. In ons onderzoek is er slechts één voorbeeld van een notatie die een structuur gebruikt die én helpt om overzicht te verkrijgen én de communicatie tussen braille- en zwartdruklezers ondersteunt.

Figuur 8.1 Overzicht van de resultaten

Hoofdstuk 7 Algemene conclusies en discussie

Hoofdstuk 7 begint met de samenvatting en conclusies van het proefschrift. In dit hoofdstuk worden ook de theoretische en methodologische inzichten en de implicaties en aanbevelingen voor de onderwijspraktijk, die deze studie hebben opgeleverd, uitgewerkt. De theoretische inzichten hebben betrekking op metacognitie en op het TPACK-model. In onze studie betreft metacognitief bewustzijn het begrip en herkennen van de wiskundige structuur van expressies en vergelijkingen. Metacognitieve controle heeft betrekking op de vaardigheid om gebruik te maken van deze structuur bij het lezen en begrijpen van wiskundige expressies. Braillelezers die metacognitieve controle hebben gebruiken tactiele leesstrategieën met een focus op de structuur. De resultaten van de interventie die wordt beschreven in *Hoofdstuk 3*, laten zien dat aandacht voor metacognitie een positief effect heeft op het tempo van de braillelezers en helpt om bepaalde structuurkenmerken sneller op te pakken. Een ander theoretisch inzicht heeft betrekking op het TPACK-model. We hebben dit model aangepast voor braillelezers. Volgens dit model zijn hulpmiddelen niet neutraal. Ze beïnvloeden het lesgeven aan en leren van leerlingen. Dat is een belangrijk inzicht dat kan helpen om het wiskundeonderwijs aan braillelezers te verbeteren.

Naast theoretische inzichten heeft deze studie ook een aantal methodologische inzichten opgeleverd. Het eerste inzicht heeft betrekking op finger-tracking. Het gebruik van finger-tracking technologie geeft inzicht in hoe braillelezers wiskundige expressies lezen en begrijpen en hoe ze dat kunnen verbeteren. De resultaten van onze studie laten zien dat dit een heel geschikte methode is. Het tweede inzicht heeft betrekking op wiskundebraillenotaties. In onze studie hebben we deze notaties onderzocht door vooral te kijken naar de wiskundige structuur van de getransformeerde expressies. Van belang was hoe deze structuur ondersteuning biedt. Deze methode is gemakkelijk uit te breiden naar meer expressies, vergelijkingen en wiskundebraillenotaties. Als professionals het eens zouden kunnen worden over welke type ondersteuning van de wiskundige structuur – bijvoorbeeld verkrijgen van overzicht over de expressies of communicatie tussen braille- en zwartdruklezers – de hoogste prioriteit heeft, zou dat een allereerste stap kunnen zijn naar een universele wiskundebraillenotatie.

Tenslotte worden implicaties en aanbevelingen voor de onderwijspraktijk besproken. De resultaten van onze studie laten zien dat het verbeteren van de braillevaardigheden van braillelezers cruciaal is voor het verbeteren van hun wiskundeprestaties. De wiskundedocenten spelen daarbij een belangrijk rol. Zij moeten voldoende TPACK-kennis en -vaardigheden hebben om de braillelezers goed te kunnen begeleiden en zich daar ook verantwoordelijk voor voelen. *Hoofdstuk 7* eindigt met suggesties voor verder onderzoek. Een van de voorstellen is om onderzoek te doen naar hoorbare en tactiele grafieken. Bij het onderzoek naar tactiele grafieken zou finger-tracking technologie ingezet kunnen worden.

Overview of publications related to this thesis

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Curriculum Vitae

Annemiek van Leendert was born on January 20, 1959, in Sittard (the Netherlands). She went to the St. Ludgergymnasium in Doetinchem. In 1987, Annemiek graduated from the Arnhem-Nijmegen University of Applied Sciences, with a bachelor's degree in Mathematics and Physics. In 2003, she obtained master's degrees in Mathematics Education from Utrecht University of Applied Sciences (Master of Education) and from the University of Greenwich (Master of Arts of Mathematical Education).

In 1991, she started working at what is now Royal Visio, in the field of mathematics education for students with a visual impairment. She worked as a mathematics teacher, visiting teacher and educational researcher. She also has international experiences in conducting projects and organizing conferences. In 2015, Annemiek started her PhD research in collaboration with the Freudenthal Institute and Erasmus MC. This research was conducted under the supervision of dr. Michiel Doorman, dr.ir. Johan Pel, prof. dr. Paul Drijvers and prof. dr. Hans van der Steen. It was funded by the Dutch Research Council (NWO, Project 023004048). During this project, Annemiek published several articles about her research project and presented her research results at national and international conferences.

Motivation for writing this thesis

In 1991, I started teaching mathematics at the school for visually impaired students in Rotterdam. At that time, the students at this school did not use braille. During my education, and that still applies to all mathematics teachers in the Netherlands, I had no training in teaching students with a visual impairment. Therefore, I always tried to see myself as a potential learner in my interaction with braille readers. That helped to extend my teaching practice.

In the mid-1990s, the student population was expanded with braille readers. That was a big challenge. My colleagues and I were first introduced to braille, the mathematical notation for braille readers, tactile graphs and other assistive devices. It was an exciting, challenging and fun time. I was lucky enough to always be able to ask Ans van Helden for help. She worked as a mathematics teacher at Bartiméus, at the school for visually impaired students in Zeist, and had a lot of experience teaching braille readers.

From the early 90s, I worked in both special and mainstream – vocational and higher – education. I taught mathematics, physics and chemistry. It was an interesting, busy and inspiring time for me. In 2000, I stopped teaching in mainstream education and started to work as a visiting teacher of braille readers who went to mainstream schools. My focus was on the science subjects, especially on mathematics. I enjoyed my work very much. I felt that I benefited greatly from my many years of experience in both mainstream and special education.

I had the feeling that I was making good progress, but at the same time, I realized that I had too little knowledge and skills to really help the braille readers and their mathematics teachers. I often felt that they underperformed in mathematics because I (we) could not support them properly. Slowly but surely, I more and more felt the need to do research.

In 2007, I was awarded a grant for a two-year research project from the Dutch Research Council (NWO). I conducted this research at the Freudenthal Institute under the guidance of Michiel Doorman. My research focused on stimulating and promoting collaboration between braille readers and sighted students and on exploring tactile graphs. During that period, the foundation was laid for my PhD research. Jan van Maanen, director of the FI at the time, and Michiel Doorman, played a central role in this.

In 2014, I submitted an application for PhD research to the NWO. During the writing of the application, our small group – consisting of Jan, Michiel and I – was expanded with Hans van der Steen and Johan Pel. They both work at the Department of Neuroscience of Erasmus MC in Rotterdam. What a multidisciplinary team! In 2015, I started my PhD project. Unfortunately, Jan van Maanen had to withdraw due to health reasons. Luckily, Paul Drijvers could take over his tasks. It was the start of a beautiful and interesting adventure.

Now, at the end of my PhD, I am proud on the one hand because I have gained much more insight into a number of things. That insight helps me and my colleagues in supporting braille readers. On the other hand, I also realize very well that there is still a lot of work to be done. I sincerely hope that I can contribute to that as well.

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The main research question guiding this study is: How can braille readers improve reading and comprehending of mathematical expressions? We have studied that question from different perspectives. The emphasis was on research into tactile perception, professionalization of mathematics teachers and the representation of mathematical expressions in braille.

Braille readers read expressions by moving their fingertips over the braille characters. We have gained a lot of insight into this with the help of finger tracking technology. The results show that braille readers have developed their own 'reading strategies', making little use of the mathematical structure of expressions. Therefore, we developed an intervention teaching braille readers to use finger movements that focus on the structure of the expressions. As a result, they picked up on the features of the mathematical structure more quickly. We then gave a professionalization course to mathematics teachers of braille readers. Tactile and auditory perception played a crucial role in this course. The results show that teaching mathematics to braille readers is not just a matter of translating teaching materials into braille. Reading in braille also has repercussions for teaching and learning mathematics. Finally, we examined mathematical braille notations in various countries. The results show that the expressions are usually represented in such a way that they support the braille readers themselves in obtaining a mathematical overview, or that they support communication between braille readers and readers who do not use braille.