Contact Structures & Codimension-one Symplectic Foliations

Contact Structuren & Symplectische Foliaties van Codimensie één

(met een samenvatting in het Nederlands)

Proefschrift

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Introduction

Parts of this thesis are based on joint work; Chapter 2 is joint with F. Presas, and Chapter 3 is joint with A. del Pino.

Differentiable manifolds are topological spaces which can be studied using the tools of calculus. We can derive/integrate functions and consider objects such as distributions, allowing us to define and study differential equations. The notions used to define differential equations are infinitesimal in nature, making them relatively easy to manipulate using algebraic methods. On the other hand their solutions usually reflect the global properties of the manifolds. Exploiting this interaction between local (infinitesimal) and global (topological) turns out to be extremely fruitful.

We apply this philosophy to the study of geometric structures, and in particular their topological properties. There are many interesting questions one usually poses in this setting. Some of the most fundamental ones are:

- Which manifolds admit a geometric structure of a given type?
- Can we classify all geometric structures of a given type on a fixed manifold?
- Does the existence of a structure of type A imply the existence of a structure of type B?

More often than not, such questions are surprisingly hard to answer. And, more importantly, trying to answer them provides many interesting insights.

In this thesis we restrict ourselves to two particular types of geometric structures: contact structures and codimension-one symplectic foliations (a very special kind of Poisson structure). The motivating question is:

What is the interaction between contact structures and codimension-one symplectic foliations?

Before examining what makes this question interesting, let us briefly discuss the definition of these structures. A hyperplane distribution ξ on a manifold M is a collection of codimension-one subspaces of the tangent space

$$\xi_p \subset T_pM, \quad p \in M,$$

depending smoothly on p. The class of all distributions is too large to study at once. To select smaller classes we impose additional conditions in terms of the curvature of the distribution. To be precise, the **curvature** is a map $c_\xi : \xi \times \xi \to TM/\xi$, which is defined in terms of the Lie bracket by the formula:

$$(X, Y) \mapsto [X, Y] \bmod \xi, \quad \forall X, Y \in \Gamma(\xi).$$

We can think of it as the derivative of the distribution. There are two conditions on the curvature (or "differential relations") whose solutions are particularly interesting:

- A (codimension-one) **foliation** is a hyperplane distribution whose curvature is zero. A famous theorem of Frobenius states that a foliation induces a partition of M into (smooth, immersed, codimension-one) submanifolds, called the leaves of the foliation. Moreover, the decomposition locally looks like the decomposition

  $$\mathbb{R}^n = \bigcup_{z \in \mathbb{R}} \mathbb{R}^{n-1} \times \{z\},$$

  where each copy of $\mathbb{R}^{n-1}$ is a leaf.

  Since the leaves are manifolds they can be endowed with additional structure. A **symplectic structure** on a manifold M is a differential form $\omega \in \Omega^2(M)$ which is closed and non-degenerate. That is, it satisfies:

  $$\mathrm{d}\omega = 0, \quad \omega^n := \omega \wedge \cdots \wedge \omega \neq 0,$$

  where $\mathrm{d}\omega$ denotes the de Rham differential, and the dimension of M equals $2n$. A **symplectic foliation** (or SF-structure for short) is a pair $(\mathcal{F}, \omega)$. It consists of a foliation $\mathcal{F}$, together with a leafwise symplectic form $\omega \in \Omega^2(\mathcal{F})$. Note that a symplectic structure on a surface is the same thing as an area form. Hence, for (oriented) 3-manifolds any orientable foliation is automatically symplectic.

- A **contact structure** is a hyperplane distribution ξ for which the curvature is maximally non-degenerate. This means that it is "as far away from zero as possible". If TM/ξ is trivial (which we usually assume) then the curvature can be interpreted as a differential form $c_\xi \in \Omega^2(\xi)$ and the non-degeneracy condition is equivalent to:

  $$c_\xi^n \neq 0, \tag{1}$$

  where the dimension of M is $2n + 1$.

Equation 1 is one of the reasons why we consider foliations with leafwise symplectic structures. It implies that if a distribution ξ is contact, then the curvature defines a non-degenerate 2-form $\omega \in \Omega^2(\xi)$. On the other hand, for a foliation the curvature vanishes. So, to save the analogy with the contact case we impose the existence of a leafwise non-degenerate form.

Thus, in some sense contact structures and (symplectic) foliations are complete opposites. However, they turn out to be more similar than their definition suggests. For example:

- They have the same underlying "algebraic structure". For simplicity (although it is not necessary) let us only consider distributions ξ for which TM/ξ is trivializable. For any such distribution there exists a nowhere vanishing form $\alpha \in \Omega^1(M)$ such that $\xi = \ker \alpha$.

  With this, an SF-structure can be encoded in a pair $(\alpha, \omega) \in \Omega^1(M) \times \Omega^2(M)$ satisfying:
  $$\alpha \wedge \omega^n \neq 0, \quad \alpha \wedge \mathrm{d}\alpha = 0, \quad \alpha \wedge \mathrm{d}\omega = 0.$$
  Similary, a contact structure may be interpreted as a pair $(\alpha, \omega) \in \Omega^1(M) \times \Omega^2(M)$ satisfying:
  $$\alpha \wedge \omega^n \neq 0, \quad \mathrm{d}\alpha = \omega.$$
  Although the equations involving the de Rham differential are different, in both cases we have a pair (α, ω) satisfying the equation:
  $$\alpha \wedge \omega^n \neq 0. \tag{2}$$
  Thus, the underlying algebraic equations (i.e. the ones not involving the differential) are identical. As a consequence, both structures have the same "formal obstructions" to their existence.

- Both structures have no local invariants. Let $(x_1, y_1, \ldots, x_n, y_n, z)$ denote Euclidean coordinates on $\mathbb{R}^{2n+1}$. Any contact manifold is locally isomorphic to
  $$\left(\mathbb{R}^{2n+1}, \quad \alpha := \mathrm{d}z + \sum x_i \mathrm{d}y_i\right),$$
  while any SF-manifold locally looks like
  $$\left(\mathbb{R}^{2n+1},\ \alpha := \mathrm{d}z,\ \omega := \sum \mathrm{d}x_i \wedge \mathrm{d}y_i\right).$$
  On the other hand, non-isomorphic contact (and SF) structures on the same manifold do exist. So, they have global properties distinguishing them.

  The local models above demonstrate another interesting phenomenon. On $\mathbb{R}^{2n+1}$ consider the 1-parameter family of pairs:
  $$\alpha_t := \mathrm{d}z + t\sum x_i \mathrm{d}y_i,\ \omega_t := \sum \mathrm{d}x_i \wedge \mathrm{d}y_i, \quad t \in [0,1]. \tag{3}$$
  Observe that (α_t, ω_t) satisfies Equation 2 for all $t \in [0,1]$. Furthermore, α_1 coincides with the local model for contact structures (and $\omega_1 = \mathrm{d}\alpha_1$) while (α_0, ω_0) equals the one for SF-structures. Hence, the local models can be deformed into each other. Moreover, the deformation is contained in the space of pairs satisfying Equation 2.

- Many interesting manifolds that admit an SF-structure also admit a contact structure, and vice versa. Even more, constructions which are almost immediate on one side often become highly non-trivial on the other side.

  For example, given an SF-manifold $(M, \mathcal{F}, \omega)$ we can take the product:
  $$\left(M \times \mathbb{S}^2,\ \widetilde{\mathcal{F}} := \mathcal{F} \times \mathbb{S}^2,\ \widetilde{\omega} := \omega + \omega_{\mathbb{S}^2}\right),$$

where $\omega_{\mathbb{S}^2}$ denotes the area form on $\mathbb{S}^2$. To be precise, $\widetilde{\mathcal{F}}$ is the product foliation, whose leaves are $L \times \mathbb{S}^2$ with L a leaf of $\mathcal{F}$. On the other hand, given a contact manifold (M, ξ) it is highly non-trivial to show that $M \times \mathbb{S}^2$ admits a contact structure. The proof can be found in a paper by Bowden, Crowley and Stipsicz [19].

An example in the other direction is given by the (odd-dimensional) spheres. Let $(x_1, y_1, \ldots, x_n, y_n)$ denote Euclidean coordinates on $\mathbb{R}^{2n}$. The restriction of the form

$$\alpha := \sum x_i \mathrm{d}y_i \tag{4}$$

to the unit sphere $\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$ defines a contact form. Thus, all spheres are contact manifolds. The analogous question for SF-structures is still (mostly) open. The only spheres which are known to have SF-structures are $\mathbb{S}^3$ as shown by Reeb [100], and $\mathbb{S}^5$ as shown by Mitsumatsu [89].

Another interesting parallel is that when constructing symplectic foliations, there is often a natural contact structure around. For example, the contact structure from Equation 4 plays an important role in Mitsumatsu's construction on $\mathbb{S}^5$. An example in the opposite direction is given by the main result of [54]. It states that for any 4-manifold M the product $M \times \mathbb{S}^1$ is contact. The key obsevation is that the product can be obtained as a gluing:

$$M \times \mathbb{S}^1 = (W_1 \times \mathbb{S}^1) \cup (W_2 \times \mathbb{S}^1),$$

where (W_i, ω_i), $i = 1, 2$ is a symplectic manifold with boundary. Thus each of the pieces is naturally a SF-manifold, whose foliation equals

$$\mathcal{F}_i := \bigcup_{z \in \mathbb{S}^1} W_i \times \{z\},$$

and with leafwise symplectic form ω_i.

In conclusion, although their definitions are "opposite" there exist many parallels between contact structures and symplectic foliations. This interaction/duality is an interesting subject of study on its own. Furthermore, it provides, at least on an intuitive level, a dictionary to translate between the two worlds. We expect this can be used as a tool to answer questions on one side using results from the other.

The thesis is divided into three chapters, each approaching the main question from a different perspective. Below we briefly illustrate these approaches.

I. Constructions

Arguably the most fundamental question to answer about any geometric structure is that of its existence. A classical theorem by Martinet [82] states that any 3-manifold admits a contact structure. The proof is based on a result by Lickorish [76] saying that any 3-manifold can be obtained from $\mathbb{S}^3$ (which is contact) by surgery along a codimension-2 submanifold B. That is, any 3-manifold M can be decomposed as:

$$M = (B \times \mathbb{D}^2) \cup (\mathbb{S}^3 \backslash B). \tag{5}$$

Each of the components is a contact manifold and they can be glued (in a non-trivial way) to obtain a contact structure on M.

This motivates the definition of an **open book decomposition** of a manifold M. It consists of two pieces of data:

- A codimension-2 submanifold $B \subset M$, called the binding.
- A fibration on the complement of B, $\pi : M\backslash B \to \mathbb{S}^1$, whose fibers are called the pages.

The picture to have in mind is that of a book "opened so far that the front and back cover touch". This data satisfies certain compatibility conditions (precise details are given in Definition 1.9.2) that allow us to recover M as a gluing

$$M = \left(B \times \mathbb{D}^2\right) \cup (M\backslash B).$$

Improving on Marinet's result, Giroux showed that there is a 1-1 correspondence between open books and contact structures:

Classic Result 1 ([57])**.** *Let M be a compact oriented 3-manifold. Then there is a 1-1 correspondence between contact structures on M (up to isotopy) and open book decompositions of M (up to positive stabilization).*

Even though this result is 3-dimensional in nature, it has had a marked influence on the study of higher dimensional manifolds. For example, open books have been used to obtain contact structures on circle bundles [32] and simply connected 5-manifolds [107]. Later, using a different set of techniques called h-principles, the existence (and part of the classification) question has been answered in all dimensions [36, 15].

On the side of SF-structures much less is known. The first non-trivial example was given by Reeb [100] who showed that $\mathbb{S}^3$ admits a (symplectic) foliation. His argument uses that the sphere is the union of two solid tori

$$\mathbb{S}^3 = (\mathbb{S}^1 \times \mathbb{D}^2) \cup (\mathbb{D}^2 \times \mathbb{S}^1).$$

Each solid torus has an obvious foliation by disks, however they do not match along the common boundary. Hence, we apply a trick to glue them; we change each foliation by "spinning the leaves along the $\mathbb{S}^1$-direction" so they become tangent to the boundary. This procedure is called **turbulization**. The upshot is that both foliations have the boundary torus as a leaf and thus can be glued. The resulting foliation on $\mathbb{S}^3$ is called the Reeb foliation.

Following these ideas, Lawson constructed foliations on $\mathbb{S}^5$ and $\mathbb{S}^{2^k+3}$ for $k > 1$. Later, Thurston showed that a compact manifold admits a foliation if and only if its Euler characteristic vanishes. Unlike the 3-dimensional case these foliations are

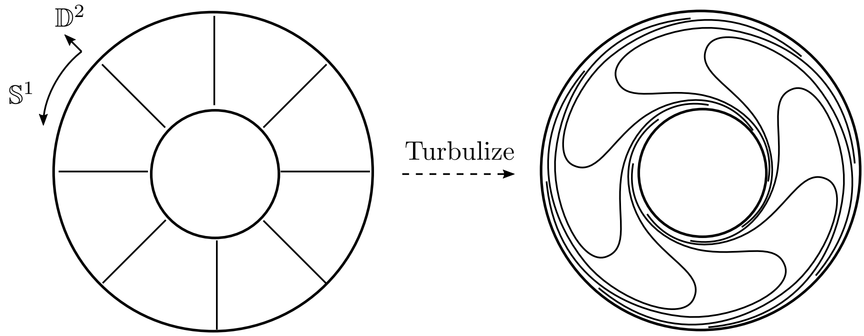

Figure 1: The foliation by disks on the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$, before and after turbulization.

not automatically symplectic. Constructing leafwise symplectic forms is an intricate problem, depending heavily on the topology of the leaves. The main issue is that constructing symplectic structures (in the non-foliated case) is already difficult. In [89] Mitsumatsu proved that the Lawson foliation on $\mathbb{S}^5$ admits a leafwise symplectic form. However, it can be shown that the Lawson foliations on $\mathbb{S}^{2^k+3}$, $k > 1$, cannot be made symplectic. Thus the existence question remains open for the higher dimensional spheres.

In Chapter 1 we focus on the construction of symplectic foliations and contact structures. Motivated by the above ideas we try to construct them by decomposing manifolds into smaller pieces. This approach requires a good understanding of gluing constructions for contact/SF- manifolds with boundary.

To this end we start by studying the behaviour of these structures near the boundary. We distinguish several special types of boundaries, analogous to the well-known contact/cosymplectic boundaries of symplectic manifolds. Furthermore, we obtain explicit normal forms and use them to describe general gluing constructions. Let us elaborate; consider a SF-structure $(\mathcal{F}, \omega)$ on a manifold M with boundary. If $\mathcal{F}$ is transverse to the boundary then the restriction

$$\mathcal{F}_\partial := \mathcal{F} \cap T\partial M, \quad \omega_\partial := \omega|_{\mathcal{F}_\partial},$$

is again a foliation, called a **∂-SF structure**. Note that the leafwise 2-form ω_∂ is still closed, but now has 1-dimensional kernel since the leaves of $\mathcal{F}_\partial$ are odd-dimensional. This is equivalent to the existence of a foliated form $\beta \in \Omega^1(\mathcal{F}_\partial)$ such that $\beta \wedge \omega^n \neq 0$ on the leaves of $\mathcal{F}_\partial$. We call β an admissible form for the ∂-SF structure $(\mathcal{F}_\partial, \omega_\partial)$. This data defines an SF-manifold called the local model associated to $(\mathcal{F}_\partial, \omega_\partial, \beta)$:

$$\big((-\varepsilon, 0] \times \partial M, (-\varepsilon, 0] \times \mathcal{F}_\partial, \omega_\partial + \mathrm{d}(t\beta)\big),$$

where $t \in (-\varepsilon, 0]$ denotes the interval coordinate.

Theorem (1.5.12)**.** *Let $(M, \mathcal{F}, \omega)$ be a symplectic foliation transverse to the (compact) boundary ∂M. For any choice of admissible form β there is a neighborhood of the boundary on which $(\mathcal{F}, \omega)$ is isomorphic to the local model associated to $(\mathcal{F}_\partial, \omega_\partial, \beta)$.*

The analogous statement for contact structures in given in Theorem 1.3.26.

On the other hand we have symplectic foliations which are tame at the boundary; this means that the boundary ∂M is a leaf of $\mathcal{F}$, and that both $\mathcal{F}$ and ω are "constant around the boundary". Such SF-structures are particularly convenient for gluing constructions. We adapt the classical turbulization construction to the setting of symplectic foliations. This allows us to change transverse boundaries into tame ones.

Theorem (1.7.32)**.** *Let $(\mathcal{F}, \omega)$ be an SF-structure on M, transverse to the boundary and with induced ∂-SF structure $(\mathcal{F}_\partial, \omega_\partial)$. Suppose that $\mathcal{F}_\partial$ can be defined by a closed 1-form (i.e. is unimodular). Then, given any symplectic extension $\widetilde{\omega}_\partial \in \Omega^2(\partial M)$ of ω_∂, there exists an SF-structure $(\widetilde{\mathcal{F}}, \widetilde{\omega})$ on M satisfying:*

- *(i) $(\widetilde{\mathcal{F}}, \widetilde{\omega})$ is tame at the boundary, and the induced symplectic form on the boundary leaf equals $\widetilde{\omega}_\partial$;*
- *(ii) $(\widetilde{\mathcal{F}}, \widetilde{\omega})$ agrees with $(\mathcal{F}, \omega)$ away from the boundary.*

Putting the normal forms and turbulization procedure together we build contact and SF-structures on open book decompositions. In both cases the arguments are extremely similar. In fact, we show that under suitable conditions both structures can be constructed simultaneously, and even "deformed" (as in Equation 3) into each other.
We provide a general statement (Theorem 1.8.14) which applies to any (closed, oriented) 3-manifold and $\mathbb{S}^5$. This recovers the (existence) result of Mitsumatsu [89], as well as (deformation) results by Mori [92] and Etnyre [48].

Theorem (1.9.1)**.** *The Lawson foliation on $\mathbb{S}^5$ admits a leafwise symplectic form, and the resulting symplectic folation can be deformed (in the sense of Definition 1.8.1) into a contact structure.*

II. Convergence of contact structures

We have seen in Equation 3 that the local models for contact and SF-structures can be deformed into each other. To give another example of this phenomenon let (x, y, z) denote angular coordinates on the torus $\mathbb{T}^3$ and define:

$$\alpha_t := \mathrm{d}z + t\,(\sin(z)\mathrm{d}x + \cos(z)\mathrm{d}y)\,, \quad \omega_t := \mathrm{d}x \wedge \mathrm{d}y, \quad t \in [0, 1].$$

This pair satisfies Equation 2 for all t, and α_1 is a contact form while (α_0, ω_0) defines a symplectic foliation.

The existence of such deformations is no coincidence. It turns out that, at least in dimension 3, almost any foliation can be approximated by contact structures. A hyperplane distribution $\xi := \ker \alpha$ on an oriented 3-manifold is called a **confolation** if it satisfies

$$\alpha \wedge \mathrm{d}\alpha \geqslant 0, \tag{6}$$

where, the sign is defined with respect to the orientation on M. This notion was first introduced by Eliashberg and Thurston [47], and provides a natural framework to compare contact structures ($\alpha \wedge \mathrm{d}\alpha > 0$) and symplectic foliations ($\alpha \wedge \mathrm{d}\alpha = 0$). Their main theorem states the following:

Classic Result 2 ([47]). *Any (symplectic) foliation on a closed oriented 3-manifold, different from the foliation by spheres:*

$$\mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \{z\} \times \mathbb{S}^2$$

on $\mathbb{S}^1 \times \mathbb{S}^2$, can be ($C^0$-) approximated (Definition 2.2.17) by contact structures.

Usually the limit foliation and the approximating sequence of contact structures are closely related. For example, sometimes the approximating contact structure is unique up to some suitable notion of equivalence [111], or the topological properties of the limit foliation are reflected in those of the sequence [110].

It is clear from the definition (Equation 6) that the theory of confoliations is purely 3-dimensional. In Chapter 2 we follow the same philosophy to study the relationship between contact structures and (symplectic) foliations in higher dimensions. This chapter is based on joint work with F. Presas. We define several notions of deformation, and study their relationship through explicit examples. For instance, we consider **linear deformations** ξ_t, $t \in [0,1]$, of a foliation $\mathcal{F}$. By this we mean that $\xi_0 = \mathcal{F}$, ξ_t is contact for $t > 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\xi_t$, does not depend on t.

One interesting aspect of our discussion is that conformal symplectic foliations naturally show up in several places. To illustrate this, recall that a symplectic structure consists of a differential form $\omega \in \Omega^2(M)$ satisfying

$$\omega^n \neq 0, \quad \mathrm{d}\omega = 0.$$

A **conformal symplectic structure** is a mild generalization of this, replacing the second condition by

$$\mathrm{d}\omega + \nu \wedge \omega = 0,$$

where $\nu \in \Omega^1(M)$ is some closed 1-form. We have:

Theorem (2.2.13). *A (co-oriented) foliation $\mathcal{F}$ can be linearly deformed (Definition 2.2.5) into a contact structure if and only if it admits an exact leafwise conformal symplectic structure.*

Another focus of the chapter is the search for foliations that cannot be approximated (or deformed into) contact structures. By Classic Result 2 only one such foliation exists in dimension-3. It turns out that in higher dimensions there are many foliations with this property. This can be seen using a special kind of submanifold:

Definition (2.4.4)**.** *Let $(M, \mathcal{F}, \omega)$ be a symplectic foliation. A submanifold $N \subset M$ is called an* ***almost CS-submanifold*** *if the restriction*

$$(\mathcal{F}|_N, \omega|_N),$$

defines a symplectic foliation on N.

The definition implies that if ξ approximates $(\mathcal{F}, \omega)$, then the restriction $\xi|_N$ approximates $(\mathcal{F}|_N, \omega|_N)$. Thus, if the SF-structure $(\mathcal{F}|_N, \omega|_N)$ cannot be approximated then the same holds for $(\mathcal{F}, \omega)$. Together with Classical Result 2 this implies:

Theorem (2.4.9)**.** *If a (conformal) symplectic foliation $(\mathcal{F}, \omega)$ on M contains $\mathbb{S}^1 \times \mathbb{S}^2$ (foliated by spheres) as an almost CS-submanifold, then it cannot be approximated by contact structures.*

For instance, given a symplectic manifold (M, ω_M), the symplectic foliation on $\mathbb{S}^1 \times \mathbb{S}^2 \times M$ defined by:

$$\left(\mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \{z\} \times \mathbb{S}^2 \times M,\ \omega := \omega_{\mathbb{S}^2} + \omega_M\right),$$

cannot be approximated by contact structures. We also show that (in dimension at least 7) the existence of a "formal" almost CS-submanifold $\mathbb{S}^1 \times \mathbb{S}^2$ is an obstruction to approximate by contact structures.

In light of these examples one may ask if the presence of $\mathbb{S}^1 \times \mathbb{S}^2$ is the only obstruction to approximation. A substantial part of the chapter is devoted to answering this question. Our result is stated as follows:

Theorem (2.5.38)**.** *There exists a conformal symplectic foliation on $\mathbb{S}^3 \times \mathbb{T}^2$ which does not contain an almost CS-submanifold isomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$ and cannot be approximated by contact structures.*

The proof is based on the clutching construction for contact fibrations. Consider a fibration $\pi : M \to \mathbb{S}^2$, with fiber F. We can decompose the base as the gluing of two disks, the north and south hemisphere. The restriction of the fibration to each of the disks is trivial (since $\mathbb{D}^2$ is contractible). Therefore, the fibration is completely encoded in the transition function

$$\phi : \mathbb{S}^1 \to \mathrm{Diff}(F).$$

The classical clutching construction states that this procedure yields a 1-1 correspondence between fibrations $\pi : M \to \mathbb{S}^2$ with fiber F (up to isomorphism) and loops of diffeomorphisms $\phi : \mathbb{S}^1 \to \mathrm{Diff}(F)$ (up to homotopy).

Taking this idea to the contact setting, contact structures on the total space of a fibration correspond to loops of contactomorphisms satisfying a certain condition called positivity. On one hand it is known that there are contact manifolds which do not admit any positive loops. On the other hand we show that for some foliations, any approximating contact structures would induce a positive loop. Combining these facts we obtain the desired family of (conformal sympletic) foliations on $\mathbb{S}^3 \times \mathbb{T}^2$ that cannot be approximated by contact structures.

III. Wrinkling h-principles

Let us go back to the similarities between contact and SF-structures. We have seen in Equation 2 that both structures have the same "formal structure". This statement can be made precise using the framework of h-principles.

The h-principle (short for homotopy principle) is a collection of techniques to study the solution space of a given (partial) differential equation. More precisely, we are interested in describing the homotopy type of the space of solutions, thus explaining the name. The idea is that a differential equation defines an underlying algebraic equation. Any solution of the former must in particular satisfy the latter. As a concrete example, consider the equation

$$m\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + kf = 0, \tag{7}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function, and $m, k \in \mathbb{R}$ are fixed constants. By replacing $\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}$ by an independent function $g : \mathbb{R} \to \mathbb{R}$ we obtain

$$mg + kf = 0. \tag{8}$$

This equation is purely algebraic, i.e. it does not involve taking derivatives. Secondly, any solution of Equation 7 induces a solution of Equation 8 (which we call a formal solution) by setting $g = \frac{\mathrm{d}^2 f}{\mathrm{d}t^2}$.

Rather surprisingly there are quite general conditions under which the existence of formal solutions implies the existence of genuine solutions. More abstractly, given a (partial) differential relation $\mathcal{R}$, there is an inclusion:

$$\iota : \mathrm{Sol}(\mathcal{R}) \hookrightarrow \mathrm{Sol}^f(\mathcal{R}),$$

where $\mathrm{Sol}^f(\mathcal{R})$ denotes the space of formal solutions of $\mathcal{R}$. Note that $\mathrm{Sol}^f(\mathcal{R})$ is just the space of sections whose image lies in $\mathcal{R}$. We say that $\mathcal{R}$ satisfies the (full) **h-principle** if the above inclusion is a homotopy equivalence (and thus, in particular, induces an isomorphism on homotopy groups). For instance, surjectivity in π_0 means that any formal solution is homotopic to a genuine solution.

This perspective was first described by Gromov [60], and popularized by Eliashberg and Mishachev in [43]. One of the main classical tools to establish h-principles is the "holonomic approximation theorem", which can be found in [60, 43]. It implies that if both the differential relation $\mathcal{R}$, and the manifold M (on which we want to solve $\mathcal{R}$) are open then the h-principle holds. The idea of the proof is to exploit the fact that solutions always exist locally. Moreover, being open implies that the manifold has a large region without any topology. By utilizing this "extra space" we can turn local solutions into global ones, establishing an h-principle. Using holonomic approximation it follows almost immediately that the h-principle for contact structures holds on open manifolds. The same techniques have been used for symplectic foliations on open manifolds [12, 49].

On closed manifolds these techniques break down and one needs a different approach. The so called wrinkling technique, introduced by Eliashberg and Mishashev in [40, 42, 41], is particularly suitable in this setting. The idea is that solutions become more flexible once allowed to have mild singularities. In other words, singularities allow us to "create the extra space" which is already present on open manifolds. A good analogy to have in mind is that if one wants to store a large piece of fabric (a solution) in a small box (a closed manifold) one needs to fold it (introduce singularities).

To turn wrinkled solutions into honest ones the singularities have to be resolved. Whether this is possible or not depends on the properties of $\mathcal{R}$, and in general only part of the solutions can be obtained this way. This gives a division of $\mathrm{Sol}(\mathcal{R})$ into two classes, a "flexible" one satisfying the h-principle, and a "rigid one", closely reflecting the topology of the underlying manifold. The prototypical example is the dichotomy between tight and overtwisted contact structures; a contact structure is called overtwisted if it contains a certain local model (around a disk) and tight otherwise. The latter are usually classified on a case by case basis, while overtwisted contact structures have been shown [36, 15] to satisfy the h-principle also on closed manifolds.

The third chapter is based on work in progress with A. del Pino. We study the h-principle technique of wrinkling in the setting of jet spaces. Given a (fiber) bundle over a manifold $\pi : X \to M$, the r-th jet bundle $J^r(X) \to M$ is the space of r-th order derivatives of sections of X. For a more concrete description consider a function $f : \mathbb{R} \to \mathbb{R}$ (i.e a section of the trivial bundle $\pi : \mathbb{R}^2 \to \mathbb{R}$). Its r-order jet at a point $t \in \mathbb{R}$, denoted by $j^r_t f$, is the tuple

$$\left(t, f(t), \frac{\mathrm{d}f}{\mathrm{d}t}(t), \ldots, \frac{\mathrm{d}^r f}{\mathrm{d}t^r}(t)\right).$$

The space of all such tuples, where we think of the derivatives as independent variables, is precisely the jet space $J^r(\mathbb{R}^2)$.

In general, given a section σ of $J^r(X)$ there does not exist a section $s \in \Gamma(X)$ such that $\sigma = j^r s$. When such an s exists, σ is called holonomic. This can be detected using the Cartan distribution ξ_{can} on $J^r(X)$. It is uniquely defined by the property that a section is holonomic if and only if it its image is tangent to ξ_{can}.

Our aim is to apply wrinkling techniques to prove an h-principle for integral submanifolds of the Cartan distribution. To describe the formal data of an integral submanifold we introduce the integral Grassmannian of jet spaces $\mathrm{Gr}_{\mathrm{integral}}(\xi_{\mathrm{can}}, l)$. It is the space of l-dimensional subspaces of ξ_{can}. Given an integral submanifold $f : N \to J^r(X)$ there is an associated Gauss map:

$$\mathrm{Gr}(f) : N \to \mathrm{Gr}_{\mathrm{integral}}(\xi_{\mathrm{can}}),$$

mapping a point $x \in N$ to the integral element $(\mathrm{d}f)_x(T_x N) \subset \xi_{\mathrm{can}, f(x)}$. Understanding this space and its homotopy type is crucial in the study of integral submanifolds. Although a full description is still out of reach, we describe the homotopy type of part (the so called Σ^2-free part) of this space in Section 3.5.

Roughly speaking, an **r-times differentiable multi-section** (Definition 3.6.2) is a smooth map $f : N \to J^r(X \to M)$ which is graphical over M on an open and dense set, and whose non-graphical part consists of mild singularities with respect to the projection $\pi : J^r(X) \to M$. Thus, images of holonomic multi-sections are special examples of integral submanifolds of the Cartan distribution. As a first step towards a general h-principle, we prove an analogue of the holonomic approximation theorem:

Theorem (3.8.2). *Let $\sigma : M \to J^r(X \to M)$ be an arbitrary section. Then, for any $\varepsilon > 0$, there exists a map $f : M \to J^r(X \to M)$ satisfying:*

- *f is a holonomic multi-section with fold singularities (in zig-zag position);*
- *$|f - \sigma|_{C^0} < \varepsilon$.*

An immediate consequence is that singular (i.e. folded) solutions always exist if $\mathcal{R}$ is open (even if M is closed). Although the above result only states existence, an inspection of the proof should convince the reader experienced in h-principles that a parametric and relative (both in domain and parameter) version also hold.

Our proof exploits the fact that, just like functions, multi-sections can in some sense be differentiated/integrated. As such they can be manipulated through their images under certain projections. A familiar example is given by $(J^1(\mathbb{R}^2 \to \mathbb{R}), \xi_{\text{can}})$, which is isomorphic to $\mathbb{R}^3$ endowed with the standard contact structure. Under this identification integral submanifolds correspond to Legendrian knots. In contact geometry one usually studies these knots through their Lagrangian projection. In particular, it is well-known that a knot can be recovered from its image.

In Section 3.6 we define the analogue of the Lagrangian projection in the setting of jet spaces, and show that it provides a convenient way of manipulating multi-sections. For instance we show (Proposition 3.6.28) that any (Σ^2-free) integral map can be recovered from its image under this projection.

Chapter 1

Constructions of contact structures and symplectic foliations

Chapter Contents

1.1 Overview

In this chapter we consider constructions of contact structures and symplectic foliations. The constructions we have in mind consist of breaking a manifold into (simpler) pieces, building a geometric structure on each of these pieces, and finally glueing them back together. The main example to have in mind is that of an open book decomposition (whose definition we recall in Appendix 1.9).

We start by studying boundaries of manifolds with a geometric structure. Then we use this understanding to obtain gluing constructions. The prototypical example is that of a symplectic manifold with boundary, which we discuss in the Section 1.2. It is well known that symplectic manifolds satisfy a normal form around their boundaries. Moreover, this local model depends only on the induced structure on the boundary. Hence, two symplectic manifolds can be glued if their boundaries (together with the induced structure) matches. Furthermore, there are special types of boundaries (contact type and cosymplectic type) with interesting properties, and for which the local model becomes particularly simple. Using the symplectic case as inspiration, Section 1.3 though Section 1.5, contains the analogous discussion in the setting of contact structures and symplectic foliations.

The main difference with symplectic structures is that there is a difference between a contact structure (resp. symplectic foliation) and the choice of contact form representing it. Given a nowhere vanishing form $\alpha \in \Omega^1(M)$ its kernel defines a distribution

$$\xi := \ker \alpha \subset TM,$$

in which case we say that ξ is represented by α. There are many forms representing the same distribution, and their properties can differ a lot. Although working on the level of structures is conceptually cleaner, many of the constructions depend on particular choices of forms. Hence, we treat both viewpoints separately; In Section 1.3 (resp. Section 1.5) we consider contact structures (resp. symplectic foliations), and in Section 1.4 (resp. Section 1.6) we consider contact forms (resp. symplectic foliation pairs). The main results in these sections are Theorem 1.3.26 and Theorem 1.5.12 giving explicit normal forms for contact structures and (transverse) symplectic foliations around the boundary.

In general, the position of a (symplectic) foliation relative to the boundary is too wild to obtain normal forms. Hence we restrict ourselves to foliations which are everywhere transverse to the boundary, or are "tame" and have the boundary as a leaf. In Section 1.7 we study the classical turbulization construction in the setting of symplectic foliations. This procedure changes a foliation transverse to the boundary into one which is tame at the boundary. The precise statement is given in Theorem 1.7.31.

As remarked before, boundaries of contact and symplectically foliated manifolds, as well as their respective gluing constructions, display many similarities. In fact, we discuss in Section 1.8 that sometimes it is possible to construct both structures at the same time. We show that given a suitable open book decomposition the manifold carries both a contact structure and a symplectic foliation. Moreover, these structures can be deformed into each other. The precise statement is given in Theorem 1.8.6. The hypotheses of the theorem are always satisfied for 3-dimensional manifolds (Corollary 1.8.11). Moreover, in Section 1.9 we apply our construction to $\mathbb{S}^5$ (Theorem 1.9.1). In particular, we recover the symplectic foliation on $\mathbb{S}^5$ constructed by Mitsumatsu [89].

1.1.1 Conventions

Throughout the text we will always assume all manifolds are oriented, unless explicitly stated otherwise. Given an oriented manifold M, we denote by $\overline{M}$ the same (smooth) manifold endowed with the opposite orientation. Furthermore, we use the following orientation convention:

> *For any (geometric) structure that induces an orientation, the induced orientation is assumed to match that of the underlying manifold.*

For example, if ω is a symplectic form on M, as in Definition 1.2.1, then we require the induced volume form to be positive with respect to the orientation on M, that is:

$$\omega^n > 0.$$

Similarly, for a contact form $\alpha \in \Omega^1(M)$, as in Definition 1.3.6, we require:

$$\alpha \wedge \mathrm{d}\alpha^n > 0.$$

Following the same philosophy, the product of two oriented manifolds is endowed with the product orientation. More precisely, if Ω_M and Ω_N are positive volume forms on M and N respectively, then we declare

$$\Omega_{M\times N} := \Omega_M \wedge \Omega_N > 0,$$

to be a positive volume form on $M \times N$. Furthermore, the boundary ∂M of an oriented manifold M is oriented using the "outward normal first" convention. That is, if Ω_M is a positive volume form, and $X \in \mathfrak{X}(M)$ a vector field pointing outwards along the boundary then we declare

$$(\iota_X \Omega_M)\,|_{\partial M} > 0,$$

to be a positive volume form on the boundary. Even more explicitly this means that a manifold with boundary has coordinate charts modeled on the left half space

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \leqslant 0\} \subset \mathbb{R}^n,$$

so that given an oriented boundary chart $(U, x_1, \ldots, x_n)$ on M, an oriented chart on ∂M is given by $(U \cap \partial M, x_2, \ldots, x_n)$.
For example, the products $[0,1) \times \partial M$ and $(-1,0] \times \partial M$, which model a collar neighborhood of the boundary, are both oriented using the (positive) volume form $\mathrm{d}t \wedge \Omega_{\partial M}$, where $\Omega_{\partial M}$ is a positive volume form on ∂M. In the first case an outward normal is given by $-\partial_t$, so its boundary equals $\overline{\partial M}$, oriented by $-\Omega_{\partial M}$. In the second case an outward normal is given by ∂_t so the boundary equals ∂M, oriented by $\Omega_{\partial M}$. Unless stated otherwise we will parametrize a collar neighborhood of the boundary as:

$$U \simeq (-1, 0] \times \partial M.$$

1.2 A source of inspiration: Symplectic structures and their boundaries.

In this section we consider symplectic manifolds and their boundaries. We describe the structure induced on the boundary of a symplectic manifold. A neighborhood of the boundary is completely determined by this structure, giving rise to a normal form. In turn this normal form allows us to glue symplectic manifolds along their boundaries. These results are well known, see for instance [86], but we recall them for completeness and as a source of inspiration for the discussion in subsequent sections.

1.2.1 Boundaries of symplectic manifolds

Let us start by recalling the definition and basic examples of symplectic manifolds.

Definition 1.2.1. *A* ***symplectic structure*** *on a manifold M^{2n} is a 2-form $\omega \in \Omega^2(M)$ satisfying*

$$d\omega = 0, \quad \omega^n > 0.$$

The existence of a symplectic form imposes strong topological restrictions on M. The non-degeneracy of ω implies that M is even dimensional. Furthermore, if M is closed all its cohomology groups of even degree must be non-zero. Indeed, since ω is closed it defines a cohomology class $[\omega] \in H^2(M)$. If this, or any of its wedge powers, vanishes we obtain an exact volume form. For closed symplectic manifolds this cannot happen.

Example 1.2.2. Some of the basic examples of symplectic manifolds are:

- **Euclidean space:** Let $(x_1, y_1, \ldots, x_n, y_n)$ denote the standard coordinates on $\mathbb{R}^{2n}$. The 2-form

  $$\omega := \sum_{i=1}^{n} \mathrm{d}x_i \wedge dy_i, \qquad (1.2.1.1)$$

  is called the standard symplectic structure. By the famous Darboux theorem any symplectic structure locally looks like the standard one. In particular this means that there are no local invariants, and the properties of symplectic structures are closely related to the topology of the underlying manifold.

- **Orientable surfaces:** For dimensional reasons any 2-form on a surface is automatically closed. Thus any choice of volume form defines a symplectic structure. In particular this means that the sphere $\mathbb{S}^2$ is a symplectic manifold. Note that by the discussion above the spheres $\mathbb{S}^{2n}$ for $n \neq 1$, do not admit a symplectic structure since their second cohomology groups are trivial.

- **Tori:** Let $(x_1, y_1, \ldots, x_n, y_n)$ denote the standard angular coordinates on the $2n$-dimensional torus T^{2n}. Then the same formula as in Equation 1.2.1.1 defines a symplectic form. Observe that the standard symplectic structure on Euclidean space is exact, which is possible since $\mathbb{R}^{2n}$ is an open manifold. On the other hand the standard symplectic structure on $\mathbb{T}^{2n}$ is not exact as the torus is closed.

- **Cotangent bundles:** Given any smooth manifold M, the cotangent bundle T^*M caries a canonical exact symplectic structure whose primitive is the so called tautological form $\lambda \in \Omega^1(T^*M)$. It is defined by the rule

  $$\lambda_\alpha := \alpha \circ \mathrm{d}\pi, \quad \forall \alpha \in T^*M, \qquad (1.2.1.2)$$

  where $\pi : T^*M \to M$ denotes the projection. In local coordinates $(q, p) \in T^*M$, where q denotes the base, and p the fiber coordinates, we have:

  $$\lambda = \sum_{i=1}^{n} p_i \mathrm{d}q_i,$$

  where $\dim M = n$. Cotangent bundles together with their symplectic structure play an important role in the description of classical mechanics where they serve as a model for the phase space of a particle.

- **Products:** The simplest way of producing new symplectic manifolds out of old ones is by taking their product. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds then

  $$(M := M_1 \times M_2, \omega := \omega_1 + \omega_2)$$

  is again a symplectic manifold.

△

Given a symplectic manifold (M, ω) with boundary, the restriction $\omega|_{\partial M}$ is still closed but has a one dimensional kernel. This gives rise to the following definition:

Definition 1.2.3. *A **∂-symplectic structure** on a manifold N^{2n+1} is a 2-form $\eta \in \Omega^2(N)$ satisfying*

$$d\eta = 0, \quad \dim \ker \eta = 1.$$

For the boundary of a symplectic manifold, the kernel of the ∂-symplectic form gets paired nondegenerately with a line transverse to the boundary. More precisely, there exists $X \in (M)$, $X \pitchfork \partial M$ and $Y \in \mathfrak{X}(\partial M)$ such that

$$\omega_p(X, Y) > 0, \quad \forall p \in \partial M.$$

The line spanned by X is not determined by η but can be chosen. This corresponds to the choice of a 1-form on N.

Definition 1.2.4. *An **admissible form** for a ∂-symplectic manifold (N^{2n+1}, η) is a 1-form $\theta \in \Omega^1(N)$ satisfying*

$$\theta \wedge \eta^n > 0.$$

By a **∂-symplectic pair** (or just ∂-pair) (θ, η) we mean a ∂-symplectic structure together with a fixed choice of admissible form.

Lemma 1.2.5. *If (N^{2n+1}, η) is a ∂-symplectic manifold then the following hold:*

(i) There exist admissible forms for η;

(ii) Given a fixed admissible form β there is a 1-1 correspondence between admissible forms θ and pairs (f, X), with $f \in C^\infty(N)$ strictly positive, and $X \in \mathfrak{X}(N)$ satisfying $X \in \ker \beta$, given by the formula:

$$\theta = f\beta + \iota_X \eta;$$

(iii) If $N := \partial M$ and $\eta := \omega|_{\partial M}$ for a symplectic manifold (M, ω), then for any $X \in \mathfrak{X}(M)$ such that $X \pitchfork \partial M$ pointing outwards,

$$\theta := \iota_X \omega|_{\partial M}$$

is an admissible form. Conversely, any admissible form is obtained this way.

Proof. (i) Locally, on an oriented coordinate chart $(U, x_1, \ldots, x_{2n+1})$ we have

$$\eta^n = \sum_i f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \ldots dx_{2n+1},$$

where $f_i \in C^\infty(U)$ are such that at each point of U at least one of them is non-zero. Now, define

$$\theta_U := \sum_i (-1)^{i+1} f_i dx_i,$$

then

$$\eta^n \wedge \theta_U = \sum_i f_i^2 dx_1 \wedge \cdots \wedge dx_{2n+1} > 0.$$

Next, choose an atlas $\mathcal{U} = \{U_j\}_{j\in J}$ on M, and $\{\rho_j\}_{j\in J}$ a partition of unity subordinate to it. Construct θ_j as above on each U_j. Then the form

$$\theta := \sum_j \rho_j \theta_j,$$

satisfies $\theta \wedge \eta^n > 0$ globally on M.

(ii) It is easy to check that if β is admissible so is $f\beta + \iota_X\eta$. Conversely, assume θ and β are both admissible forms, then there exists a strictly positive $f \in C^\infty(N)$ such that

$$\eta^n \wedge \theta = f\eta^n \wedge \beta.$$

Let $R \in \mathfrak{X}(N)$ be in the kernel of η. By contracting the above equation with R we find $\theta(R) = f\beta(R)$. Therefore, $\theta - f\beta$ vanishes on R, so there is a unique $X \in \ker\beta$ (note that $\eta|_{\ker\beta}$ is non-degenerate) such that

$$\iota_X\eta = \theta - f\beta.$$

(iii) By assumption $\omega^n > 0$ which implies $\iota_X\omega^n|_{\partial M} > 0$ if $X \in \mathfrak{X}(M)$ is pointing outwards along the boundary. Hence $\theta := \iota_X\omega|_{\partial M}$ is an admissible form for η since:

$$\theta \wedge \eta^n = \frac{1}{n}\iota_X\omega^n|_{\partial M} > 0.$$

The second part of the statement follows immediately from part (ii).

□

Thinking of $\ker\eta$ as a subbundle of TN, we can consider the quotient bundle $TN/\ker\eta$, which has a symplectic vector bundle structure induced by η. The choice of admissible form corresponds to a splitting

$$TN = \ker\eta \oplus (TN/\ker\eta)\,.$$

Observe that the orientations on N and $TN/\ker\eta$ (induced by η) induce an orientation on $\ker\eta$, and any choice of admissible form is compatible with this orientation.

Conversely, associated to each admissible form we have a special vector field spanning $\ker\eta$ and compatible with the orientation:

Definition 1.2.6. *The Reeb vector field associated to the admissible form θ is the (unique) vector field $R \in \mathfrak{X}(N)$ satisfying*

$$\theta(R) = 1, \quad \iota_R\eta = 0.$$

Through the Reeb vector field the admissible form θ gives a decomposition of the tangent bundle,

$$TN = \langle R\rangle \oplus \ker\theta,$$

into the kernel of η and a distribution on which η is non-degenerate.

Definition 1.2.7. *An* ***admissible decomposition*** *of a ∂-symplectic manifold (N, η) is a pair $(R, \mathcal{D})$ where $R \in \mathfrak{X}(N)$ spans the kernel of η, the subbundle $\mathcal{D} \subset TN$ defines a splitting:*

$$TN = \langle R \rangle \oplus \mathcal{D},$$

and $\eta|_{\mathcal{D}}$ is non-degenerate.

In fact, such a decomposition is equivalent to the choice of admissible form:

Lemma 1.2.8. *Given an ∂-symplectic manifold (N, η) there is a 1-1 correspondence between admissible forms θ and admissible decompositions $(R, \mathcal{D})$ given by*

$$\theta \mapsto (R, \ker \theta).$$

Proof. Given an admissible form θ, its Reeb vector field R and kernel $\mathcal{D}$ define an admissible decomposition. Conversely, given an admissible decomposition $(R, \mathcal{D})$ there is a unique differential form θ satisfying

$$\ker \theta = \mathcal{D}, \quad \theta(R) = 1.$$

It follows that θ is an admissible form for η. ☐

1.2.2 Special types of ∂-symplectic manifolds

In many cases, the symplectic form has special behavior around the boundary, which for the ∂-symplectic structure translates into the existence of an admissible form with extra properties. The most important examples are:

Definition 1.2.9. *A ∂-symplectic pair (θ, η) on N^{2n+1} (Definition 1.2.3) is said to be of:*

- ***contact type*** *if*
$$\mathrm{d}\theta = \eta;$$
- ***cosymplectic type*** *if*
$$\mathrm{d}\theta = 0;$$

If a ∂-symplectic manifold (N, η) has an admissible form θ of contact type, then $\xi := \ker \theta$ defines a contact structures on N. Similarly, if θ is of cosymplectic type then (θ, η) defines a cosymplectic structure, see Example 1.5.5.
Of course, there are more types of boundaries than the two above, which could be called special. For example, we have a boundary of **foliation type** if

$$\theta \wedge \mathrm{d}\theta = 0.$$

In this case, η is a globally closed 2-form which is non-degenerate on the leaves of the foliation. In dimension 3 such foliations are called taut, while in higher dimensions they are also called 2-calibrated, see [83].

Definition 1.2.10. *We say that a symplectic manifold (M, ω) has boundary of* ***right $\mathcal{S}$-type*** *(resp.* ***left $\mathcal{S}$-type****), for $\mathcal{S}$ in the above list, if in some collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ (resp. $[0, \varepsilon) \times \partial M$) we have*

$$\omega = \eta + \mathrm{d}(t\theta),$$

where (θ, η) is a ∂-symplectic pair of $\mathcal{S}$-type.

When we talk about boundary of $\mathcal{S}$-type without specifying the side, we always mean right $\mathcal{S}$-type. The above names are meant to emphasize that we think of these boundaries as the left and right boundaries of a cobordism as in Section 1.2.4.
By our conventions, the boundary of a collar of right $\mathcal{S}$-type is oriented as ∂M, while a collar of left $\mathcal{S}$-type has oriented boundary $\overline{\partial M}$. Since we require these orientations to match the ones induced by the symplectic structure, the two types of boundaries are usually not equivalent. For example, if the boundary is of contact type then our definition of left/right boundaries coincides with the usual notions of concave/convex boundary.

Remark 1.2.11. Below, see Theorem 1.2.16, we prove a normal form for boundaries of symplectic manifolds. A consequence of this theorem is that the existence of an admissible form of $\mathcal{S}$-type automatically implies the boundary is of $\mathcal{S}$-type. That is, if (M, ω) is a symplectic manifold and the induced ∂-symplectic manifold $(\partial M, \omega_\partial)$ admits an admissible form θ of $\mathcal{S}$-type, then Theorem 1.2.16 implies there is a collar neighborhood of $\mathcal{S}$-type conform definition 1.2.10. This makes precise our claim that the admissible form encodes the behavior of ω on a neighborhood of the boundary. △

Example 1.2.12. Let (M, ω) be a symplectic manifold endowed with a free Hamiltonian (left) $\mathbb{S}^1$-action and corresponding moment map $\mu : M \to \mathbb{R}$. For any $c \in \mathbb{R}$, the symplectic manifold $M_{\geqslant c} := \mu^{-1}([c, \infty))$ has a smooth boundary $M_c := \mu^{-1}(c)$. Moreover, the usual symplectic reduction, see [86], implies that the quotient manifold inherits a symplectic structure

$$\left(\widetilde{M}_c := M_c/\mathbb{S}^1, \tilde{\omega}\right).$$

Since the action restricts to M_c, the quotient map defines (after changing to a right action) a principal $\mathbb{S}^1$-bundle $\pi : M_c \to \widetilde{M}_c$. It turns out that the topology of this bundle determines the behavior of ω around the hypersurface M_c.

To see this, recall that for a principal $\mathbb{S}^1$-bundle $\pi : P \to B$, any connection form $\theta \in \Omega^1(P)$ satisfies

$$\mathrm{d}\theta = \pi^*(\sigma),$$

for some $\sigma \in \Omega^2(B)$ called the curvature of θ. Moreover, the cohomology class $c_1(P) := [\sigma] \in H^2(B; \mathbb{R})$ depends only on (the isomorphism class) of P, and is referred to as the **(real) Chern class** of P.

Going back to our example, the infinitesimal vector field of the $\mathbb{S}^1$-action on M_c spans the kernel of $\omega|_{M_c}$. Thus, any connection form $\theta \in \Omega^1(M_c)$, is an admissible form for the ∂-symplectic boundary of $M_{\geqslant 0}$. It follows that the boundary of $M_{\geqslant c}$ is of:

- Cosymplectic type if the Chern class of M_c is zero;

- Contact type if the reduced symplectic structure $\widetilde{\omega}$ represents the Chern class of M_c.

In Example 1.2.21 below we continue this example and use the above setup to describe the symplectic cut construction. △

Example 1.2.13. Let (M, ω) be a symplectic submanifold and (B, ω_B) a codimension-2 symplectic submanifold. The ω-orthogonal of TB provides a model for the normal bundle

$$\nu := TB^{\omega} \subset TM|_B,$$

which inherits a fiberwise symplectic form ω_ν. Hence, ν becomes a rank-2 symplectic vector bundle and we can talk about its first Chern class, as explained below. Similar to the previous example, we claim that B admits a neighborhood with a boundary of:

- Cosymplectic type if the Chern class of (ν, ω_ν) vanishes;
- Contact type if ω_B represents the Chern class of (ν, ω_ν).

We recall the following facts:

(i) Any symplectic vector bundle admits a compatible fiberwise complex structure J, and the space of such complex structures is contractible.

(ii) We define the first Chern class of a symplectic vector bundle (E, ω) as that of (E, J) where J is any choice of complex structure compatible with ω. For any such choice, two symplectic vector bundles are isomorphic if and only if they are isomorphic as complex vector bundles. Hence, the first Chern class of (E, ω) is well-defined, and for rank-2 bundles it determines the bundle up to isomorphism.

(iii) A neighborhood of a symplectic submanifold (B, ω_B) in a symplectic manifold (M, ω) is determined (up to isomorphism) by the symplectic form ω_B and the symplectic normal bundle (ν_B, ω_ν).

(iv) There is a 1-1 correspondence between principal $\mathbb{S}^1$-bundles and complex line bundle over B, by sending P to

$$P \times_{\mathbb{S}^1} \mathbb{C} := (P \times \mathbb{C})/\mathbb{S}^1,$$

where the (right) $\mathbb{S}^1$-action on the product is defined by

$$(p, z) \cdot \lambda := (p \cdot \lambda, \lambda^{-1} z), \quad \forall p \in P, z \in \mathbb{C}, \lambda \in \mathbb{S}^1. \tag{1.2.2.1}$$

Moreover, if $[\sigma] \in H^2(B)$ is the Chern class of $P \times_{\mathbb{S}^1} \mathbb{C}$, then there exists a connection form $\theta \in \Omega^1(P)$ such that

$$\mathrm{d}\theta = \pi^* \sigma.$$

In conclusion, we also have a 1-1 correspondence between rank-2 symplectic vector bundles and principal $\mathbb{S}^1$-bundles.

Going back to the example, let $\pi : P \to B$ the principal $\mathbb{S}^1$-bundle corresponding to the symplectic normal bundle of B. Furthermore, let $\theta \in \Omega^1(P)$ be a connection form satisfying $\mathrm{d}\theta = \pi^*\sigma$, where $[\sigma] \in H^2(B, \mathbb{R})$ is the Chern class of the symplectic normal bundle. On $P \times \mathbb{C}$ we define the 2-form

$$\Omega := \pi^*(\omega_B - \sigma) + \mathrm{d}\left((1 + r^2)\theta + r^2\mathrm{d}\phi\right), \tag{1.2.2.2}$$

where $(r, \phi) \in \mathbb{C}$ denote polar coordinates. Observe that Ω is basic with respect to the $\mathbb{S}^1$-action from Equation 1.2.2.1 and descends to a symplectic form on the quotient

$$\left(P \times_{\mathbb{S}^1} \mathbb{C}, \widetilde{\Omega}\right). \tag{1.2.2.3}$$

Observe that the submanifold $P \times \{0\} \subset P \times \mathbb{C}$ is invariant under the $\mathbb{S}^1$-action and hence defines a submanifold of the quotient $P \times_{\mathbb{S}^1} \mathbb{C} \subset P \times_{\mathbb{S}^1} \mathbb{C}$ which can be identified with B. Moreover, under this identification we have

$$\widetilde{\Omega}|_{P \times_{\mathbb{S}^1} \mathbb{C}} = \omega_B,$$

so that $P \times_{\mathbb{S}^1} \mathbb{C}$ is a symplectic submanifold. To describe the induced symplectic normal bundle observe that

$$\nu(B) = \nu(P \times \{0\})/\mathbb{S}^1 = (P \times \mathbb{C})/\mathbb{S}^1 = P \times_{\mathbb{S}^1} \mathbb{C},$$

since the induced $\mathbb{S}^1$-action on $\nu(P \times \{0\})$ is just the one from Equation 1.2.2.1. Furthermore, the restriction

$$\Omega|_{\nu(P \times \{0\})} = 2r\mathrm{d}r \wedge \mathrm{d}\phi,$$

is invariant under the $\mathbb{S}^1$-action. Hence, the symplectic normal bundle to B equals:

$$\left(P \times_{\mathbb{S}^1} \mathbb{C}, 2r\mathrm{d}r \wedge \mathrm{d}\phi\right),$$

which is compatible with the standard complex structure. This implies that its Chern class is equal to that of P, which in turn equals that of the symplectic normal bundle of $(B, \omega_B) \subset (M, \omega)$. We conclude that a neighborhood of B in M, is isomorphic to normal form of Equation 1.2.2.3.

A tubular neighborhood of B can be identified with $P \times_{\mathbb{S}^1} \mathbb{D}^2$, which has boundary $P \times_{\mathbb{S}^1} \mathbb{S}^1 \simeq P$, with the ∂-symplectic form

$$\widetilde{\Omega}_\partial = \pi^*(\omega_B - \sigma),$$

for which θ is an admissible form. Thus, the boundary is of cosymplectic type if $[\sigma] = 0$ and of contact type if $[\sigma] = [\omega_B]$. In Example 1.2.22 below, we continue this discussion to describe the Gompf connected sum for symplectic manifolds. △

Going back to the main story, we observed in Remark 1.2.11 that the admissible form encodes the behavior of the symplectic form around the boundary. In general we need to invoke the normal form, Theorem 1.2.16, but for boundaries of contact and cosymplectic type (Definition 1.2.9) this can be proven by elementary means. Furthermore, in this cases the existence of special admissible forms can be detected using vector fields.

Lemma 1.2.14. *Let (M,ω) be a symplectic manifold with boundary, and let $\omega_\partial := \omega|_{\partial M}$ denote the induced ∂-symplectic form on ∂M. Then the following are equivalent:*

(i) The symplectic form ω has boundary of right contact type (Definition 1.2.10);

(ii) The ∂-symplectic form ω_∂ has an admissible form of contact type (Definition 1.2.9);

(iii) There exists a vector field $X \in \mathfrak{X}(M)$, pointing outwards along the boundary and satisfying

$$(\mathcal{L}_X\omega)\,|_{\partial M} = \omega|_{\partial M};$$

(iv) There exists a vector field $X \in \mathfrak{X}(M)$, pointing outwards along the boundary and a neighborhood U of ∂M satisfying

$$\mathcal{L}_X\omega|_U = \omega|_U.$$

Proof. Assuming that (i) holds there exists a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which

$$\omega = \mathrm{d}\left((1+t)\theta\right),$$

for $\theta \in \Omega^1(\partial M)$ satisfying $\mathrm{d}\theta = \omega_\partial$. Since θ is admissible this immediately implies (ii), and (iii) follows from taking $X = \partial_t$. For (iv) we take $X = (1+t)\partial_t$.

Next, assume that (ii) holds, so there exists an admissible form θ of contact type. By Lemma 1.2.5 it is of the form $\theta = \iota_X\omega|_{\partial M}$ for some vector field $X \in \mathfrak{X}(M)$ pointing outwards along the boundary. Hence,

$$\mathcal{L}_X\omega|_{\partial M} = \mathrm{d}\iota_X\omega|_{\partial M} = \mathrm{d}\theta = \omega|_{\partial M},$$

proving (iii).

If (iii) is true, then we can use the vector field X to define a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$, and define

$$\theta := \iota_X\omega|_{\mathcal{U}} \in \Omega^1(\mathcal{U}).$$

Then both $\mathrm{d}\theta$ and ω are closed forms on $\mathcal{U}$ whose restrictions to ∂M agree. This implies they differ by an exact form which vanishes on the boundary, that is:

$$\omega - \mathrm{d}\theta = \mathrm{d}\beta,$$

for some $\beta \in \Omega^1(\mathcal{U})$ satisfying $\beta|_{\partial M} = 0$.
To see this, let $\mu \in \Omega^2(\mathcal{U})$ be a closed form satisfying $\mu|_{\partial M} = 0$. Then on the collar neighborhood we can write

$$\mu = \mu_t + \mathrm{d}t \wedge \nu_t,$$

with $\mu_t \in \Omega^2(\partial M)$ and $\nu_t \in \Omega^1(\partial M)$ for $t \in (-\varepsilon, 0]$. Since $\mathrm{d}\mu = 0$ (and $\mu_0 = 0$) it follows that

$$\mu_t = \int_0^t \mathrm{d}\nu_s\,\mathrm{d}s.$$

In turn this implies that $\mu = \mathrm{d}\left(\int_0^t \nu_s \mathrm{d}s\right)$ and the primitive vanishes on ∂M, proving the claim. Using the non-degeneracy of ω there is a unique $Y \in \mathfrak{X}(M)$

$$\iota_Y \omega = \theta + \beta.$$

It is easy to check that $\mathcal{L}_Y \omega|_{\mathcal{U}} = \omega|_{\mathcal{U}}$ and that it points outwards along the boundary. proving (iv).

Finally assume (iv) holds. We can use X to define a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which we identify $X = \partial_t$ and write:

$$\omega = \eta_t + \theta_t \wedge \mathrm{d}t. \tag{1.2.2.4}$$

The condition $\mathcal{L}_X \omega = \omega$ implies:

$$\eta_t = e^t \eta, \quad \theta_t = e^t \theta,$$

for some $\eta \in \Omega^2(\partial M)$ and $\theta \in \Omega^1(\partial M)$. Together with $\mathrm{d}\omega = 0$ this means that

$$\eta_t = \dot{\eta}_t = \mathrm{d}\theta_t = \mathrm{d}\theta.$$

Finally, substituting these identities in Equation 1.2.2.4 and changing coordinates $s = e^t - 1$ around $t = 0$ we obtain:

$$\omega = \mathrm{d}\left((1 + s)\beta\right),$$

proving (i). □

The analogous statement for boundaries of cosymplectic type is:

Lemma 1.2.15. *Let (M, ω) be a symplectic manifold with boundary, and let $\omega_\partial := \omega|_{\partial M}$ denote the induced ∂-symplectic form. Then the following are equivalent:*

- *(i) The symplectic form ω has boundary of right cosymplectic type (Definition 1.2.10);*
- *(ii) The ∂-symplectic form ω_∂ has an admissible form of cosymplectic type(Definition 1.2.9);*
- *(iii) There exists a vector field $X \in \mathfrak{X}(M)$, pointing outwards along the boundary and satisfying*
$$(\mathcal{L}_X \omega)|_{\partial M} = 0;$$
- *(iv) There exists a vector field $X \in \mathfrak{X}(M)$, pointing outwards along the boundary and a neighborhood U of ∂M satisfying*
$$(\mathcal{L}_X \omega)|_U = 0.$$

Proof. Assuming that (i) is true, there exists a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which

$$\omega = \eta + \mathrm{d}t \wedge \theta,$$

for some $\theta \in \partial M$ satisfying $\mathrm{d}\theta = 0$. This immediately implies (ii), (iii), and (iv) hold.

Next, if (ii) holds then there exists an admissible form θ of cosymplectic type and by Lemma 1.2.5 it can be written as $\theta = (\iota_X\omega)\,|_{\partial M}$, for $X \in \mathfrak{X}(M)$ a vector field pointing outwards along the boundary. Hence,

$$(\mathcal{L}_X\omega)\,|_{\partial M} = \mathrm{d}\iota_X\omega|_{\partial M} = \mathrm{d}\theta = 0,$$

proving (iii).

If (iii) holds then $\theta := \iota_X\omega|_{\partial M}$ is a closed form on ∂M. Extend it to a closed form on a collar neighborhood $\mathcal{U}$ (still denoted by θ), and define $Y \in \mathfrak{X}(\mathcal{U})$ by

$$\iota_Y\omega = \theta.$$

Then $\mathcal{L}_Y\omega = \mathrm{d}\theta = 0$, proving (iv).

Finally, if (iv) is true, then we can identify $X = \partial_t$ on a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ and write

$$\omega = \eta_t + \theta_t \wedge \mathrm{d}t, \tag{1.2.2.5}$$

with $\eta_t \in \Omega^2(\partial M)$, $\theta_t \in \Omega^1(\partial M)$ for $t \in (-\varepsilon, 0]$. Then, $\mathcal{L}_X\omega = 0$ implies $\dot{\eta}_t = 0$ and $\dot{\theta}_t = 0$, so that $\eta_t = \eta$ and $\theta_t = \theta$ are independent of t. Thus, Equation 1.2.2.5 becomes:

$$\omega = \eta + \theta \wedge \mathrm{d}\theta,$$

from which it is easily seen that θ is a closed admissible for ω_∂, proving (i). □

1.2.3 Normal form around the boundary of symplectic manifolds

To prove normal forms around the boundary (or other types of submanifolds), we will use the following general strategy. Let M be a manifold with boundary ∂M, endowed with some geometric structure $\mathscr{S}$. In this section $\mathscr{S}$ will be a symplectic structure, and in the sections below a contact structure respectively a symplectic foliation.

Constructing a normal form for $\mathscr{S}$ breaks down in the following steps:

- **Induced structure on ∂M:** The first step is to identify what structure is induced on the boundary by considering the restriction $\mathscr{S}|_{\partial M}$. The induced structure is there canonically, without any choices, but forgets about the "information in the direction transverse to ∂M".
- **Local model**: Starting from $(\partial M, \mathscr{S}|_{\partial M})$ we build a local model $(M_{loc}, \mathscr{S}_{loc})$. Since, in passing from $\mathscr{S}$ to $\mathscr{S}|_{\partial M}$ we forgot some information, the construction of $\mathscr{S}_{loc}$ usually involves some choices.
- **Normal form:** The final step is to prove a result saying that locally around ∂M, there is an isomorphism

  $$(M, \mathscr{S}) \simeq (M_{loc}, \mathscr{S}_{loc}),$$

  and that, up to isomorphism, $(M_{loc}, \mathscr{S}_{loc})$ is independent of the choices made in the previous step.

For a symplectic manifold (M, ω) the first step amounts to passing to the ∂-symplectic manifold $(\partial M, \omega|_{\partial M})$, as in Definition 1.2.3. The extra data needed for the second step consists of a choice of admissible form as in Definition 1.2.4.

The local model can be defined for any ∂-symplectic manifold, not only the boundary of a symplectic manifold. Given a ∂-symplectic manifold (N, η) and an admissible form θ, the **local model** is defined by:

$$\big((-\varepsilon, 0] \times N, \eta + d(t\theta)\big),$$

which is symplectic for $\epsilon > 0$ small enough. Finally, in the symplectic case the normal form is well known, see e.g. [86]. We recall the proof for completeness and as a source of inspiration.

Theorem 1.2.16. *For any symplectic manifold (M, ω) a neighborhood of its boundary is isomorphic to the local model associated to $(\partial M, \omega|_{\partial M})$.*

In particular, up to isomorphism the local model does not depend on the choice of admissible form. The proof of the theorem is a direct consequence of the following more technical lemma:

Lemma 1.2.17. *Let (M, ω) be a symplectic manifold with boundary and write $\eta := \omega|_{\partial M}$. Let $\theta \in \Omega^1(\partial M)$ be any admissible form then there exists a collar neighborhood of the boundary $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which*

$$\omega = \eta + \mathrm{d}(t\theta).$$

Proof. On a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ of the boundary we can write

$$\omega_0 := \omega = \eta_t + \beta_t \wedge \mathrm{d}t, \quad \omega_1 := \eta + \mathrm{d}(t\theta),$$

for $\eta_t \in \Omega^2(\partial M)$, $\beta_t \in \Omega^1(\partial M)$ and $t \in (-\varepsilon, 0]$. Define a path of closed forms joining ω_0 and ω_1 by:

$$\omega_s := (1-s)\omega_0 + s\omega_1.$$

Following the standard Moser trick, we look for an isotopy such that

$$\phi_s^* \omega_s = \omega_0,$$

so that ϕ_1 provides the desired change of coordinates.

Since, ω_s is closed and $\omega_0|_{\partial M} = \omega_1|_{\partial M}$, it follows that

$$\omega_1 - \omega_0 = \mathrm{d}\lambda,$$

for some $\lambda \in \Omega^1(\mathcal{U})$ (see the proof of Lemma 1.2.14). Differentiating the above equation we find

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\phi_s^*(\omega_s) = \phi_s^*\left(\mathcal{L}_{X_s}\omega_s + \dot{\omega}_s\right) = \phi_s^*\mathrm{d}\left(\iota_{X_s}\omega_s + \lambda\right),$$

so it suffices to solve

$$\iota_{X_s}\omega = -\lambda. \tag{1.2.3.1}$$

At points in the boundary we have

$$\omega_s^n = n(1-s)\mathrm{d}t \wedge \beta_0 \wedge \eta^{n-1} + ns\mathrm{d}t \wedge \theta \wedge \eta^{n-1} > 0,$$

since both summands are positive volume forms. Hence, ω_s is symplectic for all $s \in [0,1]$ on a neighborhood of ∂M. Therefore there is a unique solution to Equation 1.2.3.1, completing the proof. □

1.2.4 Gluing symplectic manifolds

Gluing operations are extremely useful for constructing geometric structures on a given manifold. They allow us to reduce the problem by cutting a manifold into smaller pieces. It is usually much simpler to show existence on these pieces and gluing them back together solving the original problem.
There are many flavors of such gluing operations, but the most elementary is gluing two manifolds along their boundaries. That is, given two manifolds M_1 and M_2, together with a diffeomorphism $\phi : \partial M_1 \to \partial M_2$, identifying their boundaries, we define

$$M_1 \cup_\phi M_2 := (M_1 \sqcup M_2)/(x \sim \phi(x)) \quad x \in \partial M_1. \tag{1.2.4.1}$$

It is clear that $M_1 \cup_\phi M_2$ canonically is a topological space. However, endowing it with a smooth or symplectic structure is slightly more subtle. To make things more transparent we first recall some of the basics for gluing oriented manifolds, and then consider the symplectic case.

1.2.4.1 Gluing oriented manifolds

Recall that given an oriented manifold M, the boundary is oriented according to the "outward normal first" convention. Thus, if we have a collar neighborhood of the form

$$\mathcal{U} := (-\varepsilon, 0] \times \partial M,$$

then these conventions imply $\partial\mathcal{U} = \partial M$ as oriented manifolds.

To obtain collar neighborhoods we use the following construction. Let M be a manifold with boundary and $X \in \mathfrak{X}(M)$ be a vector field pointing outwards along the boundary. Denote its flow by ϕ^t and define an embedding

$$\phi : (-\varepsilon, 0] \times \partial M \to M, \quad (t,x) \mapsto \phi^t(x).$$

Since $\mathrm{d}\phi(\partial_t) = X$ points outwards, ϕ is orientation preserving.
Similarly, using a vector field $W \in \mathfrak{X}(M)$ pointing inwards with flow ψ^t we obtain an (orientation preserving) embedding

$$\psi : [0, \varepsilon) \times \overline{\partial M} \to M, \quad (t,x) \mapsto \psi^t(x).$$

Remark 1.2.18. Note that the diffeomorphism

$$F : (-\varepsilon, 0] \times \partial M \to [0, \varepsilon) \times \overline{\partial M}, \quad (t, x) \mapsto (-t, x)$$

is orientation preserving and satisfies $F \circ \phi = \psi$. Hence we can identify the two types of collar neighborhoods defined above. △

Let M_1 and M_2 be oriented manifolds whose boundaries are non-empty and diffeomorphic by an (orientation preserving) diffeomorphism

$$\phi : \partial M_1 \xrightarrow{\sim} \overline{\partial M_2},$$

and define, as before,

$$M_1 \cup_\phi M_2 := (M_1 \sqcup M_2)/(x \sim \phi(x)) \quad x \in \partial M_1.$$

To define a smooth structure, choose collar neighborhoods

$$k_1 : (-1, 0] \times \partial M_1 \to M_1, \quad k_2 : [0, 1) \times \overline{\partial M_2} \to M_2, \tag{1.2.4.2}$$

as above. We use the parametrization $[0, \varepsilon)$ instead of $(-\varepsilon, 0]$ for M_2 to avoid unnecessary signs and to indicate we picture M_1 on the left and M_2 on the right.

The two collar neighborhoods k_1 and k_2 define a map

$$k_1 \cup_\phi k_2 : \partial M_1 \times (-1, 1) \to M_1 \cup_\phi M_2, \quad (x, t) \mapsto \begin{cases} k_1(x, t) & t \leqslant 0 \\ k_2(\phi(x), t) & t \geqslant 0 \end{cases} \tag{1.2.4.3}$$

and we obtain a unique smooth structure on $M_1 \cup_\phi M_2$ by requiring this map to be smooth. The following lemma is immediate:

Lemma 1.2.19. *The space $M := M_1 \cup_\phi M_2$ admits a unique smooth structure and orientation, with the property that the inclusions $M_i \hookrightarrow M$ are oriented embeddings and $k_1 \cup_\phi k_2$ is smooth and orientation preserving.*
The resulting structure depends on k_1, k_2 and on ϕ but its diffeomorphism class does not.

1.2.4.2 Gluing symplectic manifolds

Using the normal form from Theorem 1.2.16 we now adapt the gluing operation above to symplectic manifolds. The precise statement is:

Proposition 1.2.20. *Let (M_i, ω_i), $i = 1, 2$ be symplectic manifolds with boundary and induced ∂-symplectic forms $\eta_i := \omega_i|_{\partial M_i}$, as in Definition 1.2.3. Given an orientation reversing diffeomorphism $\phi : \partial M_2 \to \partial M_1$ satisfying*

$$\phi^* \eta_1 = \eta_2.$$

Then, $M_1 \cup_\phi M_2$ admits a symplectic structure ω which restricts to ω_i on M_i.

Proof. Choose an admissible form $\theta_1 \in \Omega^1(\partial M_1)$ for η_1, as in Definition 1.2.4. By Lemma 1.2.17 we know that locally around the boundary (M_1, ω_1) is isomorphic to

$$((-\varepsilon, 0] \times \partial M_1, \eta_1 + \mathrm{d}(t\theta_1)) . \tag{1.2.4.4}$$

Define $\theta_2 := \phi^*(\theta_1)$ and note that $-\theta_2$ is an admissible form on ∂M_2. Thus, locally around the boundary (M_2, ω_2) is isomorphic to

$$((-\varepsilon, 0] \times \partial M_2, \eta_2 - \mathrm{d}(t\theta_2)) .$$

Now, sending $t \mapsto -t$, and using ϕ to identify $(\partial M_1, \eta_1, \theta_1) \simeq (\overline{\partial M_2}, \eta_2, \theta_2)$, the above collar is isomorphic to

$$([0, \varepsilon) \times \partial M_1, \eta_1 + \mathrm{d}(t\theta_1)) ,$$

which glues smoothly to the collar from Equation 1.2.4.4. □

Example 1.2.21. Recall the setup from Example 1.2.12, where we showed that the boundary of the symplectic manifold $(M_{\geqslant c}, \omega)$ is a principal $\mathbb{S}^1$-bundle. We use it to describe the standard **symplectic cut** construction from [74, 86], which is a generalization of the symplectic blowup.

Thus, consider a (left) $\mathbb{S}^1$-action $\rho : \mathbb{S}^1 \times M \to M$ on a symplectic manifold (M^{2n}, ω), with moment map $\mu : M \to \mathbb{R}$. As in Example 1.2.12, suppose that $c \in \mathbb{R}$ is a regular value of μ and that the $\mathbb{S}^1$-action on the submanifold $M_c := \mu^{-1}(c) \subset M$ is free. Then $\pi : M_c \to \widetilde{M}_c$ is a principal $\mathbb{S}^1$-bundle, where the right action is defined by:

$$x \cdot \lambda := \rho_{\lambda^{-1}}(x), \quad \forall x \in M_c, \lambda \in \mathbb{S}^1.$$

Thus if $\theta \in \Omega^1(M_c)$ is a connection 1-form and $\omega_c := \omega|_{M_c} = \pi^*\widetilde{\omega}$ then the orientation on M_c is defined by declaring

$$\theta \wedge \omega_c^{n-1} > 0.$$

We claim that with our usual orientation conventions, the boundary of $M_{\geqslant c}$ is oriented as $\overline{M_c}$. To see this note that the infinitesimal generator $X \in \mathfrak{X}(M_c)$ of the (right) action satisfies

$$\iota_X \omega_c = -\mathrm{d}\mu.$$

Choose a vector field $Y \in \mathfrak{X}(M)$ satisfying

$$\mathrm{d}\mu(Y)|_{M_c} = 1,$$

which in particular implies that Y is pointing inwards along the boundary of $M_{\geqslant c}$. Define $\theta := \iota_Y \omega|_{M_c} \in \Omega^1(M_c)$ and observe that

$$\theta(X) = \omega(Y, X) = \mathrm{d}\mu(Y) = 1.$$

Hence, θ is a connection 1-form and

$$\iota_Y \omega^n|_{M_c} > 0.$$

Since Y is pointing inwards, this means that the induced ∂-symplectic boundary of $(M_{\geqslant c}, \omega)$ equals

$$\left(\overline{M}_c, \omega_c = \pi^*\widetilde{\omega}\right) .$$

Since $\pi : M_c \to \widetilde{M}_c$ is a principal $\mathbb{S}^1$-bundle, we can consider the symplectic manifold

$$\left(M_c \times_{\mathbb{S}^1} \mathbb{D}^2, \widetilde{\Omega}\right),$$

defined similarly to the manifold from Equation 1.2.2.3 in Example 1.2.12, using the formula from Equation 1.2.2.2 with $\sigma = 0$. Then the induced ∂-symplectic boundary is

$$(M_c, \pi^*\widetilde{\omega}).$$

Hence, Proposition 1.2.20 applies and we can glue $(M_{\geqslant c}, \omega|_{M_c})$ to $(M_c \times_{\mathbb{S}^1} \mathbb{D}^2, \widetilde{\Omega})$, and the resulting symplectic manifold is, by definition, the symplectic cut of (M, ω, μ) along c. △

Example 1.2.22. Using the above gluing construction together and the discussion from Example 1.2.13, we describe the standard **Gompf connected sum**, from [59]. Let (M_i, ω_i), $i = 1, 2$ be symplectic manifold with codimension-2 submanifolds (B_i, ω_{B_i}) and $\phi : B_1 \xrightarrow{\sim} B_2$ an orientation preserving diffeomorphism. Moreover, suppose that

(i) $\omega_{B_1} = \phi^*\omega_{B_2}$;

(ii) $\phi^* c_1(\nu_{B_2}) = -c_1(\nu_{B_1}) \in H^2(B_1; \mathbb{Z})$;

where $c_1(\nu_{B_i})$ denotes the first Chern class of the symplectic normal bundle as in Example 1.2.13. The condition on the Chern class implies that P_2 is isomorphic to $\overline{P}_1$. As before, a neighborhood of B_i is isomorphic to

$$(P_i \times_{\mathbb{S}^1} \mathbb{C}, \overline{\Omega}_i),$$

following Equation 1.2.2.3.

The above conditions imply that the ∂-symplectic boundary of the neighborhood of B_2 is given by

$$(P_2, \pi^*\omega_{B_2}) \simeq \left(\overline{P}_1, \pi^*\omega_{B_1}\right).$$

Thus, the symplectic manifolds $M_i \backslash P_i \times_{\mathbb{S}^1} \mathbb{D}^2$, $i = 1, 2$, satisfy the conditions of Proposition 1.2.20 and can be glued along their boundaries. Hence, we conclude that the **Gompf connected sum**,

$$(M_1, B_1)\#(M_2, B_2) := \left(M_1 \backslash P_1 \times_{\mathbb{S}^1} \mathbb{D}^2\right) \cup_\psi \left(M_2 \backslash P_2 \times_{\mathbb{S}^1} \mathbb{D}^2\right),$$

where $\psi : P_1 \xrightarrow{\sim} \overline{P_2}$ is induced by ϕ, carries a symplectic form ω which restricts to ω_i on each of the pieces. △

Remark 1.2.23. The above condition on the Chern classes can be slightly weakened, as it suffices that

$$\phi^* c_1(\nu_{B_2}) = -n c_1(\nu_{B_1}),$$

for some $n \in \mathbb{N}$. In this case we consider $L^{\otimes n} := P \times_{\mathbb{S}^1} \mathbb{C}$, where now the $\mathbb{S}^1$-action on $\mathbb{C}$ is given by

$$\lambda \cdot z := \lambda^n z.$$

Then $c_1(L^{\otimes n}) = n c_1(P)$, and the rest of the construction goes through as before. △

1.3 Contact structures and their boundaries

1.3.1 Contact structures

In this section we recall the basic definitions from contact geometry. We take some care in separating the notions of "contact form" and "contact structure" in our discussion. This is convenient for studying boundaries of contact manifold and gluing constructions, since the extra freedom of working with structures allows us to prove more general results. Secondly, since we always work with oriented manifolds, we point out in which cases a contact structure canonically induces an orientation on the underlying manifold.

Definition 1.3.1. *A* ***contact structure*** *on M^{2n+1} is a codimension one distribution $\xi \subset TM$ such that the associated curvature map $c_\xi : \Lambda^2\xi \to TM/\xi$, which on sections is given by*

$$X \wedge Y \mapsto [X, Y] \bmod \xi, \quad X, Y \in \Gamma(\xi), \tag{1.3.1.1}$$

is non-degenerate.

In general, even though M is always assumed to be oriented, we do not make any assumptions about the (co-)orientability of ξ.

Definition 1.3.2. *A contact structure $\xi \subset TM$ is said to be* ***coorientable/cooriented*** *if TM/ξ is orientable/oriented.*

By a coorientation we mean a trivialization of $(TM/\xi)^*$, i.e. a nowhere vanishing section. This is equivalent to a choice of orientation σ on TM/ξ.

Lemma 1.3.3. *Given a contact structure ξ on M^{2n+1}, the curvature map $c_\xi : \Lambda^2\xi \to TM/\xi$ induces an isomorphism of vector bundles*

$$\Lambda^{2n}\xi \xrightarrow{\sim} \otimes^n(TM/\xi).$$

Proof. Given vector bundles E, F over M denote by $\mathcal{P}_k(E, F)$ the vector bundle with fiber

$$\mathcal{P}_k(E, F)_x = \mathcal{P}_k(E_x, F_x),$$

the space of homogeneous polynomial maps of order k. That is, the space of maps $f : E_x \to F_x$ satisfying $f(tv) = t^k f(v)$, for all $v \in E_x$ and $t \in \mathbb{R}$.
Since the curvature $c_\xi : \Lambda^2\xi \to TM/\xi$ is non-degenerate it induces a bundle map $(TM/\xi)^* \to (\Lambda^{2n}\xi)^*$ given by

$$\sigma \mapsto \Lambda^n(\sigma \circ c_\xi) \in (\Lambda^{2n}\xi)^*, \quad \forall \sigma \in (TM/\xi)^*.$$

This map is homogeneous of degree n and a fiberwise isomorphism so it corresponds to a nowhere vanishing section of $\mathcal{P}_n((TM/\xi)^*, \Lambda^{2n}\xi^*)$. In turn this bundle is canonically isomorphic to the bundle $\otimes^n(TM/\xi) \otimes \Lambda^{2n}\xi^* \simeq \mathrm{Hom}(\Lambda^{2n}\xi, \otimes^n TM/\xi)$, proving the claim. □

Using this lemma we obtain the following:

Corollary 1.3.4. *Let ξ be a coorientable contact structure on a manifold M^{2n+1} then:*

- *If n is even:*
  - *(i) ξ has a canonical orientation;*
  - *(ii) There is a canonical correspondence between coorientations of ξ and orientations on M.*
- *If n is odd:*
  - *(iii) TM has a canonical orientation;*
  - *(iv) There is a canonical correspondence between coorientations of ξ and orientations of ξ.*

Proof. Recall that given a vector bundle E we have the associated determinant bundle $\det(E) := \Lambda^{top}E$, so that orientations of E correspond to nowhere vanishing sections of $\det(E)$ up to scaling by a positive conformal factor.
Choosing a splitting $TM = \xi \oplus TM/\xi$, we have an isomorphism

$$\det(\xi) \otimes TM/\xi \xrightarrow{\sim} \det(TM), \quad (X_1 \wedge \cdots \wedge X_n) \otimes Y \mapsto X_1 \wedge \cdots \wedge X_n \wedge Y.$$

Observe that this isomorphism does not depend on the choice of splitting. Indeed, any two right splittings of

$$0 \to \xi \to TM \to TM/\xi \to 0,$$

differ by a section of ξ. This contribution gets killed under the above map since it corresponds to an element in $\Lambda^{n+1}\xi = 0$. Together with the isomorphism

$$\det(\xi) \simeq \otimes^n TM/\xi,$$

from Lemma 1.3.3 this yields a canonical isomorphism

$$\det(TM) \simeq \otimes^{n+1} TM/\xi. \tag{1.3.1.2}$$

Any nowhere vanishing section $X \in \Gamma(TM/\xi)$ gives a nowhere vanishing section $\tilde{X} := \otimes^n X \in \Gamma(\otimes^n TM/\xi) \simeq \Lambda^{2n}\xi$, satisfying

$$\widetilde{-X} = (-1)^n \tilde{X}.$$

Hence, if n is even any coorientation of ξ induces the same orientation on ξ while opposite coorientations of ξ induce opposite orientations on TM. The proof is similar when n is odd. □

Remark 1.3.5. Recall from Section 1.1.1 that we always assume M to be oriented, and that its orientation agrees with the one induced by ξ. By Corollary 1.3.4 this means that on a manifold M of dimension $2n + 1$ we have:

- If n is odd and ξ is a contact structure on M, then $\overline{M}$ does not admit any contact structure (conform the orientation conventions above). Furthermore if ξ is coorientable, we are free to choose the coorientation, since both choices induce the same orientation on M. Therefore, if we want to be precise, a cooriented contact structure is denoted by a pair (ξ, σ), where σ is an orientation on TM/ξ.
- If n is even ξ only induces an orientation on M after we choose an orientation on TM/ξ. Thus, there is only one possible choice of coorientation so that the induces orientation matches that of M. This also implies that (ξ, σ) is a contact structure on M if and only if $(\xi, -\sigma)$ is a contact structure on $\overline{M}$.

△

The definition of a contact structure can be rephrased in terms of differential forms which are often easier to handle than distributions. For a general contact structure ξ the projection map

$$TM \xrightarrow{\pi} TM/\xi,$$

can be interpreted as a bundle valued differential form $\pi \in \Omega^1(M, TM/\xi)$ satisfying $\xi = \ker \pi$. If ξ is coorientable then there exists a nowhere vanishing section $s \in \Gamma(TM/\xi)^*$ and the composition

$$TM \xrightarrow{\pi} TM/\xi \xrightarrow{s} M \times \mathbb{R},$$

defines a form $\alpha \in \Omega^1(M)$ satisfying $\xi = \ker \alpha$. Conversely, observe that any such form defines a trivialization $TM/\xi \xrightarrow{\sim} M \times \mathbb{R}$ by:

$$X \mapsto \alpha(X), \quad \forall X \in \Gamma(TM/\xi). \tag{1.3.1.3}$$

If ξ is cooriented, we will always assume that α is chosen so that the map above is an oriented isomorphism, where $M \times \mathbb{R}$ has the standard orientation.

For $X, Y \in \Gamma(\xi)$ we have

$$d\alpha(X, Y) = -\alpha([X, Y]) = c_\xi(X, Y),$$

using the above trivialization of TM/ξ. Hence, the condition that c_ξ is non-degenerate translates into

$$\alpha \wedge d\alpha^n \neq 0.$$

Moreover, if ξ is cooriented, then the trivialization from Equation 1.3.1.3 is oriented if and only if

$$\alpha \wedge \mathrm{d}\alpha^n > 0.$$

The sign in the above equation makes sense since $\alpha \wedge \mathrm{d}\alpha^n$ is a volume form and can be compared to any positive volume form on M.

Definition 1.3.6. *A **contact form** for a (coorientable) contact structure ξ on M is a form $\alpha \in \Omega^1(M)$ satisfying*

$$\xi = \ker \alpha.$$

If we talk about a contact form α, without reference to any contact structure, then it is understood that we consider $\xi := \ker \alpha$ together with the coorientation induced by α.

Remark 1.3.7. If the contact structure ξ comes with a fixed coorientation, and α is a contact form for ξ, then we always assume the coorientation of ξ matches the one induced by α as in Equation 1.3.1.3. Note that with these conventions a contact form for a cooriented contact structure (or a contact form without reference to a contact structure) always satisfies

$$\alpha \wedge \mathrm{d}\alpha^n > 0.$$

△

Example 1.3.8. Some of the basic examples of contact manifolds are:

- **Euclidean space:** Let $(x_1, y_1, \ldots, x_n, y_n, z)$ denote the standard coordinates on $\mathbb{R}^{2n+1}$. The form
$$\alpha := \mathrm{d}z + \sum_{i=1}^{n} x_i \mathrm{d}y_i,$$
is called the standard contact form. The contact analogue of Darboux's theorem, as stated for example in [8], says that any contact form locally looks like the standard one. Thus, contact structures have no local invariants.

- **Tori:** Let (x, y, z) denote the standard angular coordinates on $\mathbb{T}^3$. Then, for each $k \in \mathbb{N}$ the form
$$\alpha_k := \mathrm{d}z + \sin(kz)\mathrm{d}x + \cos(kz)\mathrm{d}y,$$
defines a contact structure. The naive generalization of this formula to higher dimensional tori does not define a contact form. Nevertheless, it was shown by Bourgeois, see [16], that all odd dimensional tori admit a contact structure. His result states that given a contact manifold (M, ξ) with $\dim M \geqslant 3$, the product $M \times \Sigma_g$ admits a contact structure, for any surface Σ_g of genus at least one.

- **Products:** For dimensional reasons the product of two contact manifolds cannot be contact again. Instead, let (M, α) be a contact manifold and $(W, \mathrm{d}\lambda)$ an exact symplectic manifold. Then,
$$\left(M \times W, \tilde{\alpha} := \alpha + \lambda\right),$$
is again contact. For example, interpreting $(\mathbb{S}^1, \mathrm{d}z)$ as a contact manifold, it follows that for any exact symplectic manifold $(W, \mathrm{d}\lambda)$, the product $\mathbb{S}^1 \times W$ is contact.

- **Spheres:** Let (M, ω) be a symplectic manifold and $\Sigma \subset M$ a hypersurface. Assume there exists a vector field $X \in \mathfrak{X}(M)$ which is transverse to Σ and satisfies $\mathcal{L}_X \omega = \omega$. Then, the form
$$\alpha := (\iota_X \omega)\,|_\Sigma, \tag{1.3.1.4}$$
defines a contact structure on Σ. In this case we say that Σ is a **hypersurface of contact type**.
In particular this applies to the spheres $\mathbb{S}^{2n+1} \subset \mathbb{R}^{2n}$. Indeed, let ω be the

standard symplectic form on $\mathbb{R}^{2n}$as in Equation 1.2.1.1, and observe that the Euler vector field

$$X := \sum_i x_i \partial_{x_i} + y_i \partial_{y_i},$$

is transverse to $\mathbb{S}^{2n-1}$ and satisfies $\mathcal{L}_X \omega = \omega$.
In fact, any contact manifold can obtained as a hypersurface of contact type; given a contact form α on M, consider the symplectic manifold

$$\left(\mathbb{R} \times M, \omega := \mathrm{d}(e^t \alpha)\right),$$

where t denotes the coordinate on $\mathbb{R}$, called the **symplectization** of (M, α). Then, ∂_t is transverse to $\{0\} \times M$, satisfies $\mathcal{L}_{\partial_t} \omega = \omega$ and the induced contact form is α.

- **Contact elements:** A contact element on a manifold M is a hyperplane $\xi_p \in T_p M$ for some $p \in M$. Any contact element can be written as the kernel of a non-zero covector $\alpha_p \in T_p^* M$, which is unique up to scaling by a non-zero constant. Thus the space of all contact elements can be identified with $\mathbb{P}T^*M$, the projectivized cotangent bundle. It comes equipped with a canonical contact structure defined by the rule

$$\xi_{[\alpha]} := \ker(\alpha \circ \mathrm{d}\pi) \subset T_{[\alpha]} \mathbb{P}T^*M,$$

where $\pi : T^*M \to M$ denotes the projection.
The above formula resembles that of the tautological form λ from Equation 1.2.1.2. Viewing the unit sphere bundle $\mathbb{S}T^*M$ as a hypersurface in the symplectic manifold $(T^*M, \mathrm{d}\lambda)$, the argument from the previous example shows that $\lambda|_{\mathbb{S}T^*M}$ defines a contact structure. Moreover, this contact structure descends to the quotient $\mathbb{P}T^*M$ and equals the one from the previous equation.

△

The choice of contact form α for a given contact structure ξ is not unique. Indeed, let $f \in C^\infty(M)$ be nowhere vanishing, then $\ker f\alpha = \ker \alpha$ and

$$(f\alpha) \wedge \mathrm{d}(f\alpha)^n = f^{n+1} \alpha \wedge \mathrm{d}\alpha^n \neq 0.$$

Thus, a contact form is unique up to multiplication by a nowhere vanishing function, or a strictly positive function if we want to preserve the coorientation. Given a differential form $\alpha \in \Omega^1(M)$ we denote by $[\alpha]$ the equivalence class of the equivalence relation

$$\alpha \sim \alpha' \iff \alpha' = f\alpha,$$

for a nowhere vanishing function $f : M \to \mathbb{R} \backslash \{0\}$. Similarly, we denote by $[\alpha]_+$ the equivalence class where we only allow multiplication by positive functions. Then the above discussion implies:

Corollary 1.3.9. *Given a manifold M there are a one-to-one correspondences between:*

(i) Coorientable contact structures ξ and equivalence classes $[\alpha]$ where α is a contact form for ξ;

(ii) Cooriented contact structures ξ and equivalence classes $[\alpha]_+$ where α is a cooriented contact form for ξ. Under this correspondence changing the coorientation of ξ is the same thing as changing $[\alpha]_+$ to $[-\alpha]_+$.

Although equivalent contact forms induce the same contact structure, they can have very different properties. For example, any contact form has a distinguished vector field associated to it, spanning the kernel of $\mathrm{d}\alpha$, and which is not preserved under equivalence.

Definition 1.3.10. *The **Reeb vector field** of a contact form $\alpha \in \Omega^1(M)$ is the unique vector field satisfying*

$$\alpha(R) = 1, \quad \iota_R \mathrm{d}\alpha = 0.$$

As claimed above the Reeb vector field is not preserved under equivalence, and the change can be computed as follows. If $\alpha' = f\alpha$ for a function $f : M \to \mathbb{R}\backslash\{0\}$ then

$$R_{\alpha'} = \frac{1}{f} R_\alpha + V, \tag{1.3.1.5}$$

where $V \in \mathfrak{X}(M)$ is the unique vector field satisfying

$$\alpha(V) = 0, \quad \iota_V d\alpha = \frac{df - (\mathcal{L}_{R_\alpha} f)\alpha}{f^2}.$$

1.3.2 Contact structures with transverse boundaries

Let ξ be a contact structure on a manifold M with boundary, and consider the intersection with the tangent space of the boundary

$$\zeta := \xi \cap T(\partial M).$$

In general ζ is a singular distribution in the sense that it does not have constant rank.

Definition 1.3.11. *We say that a contact manifold (M, ξ) has **transverse boundary** if $\xi \pitchfork \partial M$.*

In the transverse case, ζ is an honest codimension-1 distribution on ∂M. The associated curvature $c_\zeta : \Lambda^2\zeta \to T(\partial M)/\zeta$, is defined as in Equation 1.3.1.1. Since $\xi \pitchfork \partial M$ there is short exact sequence

$$0 \to \zeta \to T\partial M \to (TM/\xi)|_{\partial M} \to 0,$$

giving a canonical isomorphism

$$T(\partial M)/\zeta \simeq (TM/\xi)|_{\partial M}, \tag{1.3.2.1}$$

and, under this identification, c_ζ is just the restriction of c_ξ to ζ. This implies that c_ζ has one-dimensional kernel so that the induced structure on the transverse boundary of contact manifold is the following:

Definition 1.3.12. *A* ***∂-contact structure*** *on* N^{2n} *is a codimension one distribution* $\zeta \subset TN$ *for which the curvature* c_ζ *is maximally non-degenerate.*

Because ζ is odd dimensional, this is equivalent to c_ζ having 1-dimensional kernel. As for contact structures, we make no a priori assumptions on the orientability of TN/ζ.

Definition 1.3.13. *An ∂-contact structure* $\zeta \subset TN$ *is said to be* ***coorientable (resp.cooriented)*** *if* TN/ζ *is orientable (resp. oriented).*

Note that, by Equation 1.3.2.1, the ∂-contact structure on the boundary of a contact manifold inherits a coorientation from ξ.

For an abstract ∂-contact structure (N, ζ), a coorientation can be defined using a differential form, analogous to the discussion of the previous section. That is, if TN/ζ is orientable then so is its dual, and any nowhere vanishing section of $(TN/\zeta)^*$ defines a form $\beta \in \Omega^1(N)$ satisfying $\zeta = \ker \beta$. As in Equation 1.3.1.3, such a form induces a coorientation on ζ by requiring the isomorphism

$$TN/\zeta \xrightarrow{\sim} N \times \mathbb{R}, \tag{1.3.2.2}$$

to be orientation preserving, where $N \times \mathbb{R}$ has the standard orientation.

Definition 1.3.14. *A* ***∂-contact form*** *for a (coorientable) ∂-contact structure* ζ *on* N *is a (nowhere vanishing) form* $\beta \in \Omega^1(N)$ *satisfying*

$$\zeta = \ker \beta, \quad \dim \ker \mathrm{d}\beta|_\zeta = 1.$$

If we talk about a ∂-contact form β*, without reference to any ∂-contact structure, then it is understood that we consider* $\zeta := \ker \beta$ *together with the coorientation induced by* β*.*

Remark 1.3.15. As in Remark 1.3.7, if the ∂-contact structure ζ is cooriented, and β is a ∂-contact form for ζ, then we assume the coorientation of ζ matches the one induced by β, as in Equation 1.3.2.2. △

The above condition allows both for $\mathrm{d}\beta^n = 0$ and $\mathrm{d}\beta^n \neq 0$ to happen. Recall that the Reeb vector field of a contact form α, as defined in Defintion 1.3.10 spans the 1-dimensional kernel of $\mathrm{d}\alpha$. Hence, if $(M, \xi := \ker \alpha)$ is a contact manifold with boundary, then the induced ∂-contact form $\beta := \alpha|_{\partial M}$ satisfies $\mathrm{d}\beta^n = 0$ if and only if the Reeb vector field is tangent to the boundary. If the contact structure ξ is transverse to the boundary such contact forms always exist, as shown in the following lemma. This is very convenient since many computations simplify if the top power of $\mathrm{d}\beta$ vanishes.

Lemma 1.3.16. *Let* (M, ξ) *be a (cooriented) contact manifold with* $\xi \pitchfork \partial M$*. Then there exists a contact form* α *such that* $\xi = \ker \alpha$ *and* R_α *is tangent to* ∂M*.*

Proof. Since $\xi \pitchfork \partial M$, there exists $X \in \mathfrak{X}(M)$, in the kernel of α and pointing outwards along the boundary.

Define $\alpha_\partial := \alpha|_{\partial M}$ and $\theta := \iota_X \mathrm{d}\alpha|_{\partial M}$, then

$$\theta \wedge \alpha_\partial \wedge d\alpha_\partial^{n-1} = \frac{1}{n}\iota_X(\alpha \wedge d\alpha^n)|_{\partial M} > 0.$$

Hence, we can apply Theorem 1.3.19 (to the ∂-contact manifold $(\partial M, \alpha_\partial)$ with admissible form θ) to find a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ of the boundary on which

$$\alpha = f(\alpha_\partial + t\theta),$$

for $f \in C^\infty(U)$ a smooth strictly positive function.

Then, choosing a positive function $f \in C^\infty(M)$ satisfying

$$g := \begin{cases} f & \text{on } \partial M \times [0, \frac{\epsilon}{3}) \\ 1 & \text{on } M \backslash (\partial M \times [0, \frac{2\epsilon}{3}) \end{cases},$$

we have

$$\tilde{\alpha} := \frac{1}{g}\alpha = \beta + t\theta,$$

near the boundary. Moreover, at points in the boundary, $\mathrm{d}\tilde{\alpha} = \mathrm{d}\beta + \mathrm{d}t \wedge \theta$ implying $\ker \mathrm{d}\tilde{\alpha} = \ker \mathrm{d}\beta$ which is tangent to ∂M. □

As before, multiplying a ∂-contact form by a nowhere vanishing function does not change the induced ∂-contact structure. In the notation of Corollary 1.3.9 we have:

Lemma 1.3.17. *Given a manifold N there are one-to-one correspondences between:*

(i) ∂-contact structures ζ and equivalence classes $[\beta]$ where β is a ∂-contact form for ζ;

(ii) Cooriented ∂-contact structures ζ and equivalence classes $[\beta]_+$ where β is a cooriented ∂-contact form for ζ.

In contrast with contact structures, a cooriented even contact structure does not induce an orientation on N, so that the analogue of Corollary 1.3.4 does not hold. The reason for this is that c_ζ has a 1-dimensional kernel, which does not have a canonical orientation. This is reflected in the fact that an ∂-contact form does not induce a volume form; instead we need to choose an extra piece of data:

Definition 1.3.18. *An admissible form for an ∂-contact form β on N, is a form $\theta \in \Omega^1(N)$ satisfying*

$$\theta \wedge \beta \wedge d\beta^{n-1} > 0.$$

Admissible forms will be studied more closely in Section 1.4.1. For now, it suffices to think of them as an auxillary piece of data needed to define the local model associated to the ∂-contact manifold.

1.3.2.1 Statement of the normal form

Let (M, ξ) be a contact structure with transverse boundary, $\alpha \in \Omega^1(M)$ a contact form representing ξ and denote by $\alpha_\partial := \alpha|_{\partial M}$ the induced ∂-contact form. For any choice of admissible form $\theta \in \Omega^1(\partial M)$ consider the **local model**

$$((-\varepsilon, 0] \times \partial M, \alpha := \alpha_\partial + t\theta)\,, \tag{1.3.2.3}$$

which defines a contact structure for $\varepsilon > 0$ small enough.

Theorem 1.3.19. *Any contact structure with transverse boundary (Definition 1.3.11) is isomorphic (as a contact structure) to its local model on a neighborhood of the boundary.*

In particular, up to isomorphism of contact structures, the local model is independent of the choice of the contact form α and the admissible form.

Remark 1.3.20. The previous local model around the transverse boundary of a contact manifold $(M, \xi := \ker \alpha)$, with induced ∂-contact structure $\xi_\partial := \xi \cap T(\partial M)$, can be defined more invariantly as follows. The restriction of the curvature c_ξ from Defintition 1.3.1 to ξ_∂ has a 1-dimensional kernel

$$L := \ker c_\xi|_{\xi_\partial} = \ker \mathrm{d}\alpha_\partial|_{\xi_\partial} \subset T\partial M.$$

Viewing L as a subbundle of $T\partial M$, it comes with a projection $\pi : L \to \partial M$, making it into a rank 1 vector bundle. Thus, L defines a 1-dimensional foliation $\mathcal{L}$ on ∂M, and the dual bundle L^*, can be viewed as the leafwise cotangent bundle. The total space of $\pi : L^* \to \partial M$ carries a canonical contact structure defined by

$$\alpha := \pi^*\alpha_\partial + \lambda_{can},$$

where $\lambda_{can} \in \Omega^1(T^*L)$ denotes the tautological form.

Furthermore, L has a canonical orientation, for which $V \in L_p$, $p \in \partial M$, is positive if and only if

$$(\mathrm{d}\alpha_\partial)_p(X, V) > 0,$$

where $X \in T_pM$ is any outward pointing vector. Hence, L and L^* are trivializable. A choice of vector field $X \in \mathfrak{X}(M)$ transverse to the boundary corresponds to a trivialization of T^*L, that is, a nowhere vanishing section $\beta \in \Gamma(T^*L)$ defined by

$$\beta(x) := (\iota_X \mathrm{d}\alpha)|_{T_xL}, \quad x \in \partial M.$$

In this trivialization $T^*L \simeq \mathbb{R} \times \partial M$, the contact structure ξ is represented by the local model from Equation 1.3.2.3. △

The proof of the theorem follows immediately from the following, more technical, proposition.

Proposition 1.3.21. *Let (M, ξ) be a contact manifold with transverse boundary (Definition 1.3.11), and $\alpha \in \Omega^1(M)$ a contact form representing it. Let $\alpha_\partial := \alpha|_{\partial M}$ be the induced ∂-contact form (Definition 1.3.14) and $\theta \in \Omega^1(\partial M)$ and admissible form (Definition 1.3.18). Then there exists a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which*

$$\alpha = f(\alpha_\partial + t\theta),$$

for $f \in C^\infty(\mathcal{U})$ strictly positive and satisfying $f|_{\partial M} = 1$.

1.3.2.2 Proof of the normal form

The key ingredient in the proof is the following analogue of Giroux's theorem for 3-dimensional contact manifolds from [56]. The proof in higher dimensions given below is essentially the same as that for the 3-dimensional case from [53]. We have included it here for the sake of completeness.

Theorem 1.3.22. *For $i = 0, 1$, let S_i be a closed hypersurface in a contact manifold $(M_i, \xi_i := \ker \alpha_i)$ and $\phi : S_0 \to S_1$ a diffeomorphism satisfying*

$$\phi^*(\alpha_1|_{S_1}) = \alpha_0|_{S_0}.$$

Then there exists a contactomorphism $\psi : U_0 \to U_1$ of suitable open neighborhoods of the hypersurfaces, such that $\psi|_{S_0} = \phi$.

Proof. Following our usual convention we assume that M_i and the hypersurfaces are oriented. This implies that a neighborhood of S_i can be identified with $(-\varepsilon, \varepsilon) \times S_i$ where S_i corresponds to $\{0\} \times S_i$. Extend ϕ to a diffeomorphism (still denoted by ϕ) between these open neighborhoods of S_i, and consider the contact forms α_0 and $\phi^*\alpha_1$. In the above coordinates any contact form can be written as

$$\alpha = \beta_t + u_t \mathrm{d}t,$$

where $\beta_t \in \Omega^1(S_0)$, $u_t \in C^\infty(S_0)$ and $t \in (-\varepsilon, \varepsilon)$. The contact condition then becomes:

$$\alpha \wedge (\mathrm{d}\alpha)^n = \Big(-n\beta_t \wedge \dot{\beta}_t + n\beta_t \wedge du_t + u_t \mathrm{d}\beta_r\Big) \wedge (\mathrm{d}\beta_r)^{n-1} \wedge dr > 0. \quad (1.3.2.4)$$

Note that this equation is linear in $\dot{\beta}_t$ and u_t. Hence, convex linear combinations of solutions of Equation 1.3.2.4 with the same β_0 (and $d\beta_0$) will again be solutions for small $|t|$. Hence taking ε small enough,

$$\alpha_s := (1 - s)\alpha_0 + s\phi^*\alpha_1, \quad s \in [0, 1],$$

is a solution for all s. We now use Moser's trick to find an isotopy ψ_s such that $\psi_s^*\alpha_s = \lambda_s\alpha_0$. Differentiating the above equation and setting $\mu_s := (\frac{\mathrm{d}}{\mathrm{d}s} \log \lambda_s) \circ \psi_s^{-1}$ we see that we have to find a vector field X_s satisfying

$$\dot{\alpha}_s + \mathcal{L}_{X_s}\alpha_s = \mu_s\alpha_s. \quad (1.3.2.5)$$

Furthermore, we want $X_s|_{S_0} = 0$ which ensures both that X_s can be integrated up to time one around S_0 and that $\psi|_{S_0} = \phi$. Write

$$X_s = H_s R_s + Y_s,$$

with R_s the Reeb vector field of α_s, $Y_s \in \ker \alpha_s$ and H_s a family of smooth functions. Then, Equation 1.3.2.5 becomes

$$\dot{\alpha}_s + \mathrm{d}H_s + \iota_{Y_s} d\alpha_s = \mu_t\alpha_s.$$

For a fixed H_s this equation is solved by first applying it to R_s, giving μ_s, and then noting that we find a unique $Y_s \in \ker \alpha_s$ by non-degeneracy of $\mathrm{d}\alpha_s|_{\ker \alpha_s}$.

We want to choose H_s in such a way that so that $X_s|_{S_0} = 0$. This condition translates into $H_s|_{S_0} = 0$ and $Y_s|_{S_0} = 0$. The latter can be satisfied by requiring

$$\dot{\alpha}_s + \mathrm{d}H_s = 0, \quad \text{on } S_0$$

which is automatically satisfied if $H_s|_{S_0} = 0$ since $\dot{\alpha}_s|_{TS_0} = 0$. Therefore it is possible to find a suitable H_s and we get a solution X_s which by compactness of S_0 can be integrated up to time one on a neighborhood of S_0. The desired map is given by $\psi := \phi \circ \psi_1$. □

The proof of the normal form now follows almost immediately.

Proof of Proposition 1.3.21. Observe that $\tilde{\alpha} := \alpha_\partial + t\beta$ is contact since

$$\tilde{\alpha} \wedge \mathrm{d}\tilde{\alpha}^n = n\mathrm{d}t \wedge \beta \wedge \alpha_\partial \wedge (\mathrm{d}\alpha_\partial + t\mathrm{d}\beta)^{n-1} > 0,$$

and $\tilde{\alpha}|_{\partial M} = \alpha_\partial$. Thus we can apply Theorem 1.3.22 with $\psi = \mathrm{id}$, to obtain the required collar neighborhood. □

1.3.3 Contact structures with singular boundaries

Let ξ be a contact structure on M, and denote the intersection with the boundary by

$$\zeta := \xi \cap T(\partial M).$$

In the previous section we assumed the boundary was regular so that ζ defines a codimension-1 distribution on ∂M. However, in general ξ can have points where it is tangent to ∂M. If this happens we say that ξ has **singular boundary**, to distinguish it from the previous situation.

For a singular boundary, ζ does not define a distribution in the classical sense. However, if α is a contact form representing ξ, then the restriction $\alpha_\partial := \alpha|_{\partial M}$ makes sense both in the regular and singular case. Thus, for singular boundaries we work only with differential forms, and make the following definition:

Definition 1.3.23. *A* ***(singular) ∂-contact form*** *on N^{2n} is a one form $\beta \in \Omega^1(N)$ such that $d\beta|_{\ker\beta}$ is maximally nondegenerate.*

Note that, in case β is nowhere vanishing this recovers Definition 1.3.14. On the other hand, in the above definition β is allowed to vanish, so if $p \in N$ then:

(i) if $\beta_p = 0$ then $(\mathrm{d}\beta)_p$ is nondegerate on T_pN or equivalently $(\mathrm{d}\beta)_p^n \neq 0$. In particular

$$\dim\ker \mathrm{d}\beta|_{\ker\beta} = 0;$$

(ii) if $\beta_p \neq 0$ then $\beta_p \wedge \mathrm{d}\beta_p^{n-1} \neq 0$ or equivalently

$$\dim\ker \mathrm{d}\beta|_{\ker\beta} = 1.$$

This means that we have inclusions:

$$\{p \in N \mid \beta_p \wedge \mathrm{d}\beta_p^{n-1} = 0\} \subset \{p \in N \mid \beta_p = 0\} \subset \{p \in N \mid \mathrm{d}\beta_p^n \neq 0\}$$

In order to write down a local model we need to make some choices, analogous to the choice of admissible form for a regular ∂-contact form.

Definition 1.3.24. *Given a singular even contact form β on a manifold N^{2n} an* ***admissible pair*** (θ, u) *for β consists of :*

(i) A form $\theta \in \Omega^1(N)$ satisfying

$$\theta \wedge \beta \wedge \mathrm{d}\beta^{n-1} \geqslant 0, \quad (\theta \wedge \beta \wedge \mathrm{d}\beta^{n-1})_p > 0 \iff \beta_p \neq 0.$$

(ii) A function $u \in C^\infty(N)$ satisfying

$$u\mathrm{d}\beta^n \geqslant 0, \quad \beta_p = 0 \implies (u\mathrm{d}\beta^n)_p > 0.$$

Just as for non-singular even contact forms admissible pairs always exist:

Lemma 1.3.25. *For any singular even contact manifold (N, β) there exists an admissible pair (θ, u).*

Proof. Fix a volume form Ω on N, compatible with the orientation on N, giving an isomorphism $\mathfrak{X}(N) \xrightarrow{\sim} \Omega^{2n-1}(N)$ by $X \mapsto \iota_X\Omega$. Hence, we can find $V \in \mathfrak{X}(N)$ satisfying

$$\iota_V\Omega = \beta \wedge d\beta^{n-1}.$$

Pick a metric $\langle\cdot,\cdot\rangle$ on N and define $\theta \in \Omega^1(N)$ by

$$\theta := \langle V, \cdot\rangle.$$

Then $\theta(V) \geqslant 0$ and $\theta(V) > 0$ at points where $\beta_p \wedge d\beta_p^{n-1} \neq 0$. In particular at points where $\beta_p \neq 0$. This implies that

$$\theta \wedge \beta \wedge d\beta^{n-1} = \theta \wedge \iota_V\Omega = -\iota_V(\theta \wedge \Omega) + \theta(V)\Omega = \theta(V)\Omega \geqslant 0,$$

and $\theta \wedge \beta \wedge d\beta^{n-1} > 0$ at points where $\beta_p \neq 0$.

For the second part define $u \in C^\infty(N)$ by

$$u\Omega = (d\beta)^n.$$

Then

$$u d\beta^n = u^2\Omega \geqslant 0, \tag{1.3.3.1}$$

and $ud\beta^n > 0$ at points where $u \neq 0$ or equivalently at points where $d\beta^n \neq 0$. □

1.3.3.1 Statement of the normal form

Let β be a singular even contact form on N. Then, for any admissible pair (u, β) consider the **local model**,

$$((-\varepsilon, 0] \times N, \alpha := \beta + t\theta + \mathrm{d}(tu)),$$

which is a contact for ε small enough. Indeed:

$$\begin{aligned} \alpha \wedge d\alpha^n|_{t=0} &= (\beta + udt) \wedge (d\beta + \mathrm{d}t \wedge \theta)^n \\ &= (\beta + udt) \wedge (d\beta^n + nd\beta^{n-1} \wedge \mathrm{d}t \wedge \theta) \\ &= n\mathrm{d}t \wedge \theta \wedge \beta \wedge d\beta^{n-1} + u\mathrm{d}t \wedge d\beta^n \\ &= \mathrm{d}t \wedge \left(n\theta \wedge \beta \wedge d\beta^{n-1} + u^2\Omega\right) \geqslant 0, \end{aligned} \tag{1.3.3.2}$$

where we used Equation 1.3.3.1. Hence, Equation 1.3.3.2 is zero if both $\theta \wedge \beta \wedge d\beta^{n-1} = 0$ and $u = 0$. However, these conditions are equivalent to $\mathrm{d}\beta_p$ being non-degenerate, and β_p being zero respectively, which cannot happen at the same time. Thus, $\alpha \wedge \mathrm{d}\alpha^n > 0$ adt $t = 0$ and hence also for $t \in (-\varepsilon, 0]$ if $\varepsilon > 0$ is small enough.

Theorem 1.3.26. *Any contact structure with (singular) boundary is isomorphic to its local model on a neighborhood of the boundary.*

Of course, this theorem also covers regular boundaries. In this case we can choose any function u for the admissible pair (θ, u), in particular $u = 0$ which recovers the regular local model. The proof is a direct consequence of the following.

Lemma 1.3.27. *Let α be a contact form on a manifold with boundary M, and $\alpha_\partial := \alpha|_{\partial M}$ the induced (singular) ∂-contact form. Then, for any choice of admissible pair (u, θ) there exists a collar neighborhood of the boundary $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$, on which*

$$\alpha = f\left(\alpha_\partial + t\theta + \mathrm{d}(tu)\right),$$

for a positive function $f \in C^\infty(\mathcal{U})$ satisfying $f|_{\partial M} = 1$.

Proof. We checked in Equation 1.3.3.2 that $\tilde{\alpha} := \alpha_\partial + t\theta + \mathrm{d}(tu)$ defines a contact structure and by definition $\tilde{\alpha}|_{\partial M} = \alpha_\partial$. Hence, the proof follows by applying Theorem 1.3.22 with $\psi = \mathrm{id}$. ☐

1.3.4 Gluing contact structures

Using the normal form for boundaries of contact manifolds we can glue contact manifolds along their (possibly singular) boundaries. Recall from Section 1.2.4.1 that, given manifolds M_i, $i = 1, 2$, and an orientation reversing diffeomorphism $\phi : \partial M_1 \to \partial M_2$, we obtain a manifold

$$M_1 \cup_\phi M_2 := (M_1 \sqcup M_2)/x \sim \phi(x), \quad \forall x \in \partial M_1.$$

The resulting smooth structure on the gluing depends on ϕ and the choice of collar neighborhoods $k_i : (-\varepsilon, 0] \times \partial M_i \to M_i$. Note that in the statement below we have surpressed these choices from the notation.

Proposition 1.3.28. *Let (M_i, ξ_i), $i = 1, 2$, be a contact manifold with non-empty boundary, and induced ∂-contact structure $\xi_{\partial,i} := \xi_i \cap T\partial M_i$, as in Definition 1.3.12. Assume there exists an orientation reversing diffeomorphism $\phi : \partial M_1 \to \partial M_2$, such that*

$$\phi_* \xi_{\partial,1} = \xi_{\partial,2}.$$

Then there exists a contact structure ξ on

$$M_1 \cup_\phi M_2 := (M_1 \sqcup M_2)/x \sim \phi(x), \quad x \in \partial M_1,$$

which restricts to ξ_i on M_i.

Proof. Choose contact forms α_i for ξ_i, and denote $\beta_i := \alpha_i|_{\partial M_i}$. Then, because ϕ preserves the ∂-contact structures, we have

$$\phi^* \beta_2 = f \beta_1,$$

for some positive function $f \in C^\infty(\partial M_1)$. By Lemma 1.3.27 and rescaling α_2, we can find an admissible pair (θ_2, u_2) and a collar neighborhood isomorphic to

$$\big((-\varepsilon, 0] \times \partial M_2,\ \beta_2 + t\theta_2 + \mathrm{d}(tu_2)\big).$$

Under the map $t \mapsto -t$ this is isomorphic to

$$\big([0, \varepsilon) \times \overline{\partial M_2},\ \beta_2 + t(-\theta_2) + \mathrm{d}(t(-u_2))\big). \tag{1.3.4.1}$$

More precisely, a computation similar to Equation 1.3.3.2 shows that for $\tilde{\alpha} := \beta_2 - t\theta_2 - \mathrm{d}(tu_2)$ we have

$$\tilde{\alpha} \wedge \mathrm{d}\tilde{\alpha}^n = -\mathrm{d}t \wedge (n\theta_2 \wedge \beta_2 \wedge \mathrm{d}\beta_2^{n-1} + u_2^2 \Omega),$$

where Ω is a volume form on ∂M_2 and thus a negative volume form on $\overline{\partial M_2}$. Together with the fact that $\theta_2 \wedge \beta_2 \wedge d\beta_2^{n-1} \leqslant 0$ on $\overline{\partial M_2}$ and all intervals in $\mathbb{R}$ are oriented by ∂_t, we see that $\tilde{\alpha} \wedge \mathrm{d}\tilde{\alpha}^n > 0$ on $[0, \varepsilon) \times \overline{\partial M_2}$.

Rescaling α_1, and thus β_1, we can assume that $\phi^*(\beta_2) = \beta_1$. If we denote

$$(\theta_1 := \phi^*(-\theta_2), u_1 := \phi^*(-u_2)),$$

then using ϕ we can identify the neighborhood in Equation 1.3.4.1 with

$$\big([0, \varepsilon) \times \partial M_1, \alpha_r := \beta_1 + t\theta_1 + \mathrm{d}(tu_1)\big).$$

It follows directly from Definition 1.3.24 that if (θ, u) is an admissible pair for a singular ∂-contact form β on N then $(-\theta, -u)$ is an admissible pair for β on $\overline{N}$. Hence, since $(-\theta_2, -u_2)$ is admissible for β_2 on $\overline{\partial M_2}$, it follows that (θ_1, u_1) is admissible for

β_1 on ∂M_1. By Lemma 1.3.27, and possibly rescaling α_1, we find an isomorphism between an open neighborhood of the boundary of ∂M_1 and

$$\big((-\varepsilon, 0] \times \partial M_1, \alpha_l := \beta_1 + t\theta_1 + \mathrm{d}(tu_1)\big).$$

We glue the collar neighborhoods and define a cooriented contact structure on it by

$$\Big((-\varepsilon, \varepsilon) \times \partial M_1, \alpha := \beta_1 + t\theta_1 + \mathrm{d}(tu_1)\Big).$$

□

1.4 Contact forms and their boundaries

In this section we consider boundaries of manifolds endowed with a contact form, and gluing such manifolds. Unlike for symplectic and contact structures, for contact forms a neighborhood of the boundary is not determined only by the data induced on the boundary. Thus, there is no general normal form, and instead we distinguish several special kind of boundaries. For every type the structure on the boundary can be encoded in a pair of differential forms, and can therefore be treated in a uniform manner.

The lack of a normal form makes gluing contact forms much harder. We need to impose that their boundaries are of the special types mentioned before, and the types need to match. However, the analogy with the symplectic case can be partially saved. We define a notion of contact cobordism and show that by gluing topologically trivial cobordisms we can pass from one type of boundary to another.

Lastly, these observations are used to construct contact forms on abstract open book decompositions. The reader unfamiliar with open book decompositions is refered to Appendix 1.9 for the definition and their basic properties.

1.4.1 Contact forms with regular boundaries

Consider a contact form α on a manifold with boundary and assume that $\ker \alpha \pitchfork \partial M$. Using a vector field $X \in \mathfrak{X}(M)$ in the kernel of α and transverse to ∂M we define a collar neighborhood $(-\varepsilon, 0] \times \partial M$ (on which X is identified with ∂_t). Using these coordinates we can write down the Taylor expension of α in the interval coordinate $t \in (-\varepsilon, 0]$ at $t = 0$. This gives:

$$\alpha = v + tu + \mathcal{O}(t^2), \tag{1.4.1.1}$$

for some $v, u \in \Omega^1(\partial M)$ and $f \in C^\infty(\partial M)$. Note that there are no terms containing $\mathrm{d}t$ since $X \in \ker \alpha$. In terms of this expansion the contact condition for α becomes:

$$0 < \alpha \wedge \mathrm{d}\alpha^n = n\mathrm{d}t \wedge u \wedge v \wedge \mathrm{d}v^{n-1} + \mathcal{O}(t).$$

From this perspective, the simplest possible contact forms are those with a linear Taylor expansion. Indeed, all the terms except the constant term u are zero, in the expansion of Equation 1.4.1.1, then α cannot satisfy the contact condition. Furthermore, close to the boundary the first summand in the above equation dominates the terms of order $\mathcal{O}(t)$, so that the contact condition can be satisfied.

Definition 1.4.1. *A contact form α on M is* ***regular*** *at the boundary, if there exists a collar neighborhood $U \simeq (-\varepsilon, 0] \times \partial M$ on which we have:*

$$\alpha = tu + v, \tag{1.4.1.2}$$

for some $u, v \in \Omega^1(\partial M)$, and where s denotes the coordinate on $(-\varepsilon, 0]$.

As observed above, the contact condition for a regular contact form implies:

$$\alpha \wedge \mathrm{d}\alpha^n = n\mathrm{d}t \wedge u \wedge v \wedge (t\mathrm{d}u + \mathrm{d}v)^{n-1} > 0.$$

Since this is an open condition it suffices to require it at points in the boundary, where $t = 0$. Then, by shrinking the collar neighborhood, it holds everywhere. Therefore, the conditions on u and v can be packed into the following definition, which does not make reference to a boundary:

Definition 1.4.2. *A ∂-**contact pair** (u, v) on a manifold N^{2n} is a pair of forms $u, v \in \Omega^1(N)$ satisfying*

$$u \wedge v \wedge \mathrm{d}v^{n-1} > 0.$$

Remark 1.4.3. A ∂-contact pair is similar to the data induced on the boundary of a (regular) symplectic foliated manifolds, see Definition 1.6.2. In both cases, the data can be encoded in a triple (u, v, η), with $u, v \in \Omega^1(N)$ and $\eta \in \Omega^2(N)$, satisfying

$$u \wedge v \wedge \eta^{n-1} > 0.$$

Depending on the situation the forms can be closed, exact or have various other relations between them. However, the essential structure is that of a codimension-2 almost symplectic distribution (ξ, ω) defined by

$$\xi := \ker u \cap \ker v, \quad \omega := \eta|_\xi.$$

△

Remark 1.4.4. For later reference we compute explicitely the Reeb vector field, as in Definition 1.3.10, of a regular contact form. Let (u, v) be a ∂-contact pair on N^{2n}, and consider $M := (-\varepsilon, 0] \times N$ with the contact form

$$\alpha = tu + v.$$

Let $R_u, R_v \in \mathfrak{X}(N)$ be defined by:

$$\iota_{R_u} u = 1, \quad \iota_{R_u} v = 0, \quad \iota_{R_u} \mathrm{d}v = 0, \quad \text{and} \quad \iota_{R_v} u = 0, \quad \iota_{R_v} v = 1, \quad \iota_{R_u} \mathrm{d}v = 0.$$

The Reeb vector field R of α can be computed explicitely in the following cases.

(i) If $\mathrm{d}v^n = 0$ then

$$R = R_v + X_t + f_t\partial_t,$$

where $X_t \in \ker u \cap \ker v$ is uniquely defined by

$$\iota_{X_t}(\mathrm{d}v + t\mathrm{d}u) = t\iota_{R_v}\mathrm{d}u, \qquad \text{on } \ker u \cap \ker v,$$

and

$$f_t := t\mathrm{d}u(R_u, R_v + X_t).$$

(ii) If $\mathrm{d}v^n > 0$ then

$$R = \frac{1}{f}\left(X_t - \partial_t\right),$$

where $f \in C^\infty(N)$ and $X_t \in \mathfrak{X}(N)$ are uniquely defined by

$$\iota_{X_t}\left(\mathrm{d}v + t\mathrm{d}u\right) = u, \quad f := v(X_t).$$

Observe that the Reeb vector field is tangent to the boundary if and only if $(\mathrm{d}v)^n = (\mathrm{d}\alpha|_{\partial M})^n = 0$. △

Note that v is a ∂-contact form as in Definition 1.3.14, and that u is an admissible form as in Definition 1.3.18. Given a fixed ∂-contact form, there are many admissible forms completing it to a ∂-contact pair. The following is analogous to Lemma 1.2.5 for symplectic structures.

Lemma 1.4.5. *If v is a ∂-contact form on N^{2n}, then:*

(i) There exists an admissible form u;

(ii) Given a fixed admissible form u, there is a 1-1 correspondence between admissible forms and triples (f, g, X), where $f, g \in C^\infty(N)$ with $g > 0$, and $X \in \mathfrak{X}(N)$ with $X \in \ker u \cap \ker v$, given by the formula:

$$\theta = fv + gu + \iota_X\mathrm{d}v.$$

Proof. The proof is analogous to that of Lemma 1.2.5 and Lemma 1.6.3. □

For regular contact forms Equation 1.4.1.2 implies that the admissible form is "the variation of α transverse to the boundary", that is,

$$u = \mathcal{L}_{\partial_t}\alpha|_{\partial M}.$$

In most of the cases we consider, this property also holds for non-regular boundaries.

Lemma 1.4.6. *Let α be a contact form on M^{2n+1} such that $\ker\alpha \pitchfork \partial M$, and $v := \alpha_\partial$ the induced ∂-contact form. Then, for any vector field $X \in \mathfrak{X}(M)$ satisfying $X \in \ker\alpha$ and transverse to the boundary,*

$$u := \iota_X\mathrm{d}\alpha|_{\partial M},$$

is an admissible form for v.

Conversely, assuming that the Reeb vector field R of α is everywhere tangent to ∂M, for any admissible form u there exists a vector field $X \in \mathfrak{X}(M)$ such that:

$$u = \iota_X\mathrm{d}\alpha|_{\partial M}, \quad X \pitchfork \partial M.$$

Proof. The proof of Lemma 1.6.6 is purely linear algebra for a triple (u, v, w) satisfying $u \wedge v \wedge w^{n-1} > 0$. Thus, taking $w = \mathrm{d}v$, it carries over to the contact case. □

To phrase the regularity condition (Definition 1.4.1) in a coordinate invariant way, recall that a choice of collar neighborhood $U \simeq (-\varepsilon, 0] \times \partial M$ is equivalent to a choice of vector field $X \in \mathfrak{X}(M)$ transverse to the boundary. In the collar neighborhood coordinates X is identified with ∂_t. It follows directly from Equation 1.4.1.2 that if (M, α) has regular boundary, then there exists a vector field $X \in \mathfrak{X}(M)$, transverse to the boundary, and satisfying

$$\iota_X \alpha = 0, \quad \mathcal{L}_X \mathcal{L}_X \alpha = 0. \tag{1.4.1.3}$$

By the following lemma the converse is also true, and thus this equation characterizes regular contact boundaries. Since the proof does not use the contact conditions we state the lemma for general 1-forms.

Lemma 1.4.7. *Let M be a manifold with boundary, and $\alpha \in \Omega^1(M)$ nowhere vanishing. Then, there exists a collar neighborhood $U \simeq (-\varepsilon, 0] \times \partial M$ and $u, v \in \Omega^1(\partial M)$ nowhere vanishing, for which*

$$\alpha = tu + v, \quad t \in (-\varepsilon, 0],$$

if and only if there exists a vector field $X \in \mathfrak{X}(M)$, transverse to the boundary and satisfying

$$\iota_X \alpha = 0, \quad \mathcal{L}_X \mathcal{L}_X \alpha = 0,$$

on an open neighborhood of the boundary. Moreover, in the collar neighborhood X is identified with ∂_t.

Note that using a bump function, the above conditions on the vector field only needs to be satisfies locally around the boundary. One way of interpreting the conditions in the above lemma, is that there exists a direction, transverse to ∂M and tangent to $\ker \alpha$, in which the contact form is linear, i.e. has no second order information.

Proof. By Equation 1.4.1.3 above, it suffices to prove the if implication. Thus assume that X is a vector field satisfying the above conditions. In the induced collar neighborhood $(-\varepsilon, 0] \times \partial M$, we can write

$$\alpha = \alpha_t + f_t \mathrm{d}t, \quad t \in (-\varepsilon, 0]$$

for $\alpha_t \in \Omega^1(\partial M)$ and $f_t \in C^\infty(\partial M)$, and identify X with ∂_t. The first condition on X implies

$$\iota_{\partial_t} \alpha = f_t = 0,$$

and therefore the second condition gives

$$\ddot{\alpha}_t = 0, \tag{1.4.1.4}$$

by which we mean that the second derivative in the parameter t is zero. Observe that,

$$\alpha_t = \alpha_0 + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s} \alpha_{st} \,\mathrm{d}s = \alpha_0 + t \int_0^1 \dot{\alpha}_{st} \,\mathrm{d}s = \alpha_0 + t\beta_t.$$

where we defined

$$\beta_t := \int_0^1 \dot{\alpha}_{st}\,\mathrm{d}s.$$

Now observe that by Equation 1.4.1.4 we have

$$\dot{\beta}_t = \int_0^1 s\ddot{\alpha}_{st}\,\mathrm{d}s = 0,$$

so that β_t does in fact not depend on t. Putting this together we conclude

$$\alpha = \alpha_0 + t\beta,$$

as desired. □

1.4.2 Special boundaries of contact forms

As a consequence of the normal form of Theorem 1.3.19, the contact *structure* around the boundary is, up to equivalence, completely determined by the induced ∂-contact *structure* on the boundary. In particular, the choice of admissible form (or admissible pair) is of little importance since, up to isomorphism, the local model does not depend on it.

On the level of *forms* there is no general normal form, and instead we have to impose it, as in Definition 1.4.1. As a consequence, we are not free to choose the admissible form anymore, and it is part of the definition of a ∂-contact pair, see Definition 1.4.2. In fact, the behaviour of the contact form around the boundary is mostly determined by the admissible form. Understanding their properties makes several gluing constructions from the literature more transparent.

A ∂-contact pair (N^{2n}, u, v) is said to be of:

- **Liouville type** if
$$\mathrm{d}u = \mathrm{d}v;$$
- **Unimodular type** if
$$\mathrm{d}u = 0;$$
- **Foliation type** if
$$u \wedge \mathrm{d}u = 0;$$
- **Principal type** if
$$u \wedge v \wedge \mathrm{d}u^k \wedge \mathrm{d}v^{n-k-1} \geqslant 0,$$
for all $k = 0, \ldots, n-1$.

The above list is ordered from strong to weak. More precisely, N admits a ∂-contact structure of Liouville type if and only if it admits one of Unimodular type, and that the latter is a special case of Foliation type. Indeed, if (u, v) is of Liouville type then $(u' := u - v, v' := v)$ is of unimodular type. Moreover, they all satisfy the conditions

of principal type.
The existence of a special pair puts restrictions on the topology of N. For example, if (N, u, v) is of unimodular type then it follows from a theorem of Tischler [106] that N is the total space of a fibration $\pi : N \to \mathbb{S}^1$. In fact, denoting by θ the angle coordinate of $\mathbb{S}^1$, this theorem shows that $\pi^*(\mathrm{d}\theta)$ can be chosen arbitrarily close to u. Since the contact condition is open v defines a contact structures on the fibers of π. Thus, if (N, u, v) is of Unimodular type, N must admit a contact fibration over $\mathbb{S}^1$.

The following is analogous to Definition 1.2.10.

Definition 1.4.8. *We say that a contact manifold (M, α) has boundary of **right** $\mathcal{S}$**-type** (resp. **left** $\mathcal{S}$**-type**), for $\mathcal{S}$ in the above list, if in some collar neighborhood $U \simeq (-\varepsilon, 0] \times \partial M$ (resp. $U \simeq [0, \varepsilon) \times \partial M$) we have*

$$\alpha = tu + v,$$

where (u, v) is a ∂-contact pair of $\mathcal{S}$-type.

The left and right versions of each type only differ in the orientations induced on the boundary. In line with our orientation conventions, the boundary of a manifold with the standard orientation is always a right boundary. However, these names are particularly useful when considering cobordisms, where we think of these models as the left or right side of a cobordism as in Section 1.4.3.

Example 1.4.9. Let $(\Sigma, \mathrm{d}\lambda)$ be an exact symplectic manifold with boundary of contact type $(B := \partial\Sigma, \lambda_B := \lambda|_{\partial\Sigma})$. The product $\Sigma \times \mathbb{S}^1$ admits a contact form

$$\alpha := \lambda + \mathrm{d}z,$$

which has regular boundary. More precisely, as in Definition 1.2.9 and Definition 1.2.10, there is a collar neighborhood $(-\varepsilon, 0] \times \partial\Sigma$ in Σ on which

$$\lambda = (1 + t)\lambda_B.$$

In turn, this gives a collar neighborhood $(-\varepsilon, 0] \times B \times \mathbb{S}^1$ such that

$$\alpha = t\lambda_B + \lambda_B + \mathrm{d}z.$$

Hence, the induced ∂-contact boundary $\big(B \times \mathbb{S}^1, u := \lambda_B, v := \lambda_B + \mathrm{d}z\big)$ is of Liouville type.

Similarly, the product $B \times \mathbb{D}^2_\delta$, where $\mathbb{D}^2_\delta$ denotes the disk of radius δ, admits a contact form

$$\alpha := \lambda_B + r^2 \mathrm{d}\theta.$$

Reparametrizing the r-coordinate yields a collar neighborhood $(-\varepsilon, 0] \times \overline{B \times \mathbb{S}^1}$ on which

$$\alpha = s\mathrm{d}\theta + \lambda_B + \delta^2 \mathrm{d}\theta.$$

Thus the induced ∂-contact boundary $(\overline{B \times \mathbb{S}^1}, u := \mathrm{d}\theta, v := \lambda + \delta^2 \mathrm{d}\theta)$, is of unimodular type.

These two pieces form the inside and outside component of an abstract open book (with trivial monodromy), and we will see how they can be glued in Section 1.4.3 below.

Both examples above are products of an exact symplectic manifold and a contact manifold. As explained in Example 1.3.8, such products always admit a contact structure. Let (N, β) be a closed contact manifold and $(W, \mathrm{d}\lambda)$ an exact symplectic manifold then the product manifold $M := N \times W$ admits a contact form

$$\alpha := \beta + \lambda.$$

If W has boundary ∂W, then (M, α) has regular boundary, and the induced strict ∂-contact structure is given by

$$u := \gamma, \quad v := \beta + \lambda|_{\partial W},$$

where $\gamma \in \Omega^1(\partial W)$ is an admissible form for $\mathrm{d}\lambda|_{\partial W}$, see Definition 1.2.4. As usual, the behaviour of $\mathrm{d}\lambda$ on the boundary ∂W, encoded in γ, determines the type of (u, v). △

Example 1.4.10. The following situation is considered in [33, 54] to construct and classify invariant contact structures on principal circle bundles. Let $\pi : M \to W$ be a principal $\mathbb{S}^1$-bundle, and denote by $\partial_\theta \in \mathfrak{X}(M)$ the infinitesimal generator of the $\mathbb{S}^1$-action. Recall that a connection on M is a form $\gamma \in \Omega^1(M)$ satisfying

$$\mathcal{L}_{\partial_\theta}\gamma = 0, \quad \iota_{\partial_\theta}\gamma = 1.$$

These conditions imply that $\mathrm{d}\gamma = \pi^*\omega$ for some closed form $\omega \in \Omega^2(W)$, called the curvature. The class $[\omega] \in H^2(W)$ is called the Chern class of M. In the case that M and W have boundary, we assume that $\partial M = \pi^{-1}(\partial W)$ and write

$$\gamma_\partial := \gamma|_{\partial M}, \quad \omega_\partial := \omega|_{\partial W}.$$

Now, let $\beta \in \Omega^1(\partial W)$ be a contact form, and assume that the curvature $\omega \in \Omega^2(W)$ is a symplectic form satisfying

$$\beta \wedge \mathrm{d}\beta^k \wedge \omega_\partial^{n-k-1} > 0, \quad k = 0, \ldots, n.$$

Then, the conclusion of Lemma 4.2 and Lemma 4.5 in [33] is that M admits an $\mathbb{S}^1$-invariant contact form α, and a collar neighborhood $U \simeq (-\varepsilon, 0] \times \partial M$ on which we have

$$\alpha = s\pi^*(\beta) + \gamma_\partial + \pi^*\beta.$$

That is, it has boundary of right principal type with

$$(u, v) = (\pi^*\beta, \pi^*\beta + \gamma_\partial).$$

△

Example 1.4.11. Here we consider the contact analogue of Example 1.2.13, where we considered the normal form around a codimension-2 symplectic submanifold. We will use the facts about complex line bundles stated there. Let $(M^{2n+1}, \xi := \ker\alpha)$ be a contact manifold and $(B^{2n-1}, \xi_B := \ker \alpha_B)$ a codimension-2 contact submanifold. That is,

$$\xi_B := TB \cap \xi|_B,$$

is a contact structure on B, defined by $\alpha_B := \alpha|_B$. Hence, the dα-orthogonal of ξ_B provides a model for the normal bundle of B in M:

$$\nu := (TB \cap \xi)^{\mathrm{d}\alpha} \subset TM|_B.$$

The restriction of dα to ν makes it into a symplectic vector bundle $(\nu, \mathrm{d}\alpha|_\nu)$, and we note that the conformal class of $\mathrm{d}\alpha|_\nu$, only depends on ξ. As in Example 1.2.13, we can talk the first Chern class of $(\nu, \mathrm{d}\alpha|_\nu)$. We claim that there exists a neighborhood of B, endowed with a contact form representing ξ, which has boundary of:

- Liouville type if the Chern class of $(\nu, \mathrm{d}\alpha|_\nu)$ vanishes;
- Unimodular type if the Chern class of $(\nu, \mathrm{d}\alpha|_\nu)$ vanishes;
- Principal type if the Chern class $(\nu, \mathrm{d}\alpha|_\nu)$ has a representative $\sigma \in \Omega^2(B)$ satisfying
$$\alpha_B \wedge \sigma^k \wedge \mathrm{d}\alpha_B^{n-k-1} \geqslant 0, \quad \forall k = 0, \dots, n-1.$$

To construct the local model around B, let $[\sigma] \in H^2(B)$ be the Chern class of $(\nu, \mathrm{d}\alpha|_\nu)$. Furthermore, let $\pi : P \to B$ be the associated principal $\mathbb{S}^1$-bundle, endowed with a connection form $\theta \in \Omega^1(P)$ (satisfying $\mathrm{d}\theta = \pi^*\sigma$). On $P \times \mathbb{C}$, we define a 1-form

$$A := \pi^*(\alpha_B) + r^2(\mathrm{d}\phi + \theta),$$

where $(r, \phi) \in \mathbb{C}$ denote polar coordinates.

Observe that A is basic with respect to the (right) $\mathbb{S}^1$-action from Equation 1.2.2.1, and descends to the quotient

$$\left(P \times_{\mathbb{S}^1} \mathbb{C}, \widetilde{A}\right). \tag{1.4.2.1}$$

Furthermore, the form $\theta - \mathrm{d}\phi \in \Omega^1(P \times \mathbb{C})$ is dual to the infinitesimal generator of the $\mathbb{S}^1$-action and a straightforward computation shows:

$$(\theta - \mathrm{d}\phi) \wedge A \wedge \mathrm{d}A^{n+1} = 4(n+1)\theta \wedge \pi^* \left(\alpha_B \wedge (\mathrm{d}\alpha_B + r^2\sigma)^n\right) \wedge r\mathrm{d}r \wedge \mathrm{d}\phi$$

Hence, $\widetilde{A}$ defines a contact form on a neighborhood of $B = P \times_{\mathbb{S}^1} \{0\} \subset P \times_{\mathbb{S}^1} \mathbb{C}$, for which (B, α_B) is a contact submanifold. Furthermore, if the Chern class of $(\nu, \mathrm{d}\alpha|_\nu)$ is of principal type then $\widetilde{A}$ is contact on the whole of $P \times_{\mathbb{S}^1} \mathbb{C}$.

The same argument as in Example 1.2.13 shows that the induced symplectic normal bundle of $B \subset P \times_{\mathbb{S}^1} \mathbb{C}$ equals

$$(B \times \mathbb{C}, 2r\mathrm{d}r \wedge \mathrm{d}\phi),$$

so that its Chern class equals that of P, which in turn equals that of $(B, \alpha_B) \subset (M, \alpha)$. Thus, by the standard normal form theorem for contact submanifolds, see for example

[53], we conclude that a neighborhood of B in M is contact isomorphic to the model of Equation 1.4.2.1.

A tubular neighborhood of B can be identified with $P \times_{\mathbb{S}^1} \mathbb{D}^2$ with boundary $P \simeq P \times_{\mathbb{S}^1} \mathbb{S}^1$. Observe that

$$\iota_{\frac{1}{r}\partial_r}\widetilde{A} = 0, \quad \mathcal{L}_{\frac{1}{r}\partial_r}\mathcal{L}_{\frac{1}{r}\partial_r}\widetilde{A} = 0,$$

so that by Lemma 1.4.7 the boundary is regular. The restriction

$$A_\partial := A|_{P\times\mathbb{S}^1} = \pi^*(\alpha_B) + \mathrm{d}\phi + \theta,$$

is again basic with respect to the $\mathbb{S}^1$-action. Its reduction $\widetilde{A}_\partial$ equals the ∂-contact form induced by $\widetilde{A}$ on $P \times_{\mathbb{S}^1} \mathbb{S}^1$. Moreover, under the identification with P the induced ∂-contact form equals:

$$\widetilde{A}_\partial = \pi^*(\alpha_B) + \theta.$$

Hence:

- if the Chern class vanishes we can choose $\sigma = 0$, implying $\mathrm{d}\theta = 0$. Then, θ is an admissible form for which the boundary is of Unimodular type;
- if the Chern class vanishes we can choose $\sigma = 0$, implying $\mathrm{d}\theta = 0$. Then, $\alpha_B + 2\theta$ is an admissible form for which the boundary is of Liouville type;
- if the Chern class satisfief $\alpha_B \wedge \mathrm{d}\alpha_B^k \wedge \sigma^{n-k-1} \geqslant 0$ then θ is an admissible form for which the boundary is of Principal type.

△

Example 1.4.12. Combining the gluing construction from Section 1.3.4 with the local model from the previous example, we can move the Gompf connected sum from Example 1.2.22 to the contact setting.

Let (M_i, ξ_i), $i = 1, 2$, be contact manifolds with codimension-2 contact submanifolds (B_i, ξ_{B_i}) as in Example 1.4.11. Suppose there exists an orientation preserving diffeomorphism $\phi : B_1 \to B_2$ satisfying:

(i) $\phi^*(\xi_{B_2}) = \xi_{B_1}$;

(ii) $\phi^* c_1(\nu_{B_2}) = -c_1(\nu_{B_1}) \in H^2(B_1)$;

where $c_1(\nu_{B_i})$ is the Chern class of the symplectic normal bundle as in Example 1.4.11. As before, a neighborhood of B_i is contact isomorphic to

$$\left(P_i \times_{\mathbb{S}^1} \mathbb{C}, \widetilde{A}_i\right),$$

as in Equation 1.4.2.1. The above conditions imply that the induced ∂-contact structures on the boundaries of these neighborhoods satisfy the conditions of Proposition

1.3.28, so that they can be glued along their boundaries. Hence we conclude that the **Gompf connected sum**,

$$(M_1, B_1)\#(M_2, B_2) := \left(M_1\backslash P_1 \times_{\mathbb{S}^1} \mathbb{D}^2\right) \cup_\psi \left(M_2\backslash P_2 \times_{\mathbb{S}^1} \mathbb{D}^2\right),$$

where $\psi : P_1 \xrightarrow{\sim} P_2$ is induced by ϕ, carries a contact structure which restricts to ξ_i on each of the pieces. △

Just as for boundaries of symplectic manifolds, the Liouville and unimodular types can be easily recognized in terms of existence of a special vector field.

Lemma 1.4.13. *Let (M, α) be a contact manifold boundary. Then then the boundary is of:*

(i) Liouville type if and only if there exists a vector field $X \in \mathfrak{X}(M)$, pointing out along ∂M, and satisfying

$$\iota_X\alpha = 0, \quad \mathcal{L}_X \mathrm{d}\alpha = \mathrm{d}\alpha,$$

on a neighborhood of the boundary.

(ii) Unimodular type if and only if there exists a vector field $X \in \mathfrak{X}(M)$, transverse to ∂M, and satisfying

$$\iota_X\alpha = 0, \quad \mathcal{L}_X \mathrm{d}\alpha = 0,$$

on a neighborhood of the boundary.

Moreover, if X satisfies $\mathcal{L}_X\mathcal{L}_X\alpha = 0$ then in each of the cases above it suffices to require the conditions only at points in the boundary of M.

Proof. (i) Assume that the boundary is of Liouville type so that the associated ∂-contact structure (u, v) satisfies $\mathrm{d}u = \mathrm{d}v$. On the collar neighborhood $(-\varepsilon, 0] \times \partial M$ the vector field $X := (1 + t)\partial_t$ is in the kernel of α and satisfies:

$$\begin{aligned}\mathcal{L}_X \mathrm{d}\alpha &= \mathrm{d}\iota_{(1+t)\partial_t}\mathrm{d}(tu + v)\\ &= \mathrm{d}t \wedge u + (1 + t)\mathrm{d}u = \mathrm{d}\alpha.\end{aligned}$$

Conversely, assume a vector field X satisfying the above conditions exists, and let $(-\varepsilon, 0] \times \partial M$ the resulting collar neighborhood on which we identify $X = \partial_t$. Then define $Y := e^{-t}\partial_t$, which is in the kernel of α and satisfies:

$$\begin{aligned}\mathcal{L}_Y\mathcal{L}_Y\alpha &= \iota_Y \mathrm{d}\iota_Y \mathrm{d}\alpha\\ &= \iota_Y \mathrm{d}(e^{-t}\iota_X \mathrm{d}\alpha)\\ &= \iota_Y(-e^{-t}\mathrm{d}t \wedge \iota_X\mathrm{d}\alpha + e^{-t}\mathrm{d}\alpha)\\ &= -e^{-2t}\iota_X\mathrm{d}\alpha + e^{-2t}\iota_X\mathrm{d}\alpha = 0.\end{aligned}$$

Hence, by Lemma 1.4.7, we have

$$\alpha = tu + v,$$

in the collar neighborhood induced by Y. Observe that

$$\mathcal{L}_Y \mathrm{d}\alpha|_{\partial M} = e^{-t}\mathcal{L}_X \mathrm{d}\alpha|_{\partial M} = \mathrm{d}\alpha|_{\partial M},$$

and since in the above collar we have

$$\mathrm{d}\alpha|_\partial = \mathrm{d}v, \quad (\mathcal{L}_{\partial_t}\mathrm{d}\alpha)\,|_\partial = \mathrm{d}u,$$

the boundary is of Liouville type. Finally, if the vector field X already satisfies $\mathcal{L}_X\mathcal{L}_X\alpha = 0$, then the above argument shows it suffices to ask $\mathcal{L}_X\mathrm{d}\alpha|_{\partial M} = \mathrm{d}\alpha|_{\partial M}$.

(ii) If α is of unimodular type we simply check that the above conditions hold for $X := \partial_t$. Conversely, if such an X exists then it also satisfies

$$\mathcal{L}_X\mathcal{L}_X\mathrm{d}\alpha = 0.$$

In the collar neighborhood of Lemma 1.4.7 the condition $\mathcal{L}_X\mathrm{d}\alpha = 0$ is equivalent to $\mathrm{d}u = 0$.

□

1.4.3 Cobordisms between ∂-contact manifolds

We now consider gluing contact manifolds with fixed contact forms, where we ask that the gluing preserves the chosen forms. This might seem superfluous in light of the gluing construction of Section 1.3.4. However, it allows us to phrase the technical arguments needed in Section 1.8 in a more conceptual way.

The following glueing construction for contact forms regular at the boundary, follows directly from the definitions:

Lemma 1.4.14. *Let α_i be a contact manifold with regular boundary, as in Definition 1.4.1, on a manifold M_i, $i = 1, 2$, and denote by (u_i, v_i) the induced ∂-contact pair on ∂M_i, as in Definition 1.4.2. If there exists an orientation reversing diffeomorphism $\phi : \partial M_1 \to \partial M_2$, satisfying*

$$\phi^* u_2 = -u_1, \quad \phi^* v_2 = v_1,$$

then the manifold $M_1 \cup_\phi M_2$, admits a contact form α which restricts to α_i on M_i.

Proof. By Definition 1.4.1 there exist collar neighborhoods $(-\varepsilon, 0] \times \partial M_i$ on which

$$\alpha_i = tu_i + v_i, \quad i = 1, 2.$$

Observe that $(-\varepsilon, 0] \times \partial M_2 \simeq [0\varepsilon) \times \overline{\partial M_2}$ by sending $t \mapsto -t$. Hence under this isomorphism

$$\alpha_2 = t(-u_2) + v_2.$$

Using ϕ to identify $\overline{\partial M_2} \simeq \partial M_1$, $-u_2 = u_1$, and $v_2 = v_1$ the two collars can be matched along their boundary. □

Of course, in most situations the difficulty comes from the search for the required diffeomorphism. Even more, since we want to preserve the contact forms on each of the manifolds we want to glue, we have no freedom to change the contact forms, in order to make them match along the boundaries. To solve this problem we can, instead of gluing the manifolds directly to each other glue an extra piece, called a cobordism, in between. In many cases it suffices to consider topologically trivial cobordisms $[0,1] \times \partial M$, so that the topology of the gluing is not affected. However, for completeness we introduce the general notion of cobordism, both on the level of structures and forms.

Definition 1.4.15. *Let (N_i, ζ_i), $i = 1, 2$, be ∂-contact manifolds as in Definition 1.3.12. A **contact cobordism** $(N_1, \zeta_1) \prec_{(M,\xi)} (N_2, \zeta_2)$ is a contact manifold (M, ξ) with*

$$\partial M = \overline{N_1} \sqcup N_2,$$

and inducing ζ_i on the boundary. In the cooriented case we additionally require the coorientations to match.

Remark 1.4.16. Strictly speaking the identification of the boundary ∂M with N_1 and N_2, is only up to diffeomorphism. That is, we have $\partial M = \partial M_1 \sqcup \partial M_2$, where ∂M_i, $i = 1, 2$, denotes a (collection) of connected components of ∂M. Then, we require there exists an orientation reversing diffeomorphism $\phi_1 : N_1 \to \partial M_1$, and an orientation preserving diffeomorphism $\phi_2 : N_2 \to \partial M_2$. The choice of diffeomorphisms is usually clear from the context, so we suppress them in the notation for the sake of readability. △

Example 1.4.17. The notion of contact cobordism is very convenient to keep track of gluings, since it automatically takes care of the conventions for left and right boundaries. That is, a contact manifold (M, ξ) with non-empty right boundary, is the same thing as a cobordism

$$\varnothing \prec_{(M,\xi)} (\partial M, \xi_\partial := \xi \cap T\partial M).$$

Similarly, a contact manifold with left boundary is the same thing as a contact cobordism

$$(\partial M, \xi_\partial) \prec_{(M,\xi)} \varnothing.$$

The gluing construction from Section 1.3.4 implies that cobordisms can be composed. That is, given $(N_1, \zeta)_1 \prec_{(M,\xi)} (N_2, \zeta_2)$, and $(N_2, \zeta_2) \prec_{(\widetilde{M},\tilde{\xi})} (N_3, \zeta_3)$, the composition gives a cobordism

$$(N_1, \zeta_2) \prec_{(M \cup \widetilde{M}, \xi \cup \tilde{\xi})} (N_3, \zeta_3).$$

In particular, gluing contact manifolds (M_i, ξ_i), $i = 1, 2$, with isomorphic ∂-contact boundaries gives a cobordism from the emptyset to itself:

$$\varnothing \prec_{(M_1,\xi_1)} (\partial M_1, \xi_{1,\partial}) \prec_{(M_2,\xi_2)} \prec \varnothing,$$

where, as in Remark 1.4.16, the isomorphism of ∂-contact manifolds $\phi : (\partial M_1, \xi_{1,\partial}) \to (\partial M_2, \xi_{2,\partial})$ is implicit in the notation. △

For ∂-contact pairs we consider the following notion of cobordism:

Definition 1.4.18. *Let (u_i, v_i) be a ∂-contact pair on N_i, for $i = 1, 2$, as in Definition 1.4.2. A* ***regular contact cobordism*** $(N_1, u_1, v_1) \prec_{(M,\alpha)} (N_2, u_2, v_2)$*, consists of a contact manifold (M, α) with*

$$\partial M = \overline{N_1} \sqcup N_2,$$

and such that, in the notation of Definition 1.4.8, α has:

(i) Regular left boundary N_1, with induced ∂-contact pair (u_1, v_1);

(ii) Regular right boundary N_2, with induced ∂-contact pair (u_2, v_2).

Of course, any regular contact cobordism induces a contact cobordism as above. As stated before, regular contact cobordisms allow us to change the type of boundary, as in Definition 1.4.8, of a contact form. The precise conditions under which this is possible are as follows:

Lemma 1.4.19. *Let (N^{2n}, u, v) be a ∂-contact manifold and $a, b, c, d \in \mathbb{R}$ satisfying $ad - bc > 0$. Then*

$$u' := au + bv, \quad v' := cu + dv,$$

defines a ∂-contact structure in any of the following cases:

(i) (u, v) is of Liouville type and $(c + d)^{n-1} > 0$;

(ii) (u, v) is of Foliation type and $d^{n-1} > 0$;

(iii) (u, v) is of Principal type, $c > 0$, $d > 0$, and not both equal to zero.

In these cases there exists a contact form on $[0, 1] \times N$ giving a regular contact cobordism from (N, u, v) to (N, u', v').

Proof. Consider the trivial cobordims $[0, 1] \times N$, endowed with the 1-form

$$\alpha := f(s)u + g(s)v,$$

for $f, g : [0, 1] \to \mathbb{R}$ suitable functions to be chosen later. The contact condition for α reads

$$\alpha \wedge \mathrm{d}\alpha^n = n(\dot{f}g - f\dot{g})\mathrm{d}s \wedge u \wedge v \wedge (f\mathrm{d}u + g\mathrm{d}v)^{n-1}. \tag{1.4.3.1}$$

Thus, under the assumption that (u, v) is of principal type, α will be contact if

$$\dot{f}g - f\dot{g} > 0, \quad g > 0, \quad f \geqslant 0. \tag{1.4.3.2}$$

In case (u, v) is of Liouville or foliation type the above conditions can be slightly relaxed, giving the other statements in the lemma.

Observe that

$$(s - 1)u' + v' = (sa - a + c)u + (sb - b + d)v.$$

Hence, if we want α to induce (u, v) on the left boundary, and (u', v') on the right boundary then we additionally need to require that

$$\begin{pmatrix} f & \dot{f} \\ g & \dot{g} \end{pmatrix} = \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix}, \quad s \in [0, \varepsilon), \quad \text{and} \quad \begin{pmatrix} f & \dot{f} \\ g & \dot{g} \end{pmatrix} = \begin{pmatrix} (s-1)a + c & a \\ (s-1)b + d & b \end{pmatrix}, \quad s \in (1-\varepsilon, 1]. \tag{1.4.3.3}$$

In order for Equation 1.4.3.2 to be satisfied at the right boundary we need

$$ad - bc > 0, \quad c \geqslant 0, \quad d > 0. \tag{1.4.3.4}$$

An extension of the functions f and g satisfying Equation 1.4.3.2, can be viewed as a path

$$\begin{aligned} \lambda : [0, 1] &\to \mathbb{R}^2 \\ t &\mapsto (f(t), g(t)), \end{aligned} \tag{1.4.3.5}$$

into the upper right quadrant of $\mathbb{R}^2$, and such that $(\dot{\lambda}(t), \lambda(t))$ defines an oriented frame. Such a path exists, see Figure 1.1, provided the conditions in Equation 1.4.3.4 and $c > 0$ are satisfied. □

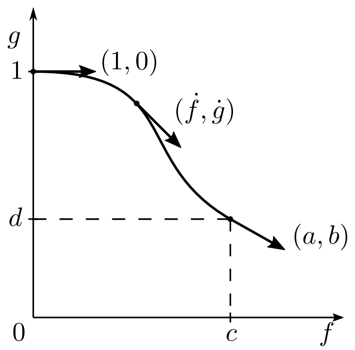

Figure 1.1: Functions f and g satisfying the conditions in Equation 1.4.3.2 and Equation 1.4.3.3.

Remark 1.4.20. The conditions on the coefficients (a, b, c, d) are necessary in the most general case. However, they can be relaxed in many specific examples, where u and v are explicitly given. For example, if $n = 1$ then Equation 1.4.3.1 simplifies, and the only remaining condition is $ac - bd > 0$. This gives extra freedom, since the path λ from Equation 1.4.3.5 is now allowed to make a loop around the origin. In fact, in this case any two points can be connected, see Figure 1.2. △

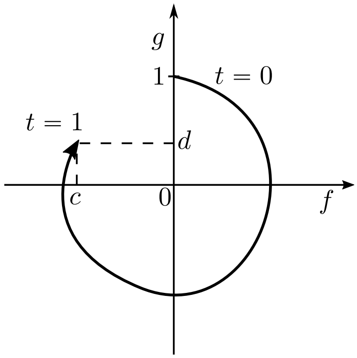

Figure 1.2: Functions f and g satisfying the conditions in Equation 1.4.3.2 and Equation 1.4.3.3, in the case $n = 1$.

1.4.4 Contact open books

An open book decomposition of a manifold M consist of a codimension-2 submanifold B whose normal bundle is trivial, and a fibration on the complement $\pi : M\backslash B \to \mathbb{S}^1$. The definition and basic properties of open books are discussed in Appendix 1.9. As discussed there we consider two points of view; as a way to decompose a given manifold into simpler pieces, called geometric open book, and as way of constructing new manifolds out of simpler data, called an abstract open book.

In this section we consider open book decompositions for contact manifolds. Imposing compatibility conditions between the contact structure and the open book ensures that both pieces of the decomposition inherit natural contact structures. These conditions translate into conditions on the associated abstract open book, and we show that the contact manifold can be recovered from this data.

Definition 1.4.21. *A contact form α on M is said to be* ***adapted*** *to a (geometric) open book (B, π) if:*

(i) The binding $(B, \alpha_B := \alpha|_B)$ is a contact submanifold;

(ii) Away from the binding Reeb vector field R is positively transverse to the (open) pages, that is:
$$\pi^*(\mathrm{d}\theta)(R) > 0.$$

In turn, a contact structure ξ on M is adapted to (B, π) if there exist an adapted contact form α representing it.

Remark 1.4.22. Recall that a **symplectic fibration** is defined to be fibration $\pi : M \to B$ together with a symplectic form on each fiber. That is, a leafwise symplectic form $\omega \in \Omega^2(\mathcal{F})$ on the foliation $\mathcal{F}_\pi := \ker d\pi \subset TM$ induced by the fibration. An extension $\eta \in \Omega^2(M)$ of ω defines a connection by

$$\mathcal{H} := (\ker d\pi)^{\perp\eta} \subset TM. \tag{1.4.4.1}$$

Indeed, since $\eta|_{\mathcal{F}_\pi}$ is non-degenerate, $\mathcal{H}$ is a horizontal distribution. Moreover if η is closed, then the parallel transport of $\mathcal{H}$ preserves the fiberwise symplectic form, see [86].

Now if α is a contact form on M adapted to an open book (B, π), then the condition that the Reeb vector field is positively transverse to the page is equivalent to requiring

$$d\alpha^n \wedge \pi^* d\theta > 0,$$

where $\theta \in \mathbb{S}^1$ denotes the usual angle coordinate. In turn this means that $\pi : M \backslash B \to \mathbb{S}^1$, becomes a symplectic fibration with induced symplectic foliation $(\mathcal{F}_\pi, d\alpha|_{\mathcal{F}_\pi})$. △

The above conditions depend on the Reeb vector field, and thus on the choice of contact form. Hence, even if ξ is adapted to (B, π) there exist contact forms representing ξ which are not adapted.

Example 1.4.23. Let $(r, \theta, z) \in \mathbb{R}^3$ denote the standard cylindrical coordinates. Then, the standard open book decomposition is given by $B := \{r = 0\}$ and

$$\pi : \mathbb{R}^3 \backslash B \to \mathbb{S}^1, \quad (r, \theta, z) \mapsto \theta.$$

The standard contact form $\alpha := dz + r^2 d\theta$ is not adapted to (B, π). Indeed, although B is a contact submanifold for α, its Reeb vector field equals ∂_z which is tangent to the fibers of π. However, the contact form $e^{-r^2}\alpha$ has Reeb vector field

$$R = (1 - r^2)e^{r^2}\partial_z + e^{r^2}\partial\theta,$$

which is positively transverse to the pages. Hence, the standard contact structure $\xi := \ker \alpha$ is adapted to (B, π).

This example generalizes to $B \times \mathbb{R}^2$, with the obvious open book decomposition and the contact structure $\xi := \ker\left(\alpha_B + r^2 d\theta\right)$, for $\alpha_B \in \Omega^1(B)$ a contact form on B. As shown in Equation 1.4.4.2 in the proof of Theorem 1.4.26 below, a contact structure adapted to an open book looks like this one. Hence, any contact structure adapted to an open book has representing contact forms which are not adapted. △

Although the definition of a contact form adapted to a geometric open book looks quite restrictive we have the following:

Theorem 1.4.24 ([57][99])**.** *Let (M, ξ) be a contact manifold. There exists a (geometric) open book decomposition (B, π) of M to which ξ is adapted as in Definition 1.4.21.*

Recall that any geometric open book decomposition (B, π) has an associated abstract open book (Σ, ϕ), as in Lemma 1.9.6. If α is a contact form adapted to (B, π) then the restriction of $d\alpha$ to the page P is non-degenerate, since the Reeb vector field is transverse to the page. Moreover, as we will see below, in this case the monodromy of the fibration $\pi : M \backslash B \to \mathbb{S}^1$ can be chosen to preserve $d\alpha|_P$. Thus we arrive at the following definition.

Definition 1.4.25. *An (abstract)* ***contact open book*** *consists of an exact symplectic manifold* $(\Sigma, \mathrm{d}\lambda)$ *with boundary of Liouville type, as in Definition 1.2.10, together with a symplectomorphism* $\phi : (\Sigma, \mathrm{d}\lambda) \to (\Sigma, \mathrm{d}\lambda)$ *which is the identity on a neighborhood of the boundary.*

As claimed above, Definition 1.4.21 and Definition 1.4.25 correspond to each other under the identifcations of Lemma 1.9.6 and Lemma 1.9.5. We first show that a geometric contact open book induces an abstract contact open book. The non-trivial part is finding the required monodromy. Let $\alpha \in \Omega^1(M)$ and (B, π) are as in Definition 1.4.21. Then the rescaling of the Reeb vector field;

$$X := \frac{1}{\pi^*(\mathrm{d}\theta)(R)} R \in \mathfrak{X}(M \backslash B),$$

satisfies $\mathcal{L}_X \mathrm{d}\alpha = 0$, so its time one flow is a symplectomorphism ϕ of the page $(P, \mathrm{d}\alpha|_P)$. However, ϕ need not be equal to the identify on a neighborhood of the boundary ∂P. The proof of the following lemma shows that, by modifying X close to the binding, this additional condition can be satisfied.

Theorem 1.4.26. *Let* ξ *be a contact structure on* M *adapted to an open book decomposition* (B, π)*. Then there exists an adapted contact form* $\alpha \in \Omega^2(M)$ *for* ξ*, and a symplectomorphism* ϕ *on* $(P, \mathrm{d}\alpha|_P)$ *such that* $(P, \mathrm{d}\alpha|_P, \phi)$ *is an abstract contact open book.*

Proof. Fix an adapted contact form α for ξ, and denote by α_B the induced contact form on the binding B. Since the normal bundle of B is trivial, the normal form from Example 1.4.11 implies there exists a neighborhood $B \times \mathbb{D}^2$ of the binding on which

$$\alpha = f(\alpha_B + r^2 \mathrm{d}\theta), \tag{1.4.4.2}$$

where $(r, \theta) \in \mathbb{D}^2$, and $f \in C^\infty(B \times \mathbb{D}^2)$ is a positive function satisfying $f|_{B \times \{0\}} = 1$. Furthermore, the codnition that α is adapted implies that

$$\mathrm{d}\alpha^n \wedge \mathrm{d}\theta = -n f^{n-1} \partial_r f \alpha_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}r \wedge \mathrm{d}\theta > 0.$$

Thus $\partial_r f < 0$ for $r > 0$, in fact since f is smooth it is of the form

$$f = 1 - r^2 g,$$

for some smooth function g. We can choose a function $\tilde{f} \in C^\infty(B \times \mathbb{D}^2)$ satisfying

(i) $\tilde{f}|_{B \times \{0\}} = 1$ and $\partial_r \tilde{f} < 0$;

(ii) $\tilde{f}$ agrees with f on a neighborhood of the boundary $\partial(B \times \mathbb{D}^2)$;

(iii) $\mathrm{d}\tilde{f}|_B = 0$ and on a neighborhood of the binding $B \times \{0\}$ we have

$$\partial_\theta \tilde{f} = 0.$$

Then,

$$\widetilde{\alpha} := \widetilde{f}(\alpha_B + r^2 \mathrm{d}\theta),$$

is again an adapted contact form for ξ, and agrees with α away from the binding. A straightforward computation shows that its Reeb vector field equals

$$R_{\widetilde{\alpha}} = \left(\frac{1}{\tilde{\tilde{f}}} + \frac{r\partial_r \tilde{f}}{2\tilde{f}^2}\right) R_{\alpha_B} - \frac{\partial_r \tilde{f}}{2r\tilde{\tilde{f}}^2}\partial_\theta.$$

Choose a bump function $\lambda : [0,1] \to [0,1]$ which is constant equal to 0 around zero and constant equals to 1 around one. Then, the vector field

$$X := \lambda(r)\left(\frac{1}{f} + \frac{r\partial_r f}{2f^2}\right) R_{\alpha_B} - \frac{\partial_r f}{2rf^2}\partial_\theta \in \mathfrak{X}(B \times \mathbb{D}^2),$$

agrees with $R_{\widetilde{\alpha}}$ away from the binding and is a multiple of ∂_θ near the boundary. Observe that X is everywhere positively transverse to the pages. That is, the function

$$g := \pi^* \mathrm{d}\theta(X) \in C^\infty(M \backslash B),$$

is strictly positive. Hence, the time one flow of $\frac{1}{g}X$ defines the monodromy $\phi \in \mathrm{Diff}(P)$ of the open book, and we claim it preserves $\mathrm{d}\widetilde{\alpha}|_P$. Away from the binding the follows since X equals the Reeb vector field of $\widetilde{\alpha}$ so that

$$\mathcal{L}_{\frac{1}{g}X} \mathrm{d}\alpha = 0.$$

On the collar neighborhood it follows from the normal form above since there $\frac{1}{g}X$ is of the form $a(r)R_B + b(r)\partial_\theta$ for functions $a, b \in C^\infty(B \times \mathbb{D}^2)$. Hence, we have

$$\begin{aligned}\mathcal{L}_{aR_B + b\partial_\theta} \mathrm{d}\alpha &= \mathrm{d}\iota_{aR_B + b\partial_\theta} \mathrm{d}\alpha \\ &= -\mathrm{d}\left(a\partial_r f \mathrm{d}r + b(r^2\partial_r f + 2rf)\mathrm{d}r\right) = 0,\end{aligned}$$

since all the functions depend only on r. Lastly, since $\frac{1}{g}X$ is a multiple of ∂_θ near the binding, the monodromy is the identity near the boundary ∂P. □

Conversely, using Lemma 1.9.5, we can construct a manifold $M(\Sigma, \phi)$ out of an abstract open book (Σ, ϕ). Given a contact open book $(\Sigma, \phi, \mathrm{d}\lambda)$ then (M, Σ, ϕ) carries a contact structure adapted to the induced geometric open book.

Lemma 1.4.27. *Given an abstract contact open book* $(\Sigma, \phi, \mathrm{d}\lambda)$ *with compact* Σ*, the manifold* $M(\Sigma, \phi)$ *with its canonical open book decomposition admits an adapted contact form.*

Proof. The main technicality of the proof is constructing the contact form on the outside component of the open book. We first consider the case that $\phi = \mathrm{id}$, implying that the outside component is just the product $\Sigma \times \mathbb{S}^1$. From Example 1.4.9 we have the contact manifold

$$\left(B \times \mathbb{D}^2_\delta, \lambda + r^2 \mathrm{d}\theta\right),$$

for $0 < \delta < 1$, with induced ∂-contact boundary $(\overline{B \times \mathbb{S}^1}, u = \mathrm{d}\theta, v = \lambda_B + \delta \mathrm{d}\theta)$. We also have the contact manifold

$$\left(\Sigma \times \mathbb{S}^1, \lambda_B + \mathrm{d}z\right),$$

with induced ∂-contact boundary $(B \times \mathbb{S}^1, u' = \lambda_B, v' = \lambda_B + \mathrm{d}z)$. Observe that

$$\begin{pmatrix} -u' \\ v' \end{pmatrix} = \begin{pmatrix} \delta & -1 \\ 1-\delta & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

and this matrix has determinant 1. Hence, the conditions of Lemma 1.4.19 are satisfied and we find a contact form on $[\delta, 1] \times \overline{B \times \mathbb{S}^1}$ which can be glued in between the two pieces. Together constructs a contact form on the filled mapping cylinder

$$M(\Sigma, \phi) = B \times \mathbb{D}^2 \cup_{B \times \mathbb{S}^1} \Sigma \times \mathbb{S}^1. \tag{1.4.4.3}$$

If the monodromy ϕ is non-trivial the construction on the outside component changes as follows. The monodromy is isotopic through symplectomorphisms equal to the identity near $\partial\Sigma$, to an exact symplectomorphism $\tilde{\phi}$(see for example Lemma 7.3.4 in [52]). The mapping cylinders $M(\Sigma, \phi)$ and $M(\Sigma, \tilde{\phi})$ are isomorphic (see for example Lemma 7.3.1 in [52]) so that we can assume that ϕ is an exact symplectomorphism of $(\Sigma, \mathrm{d}\lambda)$. Hence there exists a function $f \in C^\infty(\Sigma)$ satisfying

$$\phi^* \lambda = \lambda + \mathrm{d}f.$$

Note that adding a constant to f does not change the above equation. Hence, by compactness of Σ we can assume f is strictly positive. Then, we form the "scaled mapping cylinder" given by:

$$M \times_{\mathbb{Z}, f} \mathbb{R} := M \times \mathbb{R}/(x, z) \sim \Phi(x, z),$$

the quotient of $M \times \mathbb{R}$ under the $\mathbb{Z}$-action generated by:

$$\Phi : M \times \mathbb{R} \to M \times \mathbb{R}, \quad (x, z) \mapsto (\phi(x), z - f(x)).$$

Observe that the scaled mapping cylinder is diffeomorphic to the usual (non-scaled) one. The map

$$\Psi : M \times_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} M \times_{\mathbb{Z}, f} \mathbb{R}, \quad [(x, z)] \mapsto [(x, f(x)z)],$$

gives a diffeomorphism since f is strictly positive. The contact form $\alpha := \mathrm{d}z + \lambda$, is preserved under the action, and hence descends to the quotient. Furthermore, around the boundary $\phi = \mathrm{id}$ so the outside component can be glued to the inside component as in the trivial case. □

1.5 Symplectic foliations and their boundaries

1.5.1 Codimension one symplectic foliations

Recall that a codimension-k foliation $\mathcal{F}$ on an n-dimensional manifold M is a decomposition

$$M = \bigcup_{x \in M} L_x,$$

into a disjoint union of connected immersed submanifolds of codimension-k, called the leaves of $\mathcal{F}$, such that around each point there exists a local coordinate chart $U \simeq \mathbb{R}^n$ in which the decomposition equals:

$$\mathbb{R}^n := \bigcup_{x \in \mathbb{R}^k} \mathbb{R}^{n-k} \times \{x\}.$$

By the famous Frobenius theorem, this is equivalent to a distribution $T\mathcal{F} \subset TM$, of corank-k, which is involutive in the sense that

$$[X, Y] \in \Gamma(T\mathcal{F}), \quad \forall X, Y \in \Gamma(T\mathcal{F}).$$

From now on we will only consider foliations of codimension-1, unless explicitly stated otherwise.

Since the leaves of $\mathcal{F}$ are submanifolds, the complex of leafwise differential forms

$$\Omega^\bullet(\mathcal{F}) := \Gamma(\Lambda^\bullet T^*\mathcal{F}).$$

admits a differential $\mathrm{d}_\mathcal{F}$ (usually denoted by d if there is not risk of confusion), which is just the leafwise deRham differential. The usual Koszul formula gives an explicit description of $\mathrm{d}_\mathcal{F}$ given by:

$$\begin{aligned}(\mathrm{d}_\mathcal{F}\alpha)(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \mathcal{L}_{X_i}(\alpha(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}),\end{aligned}$$

for $\alpha \in \Omega^k(\mathcal{F})$. In this language a symplectic foliation is defined as follows:

Definition 1.5.1. *A* ***symplectic foliation*** $(\mathcal{F}, \omega)$ *on a manifold* M *is a (codimension-1) foliation* $\mathcal{F}$ *endowed with a leafwise form* $\omega \in \Omega^2(\mathcal{F})$ *that is (leafwise) closed and non-degenerate.*

To highlight the analogy between contact structures and symplectic foliations in the notation, we will often abbreviate symplectic foliation by SF. That is, we talk about **SF-structures** and SF-manifolds, and similarly for the notions of SF-pair, ∂-SF structure, etc, introduced below.

Since a symplectic foliation is in particular a codimension-1 distribution, much of the discussion for contact structures from Section 1.3.1, also applies here. As there, although M is always assumed to be oriented, we do not necessarily require $\mathcal{F}$ to be (co-) oriented.

Definition 1.5.2. *A foliation* $\mathcal{F}$ *is said to be* ***coorientable/cooriented*** *if* $TM/\mathcal{F}$ *is orientable/oriented.*

On an oriented manifold M endowed with a foliation $\mathcal{F}$ we have a one-to-one correspondence between orientations of the leaves of $\mathcal{F}$ and coorientation (i.e. orientations of $TM/\mathcal{F}$) of $\mathcal{F}$. Hence, as a consequence of our convention, that unless explicitly stated otherwise all manifolds are oriented, any symplectic foliation $(\mathcal{F}, \omega)$ has a canonical coorientation by declaring $\omega^n > 0$ along the leaves of $\mathcal{F}$.

For various constructions it is useful to rephrase the above definition in terms of global forms on M. This motivatives the following definition, whose terminology is inspired by that for contact structures from Definition 1.3.6.

Definition 1.5.3. *A **symplectic foliation pair** (SF-pair for short) on M^{2n+1} is a pair $(\gamma, \eta) \in \Omega^1(M) \times \Omega^2(M)$ satisfying*

$$\gamma \wedge \mathrm{d}\gamma = 0, \quad \gamma \wedge \eta^n > 0, \quad \gamma \wedge \mathrm{d}\eta = 0.$$

Note that any SF-pair induces a symplectic foliation by $(\mathcal{F} := \ker \gamma, \omega := \eta|_{T\mathcal{F}})$. When we study SF-pairs the emphasis is on the specific choice (γ, η), and we do not consider the induced symplectic foliation. On the other hand, when we study symplectic foliations it is often convenient to represent them by a pair (γ, η) as above. To stress the difference, in this situation we will refer to (γ, η) as a **symplectic foliation pair** representing $(\mathcal{F}, \omega)$. As observed above, such representing pairs always exist. However, the choice of representative is only unique up to an element of

$$G := \{(f, \beta) \in C^\infty(M) \times \Omega^1(M) \mid f > 0\}.$$

Indeed, it is not hard to see that two SF-pairs induce the same symplectic foliation they are equivalent in the following sense:

Definition 1.5.4. *Two SF-pairs (γ, η) and $(\tilde{\gamma}, \tilde{\eta})$, as in Definition 1.5.3, are **equivalent** if*

$$\tilde{\gamma} = f\gamma, \quad \tilde{\eta} = \eta + \beta \wedge \gamma,$$

for some $(f, \beta) \in G$. In this case we write $(\gamma, \eta) \sim (\tilde{\gamma}, \tilde{\eta})$.

In fact, G is a group under the multiplication

$$(f, \alpha) \cdot (g, \beta) := (fg, \beta + g\alpha),$$

and the above equivalence classes are precisely the orbits of the induced G action. In case the manifold M has a non-empty boundary ∂M we talk about **equivalence adapted to the boundary**, denoted by $(\gamma, \eta) \sim_\partial (\gamma', \eta')$ if the extra conditions

$$f|_{\partial M} = 1, \quad \beta_p = 0, \quad \forall p \in \partial M \tag{1.5.1.1}$$

are satisfied.

Example 1.5.5. The following are some of the basic examples of SF-manifolds. Its interesting to compare this list with the one from Example 1.3.8 for contact structures.

- **Euclidean space**: Let $(x_1, y_1, \ldots, x_n, y_n, z)$ denote the standard coordinates in $\mathbb{R}^{2n+1}$. The forms

$$\gamma := \mathrm{d}z, \quad \eta := \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}y_i,$$

are called the standard SF-pair. The analogue of Darboux's theorem, which follows for example from Weinstein's splitting theorem [112], stated that locally any symplectic foliation looks like the standard one. Hence, similar to contact and symplectic structures, SF-manifolds have no local invariants.

- **Tori:** Let $(x_1, y_1, \dots, x_n, y_n, z)$ denote the standard angular coordinates on $\mathbb{T}^{2n+1}$. Then, the forms

$$\gamma := \mathrm{d}z, \quad \eta := \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i,$$

define an SF-pair. Comparing with Example 1.3.8, we see that the tori $\mathbb{T}^{2n+1}$, both admit a contact structure and a symplectic foliation. On the symplectic foliation side this is immediate, whereas on the contact side it is a rather non-trivial result.

- **Products:** For dimensional reasons, the product of two SF-manifolds cannot admit an SF-structure. Instead let $(M, \mathcal{F}, \omega)$ be an SF-manifold and (W, ω) a symplectic manifold. Then, the product $M \times W$ admits a foliation, called the product foliation, defined by:

$$\mathcal{F} \times W := \bigcup_{x \in M} L_x \times W,$$

where L_x denotes the leaf of $\mathcal{F}$ through x. Thus, the corresponding distribution equals:

$$T(\mathcal{F} \times W) = T\mathcal{F} \oplus TW \subset T(M \times W),$$

endowed with the leafwise symplectic form

$$\eta' := \eta + \omega.$$

In particular given any symplectic manifold (W, ω), the product

$$\left(\mathbb{S}^1 \times W, \mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \{z\} \times W, \omega\right),$$

is an SF-manifold. Similarly, given any SF manifold $(M, \mathcal{F}, \omega)$, the product

$$\left(M \times \Sigma_g, \mathcal{F} \times \Sigma_g, \eta + \omega\right),$$

is an SF-manifold, where ω is any volume form on Σ_g, the surface of genus g.

- **Cosymplectic structures:** A cosymplectic structure on a manifold M^{2n+1} is a pair $(\gamma, \eta) \in \Omega^1(M) \times \Omega^2(M)$ satisfying

$$\mathrm{d}\gamma = 0, \quad \mathrm{d}\eta = 0, \quad \gamma \wedge \eta^n > 0.$$

In this case $(\mathcal{F} := \ker \gamma, \omega|_{\mathcal{F}})$ is a symplectic foliation. Such foliations behave a lot like the products from the previous example, in the sense that both the foliation and the leafwise symplectic form are "constant". That is the forms being (globally) closed implies they do not vary, in a sense we will make precise later in Definition 1.7.16 and Definition 1.7.22.

- **Spheres:** The so called Reeb foliation on $\mathbb{S}^3$ [100] played an important role in the development of foliation theory. Recall that the sphere $\mathbb{S}^3$ can be decomposed as two solid tori intersecting along their boundary:

  $$\mathbb{S}^3 = \mathbb{S}^1 \times \mathbb{D}^2 \cup_{\mathbb{T}^2} \mathbb{D}^2 \times \mathbb{S}^1. \tag{1.5.1.2}$$

  On the interior of the solid torus, $\operatorname{int}(\mathbb{S}^1 \times \mathbb{D}^2)$ consider the image of the product foliation $\bigcup_{z\in\mathbb{S}^1}\{z\} \times \mathbb{D}^2$, under the diffeomorphism

  $$\phi : \operatorname{int}(\mathbb{S}^1 \times \mathbb{D}^2) \to \operatorname{int}(\mathbb{S}^1 \times \mathbb{D}^2), \quad (z, r, \theta) \mapsto (z + \frac{1}{1-r^2}, r, \theta).$$

  This foliation can be smoothly extended to the solid torus, by taking the boundary $\mathbb{T}^2 = \partial(\mathbb{S}^1 \times \mathbb{D}^2)$ as a leaf. Since each leaf is an orientated surface $\mathcal{F}_{Reeb}$ is in fact a symplectic foliation. The resulting SF-manifold $(\mathbb{S}^1 \times \mathbb{D}^2, \mathcal{F}_{Reeb})$ is called a **Reeb component**. Endowing each of the pieces of the decomposition in Equation 1.5.1.2 with this foliation gives the Reeb foliation on $\mathbb{S}^3$.

  Reeb components are a special example of the more general turbulization construction which we will discuss in detail in Section 1.6.2 below. We use it to give a general construction for symplectic foliations on manifolds that admit a suitable open book decomposition. This is the SF version of the construction from Section 1.4.4. In particular, in Theorem 1.9.1 we use this to recover a result by Mitsumatsu [89] showing that the Lawson foliation [72] on $\mathbb{S}^5$ is part of a SF-structure.

  The existence question for SF-structures on the higher dimensional spheres is still open. The techniques of Lawson can be used to show that all the odd dimensional spheres admit a codimension-1 foliation. However, the compact leaves do not admit a symplectic structure. Again, compare this with Example 1.3.8, showing that all odd-dimensional spheres admit a contact structure.

$\triangle$

1.5.2 Symplectic foliations with transverse boundaries

Let $(\mathcal{F}, \omega)$ be a symplectic foliation on a manifold with boundary M. In general, the intersection of the foliation with the tangent space to the boundary can be extremely complicated, and impossible to put into a normal form. However, there are two situations which are relatively easy to understand: when $\mathcal{F}$ is transverse to the boundary, and when ∂M is a leaf. In this section we consider the first case, and show that both the foliation and the leafwise symplectic form can be put in normal form.

Definition 1.5.6. *The boundary of a symplectically foliated manifold $(M, \mathcal{F}, \omega)$ is called **transverse** if $\mathcal{F} \pitchfork \partial M$.*

In this case, the boundary inherits a foliation with a leafwise 2-form:

$$\mathcal{F}_\partial := \mathcal{F} \cap T(\partial M), \quad \omega_\partial := \omega|_{\mathcal{F}_\partial}.$$

Although ω_∂ is still closed, it is degenerate and has a 1-dimensional kernel.

Definition 1.5.7. *A* ∂***-symplectic foliation*** *(or* ∂*-SF structure for short)* $(\mathcal{F}, \omega)$ *on a manifold* N^{2n} *is a (codimension-1) foliation* $\mathcal{F}$ *endowed with a leafwise form* $\omega \in \Omega^2(\mathcal{F})$ *that is closed and maximally non-degenerate, i.e. has 1-dimensional kernel.*

As before, this structure can be represented by global differential forms, and the representatives are unique up to the same equivalence relation as in Definition 1.5.4.

Definition 1.5.8. *A* ∂***-symplectic foliation pair*** (γ, η) *on a manifold* N^{2n} *is a pair* $(\gamma, \eta) \in \Omega^1(N) \times \Omega^2(N)$ *satisfying*

$$\gamma \wedge \mathrm{d}\gamma = 0, \quad \gamma \wedge \mathrm{d}\eta = 0, \quad \dim\left(\ker \gamma \wedge \eta^{n-1}\right) = 1. \tag{1.5.2.1}$$

As for symplectic foliations, a ∂-SF pair (γ, η) induces a ∂-SF structure by

$$\mathcal{F} := \ker \gamma, \quad \omega := \eta|_{\mathcal{F}},$$

in which case we say that (γ, η) represents $(\mathcal{F}, \omega)$. The choice of representing pair is not unique and induces an equivalence relation, analogous to Definition 1.5.4. We say two pairs are equivalent, denoted $(\gamma, \eta) \sim (\tilde{\gamma}, \tilde{\eta})$, if

$$\tilde{\gamma} = f\gamma, \quad \tilde{\eta} = \eta + \alpha \wedge \gamma,$$

for some $\alpha \in \Omega^1(N)$ and $f \in C^\infty(N)$ strictly positive.

Note that the conditions in Equation 1.5.2.1 do not say wether or not η^n is non-zero. In fact, by Lemma 1.7.11 below, both can happen in the same equivalence class. Hence, unlike an SF-pair, in general a ∂-SF pair does not induce a canonical volume form. However, $\gamma \wedge \eta^{n-1}$ can always completed to a volume form by making an extra choice:

Definition 1.5.9. *An* ***admissible form*** *for a* ∂*-symplectic foliation pair* (γ, η)*, is a 1-form* $\beta \in \Omega^1(N)$ *such that*

$$\beta \wedge \gamma \wedge \eta^{n-1} > 0.$$

By the following lemma it makes sense to call β an admissible form for a ∂-SF structure.

Lemma 1.5.10. *Let* (γ, η) *and* $(\tilde{\gamma}, \tilde{\eta})$ *be two* ∂*-SF pairs, representing the same* ∂*-SF structure* $(\mathcal{F}, \omega)$ *on* N*. Then* $\beta \in \Omega^1(N)$ *is admissible for* (γ, η) *if and only if it is admissible for* $(\tilde{\gamma}, \tilde{\eta})$*.*

Proof. Since $(\gamma, \eta) \sim (\tilde{\gamma}, \tilde{\eta})$, there exist $\alpha \in \Omega^1(N)$ and $f \in C^\infty(N)$ strictly positive so that

$$\tilde{\gamma} = f\gamma, \quad \tilde{\eta} = \eta + \alpha \wedge \gamma.$$

Hence, assuming that $\beta \wedge \gamma \wedge \eta^{n-1} > 0$ we have:

$$\beta \wedge \tilde{\gamma} \wedge \tilde{\eta}^{n-1} = f\beta \wedge \gamma \wedge (\eta^{n-1} + (n-1)\eta^{n-2} \wedge \alpha \wedge \gamma) = f\beta \wedge \gamma \wedge \eta^{n-1} > 0.$$

□

As stated above, given a ∂-SF pair (γ, η) representing $(\mathcal{F}, \omega)$ on N^{2n}, the form η^n does not contain any information about $(\mathcal{F}, \omega)$. In fact, up to equivalence, it can equal any top-degree form, as shown in the following lemma.

Lemma 1.5.11. *Let $(\mathcal{F}, \omega)$ be a ∂-SF structure on N^{2n}. Let $\Omega \in \Omega^{2n}(N)$ be a volume form and $f \in C^\infty(N)$ a function. Then there exists a ∂-SF pair (γ, η) representing $(\mathcal{F}, \omega)$ such that*

$$\eta^n = f\Omega.$$

In particular, there exist ∂-SF pairs (γ_i, η_i), $i = 1, 2$, representing $(\mathcal{F}, \omega)$ and such that

$$\eta_1^n = 0, \quad \eta_2^n > 0.$$

Proof. Let (η, γ) be any ∂-SF pair representing $(\mathcal{F}, \omega)$ and β an admissible form. Since, $\beta \wedge \gamma \wedge \eta^{n-1}$ is a positive volume form, there exists a functions $g, h \in C^\infty(N)$, with h strictly positive, so that

$$\eta^n = g\Omega, \quad \beta \wedge \gamma \wedge \eta^{n-1} = h\Omega.$$

Then, (γ, η) is equivalent to

$$\tilde{\gamma} = \gamma, \quad \tilde{\eta} = \eta + \frac{f - g}{hn}\beta \wedge \gamma,$$

and it follows that

$$\tilde{\eta}^n = \eta^n + \frac{f - g}{h}\beta \wedge \gamma \wedge \eta^{n-1} = g\Omega + (f - g)\Omega = f\Omega.$$

□

Admissible forms will be important for understanding special boundaries of strict symplectic foliations, later in Section 1.6.1. For now we consider them as an auxilliary piece of data that is handy for writing down the normal form of transverse symplectic foliations.

1.5.2.1 Statement of the normal form

Let $(M, \mathcal{F}, \omega)$ be a symplectic foliation with transverse boundary, (γ, η) any representing pair and

$$\gamma_\partial := \gamma|_{\partial M}, \quad \eta_\partial := \eta|_{\partial M}$$

the induced ∂-symplectic foliation pair on the boundary. For any choice of admissible form $\beta \in \Omega^1(\partial M)$, consider

$$((-\varepsilon, 0] \times \partial M, \gamma := \gamma_\partial, \eta := \eta_\partial + \mathrm{d}(t\beta)), \tag{1.5.2.2}$$

which defines a symplectic foliation for $\varepsilon > 0$ small enough. We call this the **local model** for $(\mathcal{F}, \omega)$.

Theorem 1.5.12. *Any symplectic foliation with transverse boundary, as in Definition 1.5.6, is isomorphic, as SF-structures, to its local model on a neighborhood of the boundary.*

In particular, up to isomorphism the local model is independent of the choice of admissible form and the representing SF-pair (γ, η). This theorem is an immediate consequence of the more technical statement in Proposition 1.5.14 below.

Remark 1.5.13. Analogous to Remark 1.3.20, the above local model for a symplectic foliation $(\mathcal{F}, \omega)$ can be stated more invariantly as follows. The leaves of the induced foliation $\mathcal{F}_\partial$ on the boundary, are odd-dimensional. Hence the restriction of the 2-form ω to $\mathcal{F}_\partial$ has a 1-dimensional kernel:

$$L := \ker \omega_\partial \subset T\mathcal{F}_\partial.$$

We view L as a subbundle of $T\partial M$, making it into a rank-1 vector bundle $\pi : L \to \partial M$. In particular, since every rank-1 distribution is involutive, it defines a foliation $\mathcal{L}$ with 1-dimensional leaves on ∂M, such that

$$L = T\mathcal{L}.$$

Hence, the dual bundle L^* can be identified with the leafwise cotangent bundle

$$\pi : T^*\mathcal{L} \to \partial M.$$

The total space carries a canonical symplectic foliation defined by:

$$\mathcal{F} := \pi^*\mathcal{F}_\partial, \quad \omega := \pi^*\omega_\partial + \mathrm{d}\lambda_{can},$$

where $\lambda_{can} \in \Omega^1(T^*\mathcal{L})$ denotes the tautological form.

Moreover, L has a canonical orientation, for which $V \in L_p$, $p \in \partial M$ is positive if and only if

$$\omega_p(X, V) > 0,$$

where $X \in T_p\mathcal{F}$ is any outward pointing vector. In particular, $T\mathcal{L}$ and $T^*\mathcal{L}$ are trivializable. A choice of vector field $X \in \mathfrak{X}(M)$ transverse to the boundary corresponds to a trivialization of $T^*\mathcal{L}$, that is, a nowhere vanishing section $\beta \in \Gamma(T^*\mathcal{L})$ defined by

$$\beta(x) := (\iota_X \omega)|_{T_x L}, \quad x \in \partial M.$$

In this trivialization $T^*\mathcal{L} \simeq \mathbb{R} \times \partial M$, the symplectic foliation $(\mathcal{F}, \omega)$ is represented by (γ, η) as in Equation 1.5.2.2. △

The following proposition is the technical version of Theorem 1.5.12.

Proposition 1.5.14. *Suppose $(M^{2n+1}, \mathcal{F}, \omega)$ is a symplectic foliation transverse to the boundary, represented by (γ, η). Let $(\gamma_\partial, \eta_\partial)$ be the induced ∂-SF pair, and $\beta \in \Omega^1(\partial M)$ an admissible form, as in Definition 1.5.9. Then there exists a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which (γ, η) is equivalent to the local model, as in Definition 1.5.4;*

$$(\gamma, \eta) \sim (\gamma_\partial, \eta_\partial + \mathrm{d}(t\beta)).$$

Moreover, if additionally

$$\beta = \iota_X \eta|_\partial,$$

for a vector field $X \in \mathfrak{X}(M)$ transverse to the boundary and tangent to $\mathcal{F}$, then the equivalence can be made adapted to the boundary, as in Equation 1.5.1.1.

Note that, admissible forms β satisfying the addition hypothesis always exist. Moreover, if the SF-pair (γ, η) representing $(\mathcal{F}, \omega)$ is chosen carefully, any admissible form satisfies the additional hypothesis, see Lemma 1.6.6.

Proof of Proposition 1.5.14. Since the foliation is assumed to be transverse to the boundary, we can use Proposition 1.6.14 to find a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which we have:

$$\gamma = f\gamma_\partial,$$

for some $f \in C^\infty(\mathcal{U})$ with $f|_{\partial M} = 1$. Furthermore, in these coordinates we also have

$$\eta = \eta_t + \mathrm{d}t \wedge \nu_t,$$

for some $\eta_t \in \Omega^2(\partial M)$, and $\nu_t \in \Omega^1(\partial M)$. That is, the foliation locally looks like a product foliation and we only need to put the leafwise symplectic structure in normal form. We apply a leafwise Moser argument to obtain the normal form for η. That is, we define the linear path

$$\eta^s := (1-s)\left(\eta_\partial + \mathrm{d}(t\beta)\right) + s\left(\eta_t + \mathrm{d}t \wedge \nu_t\right), \quad s \in [0,1]$$

and look for a time dependent vector field $X_s \in \mathfrak{X}(\mathcal{U})$ in the kernel of γ and such that its flow ϕ_s satisfies

$$\phi_s^*\eta^s = \eta^0, \quad \phi_s(\partial M) = \partial M. \tag{1.5.2.3}$$

By Lemma 1.6.15 the flow of X_s preserves the normal form of the foliation, and by definition ϕ_1 defines a changes of coordinate on $\mathcal{U}$ giving the desired normal form for η.

Differentiating the above equation we see that X_s should satisfy

$$\phi_s^*\left(\mathcal{L}_{X_s}\eta^s + \dot{\eta}^s\right) = 0.$$

We will solve this equation restricted to $\ker\gamma$, where η is closed so it suffices to solve

$$\mathrm{d}\iota_{X_s}\eta^s + \dot{\eta}^s = 0. \tag{1.5.2.4}$$

We claim that $\dot{\eta}^s$ is an exact form, and a primitive is defined by

$$\lambda_t := \int_0^1 t\nu_{st}\,\mathrm{d}s + t\beta.$$

To see this denote by $\bar{\mathrm{d}}$ the deRham differential on ∂M. Then, since η is closed on $\ker\gamma$,

$$0 = \mathrm{d}\left(\eta_t + \mathrm{d}t \wedge \nu_t\right) = \mathrm{d}t \wedge \dot{\eta}_t + \bar{\mathrm{d}}\eta_t + \mathrm{d}t \wedge \bar{\mathrm{d}}\nu_t,$$

implying $\dot{\eta}_t = \bar{\mathrm{d}}\nu_t$. Hence,

$$\begin{aligned}
\mathrm{d}\lambda &= \bar{\mathrm{d}}\left(\int_0^1 t\nu_{st}\,\mathrm{d}s\right) + \mathrm{d}t \wedge \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 t\nu_{st}\,\mathrm{d}s\right) + \mathrm{d}(t\beta)\\
&= \int_0^1 t\bar{\mathrm{d}}\nu_{st}\,\mathrm{d}s + \mathrm{d}t \wedge \left(\int_0^1 \nu_{st}\,\mathrm{d}s + \int_0^1 ts\dot{\nu}_{st}\mathrm{d}s\right) + \mathrm{d}(t\beta)\\
&= \int_0^1 t\dot{\eta}_{st}\,\mathrm{d}s + \mathrm{d}t \wedge \left(\int_0^1 \nu_{st}\,\mathrm{d}s + \int_0^1 s\frac{\mathrm{d}}{\mathrm{d}s}\nu_{st}\,\mathrm{d}s\right) + \mathrm{d}(t\beta)\\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\eta_{st}\,\mathrm{d}s + \mathrm{d}t \wedge \left(\int_0^1 \nu_{st}\,\mathrm{d}s + s\nu_{st}|_{s=0}^{s=1} - \int_0^1 \nu_{st}\,\mathrm{d}s\right) + \mathrm{d}(t\beta)\\
&= \eta_t - \eta_0 + \mathrm{d}t \wedge \nu_t + \mathrm{d}(t\beta) = \dot{\eta}^s,
\end{aligned}$$

proving the claim. As a consequence, Equation 1.5.2.4 further simplifies to

$$\iota_{X_s}\eta^s + \lambda = 0.$$

Because η^s is non-degenerate on $\ker\gamma$ the above equation has a unique solution for X_s. Furthermore, since $\lambda = 0$ at points in the boundary, we have $X_s|_{\partial M} = 0$, so its flow ϕ_s fixes the boundary pointwise.

Now recall that we only solved Equation 1.5.2.4 restricted to $\ker\gamma$ so that

$$\phi_1^*\eta = \tilde{\eta} + \rho \wedge \gamma_\partial, \tag{1.5.2.5}$$

for some $\rho \in \Omega^1(\mathcal{U})$.

Moreover, if we have

$$\beta = \iota_X\eta|_\partial, \tag{1.5.2.6}$$

for a vector field $X \in \mathfrak{X}(M)$ transverse to the boundary and tangent to $\mathcal{F}$, then, in the beginning of the proof, we could have applied Proposition 1.6.14 with this vector field. Thus in our collar neighborhood X can be identified with ∂_t. Using the collar neighborhood, write $\rho = \rho_t + g_t\mathrm{d}t$, $t \in (-\varepsilon, 0]$, so that at points in the boundary Equation 1.5.2.5 becomes

$$\eta_\partial + \mathrm{d}t \wedge \nu_0 = \eta_\partial + \mathrm{d}t \wedge \beta + (\rho_0 + g_0\mathrm{d}t) \wedge \gamma_\partial,$$

so that $\rho_0 = 0$ and $\nu_0 = \beta + g_0\gamma_\partial$. By Equation 1.5.2.6, $\beta = \nu_0$ from which it follows that $g_0 = 0$. □

1.6 Symplectic foliation pairs and their boundaries

Suppose we are given a SF-pair (γ, η), representing a symplectic foliation $(\mathcal{F}, \omega)$, as in Definition 1.5.3. Sometimes we are interested in the SF-pair and not only in the induced foliation. In particular, we want understand its properties on a neighborhood of the boundary ∂M, and be able to glue such manifolds.

The normal form from the previous section describes $(\mathcal{F}, \omega)$ near the boundary in terms of the structure induced on the boundary. However, the representing pair is only recovered up to equivalence, see Proposition 1.5.14, and so does not provide a normal form on the level of SF-pairs. As for contact forms, see Section 1.4.1, we deal with this by restricting to a smaller class of SF-pairs which satisfy a normal form by definition.

Similar to Definition 1.4.1, the class we consider is motivated by looking at the Taylor expansion of the 2-form in a SF-pair. That is, let (γ, η) be a SF-pair, and assume there exists a collar neighborhood $(-\varepsilon, 0] \times \partial M$ on which

$$\gamma = \gamma|_{\partial M}.$$

We interpret this equation as a Taylor polynomial with only the constant term non-zero. Furthermore, using the collar neighborhood we can write

$$\eta = w_t + \mathrm{d}t \wedge v_t,$$

for some $w_t \in \Omega^1(\partial M)$ and $v_t \in \Omega^1(\partial M)$, and where t denotes the coordinate on $(-\varepsilon, 0]$. The Taylor expansion in the t-coordinate equals:

$$\eta = w_0 + t\dot{w}_0 + \mathcal{O}(t^2) + \mathrm{d}t \wedge \left(v_0 + t\dot{v}_0 + \mathcal{O}(t^2)\right),$$

where we use the shorthand notation

$$\dot{w}_0 := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} w_t.$$

In terms of this expansion the condition that (γ, η) is an SF-pair, as in Definition 1.5.3, reads:

$$0 = \gamma \wedge \mathrm{d}\eta = \gamma_\partial \wedge (\mathrm{d}w_0 + t\mathrm{d}\dot{w}_0 - \mathrm{d}t \wedge \mathrm{d}v_0 - t\mathrm{d}t \wedge \mathrm{d}\dot{v}_0) + \mathcal{O}(t^2), \tag{1.6.0.1}$$

and

$$0 < \gamma \wedge \eta^n = n\gamma \wedge (w_0 + t\dot{w}_0)^{n-1} \wedge \mathrm{d}t \wedge (v_0 + t\dot{v}_0) + \mathcal{O}(t^2). \tag{1.6.0.2}$$

The two simplest cases of interest are:

- If the only non-zero terms are w_0 and v_0, then the above conditions are satisfied if and only if
$$v_0 \wedge \gamma_\partial \wedge w_0^{n-1} > 0, \quad \mathrm{d}w_0 = 0, \quad \mathrm{d}v_0 = 0.$$
In this case we have that
$$\eta = w_0 + \mathrm{d}(tv_0).$$
- In the previous case η is globally closed. Requiring that a foliation admits a globally closed 2-form which is leafwise nondegenerate is very restrictive and we also want to consider SF-pairs for which this is not the case. Firstly, note that allowing higher order terms in the Taylor expansion to be non-zero does not change the condition from Equation 1.6.0.2 above, since we only consider it for small t. Secondly, allowing $\dot{v}_0 \neq 0$, does not change any of the conditions

following from Equation 1.6.0.1 since it shows up in the only term containing tdt.
Hence, the next simplest case to consider is $\dot{v}_0 = 0$, but with v_0, w_0 and $\dot{w}_0$ non-zero. In this case, the first equation is satisfied provided that

$$\dot{w}_0 = d\dot{v}_0,$$

so that we have

$$\eta = w_0 + \mathrm{d}(tv_0).$$

Motivated by the above discussion we make the following definition:

Definition 1.6.1. *A SF-pair (γ, η) on M is said to be* ***regular at the boundary*** *if there exists a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which we have:*

$$\gamma = u, \quad \eta = w + \mathrm{d}(tv),$$

for some $u, v \in \Omega^1(\partial M)$ and $w \in \Omega^2(\partial M)$, and where t denotes the coordinate on $(-\varepsilon, 0]$.

As observed above, the conditions that (γ, η) is a SF-pair translates into the following conditions on u, v, and w.

$$u \wedge \mathrm{d}u = 0, \quad u \wedge dw = 0, \quad v \wedge u \wedge (w^{n-1} + t\mathrm{d}v)^{n-1} > 0.$$

Since the non-degeneracy condition is open, it suffices to require it at points in the boundary, where $t = 0$. Then, by choosing ε small enough it holds everywhere. Thus the conditions of (u, v, w) can be packed into the following definition, which does not make reference to a boundary.

Definition 1.6.2. *A* ***∂-symplectic foliation triple*** *(u, v, w) on a manifold N^{2n} consists of forms $u, v \in \Omega^1(N)$ and $w \in \Omega^2(N)$ satisfying:*

$$u \wedge \mathrm{d}u = 0, \quad u \wedge \mathrm{d}w = 0, \quad v \wedge u \wedge w^{n-1} > 0.$$

Thus for any SF-pair (γ, η) on M which is regular at the boundary, we have an **induced ∂-SF triple** (u, v, w) on the boundary ∂M. Note that the above definition puts no condition on w^n, in particular w can be non-degenerate.

Given a ∂-SF triple (u, v, w), observe that (u, w) is a ∂-symplectic foliation pair as in Definition 1.5.7 and v is an admissible form for (u, w), conform Definition 1.5.9. For a ∂-SF pair there are many admissible forms completing it to a ∂-triple.

Lemma 1.6.3. *If (u, w) is a ∂-symplectic foliation pair on N^{2n} then:*

(i) There exists an admissible form v;

(ii) Given a fixed admissible form v, there is a 1-1 correspondence between admissible forms and triples (f, g, X), where $f, g \in C^\infty(N)$ with $g > 0$ and $X \in \mathfrak{X}(N)$ with $X \in \ker u \wedge v$, by sending:

$$(f, g, X) \mapsto fu + gv + \iota_X w.$$

Proof. (i) Since $\dim\ker u\wedge w^{n-1} = 1$, on any oriented coordinate chart $(U, x_1, \ldots, x_{2n})$ we have

$$u \wedge w^{n-1} = \sum_i f_i \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \ldots \mathrm{d}x_{2n},$$

where $f_i \in C^\infty(N)$ are such that at each point at least one of the f_i is non-zero. So, define $v := \sum_i (-1)^i f_i \mathrm{d}x_i$ then

$$v \wedge u \wedge w^{n-1} = \sum_i f_i^2 \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_{2n} > 0.$$

Next, choose an atlas $\mathcal{U} = \{U_j\}_{j\in J}$ on M and $\{\rho_j\}_{j\in J}$ a partition of unity subordinate to it. Construct v_j on each U_j as above. Then,

$$v := \sum_j \rho_j v_j,$$

satisfies $v \wedge u \wedge w^{n-1} > 0$ globally on M.

(ii) For any $fu + gv + \iota_X w$ as above defines an admissible form. Conversely, let $\tilde{v}$ be any admissible form. Observe that v induces a splitting

$$TN = \ker u \wedge v \oplus \langle R_u \rangle \oplus \langle R_v \rangle,$$

where the vector field R_u and R_v are uniquely defined by the equations:

$$u(R_u) = 1, \quad v(R_u) = 0, \quad \iota_{R_u} w|_{\ker v} = 0,$$

and

$$u(R_v) = 0, \quad v(R_v) = 1,, \quad \iota_{R_v} w|_{\ker u} = 0.$$

Define functions

$$f := \tilde{v}(R_u), \quad g = \tilde{v}(R_v).$$

Then, $\tilde{v} - fu - gv$ descends to $\ker u \wedge v$, on which w is non-degenerate. Hence we find a unique $X \in \Gamma(\ker u \wedge v)$ such that

$$\tilde{v} = fu + gv + \iota_X w.$$

□

Given a ∂-SF manifold (N, u, w), we can define the associated local model from Equation 1.5.2.2 for any choice of admissible form v. The model has the property that the admissible form can be recovered from the 2-form η since

$$v = \iota_{\partial t}\eta|_{\partial M}.$$

When (N, u, w) is the boundary of an (not necessarily regular) SF-pair (γ, η) on a manifold M, this property is still true, provided $\ker \eta$ is compatible with the boundary, as we show in Lemma 1.6.6 below. To give the precise statement we first define the Reeb vector field of a SF-pair, analogous to Definition 1.3.10 for contact forms.

Definition 1.6.4. *The* ***Reeb vector field*** *of an SF-pair* (γ, η)*, as in Definition 1.5.3, on* M *is the unique vector field* $R \in \mathfrak{X}(M)$ *satisfying:*

$$\gamma(R) = 1, \quad \iota_R \eta = 0.$$

Example 1.6.5. We compute here the Reeb vector field on the local model $(\varepsilon, 0] \times N$ with

$$\gamma := u, \quad \eta := w + \mathrm{d}(tv),$$

associated to a ∂-triple (u, v, w) on N, as in Equation 1.5.2.2. First, note that the ∂-triple induces a splitting

$$T^*N = \langle u \rangle \oplus \langle v \rangle \oplus (\ker u \cap \ker v)^*.$$

In turn this induces a dual splitting:

$$TN = \langle R_u \rangle \oplus \langle R_v \rangle \oplus (\ker u \cap \ker v),$$

which we use as the definition of the vector fields $R_u, R_v \in \mathfrak{X}(N)$. That is, R_u satisfies

$$u(R_u) = 1, \quad v(R_u) = 0, \quad \beta(R_u) = 0, \quad \forall \beta \in (\ker u \cap \ker v)^*,$$

and similarly for R_v.

The Reeb vector field R of (γ, η) can be explicitly computed in the following cases.

(i) If $w^n = 0$ then $\iota_{R_u} w = 0$, $\iota_{R_v} w = 0$, and the Reeb vector field is given by

$$R = R_u + X_t + f_t \partial_t,$$

where $X_t \in \ker u \cap \ker v$ is uniquely defined by

$$\iota_{X_t}(w + t\mathrm{d}v) = t\, \iota_{R_u} \mathrm{d}v, \qquad \text{on } \ker u \cap \ker v,$$

and

$$f_t := t\mathrm{d}v(R_v, R_u + X_t).$$

In particular, at points in the boundary $R = R_u$, which is tangent to the boundary as expected.

(ii) If $w^n > 0$ then

$$R = \frac{1}{f}\left(X_t - \partial_t\right),$$

where $f \in \mathbb{C}^\infty(N)$ and $X_t \in \mathfrak{X}(N)$ are uniquely defined by

$$\iota_{X_t}(w + t\mathrm{d}v) = v, \quad f := u(X_v).$$

To see that R is well-defined note that $w + t\mathrm{d}v$ is non-degenerate for t small enough. Furthermore, assume $u(X_v) = 0$ at some point, then

$$\iota_{X_v}\left(v \wedge u \wedge w^{n-1}\right) = 0, \tag{1.6.0.3}$$

contradicting that $v \wedge u \wedge w^{n-1} > 0$.

△

Continuing the discussion from before Definition 1.6.4, since R spans the kernel of η, the compatibility condition mentioned before requires the Reeb vector field to be everywhere tangent or everywhere transverse to ∂M. Note that this is equivalent to $w^n = 0$ or $w^n > 0$ everywhere on the boundary. In this case, the following lemma shows that any admissible form is obtained by contracting η with a vector field transverse to the boundary. Recall that this property allowed us to obtain the normal form up to equivalence adapted to the boundary, as stated in Proposition 1.5.14.

Lemma 1.6.6. *Let (γ, η) be a symplectic foliation pair on a manifold M^{2n+1}, and $(\partial M, u := \gamma_\partial, w := \eta_\partial)$ the induced ∂-pair. Then, for any vector field $X \in \mathfrak{X}(M)$ satisfying $X \in \ker\gamma$ and transverse to the boundary,*

$$v := \iota_X \eta|_{\partial M},$$

is an admissible form for (u, w).

Conversely, assuming that the Reeb vector field R is everywhere transverse or tangent to ∂M. Then, for any admissible form v, there exists a vector field $X \in \mathfrak{X}(M)$ such that

$$X \in \ker u, \quad v = (\iota_X \eta)|_{\partial M}, \quad X \pitchfork \partial M.$$

Proof. For the first implication note that since $\gamma \wedge \eta^n > 0$ we have

$$0 < \iota_X(\gamma \wedge \eta^n)|_{\partial M} = nv \wedge u \wedge w^{n-1},$$

proving that v is admissible.

For the converse, first assume R is everywhere tangent to ∂M. As remarked in Example 1.6.5, this implies that $R = R_u$, at the boundary. Extend v to a form on a neighborhood of the boundary. Then, $v - v(R)u$ descends to $\ker u$ on which η is non-degenerate. Hence, we find a unique $X \in \ker u$ such that

$$v = v(R)u + \iota_X \eta.$$

We claim that X points outwards along the boundary. To see this, assume that X is tangent to the boundary, then the above equation can be restricted to and we obtain

$$v = v(R_u)u + \iota_X w.$$

Evaluating on R_v, gives $1 = w(X, R_v)$ which is a contradiction, since as we have seen in Example 1.6.5 that $w^n = 0$ implies $\iota_{R_v} w = 0$. Hence, X is transverse to the boundary. To see it points outwards observe that

$$v \wedge u \wedge \eta^{n-1} = \frac{1}{n}\iota_X(\gamma \wedge \eta^n)|_{\partial M}. \tag{1.6.0.4}$$

Since $v \wedge u \wedge w^{n-1}$ and $\gamma \wedge \eta^n$ are positive volume forms (on ∂M and M respectively) it follows that X is pointing outwards.

Secondly, assume that R is everywhere transverse to ∂M. Then, $w^n > 0$ and there exists a unique $Y \in \mathfrak{X}(\partial M)$ such that

$$\iota_Y w = v.$$

Extend Y to a vector field on a neighborhood of the boundary and define $f := \gamma(Y)$. Then

$$X := Y - fR,$$

satisfies $\gamma(X) = 0$, and $\iota_X \eta|_{\partial M} = v$. Again, we claim that X is pointing outwards along the boundary. If we assume that X is tangent to the boundary then

$$0 < v \wedge u \wedge w^{n-1} = \frac{1}{n}\iota_X(u \wedge w^n),$$

which is a contradiction since $u \wedge w^n = 0$. Hence X transverse to the boundary and the same argument as in Equation 1.6.0.4 shows its pointing outwards.

□

We finish this section with another property of the Reeb vector field that will be useful later. Recall that for a contact form $\alpha \in \Omega^1(M)$, the flow of the Reeb vector field preserves α. That is,

$$\mathcal{L}_R\alpha = 0, \quad \text{and} \quad \mathcal{L}_R \mathrm{d}\alpha = 0.$$

For a SF-pair a similar phenomenon happens if and only if (γ, η) is a cosymplectic structure:

Lemma 1.6.7. *Let (γ, η) be a symplectic foliation pair on M, and R the associated Reeb vector field. Then,*

(i) $\mathcal{L}_R\gamma = 0$ if and only if $\mathrm{d}\gamma = 0$;

(ii) $\mathcal{L}_R\eta = 0$ if and only if $\mathrm{d}\eta = 0$.

Proof. We prove the second statement; the proof of the first one is analogous. The condition $\gamma \wedge \mathrm{d}\eta = 0$ is equivalent to

$$\mathrm{d}\eta = \mu \wedge \gamma,$$

where $\mu \in \Omega^2(M)$. By taking $\tilde{\mu} := \mu + (\iota_R\mu) \wedge \gamma$ we can assume that $\iota_R\mu = 0$. Since $\iota_R\eta = 0$ we thus find

$$\mathcal{L}_R\eta = \iota_R \mathrm{d}\eta = \mu,$$

which vanishes if and only if η is closed (since $\iota_R\mu = 0$). □

1.6.1 Special boundaries of symplectic foliation pairs

In many cases, a symplectic foliation pair has a boundary which is even more special than being regular. In these cases, the admissible forms for the induced ∂-symplectic foliation pair have extra properties. The following discussion is analogous to that in Section 1.4.2 for the contact case.

Recall from Definition 1.6.2, that a ∂-SF triple (u, v, w) on a manifold N^{2n} consists of differential forms $u, v \in \Omega^1(N)$, and $w \in \Omega^2(N)$ satisfying:

$$u \wedge \mathrm{d}u = 0, \quad u \wedge \mathrm{d}w = 0, \quad v \wedge u \wedge w^{n-1} > 0.$$

Definition 1.6.8. *A ∂-SF triple (u, v, w) is said to be of:*

- ***Liouville type*** *if*
$$w = \mathrm{d}v;$$
- ***Unimodular type*** *if*
$$\mathrm{d}v = 0.$$
- ***Cosymplectic type*** *if*
$$\mathrm{d}u = 0, \quad \mathrm{d}v = 0, \quad \mathrm{d}w = 0.$$
- ***Tameable*** *if*
$$\mathrm{d}u = 0, \quad , \quad w^n \geqslant 0, \quad \mathrm{d}w + u \wedge \mathrm{d}v = 0.$$

Definition 1.6.9. *We say that a symplectic foliation pair (γ, η) on M, has boundary of **right $\mathcal{S}$-type** (resp. **left $\mathcal{S}$-type**), for $\mathcal{S}$ in the above list, if in some collar neighborhood $U \simeq (-\varepsilon, 0] \times \partial M$ (resp. $U \simeq [0, \varepsilon) \times \partial M$) we have*

$$\gamma = u, \quad \eta = w + \mathrm{d}(tv).$$

where (u, v, w) is a ∂-symplectic foliation triple of $\mathcal{S}$-type.

The left and right versions of each type only differ in the orientations induced on the boundary. In line with our conventions, see Section 1.1.1, the boundary of a manifold with the standard orientation is always a right boundary. However, these names are particularly useful when considering cobordisms, where we can think of these models as the left or right side of a cobordism as in Section 1.6.2.

The following two examples are the SF-analogue of Example 1.4.9 for contact structures.

Example 1.6.10. Let (Σ, ω) be a symplectic manifold. Then the product $\Sigma \times \mathbb{S}^1$ admits a symplectic foliation pair

$$\gamma = \mathrm{d}\theta, \quad \eta = \omega.$$

In fact since both forms are closed they define a cosymplectic structure.

If the symplectic manifold (Σ, ω) has cosymplectic type boundary $(\partial\Sigma, \omega_\partial, \beta)$, in the sense of Definition 1.2.10, then the SF-pair (γ, η) has regular boundary of cosymplectic type. To see this note that there exists a collar neighborhood $(-\varepsilon, 0] \times \partial\Sigma \subset \Sigma$ on which

$$\omega = \omega_\partial + \mathrm{d}t \wedge \beta.$$

In turn, this provides a collar $(-\varepsilon, 0] \times B \times \mathbb{S}^1 \subset \Sigma \times \mathbb{S}^1$ on which

$$\gamma = \mathrm{d}\theta, \quad \eta = \omega_\partial + \mathrm{d}t \wedge \beta.$$

Similarly, one sees that if (Σ, ω) has contact boundary then the symplectic foliation has boundary of Liouville type. △

Example 1.6.11. Let (γ_B, η_B) be a symplectic foliation pair on a closed manifold B. Then the product $B \times \mathbb{D}^2_\delta$, where $\mathbb{D}^2_\delta$ denotes the disk of radius δ, admits a symplectic foliation pair

$$\gamma := \gamma_B, \quad \eta := \eta_B + \mathrm{d}\left(r^2 \mathrm{d}\theta\right).$$

Reparametrizing the r-coordinate we obtain a collar neighborhood $(-\varepsilon, 0] \times \overline{B \times \mathbb{S}^1}$ on which

$$\gamma = \gamma_B, \quad \eta = \eta_B + \mathrm{d}(t\delta \mathrm{d}\theta).$$

The the induced ∂-symplectic foliation boundary equals

$$\left(\overline{B \times \mathbb{S}^1}, u := \gamma_B, v = \delta \mathrm{d}\theta, w = \eta_B\right),$$

which is of Unimodular type. △

The special boundary types above can be phrased in terms of vector fields. If (γ, η) has regular boundary of Liouville type or cosymplectic type, then η is a closed on a neighborhood of the boundary, which by Lemma 1.6.7 is equivalent to $\mathcal{L}_R\eta = 0$.

Lemma 1.6.12. *Let (γ, η) be a symplectic foliation on a manifold with boundary M. Then the boundary is of:*

(i) Liouville type if and only if $\mathcal{L}_R\eta = 0$ and there exists a vector field $X \in \mathfrak{X}(M)$ transverse to the boundary and satisfying

$$\mathcal{L}_X\gamma = 0, \quad \mathcal{L}_X\eta = \eta,$$

on a neighborhood of the boundary.

(ii) Unimodular type if and only if there exists a vector field $X \in \mathfrak{X}(M)$, transverse to the boundary and satisfying

$$\mathcal{L}_X\gamma = 0, \quad \mathcal{L}_X\eta = 0,$$

on a neighborhood of the boundary.

Proof. In both, cases use X to define a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which we identify $X = \partial_t$ and

$$\gamma = u_t, \quad \eta = w_t + \mathrm{d}t \wedge v_t,$$

for $f_t \in \mathbb{C}^\infty(\partial M)$, $u_t, v_t \in \Omega^1(\partial M)$ and $w_t \in \Omega^2(\partial M)$. Then, the condition $\mathcal{L}_X \gamma = 0$ translates into $\dot{u}_t = 0$, so that $\gamma = u$.

(i) If $\mathcal{L}_X \eta = \eta$ then $\dot{w}_t = w_t$ and $\dot{v}_t = v_t$, so

$$w_t = e^t w, \quad v_t = e^t v.$$

Together with $\mathrm{d}\eta = 0$, this implies

$$w_t = \dot{w}_t = \mathrm{d}v_t = e^t \mathrm{d}v.$$

Change coordinates $s = e^t - 1$ around $t = 0$ gives

$$\eta = \mathrm{d}\left((1+s)v\right) = \mathrm{d}v + \mathrm{d}(sv),$$

so (γ, η) is of Liouville type.

(ii) If $\mathcal{L}_X \eta = 0$ then $\dot{w}_t = 0$ and $\dot{v}_t = 0$, so that

$$\eta = w + \mathrm{d}t \wedge v.$$

Using Cartan's formula for the Lie derivative we see

$$\mathcal{L}_X \eta = 2\mathrm{d}v = 0,$$

meaning (γ, η) is of unimodular type.

□

Example 1.6.13. The construction from Example 1.4.11 and Example 1.2.13, also applies to symplectic foliations. Let $(M, \mathcal{F}, \omega)$ be a SF-manifold, and $(B, \mathcal{F}_B, \omega_B)$ a codimension-2 SF-submanifold, by which we mean that $B \pitchfork \mathcal{F}$ and

$$\mathcal{F}_B := \mathcal{F} \cap TB, \quad \omega_B := \omega|_{\mathcal{F}_B},$$

defines a SF-structure on B. As before, the ω-orthogonal defines a model for the normal bundle

$$\nu := \mathcal{F}_B^\omega \subset \mathcal{F},$$

and the restriction $\omega_\nu := \omega|_\nu$, makes it into a symplectic vector bundle.

Let $\pi : P \to B$ be the principal $\mathbb{S}^1$-bundle associated to the symplectic normal bundle (ν, ω_ν), and let $\theta \in \Omega^1(P)$ be a connection form satisfying $\mathrm{d}\theta = \pi^*\sigma$ where $[\sigma] \in H^2(B; \mathbb{R})$ is the Chern class of the symplectic normal bundle. Fix a SF-pair (γ, η) representing $(\mathcal{F}, \omega)$ and consider

$$\Gamma := \pi^*(\gamma_B), \quad \Omega := \pi^*(\eta_B) + \mathrm{d}\left(r^2(\mathrm{d}\phi + \theta)\right)$$

on $P \times \mathbb{C}$.

Using arguments similar to that of Example 1.2.13 (including a normal form as in (iii) there) is possible to show that these forms induce a SF-manifold

$$\left(P \times_{\mathbb{S}^1} \mathbb{C}, \widetilde{\Gamma}, \widetilde{\Omega}\right), \tag{1.6.1.1}$$

isomorphic as SF-manifolds to a neighborhood of $(B, \mathcal{F}_B, \omega_B) \subset (M, \mathcal{F}, \omega)$.

The boundary of this neighborhood is isomorphic to P, and the induced ∂-symplectic folation pair (Definition 1.5.8) equals:

$$\left(P, \widetilde{\Gamma}_\partial = \pi^*(\gamma_B), \widetilde{\Omega}_\partial = \pi^*(\omega_B + \varepsilon^2\sigma\right).$$

Thus, θ is an admissible form and using Lemma 1.6.12 we conclude that the neighborhood has regular boundary of:

- Unimodular type, if $\sigma = 0$, which happens if the Chern class of (ν, ω_ν) vanishes. Furthermore, if additionally $\mathcal{F}_B$ is unimodular, so that we can assume $\mathrm{d}\gamma_B = 0$, the boundary is of cosymplectic type;
- Liouville type if $\sigma = \eta_B$. This happens if ω_B admits a closed extension representing the Chern class of (ν, ω_ν);

△

1.6.2 Gluing and cobordisms of symplectic foliations

Gluing manifolds with symplectic foliations along their boundaries in particular entails gluing the underlying foliations. Therefore, we start this section by recalling how to glue foliated manifolds with boundary. We will consider two types of foliations, those which are everywhere transverse to the boundary, and those for which the boundary is a leaf. Next, we adapt the above story to the case of symplectic foliations. Again we consider two types of boundaries, transverse and tangent, and give gluing constructions for each of them.

1.6.2.1 Gluing foliated manifolds

Recall from Section 1.2.4 that to glue manifolds with geometric structures, one usually needs a normal form on a collar neighborhood around the boundary. These collar neighborhoods can then be matched, as in the smooth case above, and the normal form ensures that the structures glue.

1.6.2.2 Foliations transverse to the boundary

As stated above, the key ingredient in gluing manifolds with extra structure is the existence of a normal form around the boundary. Let $(M, \mathcal{F})$ be a foliated manifold such that $\mathcal{F}$ is everywhere transverse to the boundary. Then the intersection

$$\mathcal{F}_\partial := \mathcal{F} \cap T(\partial M),$$

is a foliation on the boundary ∂M. The normal form states that locally around the boundary $(M, \mathcal{F})$ looks like the product foliation

$$((-\varepsilon, 0] \times \partial M, (-\varepsilon, 0] \times \mathcal{F}_\partial)\,.$$

More precisely, we have:

Proposition 1.6.14. *Let $\gamma \in \Omega^1(M)$ defines a foliation on M transverse to the boundary ∂M. If $X \in \mathfrak{X}(M)$ is a vector field satisfying $X \in \ker\gamma$ and $X \pitchfork \partial M$, then, there exists a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which*

$$\gamma = f\gamma_\partial, \quad X = \partial_s,$$

for a positive function $f \in C^\infty(\mathcal{U})$ satisfying $f|_{\partial M} = 1$.

If we define a collar neighborhood using the flow of X, then the proof follows from applying the following lemma.

Lemma 1.6.15. *Let $\gamma \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$ a vector field with flow ϕ_t, $t \in [0, 1]$. Then the following are equivalent:*

(i) There exists strictly positive functions $f_t \in C^\infty(M)$, $f_0 = 1$, satisfying:

$$\phi_t^*\gamma = f_t\gamma;$$

(ii) There exists a function $g \in C^\infty(M)$ such that

$$\mathcal{L}_X\gamma = g\gamma.$$

If the above is satisfied then $g = \dot{f}_0$.

Proof. Differentiating the first equation at time $t = 0$, gives the second equation with $g = \dot{f}_0$. Moreover,

$$\dot{f}_t\gamma = \frac{\mathrm{d}}{\mathrm{d}t}f_t\gamma = \frac{\mathrm{d}}{\mathrm{d}t}\phi_t^*\gamma = \phi_t^*(\mathcal{L}_X\gamma) = \phi_t^*(g\gamma) = (g \circ \phi_t)f_t\gamma.$$

This defines a differential equation

$$\dot{f}_t = (g \circ \phi_t)f_t, \quad f_0 = 1$$

whose solution is given by

$$f_t = e^{\int_0^t (g\circ\phi_s)f_s\,\mathrm{d}s}.$$

Hence, provided the second equation holds, f_t can be recovered from g and ϕ_t alone, proving the first condition holds. □

Let $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ be foliated manifolds with transverse boundary, and assume there exists an orientation reversing isomorphism of foliated manifolds

$$\phi : (\partial M_1, \mathcal{F}_{1,\partial}) \to (\partial M_2, \mathcal{F}_{2,\partial}).$$

Using the above proposition we find collar neighborhoods

$$k_1 : ((-\varepsilon, 0] \times \partial M_1, (-\varepsilon, 0] \times \mathcal{F}_{1,\partial}) \to (M_1, \mathcal{F}_1),$$

and

$$k_2 : \big([0, \varepsilon) \times \overline{\partial M}_2, [0, \varepsilon) \times \mathcal{F}_{2,\partial}\big) \to (M_2, \mathcal{F}_2).$$

Together these two collar neighborhoods define a map

$$k_1 \cup_\phi k_2 : (-\varepsilon, \varepsilon) \times \partial M \to M_1 \cup_\phi M_2,$$

as in Equation 1.2.4.3 and we obtain:

Lemma 1.6.16. *The space $M_1 \cup_\phi M_2$ admits a foliation, denoted by $\mathcal{F}_1 \cup_\phi \mathcal{F}_2$, which is the unique foliation with the properties that its restriction to M_i equals $\mathcal{F}_i$, and the map*

$$k_1 \cup_\phi k_2 : (-\varepsilon, \varepsilon) \times \partial M \to M_1 \cup_\phi M_2,$$

is a foliated embedding. The resulting structure (of a foliated smooth manifold) depends on k_1, k_2 and ϕ, but its isomorphism class does not.

1.6.2.3 Foliations tangent to the boundary

Unlike the case of transverse boundaries, the condition of having the boundary as a leaf does not control the behavior of the foliation in the direction normal to the boundary. As a consequence there does not exist a normal form depending only on the induced structure on the boundary. Instead we have to impose an extra condition.

Let $(M, \mathcal{F})$ be a foliated manifold whose boundary is a leaf. For any collar neighborhood $k : (-\varepsilon, 0] \times \partial M \to M$, we define the foliated manifold

$$\big(M_{k,\infty} := M \cup_{\partial M} [0, \infty) \times \partial M,\ \mathcal{F}_{k,\infty} := \mathcal{F} \cup_{\partial M} \mathcal{F}_{[0,\infty)}\big)\,, \tag{1.6.2.1}$$

where we glue using k, and the foliation on the semi-infinite cylinder is defined by

$$\mathcal{F}_{[0,\infty)} := \bigcup_{t \in [0,\infty)} \{t\} \times \partial M.$$

Although the manifold M_∞ is always smooth, $\mathcal{F}_\infty$ is in general only continuous at the hypersurface $\partial M \subset M_\infty$. The collar neighborhood is said to be **adapted** if $(M_{k,\infty}, \mathcal{F}_{k,\infty})$ defines a smooth extension of $(M, \mathcal{F})$.

Definition 1.6.17. *A foliation $\mathcal{F}$ on M is said to be* ***tame at the boundary*** *if there exists an adapted collar neighborhood k as above.*

Remark 1.6.18. To check the tameness condition in practice, write $\mathcal{F} = \ker\theta$ for $\theta \in \Omega^1(M)$, choose a collar neighborhood $(-\varepsilon, 0] \times \partial M$, and write

$$\theta = \theta_t + f_t \mathrm{d}t, \tag{1.6.2.2}$$

for some $\theta_t \in \Omega^1(\partial M)$ and $f_t \in C^\infty(\partial M)$. Rescaling θ we can assume $f_t = 1$, so that $\mathcal{F}$ is tame if and only if θ_t vanishes up to infinite order at the boundary. △

The gluing construction follows the same pattern as befores. Let $\mathcal{F}_i$ be a foliation tangent to the boundary on M_i, $i = 1, 2$, and assume there exists an orientation reversing diffeomorphism

$$\phi : \partial M_1 \to \partial M_2.$$

Choose adapted collar neighborhoods

$$k_1 : (-\varepsilon, 0] \times \partial M_1 \to M_1, \quad k_2 : [0, \varepsilon) \times \partial M_2 \to M_2.$$

The tameness condition ensures that, putting these collars together, the foliations glue smoothly.

Lemma 1.6.19. *Let $\mathcal{F}_i$ be a foliation tame at the boundary on M_i, $i = 1, 2$. Then $M_1 \cup_\phi M_2$ admits a unique foliation, denoted by $\mathcal{F}_1 \cup_\phi \mathcal{F}_2$, whose restriction to M_i equals $\mathcal{F}_i$, and such that*

$$k_1 \cup_\phi k_2 : (-\varepsilon, \varepsilon) \times \partial M_1 \to M_1 \cup_\phi M_2,$$

is a (foliated) embedding. The resulting structure depends on k_1, k_2 and ϕ, but its isomorphism class does not.

1.6.2.4 Gluing symplectic foliations tangent to the boundary

To state the analogue of Lemma 1.6.19 above for symplectic foliations, we need to impose the following condition:

Definition 1.6.20. *Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M whose boundary ∂M is a leaf. Choose a collar neighborhood of the boundary $k : (-\varepsilon, 0] \times \partial M \to M$, and use it to define the manifold*

$$M_\infty := M \cup_{\partial M} [0, \infty) \times \partial M.$$

On $[0, \infty) \times \partial M$ define an extension of $(\mathcal{F}, \omega)$ by

$$\mathcal{F}_\infty := \bigcup_{t \in [0,\infty)} \{t\} \times \partial M, \quad \omega_\infty := \omega_\partial,$$

where $\omega_\partial := \omega|_{\partial M}$. If this extension is smooth we say that the collar neighborhood is adapted. The symplectic foliation $(\mathcal{F}, \omega)$ is said to be ***tame at the boundary*** *if there exists an adapted collar.*

Symplectic foliations with tame boundaries can be glued essentially by definition. Indeed, choose collar neighborhoods as in Definition 1.6.20, then the tameness condition implies that the foliations on each collar neighborhood together with their leafwise symplectic form match smoothly along the boundary leaf. Thus we have:

Proposition 1.6.21. *Let $(\mathcal{F}_i, \omega_i)$ be a symplectic foliation on M_i, $i = 1, 2$, tame at the boundary, and denote $\omega_{i,\partial} := \omega_i|_{\partial M_i}$. Assume there exists an orientation reversing diffeomorphism $\phi : \partial M_1 \to \partial M_2$, such that*

$$\phi^* \omega_{2,\partial} = \omega_{1,\partial}.$$

Then, there exists a symplectic foliation $(\mathcal{F}, \omega)$ on

$$M_1 \cup_\phi M_2 := (M_1 \sqcup M_2)/x \sim \phi(x), \quad x \in \partial M_1,$$

which restricts to $(\mathcal{F}_i, \omega_i)$ on M_i.

1.6.2.5 Gluing symplectic foliations transverse to the boundary

Let $(\mathcal{F}, \omega)$ be a symplectic foliation on a manifold with boundary M. Recall that if the foliation is transverse to the boundary, it inherits a ∂-symplectic foliation $(\mathcal{F}_\partial, \omega_\partial)$ as in Definition 1.5.7.

Proposition 1.6.22. *Let $(\mathcal{F}_i, \omega_i)$ be a symplectic foliation transverse to the boundary on M_i, $i = 1, 2$. Assume there exists a orientation reversing diffeomorphism $\phi : \partial M_1 \to \partial M_2$, such that*

$$(\mathcal{F}_{1,\partial}, \omega_{1,\partial}) = (\phi^*(\mathcal{F}_{2,\partial}), \phi^*(\omega_{2,\partial})).$$

Then the manifold

$$M_1 \cup_\phi M_2 := (M_1 \sqcup M_2)/x \sim \phi(x), \quad x \in \partial M_1,$$

admits a symplectic foliation $(\mathcal{F}, \omega)$ that restricts to $(\mathcal{F}_i, \omega_i)$ on M_i.

Proof. The proof follows the same pattern as that of Lemma 1.6.16, but now using the normal form for symplectic foliations from Theorem 1.5.12. Let $(\gamma_\partial, \eta_\partial)$ be a ∂-symplectic foliation pair representing $(\mathcal{F}_{1,\partial}, \omega_{1,\partial})$ and $\beta \in \Omega^1(\partial M)$ any admissible form. Using ϕ to identify ∂M_1 and ∂M_2, we see that $(-\beta, \gamma_\partial, \eta_\partial)$ is a representing ∂-SF-triple for $(\mathcal{F}_{2,\partial}, \omega_{2,\partial})$.

Applying Theorem 1.5.12 we find a collar neighborhood on which $(\mathcal{F}_1, \omega_1)$ looks like its local model

$$(-\varepsilon, 0] \times \partial M_1, \quad \mathcal{F}_1 = \ker \gamma_\partial, \quad \omega_1 = \eta_\partial + \mathrm{d}(t\beta).$$

Similarly, again using ϕ to identify ∂M_1 and $\overline{\partial M_2}$, we find a collar neighborhood in M_2, on which $(\mathcal{F}_2, \omega_2)$ looks like its local model

$$(-\varepsilon, 0] \times \overline{\partial M_1}, \quad \mathcal{F}_2 = \ker \gamma_\partial, \quad \omega_2 = \eta_\partial - \mathrm{d}(t\beta),$$

which is isomorphic to

$$[0,\varepsilon)\times\partial M_1,\quad \mathcal{F}_2=\ker\gamma_\partial,\quad \omega_2=\eta_\partial+\mathrm{d}(t\beta).$$

Putting these neighborhoods together we obtain a symplectic foliation,

$$(-\varepsilon,\varepsilon)\times\partial M_1,\quad \mathcal{F}:=\ker\gamma_\partial,\quad \omega=\eta_\partial+\mathrm{d}(t\beta).$$

By construction $(\mathcal{F},\omega)$ satisfies the required properties. □

Example 1.6.23. The Gompf connected sum construction from Example 1.2.22 also works for SF-manifolds. Let $(M_i,\mathcal{F}_i,\omega_i)$, $i=1,2$, be SF-manifolds with codimension-2 SF-submanifolds $(B_i,\mathcal{F}_{B_i},\omega_{B_i})$, as in Example 1.6.13. Suppose there exists an orientation preserving diffeomorphism $\phi: B_1\to B_2$ satisfying:

(i) $\phi_*(\mathcal{F}_{B_1},\omega_{B_1})=(\mathcal{F}_{B_2},\omega_{B_2})$;

(ii) $\phi^*c_1(\nu_{B_2})=-c_1(\nu_{B_1})\in H^2(B_1)$,

where $c_1(\nu_{B_i})$ denotes the Chern class of the symplectic normal bundle as in Example 1.6.13. Recall that a neighborhood of B_i is isomorphic to

$$\left(P_i\times_{\mathbb{S}^1}\mathbb{C},\widetilde{\Gamma}_i,\widetilde{\Omega}_i\right),$$

as in Equation 1.6.1.1 By the conditions above, the induced ∂-SF structures satisfy the hypothesis of Proposition 1.6.22, so that the complement of these neighborhoods can be glued along their boundary. Hence, the **Gompf connected sum**

$$(M_1,B_1)\#(M_2,B_2):=\left(M_1\backslash P_1\times_{\mathbb{S}^1}\mathbb{D}^2\right)\cup_\psi\left(M_2\backslash P_2\times_{\mathbb{S}^1}\mathbb{D}^2\right),$$

where $\psi: P_1\to P_2$ is induced by ϕ, admits a symplectic foliation which restricts to $(\mathcal{F}_i,\omega_i)$ on each of the pieces. △

As in Section 1.4.3, it is convenient to phrase gluing of SF-manifolds in terms of cobordisms. This automatically takes care of the induced orientations, and gives us more freedom to change the symplectic foliations to make them match.

Definition 1.6.24. *Let $(N_i,\mathcal{F}_i,\omega_i)$, $i=1,2$, be ∂-SF manifolds as in Definition 1.5.7. A **SF-cobordism** $(N_1,\mathcal{F}_1,\omega_1)\prec_{(M,\mathcal{F},\omega)}(N_2,\mathcal{F}_2,\omega_2)$ is an SF-manifold $(M,\mathcal{F},\omega)$ with*

$$\partial M=\overline{N_1}\sqcup N_2,$$

and inducing $(\mathcal{F}_i,\omega_i)$ on the boundary.

In particular, the foliation $(\mathcal{F},\omega)$ on the cobordism M, is transverse to the boundary. Thus, Proposition 1.6.22 says that SF-cobordisms can be composed. Analogous to Example 1.4.17, any SF-manifold $(M,\mathcal{F},\omega)$ can be interpreted as a cobordism

$$\varnothing\prec_{(M,\mathcal{F},\omega)}(\partial M,\mathcal{F}_\partial:=\mathcal{F}\cap T\partial M,\omega_\partial:=\omega|_{\mathcal{F}_\partial}).$$

Furthermore, gluing two SF-manifolds $(M_i, \mathcal{F}_i, \omega_i)$, $i = 1, 2$, using Proposition 1.6.22, is equivalent to a composition of cobordisms:

$$\varnothing \prec_{(M_1,\mathcal{F}_i,\omega_1)} (\partial M_1, \mathcal{F}_{1,\partial}, \omega_{1,\partial}) \prec_{(M_2,\mathcal{F}_2,\omega_2)} \varnothing.$$

Note that here we implicitly use that there exists an isomorphism of ∂-SF manifold $\phi : (\partial M_1, \mathcal{F}_{1,\partial}, \omega_{1,\partial}) \to (\partial M_2, \mathcal{F}_{2,\partial}, \omega_{2,\partial})$, see Remark 1.4.16.

We also consider the following (stronger) type of cobordism for ∂-SF triples:

Definition 1.6.25. *Let (u_i, v_i, w_i), $i = 1, 2$, be ∂-SF triples on a manifold N_i as in Definition 1.6.2. A* ***regular SF-cobordism*** $(N_1, u_1, v_1, w_1) \prec_{(M,\gamma,\eta)} (N_2, u_2, v_2, w_2)$, *is a manifold M endowed with a SF-pair (γ, η) satisfying*

$$\partial M = \overline{N_1} \sqcup N_2,$$

and, in the notation of Definition 1.6.9, (γ, η) has:

(i) Regular left boundary N_1, for the ∂-SF triple (u_1, v_1, w_1);

(ii) Regular right boundary N_2, for the ∂-SF triple (u_2, v_2, w_2).

It is not always possible to glue two SF-manifolds $(M_i, \mathcal{F}_i, \omega_i)$, $i = 1, 2$, to each other directly, or equivalently, to compose them as cobordisms. In this case, we can use an intermediate cobordism $(M, \mathcal{F}, \omega)$ to interpolate between the ∂-SF structures on ∂M_i, and form the composition:

$$\varnothing \prec_{(M_1,\mathcal{F}_1,\omega_1)} (\partial M_1, \mathcal{F}_{1,\partial}, \omega_{1,\partial}) \prec_{(M,\mathcal{F},\omega)} (\partial M_2, \mathcal{F}_{2,\partial}, \omega_{2,\partial}) \prec \varnothing.$$

Often the cobordism is topologically just the trivial cobordism $[0, 1] \times \partial M$, in which case the manifold obtained in the above decomposition is diffeomorphic to the gluing of M_1 and M_2.

Following this strategy, we give below another type of gluing for symplectic foliations transverse to the boundary, which is often more useful than Proposition 1.6.22. The reason for this is that, although the previous result works in general, its main downside is that the diffeomorphism used to identify the boundaries needs to be an isomorphism of ∂-symplectic foliations. This poses a problem in practical situations, as usually there is no freedom in choosing the gluing diffeomorphism ϕ. The construction below imposes more conditions on the symplectic foliation, but as a tradeof weakens the requirements on ϕ.

The symplectic version of the turbulization construction, discussed in the next section, allows such foliations to be changed close to the boundary so that they become tame the boundary and can be glued. This gives a way of gluing manifolds with symplectic foliations transverse to the boundary:

Theorem 1.6.26. *Let $(\mathcal{F}_i, \omega_i)$, $i = 1, 2$, be SF-structures on M_i, transverse to the boundary, and denote by $(\mathcal{F}_{\partial_i}, \omega_{\partial,i})$ the induced ∂-SF structures on the boundary, conform Definition 1.5.7. Assume that $\mathcal{F}_{\partial,i}$ is unimodular, and that there exists:*

(i) An orientation reversing diffeomorphism $\phi : \partial M_1 \to \partial M_2$;

(ii) Symplectic extensions $\widetilde{\omega}_i$ *of* $\omega_{\partial,i}$ *on* ∂M_i*;*

(iii) A family of symplectic forms ω_t*,* $t \in [0,1]$ *on* ∂M_1 *satisfying:*

$$\omega_0 = \widetilde{\omega}_0, \quad \omega_1 = \phi^*(\widetilde{\omega}_1).$$

Then, there exists a symplectic foliation $(\mathcal{F}, \omega)$ *on*

$$M_1 \cup_\phi M_2 := M_1 \sqcup M_2 / x \sim \phi(x), \quad x \in \partial M_1,$$

whose restriction to M_i *agrees with* $(\mathcal{F}_i, \omega_i)$ *away from the boundary.*

The main ingredient in the proof is the turbulization construction for symplectic foliations given in Theorem 1.7.31 which is proved in the next section. We also need the following cobordism, which allows for the interpolation ω_t between $\widetilde{\omega}_0$ and $\phi^*(\widetilde{\omega}_1)$. Note, that if such an interpolation is necessary, the resulting SF-structure on $M_1 \cup_\phi M_2$ has a family of compact leaves.

Lemma 1.6.27. *Let* ω_t*,* $t \in [0,1]$ *be a family of symplectic forms on* N^{2n}*. Then, there exists a SF-structure* $(\mathcal{F}, \omega)$ *on the trivial cobordism* $[0,1] \times N$ *which is:*

(i) Tame at the left boundary, and the induced symplectic form on the boundary leaf is ω_0*;*

(ii) Tame at the right boundary, and the induced symplectic form on the boundary leaf is ω_1*.*

Proof. Let $\lambda : [0,1] \to [0,1]$ be a bump function, satisfying

$$\lambda(t) = \begin{cases} 0 & \text{for } t \text{ near } 0 \\ 1 & \text{for } t \text{ near } 1 \end{cases}.$$

Then, the required SF-structure is given by:

$$\Big(\mathcal{F} := \bigcup_{t \in [0,1]} \{t\} \times N, \quad \omega := \omega_{\lambda(t)}\Big).$$

□

Proof of Theorem 1.6.26. Using Theorem 1.7.31 we change the SF-structures on M_i so that it is tame at the boundary and has a symplectic leaf

$$(\partial M_i, \widetilde{\omega}_i).$$

Then, using Proposition 1.6.21, we can connect the two pieces by glueing a cobordism as in Lemma 1.6.27 in between. □

1.7 Turbulization

In certain cases, a foliation with transverse boundary can be changed, locally around the boundary, so it becomes tangent to the boundary. This construction is called turbulization, and the resulting foliation generalizes the Reeb components used to obtain the Reeb foliation on $\mathbb{S}^3$ from Example 1.5.5. The main focus of this section is to adapt this construction to the setting of symplectic foliations.

We start by recalling the classical turbulization for foliations. Next, we give two versions of turbulization for SF-pairs with regular boundary. The first one, stated in Lemma 1.7.3, requires slightly stronger hypothesis, but suffices for most applications. The second one, stated in Lemma 1.7.14, requires minimal hypothesis but the proof becomes more involved.
Then, we consider turbulization on the level of symplectic foliations (without a prefered SF-pair). The main result is Theorem 1.7.31, which is based on Lemma 1.7.14. Finally,in Section 1.7.1, we apply turbulization to construct symplectic foliations on manifolds which admit an open book decomposition.

1.7.0.1 Turbulizing foliations

When we try to construct a symplectic foliation on a given manifold M, the gluing construction for foliations with transverse boundaries is often less useful than the one for foliations tame at the boundary. The reason is that when we cut the manifold M into pieces, we have to remember how to glue them back together recover M. In this case we have no freedom in choosing the diffeomorphism $\phi : \partial M_1 \to \partial M_2$. Hence, we need to construct the foliations on each of the pieces so that the given ϕ preserves the induced structures on the boundary. Therefore, the constructions on each of the pieces depend on each other, and we are essentially doing a global construction.
The gluing construction for tame boundaries does not have this problem, since any ϕ automatically preserves the boundary leaf. Therefore, the foliations on each of the pieces can be constructed independently of each other, reducing the global construction problem to several local ones.

The turbulization construction allows us to change a foliation that is transverse to the boundary into one that is tame at the boundary.

Proposition 1.7.1. *Let $\mathcal{F}$ be a foliation on M transverse to the boundary, and denote by $\mathcal{F}_\partial := \mathcal{F} \cap T\partial M$ the induced foliation on the boundary. If $\mathcal{F}_\partial$ can be defined by a closed form (i.e. is unimodular), then there exists a foliation $\widetilde{\mathcal{F}}$ on M such that*

(i) $\widetilde{\mathcal{F}}$ is tame at the boundary;

(ii) $\widetilde{\mathcal{F}}$ agrees with $\mathcal{F}$ away from the boundary.

Proof. Let $\theta_\partial \in \Omega^1(\partial M)$ be a closed form so that $\mathcal{F}_\partial = \ker \theta_\partial$. By Lemma 1.6.14 we can find a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which $\mathcal{F} = \ker \theta_\partial$. On the collar we define a new foliation by

$$\theta := f(r)\theta_\partial + g(r)\mathrm{d}r,$$

where $f, g : (-\varepsilon, 0] \to \mathbb{R}$. If we want $\ker\theta$ to be tame at the boundary and agree with $\mathcal{F}$ away from the boundary we choose the functions to satisfy

$$f = \begin{cases} 1 & \text{for } r \text{ near } -\varepsilon \\ 0 & \text{for } r \text{ near } 0 \end{cases}, \quad g = \begin{cases} 0 & \text{for } r \text{ near } -\varepsilon \\ 1 & \text{for } r \text{ near } 0 \end{cases}, \quad f^2 + g^2 > 0. \tag{1.7.0.1}$$

See Figure 1.3 for an example of functions satisfying these conditions. □

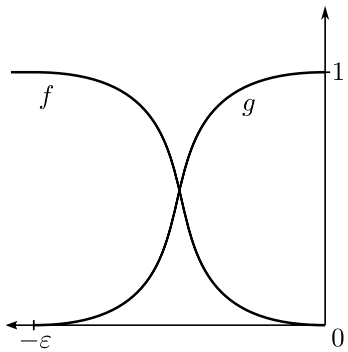

Figure 1.3: Functions f and g satisfying the conditions in Equation 1.7.0.1.

Instead of changing the foliation close to the boundary, we can change the foliation by gluing a trivial cobordism. This does not change the diffeomorphism type of the manifold, so that the resulting foliation is isomorphic to the one above.

Lemma 1.7.2. *Let θ be a closed, nowhere vanishing 1-form on a N^{2n} and denote by $\mathcal{F}_N := \ker\theta$ the induced foliation. Then, there exists a foliation $\mathcal{F}$ on the trivial cobordism $[0, 1] \times N$ such that:*

(i) $\mathcal{F}$ is transverse to the left boundary and $\mathcal{F} \cap T(\{0\} \times N) = \mathcal{F}_N$;

(ii) $\mathcal{F}$ is tame at the right boundary.

Proof. The proof is exactly the same as that of Proposition 1.7.1. □

1.7.0.2 Turbulization for ∂-SF triples of Cosymplectic type

The following Lemma is the SF-analogue of construction for foliations from Lemma 1.7.2. We use it, in Proposition 1.7.5 below, to obtain the SF-analogue of the turbulization construction from Proposition 1.7.1.

Lemma 1.7.3. *Let (u, v, w) be a ∂-SF triple of cosymplectic type on N^{2n}, as in Definition 1.6.8. Then, there exists a symplectic foliation pair (γ, η) on the trivial cobordism $[0, 1] \times N$ which has:*

(i) Regular left boundary of cosymplectic type (Definition 1.6.9) for the ∂-SF triple (u, v, w);

(ii) Tame right boundary (Definition 1.6.20) with symplectic leaf $(N, w + Cv \wedge u)$ for a constant $C > 0$ large enough;

Moreover, if $w^n \geqslant 0$, then we can take $C = 1$.

Remark 1.7.4. The proof is symmetric in the interval coordinate of $[0, 1] \times N$. That is, the same argument shows that we can obtain a cobordism $[0, 1] \times N$ which has

(i) Tame left boundary with symplectic leaf $(N, w + Cu \wedge v)$ for a constant $C > 0$ large enough;

(ii) Regular right boundary of cosymplectic type for the ∂-SF triple (u, v, w).

Note that here the symplectic form on the boundary leaf differs from the one in Lemma 1.7.3 by changing the order of u and v. This is necessary to take into account the change in orientation between the left and right boundary of the cobordism $[0, 1] \times N$. △

Proof. First note that the non-degeneracy condition for $w + Cv \wedge u$ is given by

$$(w + Cv \wedge u)^n = w^n + nCw^{n-1} \wedge v \wedge u. \tag{1.7.0.2}$$

This will always be positive for $C > 0$ large enough, and when $w^n \geqslant 0$ it suffices to take $C = 1$.

Choose functions $f, g : [0, 1] \to \mathbb{R}_{\geqslant 0}$ satisfying

(i) $f = 1$ near $t = 1$ and $f = 0$ near $t = 0$;

(ii) $g = 0$ near $t = 1$ and $g = 1$ near $t = 0$;

(iii) $f^2 + Cg^2 \gg g$, for some constant $C \gg 0$ as in Equation 1.7.0.2

Note this can always be achieved by letting f and g having graphs as in Figure 1.4. The differential forms

$$\gamma := fu + g\mathrm{d}t, \quad \eta := w + f\mathrm{d}t \wedge v + gCv \wedge u,$$

define a symplectic foliation since:

$$\begin{aligned}
\gamma \wedge \mathrm{d}\eta &= (\mathrm{d}w - \dot{g}C\mathrm{d}t \wedge u \wedge v) \wedge (fu + g\mathrm{d}t) = 0 \\
\gamma \wedge \mathrm{d}\gamma &= (fu + g\mathrm{d}t) \wedge (\dot{f}\mathrm{d}t \wedge u) = 0 \\
\gamma \wedge \eta^n &= \left(w^n + n(f\mathrm{d}t - gCu) \wedge v \wedge w^{n-1}\right) \wedge (fu + g\mathrm{d}t) \\
&= g\mathrm{d}t \wedge w^n + n(f^2 + g^2C)\mathrm{d}t \wedge v \wedge u \wedge w^{n-1} > 0.
\end{aligned}$$

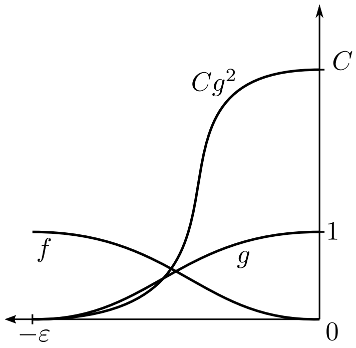

Figure 1.4: Functions f and g satisfying the conditions in the proof of Lemma 1.7.3.

For the last computation we use that the second summand dominates the first by condition (iii) above. Furthermore, conditions (i) and (ii) above imply that (γ, η) has the correct type of boundary. Indeed, around the left boundary we have

$$\gamma = u, \quad \eta = w + \mathrm{d}t \wedge v,$$

which is regular of cosymplectic type, while for points in the right boundary we have

$$\gamma = \mathrm{d}t, \quad \eta = w + Cv \wedge u,$$

inducing the desired symplectic leaf. □

Let (M, γ, η) be a symplectic foliation air with regular boundary of cosymplectic type conform Definition 1.6.9. By the previous result, the manifold

$$\widetilde{M} := M \cup_{\{0\}\times\partial M} [0,1] \times \partial M,$$

admits a symplectic foliation pair $(\tilde{\gamma}, \tilde{\eta})$ such that the boundary is a symplectic leaf

$$(\partial M, w + Cu \wedge v)\,.$$

Observe that $\widetilde{M} \simeq M$, and in fact it is not hard to see that the construction in Lemma 1.7.3 above can be realized inside a collar neighborhood of the boundary of M. Thus we conclude:

Proposition 1.7.5. *Let (γ, η) be a SF-pair on a manifold M with regular boundary of cosymplectic type (Definition 1.6.9) for the ∂-SF triple (u, v, w). Then, there exists a symplectic foliation pair $(\tilde{\gamma}, \tilde{\eta})$ on M such that:*

(i) $(\tilde{\gamma}, \tilde{\eta})$ agrees with (γ, η) away from the boundary ∂M;

(ii) $(\tilde{\gamma}, \tilde{\eta})$ *has tame boundary (Definition 1.6.20) with symplectic leaf* $(\partial M, w + Cu \wedge v)$ *for a sufficiently large constant* $C > 0$.

Note that strictly speaking the above lemma does not produce an SF-cobordism as in Definition 1.6.24, since the foliation is not transverse to the boundary. The transversality condition in Definition 1.6.24 ensures that cobordisms can be composed. However, using the gluing construction from Proposition 1.6.21 "cobordisms" as in Proposition 1.7.5 can also be composed. Hence, at least intuitively we still think of them as SF-cobordisms. In fact, as the following lemma shows that by applying the above construction twice we obtain an honest SF-cobordism.

Lemma 1.7.6. *For* $i = 0, 1$ *let* (u_i, v_i, w_i) *be a* ∂*-SF triple of cosymplectic type on* N^{2n} *(Definition 1.6.8) and assume in addition that* $w_i^n \geqslant 0$. *If there exists an orientation reversing diffeomorphism* $\phi : N \to N$, *such that*

$$\phi^*(w_1 + u_1 \wedge v_1) = w_0 + v_0 \wedge u_0,$$

then there exists a regular SF-cobordism:

$$(N, u_0, v_0, w_0) \prec_{([0,1]\times N, \gamma, \eta)} (N, u_1, v_1, w_1).$$

Moreover, the SF-cobordism has a single compact leaf diffeomorphic to N, *and with leafwise symplectic form* $w_0 + v_0 \wedge u_0$.

Proof. By Lemma 1.7.3 and Remark 1.7.4, we find two trivial cobordisms, both diffeomorphic to $[0, 1] \times N$, endowed with SF-pairs (γ_i, η_i), $i, = 0, 1$, respectively. The proof of Lemma 1.7.3 shows that the first cobordism contains a collar neighborhood isomorphic to

$$((-\varepsilon, 0] \times N, \gamma_0 = \mathrm{d}t, \eta_0 = w_0 + v_0 \wedge u_0),$$

while the second contains a collar

$$\big([0, \varepsilon) \times \overline{N}, \gamma_1 = \mathrm{d}t, \eta_1 = w_1 + u_1 \wedge v_1\big).$$

Under the identifications made by ϕ these collars can be matched smoothly giving the desired gluing. The resulting SF-cobordism $([0, 1] \times N, \gamma, \eta)$ is regular, and induces the required ∂-SF triples on the boundary. □

Remark 1.7.7. The hypothesis in the lemma above are chosen to obtain the simplest statement which suffices for our later applications. However, there are several ways in which they can be changed obtaining a slightly stronger statement:

- The condition that $w_i \geqslant 0$, is not necessary. As in Lemma 1.7.3, it can be removed if we require instead that there exists $C_i > 0$ sufficiently large, and an orientation reversing diffeomorphism $\phi : N \to N$

$$\phi^*(w_1 + C_1 v_1 \wedge u_1) = w_0 + C_0 v_0 \wedge u_0. \qquad (1.7.0.3)$$

- The condition that ϕ preserves the symplectic forms, as in Equation 1.7.0.3, can be weakened as follows. It suffices to ask there exists an orientation reversing diffeomorphism ϕ and a 1-parameter family of symplectic forms ω_t, $t \in [0,1]$ on N such that
$$\omega_0 = w_0 + v_0 \wedge u_0, \quad \omega_1 = w_1 + v_1 \wedge u_1.$$
Then, the single compact leaf of the SF-pair on $[0,1] \times N$ can be replaced by the product foliation
$$\left(\bigcup_{t \in [0,1]} \{t\} \times N, \omega_t \right).$$
- Instead of requiring the ∂-SF triples (u_i, v_i, w_i) to be of cosymplectic type, the same proof goes through when we require them to be Tameable, see Definition 1.6.8. In this case, the proof uses Lemma 1.7.14, instead of Lemma 1.7.3.

△

The following is a simple application of Lemma 1.7.6.

Example 1.7.8. Consider a fibration $\pi : N \to \mathbb{T}^2$, and a closed 2-form $\omega \in \Omega^2(N)$ which is non-degenerate on the fibers of π. Denote the standard angular coordinates on $\mathbb{T}^2$ by (θ_1, θ_2) and define $\gamma_i := \pi^* \mathrm{d}\theta_i \in \Omega^1(N)$, $i = 1, 2$. This induces two ∂-SF structures on N, as in Definition 1.5.7:
$$(\mathcal{F}_i := \ker \gamma_i, \omega_i := \omega|_{\mathcal{F}_i}), \quad i = 1, 2.$$
We want to construct a symplectic foliation on $[0,1] \times N$ which is transverse to the boundary, and induces the above ∂-SF structures on its boundary components. Note that the naive approach, of "interpolating" between the foliations above does not work. Indeed, choose functions $f, g : [0,1] \to \mathbb{R}$ and consider
$$\gamma := f(t)\gamma_1 + g(t)\gamma_2.$$
Then the condition that γ defines a foliation reads:
$$0 = \gamma \wedge \mathrm{d}\gamma = (\dot{f}g - f\dot{g})\mathrm{d}t \wedge \gamma_1 \wedge \gamma_2.$$
Furthermore, we want γ to be nowhere vanishing, and agree with γ_1 near the left boundary and with γ_2 near the right boundary. Thus, we obtain additional conditions on f and g:
$$\dot{f}g - f\dot{g} = 0, \quad f^2 + g^2 > 0, \quad f = \begin{cases} 1 & t \text{ near } 0 \\ 0 & t \text{ near } 1 \end{cases} \quad g = \begin{cases} 0 & t \text{ near } 0 \\ 1 & t \text{ near } 1 \end{cases},$$
and it is not hard to see that these conditions cannot be simultaneously satisfied.

Instead, we observe that the triples $(\gamma_1, \gamma_2, \omega)$ and $(\gamma_2, \gamma_1, \omega)$ are ∂-SF triples of cosymplectic type as in Definition 1.6.8. Thus applying the following corollary, which follows directly from Lemma 1.7.6, we obtain the desired cobordism.
Finally, observe that the "naive approach" (which does not work) would produce a foliation without any closed leaves, while using the turbulization construction does produce a closed leaf.

Corollary 1.7.9. *Let (u, v, w) be a ∂-SF triple on N^{2n} satisfing $\mathrm{d}u = 0$ and $\mathrm{d}v = 0$. Then there exists a regular SF-cobordism $([0,1] \times N, \gamma, \eta)$ which has:*

(i) Regular left boundary with induced ∂-SF triple (u, v, w);

(ii) Regular right boundary with induced ∂-SF triple (v, u, w);

(iii) A single closed leaf diffeomorphism to N, with leafwise symplectic form $w + Cv \wedge u$ for some $C > 0$ large enough.

△

1.7.0.3 ∂-SF triples of Tameable type

Let (γ, η) be a SF-pair on M with regular boundary (Definition 1.6.1) and denote the induced ∂-SF triple by (u, v, w) (Definition 1.6.2). Recall that Lemma 1.7.3 says that (γ, η) can be turbulized provided (u, v, w) is of cosymplectic type. It turns out that this condition can be weakened, and the minimal conditions the triple needs to satisfy in order to turbulize are as follows.

Firstly, forgetting the leafwise symplectic structure, we need $\mathrm{d}u = 0$ to turbulize the foliation as in Proposition 1.7.1. Secondly, provided that $w^n = 0$, the form

$$\eta := w + v \wedge u,$$

is non-degenerate, so it defines a symplectic form if it is closed, i.e.

$$\mathrm{d}w + \mathrm{d}v \wedge u = 0. \tag{1.7.0.4}$$

These necessary conditions precisely mean that (u, v, w) is of Tameable type as in Definition 1.6.8, and it turns out they are also sufficient. That is, we have following result analogous to Proposition 1.7.5.

Proposition 1.7.10. *Let (γ, η) be a SF-pair on M with regular boundary of Tameable type (Definition 1.6.9) and denote the induced ∂-SF triple by (u, v, w). Then, there exists an SF-pair $(\tilde{\gamma}, \tilde{\eta})$ on M such that:*

(i) $(\tilde{\gamma}, \tilde{\eta})$ agrees with (γ, η) away from the boundary ∂M;

(ii) $(\tilde{\gamma}, \tilde{\eta})$ has tame boundary with symplectic leaf $(\partial M, w + v \wedge u)$.

The proof follows from Lemma 1.7.14 below, since the cobordism constructed there can be realized in a collar neighborhood of the boundary. The remainder of this section is devoted to proving this lemma.

Comparing with Lemma 1.7.3, being Tameable includes the condition $w^n = 0$ while the former only requires $w^n \geqslant 0$. However, as we now explain, by adding a trivial cobordism to the manifold this condition can always be satisfied.

Lemma 1.7.11. *Let $(\mathcal{F}, \omega)$ be a symplectic foliation transverse to the boundary on M^{2n+1}, representing by SF-pair (γ, η). Then there exists equivalent pairs $(\gamma_i, \eta_i) \sim (\gamma, \eta)$, $i = 1, 2$, as in Definition 1.5.4, satisfying:*

(i) The forms η_i and η agree away from the boundary, and $\gamma_i = \gamma$ everywhere;

(ii) The Reeb vector field R_1 of (η_1, γ_1) is tangent to the boundary, i.e. $\eta_1^n = 0$;

(iii) The Reeb vector field R_2 of (η_2, γ_2) is everywhere transverse to the boundary pointing outwards, i.e. $\eta_2^n > 0$.

Proof. We denote the induced ∂-SF pair by

$$\gamma_\partial := \gamma|_{\partial M}, \quad \eta_\partial := \eta|_{\partial M},$$

and and choose an admissible form $\beta \in \Omega^1(\partial M)$ (Definition 1.5.9). By Proposition 1.5.14 there exists a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ on which

$$\gamma = f\gamma_\partial, \quad \eta = \eta_\partial + \mathrm{d}(t\beta) + \rho \wedge \gamma_\partial, \tag{1.7.0.5}$$

for a function $f \in C^\infty(\mathcal{U})$ and an admissible form $\beta \in \Omega^1(\partial M)$.

Since β and γ_∂ are linearly independent, we can find a vector field $X \in \mathfrak{X}(\partial M)$ for which

$$\gamma_\partial(X) = 1, \quad \beta(X) = 0.$$

Use this to define a form $\rho_1 := \iota_X \eta_\partial \in \Omega^1(\partial M)$ and note that, at points in the boundary,

$$\gamma_1 := \gamma_\partial, \quad \eta_1 := \eta_\partial + \mathrm{d}(t\beta) + \rho_1 \wedge \gamma,$$

has Reeb vector field X. Indeed,

$$\iota_X \gamma_\partial = 1, \quad \iota_X \left(\eta_\partial + \mathrm{d}t \wedge \beta + (\iota_X \eta_\partial) \wedge \gamma_\partial\right) = 0.$$

Furthermore, (γ, η) and (γ_1, η_1) are equivalent so we can interpolate from one to the other as explained in Lemma 1.7.13 below.

By the above argument, we can assume that we have a collar neighborhood as in Equation 1.7.0.5 for which $\eta_\partial^n = 0$. Hence, we can find a vector field $R_\gamma \in \mathfrak{X}(\partial M)$ satisfying

$$\gamma(R_\gamma) = 1, \quad \beta(R_\gamma) = 0, \quad \iota_{R_\gamma} \eta_\partial = 0.$$

Define $\rho_2 := \beta \in \Omega^1(\partial M)$ and

$$\gamma_2 := \gamma_\partial, \quad \eta_2 := \eta_\partial + \mathrm{d}(t\beta) + \rho_2 \wedge \gamma_\partial.$$

At points in the boundary, the Reeb vector field of (γ_2, η_2) equals $R_2 = \partial_t + R_\gamma$ which points outwards along the boundary. Indeed, we have

$$\gamma(R_2) = \gamma_\partial(R_\gamma) = 1, \quad \iota_{R_2}\eta_2 = \iota_{R_2}(\eta_\partial + \mathrm{d}t \wedge \beta + \rho_2 \wedge \gamma_\partial) = \beta - \beta = 0.$$

Again, the proof concludes by applying Lemma 1.7.13 below. ∎

Remark 1.7.12. Note that the proof below actually shows which vector fields can be obtained as the Reeb vector field by replacing the SF-pair by an equivalent one, as in Definition 1.5.4. Indeed defining $\gamma_\partial = \iota_V \eta_\partial$ works for any vector field $V \in \mathfrak{X}(\partial M)$. We only need to make sure that V already satisfies the conditions $\theta_\partial(V) = 1$ and $\beta(V) = 0$. Fixing such a vector field V any other vector field satisfying these conditions is of the form

$$V' = V + X,$$

where $X \in \partial(M)$ satisfies $X \in \ker \beta \cap \ker \theta_\partial$. △

To complete the proof of the lemma above, we need the following result, stating that close to a hypersurface any SF-pair representing a symplectic foliation can be changed to an equivalent pair.

Lemma 1.7.13. *Let $(\mathcal{F}, \omega)$ be symplectic foliation on M, and (γ_i, η_i), $i = 0, 1$, two SF-pairs representing $(\mathcal{F}, \omega)$. Then, there exists an SF-pair (γ, η) representing $(\mathcal{F}, \omega)$ such that:*

(i) The pair (γ, η) agrees with (γ_1, η_1) on a neighborhood of the boundary;

(ii) The pair (γ, η) agrees with (γ_0, η_0) away from the boundary.

Proof. Let $\mathcal{U} \simeq (-\varepsilon, 0] \times \partial M$ be a collar neighborhood of the boundary. Since $(\gamma_1, \eta_1) \sim (\gamma_0, \eta_0)$ we have:

$$\gamma_1 = f\gamma_0, \quad \eta_1 = \eta_0 + \beta \wedge \gamma_0,$$

for a function $f : \mathcal{U} \to \mathbb{R}_{>0}$ and $\beta \in \Omega^1(\mathcal{U})$. Choose a bump function $\rho : (-\varepsilon, 0] \to \mathbb{R}_{\geqslant 0}$, satisfying

$$\rho(t) = \begin{cases} 0 & \text{for } t \text{ near } -\varepsilon \\ 1 & \text{for } t \text{ near } 0 \end{cases}.$$

Then,

$$\gamma := (1 + \rho(t) f)\, \gamma_0, \quad \eta := \eta_0 + \rho(t)\beta \wedge \gamma,$$

is a SF-pair representing $(\mathcal{F}, \omega)$, which agrees with (γ_1, η_1) on a neighborhood of the boundary, and with (γ_0, η_0) away from the boundary. □

Now that we have proved the preparatory lemmas we return to the proof of Proposition 1.7.10. As stated before it follows immediately from the following lemma since the cobordism can be realized in a tubular neighborhood of the boundary.

Lemma 1.7.14. *Let (u, v, w) be a ∂-SF triple of Tameable type on N^{2n}, see Definition 1.6.8. Then, there exists a SF-pair (γ, η) on the trivial cobordism $[0, 1] \times N$ which has:*

(i) Regular left boundary of Tameable type with induced ∂-triple (u, v, w);

(ii) Tame right boundary with symplectic leaf $(\partial M, w + v \wedge u)$.

Proof. Consider the pair $(\gamma, \eta) \in \Omega^1([0,1] \times N) \times \Omega^2([0,1] \times N)$, given by:

$$\gamma := fu + g\mathrm{d}t, \quad \eta := w + f\mathrm{d}t \wedge v + gv \wedge u + h\mathrm{d}v,$$

for functions $f, g, h : [0,1] \to \mathbb{R}$ which will be defined later.

First we want that γ describes the turbulization foliation from Lemma 1.7.2. This means that we have to choose $f, g : [0,1] \to \mathbb{R}$ satisfying:

(i) $f = 1$ near $t = 0$ and $f = 0$ near $t = 1$;

(ii) $g = 0$ near $t = 0$ and $g = 1$ near $t = 1$;

(iii) $f^2 + g^2 > 0$.

It remains to choose h such that (γ, η) becomes the required SF-pair. If we choose h to satisfy

$$h(t) = \begin{cases} t & t \text{ near } 0 \\ 0 & t \text{ near } 0 \end{cases}, \tag{1.7.0.6}$$

then (γ, η) has regular left boundary of tameable type, and tame right boundary. Furthermore, if $|h|$ is so small that it can be treated as zero in the computation, then the conditions for (γ, η) to define a SF-pair become:

$$\begin{aligned}
\gamma \wedge \mathrm{d}\gamma &= (fu + g\mathrm{d}t) \wedge (\dot{f}\mathrm{d}t \wedge u) = 0 \\
\mathrm{d}\eta &= w - f\mathrm{d}t \wedge \mathrm{d}v + \dot{g}\mathrm{d}t \wedge v \wedge u + g\mathrm{d}v \wedge u + \dot{h}\mathrm{d}t \wedge dv \\
\gamma \wedge \eta &= \left(h^n \mathrm{d}v^n + (f\mathrm{d}t - gu) \wedge v \wedge (w + h\mathrm{d}v)^{n-1}\right) \wedge (fu + g\mathrm{d}t) \\
&= h^n \mathrm{d}v^n \wedge (fu + g\mathrm{d}t) + (f^2 + g^2)\mathrm{d}t \wedge v \wedge u \wedge (w + hdv)^{n-1} > 0, \\
\gamma \wedge \mathrm{d}\eta &= g\mathrm{d}t \wedge dw + -f^2 u \wedge \mathrm{d}t \wedge dv + g^2 \mathrm{d}t \wedge dv \wedge u + f\dot{h}u \wedge \mathrm{d}t \wedge dv \\
&= (f^2 + g^2 - g - f\dot{h})\mathrm{d}t \wedge dv \wedge u
\end{aligned}$$

Hence, we want h to satisfy $-g + f^2 + g^2 - f\dot{h} = 0$. We use this equation to define h as follows:

$$h := \int_0^t \frac{f^2 + g^2 - g}{f} dx.$$

If we assume that the (closed) set $\{t \in [0,1] \mid f(t) = 0\}$ is strictly contained in the (closed) set $\{t \in [0,1] \mid g(t) = 1\}$ then the integral is well-defined. It remains to check that with this definition h can be chosen satisfying Equation 1.7.0.6 and such that $|h|$ is sufficiently small.

Observe that for t near 0 we have $f(t) = 1$ and $g(t) = 0$ implying:

$$h(t) = \int_0^t 1\mathrm{d}x = t,$$

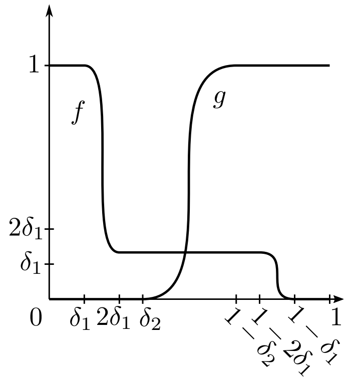

Figure 1.5: Functions f and g satisfying the conditions in the proof of Lemma 1.7.14.

as desired. Furthermore, for t near 1, the integrand is zero so that $h(t)$ is constant. Now we describe how to choose the functions f and g so that $h(1) = 0$ and $|h|$ is arbitrarily small, see also Figure 1.5.

Let $0 < \delta_1$ be a small constant and choose f such that

$$f(t) = \begin{cases} 1 & t \in [0, \delta_1] \\ \delta_1 < f(t) < 2\delta_1 & t \in [2\delta_1, 1 - 2\delta_1] \\ 0 & t \in [1 - \delta_1, 1] \end{cases}.$$

Then, the integral

$$\int_0^1 f(x)\mathrm{d}x,$$

is a positive constant $C_1(\delta_1) > 0$ which can be made arbitrarily small by choosing δ_1 small. Next, let $0 < \delta_1 < \delta_2 < 1/2$ be another constant and choose g such that

$$g(t) = \begin{cases} 0 & t \in [0, \delta_2] \\ 1 & t \in [1 - \delta_2, 1] \end{cases}.$$

With these choices the integral

$$\int_0^1 \frac{g^2 - g}{f}\mathrm{d}x,$$

is well-defined and equal to a negative constant (since $g^2 - g < 1$ for $0 < g < 1$) $C_2(\delta_1, \delta_2) < 0$. Given a fixed δ_1, we can choose δ_2 so that $C_1 = -C_2$ implying that $h(1) = 0$ as desired. Moreover, if we choose δ_1 sufficiently small and δ_2 sufficiently large then $|h|$ can be made arbitrarily small.

□

1.7.0.4 Turbulization for symplectic foliations

If we work with symplectic foliations $(\mathcal{F}, \omega_{\mathcal{F}})$ and do not fix an SF-pair representing it, the necessary conditions to apply turbulization live in cohomology. To describe these cohomology classes we start by recalling the definition of the foliated cohomology, $H^k(\mathcal{F})$ and the foliated cohomology with coefficients in the conormal bundle, denoted by $H^k(\mathcal{F}, \nu^*)$.

Given a foliation $\mathcal{F}$ on M, the inclusion $\iota : T\mathcal{F} \to TM$, induces a short exact sequence

$$0 \to T\mathcal{F} \xrightarrow{\iota} TM \to \nu := TM/T\mathcal{F} \to 0,$$

where ν is the normal bundle of $\mathcal{F}$. Dually, this induces a short exact sequence of complexes:

$$0 \to \Omega^\bullet_{\mathcal{F}}(M) \to \Omega^\bullet(M) \xrightarrow{r} \Omega^\bullet(\mathcal{F}) \to 0, \tag{1.7.0.7}$$

where:

- $\Omega^\bullet(M) := \Gamma(\Lambda^\bullet T^*M)$ is the usual complex of differential forms on M;
- $\Omega^\bullet(\mathcal{F}) := \Gamma(\Lambda^\bullet T^*\mathcal{F})$, is the complex of **foliated forms**;
- $\Omega^\bullet_{\mathcal{F}}(M) := \Gamma(\Lambda^\bullet(TM/T\mathcal{F})^*$, is the complex of **$\mathcal{F}$-relative forms**. Since, $\Omega^\bullet_{\mathcal{F}}(M)$ is the kernel of the restriction map r, we have an identification:

$$\Omega^\bullet_{\mathcal{F}}(M) = \{\alpha \in \Omega^\bullet(M) \mid \alpha|_{\mathcal{F}} = 0\}.$$

Since $T\mathcal{F}$ is involutive, $\Omega^\bullet(\mathcal{F})$ comes with a differential $\mathrm{d}_{\mathcal{F}}$ (which is just the leafwise deRham differential) defining the **foliated cohomology**

$$H^\bullet(\mathcal{F}) := H(\Omega^\bullet(\mathcal{F}), \mathrm{d}_{\mathcal{F}}).$$

The involutivity condition also implies that the usual deRham differential d on $\Omega^\bullet(M)$ preserves the subcomplex $\Omega^\bullet_{\mathcal{F}}(M)$, definining the **$\mathcal{F}$-relative cohomology**

$$H^\bullet_{\mathcal{F}}(M) := H(\Omega^\bullet_{\mathcal{F}}(M), \mathrm{d}).$$

Remark 1.7.15. By Equation 1.7.0.7 these cohomology groups fit in a short exact sequence

$$0 \to H^\bullet_{\mathcal{F}}(M) \to H^\bullet(M) \to H^\bullet(\mathcal{F}) \to 0,$$

which in turn induces a long exact sequence in cohomology:

$$\cdots \to H^k_{\mathcal{F}}(M) \to H^k(M) \to H^k(\mathcal{F}) \xrightarrow{\delta} H^{k+1}_{\mathcal{F}}(M) \to H^{k+1}(M) \to \dots. \tag{1.7.0.8}$$

The connecting homomorphism $\delta : H^\bullet(\mathcal{F}) \to H^{\bullet+1}_{\mathcal{F}}(M)$, can be explicitly described by

$$\delta([\alpha]) = [\mathrm{d}\tilde{\alpha}], \tag{1.7.0.9}$$

where $\alpha \in \Omega^k(\mathcal{F})$ is a closed foliated form and $\tilde{\alpha} \in \Omega^k(M)$ any extension of α. △

We can now define the modular class of a (coorientable) foliation $\mathcal{F}$ on M. Let $\gamma \in \Omega^1(M)$ be any differential form satisfying $\mathcal{F} = \ker \gamma$. Recall that the integrability condition on $\mathcal{F}$ translates into the differential condition

$$\gamma \wedge \mathrm{d}\gamma = 0.$$

In turn, this is equivalent to

$$\mathrm{d}\gamma = \gamma \wedge \mu, \tag{1.7.0.10}$$

for some $\mu \in \Omega^1(M)$. It is not hard to see that $(\mathrm{d}_{\mathcal{F}}\mu|_{\mathcal{F}}) = 0$, and that the cohomology class $[\mu|_{\mathcal{F}}] \in H^1(\mathcal{F})$ is independent of the choice of γ and μ.

Definition 1.7.16. *The* ***modular class*** *of a (coorientable) foliation $\mathcal{F}$ is the cohomology class*

$$\mathrm{mod}_{\mathcal{F}} := [\mu|_{\mathcal{F}}] \in H^1(\mathcal{F}).$$

The modular class measures if $\mathcal{F}$ is unimodular, i.e. if it can be defined by a closed 1-form.

Lemma 1.7.17. *The foliation $\mathcal{F}$ is unimodular (i.e. can be defined by a closed form) if and only if* $\mathrm{mod}_{\mathcal{F}} = 0$.

Proof. If $\mathcal{F}$ can be defined by a closed 1-form γ, then it is clear that $\mu = 0$ satisfies $\mathrm{d}\gamma = \gamma \wedge \mu$. Conversely, if $\mathrm{mod}_{\mathcal{F}} = 0$, choose any $\gamma \in \Omega^1(M)$ such that $\mathcal{F} = \ker \gamma$. Then there exists $\mu \in \Omega^1(M)$ and $f \in C^\infty(M)$, such that

$$\mathrm{d}\gamma = \gamma \wedge \mu, \quad \mu \wedge \gamma = \mathrm{d}f \wedge \gamma.$$

Then, $\mathcal{F} = \ker(e^f\gamma)$ and

$$\mathrm{d}(e^f\gamma) = e^f \mathrm{d}f \wedge \gamma - e^f \mu \wedge \gamma = 0.$$

□

Before we define the foliated cohomology with values in the conormal bundle, let us recall the definition of differential forms with values in a (real) line bundle. Given a line bundle $\pi : L \to M$, consider the complex of **L-valued differential forms on M**

$$\Omega^\bullet(M, L) := \Gamma((\Lambda^\bullet T^*M) \otimes L).$$

Given a flat connection $\nabla : \mathfrak{X}(M) \times \Gamma(L) \to \Gamma(L)$ on L, the usual Koszul formula defines a differential d_∇ on $\Omega^\bullet(M, L)$; for $\alpha \in \Omega^k(M, L)$ and $\mathfrak{X}_1, \ldots, X_k \in \mathfrak{X}(M)$ we have:

$$\begin{aligned}(\mathrm{d}_\nabla\alpha)(X_1, \ldots, X_{k+1}) := &\sum_{i=1}^{k+1} (-1)^{i+1} \nabla_{X_i}(\alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}).\end{aligned}$$

Observe that since d_∇ satisfies the Leibniz identity, it is uniquely determined by what it does on $\Omega^0(M, L) = \Gamma(L)$, where it is defined as

$$(\mathrm{d}_\nabla \sigma)(X) = \nabla_X \sigma, \quad \forall \sigma \in \Gamma(L), X \in \mathfrak{X}(M). \tag{1.7.0.11}$$

The connection being flat is equivalent to $\mathrm{d}_\nabla^2 = 0$, giving rise to the cohomology groups:

$$H^\bullet(M, L) := H(\Omega^\bullet(M, L), \mathrm{d}_\nabla).$$

If the line bundle L is trivializable, then the L-valued differential forms can be related to the usual real valued differential forms on M. In this case there exists a nowhere vanishing section $s \in \Gamma(L)$, and ∇ is completely determined by the differential form $\beta \in \Omega^1(M)$ defined by

$$\nabla_X s = \beta(X)s, \quad \forall X \in \mathfrak{X}(M). \tag{1.7.0.12}$$

Note that under this identification, $\mathrm{d}_\nabla^2 = 0$ if and only if $\mathrm{d}\beta = 0$. Clearly, β depends on the section s, used to trivialize L, and if $\tilde{s} := fs$, $f \in C^\infty(M)$, is any other nowhere vanishing section, then

$$\tilde{\beta} = \beta + \mathrm{d}f.$$

Hence, the class $[\beta] \in H^1(M)$, depends only on ∇, and we have:

Lemma 1.7.18. *Given an orientable line bundle $\pi : L \to M$ there is a one-to-one correspondence between flat connections ∇ on L, and cohomology classes in $H^1(M)$, sending ∇ to $[\beta]$ as Equation 1.7.0.12.*

Given a differential form $\beta \in \Omega^1(M)$ we define the **twisted differential** $\mathrm{d}_\beta : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$, by:

$$\mathrm{d}_\beta \alpha := \mathrm{d}\alpha + \beta \wedge \alpha, \quad \forall \alpha \in \Omega^\bullet(M). \tag{1.7.0.13}$$

It is easily checked that $\mathrm{d}_\beta^2 = 0$ if and only if $\mathrm{d}\beta = 0$, giving rise to the **twisted cohomology** groups

$$H^\bullet_\beta(M) := H(\Omega^\bullet(M), \mathrm{d}_\beta).$$

Any nowhere vanishing section $s \in \Gamma(L)$ induces an isomorphism of differential complexes

$$\phi_s : (\Omega^\bullet, \mathrm{d}_\beta) \xrightarrow{\sim} (\Omega^\bullet(M, L), \mathrm{d}_\nabla), \quad \alpha \mapsto \alpha \otimes s, \tag{1.7.0.14}$$

where β is as in Equation 1.7.0.12. In particular ϕ_s induces an isomorphism in cohomology

$$\phi_s : H^\bullet_\beta(M) \xrightarrow{\sim} H^\bullet(M, L).$$

The above discussion applies to the (co)normal bundle of a foliation $\mathcal{F}$ on M. Recall that the normal bundle ν of a foliation $\mathcal{F}$, is canonically equipped with a flat $T\mathcal{F}$-connection called the **Bott connection**. It is defined by

$$\nabla : \Gamma(T\mathcal{F}) \times \Gamma(\nu) \to \Gamma(\nu), \quad \nabla_X \overline{N} := \overline{[X, Y]}, \quad \forall N \in \mathfrak{X}(M), X \in \Gamma(T\mathcal{F}),$$

where $\overline{N} := N \bmod T\mathcal{F}$. Dually, it induces a connection $\nabla^* : \Gamma(T\mathcal{F}) \times \Gamma(\nu^*) \to \Gamma(\nu^*)$ on ν^*, defined uniquely by the formula:

$$(\nabla^*_X \alpha)(\overline{N}) = \mathcal{L}_X(\alpha(\overline{N})) - \alpha(\nabla_X \overline{N}),$$

for any $\alpha \in \Gamma(\nu^*), N \in \mathfrak{X}(M)$, and $X \in \Gamma(T\mathcal{F})$. As above, this connection defines a differential

$$\mathrm{d}_{\mathcal{F}} : \Omega^\bullet(\mathcal{F}, \nu^*) \to \Omega^{\bullet+1}(\mathcal{F}, \nu^*), \tag{1.7.0.15}$$

on the complex $\Omega^\bullet(\mathcal{F}, \nu^*)$ and associated cohomology

$$H^\bullet(\mathcal{F}, \nu^*) := H(\Omega^\bullet(\mathcal{F}, \nu^*), \mathrm{d}_{\mathcal{F}}).$$

If $\mathcal{F}$ is coorientable, a nowhere vanishing section of ν^* is the same thing as a form $\gamma \in \Omega^1(M)$ for which $\mathcal{F} = \ker \gamma$. As before, this induces an isomorhism $\Omega^\bullet(\mathcal{F}, \nu^*) \simeq \Omega^\bullet(\mathcal{F})$, and under this identification $\mathrm{d}_{\mathcal{F}}$ is described as follows.

Lemma 1.7.19. *Let $(\mathcal{F}, M)$ be a foliated manifold and $\gamma \in \Omega^1(M)$ such that $\mathcal{F} = \ker \gamma$. Recall from Equation 1.7.0.10 that*

$$\mathrm{d}\gamma = \gamma \wedge \mu$$

for some $\mu \in \Omega^1(M)$. Then the following statements hold:

(i) Under the identification $\Omega^\bullet(\mathcal{F}, \nu^*) \simeq \Omega^\bullet(\mathcal{F})$ induced by γ, the differential $\mathrm{d}_{\mathcal{F}}$ from Equation 1.7.0.15 corresponds to d_μ, as in Equation 2.2.2.4;

(ii) Under the correspondence from Lemma 1.7.18, the Bott connection on ν^* corresponds to $\mathrm{mod}_{\mathcal{F}} \in H^1(\mathcal{F})$ as in Definition 1.7.16.

Proof. It follows directly from the definitions that

$$(\mathrm{d}_F \gamma)(X) = \beta(X)\gamma, \quad \forall X \in \mathfrak{X}(M)$$

Hence, let $N \in \mathfrak{X}(M)$ be such that $\gamma(N) = 1$, and compute:

$$\begin{aligned}(\mathrm{d}_{\mathcal{F}}\gamma)(X)(\overline{N}) &= (\nabla^*_X \gamma)(\overline{N}) \\ &= \mathcal{L}_X(\gamma(N)) - \gamma(\nabla_X(\overline{N})) \\ &= -\gamma([X, N]) \\ &= -\mathrm{d}\gamma(X, N) \\ &= -\gamma \wedge \beta(X, N) = \beta(X).\end{aligned}$$

□

The foliated cohomology allows us to give a rigorous definition of the "the variation of a foliated form in the direction transverse to the leaves". Before giving the definition, observe that there is a map:

$$p : \Omega^{k+1}_{\mathcal{F}}(M) \to \Omega^k(\mathcal{F}, \nu^*), \quad p(\alpha)(X_1, \ldots, X_k)(\overline{N}) := \alpha(X_1, \ldots, X_k, N). \tag{1.7.0.16}$$

Suppose that $\mathcal{F} = \ker\gamma$ for $\gamma \in \Omega^1(M)$. Then, similar to Equation 1.7.0.10, any $\alpha \in \Omega^{k+1}(M)$ satisfies $\alpha|_{\mathcal{F}} = 0$ if and only if

$$\alpha = \gamma \wedge \mu,$$

for some $\mu \in \Omega^k(M)$. This gives an isomorphism of differential complexes

$$\psi_\gamma : \left(\Omega^k(\mathcal{F}), \mathrm{d}_\beta\right) \xrightarrow{\sim} \left(\Omega^{k+1}_{\mathcal{F}}(M), \mathrm{d}\right), \quad \alpha \mapsto \tilde{\alpha} \wedge \gamma,$$

where $\tilde{\alpha} \in \Omega^\bullet(M)$ is any extension of α. Together with the isomorphism ϕ_γ from Equation 1.7.0.14 this gives a commutative diagram:

$$\begin{array}{ccc} \left(\Omega^{\bullet+1}_{\mathcal{F}}(M), \mathrm{d}\right) & \xrightarrow[p]{} & (\Omega^\bullet(\mathcal{F}, \nu^*), \mathrm{d}_{\mathcal{F}}) \\ \uparrow{\scriptstyle \psi_\gamma} & & {\scriptstyle \phi_\gamma}\uparrow \\ (\Omega^\bullet(\mathcal{F}), \mathrm{d}_\beta) & \xrightarrow{\mathrm{id}} & (\Omega^\bullet(\mathcal{F}), \mathrm{d}_\beta) \end{array}$$

In particular, p induces an isomorphism in cohomology.

Definition 1.7.20. *The* ***transverse differential*** $\mathrm{d}_\nu : H^\bullet(\mathcal{F}) \to H^\bullet(\mathcal{F}, \nu^*)$ *is the defined as the composition*

$$H^\bullet(\mathcal{F}) \xrightarrow{\delta} H^{\bullet+1}_{\mathcal{F}}(M) \xrightarrow{p} H^\bullet(\mathcal{F}, \nu^*),$$

where p is defined in Equation 1.7.0.16 and δ is the connecting homomorphism from Equation 1.7.0.8.

Note that, using the description of the connection homomorphism in Equation 1.7.0.9, we have

$$\mathrm{d}_\nu[\alpha] = [p(\mathrm{d}\tilde{\alpha})],$$

for any foliated form $\alpha \in \Omega^k(\mathcal{F})$, and any extension $\tilde{\alpha} \in \Omega^1(M)$ of α.

Remark 1.7.21. Given a nowhere vanishing section $\gamma \in \nu^*$, we can use ϕ_γ from Equation 1.7.0.14, to interpret the transverse differential as a map

$$\mathrm{d}_\nu : H^\bullet(\mathcal{F}) \to H^\bullet_\beta(\mathcal{F}).$$

Explicitely, given $\alpha \in \Omega^k(\mathcal{F})$, and any extension $\tilde{\alpha} \in \Omega^k(M)$, we have that $\mathrm{d}\tilde{\alpha}|_{\mathcal{F}} = 0$ so that

$$\mathrm{d}\tilde{\alpha} = \rho \wedge \gamma,$$

for some $\rho \in \Omega^{k-1}(M)$. Then it follows that $\mathrm{d}_\beta\rho = 0$, and

$$\mathrm{d}_\nu[\alpha] = [\rho] \in H^k_\beta(\mathcal{F}).$$

△

For a symplectic foliation the transverse differential allows us to measure the variation of the leafwise symplectic form.

Definition 1.7.22. *Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M. The **variation** of ω is the cohomology class*

$$\mathrm{var}_\omega := \mathrm{d}_\nu[\omega] \in H^2(\mathcal{F}, \nu^*).$$

*Moreover, $(\mathcal{F}, \omega)$ is called **tame** if $\mathrm{var}_\omega = 0$.*

The variation plays the same role for the leafwise symplectic form, as the modular class from Definition 1.7.16 does for the foliation. That is, it measures if ω can be extended to a globally closed form. First note that from the long exact sequence in Equation 1.7.0.8, and the definition of d_ν we have an exact sequence

$$\cdots \to H^2(M) \xrightarrow{r} H^2(\mathcal{F}) \xrightarrow{\mathrm{d}_\nu} H^2(\mathcal{F}, \nu^*) \to \ldots .$$

Hence, if $\mathrm{var}_\omega = 0$ then in cohomology $[\omega]$ comes from a class in $H^2(M)$. By the following lemma this also holds for any representative.

Lemma 1.7.23. *A (coorientable) symplectic foliation $(\mathcal{F}, \omega)$ is tame if and only if the leafwise symplectic form admits a closed extension $\widetilde{\omega} \in \Omega^2(M)$.*

Proof. Let $\gamma \in \Gamma(\nu^*)$ be a nowhere vanishing section, and use it to interpret the transverse variation as a map $\mathrm{d}_\nu : H^\bullet(\mathcal{F}) \to H^\bullet_\beta(\mathcal{F})$, as explained in Remark 1.7.21. Then, the assumption that $\mathrm{var}_\omega = 0$, means that given any extension $\eta \in \Omega^2(M)$ of ω, we can write

$$\mathrm{d}\eta = \mu \wedge \gamma,$$

where $\mu = \mathrm{d}_\beta \rho$ for some $\rho \in \Omega^1(M)$. Define $\widetilde{\omega} := \eta - \rho \wedge \gamma$, and note that $\widetilde{\omega}|_{\mathcal{F}} = \omega$, and

$$\mathrm{d}\widetilde{\omega} = \mathrm{d}\eta - \mathrm{d}\rho \wedge \gamma + \rho \wedge \mathrm{d}\gamma = \mathrm{d}_\beta\rho \wedge \gamma - (\mathrm{d}\rho + \beta \wedge \rho) \wedge \gamma = 0.$$

□

Example 1.7.24. Recall that given a contact structure ξ on a manifold M, and α a contact form for ξ, we can define the symplectic manifold

$$\left(\mathbb{R} \times M, \omega := \mathrm{d}(e^t \alpha)\right).$$

The definition of ω requires us to choose a contact form α, however for any choice of contact form the above formula defines a symplectic structure.

The analogous construction for symplectic foliations does not work in general. Let (γ, η) be an SF-pair representing a symplectic foliation $(\mathcal{F}, \omega)$ on M. Then, consider

$$(\mathbb{R} \times M, \omega := \eta + \mathrm{d}t \wedge \gamma).$$

Although, ω is always non-degenerate, it is closed if and only if (γ, η) is a cosymplectic structure. Such a representing pair exists if and only if $(\mathcal{F}, \omega)$ is tame and unimodular. △

The above discussion also applies to ∂-SF manifolds, and thus in particular, to transverse boundaries of SF-manifolds.

Definition 1.7.25. *Let $(\mathcal{F}, \omega)$ be a ∂-SF structure on N^{2n}, as in Definition 1.5.7. The **variation** of ω is the cohomology class*

$$\mathrm{var}_\omega := \mathrm{d}_\nu[\omega] \in H^2(\mathcal{F}, \nu^*).$$

*Moreover, $(\mathcal{F}, \omega)$ is called **tame** if $\mathrm{var}_\omega = 0$.*

Consider a ∂-SF structure $(\mathcal{F}, \omega)$ on a manifold N^{2n} (Definition 1.5.7). We want to characterize the set of symplectic forms on N which extend the leafwise form $\omega \in \Omega^2(\mathcal{F})$. We start by observing that not every manifold with a ∂-SF structure is symplectic.

Example 1.7.26. Consider the sphere $\mathbb{S}^3 \subset (\mathbb{R}^4, \omega_{can})$ where ω_{can} is the standard symplectic structure. The product $\mathbb{D}^4 \times \mathbb{S}^1$ has a symplectic foliation:

$$\left(\mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \mathbb{D}^4 \times \{z\}, \omega := \omega_{can}\right).$$

It is transverse to the boundary, and the induced ∂-SF manifolds equals $\mathbb{S}^3 \times \mathbb{S}^1$ with ∂-SF structure

$$\left(\mathcal{F}_\partial = \bigcup_{z \in \mathbb{S}^1} \mathbb{S}^3 \times \{z\}, \omega_\partial = \omega_{can}|_{\mathbb{S}^3}\right).$$

However, since $H^2(\mathbb{S}^3 \times \mathbb{S}^1) = 0$, it does not admit a symplectic structure. △

Lemma 1.7.27. *Let $(\mathcal{F}, \omega)$ be a ∂-SF structure on N (Definition 1.5.7), then the following are equivalent:*

(i) There exists a symplectic form $\widetilde{\omega} \in \Omega^2(N)$ such that

$$\widetilde{\omega}|_\mathcal{F} = \omega;$$

(ii) There exists a ∂-SF pair (γ, η) (Definition 1.5.8) representing $(\mathcal{F}, \omega)$ and an admissible form $\beta \in \Omega^1(N)$ satisfying

$$\eta^n = 0, \quad \mathrm{d}\eta = -\mathrm{d}_\mu \beta \wedge \gamma,$$

where $\mu \in \Omega^1(N)$ is the modular form of γ as in Equation 1.7.0.10.

Moreover, if either of the above holds, then $(\mathcal{F}, \omega)$ is tame conform definition 1.7.25. In this case, if $\mathcal{F}$ is also unimodular, then (γ, β, η) can be chosen so they form a ∂-SF triple of tameable type (Definition 1.6.8).

Proof. By Lemma 1.5.11 we can find a ∂-SF pair (γ, η) representing $(\mathcal{F}, \omega)$ such that $\eta^n = 0$. If $\widetilde{\omega}$ is a symplectic extension of ω, then $\widetilde{\omega}|_\mathcal{F} = \eta|_\mathcal{F}$. Hence, there exists $\beta \in \Omega^1(N)$ such that

$$\widetilde{\omega} = \eta + \beta \wedge \gamma.$$

Then, the non-degeneracy of $\widetilde{\omega}$ implies:

$$\widetilde{\omega}^n = n\beta \wedge \gamma \wedge \eta^{n-1} > 0,$$

so that β is an admissible form for (γ, η). Furthermore, since $\widetilde{\omega}$ is closed we have:

$$\mathrm{d}\widetilde{\omega} = \mathrm{d}\eta + \mathrm{d}\beta \wedge \gamma - \beta \wedge \mathrm{d}\gamma = \mathrm{d}\eta + (\mathrm{d}\beta + \mu \wedge \beta) \wedge \gamma = \mathrm{d}\eta + \mathrm{d}_\mu \beta \wedge \gamma = 0,$$

proving the first implication. For the converse, let (γ, η) and β be such that $\eta^n = 0$ and $\mathrm{d}\eta = -\mathrm{d}_\mu \beta \wedge \gamma$, then it follows immediately that

$$\widetilde{\omega} := \eta + \beta \wedge \gamma, \tag{1.7.0.17}$$

is a symplectic extension of ω. By Remark 1.7.21, the condition $\mathrm{d}\eta = -\mathrm{d}_\mu \beta \wedge \gamma$, is precisely saying that $\mathrm{var}_\omega = 0 \in H^2(\mathcal{F}, \nu^*)$ so that $(\mathcal{F}, \omega)$ is tame. Furthermore, if $\mathcal{F}$ is unimodular then γ can be chosen closed. In this case Equation 1.7.0.17 implies that (γ, β, η) is a ∂-SF triple of tameable type. □

The following discussion characterizes all possible symplectic extensions, if they exist. Thus suppose that $\widetilde{\omega} \in \Omega^2(N)$ is symplectic and satisfies

$$\widetilde{\omega}|_{\mathcal{F}} = \omega.$$

This implies that ω is closed, and has 1-dimensional kernel. Choose a leafwise vector field $X \in \mathfrak{X}(\mathcal{F})$ which is nowhere vanishing, and spans the kernel of ω. Furthermore, let $Y \in \mathfrak{X}(N)$ be nowhere vanishing, and transverse to $\mathcal{F}$. This induces a splitting

$$TN = T\mathcal{F} \oplus \langle X \rangle \oplus \langle Y \rangle.$$

Since, ω is determined on $T\mathcal{F}$, the extension is completely determined by the function

$$f := \widetilde{\omega}(X, Y) \in C^\infty(N).$$

Hence, if a symplectic extension exists, it is unique up to a function. The integral

$$\mathrm{vol}_{\widetilde{\omega}} := \int_N f \in \mathbb{R},$$

defines an invariant of $\widetilde{\omega}$. The following lemma says this constant determines the extension, up to symplectomorphism. In particular, the space of symplectic extensions of ω is either empty or one-dimensional.

Lemma 1.7.28. *Let $(\mathcal{F}, \omega)$ be a ∂-SF structure on (a compact manifold) N^{2n}. Given two symplectic extensions $\omega_0, \omega_1 \in \Omega^2(N)$ of ω, the following are equivalent:*

(i) The extensions are in the same cohomology class, $[\omega_1] = [\omega_0] \in H^2(N)$;

(ii) The extensions induce the same volume

$$\int_N \omega_1^n = \int_N \omega_0^n;$$

(iii) There exists a isotopy $\phi : N \to N$ such that

$$\phi^* \omega_1 = \omega_0.$$

Moreover, if $H^1(\mathcal{F}, \nu^*) = 0$, then any two extensions satisfy the above conditions, so that (if it exists) the extension is unique up to symplectomorphism.

Proof. The implications $(i) \implies (ii)$ and $(iii) \implies (i)$ are immediate, so we only prove $(ii) \implies (iii)$. Fix $\gamma \in \Omega^1(N)$ such that $\mathcal{F} = \ker \gamma$, and let $\mu \in \Omega^2(N)$ be its modular form as in Equation 1.7.0.10. Now assume we have two symplectic forms $\omega_0, \omega_1 \in \Omega^2(N)$ satisfying

$$\omega_1|_{\mathcal{F}} = \omega_0|_{\mathcal{F}} = \omega.$$

Then there exists $\beta \in \Omega^1(M)$ so that

$$\omega_1 = \omega_0 + \beta \wedge \gamma.$$

Since, $\mathrm{d}\omega_1 = \mathrm{d}\omega_0 = 0$, we have

$$\mathrm{d}(\beta \wedge \gamma) = (\mathrm{d}_\mu \beta) \wedge \gamma = 0.$$

Furthermore, since ω_0 and ω_1 have the same volume we find:

$$\int_N \omega_1^n = \int_N \omega_0^n + n \int_N \omega_0^{n-1} \wedge \beta \wedge \gamma. \tag{1.7.0.18}$$

Recall that deRham's theorem states that the intersection pairing $\langle \cdot, \cdot \rangle : H^k(N) \times H^{2n-k}(N) \to \mathbb{R}$, is nondegenerate. By the equation above,

$$\langle [\omega_0^{n-1}], [\beta \wedge \gamma] \rangle = 0,$$

implying that $\beta \wedge \gamma$ is exact. Define the 1-parameter family

$$\omega_t := \omega_0 + t\beta \wedge \gamma, \quad t \in [0, 1].$$

To check non-degeneracy, let $\Omega \in \Omega^{2n}(M)$ be a positive volume form. Then

$$\omega_t^n = \omega_0^n + tn\omega_0^{n-1} \wedge \beta \wedge \gamma = (tf + (1-t)(f+g))\Omega,$$

for some functions $f, g \in C^\infty(N)$. Since ω_0 and ω_1 are symplectic, it follows that f and $f + g$ are strictly positive, but then so is $tf + (1-t)(f+g)$ for all $t \in [0, 1]$. Therefore, ω_t is a path of symplectic forms, constant in cohomology. A standard Moser argument then gives the required isotopy.

Next we prove the second statement that if $H^1(\mathcal{F}, \nu^*) = 0$ then the symplectic extension is unique (if it exists). Note that if ω_0 and ω_1 are two non-symplectomorphic, then in particular they must have different volumes. Then, it follows from Equation 1.7.0.18 that

$$\int_N \omega_0^{n-1} \beta \wedge \gamma = \langle [\omega_0]^{n-1}, [\beta \wedge \gamma] \rangle > 0.$$

Therefore, $\beta \wedge \gamma$ is closed but not exact. Since,

$$\mathrm{d}(\beta \wedge \gamma) = (\mathrm{d}_\mu \beta) \wedge \gamma,$$

this implies that $[\beta] \in H^1_\mu(\mathcal{F})$ is non-zero. Conversely, if $H^1(\mathcal{F}, \nu^*) \simeq H^1_\mu(\mathcal{F}) = 0$, then any two symplectic extensions must have the same volume, and thus be symplectomorphic. □

By the above lemma, a ∂-SF structure $(\mathcal{F}, \omega)$, with $H^1(\mathcal{F}, \nu^*) = 0$ admits at most one symplectic extension. The following example shows that conversely if $H^1(\mathcal{F}, \nu^*) \neq 0$, then there exist many such extensions.

Example 1.7.29. Let $(\mathcal{F}, \omega)$ be a ∂-SF structure on N^{2n} and suppose it can be represented by a ∂-SF triple of cosymplectic type (u, v, w), as in Definition 1.6.8. Then, for any positive constant C there exists a symplectic extension of ω, with

$$\int_N \omega_C^n = C.$$

Indeed, the required extension is given by

$$\omega_C := w + Cv \wedge u.$$

Note that by Lemma 1.7.28 any symplectic extension is symplectomorphic to one of the above. △

Remark 1.7.30. A priori, the situation in the previous example seems to give some extra freedom in using turbulization to glue SF-manifolds. It shows that (for cosymplectic ∂-SF triples) we can define symplectic extensions with any volume. Hence by Lemma 1.7.28 this should increase the chances of finding a gluing diffeomorphism $\phi : \partial M_1 \to \partial M_2$.

However, for any two symplectic extensions ω_{C_0} and ω_{C_1}, there is a path of symplectic forms connecting them:

$$\omega_t := w + (tC_1 + (1-t)C_0)v \wedge u.$$

It turns out, that to glue two SF-manifolds with transverse boundary, it suffices that the symplectic extensions on the boundary of each pieces can be connected by a path of symplectic forms. Hence, the symplectic volumes never forms an obstruction to gluing △

The turbulization construction for symplectic foliations is stated as follows:

Theorem 1.7.31. *Let $(M, \mathcal{F}, \omega)$ be an SF-manifold such that $\mathcal{F}$ is transverse to the boundary, and denote by $(\mathcal{F}_\partial, \omega_\partial)$ the induced ∂-SF structure (Definition 1.5.6). If $\mathcal{F}_\partial$ is unimodular (i.e. can be defined by a closed form), then given any symplectic extension $\widetilde{\omega}_\partial \in \Omega^2(\partial M)$ of ω_∂, there exists an SF-structure $(\widetilde{\mathcal{F}}, \widetilde{\omega})$ on M satisfying:*

- *(i) $(\widetilde{\mathcal{F}}, \widetilde{\omega})$ is tame at the boundary, and the induced symplectic form on the boundary is $\widetilde{\omega}_\partial$;*
- *(ii) $(\widetilde{\mathcal{F}}, \widetilde{\omega})$ agrees with $(\mathcal{F}, \omega)$ away from the boundary.*

The main ingredient of the proof, given below, is the following cobordism, based on the turbulization construction for ∂-SF triples of tameable type, see Lemma 1.7.14.

Lemma 1.7.32. *Let $(N, \mathcal{F}_N, \omega_N)$ be a unimodular ∂-SF manifold and $\widetilde{\omega} \in \Omega^2(N)$ a symplectic extension of ω_N. Then there exists an SF-structure $(\mathcal{F}, \omega)$ on the trivial cobordism $[0,1] \times N$ such that:*

(i) $\mathcal{F}$ *is transverse to the left boundary, and the induced* ∂*-SF structure is* $(\mathcal{F}_N, \omega_N)$*;*

(ii) $(\mathcal{F}, \omega)$ *has tame right boundary, and the induced symplectic form on the boundary leaf is* $\widetilde{\omega}$*.*

Proof. By Lemma 1.7.27, there exists a ∂-SF triple of tameable type (u, v, w) such that

$$\widetilde{\omega} = w + v \wedge u.$$

Then, applying Lemma 1.7.14 gives the required SF-structure. □

Proof of Theorem 1.7.31. By assumption the boundary of $(M, \mathcal{F}, \omega)$ with its induced ∂-SF structure $(\mathcal{F}_\partial, \omega_\partial)$ satisfies the hypotheses of Lemma 1.7.32. Thus we obtain a cobordism $[0, 1] \times \partial M$ that can be glued to M using Proposition 1.6.22. The resulting SF-manifold is isomorphic to M and has tame boundary with symplectic leaf $(\partial M, \widetilde{\omega})$. □

1.7.1 Symplectic foliated open books

Open book decompositions, as discussed in Appendix 1.9 and Section 1.4.4, can also be used to construct symplectic foliations. The construction is based on the symplectic turbulization from the previous section. As we have seen, turbulization requries rather strong conditions on the symplectic foliations under consideration. Thus, the notion of an open book decomposition adapted to a SF-structure is much more restrictive than Definition 1.4.21, in the contact setting. The following definition is the SF-analogue of Remark 1.4.22.

Definition 1.7.33. *An SF-pair* (γ, η) *in* M *is* ***adapted*** *to an open book* (B, π) *if*

(i) The binding B *is a cosymplectic submanifold, i.e. the restriction* $(\gamma, \eta)|_B$ *is a cosymplectic structure (in particular* $B \pitchfork \mathcal{F}$*);*

(ii) The pair $(\mathcal{F}_\pi, \eta|_{\mathcal{F}_\pi})$ *is a symplectic foliation, where* $\mathcal{F}_\pi := \ker \mathrm{d}\pi$ *is the foliation induced by* $\pi : M \backslash B \to \mathbb{S}^1$ *and furthermore*

$$\iota_{v_1} \iota_{v_2}(\mathrm{d}\eta) = 0, \quad \forall v_1, v_2 \in \ker \mathrm{d}\pi.$$

An SF-structure $(\mathcal{F}, \omega)$ *is adapted to* (B, π) *if there is an adapted SF-pair representing it.*

As in the contact case, there is an analogous notion of adapted abstract open book:

Definition 1.7.34. *An* ***abstract SF open book*** *consists of a symplectic manifold* (Σ, ω) *with boundary of cosymplectic type, as in Definion 1.2.10, together with a symplectomorphism* $\phi : (\Sigma, \omega) \to (\Sigma, \omega)$ *which is the identity on a neighborhood of the boundary.*

Given a SF-structure $(\mathcal{F},\omega)$ adapted to (B,π), any η (that is part of an adapter SF-pair) defines a symplectic connection on the symplectic fibration $\pi : M\backslash B \to \mathbb{S}^1$, see Equation 1.4.4.1. The associated parallel transport induces a symplectomorphism ϕ of the symplectic page $(P, \eta|_P)$. By the following lemma ϕ can be assumed to equal the identity near the boundary ∂P.

Lemma 1.7.35. *Let $(\mathcal{F},\omega)$ be a SF-structure on M adapted to an open book decomposition (B,π). Then there exists an SF-pair (γ,η) representing $(\mathcal{F},\omega)$, and a symplectomorphism ϕ on $(P,\eta|_P)$, so that (P,η,ϕ) is an abstract SF open book.*

Proof. Since $(\mathcal{F},\omega)$ is adapted to the open book, there exists an adapted SF-pair (γ,η). By definition this implies that η defines a symplectic connection $\mathcal{H}$ on the symplectic fibration $\pi : M\backslash B \to \mathbb{S}^1$ (conform [62]). Observe that, for dimensional reasons, $\mathcal{H}$ is spanned by the kernel of η.

By Example 1.6.13, on a neighborhood of the binding, $(\mathcal{F},\omega)$ is isomorphic to the normal form. Recall that the normal bundle of the binding is trivial. Hence, passing to forms and using equivalences (cf. Definition 1.5.4), it means that (γ,η) is equivalent to the normal form. That is,

$$\gamma = \gamma_B, \quad \eta = \eta_B + r\mathrm{d}r \wedge \mathrm{d}\theta + \rho \wedge \gamma_B,$$

for some form $\rho \in \Omega^1(B \times \mathbb{D}^2)$, satisfying $\rho|_{B\times\{0\}} = 0$. Moreover, for any function $g \in C^\infty(B \times \mathbb{D}^2)$ we have that

$$\eta + g\mathrm{d}\theta \wedge \gamma,$$

is equivalent to η, conform Definition 1.5.4. Hence, we can assume without loss of generality that $\rho(\partial_\theta) = 0$.

Since (γ,η) is adapted, we have

$$\eta^n \wedge \mathrm{d}\theta = n\eta_B^{n-1} \wedge \rho \wedge \gamma_B \wedge \mathrm{d}\theta > 0, \quad \mathrm{d}\eta \wedge \mathrm{d}\theta = \mathrm{d}\rho \wedge \gamma \wedge \mathrm{d}\theta = 0$$

implying that $\rho(\partial_r) > 0$. Since $(\gamma_B, \eta_B + \rho \wedge \gamma_B)$ defines an SF-pair on B, it has an associated Reeb vector field $R \in \mathfrak{X}(B)$. It follows from the above equation that the Reeb vector field of (γ,η) also equals R, interpreted as a vector field on $B \times \mathbb{D}^2$. In particular,

$$\mathrm{d}r(R) = 0, \quad \mathrm{d}\theta(R) = 0. \tag{1.7.1.1}$$

Choose a non-negative function $g : [0,1] \to \mathbb{R}$ satisfying

$$g(r) := \begin{cases} r^2 & r \text{ near } 0 \\ 0 & r \text{ near } 1 \end{cases}.$$

Then

$$\tilde{\gamma} := \gamma_B, \quad \tilde{\eta} := \eta + g\mathrm{d}r \wedge \gamma_B = \eta_B + r\mathrm{d}r \wedge \mathrm{d}\theta + g(r)\mathrm{d}r \wedge \gamma_B,$$

defines an SF-pair on $B \times \mathbb{D}^2$ representing $(\mathcal{F},\omega)$, and which agrees with (γ,η) away from the binding. Hence, it extends to an SF-pair representing $(\mathcal{F},\omega)$ on M. Using Equation 1.7.1.1, it follows that the Reeb vector field of $(\tilde{\gamma},\tilde{\eta})$ equals

$$\tilde{R} = R + \frac{g}{r}\partial_\theta.$$

Hence, if $h : [0,1] \to \mathbb{R}$ is a non-negative function satisfying:

$$h(r) := \begin{cases} 0 & r \text{ near } 0 \\ 1 & r \text{ near } 1 \end{cases},$$

then the vector field

$$X := h(r)R + \frac{g(r)}{r}\partial_\theta \in \mathfrak{X}(B \times \mathbb{D}^2),$$

agrees with the Reeb vector field $\widetilde{R}$ away from the binding, and is a multiple of ∂_θ near the binding. Furthermore,

$$(\mathcal{L}_X \eta) \wedge \mathrm{d}\theta = (\mathrm{d}\iota_X \widetilde{\eta})\mathrm{d}\theta = \mathrm{d}(h\mathrm{d}r) \wedge \mathrm{d}\theta = 0.$$

Therefore, the flow of X, preserves the restriction of η to the fibers of π. Thus, the time one flow ϕ of a suitable rescaling of X defines a symplectomorphism of $(P, \eta|_P)$, which equals the identity near the boundary. □

Conversely, starting from an abstract SF-open book (Σ, ω, ϕ), we can construct an adapted SF-structure on the manifold $M(\Sigma, \phi)$. The construction uses symplectic turbulization defined in the previous section. Therefore, starting from an SF-structure $(\mathcal{F}, \omega)$ on M and applying Lemma 1.7.35 and Lemma 1.7.36 successively, (in general) we do not recover the original SF-structure, see Example 1.7.37 below.

Lemma 1.7.36. *Let (Σ, ω, ϕ) be an abstract SF-open book. Then the manifold $M(\Sigma, \phi)$, constructed in Lemma 1.9.5, with its canonical open book decomposition (B, π), admits an SF-structure $(\mathcal{F}, \omega)$.*

Note that the above lemma does not say that the SF-structure is adapted as in Definition 1.7.33.

Proof. Let (Σ, ω, ϕ) be an abstract SF-open book. Thus, ω has boundary $B := \partial\Sigma$ of cosymplectic type, as in Definition 1.2.10, and we denote the induced ∂-symplectic pair by (γ_B, η_B). The SF-pair

$$\gamma := \mathrm{d}\theta, \quad \eta := \omega,$$

on $\Sigma \times \mathbb{R}$, descends to the mapping cylinder

$$\Sigma \times_{\mathbb{Z}} \mathbb{R} := \Sigma \times \mathbb{R}/(\phi(x), \theta) \sim (x, \theta + 1).$$

The boundary $\partial\Sigma \times \mathbb{S}^1$ is of cosymplectic type, as in Definition 1.6.9, with induced ∂-SF triple

$$u = \mathrm{d}z, \quad v = \gamma_B, \quad w = \eta_B.$$

As in Example 1.6.11, the manifold $B \times \mathbb{D}^2$ admits an SF-pair defined by

$$\gamma := \gamma_B, \quad \eta := \eta_B + \mathrm{d}(r^2\mathrm{d}\theta),$$

which has boundary of cosymplectic type with ∂-SF triple

$$u = \gamma_B, \quad v = \mathrm{d}\theta, \quad w = \eta_B.$$

By Corollary 1.7.9 the above pieces can be connected by a regular SF-cobordism diffeomorphic to $[0,1] \times B \times \mathbb{S}^1$, Hence, we obtain a symplectic foliation on $M(\Sigma, \phi)$. □

Example 1.7.37. Consider the manifold $\mathbb{S}^1 \times \mathbb{S}^2$, endowed with the SF-structure

$$\mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \{z\} \times \mathbb{S}^2, \quad \omega := \omega_{\mathbb{S}^2},$$

where $\omega_{\mathbb{S}^2} \in \Omega^2(\mathbb{S}^2)$ denotes the standard area form on $\mathbb{S}^2$. Clearly, all the leaves of $\mathcal{F}$ are isomorphic to $\mathbb{S}^2$. The standard embedding $\mathbb{S}^1 \times \mathbb{S}^2 \subset \mathbb{S}^1 \times \mathbb{R}^3$ is transverse to the natural open book decomposition of $\mathbb{S}^1 \times \mathbb{R}^3$, and thus induces an open book decomposition on $\mathbb{S}^1 \times \mathbb{S}^2$, as in Example 1.9.3. Explicitly, the binding equals

$$B := \left(\mathbb{S}^1 \times \mathbb{S}^2\right) \cap \left(\mathbb{S}^1 \times \{(0,0,z) \mid z \in \mathbb{R}\}\right) \subset \mathbb{S}^1 \times \mathbb{S}^2,$$

and in the coordinates $(z, \phi, \theta) \in \mathbb{S}^1 \times \mathbb{S}^2$, where $(\phi, \theta) \in \mathbb{S}^2$ denote spherical coordinates, the fibration is given by

$$\pi : \mathbb{S}^1 \times \mathbb{S}^2 \backslash B \to \mathbb{S}^1, \quad (z, \phi, \theta) \mapsto \theta.$$

The SF-pair

$$\gamma := \mathrm{d}z, \quad \eta = \sin\phi \mathrm{d}\theta \wedge \mathrm{d}\phi + \sin\phi \mathrm{d}z \wedge \mathrm{d}\phi,$$

represents $(\mathcal{F}, \omega)$ and is adapted to (B, π). Indeed,

$$\eta \wedge \mathrm{d}\theta = \sin\phi \mathrm{d}z \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi > 0, \quad \mathrm{d}\eta \wedge \mathrm{d}\theta = 0.$$

The resulting abstract open book is diffeomorphic to $\Sigma := \mathbb{S}^1 \times [0,1]$. Turbulizing at the boundary of the mapping cylinder $\Sigma \times \mathbb{S}^1$, as in the proof of Lemma 1.7.36, produces two torus leaves. Hence, the resulting SF-structure is not isomorphic to $(\mathcal{F}, \omega)$ above. △

1.8 Deformations

In an oriented 3-dimensional manifold M, any cooriented hyperplane distribution is automatically oriented. Hence, a symplectic foliation is essentially the same as a (nowhere vanishing) form $\theta \in \Omega^1(M)$ satisfying $\theta \wedge \mathrm{d}\theta = 0$. On the other hand a contact form α satisfies $\alpha \wedge \mathrm{d}\alpha > 0$. Combining these conditions gives rise to the notion of a **confoliation**, a form $\alpha \in \Omega^1(M)$ satisfying

$$\alpha \wedge \mathrm{d}\alpha \geqslant 0,$$

as introduced and studied in [47]. Hence, in dimension 3, the natural type of deformations to study are given by a path of confoliations $\alpha_t \in \Omega^1(M)$, $t \in [0,1]$ satisfying

$$\alpha_0 \wedge \mathrm{d}\alpha_0 = 0, \quad \alpha_t \wedge \mathrm{d}\alpha_t > 0, \quad t \in (0,1]. \tag{1.8.0.1}$$

In higher dimensions, we have to handle the leafwise symplectic form making the situation more involved. On the level of structures, the natural generalization of confoliations is not so clear. We investigate several notions in Chapter 2. Here we consider following type of deformation on the level of differential forms.

Definition 1.8.1. *A* ***SF-deformation*** *on a manifold M^{2n+1} consists of a pair $(\alpha_t, \omega_t) \in \Omega^1(M) \times \Omega^2(M)$, $t \in [0,1]$ satisfying*

(i) (α_0, ω_0) *is a symplectic foliation pair as in Definition 1.5.3;*

(ii) α_t *is a contact form for all* $t > 0$ *and* $\omega_1 = \mathrm{d}\alpha_1$*;*

(iii) $\alpha_t \wedge \omega_t^n > 0$ *for all* $t \in [0,1]$.

By the third condition we can think of ω_t as a path of symplectic forms on $\ker \alpha_t$ interpolating between $\mathrm{d}\alpha_1$ and ω_0. Note that if $n = 1$, we recover the notion from Equation 1.8.0.1

Example 1.8.2. Several of the basic examples of contact structures and symplectic foliations from Example 1.3.8 and Example 1.5.5, can be connected by an SF-deformation:

- **Euclidean space:** Let $(x_1, y_1, \ldots, x_n, y_n, z)$ denote the standard Euclidean coordinates on $\mathbb{R}^{2n+1}$. Then,

$$\alpha_t := \mathrm{d}z + t\sum_{i=1}^{n} x_i \mathrm{d}y_i, \quad \omega_t = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i,$$

  defines a SF-deformation from the standard SF-pair to the standard contact form.

- **Tori:** Let (x, y, z) be standard angular coordinates on $\mathbb{T}^3$. Then, for each $k \in \mathbb{N}$ the pair

$$\alpha_t := \mathrm{d}z + t\left(\sin(tkz)\mathrm{d}x + \cos(tkz)\mathrm{d}y\right), \quad \omega := \mathrm{d}x \wedge \mathrm{d}y,$$

  defines a SF-deformation. Recall that although the higher dimensional tori are easily seen to have symplectic foliations, the analogous statement for contact structures depends on a construction by Bourgeois [96]. Hence, it is not immediate that the above deformation extends to higher dimensions. We consider this construction in more detail in Section 2.6.4.

- **Products:** Let M be endowed with a SF-deformation (α_t, ω_t), $t \in [0,1]$ and $(W, \mathrm{d}\lambda)$ be an exact symplectic manifold. Then, the product $M \times W$ admits a SF-deformation interpolating between the product contact and SF-structures from Example 1.3.8 and Example 1.5.5, given by:

$$\tilde{\alpha}_t := \alpha_t + t\lambda, \quad \omega_t = \omega_t + \mathrm{d}\lambda.$$

△

In all the examples above the deformations are given by affine paths. The following definition gives sufficient conditions for such a deformation to exist.

Definition 1.8.3. *Given a manifold M endowed with a*

(i) symplectic foliation pair (γ, η), as in Definition 1.5.3;

(ii) contact form α;

we say that (γ, η) is ***friendly*** *to α if:*

(i) $\alpha \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k} \geqslant 0$ for all $k = 0, \ldots, n$;

(ii) $\gamma \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k} \geqslant 0$ for all $k = 0, \ldots, n$;

(iii) $\alpha \wedge \mathrm{d}\alpha^{n-1} \wedge \mathrm{d}\gamma \geqslant 0$.

The proof of the following lemma is a straightforward computation.

Lemma 1.8.4. *Let (γ, η) be a symplectic folation pair friendly to a contact form α on M. Then, the affine path joining them;*

$$\gamma_t := (1-t)\gamma + t\alpha, \quad \eta_t := (1-t)\eta + t\mathrm{d}\alpha,$$

defines a SF-deformation.

Proof. By definition (γ_0, η_0) defines a symplectic foliation pair, and $\eta_1 = \mathrm{d}\gamma_1$. To see that γ_t is contact form $t > 0$, note that, since γ defines a foliation, $\mathrm{d}\gamma \wedge \mathrm{d}\gamma = 0$ and compute:

$$\begin{aligned}
\gamma_t \wedge \mathrm{d}\gamma_t^n &= ((1-t)\gamma + t\alpha) \wedge ((1-t)\mathrm{d}\gamma + t\mathrm{d}\alpha)^n \\
&= ((1-t)\gamma + t\alpha) \wedge \left(t^n \mathrm{d}\alpha^n + nt^{n-1}(1-t)\mathrm{d}\alpha^{n-1} \wedge \mathrm{d}\gamma\right) \\
&= t^n(1-t)\gamma \wedge \mathrm{d}\alpha^n + t^{n+1}\alpha \wedge \mathrm{d}\alpha^n + nt^n(1-t)\alpha \wedge \mathrm{d}\alpha^{n-1} \wedge \mathrm{d}\gamma,
\end{aligned}$$

which is positive for all $t > 0$. The condition $\gamma_t \wedge \eta_t^n > 0$ is checked by a similar computation:

$$\begin{aligned}
\gamma_t \wedge \eta_t^n &= ((1-t)\gamma + t\alpha) \wedge ((1-t)\eta + t\mathrm{d}\alpha)^n \\
&= ((1-t)\gamma + t\alpha) \wedge \left(\sum_{k=0}^{n} \binom{n}{k} (1-t)^k t^{n-k} \eta^k \wedge \mathrm{d}\alpha^{n-k}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} (1-t)^{k+1} t^{n-k} \gamma \wedge \eta^k \wedge \mathrm{d}\alpha^{n-k} + (1-t)^k t^{n-k+1} \alpha \wedge \eta^k \wedge \mathrm{d}\alpha^{n-k+1}.
\end{aligned}$$

Since (γ, η) and α are friendly, all the summands are non-negative. Moreover, if $t \neq 0$, the sum contains the strictly positive term $\alpha \wedge \mathrm{d}\alpha^n$, while if $(1-t) \neq 0$, there is the term $\gamma \wedge \eta^n$. Hence, $\gamma_t \wedge \eta_t^n > 0$ for all $t \in [0, 1]$. □

1.8.0.1 Deformations on open book decompositions

Consider an abstract open book decomposition (Σ, ϕ) of a manifold M. In Section 1.4.4 and Section 1.7.1 we have seen that if the page carries some additional structure we can construct a contact structure and a symplectic foliation on M. It turns out that under extra compatibility conditions, analogous to those in Definition 1.8.3, these two structures can be deformed into each other through an affine deformation. The precise statement is as follows:

Definition 1.8.5. *Let (Σ, ϕ) be an abstract open book, and denote the boundary of the page by $B := \partial\Sigma$. If Σ is endowed with a*

(i) symplectic form ω with cosymplectic boundary (B, γ_B, η_B), as in Definition 1.2.10, and $\phi^*\omega = \omega$;

(ii) exact symplectic form $\mathrm{d}\lambda$ with contact boundary (B, λ_B), as in Definition 1.2.10, and $\phi^*\lambda = \lambda$;

*we say that ω is **friendly** to $\mathrm{d}\lambda$ if:*

(i) $\omega^k \wedge \mathrm{d}\lambda^{n-k} \geqslant 0$ for all $k = 0, \ldots, n$;

(ii) (γ_B, η_B) is friendly to λ_B on B.

(iii) There exists a collar neighborhood $(-\varepsilon, 0] \times \partial\Sigma \subset \Sigma$ on which

$$\omega = \eta_B + \mathrm{d}t \wedge \gamma_B, \quad \lambda = (1+t)\lambda_B.$$

Observe that it is a direct consequence of Definition 1.2.10 that each of the formulas in condition (iii) above can be achieved in some collar neighborhood. However, in general these neighborhoods need not be the same. Condition (iii) requires that there is a single collar neighborhood realizing both formulas.

The main result is:

Theorem 1.8.6. *Let (Σ, ϕ) be an abstract open book, and $\omega, \mathrm{d}\lambda \in \Omega^2(\Sigma)$ symplectic forms which are friendly to each other (Definition 1.8.5). Then, the resulting manifold $M(\Sigma, \phi)$, admits a symplectic foliation pair (γ, η) and a contact form α which are friendly to each other (Definition 1.8.3).*
Moreover, the binding B is both a contact and a symplectic foliation submanifold, and the induced pair (γ_B, η_B) is friendly to α_B.

The proof follows the usual strategy of defining the required structure on each of the pieces that make up the open book manifold $M(\Sigma, \phi)$ and then gluing them together. Since the pieces are interesting on their own, we state them separately before combining them in the proof.
More precisely, the following two lemma's put together the contact structures and symplectic foliations from Example 1.4.9 and Example 1.6.11, saying that they are friendly, as in Definition 1.8.3. For the inside component we have:

Lemma 1.8.7. *Let B^{2n+1} be a closed manifold, and $0 < \delta < 1$ a constant.*

(i) If α_B is a contact form on B, then $B \times \mathbb{D}^2_\delta$ admits a contact form

$$\alpha := \alpha_B + r^2 \mathrm{d}\theta.$$

It has regular boundary of unimodular type with induced ∂-contact pair

$$(u = \mathrm{d}\theta, v = \alpha_B + \delta \mathrm{d}\theta).$$

(ii) If (γ_B, η_B) is a symplectic foliation pair on B, then $B \times \mathbb{D}^2_\delta$ admits a symplectic foliation pair

$$\gamma := \gamma_B, \quad \eta := \eta_B + 2r \mathrm{d}r \wedge \mathrm{d}\theta.$$

It has regular boundary of Cosymplectic type with induced ∂-symplectic foliation triple

$$(u = \gamma_B, v = \mathrm{d}\theta, w = \eta_B).$$

Moreover, if (γ_B, η_B) is friendly to α_B, then so are (γ, η) and α.

Proof. The existence of the contact and symplectic foliation forms follows from Example 1.4.9 and Example 1.6.11. The boundary types are also discussed there. It remains to check that (γ, η) is friendly to α as in Definition 1.8.3.

The first condition clearly holds for $k = n$, and for $k = 0$ becomes:

$$\begin{aligned} \alpha \wedge \eta^n &= \left(\alpha_B + r^2 \mathrm{d}\theta\right) \wedge \left(\eta_B^n + 2n\eta_B^{n-1} \wedge r\mathrm{d}r \wedge \mathrm{d}\theta\right) \\ &= 2n\alpha_B \wedge \eta_B^{n-1} \wedge r\mathrm{d}r \wedge \mathrm{d}\theta \geqslant 0, \end{aligned}$$

while for $1 \leqslant k \leqslant n-1$ we have:

$$\alpha \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k} = \alpha_B \wedge \left(2(n-k)\mathrm{d}\alpha_B^k \wedge \eta_B^{n-k-1} + 2k\mathrm{d}\alpha_B^{k-1} \wedge \eta_B^{n-k}\right) \wedge r\mathrm{d}r \wedge \mathrm{d}\theta \geqslant 0.$$

Similarly, the second condition clearly holds for $k = 0$, and for $k = n$ becomes:

$$\begin{aligned} \gamma \wedge \mathrm{d}\alpha^n &= \gamma_B \wedge \left(\mathrm{d}\alpha_B^n + 2n\mathrm{d}\alpha_B^{n-1} \wedge r\mathrm{d}r \wedge \mathrm{d}\theta\right) \\ &= 2n\gamma_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge r\mathrm{d}r \wedge \mathrm{d}\theta \geqslant 0, \\ \alpha \wedge \mathrm{d}\alpha^{n-1} \wedge \mathrm{d}\gamma &= (\alpha_B + r^2\mathrm{d}\theta) \wedge \left(\mathrm{d}\alpha_B^{n-1} + 2(n-1)\mathrm{d}\alpha_B^{n-2} \wedge r\mathrm{d}r \wedge \mathrm{d}\theta\right) \wedge \mathrm{d}\gamma_B \\ &= 2(n-1)\alpha_B \wedge \mathrm{d}\alpha_B^{n-2} \wedge \mathrm{d}\gamma_B \wedge r\mathrm{d}r \wedge \mathrm{d}\theta \geqslant 0 \end{aligned}$$

while for $1 \leqslant k \leqslant n-1$ the computation becomes:

$$\gamma \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k} = \gamma_B \wedge \left(2(n-k)\mathrm{d}\alpha_B^k \wedge \eta_B^{n-k-1} + 2k\mathrm{d}\alpha_B^{k-1} \wedge \eta^{n-k}\right) \wedge r\mathrm{d}r \wedge \mathrm{d}\theta \geqslant 0.$$

□

For the outside component we have:

Lemma 1.8.8. *Let (Σ^{2n}, ϕ) be an abstract open book.*

(i) Let ω be a symplectic form on Σ with cosymplectic type boundary (B, γ_B, η_B) and for which ϕ is a symplectomorphism. Then the mapping cyclinder $\Sigma \times_\phi \mathbb{R}$, admits a symplectic foliation pair (actually cosymplectic structure) induced by

$$\gamma := \mathrm{d}z, \quad \eta := \omega.$$

It has regular boundary of cosymplectic type with induced ∂-symplectic foliation triple

$$(u = \mathrm{d}\theta, v = \gamma_B, w = \eta_B).$$

(ii) Let $\mathrm{d}\lambda$ be a symplectic form on Σ with contact type boundary (B, λ_B) and such that $\phi^*\lambda = \lambda$. Then the mapping cylinder $\Sigma \times_\phi \mathbb{R}$, admits a contact form induced by

$$\alpha := \mathrm{d}z + \lambda.$$

It has regular boundary of Liouville type with induced ∂-contact pair

$$(u = \lambda_b, v = \lambda_B + \mathrm{d}z).$$

Moreover, if

$$\omega^k \wedge \mathrm{d}\lambda^{n-k} \geqslant 0, \quad \text{for all } k = 0, \dots, n,$$

then (γ, η) is friendly to α.

Proof. The existence of the contact and symplectic foliation forms follows from Example 1.4.9 and Example 1.6.10. The boundary types are also discussed there. It remains to show that (γ, η) is friendly to α as in Definition 1.8.3. This follows by observing that

$$\begin{aligned}
\alpha \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k} &= \mathrm{d}z \wedge \mathrm{d}\lambda^k \wedge \omega^{n-k} \geqslant 0 \\
\gamma \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k} &= \mathrm{d}z \wedge \mathrm{d}\lambda^k \wedge \omega^{n-k} \geqslant 0 \\
\alpha \wedge \mathrm{d}\alpha^{n-1} \wedge \mathrm{d}\gamma &= 0
\end{aligned}$$

□

The middle component allows us to glue the pieces of the two lemmas above.

Lemma 1.8.9. *Let B^{2n-1} be a closed manifold.*

(i) If α_B is a contact form on B, then the trivial cobordism $[0,1] \times \overline{B \times \mathbb{S}^1}$ admits a contact form α which has:

- *Regular left boundary of Unimodualar type with induced ∂-contact pair*

$$(u = \mathrm{d}\theta, v = \alpha_B + \delta \mathrm{d}\theta)$$

for any $0 < \delta < 1$;

- *Regular right boundary of Liouville type with induced ∂-contact pair*
$$(u = \alpha_B, v = \alpha_B + \mathrm{d}\theta).$$

(ii) If (γ_B, η_B) *is a cosymplectic pair on* B*, then the trivial cobordism* $[0,1] \times \overline{B \times \mathbb{S}^1}$ *admits a symplectic foliation pair* (γ, η) *which has:*

- *Regular left boundary of cosymplectic type with induced ∂-symplectic foliation triple*
$$(u = \gamma_B, v = \mathrm{d}\theta, w = \eta_B);$$
- *Regular right boundary of cosymplectic type with induced ∂-symplectic foliation triple*
$$(u = \mathrm{d}\theta, -\gamma_B, w = \eta_B).$$
- *A single closed leaf*
$$\left(\overline{B \times \mathbb{S}^1}, \eta_B + \mathrm{d}\theta \wedge \gamma_B\right).$$

Moreover, if (γ_B, η_B) *and* α_B *are friendly, then so are* (γ, η) *and* α*.*

Proof. The existence of the contact form follows from Lemma 1.4.19. There the contact form is described in terms of the ∂-contact pair (u, v). Here, we have an explicit description in terms of $\mathrm{d}\theta$ and α_B, and for the computations at hand it is convenient to describe the contact form in terms of these forms. Thus we consider

$$\alpha := \phi(t)\alpha_B + \psi(t)\mathrm{d}\theta,$$

for $\phi, \psi : [0,1] \to \mathbb{R}$ satisfying

$$\dot{\psi} \geqslant 0, \quad \phi > 0, \quad \phi\dot{\psi} - \dot{\phi}\psi > 0,$$

and

$$\dot{\phi} = \begin{cases} \geqslant 0 & \text{for } t \leqslant 1/2 \\ \leqslant 0 & \text{for } t \geqslant 1/2 \end{cases}, \quad \phi = \begin{cases} 1 & \text{for } t \text{ near } 0 \\ 2 - t & \text{for } t \text{ near } 1 \end{cases}, \quad \psi = \begin{cases} t + \delta & \text{for } t \text{ near } 0 \\ 1 & \text{for } t \text{ near } 1 \end{cases}.$$

The value $t = 1/2$ is special, because the symplectic foliation defined below has $\{1/2\} \times \overline{B \times \mathbb{S}^1}$ as its compact leaf.

The existence of the symplectic foliation pair follows from Lemma 1.7.6,. That is, we have

$$\gamma := f(t)\gamma_B + g(t)\mathrm{d}t + h(t)\mathrm{d}\theta, \quad \eta := \eta_B + f(t)\mathrm{d}t \wedge \mathrm{d}\theta + g(t)\mathrm{d}\theta \wedge \gamma_B + h(t)\gamma_B \wedge \mathrm{d}t,$$

for functions $f, g, h : [0,1] \to \mathbb{R}$ as in Figure 1.6.

The boundary types can be checked directly from the definitions, so it remains to show that (γ, η) is friendly to α. We split the cobordism into two pieces and check the conditions from on each of the pieces.

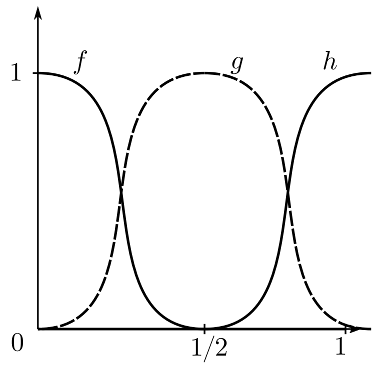

Figure 1.6: Functions f,g and h satisfying the required conditions for the proof of Lemma 1.8.9.

First we consider the part $\{t \leqslant 1/2\}$ where $h = 0$. We have the following identities:

$$\begin{aligned} \eta^k &= \eta_B^k + k\eta_B^{k-1} \wedge (f\mathrm{d}t + g\gamma_B) \wedge \mathrm{d}\theta \\ \mathrm{d}\alpha^k &= \phi^k \mathrm{d}\alpha_B^k + k\phi^{k-1}\mathrm{d}\alpha_B^{k-1}\mathrm{d}t \wedge \left(\dot{\phi}\alpha_B + \dot{\psi}\mathrm{d}\theta\right). \end{aligned}$$

The first condition of Definitions 1.8.3 clearly holds for $k = n$ and for $k = 0$ becomes:

$$\begin{aligned} \alpha \wedge \eta^n &= (\phi\alpha_B + \psi\mathrm{d}\theta) \wedge \left(\eta_B^n + n\eta_B^{n-1} \wedge (f\mathrm{d}t + g\gamma_B) \wedge \mathrm{d}\theta\right) \\ &= -n\phi f\mathrm{d}t \wedge \alpha_B \wedge \eta_B^{n-1} \wedge \mathrm{d}\theta \geqslant 0, \end{aligned}$$

while for $1 \leqslant k \leqslant n-1$ we have:

$$\begin{aligned} \alpha \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k} &= (\phi\alpha_B + \psi\mathrm{d}\theta) \wedge \Big((n-k)\phi^{n-k-1}\eta_B^k \wedge \mathrm{d}\alpha_B^{n-k-1} \wedge \mathrm{d}t \wedge \left(\dot{\phi}\alpha_B + \dot{\psi}\mathrm{d}\theta\right) \\ &\quad + k\phi^{n-k}\mathrm{d}\alpha_B^{n-k} \wedge \eta_B^{k-1} \wedge (f\mathrm{d}t + g\gamma_B) \wedge \mathrm{d}\theta\Big) \\ &= -(n-k)\left(\phi\dot{\psi} - \dot{\phi}\psi\right)\phi^{n-k-1}\mathrm{d}t \wedge \alpha_B \wedge \eta_B^k \wedge \mathrm{d}\alpha_B^{n-k-1} \wedge \mathrm{d}\theta \\ &\quad - k\phi^{n-k+1}f\mathrm{d}t \wedge \alpha_B \wedge \mathrm{d}\alpha_B^{n-k} \wedge \eta_B^{k-1} \wedge \mathrm{d}\theta \geqslant 0. \end{aligned}$$

The second condition clearly holds for $k = 0$ and for $k = n$ becomes:

$$\begin{aligned} \gamma \wedge \mathrm{d}\alpha^n &= (f\gamma_B + g\mathrm{d}t) \wedge \left(\phi^n \mathrm{d}\alpha_B^n + n\phi^{n-1}\mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}t \wedge \left(\dot{\phi}\alpha_B + \dot{\psi}\mathrm{d}\theta\right)\right) \\ &= -nf\phi^{n-1}\dot{\psi}\mathrm{d}t \wedge \gamma_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}\theta \geqslant 0. \end{aligned}$$

while for $1 \leqslant k \leqslant n-1$ we have:

$$\begin{aligned}\gamma \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k} &= (f\gamma_B + g\mathrm{d}t) \wedge \left(\eta_B^{n-k} + (n-k)\eta_B^{n-k-1} \wedge (f\mathrm{d}t + g\gamma_B) \wedge \mathrm{d}\theta\right) \\ &\quad \wedge \left(\phi^k \mathrm{d}\alpha_B^k + k\phi^{k-1}\mathrm{d}\alpha_B^{k-1} \wedge \mathrm{d}t \wedge \left(\dot{\phi}\alpha_B + \dot{\psi}\mathrm{d}\theta\right)\right) \\ &= -(n-k)f^2\phi^k \mathrm{d}t \wedge \gamma_B \wedge \mathrm{d}\alpha_B^k \wedge \eta_B^{n-k-1} \wedge \mathrm{d}\theta \\ &\quad - kf\phi^{k-1}\dot{\psi}\mathrm{d}t \wedge \gamma_B \wedge \mathrm{d}\alpha_B^{k-1} \wedge \eta_B^{n-k} \wedge \mathrm{d}\theta \geqslant 0.\end{aligned}$$

Lastly we check:

$$\begin{aligned}\alpha \wedge \mathrm{d}\alpha^{n-1} \wedge \mathrm{d}\gamma &= (\phi\alpha_B + \psi\mathrm{d}\theta) \wedge \phi^{n-1}\mathrm{d}\alpha_B^{n-1} \wedge \dot{f}\mathrm{d}t \wedge \gamma_B \\ &\quad + (\phi\alpha_B + \psi\mathrm{d}\theta) \wedge (n-1)\mathrm{d}\alpha_B^{n-2} \wedge \mathrm{d}t \wedge \left(\dot{\phi}\alpha_B + \dot{\psi}\mathrm{d}\theta\right) \wedge \dot{f}\mathrm{d}t \wedge \gamma_B \\ &= \dot{\psi}\dot{f}\phi^{n-1}\mathrm{d}t \wedge \gamma_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}\theta \geqslant 0.\end{aligned}$$

Next we consider the part $\{t \geqslant 1/2\}$ where $f = 0$. We have the following identities:

$$\eta^k = \eta_B^k + k\eta_B^{k-1} \wedge (g\mathrm{d}\theta - h\mathrm{d}t) \wedge \gamma_B,$$

while $\mathrm{d}\alpha^k$ is the same as above. The first condition clearly holds for $k = n$ and for $k = 0$ it becomes:

$$\begin{aligned}\alpha \wedge \eta^n &= (\phi\alpha_B + \psi\mathrm{d}\theta) \wedge \left(\eta_B^n + n\eta_B^{n-1} \wedge (g\mathrm{d}\theta - h\mathrm{d}t) \wedge \gamma_B\right) \\ &= -nh\psi\,\mathrm{d}t \wedge \gamma_B \wedge \eta_B^{n-1} \wedge \mathrm{d}\theta \geqslant 0\end{aligned}$$

while for $1 \leqslant k \leqslant n-1$ we have:

$$\begin{aligned}\alpha \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k} &= (\phi\alpha_B + \psi\mathrm{d}\theta) \wedge \left(\phi^k \wedge \mathrm{d}\alpha_B^k + k\phi^{k-1}\mathrm{d}\alpha_B^{k-1} \wedge \mathrm{d}\theta \wedge \left(\dot{\phi}\alpha_B + \dot{\psi}\mathrm{d}\theta\right)\right) \\ &\quad \wedge \left(\eta_B^{n-k} + (n-k)\eta_B^{n-k-1} \wedge (g\mathrm{d}\theta - h\mathrm{d}t) \wedge \gamma_B\right) \\ &= -(n-k)\psi\phi^k h \wedge \mathrm{d}t \wedge \gamma_B \wedge \eta_B^{n-k-1} \wedge \mathrm{d}\alpha_B^k \wedge \mathrm{d}\theta \\ &\quad - k\phi^{k-1}\left(\phi\dot{\psi} - \dot{\phi}\psi\right)\mathrm{d}t \wedge \alpha_B \wedge \mathrm{d}\alpha_B^{k-1} \wedge \eta_B^{n-k} \wedge \mathrm{d}\theta \geqslant 0.\end{aligned}$$

The second condition clearly holds for $k = 0$ and for $k = n$ it becomes:

$$\begin{aligned}\gamma \wedge \mathrm{d}\alpha^n &= (g\mathrm{d}t + h\mathrm{d}\theta) \wedge \left(\phi^n \mathrm{d}\alpha_B^n + n\phi^{n-1}\mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}t \wedge \left(\dot{\phi}\alpha_B + \dot{\psi}\mathrm{d}\theta\right)\right) \\ &= n\dot{\phi}h\phi^{n-1}\mathrm{d}t \wedge \alpha_B \wedge \mathrm{d}\alpha_B \wedge \mathrm{d}\theta \geqslant 0.\end{aligned}$$

while for $1 \leqslant k \leqslant n-1$ we have:

$$\begin{aligned}\gamma \wedge \mathrm{d}\alpha^k \wedge \eta^{n-k-1} &= -(n-k)\phi^k(g^2 + h^2)\mathrm{d}t \wedge \gamma_B \wedge \mathrm{d}\alpha_B^k \wedge \eta_B^{n-k-1} \wedge \mathrm{d}\theta \\ &\quad + kh\phi^{k-1}\dot{\phi}\mathrm{d}t \wedge \alpha_B \wedge \mathrm{d}\alpha_B^{k-1} \wedge \eta_B^{n-k} \wedge \mathrm{d}\theta \geqslant 0.\end{aligned}$$

Lastly we check:

$$\begin{aligned}\alpha \wedge \mathrm{d}\alpha^{n-1} \wedge \mathrm{d}\gamma = &(\phi\alpha_B + \psi \mathrm{d}\theta) \wedge \left(\phi^{n-1} \wedge \mathrm{d}\alpha_B^{n-1} + (n-1)\phi^{n-2} \wedge \mathrm{d}\alpha_B^{n-2}\right) \\ &\wedge \mathrm{d}t \wedge \left(\dot{\phi}\alpha_B + \dot{\psi}\mathrm{d}\theta\right) \wedge \dot{h}\mathrm{d}t \wedge \mathrm{d}\theta \\ = &-\phi^n \dot{h}\mathrm{d}t \wedge \alpha_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}\theta \geqslant 0.\end{aligned}$$

□

Proof of Theorem 1.8.6. Choose $0 < \delta < 1$, write $B := \partial\Sigma$, and decompose the filled mapping cylinder into three pieces:

$$M(\Sigma,\phi) = \left(B \times \mathbb{D}^2_\delta\right) \cup \left([\delta,1] \times \overline{B \times \mathbb{S}^1}\right) \cup \left(\Sigma \times \mathbb{S}^1\right).$$

Applying Lemma 1.8.7, Lemma 1.8.9, and Lemma 1.8.8 each of the pieces admits a symplectic foliation pair and a contact form friendly to each other. Hence, it follows from Lemma 1.8.4 that each of the pieces admits an affine deformation.

It remains to show that the pieces can be glued. Since the boundary types of the contact form match, it is possible to glue the contact forms on each piece to one on $M(\Sigma,\phi)$. The same holds for the symplectic foliation pair. However, recall that the gluing depends on a choice of collar neighborhood putting the contact form (resp. the symplectic foliation pair) in the required normal form. Therefore, we need to ensure that we can find a single collar on which both the structures are in normal form. For the common boundary of $B \times \mathbb{D}^2_\partial$ and $[\delta,1] \times \overline{B \times \mathbb{S}^1}$ this follows directly from the definitions. For the common boundary of $[0,1] \times \overline{B \times \mathbb{S}^1}$ and $\Sigma \times \mathbb{S}^1$, it is ensured by the last condition in Definition 1.8.5. □

Example 1.8.10. Recall Giroux correspondence stating that given a closed 3-manifold, there is a 1-1 correspondence between (cooriented) contact structures up to isotopy and open books up to stabilization. In particular, any contact 3-manifold admits an abstract contact open book $(\Sigma, \mathrm{d}\lambda)$.

Since the binding, $B := \partial\Sigma$, is 1-dimensional the boundary is simultaneously of contact and cosymplectic type. In fact, the conditions of Definition 1.8.3 are trivially satisfied, and applying Theorem 1.8.6 we obtain:

Corollary 1.8.11. *Any contact structure on a closed 3-manifold can be (affinely) deformed into a symplectic foliation.*

△

As in the 3-dimensional case, any contact structure in higher dimensions admits an adapted open book decomposition. Suppose that we start with an abstract open book (Σ,ϕ) that admits an exact symplectic form $\mathrm{d}\lambda$ with boundary of contact type. A necessary condition for Σ to also admit a symplectic structure ω with cosymplectic type boundary, is that $B := \partial\Sigma$ admits a cosymplectic structure (γ_B, η_B) for which η_B is in the image of the restriction map

$$\iota^* : H^2_\phi(\Sigma) \to H^2(\partial\Sigma).$$

Here we denote by $H^2_\phi(\Sigma) \subset H^2(\Sigma)$ those cohomology classes which can be represented by a closed 2-form that is invariant under pullback by ϕ. Note that in general this is not the same as a cohomology class that is invariant under pullback by ϕ.

Remark 1.8.12. Suppose that the monodromy satisfies $\phi^k = \mathrm{id}$ for some $k \in \mathbb{N}$. Given any class $[\omega] \in H^2(\Sigma)$ define

$$\tilde{\omega} := \frac{1}{k}\sum_{i=0}^{k}(\phi^i)^*\omega.$$

Then, $[\tilde{\omega}] \in H^2_\phi(\Sigma)$ and $\iota^*[\tilde{\omega}] = \iota^*[\omega]$ since the monodromy equals the identity on a neighborhood of the boundary. Therefore, it suffices in this case that η_B is in the image of the restriction map

$$\iota^* : H^2(\Sigma) \to H^2(\partial\Sigma).$$

△

An (affine) deformation between the resulting structures will induce a deformation on the boundary B. Hence, for such a deformation to exist we necessarily need (γ_B, η_B) and λ_B to be friendly.

Definition 1.8.13. *An abstract open book (Σ, ϕ) with boundary $B := \partial\Sigma$, is said to be of* ***deformation type*** *if:*

(i) There exists an exact symplectic form $\mathrm{d}\lambda$ on Σ, of contact type at the boundary, and such that $\phi^*\mathrm{d}\lambda = \mathrm{d}\lambda$;

(ii) There exists a cosymplectic structure (γ_B, η_B) on B such that $[\eta_B]$ is in the image of the restriction map

$$\iota^* : H^2_\phi(\Sigma) \to H^2(B);$$

(iii) The contact form λ_B is friendly to (γ_B, η_B).

In this situation we can use the following specialization of Theorem 1.8.14, to obtain a contact structure and symplectic foliation on $M(\Sigma, \phi)$ and a deformation between them.

Theorem 1.8.14. *Let (Σ, ϕ) be an abstract open book decomposition of deformation type. Then the resulting manifold $M(\Sigma, \phi)$ admits a contact form α and a symplectic foliation pair (γ, η) which are friendly to each other (Definition 1.8.3).*

The proof follows immediately from combining Theorem 1.8.6 with Lemma 1.8.18 below. We start by showing that the restriction map in cohomology can be lifted to the level of forms.

Lemma 1.8.15. *Let $\iota : N \hookrightarrow M$ be a submanifold. If $\eta \in \Omega^k(N)$ is a closed form whose cohomology class is in the image of the restriction map $\iota^* : H^k(M) \to H^k(N)$. Then there exists $\omega \in \Omega^k(M)$ such that*

$$\mathrm{d}\omega = 0, \quad \omega|_N = \eta.$$

Proof. Let $\tilde{\omega} \in \Omega^k(M)$ be such that $\iota^*[\tilde{\omega}] = [\omega]$, and $\mathcal{U}$ a tubular neighborhood of N. Then, using that $\mathcal{U}$ retracts onto N we have

$$\tilde{\omega} - \eta = \mathrm{d}\gamma,$$

for some $\gamma \in \Omega^{k-1}(\mathcal{U})$. Hence, the required form is defined by

$$\omega := \tilde{\omega} + \mathrm{d}(\rho\gamma),$$

where $\rho \in C^\infty(\mathcal{U})$ is a suitable bump function. □

Remark 1.8.16. In the above proof, we can choose the bump function ρ to be zero on a neighborhood of N. Hence, if the collar neighborhood is a product $\mathcal{U} = N \times \mathbb{D}^{m-n}$ then in these coordinates we have $\omega = \eta$ on a neighborhood of $N = N \times \{0\}$. △

For a suitable open book decomposition, the above trick allows us to extend a cosymplectic structure on the binding to a symplectic form on the page, which has cosymplectic boundary.

Lemma 1.8.17. *Let (Σ^{2n}, σ) be symplectic manifold with boundary, $\phi \in \mathrm{Symp}(\Sigma, \sigma)$ a symplectomorphism equal to the identity on a neighborhood of $\partial\Sigma$, and (γ, η) a cosymplectic structure on $\partial\Sigma$ satisfying:*

(i) The class $[\eta]$ is in the image of the restriction map $\iota^* : H^2_\phi(\Sigma) \to H^2(\partial\Sigma)$;

(ii) For $\sigma_\partial := \sigma|_{\partial\Sigma}$ we have $\gamma \wedge \sigma_\partial^k \wedge \eta^{n-k} \geqslant 0$ for all $k = 1, \ldots, n-1$.

Then, for $\varepsilon > 0$ small enough, there exists a symplectic form $\omega \in \Omega^2(\Sigma)$, for which ϕ is a symplectomorphism and with regular boundary of cosymplectic type and induced ∂-symplectic pair

$$(\gamma, \sigma + \varepsilon\eta).$$

Proof. By the previous lemma we find a closed extension $\tilde{\eta} \in \Omega^2(\Sigma)$ of η, which is invariant under pullback by ϕ. Hence, for $\varepsilon > 0$ small enough

$$\omega := \sigma + \varepsilon\tilde{\eta},$$

is symplectic, and invariant under pullback by ϕ. To see that γ is admissible for $\omega_\partial := \omega|_{\partial\Sigma}$ observe:

$$\begin{aligned}\gamma \wedge \omega_\partial^{n-1} &= \gamma \wedge (\sigma_\partial + \varepsilon\eta)^{n-1} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} \varepsilon^{n-k-1} \gamma \wedge \sigma_\partial^k \wedge \eta^{n-k-1} > 0,\end{aligned}$$

since all summands are non-negative and strictly positive for $k = 0$. □

Going back to the proof of Theorem 1.8.14; to prove Theorem 1.8.14, we want to apply Theorem 1.8.6. The symplectic form constructed in the lemma above satisfies all but one of the conditions of Definition 1.8.5. More precisely, we still need to show that there exist a collar neighborhood on which both the symplectic forms are in the required normal form. We show this in the following lemma, by modifying the construction above close to the boundary of the page.

Lemma 1.8.18. *Let $(\Sigma, \mathrm{d}\lambda, \phi)$ be an abstract contact open book, and assume the boundary $B := \partial\Sigma$ admits a cosymplectic structure (γ_B, η_B) satisfying:*

(i) The class $[\eta_B]$ is in the image of the restriction map $\iota^* : H^2(\Sigma) \to H^2(\partial\Sigma)$;

(ii) The cosymplectic pair (γ_B, η_B) is friendly to α_B.

Then, for $\varepsilon > 0$ small enough, there exists:

(i) A symplectic structure $\omega \in \Omega^2(\Sigma)$ with boundary of cosymplectic type and induced ∂-symplectic pair
$$(\gamma_B, \mathrm{d}\alpha_B + \varepsilon\eta_B)\,.$$

(ii) A collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times B$ on which
$$\alpha = (1+t)\alpha_B, \quad \omega = (\mathrm{d}\alpha_B + \varepsilon\eta_B) + \mathrm{d}t \wedge \alpha_B.$$

Proof. Since $\mathrm{d}\lambda$ has boundary of contact type we can find a collar neighborhood $\mathcal{U} \simeq (-\varepsilon, 0] \times B \subset \Sigma$ on which
$$\alpha = (1+t)\alpha_B.$$

We fix this collar neighborhood for the rest of the proof. Following Lemma 1.8.17, we can consider the symplectic form
$$\varepsilon\eta + \mathrm{d}\alpha,$$
for $\varepsilon > 0$ small enough, and $\eta \in \Omega^2(\Sigma)$ a closed extension of η_B. By Remark 1.8.16 we can assume that $\eta = \eta_B$ on $\mathcal{U}$.

By slightly altering the symplectic form above we can ensure that it has the desired normal form near the boundary. More precisely, define the closed form
$$\omega := \varepsilon\eta_B + \mathrm{d}(f\alpha_B) + \mathrm{d}(g\gamma_B),$$
for suitable functions $f, g : (-\varepsilon, 0] \to \mathbb{R}$ to be chosen later. To have the required normal form, close to the boundary we want
$$\omega = \varepsilon\eta_B + \mathrm{d}\alpha_B + \mathrm{d}t \wedge \gamma_B = \varepsilon\eta_B + \mathrm{d}\alpha_B + \mathrm{d}(t\gamma_B),$$
while away from the boundary we want
$$\omega = \varepsilon\eta + \mathrm{d}\alpha = \varepsilon\eta_B + \mathrm{d}\left((1+t)\alpha_B\right).$$

Hence, the first conditions on the functions are:

(1.8.0.2)
$$f = \begin{cases} 1 & \text{for } t \text{ near } 0 \\ 1+t & \text{for } t \text{ near } -\varepsilon \end{cases}, \quad g = \begin{cases} t & \text{for } t \text{ near } 0 \\ 0 & \text{for } t \text{ near } -\varepsilon \end{cases}.$$

The non-degeneracy condition for ω reads:

$$\begin{aligned}\omega^n &= (f\mathrm{d}\alpha_B + \varepsilon\eta_B)^n + n\,(f\mathrm{d}\alpha_B + \varepsilon\eta_B)^{n-1} \wedge \mathrm{d}t \wedge \left(\dot f\alpha_B + \dot g\gamma_B\right) \\ &= n\sum_{k=0}^{n} f^k\varepsilon^{n-k-1}\binom{n-1}{k}\mathrm{d}t \wedge \mathrm{d}\alpha_B^k \wedge \eta_B^{n-k-1} \wedge \left(\dot f\alpha_B + \dot g\gamma_B\right).\end{aligned}$$

Hence, ω will be symplectic if

(1.8.0.3)
$$\dot f^2 + \dot g^2 > 0, \quad \dot f \gg 0 \text{ whenever } \dot g \leqslant 0.$$

The last condition means that at points $t \in (-\varepsilon, 0]$ where $\dot g \leqslant 0$, we have that $\dot f$ is much larger than 0, so that the positive summand in the equation above dominates. It is not hard to see that functions f and g satisfying these conditions exist as shown in Figure 1.7. □

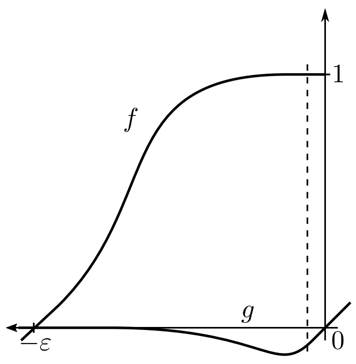

Figure 1.7: Functions f and g satisfying the conditions in Equation 1.8.0.2 and Equation 1.8.0.3.

1.9 An application: Mitsumatsu's construction on $\mathbb{S}^5$

It was shown by Lawson[72] that the spheres $\mathbb{S}^{2^k+3}$, $k = 1, 2, \ldots$ admit (codimension-1) foliations. For $\mathbb{S}^5$ Mitsumatsu[89] proved that the foliation resulting from Lawsons construction carries a leafwise symplectic form. Using the right openbook decomposition, this symplectic foliation can be obtained using Lemma 1.7.36. Moreover, using the results of the previous section we show that it is part of a SF-deformation as in Definition 1.8.1.

Theorem 1.9.1. *The Lawson foliation on $\mathbb{S}^5$ admits a leafwise symplectic form and the resulting symplectic foliation can be deformed to a contact structure, as in Definition 1.8.1.*

Proof. Consider the Hopf fibration $h : \mathbb{S}^5 \to \mathbb{CP}^2$, which is an principal $\mathbb{S}^1$ bundle. The map $f : \mathbb{S}^5 \to \mathbb{C}$ defined by restricting

$$f(z_0, z_1, z_2) := z_0^3 + z_1^3 + z_2^3, \quad (z_0, z_1, z_2) \in \mathbb{C}^3,$$

satisfies

$$f(\lambda \cdot z) = \lambda^3 f(z), \quad z \in \mathbb{C}^3, \lambda \in \mathbb{S}^1, \tag{1.9.0.1}$$

so that by the genus-degree formula its zero-locus in $\mathbb{CP}^2$ is diffeomorphic to the torus. The infinitesimal vector field of the $\mathbb{S}^1$-action is the Reeb vector field of the standard contact form α on $\mathbb{S}^5$. Therefore, f defines an adapted open book decomposition by Lemma 1.9.8, whose binding is an $\mathbb{S}^1$-bundle $h_B : B \to \mathbb{T}^2$. Note that α_B defines a principal $\mathbb{S}^1$-connection on this bundle since by definition of the Reeb vector field we have

$$\alpha_B(R) = 1, \quad \mathcal{L}_R \alpha = 0.$$

By definition of an adapted open book, we obtain an exact symplectic page $(\Sigma, \mathrm{d}\alpha)$ with boundary of contact type (B, α_B). Pulling back the standard (oriented) coframe on $\mathbb{T}^2$ we obtain closed forms $\theta_1, \theta_2 \in \Omega^1(B)$. It is straightforward to check that

$$\gamma_B := \theta_1, \quad \eta_B := \theta_2 \wedge \alpha_B,$$

defines a cosymplectic structure friendly to α_B (Definition 1.8.3). Finally, a standard Mayer-Vietoris argument shows that the restriction map $\iota^* : H^2(\Sigma) \to H^2(B)$ is surjective, see for example Lemma 5.3 in [89] or Lemma 6.4.7 in [96].
The monodromy ϕ of the open book is induced by the $\mathbb{S}^1$-action, so that it follows from Equation 1.9.0.1 that $\phi^3 = \mathrm{id}$. As explained in Remark 1.8.12 this means that it suffices to ask the restriction map $\iota^* : H^2(\Sigma) \to H^2(B)$ to be surjective when applying Theorem 1.8.14. So, there exists a symplectic foliation pair and a contact structure on $\mathbb{S}^5$ that are friendly to each other.

Observe, that Equation 1.9.0.1 actually shows that f descends to a function $\tilde{f}$ on the quotient $\mathbb{S}^5/\mathbb{Z}_3$, where the $\mathbb{Z}_3$ action is induced by ϕ. Moreover, the standard contact form also descends, and $\tilde{f}$ defines an adapted open book decomposition with trivial monodromy. The same argument as above then shows that $\mathbb{S}^5/\mathbb{Z}_3$ admits a symplectic foliation pair and a contact form that are friendly to each other. Pulling back, these structures under the quotient map $\mathbb{S}^5 \to \mathbb{S}^5/\mathbb{Z}_3$ recovers the ones from before. □

Appendix A: Open book decompositions

We recall here the definition and basic properties of open book decompositions. Morally speaking an open book decomposition of a manifold M tries to fiber the manifold over $\mathbb{S}^1$. Of course not every manifold, globally admits such a fibration, and so we divide the manifold into two pieces. The first one fibers over $\mathbb{S}^1$ and the fibers are called the pages of the open book. The fibration is extended over the other piece, called the binding, using a local model. This model "glues the pages to the binding", so that the resulting picture is that of a "book opened so that the front touches the back".

Definition 1.9.2. *An* ***(geometric) open book decomposition*** *of a manifold M, is a pair (B, π) consisting of:*

(i) A codimension-2 submanifold $B \subset M$, with trivial normal bundle ν_B;

(ii) A fibration $\pi : M \backslash B \to \mathbb{S}^1$, such that there exists a neighborhood $B \times \mathbb{D}^2 \subset M$ of B on which:

$$\pi(b, x) = \frac{x}{||x||}. \tag{1.9.0.2}$$

We refer to B as the ***binding*** *and to $P := \overline{\pi^{-1}(1)}$ as the (closed)* ***page*** *of the open book.*

The normal form in Equation 1.9.0.2 implies that P is a submanifold with boundary $\partial P = B$. The choice of $1 \in \mathbb{S}^1$, in the definition of the page is not important. For any $\phi \in \mathbb{S}^1$ we define the corresponding page

$$P_\phi := \overline{\pi^{-1}(\phi)},$$

and these are all diffeomorphic.

Example 1.9.3. The following are some basic examples of open book decompositions:

- **Euclidean space:** Let (r, ϕ, z) denote the cylindrical coordinates on $\mathbb{R}^3$. The standard open book decomposition of $\mathbb{R}^3$ is defined by taking $B := \{r = 0\}$, and
$$\pi : \mathbb{R}^3 \backslash B \to \mathbb{S}^1, \quad (r, \phi, z) \mapsto \phi.$$
Note that open books can be "pulled back" in the sense that if (B, π) is an open book decomposition of M and $f : \widetilde{M} \to M$ is a submersion then,
$$\widetilde{B} := f^{-1}(B), \quad \tilde{\pi} := \pi \circ f,$$
defines an open book decomposition on $\widetilde{M}$. Since for any $n \geqslant 2$ the projection $\mathrm{pr} : \mathbb{R}^n \to \mathbb{R}^2$ is a submersion, we obtain an open book decomposition on any Euclidean space.

- **Submanifolds:** Open books can also be "restricted" to submanifolds, in the sense that if (B, π) is an open book decomposition of M and $\widetilde{M} \subset M$ is a submanifold transverse to B and the fibers of π, then

  $$\widetilde{B} := \widetilde{M} \cap B, \quad \widetilde{\pi} := \pi|_{\widetilde{M}\backslash\widetilde{B}}, \tag{1.9.0.3}$$

  is an open book decomposition of $\widetilde{M}$.
  For example, the standard embedding of the sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is transverse to the binding and the pages of the standard open book decomposition on $\mathbb{R}^{n+1}$. Thus we obtain an open book decomposition of $\mathbb{S}^n$ with binding $\mathbb{S}^{n-2}$ and page $\mathbb{D}^{n-1}$.

- **Singularities:** Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with an isolated singuarity at the origin $0 \in \mathbb{C}^n$. Then for $\varepsilon > 0$ small enough, the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^n$ of radius ε consists of regular points of f. We obtain an open book decomposition of $\mathbb{S}^{2n+1}$ by setting

  $$B := f^{-1}(0) \cap \mathbb{S}^{2n+1}, \quad \pi(z) := \frac{f(z)}{||z||},$$

  whose projection is called the Milnor fibration of the hypersurface singularity, see [87]. For example taking

  $$f : \mathbb{C}^2 \to \mathbb{C}, \quad (z_1, z_2) \mapsto z_1 + z_2,$$

  recovers the open book decomposition on $\mathbb{S}^3$ given above. Note that this construction is a combination of the pullback and restriction in the previous examples.

- **Circle bundles:** Recall from Example 1.2.13 that there is a one-to-one correspondence between complex line bundles $L \to M$ and principles $\mathbb{S}^1$ bundles $P \to M$, by

  $$P \mapsto L := P \times_{\mathbb{S}^1} \mathbb{C}.$$

  Under this identification sections $\sigma \in \Gamma(L)$ correspond to $\mathbb{S}^1$-equivariant functions $f \in C^\infty(P, \mathbb{C})$. Explicitly, given f we can define a section by:

  $$\sigma : M \to P \times_{\mathbb{S}^1} \mathbb{C}, \quad x \mapsto [p, f(p)], \quad x \in M, p \in P_x.$$

  Note that since f is $\mathbb{S}^1$-equivariant the above formula does not depend on the choice of p, and hence σ is well-defined.

  Consider a principal $\mathbb{S}^1$-bundle $\pi : P \to M$ and let $\sigma \in \Gamma(L)$ be a section of the associated complex line bundle L, transverse to the zero section. The transversality condition implies that

  $$B := \pi^{-1}(\sigma^{-1}(0)) \subset P,$$

  is a codimension-2 submanifold with trivial normal bundle. Furthermore, if $f \in C^\infty_{\mathbb{S}^1}(P \times \mathbb{C})$ denotes the corresponding function then we obtain a fibration:

  $$\pi : P \backslash B \to \mathbb{S}^1, \quad p \mapsto \frac{f(p)}{||f(p)||}.$$

In fact, the map $f : P \to \mathbb{C}$ is an open book map as in Definition 1.9.7 below. As an example of this construction recall that the tautological bundle over $\mathbb{CP}^1$, is the complex line bundle defined by

$$\mathcal{O}(1) := \{(z, \ell) \in \mathbb{C}^2 \times \mathbb{CP}^1 \mid z \in \ell\}.$$

The associated principal bundle can be identified with those points (z, ℓ) such that $||z|| = 1$. That is, it is just the usual Hopf fibration $h : \mathbb{S}^3 \to \mathbb{CP}^1$. The $\mathbb{S}^1$-invariant map

$$f : \mathbb{S}^3 \to \mathbb{C}, \quad (z_1, z_2) \mapsto z_1 + z_2,$$

has $0 \in \mathbb{C}$ as a regular value and thus defines an open book decomposition of $\mathbb{S}^3$. Note that this open book is isomorphic to the one obtained by viewing $\mathbb{S}^3$ as a submanifold of $\mathbb{R}^4$ as in Equation 1.9.0.3 above, under the diffeomorphism

$$\psi : \mathbb{C}^2 \to \mathbb{C}^2, (z_1, z_1) \mapsto \left(\frac{1}{2}(z_1 + z_2), \frac{1}{2}(z_1 - z_2)\right).$$

Similarly, the principal $\mathbb{S}^1$-bundle associated to the tautological line bundle over $\mathbb{CP}^2$, is the Hopf fibration $h : \mathbb{S}^5 \to \mathbb{CP}^2$. The $\mathbb{S}^1$-equivariant function

$$f : \mathbb{S}^5 \to \mathbb{C}, \quad (z_1, z_2, z_2) \mapsto z_1^3 + z_2^3 + z_3^3,$$

defines an open book decomposition on $\mathbb{S}^5$, see also Theorem 1.9.1.

△

The above definition emphasizes that the manifold M is decomposed into pieces. Alternatively, we can think of M as constructed out of the open book decomposition. That is, we think of open books as a method of constructing new manifolds out of old ones. From this perspective, the above definition contains redundant information, for example we only need to know a single page since all the others are diffeomorphic. The minimal amount of information is given in the following definition:

Definition 1.9.4. *An* ***(abstract) open book*** *is a pair* (Σ, ϕ) *consisting of :*

(i) A (compact) manifold with boundary Σ*, called the* ***page****;*

(ii) A diffeomorphism $\phi \in \mathrm{Diff}(\Sigma)$*, called the* ***monodromy****, which equals the identity near the boundary.*

The two notions above are equivalent in the sense that out of an abstract open book we can construct a manifold with a geometric open book decomposition, and the vice versa. However, different abstract (resp. geometric) open books can give rise to the isomorphic geometric (resp. abstract open book). Thus we consider the following equivalences:

- An isomorphism of geometric open book decompositions (M, B, π) and $(\widetilde{M}, \widetilde{B}, \widetilde{\pi})$ is a diffeomorphism $\phi : M \to \widetilde{M}$ satisfying:

$$\phi(B) = \widetilde{B}, \quad \widetilde{\pi} = \pi \circ \phi.$$

- An isomorphism of abstract open books (Σ, ϕ) and $(\widetilde{\Sigma}, \widetilde{\phi})$ is a diffeomorphism $\psi : \Sigma \to \widetilde{\Sigma}$ satisfying:
$$\psi \circ \phi \circ \psi^{-1} = \widetilde{\phi}.$$

Lemma 1.9.5. *Given an abstract open book (Σ, ϕ) there exists a manifold $M(\Sigma, \phi)$ endowed with a geometric open book decomposition (B, π) whose page is Σ. Moreover, isomorphic abstract open books give isomorphic geometric open books.*

Proof. Out of the abstract open book decomposition we construct the mapping torus

$$\Sigma \times_{\mathbb{Z}} \mathbb{R} := \Sigma \times \mathbb{R}/(x, t) \sim (\phi(x), t-1),$$

Since ϕ is the identity near $\partial\Sigma$ the above mapping torus has boundary $\partial\Sigma \times \mathbb{S}^1$. Thus we can glue it to $\partial\Sigma \times \mathbb{D}^2$, using the identity map. This gives a smooth manifold:

$$M(\Sigma, \phi) := \left(\partial\Sigma \times \mathbb{D}^2\right) \cup_{\mathrm{id}} (\Sigma \times_{\mathbb{Z}} \mathbb{R})$$

The binding of the induced geometric open book $B := \partial\Sigma \times \{0\} \subset \partial\Sigma \times \mathbb{D}^2$ is isomorphic to $\partial\Sigma$. Furthermore, the map $\mathrm{pr}_2 : \Sigma \times_{\mathbb{Z}} \mathbb{R} \to \mathbb{S}^1$ smoothly extends to $M(\Sigma, \phi) \backslash B$, by defining it to be

$$(b, x) \mapsto \frac{x}{||x||}, \quad \forall (b, x) \in B \times \mathbb{D}^2.$$

Hence, we obtain a fibration $\pi : M(\Sigma, \phi)\backslash B \to \mathbb{S}^1$, by definition satisfies the required normal form around the binding. Finally, the page equals $\partial\Sigma \times (0, 1] \cup_{\mathrm{id}} \Sigma \simeq \Sigma$. □

To pass from geometric to abstract open books, we need to produce the monodromy map ϕ. For this let us recall the following facts about connections on manifolds with boundary.

- Associated to any fibration $f : M \to N$, (i.e. a surjective submersion) we have the **vertical bundle**
$$\mathcal{V} := \ker \mathrm{d}f \subset TM,$$
which fits inside the short exact sequence
$$0 \to \mathcal{V} \to TM \xrightarrow{\mathrm{d}f} TN \to 0.$$
An (Ehresmann) **connection** on M is a right splitting of the above sequence, that is, a subbundle $\mathcal{H} \subset TM$ such that
$$TM = \mathcal{V} \oplus \mathcal{H}.$$

- A connection is said to be **complete** if for any path $\gamma : I \to N$, and $x \in M$ there exists path $\widetilde{\gamma}_x : I \to M$ satisfying
$$f \circ \widetilde{\gamma}_x = \gamma, \quad \widetilde{\gamma}_x(0) = x, \quad \dot{\gamma}_x(t) \in \mathcal{H}_{\gamma(t)}, \quad \forall t \in I.$$
Note that in general, paths can only be lifted locally. However, if $f : M \to N$ is a proper map then it admits a complete connection.

- Given a complete connection on a fibration $f : M \to N$, any path $\gamma : I \to N$ induces a diffeomorphism

$$T_\gamma : M_{\gamma(0)} \xrightarrow{\sim} M_{\gamma(1)}, \quad x \mapsto \widetilde{\gamma}_x(1),$$

called the **parallel transport** along γ.

- In particular, any connection $\mathcal{H}$, on a fibration $f : M \to \mathbb{S}^1$ gives rise to a diffeomorphism $\phi : M_1 \xrightarrow{\sim} M_1$, called the **monodromy**, by lifting the generator of $\pi_1(\mathbb{S}^1)$. In turn this gives an isomorphism between M and the suspension

$$M \simeq M_1 \times_{\mathbb{Z}} \mathbb{R} := (M \times \mathbb{R})/(\phi(x), t) \sim (x, t-1).$$

If $\widetilde{\mathcal{H}}$ is another connection on f, with monodromy $\widetilde{\phi}$, then there exists a diffeomorphism $\psi : M_1 \to M_1$ satisfying

$$\psi \circ \widetilde{\phi} \circ \psi^{-1} = \phi.$$

Hence, the resulting suspensions are isomorphic.

- By a fibration of a manifold with boundary M into a manifold N (without boundary), we mean a fibration $f : M \to N$ such that

$$f^{-1}(y) \pitchfork \partial M, \quad \forall y \in N.$$

In this case a connection $\mathcal{H}$ is said to be **compatible** if in addiction to the conditions above it satisfies

$$\mathcal{H}_x \subset T_x(\partial M), \quad \forall x \in \partial M.$$

- If $f : M \to N$ is a fibration of a manifold with boundary, then the restriction $f_\partial := f|_{\partial M}\partial M \to N$, is again a fibration. Moreover, any connection on f_∂ can be extended to a compatible connection on f. In particular, given a fibration $f : M \to \mathbb{S}^1$ of a manifold with boundary, such that f_∂ is the trivial fibration

$$\mathrm{pr}_2 : \partial M_1 \times \mathbb{S}^1 \to \mathbb{S}^1,$$

there exists a connection on f, whose monodromy ϕ is the identity in a neighborhood of ∂M.

Going back to the main story, the following lemma shows how to pass from a geometric open book to an abstract one.

Lemma 1.9.6. *Let (B, π) be a geometric open book decomposition of a manifold M. Then there exists an abstract open book (Σ, ϕ) such that $M \simeq M(\Sigma, \phi)$, as in Lemma 1.9.5, and Σ equals the page of (B, π). Moreover, isomorphic geometric open books give rise to isomorphic abstract open books.*

Proof. The restriction of π to the complement of an open neighborhood of the binding gives a fibration of a manifold with boundary

$$\widetilde{\pi} : M \backslash \mathrm{int}(B \times \mathbb{D}^2) \to \mathbb{S}^1.$$

We define $\Sigma := \widetilde{\pi}^{-1}(1)$, which is isomorphic to the page of (B, π). Furthermore, the boundary of $M \backslash \mathrm{int}(B \times \mathbb{D}^2)$ is trivial, so we can take the trivial connection and extend it to a compatible connection. Then, the monodromy $\phi \in \mathrm{Diff}(\Sigma)$ equals the identity near the boundary, so that (Σ, ϕ) is the required abstract open book decomposition. In fact, we have

$$M \backslash \mathrm{int}(B \times \mathbb{D}^2) \simeq \Sigma \times_{\mathbb{Z}} \mathbb{R},$$

from which it follows that $M \simeq M(\Sigma, \phi)$. □

The two definitions above both involve dividing the manifold into two pieces, a neighborhood of the binding $B \times \mathbb{D}^2$, and the complement which is isomorphic to a mapping torus $P \times_{\mathbb{Z}} \mathbb{R}$. The following definition aims to avoid this decomposition and describe the open book in a global way. It emphasizes that we can think of the binding as a sort of "singularity" of a fibration.

Definition 1.9.7. *An **open book (map)** on a manifold M, is a map $f : M \to \mathbb{R}^2$, onto a neighborhood of the origin and transverse to the Euler vector field*

$$\mathcal{E} := x\partial_x + y\partial_y \in \mathfrak{X}(\mathbb{R}^2).$$

Observe that $\mathcal{E}$ vanishes at the origin. Hence, the transversality condition in particular means that $0 \in \mathbb{R}^2$ is a regular value.

This definition is equivalent to the ones above. To obtain a one-to-one correspondence we have the following notion of isomorphism. Two open book maps $f : M \to \mathbb{R}^2$, $\widetilde{f} : \widetilde{M} \to \mathbb{R}^2$ are **isomorphic** if there exists a diffeomoprhism $\psi : M \to \widetilde{M}$ such that

$$\widetilde{f} \circ \psi = f.$$

Lemma 1.9.8. *Given an open book map $f : M \to \mathbb{R}^2$ there exists a geometric open book decomposition (B_f, π_f) on M, whose binding equals $B_f := f^{-1}(0)$. Moreover, isomorphic open book maps give rise to isomorphic geometric open books.*

Proof. As remarked above, the origin $0 \in \mathbb{R}^2$ is a regular value of $f : M \to \mathbb{R}^2$. Hence, $B_f := f^{-1}(0)$ is a codimension-2 submanifold in M whose normal bundle is isomorphic to the pullback of the rank-2 vector bundle over a point $\mathbb{R}^2 \to \{0\}$ and hence is trivial. On the complement of the binding we define

$$\pi : M \backslash B \to \mathbb{S}^1, \quad x \mapsto \frac{f(x)}{||f(x)||}.$$

Since $f \pitchfork \mathcal{E}$ this map defines a submersion. □

The converse implication is as follows.

Lemma 1.9.9. *Given a geometric open book decomposition (B, π) on M, there exists an open book map $f : M \to \mathbb{R}^2$ such that $(B, \pi) = (B_f, \pi_f)$ as in Lemma 1.9.8. Moreover, isomorphic geometric open books give rise to isomorphic open book maps.*

Proof. Using a Riemannian metric, we can define a distance function $\rho : M \to \mathbb{R}_{\geqslant 0}$ to the binding B. Then we define

$$f : M \to \mathbb{R}^2, \quad x \mapsto \begin{cases} 0 & \text{if } \rho(x) = 0 \\ (\rho(x), \pi(x)) & \text{if } \rho(x) > 0 \end{cases},$$

where we use polar coordinates (r, ϕ) on $\mathbb{R}^2$. The normal form of π ensures that $0 \in \mathbb{R}^2$ is a regular value of f, and transverse to the Euler vector field away from the binding. □

Chapter 2

Convergence of contact structures

Chapter Contents

2.1 Overview

This chapter is based on joint work with F. Presas. The theory of confoliations, introduced by Eliashberg and Thurston [47], unites contact structures and codimension-one foliations on 3-manifolds in a single framework. Following the same philosophy we investigate the relationship between contact structures and (symplectic) foliations in higher dimensions.

An essential property of the space of confoliations is that it contains the closure of the space of contact structures. In Section 2.2 we generalize confoliations to higher dimensions preserving this property. In more detail, given a contact structure ξ on M^{2n+1}, the curvature

$$c_\xi : \Lambda^2\xi \to TM/\xi, \quad (X,Y) \mapsto [X,Y] \bmod \xi, \quad \forall X,Y \in \Gamma(\xi),$$

is non-degenerate. Thus, any contact distribution carries a non-degenerate (bundle valued) form $c_\xi \in \Omega^2(\xi; TM/\xi)$. It turns out, see Lemma 2.4.1, that if a distribution is a limit of contact structures then it also admits such a non-degenerate form.

In light of this we consider "almost conformal symplectic hyerplane fields" (almost CS-hyperplane fields for short). These are pairs (ξ,ω) where $\xi \subset TM$ is a hyperplane field and $\omega \in \Omega^2(\xi, TM/\xi)$ is non-degenerate. They are the generalization of confoliations to higher dimensions. As expected, this notion includes contact structures and (conformal) symplectic foliations.

In Section 2.2 and Section 2.3, we define almost CS-hyperplane fields as sections of the "symplectic Grassmannian" bundle. This induces a natural topology on the space of all almost CS-hyperplane fields. This topology allows us to talk about deformations (using paths) and approximations (using sequences).

We define several types of deformations (for example "linear deformations") and translate the definitions in terms of (real valued) differential forms, which are easier to work with. One conclusion following almost directly from these definitions is that a foliation $\mathcal{F}$ admits a linear (Type I) deformation into contact structures if and only if it admits a leafwise exact conformal symplectic structure, see Theorem 2.2.13 .

In Section 2.4 we show that all the types of convergence are distinct. We also provide explicit examples for each of them. The most important concept introduced in this section is that of an almost CS-submanifold (Definition 2.4.4). A submanifold N of an almost CS-manifold (M, ξ, ω) is an almost CS-submanifold if the restriction

$$(\xi|_N, \omega|_N)$$

defines an almost CS-structure on N. Using Donaldson techniques we show that almost CS-foliations which can be linearly deformed always admit such submanifolds (Theorem 2.4.16).

In dimension-3 it is known [47] that any foliation except the product foliation on $\mathbb{S}^1 \times \mathbb{S}^2$ can be approximated by contact structures. The key property of almost CS-subamanifolds (Lemma 2.4.5) is that a deformation on the ambient manifold induces a deformation on the submanifold. Therefore there is a strong interaction between the 3-dimensional case and higher dimensional one. In particular, any symplectic foliation containing $\mathbb{S}^1 \times \mathbb{S}^2$ (endowed with the product foliation) cannot be approximated by contact structures.

It turns out that there are many foliations that cannot be approximated because they contain $\mathbb{S}^1 \times \mathbb{S}^2$. The aim of Section 2.5 is to find different obstructions to the existence of approximations. The classical clutching construction relates fibrations over $\mathbb{S}^2$ with loops of diffeomorphisms on the fiber. After discussing the clutching construction in the setting of contact fibrations we use it to produce examples of (conformal) symplectic foliations on $\mathbb{S}^3 \times \mathbb{T}^2$ which cannot be deformed into contact structures, see Theorem 2.5.38. As desired, these examples do not contain almost CS-submanifolds isomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$.

In Section 2.6 we give more examples, both of foliations which can and cannot be approximated by contact structures. We highlight the following two results.

The first one is based on the h-principle for isosymplectic embeddings. We use it to show that in dimension $\geqslant 7$, any symplectic foliation containing a "formal" almost CS-submanifold $\mathbb{S}^1 \times \mathbb{S}^2$ cannot be approximated by contact structures. The precise statement is given in Theorem 2.6.6. The second result is based on a construction by Bourgeois [16]. He showed that the product of a contact manifold with $\mathbb{T}^2$ is again contact. Under certain conditions this construction goes through for deformations. That is, if a almost CS-foliation on M can be deformed into contact structures then so can the product foliation on $M \times \mathbb{T}^2$. The details are given in Theorem 2.6.25.

Lastly, in Section 2.7 we briefly investigate fillability of almost CS-foliations. The main result is Theorem 2.7.8 stating that there exist almost CS-foliations which are not weakly fillable in the sense of [85].

2.2 Hyperplane fields

In this section we look at the problem of approximating foliations (without leafwise symplectic forms on them) by contact structures. This is possible by viewing them as part of a single space, that of hyperplane distributions:

$$\mathcal{H}yper(M) := \{\xi \subset TM : \xi \text{ is a hyperplane distribution in } TM\}.$$

There are various topologies one can consider on this space (some of which are discussed in the last part of this section), and one can even make sense of it as an infinite dimensional manifold. However, all that can be avoided when discussing deformations, because given a path

$$[0, \epsilon) \ni t \mapsto \xi_t \in \mathcal{H}yper(M)$$

one can make sense of its smoothness right away, by interpreting ξ_t as a sub-bundle of the pull-back pr^*TM via the projection $\mathrm{pr} : \mathbb{R} \times M \to M$. Or interpret ξ as a section of the pullback via pr of the Grassmannian bundle of M (recalled below). Of course, one can also weaken the smoothness condition or, thinking of ξ as a function of $(t, p) \in \mathbb{R} \times M$, one can even consider different orders of differentiability in t and p; we will be making some remarks in that direction (e.g. Remark 2.2.18) but, for simplicity, smoothness will be the overall assumption.

Definition 2.2.1. *We say that a foliation $\mathcal{F}$ can be* ***deformed into contact structures*** *if one can find a smooth path $(\xi_t)_{t\in[0,\epsilon)}$ of hyperplane distributions such that*

$$\xi_0 = \mathcal{F}, \quad \xi_t\text{-contact for all } t > 0.$$

In this case we also say that ξ_t ***is a contact deformation of*** *$\mathcal{F}$.*

It is handy to represent hyperplanes by 1-forms; this can be achieved smoothly in t:

Lemma 2.2.2. *For any smooth path $(\xi_t)_{t\in[0,\epsilon)}$ of hyperplane distributions, with ξ_0 co-orientable, one can write*

$$\xi_t = \ker(\alpha_t)$$

for some smooth path of 1-forms $\alpha_t \in \Omega^1(M)$.

Proof. Let $\widetilde{\xi}$ be the sub-bundle of pr^*TM corresponding to ξ_t and consider the resulting quotient

$$\tau := \mathrm{pr}^*TM/\widetilde{\xi}.$$

This is a line bundle over $M \times [0, \epsilon)$, hence it is isomorphic to the pull-back of $\tau|_{M\times\{0\}}$; by the co-orientability condition, it follows that τ is trivializable. A trivialization precisely means a family α_t as above. □

With the previous lemma at hand, we find ourselves in the following setting: a foliation $\mathcal{F}$ represented by some 1-form α, and then a deformation of α into contact forms.

Definition 2.2.3. *A* ***contact deformation*** *of a 1-form $\alpha \in \Omega^1(M)$ is a smooth path α_t of 1-forms, defined for t in some interval $[0, \epsilon)$ such that*

$$\alpha_0 = \alpha, \quad \alpha_t\text{-contact for all } t > 0.$$

The contact deformation is called ***linear*** *of it is of type*

$$\alpha_t = \alpha + t\beta.$$

We say that a foliation $\mathcal{F}$ can be ***linearly deformed*** *(into contact structures) if some 1-form α inducing $\mathcal{F}$ admits a linear contact deformation.*

Starting with an arbitrary deformation α_t one can define its **linearization**

$$\alpha_t^{\text{lin}} := \alpha + t\beta, \quad \text{where} \quad \alpha = \alpha_0,\ \beta := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \alpha_t. \tag{2.2.0.1}$$

2.2.1 Type I and type II contact deformations

With the notation from Definition 2.2.3, while we are interested in the case when $\xi_0 = \ker(\alpha_0)$ is a foliation and $\xi_t = \ker(\alpha_t)$ are contact structures, it is interesting to measure the "order" at which α_t are contact. More precisely, while being contact is encoded in the corresponding volume form

$$\Omega_t := \alpha_t \wedge (d\alpha_t)^n,$$

which at the limit $t = 0$ gives $\Omega_0 = 0$ (since $\mathcal{F}$ is a foliation), the question is: what is the order k at which one can write $\Omega_t = t^k \widetilde{\Omega}_t$ with $\widetilde{\Omega}_0 \neq 0$.

Lemma 2.2.4. *If (α_t) is contact deformation of an integrable 1-form α, then the corresponding volume forms Ω_t satisfy*

$$\Omega_t = \mathcal{O}(t^n).$$

In other words, if one fixes a volume form Ω on M, then

$$\alpha_t \wedge (d\alpha_t)^n = t^n f_t \Omega$$

for some smooth family of functions $f_t \in C^\infty(M)$.

Although this can be proven by a simple trick, it is interesting to interpret it via the Taylor expansion of α_t around $t = 0$:

$$\alpha_t = \alpha + t\beta + t^2\gamma + \mathcal{O}(t^3). \tag{2.2.1.1}$$

The foliation condition $\alpha \wedge \mathrm{d}\alpha = 0$ implies that $\mathrm{d}\alpha^k = 0$ for $k \geqslant 2$. Hence, if we only take into account the quadratic part of the expansion, the contact condition for α_t, $t > 0$, is :

$$\begin{aligned} 0 < \alpha_t \wedge \mathrm{d}\alpha_t^n = t^n &\Big(\alpha \wedge \mathrm{d}\beta + n\beta \wedge \mathrm{d}\alpha\Big) \wedge \mathrm{d}\beta^{n-1} \\ &+ t^{n+1}\Big(n\alpha \wedge \mathrm{d}\beta \wedge \mathrm{d}\gamma + \beta \wedge \mathrm{d}\beta^2 + n\gamma \wedge \mathrm{d}\alpha \wedge \mathrm{d}\beta \\ &+ n(n-1)\beta \wedge \mathrm{d}\alpha \wedge \mathrm{d}\gamma\Big) \wedge \mathrm{d}\beta^{n-2} + \mathcal{O}(t^{n+2}). \end{aligned} \tag{2.2.1.2}$$

This discussion is particularly interesting when one starts with a foliation $\mathcal{F}$, represented by some 1-form α, and then one tries to realise it as the limit of a sequence of contact structures/forms. We see that the best scenario is when the coefficient of t^n in the previous formula is already strictly positive.

Definition 2.2.5. *Given an integrable form $\alpha \in \Omega^1(M)$, a* ***type I*** *(contact) deformation of α is a contact deformation α_t of α (as in Definition 2.2.3) with the property that, writing*

$$\alpha_t \wedge \mathrm{d}\alpha_t^n = t^n f_t \Omega$$

as in the previous lemma, one has $f_0 > 0$.

A contact deformation ξ_t of a foliation $\mathcal{F}$ is said to be a ***deformation of type I*** *if it can be represented by a smooth path of 1-forms α_t (with α_0 inducing $\mathcal{F}$), which is of type I.*

Written more compactly (and without having to choose a volume form Ω) the type I condition reads

$$\lim_{t \to 0} \frac{1}{t^n} \alpha_t \wedge \mathrm{d}\alpha_t^n > 0.$$

On the other hand, it can also be further expanded and written as a condition up to order $n+1$:

(2.2.1.3)

$$\text{type I:} \quad \alpha_t \wedge \mathrm{d}\alpha_t^n = t^n f \Omega + \mathcal{O}(t^{n+1}) \quad \text{for some strictly positive function } f \in C^\infty(M).$$

Note that the discussion leading to the definition of "type I" was based on the Taylor expansion (2.2.1.1) where we concentrated on the first non-zero term (involving t^n); that part clearly only depends on the linearization of α_t, as defined in (2.2.0.1). We deduce that if a foliation $\mathcal{F}$ admits a type I deformation (by contact structures), then it can also be linearly deformed. Actually, here are the interesting interactions between type I and linearity.

Lemma 2.2.6. *If α_t is a smooth path of 1-forms with α_0 inducing a foliation $\mathcal{F}$, then one has*

$$\begin{array}{ccc}
\alpha_t = \textit{type I deformation} & \Longrightarrow & \alpha_t = \textit{contact deformation} \\
\Updownarrow & & \Uparrow \\
\alpha_t^{lin} = \textit{type I deformation} & \Longrightarrow & \alpha_t^{lin} = \textit{contact deformation}
\end{array}$$

Example 2.2.7. One should be aware that the horizontal implications are not equivalence even in the case of linear case. That is, there are linear contact deformations which are not of type I. For example, on $\mathbb{T}^3$ the linear path

$$\alpha_t := \mathrm{d}z + t\,(\sin(z)\mathrm{d}x + \cos(z)\mathrm{d}y)\,.$$

has

$$\alpha_t \wedge \mathrm{d}\alpha_t = t^2 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

implying that α_t is not of type I (the relevant coefficient f_0 actually vanishes!). We will see (Proposition 2.4.14) that there does not exist any $\tilde{\alpha}_t$ of type I representing $\xi_t := \ker \alpha_t$. △

Remark 2.2.8. Although α_t is a type I deformation if and only if α_t^{lin} is, in general these paths induce different contact structures. The parametric Moser trick shows that there exists an isotopy $\phi_t \in \text{Diff}(M)$ such that

$$\phi_t^*(\alpha_t^{\text{lin}}) = f_t \alpha_t,$$

for positive functions $f_t \in C^\infty(M)$. Thus, up to isotopy, the two paths of contact structures agree. △

With the previous example in mind, let us return to the Taylor expansion 2.2.1.2 and see what can happen beyond the type I case, i.e. when the t^n-term is not strictly positive. The next simplest case is when the the linear part (in t) of the function in front of t^n is strictly positive. Having in mind the characterization (2.2.1.3) for type I, type II appears as the next step:

Definition 2.2.9. *Given an integrable form $\alpha \in \Omega^1(M)$, a **type II** (contact) deformation of α is a contact deformation α_t of α (as in Definition 2.2.3) with the property that*

$$\alpha_t \wedge (\mathrm{d}\alpha_t)^n = t^n f \Omega + t^{n+1} g \Omega + \mathcal{O}(t^{n+2}), \tag{2.2.1.4}$$

for a volume form Ω, $f, g \in C^\infty(M)$ such that $f + tg$ is strictly positive for all $t > 0$ close to 0.

*And then, similar to Definition 2.2.5, one talks about **type II deformation** ξ_t of a foliation $\mathcal{F}$.*

Example 2.2.10. Of course, type I implies type II. But note that, unlike for type I, the type II condition does not imply that the linearization α_t^{lin} is made of contact forms. E.g., already on $\mathbb{R}^3$, the path

$$\alpha_t := \mathrm{d}z + t^2 x \mathrm{d}y$$

has

$$\alpha_t \wedge \mathrm{d}\alpha_t = t^2 \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z,$$

so α_t is type II. On the other hand the linearization equals $\alpha_t^{\text{lin}} = \mathrm{d}z$ which is never contact. △

Remark 2.2.11. As we have already pointed out, and is seen also in the last example, if we start with a contact deformation α_t and we linearize it, in general α_t^{lin} may fail to be contact; and the type I case removed this "problem". However, since

$$\alpha_t \wedge (\mathrm{d}\alpha_t)^n - \alpha_t^{\text{lin}} \wedge (\mathrm{d}\alpha_t^{\text{lin}})^n = \mathcal{O}(t^{n+1}),$$

we see that

$$\alpha_t \text{ is of type I} \Longrightarrow \quad \alpha_t^{\text{lin}} \text{ is a contact deformation} \Longrightarrow \quad \alpha_t \text{ is of type II}$$

In particular, the type II condition is actually necessary for achieving the the linearization is contact. △

However, there is some analogy with the type I and type II conditions, just that one has to look a bit closer. E.g., looking again at the Taylor expansion (2.2.1.2), we see that the type II condition is a condition involving just α, β and γ; i.e. only the "quadratization" α_t^{quadr} of α_t (defined completely similar to the linearization). One then obtains the analogue of the diagram from Lemma 2.2.6:

$$\begin{array}{ccc}
\alpha_t = \text{ type II deformation} & \Longrightarrow & \alpha_t = \text{ contact deformation} \\
\Updownarrow & & \Uparrow \\
\alpha_t^{\text{quadr}} = \text{ type II deformation} & \Longrightarrow & \alpha_t^{\text{quadr}} = \text{ contact deformation}
\end{array}$$

Actually, the two diagrams can be nicely merged together by making use also of the Remark 2.2.11 (while also getting more insight into the remark itself):

$$\begin{array}{ccccc}
\alpha_t = \text{ type I} & \Longrightarrow & \alpha_t = \text{ type II} & \Longrightarrow & \alpha_t = \text{ contact} \\
\Updownarrow & & \Updownarrow & & \Uparrow \\
 & & \alpha_t^{\text{quadr}} = \text{ type II} & \Longrightarrow & \alpha_t^{\text{quadr}} = \text{ contact} \\
 & & \Uparrow & & \\
\alpha_t^{\text{lin}} = \text{ type I} & \Longrightarrow & \alpha_t^{\text{lin}} = \text{ contact} & &
\end{array}$$

2.2.2 Relationship with conformally symplectic structures

We know that a contact form gives rise to a rich geometry, starting already with the basic concepts such as the induced Reeb vector fields or the induced non-degenerate 2-forms along the hyperplane distributions. How much of that is seen in the limit of a sequence of contact forms, or at $t = 0$ for a contact deformation α_t? It seems that the answer is: not so much in general.

Example 2.2.12. On $\mathbb{R}^3(x, y, z)$ consider the symplectic foliation

$$(\mathcal{F} := \ker \mathrm{d}z, \omega := \mathrm{d}x \wedge \mathrm{d}y)$$

and the following contact deformation:

$$\alpha_t := e^y \left(\mathrm{d}z - ty\mathrm{d}x\right).$$

Computing its Reeb vector field one finds

$$R_t = e^{-y} \left(\frac{1}{t}\frac{\partial}{\partial x} + (y+1)\frac{\partial}{\partial z}\right).$$

We see that, although α_t is smooth also around $t = 0$, the Reeb vector fields (defined for $t \neq 0$) do not have a limit as $t \to 0$. One can try to fix this by looking at the induced Reeb directions

$$\tau_t = \mathrm{Span}(R_t),$$

which go to $\tau_0 = \text{Span}(e^{-y}\partial_x)$. But note that, while in this example τ_0 is tangent to $\mathcal{F}$, for other simple contact deformations of $\mathcal{F}$ the limit is transverse to $\mathcal{F}$ (e.g. even $\beta_t := e^{-y}\alpha_t = \mathrm{d}z - ty\mathrm{d}x$, with constant Reeb vector field $R_t = \partial_z$).

And similar (negative) remarks hold also for the induced non-degenerated 2-forms along the hyperplane distributions. In our example we have

$$d\alpha_t = e^y\left(t(1+y)\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}y \wedge \mathrm{d}z\right)$$

which goes, as $t \to 0$, to $\eta = e^y \mathrm{d}y \wedge \mathrm{d}z$. And this is no longer nondegenerate on $\mathcal{F}$. And for the other example mentioned above, $\beta_t := \mathrm{d}z - ty\mathrm{d}x$, $\mathrm{d}\beta_t = t\mathrm{d}x \wedge \mathrm{d}y$ even goes to zero. However, rescaling β_t by $\frac{1}{t}$, the resulting 2-form in the limit is $\mathrm{d}x \wedge \mathrm{d}y$ which is nondegenerate on $\mathcal{F}$. △

In this section we would like to point out one conclusion that can be drawn in the limit, i.e. about the foliations $\mathcal{F}$: that it admits a conformally symplectic structure. Before we recall the necessary definitions, here is the precise statement we will be discussing:

Theorem 2.2.13. *A co-oriented foliation $\mathcal{F}$ admits a type I contact deformation (Definition 2.2.5) if and only if it admits an exact leafwise CS-structure (Definition 2.2.16) with coefficients in the normal bundle of $\mathcal{F}$.*

In particular, if $\mathcal{F}$ is unimodular, the condition is that $\mathcal{F}$ admits an exact leafwise symplectic structure.

Note that, in this case, Lemma 2.2.6 implies that the contact deformation can be chosen to be linear (and that is what we will do in the proof anyway). To explain the previous theorem, we start by recalling the notion of conformal symplectic structures. Very briefly, they are generalizations of symplectic structures obtained by allowing more general line bundles as coefficients. Furthermore by Corollary 2.4.17 the second statement can only applies to non-compact manifolds.

Definition 2.2.14. *A* ***conformal symplectic structure*** *(CS-structure for short) on M is a triple (L, ∇, ω) where $\pi : L \to M$ is a line bundle, ∇ is a flat connection on L, and $\omega \in \Omega^2(M, L)$ is non-degenerate and d_∇-closed.*

Here we denote by $\Omega^\bullet(M, L)$ the space of L-valued differential forms and we use that any connection $\nabla : \mathfrak{X}(M) \times \Gamma(L) \to \Gamma(L)$ gives rise to a DeRham differential d_∇ on $\Omega^\bullet(M, L)$; it can be described e.g. using the usual Koszul formula:

(2.2.2.1)

$$\begin{aligned}(\mathrm{d}_\nabla\alpha)(X_1, \ldots, X_{k+1}) := &\sum_{i=1}^{k+1}(-1)^{i+1}\nabla_{X_i}\left(\alpha(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1})\right) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}),\end{aligned}$$

for any $\alpha \in \Omega^k(M, L)$ and $X_1, \ldots, X_{k+1} \in \mathfrak{X}(M)$. Note that the connection ∇ is flat if and only if $\mathrm{d}_\nabla^2 = 0$, and that by the Leibniz identity d_∇ is uniquely determined by how it acts on sections. That is, it can be equivalently defined setting

$$(\mathrm{d}_\nabla\sigma)(X) = \nabla_X\sigma, \quad X \in \mathfrak{X}(M), \sigma \in \Gamma(L). \tag{2.2.2.2}$$

Note that if $L = M \times \mathbb{R}$, and ∇ is the flat connection induced by the usual Lie derivative, then the above definition reduces to the usual definition of a symplectic structure.

If L is oriented, the notion of conformal symplectic structure can be entirely unravelled in terms of real valued differential forms. First of all, choosing a trivialization $L \simeq M \times \mathbb{R}$, the connection itself can be identified with a 1-form

$$\nu \in \Omega^1(M)$$

via the equation $\nabla_X f = \nu(X)f$. And ∇ being flat is equivalent to η being closed.

With this, d_∇ is identified with

$$\mathrm{d}_\nu : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M), \quad \mathrm{d}_\nu\alpha := \mathrm{d}\alpha + \nu \wedge \alpha. \tag{2.2.2.3}$$

Therefore, ω will become a non-degenerate 2-form $\omega \in \Omega^2(M)$, which is d_ν-closed. Of course, one should think of (ω, ν) as representing the original conformal symplectic structure. Note that if one changes the trivialization of L, the pair will be changed to $(e^f\omega, \nu - \mathrm{d}f)$, for some $f \in C^\infty(M)$. In particular, the cohomology class $[\nu] \in H^1(M)$ is independent of the choices (and is associated to the flat line bundle L), while the non-degenerate 2-form is unique up to a "conformal factor".

Example 2.2.15. Let (M, ξ) be a contact manifold, then $M \times \mathbb{S}^1$ admits a CS-structure called the **conformal symplectization** of ξ, defined as follows. For the line bundle we take

$$L := \mathrm{pr}_1^*(TM/\xi),$$

using the projection pr $:_1$: $M \times \mathbb{R} \to M$. A section σ of L can be identified with 1-parameter family of sections σ_θ, $\theta \in \mathbb{S}^1$, of TM/ξ. Then we define a flat connection on L by

$$\nabla_X\sigma := \mathrm{d}\theta(X)\frac{\mathrm{d}}{\mathrm{d}\theta}\Big|_{\theta=0}\sigma_\theta, \quad X \in \mathfrak{X}(M).$$

The composition

$$T(M \times \mathbb{S}^1) \xrightarrow{\mathrm{dpr}_1} TM \to TM/\xi,$$

can be interpreted as a differential form $\alpha \in \Omega^1(M \times \mathbb{S}^1, L)$. It follows that

$$\omega := \mathrm{d}_\nabla\alpha \in \Omega^2(M \times \mathbb{S}^1, L),$$

is non-degenerate, and it is clearly d_∇-closed, so defines a CS-structure. Observe that if TM/ξ can be trivialized, so that $\alpha_M \in \Omega^1(M)$ is a contact form for ξ. Then,

$$(\mathrm{d}_{-\mathrm{d}\theta}\alpha_M, -\mathrm{d}\theta)\,,$$

is a CS-pair representing ω. Lastly, if $\pi : M \to \mathbb{R} \to M \times \mathbb{S}^1$ denotes the usual covering map, and $t \in \mathbb{R}$, then the pullback of the above CS-pair is given by

$$(\mathrm{d}_{-\mathrm{d}t}\alpha_M, -\mathrm{d}t)\,.$$

This is again a CS-pair, equivalent to the pair $(\mathrm{d}(e^t\alpha_M), 0)$, defining the usual symplectization of (M, α_M). △

We now move to the foliated version of the previous discussion. Let $\mathcal{F}$ be a foliation on M. Recall that by an $\mathcal{F}$-connection on a line bundle $L \to M$, we mean a map $\nabla : \Gamma(\mathcal{F}) \otimes \Gamma(L) \to \Gamma(L)$ satisfying the usual conditions for a connection. Since $\mathcal{F}$ is a foliation we can consider the complex of L-valued, leafwise differential forms $\Omega^\bullet(\mathcal{F}, L)$ together with the differential d_∇ defined as in Equation 2.2.2.1. Then the foliated version of Definition 2.2.14 is:

Definition 2.2.16. *A* ***conformal symplectic foliation*** *(CS-foliation for short) on a manifold M is a triple $(\mathcal{F}, \omega, L)$, consisting of a foliation $\mathcal{F}$, a line bundle $L \to M$ endowed with a flat $\mathcal{F}$-connection ∇, and a differential form $\omega \in \Omega^2(\mathcal{F}, L)$ which is non-degenerate and d_∇-closed.*

The same discussion as for CS-structures applies; if L is oriented then choosing a trivialization identifies ∇ with a leafwise form $\nu \in \Omega^1(\mathcal{F})$ (via the same formula $\nabla_X f = \nu(X)f$ as above) and then d_∇ with d_ν given by the formula similar to (2.2.2.4):

$$\mathrm{d}_\nu : \Omega^\bullet(\mathcal{F}) \to \Omega^{\bullet+1}(\mathcal{F}), \quad \mathrm{d}_\nu\alpha := \mathrm{d}_\mathcal{F}\alpha + \nu \wedge \alpha, \tag{2.2.2.4}$$

where $\mathrm{d}_\mathcal{F}$ is the leafwise DeRham differential. Again, ∇ being flat is equivalent to $\mathrm{d}_\nabla^2 = 0$, and to ν being leafwise closed. We see that, as before, the conformal symplectic foliation will be encoded in a pair (ω, μ) of two leafwise forms

$$\omega \in \Omega^2(\mathcal{F}),\ \nu \in \Omega^1(\mathcal{F}),$$

with ν leafwise closed and $d_\nu\omega = 0$.

We will be particularly interested in the case when L is the normal bundle

$$L := TM/\mathcal{F},$$

endowed with the canonical flat $\mathcal{F}$-connection (the Bott connection)

$$\nabla_X(\overline{V}) = \overline{[X, V]}.$$

Note that the resulting cohomology class $[\nu]$ is precisely the **modular class** of $\mathcal{F}$:

$$\mathrm{mod}(\mathcal{F}) \in H^1(\mathcal{F}).$$

Working out the previous description of the representative ν, we see that we need to start with a 1-form α inducing $\mathcal{F}$ and then choose any ν so that $\mathrm{d}\alpha = \alpha \wedge \nu$. One deduces that $\mathrm{mod}(\mathcal{F}) = 0$ if and only if one can choose a closed 1-form α representing $\mathcal{F}$, i.e. $\mathcal{F}$ is **unimodular**. With this, the statement of Theorem 2.2.13 has been explained, and we can now turn to the actual proof.

Proof of Theorem 2.2.13. Write as above $\mathcal{F} = \ker\alpha$, and $\mathrm{d}\alpha = \alpha \wedge \nu$, so that the coefficients $L = TM/\mathcal{F}$ (with the Bott connection, of course) is identified with the trivial line bundle endowed with d_ν. In particular, we are looking for non-degenerate foliated forms

$$\omega \in \Omega^2(\mathcal{F})$$

which are d_ν-closed, and exactness means that $\omega = \mathrm{d}_\nu(\beta) = \mathrm{d}_{\mathcal{F}}\beta + \nu \wedge \beta$ for some $\beta \in \Omega^1(\mathcal{F})$. We see that when this is the case then, choosing any extension of β to M, still denoted by β,

$$\alpha_t := \alpha + t\beta$$

has the desired properties. Indeed, choosing also an arbitrary extension of ν to M (just to be able to write the formulas below), one has:

$$\begin{aligned}\alpha_t \wedge (\mathrm{d}\alpha_t)^n &= (\alpha + t\beta) \wedge \big(t^n(\mathrm{d}\beta)^n + nt^{n-1}(\mathrm{d}\beta)^{n-1} \wedge \mathrm{d}\alpha\big) \\ &= t^n\alpha \wedge \big((\mathrm{d}\beta)^n + n\, \nu \wedge \beta \wedge (\mathrm{d}\beta)^{n-1}\big) + t^{n+1}\beta \wedge (\mathrm{d}\beta)^n \\ &= t^n\alpha \wedge (\mathrm{d}_\nu\beta)^n + t^{n+1}\beta \wedge (\mathrm{d}\beta)^n,\end{aligned}$$

and then (for small t) the dominating term is $\alpha \wedge (\mathrm{d}_\nu\beta)^n = \alpha \wedge \omega^n > 0$. Hence, indeed, this is a contact deformation of type I.

For the converse, we start with any type I contact deformation α_t and we look at its Taylor expansion as in (2.2.1.1) (giving rise to α and β as in that equation) and we read the type I condition from the resulting expansion (2.2.1.2):

$$(\alpha \wedge \mathrm{d}\beta + \beta \wedge \mathrm{d}\alpha) \wedge (d\beta)^{n-1} > 0.$$

But writing $\mathrm{d}\alpha = \alpha \wedge \nu$ as before, this term is

$$\alpha \wedge (\mathrm{d}\beta + \nu \wedge \beta) \wedge (\mathrm{d}\beta)^{n-1} = \alpha \wedge (\mathrm{d}_\nu\beta)^n$$

hence the leafwise restriction of β, and its leafwise differential, gives us the foliated exact symplectic structure we were looking for. □

2.2.3 Using sequences instead of paths

While a deformation of a foliation $\mathcal{F}$ into contact structures ξ_t can be thought of as approximating $\mathcal{F}$ by contact structures ($\mathcal{F} = \lim_{t\to 0}\xi_t$), it is sometime useful to allow non-continuous approximations, i.e. by sequences:

(2.2.3.1) $$\mathcal{F} = \lim_{k\to\infty} \zeta_k \quad (\zeta_k \in \mathcal{H}yper(M)).$$

Of course, when drawing the analogy between sequences ζ_k and paths ξ_t, one should think that

(2.2.3.2) $$\zeta_k = \xi_{\frac{1}{k}}, \quad \mathcal{F} = \xi_0.$$

This actually shows how to pass from paths to sequences, but the point is that not every sequence arises in this way (and sometimes it may be easier to produce sequences instead of paths).

However, this time, to make sense of such limits (2.2.3.1), we face the problem of being precise about the topology one uses on $\mathcal{H}yper(M)$. This comes with some technicalities and, for simplicity of the statements, we will often assume M to be compact.

To make sense of $\mathcal{H}yper(M)$ as a topological space, we start with the finite dimensional picture. Recall that, for any $2n+1$-dimensional vector space V and any integer d we have the Grassmannian of d-planes:

$$\mathrm{Gr}_d(V) := \{\xi \subset V \mid \dim \xi = d\},$$

which is a smooth manifold as follows. To describe the a chart around a "point" $\xi_0 \in \mathrm{Gr}_d(V)$ (fixed now for the construction of the chart), one chooses a vector subspace $\tau_0 \subset V$ that is complementary to ξ_0; the subspaces transverse to τ_0 define an open in the Grassmannian,

$$U_{\tau_0} := \{\xi \in \mathrm{Gr}_d(V) : \xi \text{ is transverse to } \tau_0\} \subset \mathrm{Gr}_d(V), \tag{2.2.3.3}$$

which serves as the domain of a chart χ, with

$$\chi^{-1} : \mathrm{Hom}(\xi_0, \tau_0) \to U_\tau, \quad \phi \mapsto \mathrm{Graph}(\phi) = \{v + \phi(v) : v \in \xi_0\}. \tag{2.2.3.4}$$

(or, a bit more conceptually: U_{τ_0} is an affine space (modelled on $\mathrm{Hom}(\mathbb{R}^d, \mathbb{R})$ and, once a point $\xi_0 \in U_{\tau_0}$ is chosen, one uses it as origin to identify the affine space with the underlying vector space).

Moving to an $2n+1$-dimensional manifold M one defines

$$\mathrm{Gr}_d(M) := \bigcup_{p \in M} \mathrm{Gr}_d(T_pM)$$

and then, combining the previous discussion with the charts of M one sees that $\mathrm{Gr}_d(M)$ inherits a canonical smooth structure that makes it into a smooth fiber bundle over M. Of course, we will be using this for $d = 2n$, when contact hyperplanes and foliations can be both interpreted as sections of this bundle:

$$\mathcal{H}yper(M) = \Gamma(\mathrm{Gr}_{2n}(M)).$$

The simplest topology one can consider on such spaces of sections is the C^0- compact-open topology, with respect to which convergence means uniform C^0-convergence on compacts. Actually, let us replace for the moment $\mathrm{Gr}_{2n}(M)$ by an arbitrary fiber bundle $P \to M$ so that the generality of the discussion is clearer. For convergences that take into account also derivatives it is useful to use jets. For any section ξ of R one can talk about its l-jet at any point $p \in M$,

$$j^l_p \in J^l_p(P)$$

and, varying p, the l-jet of ξ makes sense as a section of the l-th jet bundle $J^l(P) \to M$:

$$j^l(\xi) \in \Gamma(J^l(P)).$$

The map $\xi \mapsto j^l(\xi)$ allows one to induce a topology on $\Gamma(P)$ from the C^0-compact-open topology on $\Gamma(J^l(P))$. That is the so called C^l-compact-open topology on $\Gamma(P)$- simply called from now on **the C^l-topology** on the space of sections. Finally, one defines **the C^∞-topology on $\Gamma(P)$** as the union of the previous topologies for all l. This is the topology that we will be using by default.

Definition 2.2.17. *We say that a sequence of contact structures* $(\xi_k)_{k\geqslant 0}$ ***converges*** *to a foliation* $\mathcal{F}$ *if it converges (for* $k \to \infty$*) in* $\mathcal{H}yper(M)$ *with respect to the topology we just described.*

We say that a foliation $\mathcal{F}$ *can be* ***approximated by contact structures*** *if one can find such a sequence of contact structures converging to* $\mathcal{F}$*.*

Remark 2.2.18. Of course, using the C^l-topologies one can talk about C^l-convergence and approximations. In the analogy (2.2.3.2) with paths $\xi_t = \xi(t, p)$, this corresponds to being of class C^l in p (but still smooth in t)- called C^l-deformations by smooth paths. A slightly different notion is obtained if one requires only continuity in t. One ends up, for each l, with three related notions for a foliation $\mathcal{F}$:

- to be C^l- approximated by contact structures as in Definition 2.2.17,
- to be C^l- deformed into contact structures by smooth paths as in the last definition,
- to be C^l- deformed into contact structures by continuous paths.

Are these equivalent? We did not spend much time on this question but, at a first glance, even for $l = 0$, the situation does not appear to be have an "obvious answer".

The main reason to use the C^∞-topology is to simplify the terminology and to avoid the search for "the best l" in each argument (which may obscure the discussion). The choice that would probably be most natural would be $l = 1$. The case $l = 0$ is too weak since it disregards the derivatives. More precisely, since we are interested mainly on co-orientable hyperplanes and we will represent the ξ_k above by contact forms α_k, the problem is that convergence $\alpha_k \to \alpha$ in the C^0-topology does not imply that $\mathrm{d}\alpha_k \to d\alpha$; and the C^1-topology takes care precisely of that. △

Here is the analogue of Lemma 2.2.2. Note that we are applying the previous discussion also to the bundle $P := T^*M$ to make sense of convergence of 1-forms (again using the C^∞-topology).

Lemma 2.2.19. *Assume that* M *is compact. Then, for a sequence* $(\xi_k)_k$ *of co-oriented hyperplane distributions, and another hyperplane distribution* ξ*, the following are equivalent:*

1. $\lim_k \xi_k = \xi$.
2. $\lim_k \alpha_k = \alpha$ *for some 1-forms* α_k *representing* ξ_k *and* α *representing* ξ*.*
3. *for any 1-forms* α_k *and* α *representing* ξ_k *and* ξ*, and any vector field* R *transverse to* ξ*,*
$$\lim_k \frac{1}{\alpha_k(R)}\alpha_k = \frac{1}{\alpha(R)}\alpha.$$

Proof. Let R be a vector field that is transverse to ξ and let α the corresponding 1-form inducing ξ. We use the line-field $\tau = \mathbb{R} \cdot R$, and consider $\mathcal{U}_\tau \subset \Gamma(R)$ defined by applying the construction (2.2.3.3) at each point. This is open in $\Gamma(P)$, hence it must contain all the ξ_k for k large enough. On the other hand, this open is identified with the space of sections of

$$P_0 := \xi^*$$

(with its topology as it follows from the previous discussion applied to the bundle P_0). Explicitly, the identification is obtained by applying (2.2.3.4) and identifying τ with $\mathbb{R}$ via α; one finds:

$$\phi : \Gamma(\xi^*) \to U_\tau, \quad \theta \mapsto \xi_\theta := \{v + \theta(v) \cdot R : v \in \xi\}.$$

Denoting by $\widetilde{\theta}$ the unique extension of θ to TM which is 1 on R, one finds that ξ_θ is represented by the 1-form

$$\alpha_\theta := \widetilde{\theta} \in \Omega^1(M).$$

Applying this to each ξ_k we find the corresponding θ_k and then the desired α_k. Since the entire argument is based on passing from one fiber bundle to another, by operations that are clearly continuous, the equivalence follows. □

2.2.4 The 3-dimensional case; confoliations

The case of 3-dimensional manifolds M is rather special. Indeed, the situation is, in principle, pretty simple: if ξ is induced by a 1-form α

$$\xi \text{ is a foliation, or contact structure} \iff \alpha \wedge d\alpha = 0, \text{ or } \alpha \wedge d\alpha \neq 0, \text{ respectively.}$$

That is, the form $\alpha \wedge d\alpha$ controls ξ being a foliation (and that is not only in dimension 3), as well as being contact (only in dimension 3) when one obtains a volume form (again only in dimension 3).

With the convention of working on oriented manifolds and the subsequent compatibility conditions, the discussion is whether $\alpha \wedge d\alpha$ is zero or strictly positive. This is not an honest dichotomy, as the conditions are required globally and not pointwise; however, it clearly indicates the space in which the two structures naturally interact each other- and that brings us to the notion of confoliation.

Definition 2.2.20. *A* ***confoliation*** *on a 3-dimensional manifold M, is a cooriented hyperplane distribution $\xi = \ker \alpha$, for some $\alpha \in \Omega^1(M)$ inducing the coorientation on ξ, and satisfying*

$$\alpha \wedge d\alpha \geqslant 0. \tag{2.2.4.1}$$

A confoliation ξ on a manifold M gives a decomposition into two regions, an open set where $\xi := \ker \alpha$ is contact and closed set where ξ is a foliation:

$$M = \mathcal{C}ont(\xi) := \{x \in M \mid (\alpha \wedge d\alpha)_x > 0\} \sqcup \mathcal{F}ol(\xi) := \{x \in M \mid (\alpha \wedge d\alpha)_x = 0\}.$$

The space of confoliations is closed as a subset of $\mathcal{H}yper(M)$. Its interior consists of contact structures, and the boundary consists of ξ for which $\mathcal{F}ol(\xi) \neq \varnothing$. Indeed, if a confoliation $\xi = \ker\alpha$ admits a point $x \in M$ where $(\alpha \wedge \mathrm{d}\alpha)_x = 0$, then there exists a distribution $\zeta := \ker\beta$ arbitrarily close to ξ in $\mathcal{H}yper(M)$ and satisfying $(\beta \wedge \mathrm{d}\beta)_x < 0$.

The interesting question (addressed already by Eliashberg-Thurston [47]) is when a foliation can be deformed into, or approximated by, contact structures, through the space of confoliations. Note that the linearity of deformations, or the type I condition

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\alpha_t \wedge \mathrm{d}\alpha_t) > 0, \tag{2.2.4.2}$$

agree with the discussion from [47].

The main findings of [47] (at least for our discussion) can be summarized into the following:

Theorem 2.2.21. *In dimension 3, looking for approximations of foliations by contact structures:*

(i) The trivial foliation of $\mathbb{S}^1 \times \mathbb{S}^2$ by spheres,

$$\left(\mathbb{S}^1 \times \mathbb{S}^2, \mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \{z\} \times \mathbb{S}^2\right),$$

cannot be approximated.

(ii) Any other foliation of any other 3-dimensional manifold can be C^0-approximated.

However, a similar theory (and similar results) in higher dimensions is missing. The reason is, we believe, that finding the correct higher dimensional analogues is more subtle than it may seem at first. Already in part (i) of the previous theorem, if one looks at the trivial foliation by spheres:

$$\left(\mathbb{S}^1 \times \mathbb{S}^{2n}, \mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \{z\} \times \mathbb{S}^{2n}\right), \quad n \neq 2, 6, \tag{2.2.4.3}$$

this cannot be deformed into contact structures for much more obvious reasons: contact hyperplanes carry non-degenerate two forms and, therefore (see Remark 2.3.2), the sphere S^{2n} would then carry such forms- which is well-known not to be the case for $n \neq 2, 6$.

As already clear from this example, the non-degenerate two-forms on the hyperplanes should enter the story, and we should be looking at symplectic foliations (or variations of them) in the limit. And the reasons those two-forms are not taken into account in dimension 3 is very simple: they are there anyway, implicitly. More precisely, fixing a volume form $\Omega \in \Omega^3(M)$, one sees that any hyperplane field $\xi \subset TM$ carries an induced non-degenerate 2-form Ω_ξ: while $\Omega|_\xi = 0$ for dimensional reasons, $(v, w) \mapsto \Omega(v, w, \cdot)$ becomes a 2-forms with coefficients in the conormal direction:

$$\Omega_\xi \in \Omega^2(\xi, \nu_\xi^*). \tag{2.2.4.4}$$

2.3 Going conformal: almost CS-hyperplane fields

2.3.1 Various symplectic Grassmannians

The previous section was basically about approximating foliations, with no reference to *symplectic* foliations. I.e. we concentrated on the hyperplane distributions, disregarding the induced non-degenerate 2-forms that they carry (both in the case of contact structures as well as in that of symplectic foliations!). We now start including those two-forms into the discussion. The need for doing so was already clearly indicated at the end of the last section, in the 3-dimensional discussion; see also the next remark.

Definition 2.3.1. *An* ***almost conformal symplectic hyperplane field*** *on a manifold M is a triple (ξ, ω, L) consisting of*

- *a hyperplane distribution $\xi \subset TM$*
- *a non-degenerate 2-form $\omega \in \Omega^2(\xi, L)$ along ξ with coefficients in a line bundle $L \to M$.*

We also use the acronym ***ACS- hyperplane field****, or we even omit the reference to L and talk about the ACS-hyperplane field (ξ, ω). Furthermore, we say that a pair*

$$(\alpha, \eta) \in \Omega^1(M) \times \Omega^2(M)$$

represents *(ξ, ω), or that it is a representing pair for (ξ, ω), if there exists an trivialization $\phi : L \xrightarrow{\sim} \mathbb{R}$ of the coefficients L of ω such that*

$$\xi = \ker(\alpha), \quad \phi \circ \omega = \eta|_{\xi}.$$

Remark 2.3.2. The remark from the end of the previous section (when discussing the obvious foliation by the spheres S^{2n}) is of a more general nature and clearly shows the need for ACS-structures in higher dimensions; the remark is:

$$\xi_k \to \xi \text{ with } \xi_k = \text{contact} \implies \xi \text{ can be made into an ACS-hyperplane field}$$

(on compact manifolds) and the similar statement for deformations (on arbitrary manifolds).

Indeed, choosing a complement τ of ξ in TM and using the corresponding projection $\mathrm{pr} : TM \to \xi$, if ξ_k is close to ξ then it will still be transverse to τ for some k large enough- hence

$$\mathrm{pr}|_{\xi_k} : \xi_k \to \xi$$

is an isomorphism; therefor ξ will admit an ACS-structure as well. △

Remark 2.3.3. In a first attempt to add the 2-forms in the picture and compare contact structures to symplectic foliations "on the nose", one has to face some mess due to the fact that the various 2-forms floating around take values in different line bundles. Of course, in principle there is not much of a loss of generality to assume that those line bundles are trivializable but (as we shall see) those trivializations may affect the resulting notion of convergence and serious problems may arise. Any way, here are some types of pairs (ξ, ω) that arise, with special attention to the line bundles:

- symplectic foliations $(\mathcal{F}, \omega_{\mathcal{F}})$ for which the line bundle is always $\mathbb{R}$. And similarly contact forms α, with corresponding pair $(\ker(\alpha), d\alpha|_{\ker(\alpha)})$.
- the conformal symplectic foliations that started showing up (e.g. in Theorem 2.2.13), where the coefficients can be any flat line bundle.
- the contact structures $\xi \subset TM$ which, by their very definition, carry a non-degenerate two-form, the curvature c_ξ of ξ, with coefficients in the normal bundle $\nu_\xi := TM/\xi$:

$$c_\xi \in \Omega^2(\xi, \nu_\xi), \quad c_\xi(X, Y) := [X, Y] \bmod \xi.$$

- on three-dimensional manifolds M, fixing a volume form Ω, any ξ carries an induced non-degenerate 2-form $\Omega_\xi \in \Omega^2(\xi, \nu_\xi^*)$ as discussed at the end of the previous section (see (2.2.4.4)). Hence, in this case, $L = \nu_\xi^*$.

Each such class lives naturally in a certain "symplectic Grassmannian manifold", respectively:

$$\mathrm{ACS}\mathcal{H}yper(M, \mathbb{R}), \quad \mathrm{ACS}\mathcal{H}yper(M, \nu), \quad \mathrm{ACS}\mathcal{H}yper(M, \nu^*), \quad \mathrm{ACS}\mathcal{H}yper(M, L).$$

△

To explain these spaces, we start with the linear discussion, for an arbitrary vector space V of dimension $2n+1$; then, according to the list above, there are four types of "sympectic Grassmannians" to consider,

$$\mathrm{SGr}(V), \quad \mathrm{SGr}(V, L), \quad \mathrm{SGr}(V, \nu), \quad \mathrm{SGr}(V, \nu^*)$$

(to be explained). Or, to treat them at once, just think that one has a functor $\mathfrak{l}$ going from 1-dimensional vector spaces (with isomorphisms) to itself. For the four Grassmannians we mentioned, one may actually just think that $\mathfrak{l}$ is just the d-th power functor

$$\mathfrak{l}(V/\xi) = (V/\xi)^{\otimes d}, \tag{2.3.1.1}$$

and we are mainly interested in the cases $d = 0, 1$ and -1. At this level (and important later when describing the resulting convergences), the main difference between the different $\mathfrak{l}$'s is that when applied to the multiplication m_λ by $\lambda \neq 0$ (viewed as a linear automorphism on a/any vector space),

$$\mathfrak{l}(m_\lambda) = m_{\lambda^d}. \tag{2.3.1.2}$$

Once we fix $\mathfrak{l}$, it gives rise to

- to a version of the Grassmannian with $\mathfrak{l}$-coefficients:

$$\mathrm{SGr}(V,\mathfrak{l}) := \{(\xi,\omega) \mid \xi \in \mathrm{Gr}_{2n}(V),\ \omega \in \Lambda^2\xi^* \otimes \mathfrak{l}(V/\xi) \text{ nondegenerate}\}$$

  which has a smooth structure by arguments completely similar to those for $\mathrm{Gr}(V)$: for $(\xi_0,\omega_0) \in \mathrm{SGr}(V)$, one considers the analogue of (2.2.3.3),

$$\widetilde{U}_{\tau_0} := \{(\xi,\omega) \in \mathrm{SGr}(V,\mathfrak{l}) : \xi \text{ is transverse to } \tau_0\}$$

  which, as in (2.2.3.4) (and actually using that chart plus the and the naturality of $\mathfrak{l}$) is identified with

$$\mathrm{Hom}(\xi_0,\tau_0) \times \left(\Lambda^2\xi_0^* \otimes \mathfrak{l}(\tau_0)\right)^{\mathrm{nondeg}}.$$

- Then, moving from V to $2n+1$ dimensional manifolds M and applying the previous discussion to the tangent spaces T_pM one obtains a fiber-bundle over M

$$\mathrm{SGr}(M,\mathfrak{l}) = \bigcup_{p\in M} \mathrm{SGr}(T_pM,\mathfrak{l}) \to M.$$

- And then, looking at sections of this bundle, one obtains the space of $\mathfrak{l}$-valued symplectic hyperplanes

$$\mathrm{ACS}\mathcal{H}yper(M,\mathfrak{l}).$$

  Since it is a space of sections it comes with various topologies but, as before, we will be looking only at the C^l-compact open topologies, where we assume $l = \infty$ if not otherwise specified.

Morally, $\mathrm{ACS}\mathcal{H}yper(M,\mathfrak{l})$ is an infinite dimensional manifold (and this can even be made precise using various frameworks) and one may think of it sitting over $\mathcal{H}yper(M)$ via the obvious projection

(2.3.1.3) $$\mathrm{pr} : \mathrm{ACS}\mathcal{H}yper(M,\mathfrak{l}) \to \mathcal{H}yper(M),$$

making it into an (infinite dimensional) bundle with the fiber above $\xi \in \mathcal{H}yper(M,\mathfrak{l})$ given by

$$\mathrm{ACS}\mathcal{H}yper(M,\mathfrak{l})_\xi = \left(\Omega^2(\xi,\mathfrak{l}_\xi)\right)^{\mathrm{nondeg}}, \quad \text{with } \mathfrak{l}_\xi = \mathfrak{l}(TM,\xi).$$

Note also that the constructions of the curvature c_ξ of hyperplanes,

$$\xi \mapsto c_\xi,$$

can now be interpreted as a section of a related bundle: the slightly larger version of $\mathrm{ACS}\mathcal{H}yper(M,\nu^*)$ where the non-degeneracy is ignored. Of course, this is a continuous (and even smooth in the appropriate sense) map.

2.3.2 Convergence in the various symplectic Grassmannians

We would like to emphasise: even if one is interested (like us) in hyperplane distributions which are coorientable (for which the various line bundles showing up can be trivialised), one still has to pay attention to those line bundles (as, in each of the spaces we discussed, the resulting notion of convergences depends essentially on the actual trivializations of those bundles). Here are some details. As before, to handle the various spaces at once, we work in the context of a general l as above.

First of all, let us be precise about the passing to trivial bundles. For notational simplicity let us just assume that $\mathfrak{l}(\mathbb{R}) = \mathbb{R}$. We use the terminology **$\mathfrak{l}$-symplectic hyperplane field** (ξ, ω) for the ACS-hyperplane fields with ω having as coefficients $\mathfrak{l}(TM/\xi)$, i.e. the points in ACS$\mathcal{H}yper(M, \mathfrak{l})$. Given (ξ, ω), we say that a pair

$$(\alpha, \eta) \in \Omega^1(M) \times \Omega^2(M)$$

$\mathfrak{l}$-represents (ξ, ω) if

$$\xi = \ker(\alpha), \quad \mathfrak{l}_\alpha \circ \omega = \eta|_\xi.$$

(compare with the similar notion from Definition 2.3.1). In the last equation we interpret α as a trivialization $\alpha : TM/\xi \cong \mathbb{R}$ and we apply the functor $\mathfrak{l}$ to α to obtain $\mathfrak{l}_\alpha : \mathfrak{l}(TM/\xi) \to \mathbb{R}$; this allows us to move to one single type of coefficients.

Remark 2.3.4. Given an $\mathfrak{l}$-symplectic hyperplane field (ξ, ω), any $\mathfrak{l}$-representative (α, η) comes with an induced Reeb vector field R, uniquely characterized by

$$\alpha(R) = 1, \quad i_R\eta = 0.$$

It is immediate to see that this actually gives a 1-1 correspondence between

- $\mathfrak{l}$-representatives (α, η) of (ξ, ω),
- vector fields $R \in \mathfrak{X}(M)$ which are transverse to ξ and such that $i_R(\omega) = 0$,

Given (α, η), R is the associated Reeb vector field, defined by the condition

$$\alpha(R) = 1, \quad i_R\eta = 0.$$

△

We now work out convergence in ACS$\mathcal{H}yper(M, \mathfrak{l})$ in terms of representatives.

Lemma 2.3.5. *Consider a $\mathfrak{l}$-symplectic hyperplane field (ξ, ω), with some chosen $\mathfrak{l}$-representative (α, η), with corresponding Reeb vector field denoted by R. For notational simplicity we assume that $\mathfrak{l}$ is the d-th power (2.3.1.1) and that M is compact.*

Then for an arbitrary sequence (ξ_k, ω_k) of $\mathfrak{l}$-symplectic hyperplane fields, the following are equivalent:

1. *$(\xi_k, \omega_k) \to (\xi, \omega)$ in $ACS\mathcal{H}yper(M, \mathfrak{l})$.*

2. *for a/any* $\mathfrak{l}$*-representatives* (α_k, η_k) *for the sequence one has*

$$\frac{1}{\alpha_k(R)}\alpha_k \to \alpha \text{ in } \Omega^1(M), \quad \frac{1}{\alpha_k(R)^{d+1}} \cdot i_R(\alpha_k \wedge \eta_k) \to \eta \text{ in } \Omega^2(M).$$

3. $\xi_k \to \xi$ *as hyperplanes and, for a/any* $\mathfrak{l}$*-representatives* (α_k, η_k) *for the sequence one has*

$$\frac{1}{\alpha_k(R)^{d+1}} \cdot i_R(\alpha_k \wedge \eta_k) \to \eta \quad \text{in } \Omega^2(M).$$

4. $\xi_k \to \xi$ *as hyperplanes and, for a/any* $\mathfrak{l}$*-representatives* (α_k, η_k) *for the sequence so that* $\alpha_k \to \alpha$*, one has*

$$i_R(\alpha_k \wedge \eta_k) \to \eta \quad (\text{in } \Omega^2(M)).$$

5. $\xi_k \to \xi$ *as hyperplanes and, for a/any* $\mathfrak{l}$*-representatives* (α_k, η_k) *for the sequence so that* $\alpha_k \to \alpha$*, one has*

$$\alpha_k \wedge \eta_k \to \alpha \wedge \eta \quad (\text{in } \Omega^3(M)).$$

6. *there exist* $\mathfrak{l}$*-representatives* (α_k, η_k) *for the sequence such that*

$$\alpha_k \to \alpha \text{ in } \Omega^1(M), \quad \eta_k \to \eta \text{ in } \Omega^2(M).$$

Remark 2.3.6. When M is not compact the statement of the result is, in principle, the same- just that one has to be careful when interpreting it. The problem is that, if $\alpha_k \to \alpha$ (in the compact-open topology), $\alpha_k(R)$ will be non-zero outside any compact C only for large enough $k \geqslant k_C$, with the k_C depending on the compact. Hence, strictly speaking, dividing by $\alpha_k(R)$ as a global function, is problematic. However, this enters a limit in the compact-open topology, hence it is something to be checked on each compact C and then "the problem" becomes irrelevant. △

Remark 2.3.7. In the characterizations from 4, 5 and 6 the apparent independence of d is misleading. Indeed, it is actually hidden in the notion of $\mathfrak{l}$-representative: if (α, η) is an $\mathfrak{l}$-representative of (ξ, ω) and we change α to $f \cdot \alpha$, then η must be changed to $f^d \cdot \eta$ to get the new $\mathfrak{l}$-representative $(f \cdot \alpha, f^d \cdot \eta)$.

On the other hand, while condition 5 is nicer, 2 (and often 4 and 5 as well) is handier since it can be checked on whatever representatives one may have at hand. For instance, in the contact case, once one has chosen α_k, one would like to use $\eta_k = \mathrm{d}\alpha_k$ (which may not be the one ensured by 6!).

Finally, it is good to have in mind that the expressions of type $\alpha \wedge \eta$ encode $\eta|_{\ker \alpha}$. Hence, in some sense, the condition from 5 can be thought of as saying that "$\eta_k|_{\xi_k} \to \eta|_{\xi}$". Note the related possible condition $\eta_k|_{\xi} \to \eta|_{\xi}$ which makes sense without quotes; it can be expressed algebraically as follows, giving rise to the following variation of condition 5:

5'. $\xi_k \to \xi$ as hyperplanes and, for some $\mathfrak{l}$-representatives (α_k, η_k) for the sequence so that $\alpha_k \to \alpha$, one has $\alpha \wedge \eta_k \to \alpha \wedge \eta$ in $\Omega^3(M)$.

While 6 clearly implies 5', the converse is not true: there may exist (α_k, ω_k), represented by some (α_k, η_k), with

$$\xi_k \to \xi, \quad \eta_k|_\xi \to \eta_k|_\xi \quad \text{BUT} \quad (\xi_k, \omega_k) \nrightarrow (\xi, \omega).$$

The problem is that the Reeb vector fields R_k may behave pretty wildly; when there is some condition on the $\mathfrak{l}$-representatives, then condition 5' on those representative does imply convergence. For example, one extreme situation would be to require that all the resulting Reeb vector fields R_k coincide with R (but note that, starting with $\xi_k \to \xi$, and with $\mathfrak{l}$-representative (α, η) with corresponding R, then such representatives (α_k, η_k) can be arranged). △

Proof. The first part of the last remark immediately implies the equivalence of 2 and 3. Looking at 3 and using the fact that $\alpha_k \to \alpha$ can be arranged (cf. Lemma 2.2.19) allows rewriting the condition from 3 as the condition from 4 (since $\alpha_k(R)$ will go to 1).

$$i_R(\alpha_k \wedge \eta_k) \to \eta \quad \text{in } \Omega^2(M).$$

That 5 implies 4 is obvious, while for the converse note that

$$\alpha_k \wedge \eta_k = \frac{1}{\alpha_k(R)} \alpha_k \wedge i_R(\alpha_k \wedge \eta_k).$$

Next, if the condition from 4 is satisfied, we use the same trick and we change to

$$\eta_k^{'} = \frac{1}{\alpha_k(R)} \cdot i_R(\alpha_k \wedge \eta_k) = \eta_k - \frac{1}{\alpha_k(R)} \cdot \alpha_k \wedge i_R(\eta_k);$$

Then $(\alpha_k, \eta_k^{'})$ is still representing (ξ_k, ω_k), but now has the property required in 6. Conversely, 6 implies 4 because the condition from 4 is independent of the choice of η_k in 4.

Hence we are left with proving that 1 is equivalent to 2. For that one starts with arbitrary $\mathfrak{l}$-representatives (α_k, η_k), (α, η) (as in 2), and write the limit condition from 1 out, by looking at the topology that was explained above; the outcome is precisely the limit condition from 2. □

Of course, when looking at contact structures, the most natural choice of $\mathfrak{l}$ is the identity, so that the 2-forms on the hyperplane fields ξ are with values in TM/ξ-precisely as the curvature c_ξ of ξ. Therefore we obtain a natural inclusion of the space $\mathcal{CH}yper(M)$ of contact structures.

Proposition 2.3.8. *For $\mathfrak{l} = id$ one obtains an embedding of the space of contact structures:*

$$I : \mathcal{CH}yper(M) \hookrightarrow AC\mathcal{SH}yper(M, id), \quad \xi \mapsto (\xi, c_\xi). \tag{2.3.2.1}$$

Moreover, the closure of its image contains no $(\xi, \omega) \in AC\mathcal{SH}yper(M, id)$ with ξ a co-orientable foliation.

Proof. Let us look at the last part (the first one is similar): when does (ξ_k, c_{ξ_k}) converge to (ξ, ω)? We apply the previous lemma and we see that we can write represent (ξ, ω) by some (α, η) and we find contact forms $\alpha_k \to \alpha$ such that $\alpha_k \wedge \mathrm{d}\alpha_k \to \alpha \wedge \eta$. But, if ξ was a foliation, then $\alpha_k \wedge \mathrm{d}\alpha_k \to \alpha \wedge \mathrm{d}\alpha = 0$ giving a contradiction. ☐

Remark 2.3.9. In the previous proof we used that $\alpha_k \to \alpha$ implies $\mathrm{d}\alpha_k \to \mathrm{d}\alpha$. This would break down if we use the C^0-topology. Hence, in principle, the situation may not be so negative as in the previous proposition if we use C^0-convergence. △

The presence of other $\mathfrak{l}$'s allow for new perspectives on contact forms (in a manner in which the tempting sloppiness on identifying the coefficients right away is avoided): for each $\mathfrak{l}$ one can look inside the corresponding space $\mathrm{ACS}\mathcal{H}yper(M, \mathfrak{l})$ for elements that correspond to contact forms.

Definition 2.3.10. *We say that* $(\xi, \omega) \in ACS\mathcal{H}yper(M, \mathfrak{l})$ *is of* ***contact type*** *if it admits an* $\mathfrak{l}$*-representative of type* $(\alpha, \mathrm{d}\alpha)$ *for some contact form:*

$$\xi = \ker(\alpha), \quad \mathfrak{l}_\alpha \circ \mathrm{d}\alpha = \eta|_\xi \quad \textit{for some contact form } \alpha. \tag{2.3.2.2}$$

Looking the other way around, we obtain for each $\mathfrak{l}$ an inclusion

$$I_{\mathfrak{l}} : \Omega^1_{\mathrm{cont}}(M) \hookrightarrow \mathrm{ACS}\mathcal{H}yper(M, \mathfrak{l}) \tag{2.3.2.3}$$

which associates to a contact form α the hyperplane $\xi = \ker\alpha$ with the 2-form $l_\alpha^{-1} \circ \mathrm{d}\alpha|_\xi$. And this allows one, in principle, to approximate various ACS-hyperplane fields by contact structures. Note that the case $\mathfrak{l} = \mathrm{id}$ is pretty special:

- it is the only case when $I_{\mathfrak{l}}$ is not injective (instead it factors through $\mathcal{CH}yper(M)$ giving rise to the embedding from the previous proposition),
- despite being the most natural choice of $\mathfrak{l}$ for handling contact structures, it is the only bad choice when trying to approximate symplectic foliations.

Note that the first point is due to the remark that, if (α, η) is an $\mathfrak{l}$-representative for an element (ξ, ω), and we want to replace α by $f \cdot \alpha$ to have another $\mathfrak{l}$-representative, then one has to multiply η by f^d. For the second item, let us work out the resulting convergence.

Lemma 2.3.11. *Given* $(\omega, \xi) \in ACS\mathcal{H}yper(M, \mathfrak{l})$ *with some chosen* $\mathfrak{l}$*-representative* (α, η)*, then* (ω, ξ) *is the limit of a sequence of elements in* $ACS\mathcal{H}yper(M, \mathfrak{l})$ *of contact type, i.e. it is in the closure of the inclusion (2.3.2.3), if and only if there exists a sequence of contact forms* α_k *and a sequence of smooth nowhere vanishing functions* f_k *such that*

$$\alpha_k \to \alpha, \quad f_k^{d-1} \cdot \alpha_k \wedge \mathrm{d}\alpha_k \to \alpha \wedge \eta.$$

(where d *is associated to* $\mathfrak{l}$ *as in ((2.3.1.2))).*

We see why the truly special case is when $\mathfrak{l} = \mathrm{id}$ (i.e. $d = 1$); all the other cases can be brought to the condition

$$\alpha_k \to \alpha, \quad g_k \cdot \alpha_k \wedge \mathrm{d}\alpha_k \to \alpha \wedge \eta$$

for some sequence of nowhere zero smooth functions g_k, required to be positive if d is odd.

Proof. The hypothesis is that there are contact forms β_k such that the element in $\mathrm{ACS}\mathcal{H}yper(M, \mathfrak{l})$ that is $\mathfrak{l}$-represented by $(\beta_k, \mathrm{d}\beta_k)$ converges to (ω, ξ). We apply the characterization 2. form Lemma 2.3.5 and we find that

$$\alpha_k := \frac{1}{\beta_k(R)}\beta_k \to \alpha \text{ in } \Omega^1(M), \quad \frac{1}{\beta_k(R)^{d+1}} \cdot i_R(\beta_k \wedge \mathrm{d}\beta_k) \to \eta \text{ in } \Omega^2(M).$$

(where, as in the Lemma, R is the Reeb vector field corresponding to (α, η)). Denoting β_k by $\alpha_k(R) \cdot \alpha_k$ in the last limit we find that

$$f_k^{d-1} \cdot i_R(\alpha_k \wedge \mathrm{d}\alpha_k) \to \eta, \quad \text{where } f_k = \frac{1}{\beta_k(R)}$$

(and conversely, since we can just set $\beta_k = \frac{1}{f_k} \cdot \alpha_k$). Moreover, using again the identity

$$\alpha_k \wedge \mathrm{d}\alpha_k = \alpha_k \wedge i_R(\alpha_k \wedge \mathrm{d}\alpha_k),$$

we find the condition as written in the statement. □

Example 2.3.12. In $\mathbb{R}^3$ consider the symplectic foliation

$$(\mathcal{F} := \ker \mathrm{d}z, \omega := \mathrm{d}x \wedge \mathrm{d}y).$$

One can then take the sequence of contact forms

$$\alpha_k = \mathrm{d}z - \frac{1}{k} y \mathrm{d}x$$

which has

$$k \cdot \alpha_k \wedge \mathrm{d}\alpha_k = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \alpha \wedge \omega,$$

hence the criterion from the previous lemma is satisfied choosing each f_k to be a constant function ($d-1$-th rooth of k). Note that, for each $d \neq 1$ (or $\mathfrak{l}$) , the sequence β_k of contact forms with $I_{\mathfrak{l}}(\beta_k)$ converging to the element in $\mathrm{ACS}\mathcal{H}yper(M, \mathfrak{l})$ represented by $(\mathcal{F}, \omega)$ depends on $\mathfrak{l}$- it is:

$$\beta_k = k^{-\frac{1}{d-1}} \cdot \alpha_k.$$

△

Note that, as in the previous sections, there is a similar (and simpler) discussion which, instead of sequences one uses smooth paths

$$[0, \epsilon) \ni t \mapsto (\xi_t, \omega_t) \in \mathrm{ACS}\mathcal{H}yper(M, \mathfrak{l}).$$

To talk about smoothness one can, as mentioned above, make $\mathrm{ACS}\mathcal{H}yper(M, \mathfrak{l})$ into an infinite dimensional manifold. Or, as in the previous section, just adopt the obvious ad-hoc definition obtained by interpreting paths ξ_t as sub-bundles $\widetilde{\xi}$ of the pull-back of TM by the projection $\mathrm{pr} : \mathbb{R} \times M \to M$ and ω_t as a 2-form $\widetilde{\omega}$ on $\widetilde{\xi}$. Representing (ξ_t, ω_t) by pairs (α_t, η_t) as discussed above, the analogue of Lemma 2.3.5 gives various charaterizations of the smoothness of (α_t, ω_t) for instance the analogue of condition (2) would be a version of Lemma 2.2.2 that takes into account the 2-forms as well.

Lemma 2.3.13. *For any smooth path $(\xi_t, \omega_t)_{t\in[0,\epsilon)}$ of $\mathfrak{l}$-sympectic hyperplanes, with ξ_0 co-orientable, one can find $\mathfrak{l}$-representatives (α_t, η_t) such that both α_t and η_t are smooth paths (in $\Omega^1(M)$ and $\Omega^2(M)$, respectively).*

Of course, one has also a characterization similar to 4 from Lemma 2.3.5: if (α_t, ω_t) are arbitrary $\mathfrak{l}$-representatives of (ξ_t, ω_t), with corresponding Reeb vector field R_0 at $t = 0$, then

$$t \mapsto \frac{1}{\alpha_t(R_0)}\alpha_t, \quad t \mapsto \frac{1}{\alpha_t(R_0)^{d+1}} \cdot i_R(\alpha_t \wedge \eta_t)$$

is smooth in t.

2.3.3 Going conformal; conformal convergence

With the rather negative conclusions from Proposition 2.3.8 in mind, let us start by looking back at some of the positive results that we mentioned, on approximations of foliations by contact structures.

Remark 2.3.14 (A brief look at the 3-dimensional case again)**.** Let us briefly return to the discussion of the 3-dimensional case from the end of the previous section (subsection 2.2.4). First of all, the remark that each hyperplane field ξ comes endowed with a canonical non-degenerate 2-form Ω_ξ (see (2.2.4.4)) shows that, in this case, there is yet another inclusion

$$I_\Omega : \mathcal{H}yper(M) \hookrightarrow \mathrm{ACS}\mathcal{H}yper(M, \nu^*), \quad \xi \mapsto (\xi, \Omega_\xi)$$

(for all hyperplane fields, not only contact ones!). Unlike the other canonical inclusion (2.3.2.1), and in contrast with what happens in Proposition 2.3.8, the closure of the image of this inclusion can give rise to foliations. Indeed, since the inclusion is defined on all hyperplanes, and it is a topological embedding, the second component does not bring anything new, i.e. we are simply looking at convergence of contact hyperplanes to foliations.

In fact under the above inclusion convergence in $\mathcal{H}yper(M)$ corresponds to convergence in $\mathrm{ACS}\mathcal{H}yper(M, \nu^*)$, or more precisely:

Lemma 2.3.15. *Given hyperplane fields $\xi_k, \xi \in \mathcal{H}yper(M)$ we have:*

$$\xi_k \to \xi, \quad \textit{if and only if} \quad I_\Omega(\xi_k) \to I_\Omega(\xi).$$

Proof. In this case $\mathfrak{l}$ is the d-th power functor with $d = -1$, see Equation 2.3.1.1. Write $\xi = \ker \alpha$ and let R be any vector fields such that $\alpha(R) = 1$. Then $(\alpha, \eta := \iota_R\Omega)$ is an $\mathfrak{l}$-representative for $I_\Omega(\xi)$. Similarly, we find $\mathfrak{l}$-representatives (α_k, η_k) for $I_\Omega(\xi_k)$, and observe that

$$\alpha_k \wedge \omega_k = \alpha_k \wedge \iota_{R_k}\Omega = \Omega,$$

from which it follows (using Lemma 2.3.5) that $I_\Omega(\xi_k) \to I_\Omega(\xi)$. △

We see that the way to "fix" the negative phenomenon from Proposition 2.3.8 is very "cheap": for contact structures $\xi = \ker \alpha$, instead of considering the obvious/most natural non-degenerate 2-form $\mathrm{d}\alpha|_\xi$ (and, in higher dimensions, the only available one!), pick up a volume form and consider Ω_ξ. How different are they? Well, being in dimension 3, i.e. with ξ being 2-dimensional, we can certainly write

$$\Omega_\xi = f_\xi \cdot \mathrm{d}\alpha|_\xi$$

for some nowhere vanishing (or even strictly positive, under the appropriate orientation conventions) smooth function f_ξ. Therefore, one may say that the negative phenomena from Corollary 2.3.8 can actually be fixed by working "conformally", i.e. allowing to change $\mathrm{d}\alpha$ by multiplying by functions. And this is something that makes a lot of sense in higher dimensions as well. △

Apart from the last remark we can also say that, by now, we have already seen several manifestation of "conformal factors". Already the notion of representing pair from Definition 2.3.1 is of a conformal nature. And similarly for the convergence that was worked out in Lemma 2.3.11 (when $\mathfrak{l} \neq \mathrm{id}$). And, even when looking at the convergence of just hyperplane fields, Theorem 2.2.13 already pointed out the importance of the "conformal factor". All together, it should be clear now (in case it wasn't clear earlier!) what is the notion of convergence that is "correct" (or at least "most appropriate") for our discussion on deforming contact structures into symplectic foliations.

Definition 2.3.16. *Given an ACS-hyperplane field (ξ, ω) with $\omega \in \Omega^2(\xi, L)$, and a sequence (ξ_k, ω_k) of ACS-hyperplane fields (with line bundles L_k), one say that*

$$(\xi_k, \omega_k) \rightsquigarrow (\xi, \omega) \quad \textit{(and one reads: } \textbf{conformally converges}\textit{)}$$

if there exist line bundle isomorphisms $\phi_k : L_k \xrightarrow{\sim} L$ such that

$$(\xi_k, \phi_k \circ \omega_k) \to (\xi, \omega)$$

as ACS-hyperplane fields with fixed coefficients L (i.e. in $ACS\mathcal{H}yper(M, L)$).

Of course, this is a discussion that takes place at the level of "conformal classes". Let us formalise this.

Definition 2.3.17. *Two ACS-hyperplane fields (ξ_i, ω_i), $i = 1, 2$, are said to be* ***conformally equivalent****, and write*

$$(\xi_1, \omega_1) \sim (\xi_2, \omega_2),$$

if there exists an isomorphism $\phi : L_1 \xrightarrow{\sim} L_2$ between their coefficients such that $\omega_2 = \phi \circ \omega_1$. We denote by $[\xi, \omega]$ the resulting equivalence classes.

An equivalence class with respect to this equivalence relation is called an ***ACS-structure*** *on M, and the set of such structures is denoted $ACS(M)$. Hence*

$$ACS(M) = \left(\bigcup_L ACS\mathcal{H}yper(M, L) \right) \Big/ \sim .$$

With this, $\rightsquigarrow$ makes sense as a convergence defined on $\mathrm{ACS}(M)$ (still called conformal convergence).

Remark 2.3.18. Although we do not find it particularly enlightening, it is nice to know that $\mathrm{ACS}(M)$ admits a topology for which the corresponding convergence is precisely conformal convergence. The basic opens are constructed out of opens $U \subset \mathrm{ACS}\mathcal{H}yper(M, L)$ (for each line bundle L), by defining

$$[U] := \{[\xi, \omega] \in \mathrm{ACS}(M) : (\xi, \omega) \in U\}.$$

One should be aware that, if one has an element $u \in \mathrm{ACS}(M)$, written as $u = [\xi_0, \omega_0]$, for u to belong to $[U]$ it is not necessary that $(\omega_0, \xi_0) \in U$ (it may even happen that ω_0 has coefficients $L_0 \neq L$).

Putting all the "opens" $[U]$ together one obtains a topology basis, and then the desired topology on $\mathrm{ACS}(M)$. △

Note that all the types of structures that we discussed in Remark 2.3.3 (all giving rise to ACS-hyperplane fields) can now be seen inside $\mathrm{ACS}(M)$. Most notably, one obtains:

- $\mathrm{Cont}(M) \hookrightarrow \mathrm{ACS}(M)$, $\xi \mapsto [\xi, c_\xi]$, with the remark that in dimension 3:
$$[\xi, c_\xi] = [\xi, \Omega_\xi].$$
- symplectic foliations or, more generally, conformal symplectic foliations (cf. Definition 2.2.16) can be seen as ACS-structures on M.

In particular, this allows us to make sense of conformal approximation by contact:

Definition 2.3.19. *We say that a (conformal) symplectic foliation $(\mathcal{F}, \omega_{\mathcal{F}})$ can be **conformally approximated** by contact structures if there is a sequence of contact structures ξ_k conformally converging to $(\mathcal{F}, \omega_{\mathcal{F}})$, i.e.*

$$(\xi_k, c_{\xi_k}) \rightsquigarrow (\mathcal{F}, \omega_{\mathcal{F}}).$$

Returning to general conformal classes and their convergence, it is clear that (at least in the co-orientable case) one can pass to trivial coefficients, i.e. use representing pairs (cf. Definition 2.3.1). Note also that, in the new terminology, given an ACS-hyperplane field (ξ, ω), with $\omega \in \Omega^2(\xi, L)$, a representing pair in the sense of Definition 2.3.1 is a pair

$$(\alpha, \eta) \in \Omega^1(M) \times \Omega^2(M)$$

with the property that

$$(\xi, \omega) \sim (\alpha, \eta|_{\ker \alpha}).$$

Hence, in principle, one can just use invoke Lemma 2.3.5 for $d = 0$ to check convergence. Let us be a bit more explicit about the outcome in the case of contact sequences.

Lemma 2.3.20. *Consider an ACS-hyperplane field (ξ, ω) represented by a pair (α, η) as above, with corresponding $R \in \mathfrak{X}(M)$. Given a sequence of contact structures ξ_k, the following are equivalent:*

1. *ξ_k conformally converges to (ξ, ω) (or $(\xi_k, c_{\xi_k}) \rightsquigarrow (\xi, \omega)$).*
2. *there exists a (or, equivalently, for any) sequence $\alpha_k \to \alpha$ with $\xi_k = \ker \alpha_k$, and a sequence of nowhere vanishing smooth functions f_k such that*
$$f_k \cdot i_R(\alpha_k \wedge \mathrm{d}\alpha_k) \to \eta \quad (\textit{in } \Omega^2(M)).$$
3. *there exists a (or, equivalently, for any) sequence $\alpha_k \to \alpha$ with $\xi_k = \ker \alpha_k$, and a sequence of nowhere vanishing smooth functions f_k such that*
$$f_k \cdot \alpha_k \wedge \mathrm{d}\alpha_k \to \alpha \wedge \eta \quad (\textit{in } \Omega^3(M)).$$

Proof. As we mentioned, in principle we just apply Lemma 2.3.5 (for $d = 0$ and $\mathfrak{l}$ constant $\mathbb{R}$); the convenient items are 4 and 5 since they allow for arbitrary choice of η_k. For each (ξ_k, c_{ξ_k}) we choose a representative pair of type $(\beta_k, \mathrm{d}\beta_k)$; the problem is that the β_k may not be the α_k that will eventually converge to α (actually the β_k may even "explode"). Then items 4 and 5 of the lemma become:

4: there exists a (or, equivalently, for any) sequence $\alpha_k \to \alpha$ with $\xi_k = \ker \alpha_k$, such that
$$i_R(\alpha_k \wedge \mathrm{d}\beta_k) \to \eta \quad (\text{in } \Omega^2(M)).$$

5: there exists a (or, equivalently, for any) sequence $\alpha_k \to \alpha$ with $\xi_k = \ker \alpha_k$, such that
$$\alpha_k \wedge \mathrm{d}\beta_k \to \alpha \wedge \eta \quad (\text{in } \Omega^3(M)).$$

Writing now $\beta_k = f_k \alpha_k$, we obtain the conditions from the statement. □

Working out the definition, from the previous discussions we deduce:

Corollary 2.3.21. *A symplectic foliation $(\mathcal{F}, \omega_{\mathcal{F}})$, represented by (α, η) (hence $\mathcal{F} = \ker \alpha$, $\omega_{\mathcal{F}} = \eta|_{\mathcal{F}}$), can be conformally approximated by contact structures if and only if there exists a sequence of contact forms α_k and nowhere vanishing smooth functions f_k such that*
$$\alpha_k \to \alpha, \quad f_k \cdot \alpha_k \wedge \mathrm{d}\alpha_k \to \alpha \wedge \eta$$
(as 1-forms, and 2-forms, respectively).

Example 2.3.22. Again (as in Remark 2.3.7), while a condition of type $\alpha \wedge \eta_k \to \alpha \wedge \eta$ is equivalent to $\eta_k|_\xi \to \eta|_\xi$ ($\xi = \ker \alpha$), the condition $f_k \cdot \alpha_k \wedge \mathrm{d}\alpha_k \to \alpha \wedge \eta$ should be thought of (morally) as saying that
$$\text{"}\mathrm{d}\alpha_k|_{\xi_k} \rightsquigarrow \eta|_\xi\text{"}.$$
However (and again), this is not directly related to $\mathrm{d}\alpha_k|_\xi \rightsquigarrow \eta|_\xi$, or more precisely,
$$g_k \cdot \alpha \wedge \mathrm{d}\alpha_k \to \alpha \wedge \eta$$

for some sequence of nonzero functions g_k. For an example consider

$$(\mathcal{F} := \ker \mathrm{d}z, \omega := \mathrm{d}x \wedge \mathrm{d}y),$$

(hence $\alpha = \mathrm{d}z$, $\eta = \mathrm{d}x \wedge \mathrm{d}y$) and the sequence

$$\alpha_k = \frac{1}{k}\mathrm{d}x + e^y \mathrm{d}z.$$

One has

$$f_k \cdot \alpha_k \wedge \mathrm{d}\alpha_k = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \alpha \wedge \mathrm{d}\eta, \quad \text{with } f_k = ke^{-y}$$

hence conformal convergence holds, but $\alpha \wedge \mathrm{d}\alpha_k = 0$, hence no multiple $\mathrm{d}\alpha_k|_{\mathcal{F}}$ can converge to $\eta|_{\mathcal{F}}$. △

Definition 2.3.23. *We say that a (conformal) symplectic foliation $(\mathcal{F}, \omega_{\mathcal{F}})$, represented by (α, η), can be* ***naively approximated*** *if there exists a sequence of contact forms α_k, and a sequence of nowhere vanishing smooth functions f_k such that:*

$$\alpha_k \to \alpha, \quad f_k \alpha \wedge \mathrm{d}\alpha_k \to \alpha \wedge \eta \quad (\textit{in } \Omega^3(M)).$$

The conformal factors f_k are essential, since by Proposition 2.3.8 the definition becomes vacuous if we additionally require $f_k = 1$. Furthermore, the above condition depends on the choice of contact forms, i.e. replacing α_k by $\tilde{\alpha}_k := g_k \alpha_k$ does not preserve it.

Example 2.3.24. In $\mathbb{R}^3$ consider (again) the symplectic foliation $(\mathcal{F} := \ker \mathrm{d}z, \omega := \mathrm{d}x \wedge \mathrm{d}y)$. The sequence of contact forms

$$\alpha_k := \mathrm{d}z + \frac{1}{k} x \mathrm{d}y,$$

and the sequence of functions $f_k := k$ satisfy:

$$f_k \cdot \alpha \wedge \mathrm{d}\alpha_k = \alpha \wedge \eta,$$

hence $(\mathcal{F}, \omega)$ can be naively approximated by contact forms. This example might give the wrong impression that the Reeb vector fields R_k of α_k (or the induced linefields $\langle R_k \rangle$) must converge to a vector field transverse to $\mathcal{F}$. This is not true in general (see Proposition 2.4.31), although it is true that for all k sufficiently large R_k is transverse both to $\mathcal{F}$ and $\ker \alpha_k$. △

This type of approximations are very special and impose severe restrictions on $\mathcal{F}$. Indeed, observe that for k sufficiently large $\mathrm{d}\alpha_k|_{\mathcal{F}}$ is non-degenerate (since ω is) and hence defines an exact symplectic form on $\mathcal{F}$ that can be extended to a globally closed (even exact) form on M. In particular $\mathcal{F}$ is a taut foliation (as in Definition 2.4.21).

Finally, note also that the discussion we had so far in this section has a rather obvious version in which sequences are replaced by paths/deformations. For instance, a path (ξ_t, ω_t) of ACS-hyperplane fields is said to be **conformally smooth** if, for each t, there exists an isomorphism $\phi_t : L_t \xrightarrow{\sim} L$, such that $t \mapsto (\xi_t, \phi_t \circ \omega_t)$ is smooth. Then, for a conformal symplectic foliation $(\mathcal{F}, \omega_{\mathcal{F}})$, with some coefficients L, we say that $(\mathcal{F}, \omega_{\mathcal{F}})$ can be **conformally deformed** into contact structures if one can find a contact deformation ξ_t of $\mathcal{F}$ (smooth, as in Definition 2.2.1) and isomorphisms $\phi: TM/\xi_t \xrightarrow{\sim} L$ such that

$$t \mapsto (\xi_t, \phi_t \circ c_{\xi_t})$$

is smooth in $\mathrm{SGr}(M, L)$. Again, this can be further simplified (similar to the last Lemma) when the coefficients are all $\mathbb{R}$.

And here it is worth having a look back at Theorem 2.2.13 and its proof: one remarks right away that the argument there tells us something a bit more:

Corollary 2.3.25. *If a foliation $\mathcal{F}$ admits a type I contact deformation (Definition 2.2.5) then there exists $\omega_{\mathcal{F}} \in \Omega^2(M, TM/\mathcal{F})$ leafwise exact, making $\mathcal{F}$ into a conformal symplectic foliation with the property that $(\mathcal{F}, \omega_{\mathcal{F}})$ can be conformally deformed into contact structures.*

For the rest of this chapter we work with differential forms, and we use the following convention:

From now on, unless explicitly stated otherwise, when talking about almost CS-hyperplane fields,

$$\textbf{convergence} \equiv \textbf{conformal convergence}$$
$$\textbf{approximation} \equiv \textbf{conformal approximation}$$

as in Definition 2.3.19.

2.4 Comparison of the approximation types

In the previous section we introduced a generalization of confoliations, together with several types of convergences and approximations. The aim of this section is to provide "isolating examples" for each type of approximation. That is, foliations which can be approximated in the sense of one definition but not another.

2.4.1 Hyperplane fields and almost CS-hyperplane fields

We first compare the convergence (and approximation) of almost CS-hyperplane fields (as elements of $\mathrm{ACS}\mathcal{H}yper(M, \mathfrak{l})$) to convergence of the underlying distributions (i.e. as elements of $\mathcal{H}yper(M)$). Clearly, as was pointed out in Equation 2.3.1.3, convergence as almost CS-hyperplane fields implies (by definition) convergence of the underlying hyperplane fields. By Lemma 2.3.15 the converse also holds in dimension-3, provided we consider ν^*-coefficients. If we pass to conformal convergence then the claim holds with coefficients in any line bundle.

Going to higher dimensions, the situation is quite different. The first remark is that, as was observed in the examples of Equation 2.2.4.3, there are many hyperplane fields, in particular foliations, which do not admit any almost CS-structure. Thus (for these foliations) we cannot talk about being approximated by contact structures in the space of almost CS-hyperplane fields.

The situation is not as bad as it looks however, we can still consider approximations in $\mathcal{H}yper(M)$, as in Definition 2.2.17. In fact, as observed in Remark 2.3.2, if a sequence of contact structures ξ_k converges (in $\mathcal{H}yper(M)$) to ξ, then the latter can be made into an almost CS-hyperplane field. Clearly, the same argument holds for sequences of almost CS-hyperplane fields (ξ_k, ω_k), but we repeat it here for later reference:

Lemma 2.4.1. *Let ξ be a hyperplane field and (ξ_k, ω_k) a sequence of almost CS-hyperplane fields such that $\xi_k \to \xi$ (in $\mathcal{H}yper(M)$). Then ξ admits an almost CS-structure.*

Proof. Let (ξ_k, ω_k) be a sequence of almost CS-hyperplane fields topologically converging to a distribution ξ. Then, for k sufficiently large, there exists a line field $\tau \subset TM$, which is transverse to ξ and ξ_k. This induces, isomorphisms

$$\xi \oplus \tau \simeq TM \simeq \xi_k \oplus \tau,$$

which in turn gives an isomorphism $\pi_\tau : \xi \to \xi_k$, which we think of as "the projection of ξ onto ξ_k along τ". Then, the pullback

$$\omega := \pi^* \omega_k \in \Omega^2(\xi, L),$$

defines an almost CS-structure on ξ. Observe that if ω_τ is the (unique) extension of ω_k whose kernel is τ, then

$$\omega = \omega_\tau|_\xi. \tag{2.4.1.1}$$

□

Remark 2.4.2. The proof actually shows the hypothesis of the lemma can be weakened. Indeed, given a hyperplane field ζ it suffices there exists an almost CS-hyperplane field (ξ, ω) and a line field τ transverse both to ζ and ξ. Thus a distribution ζ which does not admit an almost CS-structure is "far away" from the space of almost CS-distributions. In the case of Example 2.2.4.3, being "far away" means that any contact structure on $\mathbb{S}^1 \times \mathbb{S}^{2n}$ must, at some point, contain the line field spanned by ∂_z. △

Remark 2.4.3. The above lemma does not say that a sequence of contact structures ξ_k on M, converging in $\mathcal{H}yper(M)$ to a foliation $\mathcal{F}$, also (conformally) converges in ACS$\mathcal{H}yper(M)$ to some almost CS-structure ω on $\mathcal{F}$. In Proposition 2.6.22, we will see an example this situation. Moreover, the sequence of contact structures ξ_k considered there, is C^0-close to another sequence of contact structure ζ_k, which does converge to an almost CS-structure on $\mathcal{F}$. △

In light of this lemma, we can wonder if any distribution that can be approximated by contact structures in $\mathcal{H}yper(M)$ can be approximated by contact structures in ACS$\mathcal{H}yper(M, \mathfrak{l})$. The following discussion shows that the answer to this question is negative in general. We introduce a class of submanifolds of manifolds endowed with an almost CS-hyperplane field, with the property that a sequence of almost CS-hyperplane fields (ξ_k, ω) converging on the ambient manifold induces a converging sequence on the submanifold. Therefore, the existence of such a submanifold endowed with an almost CS-hyperplane field which cannot be approximated, gives an obstruction to approximating the almost CS-hyperplane field on the ambient manifold.

Definition 2.4.4. *Let (ξ, ω) be an almost CS-hyperplane field on M. A submanifold $N \subset M$ is called a* ***almost CS-submanifold*** *if the restriction*

$$(\xi_N := \xi \cap TN, \quad \omega_N := \omega|_{\xi_N}),$$

defines an almost CS-hyperplane field on N.

We often consider a submanifold $N \subset M$ that is already endowed with an almost CS-hyperplane field $(\widetilde{\xi}, \widetilde{\omega})$. In this case, when we say that N is an almost CS-submanifold, we mean that $(\widetilde{\xi}, \widetilde{\omega}) = (\xi_N, \omega_N)$ where (ξ_N, ω_N) is as in the definition above.

One way of thinking about this definition is that given a manifold M, we have a "map"

$$\{ \text{ submanifold } N \subset M \ \} \times \begin{Bmatrix} \text{almost CS-hyperplane} \\ \text{fields } (\xi, \omega) \text{ on M} \end{Bmatrix} \to \begin{Bmatrix} \text{distributions } \xi_N \text{ on } N \text{, with} \\ \text{a 2-form } \omega_N \in \Omega^2(\xi_N, L) \end{Bmatrix}.$$

Of course, this is only well-defined if N is transverse to ξ. Fixing an almost CS-hyperplane field (ξ, ω) on M, a submanifold N is an almost CS-submanifold if the resulting ω_N is nondegenerate.

The property of being an almost CS-submanifold is "stable" in both arguments of the above assignment. More precisely, if $N \subset (M, \xi, \omega)$ is an almost CS-submanifold, then in particular it is transverse to ξ. But this is an open condition; if

$$\widetilde{N} := \phi(N),$$

for a C^1-small diffeomorphism $\phi : M \to M$, then $\widetilde{N}$ is also transverse to ξ. Moreover, the induced distribution and 2-form $(\xi_{\widetilde{N}}, \omega_{\widetilde{N}})$ are such that

$$\left(\phi^*(\xi_{\widetilde{N}}), \phi^*\omega_{\widetilde{N}}\right), \quad \text{and,} \quad (\xi_N, \omega_N),$$

are close in the compact-open topology on $\mathrm{ACS}\mathcal{H}yper(M, \mathfrak{l})$. Hence, $(\xi_{\widetilde{N}}, \omega_{\widetilde{N}})$ is an almost CS-distribution.

Stability in the second argument follows from the following lemma, which says that if we fix an almost CS-submanifold in the first argument, the above "map" is sequentially continuous in the second argument (with respect to the compact-open topology on $\mathrm{ACS}\mathcal{H}yper(M, \mathfrak{l})$).

Lemma 2.4.5. *Consider an almost CS-submanifold $N \subset (M, \xi, \omega)$ with induced almost CS-hyperplane field (ξ_N, ω_N). Let (ξ_k, ω_k), $k \in \mathbb{N}$, be a sequence of almost CS-hyperplane fields converging to (ξ, ω). Then, for k sufficiently large,*

$$\left(\xi_{N,k} := \xi_k \cap TN, \omega_{N,k} := \omega_k|_{\xi_{N,k}}\right),$$

defines a sequence of almost CS-hyperplane fields on N, converging to (ξ_N, ω_N).

Proof. Choose representatives (α, η) for (ξ, ω). Then, applying Lemma 2.3.5 to find I-representatives (α_k, η_k) for (ξ_k, ω_k) satisfying

$$\alpha_k \to \alpha, \quad \eta_k \to \eta.$$

The restriction $(\alpha_N := \alpha|_{TN}, \eta_N := \eta|_{TN})$ represents (ξ_N, ω_N). Therefore, restricting the above equation to N we find a sequence $(\alpha_{N,k}, \eta_{N,k})$ representing $(\xi_{N,k}, \omega_{N,k})$ and satisfying the condition of Lemma 2.3.5, proving convergence. □

Remark 2.4.6. Observe that the above lemma also holds for paths of almost CS-hyperplane fields (ξ_t, ω_t), $t \in (0,1]$ instead of sequences. Furthermore, if the path is of type I or type II, then so is the induced path on the almost CS-submanifold N. Similarly, if the sequence on M is conformally/naively converging then so is the sequence on N. This can be used, for example, to show that almost CS-submanifolds can form an obstruction to type I approximation by contact structures, see Corollary 2.4.8 below. △

Remark 2.4.7. Usually it is understood that by submanifold we mean embedded submanifold. However, Definition 2.4.4 and Lemma 2.4.5 also make sense for immersed submanifold, or even submanifolds with self-intersections. In fact, given an almost CS-hyperplane field (ξ, ω) on M, all we need to obstruct the existence of approximations, is a map $f : N \to M$ such that $(f^*\xi, f^*\omega)$ defines an almost CS-hyperplane field on N. In this case, suppose that (ξ_k, ω_k) is a sequence of almost CS-hyperplane fields on M converging to (ξ, ω). Then, essentially the same proof as that of Lemma 2.4.5, shows that for k sufficiently large, $(f^*\xi_k, f^*\omega_k)$, defines a sequence of almost CS-hyperplane fields on N converging to $(f^*\xi, f^*\omega)$.

Analogous to Corollary 2.4.8 below, this implies that the existence of such a map for which $(f^*\xi, f^*\omega)$ cannot be approximated by contact structures, is an obstruction for (ξ, ω) to be approximated.

△

As stated before, the main use of Lemma 2.4.5, is that it provides obstructions for an almost CS-foliation $(\mathcal{F}, \omega)$ to be approximable by contact structures.

Proposition 2.4.8. *Consider an almost CS-submanifold $(N, \xi_N, \omega_N) \subset (M, \xi, \omega)$. If (ξ_N, ω_N) cannot be approximated by contact structures, then (ξ, ω) cannot be approximated by contact structures.*

Combining this proposition with Theorem 2.2.21 we obtain the following corollary:

Corollary 2.4.9. *If an almost CS-hyperplane field (ξ, ω) contains*

$$\left(\mathbb{S}^1 \times \mathbb{S}^2, \mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \{z\} \times \mathbb{S}^2, \omega := \omega_{\mathbb{S}^2}\right),$$

as an almost CS-submanifold, then it cannot be approximated by contact structures.

Foliations for which this corollary applies are plentiful. In fact, we will see in Section 2.5 that it takes quite a bit of work to find examples for which there exist different obstructions to being approximable. For now we give some basic examples, more elaborate constructions are given in Section 2.6.1.

Example 2.4.10. Let (M, ω_M) be a symplectic manifold. Then the symplectic foliation

$$\left(\mathbb{S}^1 \times \mathbb{S}^2 \times M,\, \mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \{z\} \times M \times \mathbb{S}^2,\, \omega := \omega_M + \omega_{\mathbb{S}^2} \right),$$

cannot be approximated by contact structures. △

Slightly more generaly, let $(M, \mathcal{F}_M, \omega_M)$ be any almost CS-foliation. In this case, the almost CS-foliation

$$\left(M \times \mathbb{S}^2, \mathcal{F} := \mathcal{F}_M \times \mathbb{S}^2, \omega := \omega_M + \omega_{\mathbb{S}^2} \right),$$

cannot be approximated by contact structures. To see this recall the following lemma from foliation theory:

Lemma 2.4.11. *Let $\mathcal{F}$ be a coorientable foliation on a (closed) manifold M. Then there exists a closed embedded loop $\mathbb{S}^1 \subset M$ transverse to $\mathcal{F}$.*

Proof. Choose a nowhere vanishing vector field $X \in \mathfrak{X}(M)$ transverse to $\mathcal{F}$. The flowlines of X define a 1-dimensional foliation $\mathcal{F}_X$ on M, whose leaves are transverse to the leaves of $\mathcal{F}$. If $\mathcal{F}_X$ has a leaf diffeomorphic to $\mathbb{S}^1$ (i.e. when X has a periodic orbit) we are done. If not all the leaves are non-compact and we proceed as follows.

Using the flow of X and compactness of M we can find a finite covering $\{U_i\}_{i \in I}$ of M such that each U_i is a foliated chart for $\mathcal{F}_X$ isomorphic to $\mathbb{D}^{n-1} \times (-1, 1)$ (where $\dim M = n$) and $\mathbb{D}^{n-1} \times \{0\}$ is contained in a single leaf of $\mathcal{F}$.

Fix a leaf of $\mathcal{F}_X$, since it is non-compact it must intersect some chart U_{i_0} at least twice. Thus we obtain an arc J transverse to $\mathcal{F}$, whose endpoints are contained in $\mathbb{D}^{n-1} \times \{0\} \subset U_{i_0}$. In particular ∂J is contained in a single leaf of $\mathcal{F}$. This arc can be completed inside U_{i_0} to obtain a closed transverse loop (see for example Lemma 3.3.7 in [23]). □

Choosing a closed transversal for $\mathcal{F}_M$, it follows immediately that $(\mathcal{F}, \omega)$ admits $\mathbb{S}^1 \times \mathbb{S}^2$, with the usual foliation, as an almost CS-submanifold. Thus Corollary 2.4.8 applies. It is not hard to see that the above arguments also work if, instead of $\mathbb{S}^1 \times \mathbb{S}^2$, we use any any manifold M for which $\mathbb{S}^1 \times M$, with the obvious foliation, cannot be approximated by contact structures. For example, we can use $\mathbb{S}^1 \times \mathbb{S}^{2n}$, for any $n \neq 3$, as in Example 2.2.4.3.

Consider a distribution ξ on a manifold M, and ξ_k a sequence of contact structures converging to ξ in $\mathcal{H}yper(M)$. Analogous to Lemma 2.4.5, we observe that if $N \subset M$ is an (odd dimesnional) submanifold transverse to ξ, then for all k sufficiently large,

$$\xi_{N,} := \xi_k \cap TN,$$

defines a sequence of distributions on N converging to $\xi_N := \xi \cap TN$. However, unlike the situation in Lemma 2.4.5, in general the $\xi_{N,k}$ are not contact structures. Hence almost CS-submanifolds do not necessarily obstruct convergence in $\mathcal{H}yper(M)$. We exploit this observation to show that approximation by contact structures in $\mathcal{H}yper(M)$ and ACS$\mathcal{H}yper(M)$ are not equivalent in higher dimensions:

Proposition 2.4.12. *There exist almost CS-foliations which can be approximated by contact structures in $\mathcal{H}yper(M)$ but not in ACS$\mathcal{H}yper(M)$.*

Proof. Let $\mathcal{F}$ be a foliation on M, with $\dim M = 2n+1 \geqslant 7$, which can be topologically approximated by contact structures. By Lemma 2.4.1 there exists an almost CS-structure ω on $\mathcal{F}$. As explained in Lemma 2.4.13 below, it is possible change ω to another almost CS-structure $\widetilde{\omega}$ on $\mathcal{F}$, so that $(\mathcal{F}, \widetilde{\omega})$ admits $\mathbb{S}^1 \times \mathbb{S}^2$, with the usual symplectic foliation, as an almost CS-submanifold. Hence, by Proposition 2.4.8 and Theorem 2.2.21, $(\mathcal{F}, \widetilde{\omega})$ cannot be approximated by contact structures. However, since we did not change the foliation $\mathcal{F}$, it can still be topologically approximated. □

Lemma 2.4.13. *Let $(\mathcal{F}, \omega)$ be an almost CS-foliation on a manifold M of dimension $2n+1 \geqslant 7$. Then there exists an almost CS-structure ω' on $\mathcal{F}$ which admits an almost CS-submanifold $\mathbb{S}^1 \times \mathbb{S}^2$. Moreover, ω' is homotopic (through almost CS-structures on $\mathcal{F}$) to ω and agrees with ω outside a small neighborhood of $\mathbb{S}^1 \times \mathbb{S}^2$.*

Proof. Fix an embedded $\mathbb{S}^1$ transverse to $\mathcal{F}$, as in Lemma 2.4.11. There exists a tubular neighborhood $\mathbb{S}^1 \times D^{2n} \subset M$ on which

$$\mathcal{F} = \bigcup_{z \in \mathbb{S}^1} \{z\} \times D^{2n},$$

since the restriction $T\mathcal{F}|_{\mathbb{S}^1}$ is the trivial bundle. We start by making ω symplectic on a small neighborhood of $\mathbb{S}^1 \times \{0\}$. By a (linear) change of coordinates we can assume that ω agrees with the standard symplectic form

$$\omega_{st} := \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i \in \Omega^2(D^2),$$

at points $(z, 0) \in \mathbb{S}^1 \times D^{2n}$. For $\varepsilon > 0$ let $D^2_\varepsilon := \{x \in \mathbb{R}^{2n} \mid ||x|| \leqslant \varepsilon\}$, denote the disk of radius ε. Choose a function $\rho_\varepsilon : \mathbb{R}^{2n} \to [0,1]$ satisfying:

$$\mathrm{supp}(\rho_\varepsilon) \subset D^2_\varepsilon, \quad \rho_\varepsilon|_{D^2_{\varepsilon/2}} = 1.$$

For $\varepsilon > 0$ small enough,

$$\omega' := (1-\rho)\omega + \rho\omega_{st} \in \Omega^2(\mathbb{S}^1 \times D^{2n}),$$

is a leafwise symplectic form around $\mathbb{S}^1 \times \{0\}$ and homotopic through almost CS-forms to ω. From now on we assume, without loss of generality, that $\omega = \omega_{st}$ on the entire $\mathbb{S}^1 \times D^{2n}$.

Next, we construct a suitable embedding of $\mathbb{S}^1 \times \mathbb{S}^2$. Since $2n+1 \geqslant 7$ we have an embedding:

$$\mathbb{S}^2 := \{x_1^2 + x_2^2 + x_3^2 = 1, y_1 = y_2 = y_3 = 0\} \subset D^{2n}.$$

Note that the restriction $\omega|_{\mathbb{S}^2} = 0$ and that the normal bundle $\nu_{\mathbb{S}^2} \simeq \mathbb{S}^2 \times \mathbb{R}^{2n-2}$ is trivial. Denote by $\omega_{\mathbb{S}^2}$ the standard area form on $\mathbb{S}^2$ and let $\rho_\varepsilon : \mathbb{R}^{2n-2} \to [0,1]$ as above. Then for ε small enough

$$\widetilde{\omega} := \omega + \varepsilon\rho\omega_{\mathbb{S}^2},$$

is an almost CS-form which is leafwise symplectic when restricted to $\mathbb{S}^1 \times \mathbb{S}^2 \subset \mathbb{S}^1 \times D^{2n}$ and homotopic to ω through almost CS-forms. □

Consider a foliation $\mathcal{F}$ on a manifold M with $\dim M = 2n+1 \geqslant 7$ which can be approximated by contact structures in $\mathcal{H}yper(M)$. Then, by the result above, there exists at least one almost CS-structure ω on $\mathcal{F}$ for which $(\mathcal{F}, \omega)$ cannot be approximated by contact structures in ACS$\mathcal{H}yper(M)$. However, it is still possible that $(\mathcal{F}, \widetilde{\omega})$ is approximable for a different choice of almost CS-structure $\widetilde{\omega}$.

In Proposition 2.6.22 below, we given an example of the dual situation, i.e. we give a sequence of contact structures converging to a foliation $\mathcal{F}$ (in $\mathcal{H}yper(M)$), but not converging to any almost CS-structure on $\mathcal{F}$.

2.4.2 Almost CS-hyperplane fields and Type I

Next, we compare approximation in ACS$\mathcal{H}yper(M, \mathfrak{l})$ to approximation of type I in $\mathcal{H}yper(M)$. Recall from Theorem 2.2.13 that a foliation $\mathcal{F}$ has a type I deformation into contact structures if and only if it admits an exact leafwise CS-structure. Furthermore, by Corollary 2.3.25, if this happens then the CS-structure can be conformally deformed (in ACS$\mathcal{H}yper(M, (TM/\mathcal{F}))$) to contact structures.

In dimension-3 there is another characterization of type I deformations:

Proposition 2.4.14 ([47]). *If $\mathcal{F}$ is a foliation on a (compact) 3-manifold M for which either of the following hold:*

(i) $\mathcal{F}$ has a closed leaf with trivial linear holonomy;

(ii) $\mathcal{F}$ can be defined by a closed 1-form;

(iii) $\mathcal{F}$ has no holonomy.

Then, $\mathcal{F}$ does not admit a type I deformation (Definition 2.2.5) into contact structures.

Thus, not any foliation which can be deformed into contact structures in $\mathcal{H}yper(M)$ admits a type I deformation. For example, consider the foliation

$$\left(\mathbb{S}^1 \times \mathbb{T}^2, \mathcal{F} := \bigcup_{z \in \mathbb{S}^1} \{z\} \times \mathbb{T}^2\right).$$

By Theorem 2.2.21, $\mathcal{F}$ can be approximated by contact structures. However, $\mathcal{F}$ satisfies all the hypothesis of the proposition above, so it does not admit a type I approximation. It turns out that these conditions are, in dimension 3, the only obstructions to being type I approximable.

Theorem 2.4.15 ([47]). *Let $\mathcal{F}$ be a foliation on a 3-manifold M such that $\mathcal{F}$ has holonomy, and each of its closed leaves has a curve with non-trivial linear holonomy. Then $\mathcal{F}$ can be linearly approximated by contact structures.*

Using Theorem 2.2.13, the obstructions from Proposition 2.4.14 can be translated to higher dimensions. Roughly speaking, the idea is that if $\mathcal{F}$ can be type I approximated by contact structures, we can construct an almost CS-submanifold of the exact leafwise CS-structure given by Theorem 2.2.13. By Remark 2.4.6 this means we obtain a type I deformation on the almost CS-submanifold. Repeatedly applying this procedure, relates the high-dimensional approximation to the 3-dimensional case.

The construction of the required almost CS-submanifold is based on Donaldson techniques from [34, 67]. If α_t, $t \in (0,1]$ is a type I path of contact forms converging to $\mathcal{F}$, then we can produce an almost CS-submanifold for each α_t. Using that α_t converges, we can show that these almost CS-submanifolds, for t sufficiently small, are also almost CS-submanifolds for the exact leafwise CS-structure on $\mathcal{F}$ given by Theorem 2.2.13. The precise statement is as follows:

Theorem 2.4.16. *Let $\mathcal{F}$ be a foliation on M with modular class $[\nu] \in H^1(\mathcal{F})$, as in Definition 1.7.16, and $\mathrm{d}_\nu\beta$ any exact leafwise CS-structure, as in Theorem 2.2.13. Then, there exists a codimension-2 closed almost CS-submanifold, as in Definition 2.4.4 intersecting all the leaves of $\mathcal{F}$.*

Proof. Let $\mathrm{d}_\nu\beta$ for $\beta \in \Omega^1(M)$ be the leafwise CS-structure on $\mathcal{F}$ which we assume to exists. By Theorem 2.2.13 there exists a type I path of contact forms α_t, $t \in (0,1]$ conformally converging (in ACS$\mathcal{H}yper(M)$) to $(\mathcal{F}, \mathrm{d}_\nu\beta)$. The idea of the proof is to apply the theorem for existence of codimension-2 contact submanifolds from [67], to the sequence $\alpha_n := \alpha_{t_n}$ where $t_n \in [0,1]$, $n \in \mathbb{N}$ is such that $t_n \to 0$ for $n \to \infty$. We briefly recall the argument. First, we first fix a compatible almost complex structure J_n for the symplectic vector bundle $(\xi_n := \ker \alpha_n, \mathrm{d}\alpha_n)$. This defines a global Riemannian metric $g_n := \mathrm{d}\alpha(\cdot, J\cdot) + \alpha_n \otimes \alpha_n$. Then we define the topologically trivial bundle $L := M \times \mathbb{C}$ with non-trivial connection form $\nabla := \mathrm{d} - i\alpha$. Define $\alpha_{k,n} := k\alpha_n$, $k \in \mathbb{N}$, $g_{k,n} := kg_n$ and $(L_k := L^{\otimes k}, \nabla_{k,n} := \mathrm{d} - i\alpha_{k,n})$. Observe that there is a splitting $\nabla_{k,n} = \partial + \bar{\partial}$ into complex linear and complex anti-linear part, since for any section $s \in \Gamma(L_k)$ the map $\nabla_{k,n}s$ is a linear map between complex vector spaces. We have a notion of assymptotic holomorphic sequence of sections $s_{k,n} \in \Gamma(L_k)$ defined by the following set of conditions:

$$||s_{k,n}|| \leqslant C, \quad ||\nabla^r_{k,n} s_{k,n}|| \leqslant C, \quad ||\nabla^{r-1}_{k,n}\bar{\partial} s_{k,n}|| < k^{-1/2}C, \quad r = 1,2,3.$$

Here, $C > 0$ is a constant that is independent of k. However note that in principle C could depend on n. In the previous expression the norms of the higher order derivatives are measured with respect to the norm associated to $g_{k,n}$ in the source. Furthermore there is a notion of ε-transversality that reads as follows. A section $s \in \Gamma(L_k)$ is said to be ε-transverse to zero along ξ_n at $p \in M$ if at least one of the following two conditions is fulfilled:

$$||s(p)|| > \varepsilon, \quad ||(\nabla_{k,n}|_\xi)s(p)|| > \varepsilon.$$

The main result in [67] states that there is a assympotic holomorphic sequence of sections $s_{k,n}$ that are ε-transverse to ξ_n over the whole manifold. Moreover, ε does not depend on k. This immediately implies that the zero set $Z(s_{k,n})$ is smooth, transverse to ξ_n with a minimum angle $\theta_{k,n}$ depending only on ε and C, and such that $\xi_n \cap TZ(s_{k,n})$ is a contact distribution on $Z(s_{k,n})$.

What we want to show is that $Z(s_{k,n})$ is transverse to the limit foliation $\mathcal{F}$ with a minimum angle $\tilde{\theta}_{k,n}$ that does not depend on k and n. Observe that $\tilde{\theta}_{k,n} \geqslant \theta_{k,n}/2$ for n large enough. This is because by the triangle inequality $\tilde{\theta}_{k,n} \geqslant \theta_{k,n} - \eta_n$ where η_n is the angle between ξ_n and $\mathcal{F}$ and $\eta_n \to 0$ for $n \to \infty$. Hence it suffices to show that $\theta_{k,n} > \theta$ for some $\theta > 0$. Observe that the angle $\theta_{k,n}$ at a given point depends linearly on $||\nabla_\xi s_{k,n}||$ and $||\nabla_{R_k} s_{k,n}||^{-1}$ where $R_{k,n}$ is the Reeb vector field of $\alpha_{k,n}$. Furthermore we have $||\nabla_{R_{k,n}}|| \leqslant ||\nabla_{k,n}|| \leqslant C$. Hence,

$$\theta_{k,n} = c\frac{||\nabla_\xi s_{k,n}||}{||\nabla_{R_{k,n}} s_{k,n}||} \geqslant c\frac{\varepsilon}{C},$$

for a universal constant c. The standard arguments in [34, 67] show that ε and C are independent of k. From a careful reading of the proofs it is clear that ε and C are bounded by the following:

(i) The topology of M. In particular, the minimum number of Darboux charts needed to cover the manifold.

(ii) The amount of integrability of the distribution ξ_n, measured by $||\alpha_n \wedge \mathrm{d}\alpha_n^m||$. The smaller this quantity the better.

(iii) The natural extension $\tilde{J}_n$ of J_n to the symplectization $\mathbb{R} \times M$ by the formula:

$$\tilde{J}_n(\lambda_0 + \lambda_1 R_n + V) := \lambda_1 \partial_t - \lambda_0 R_n + J_n V,$$

where $\lambda_0, \lambda_1 \in C^\infty(\mathbb{R} \times M)$ and $V \in TM$. In particular, it depends continuously on the choice of J_n.

We now check that these properties are independent of n. By Gray stability all the contact structures ξ_n are contactomorphic, so the minimum number of Darboux charts needed to cover M is independent of n. The second property is clearly satisfied since ξ_n converges to a foliation for $n \to \infty$.

To check the third property recall that by Theorem 2.2.13 convergence of $\alpha_t = \alpha + t\beta$ to $\mathcal{F} = \ker \alpha$ is equivalent to conformal convergence of $(\xi_t, \mathrm{d}\alpha_t)$ to $(\mathcal{F}, \mathrm{d}_\eta \beta)$. This just means that there exist extensions ω_t of $\mathrm{d}\alpha_t|_{\xi_t}$ conformally converging to $\mathrm{d}_\eta\beta$ along $\mathcal{F}$. Compatibility of J does not depends on the choice of representative of the conformal class. Hence, we find a family J_t, $t \in [0,1]$ of compatible complex structures and in particular $J_n := J_{t_n}$ is a sequence inside a compact set and so converges to J_∞. Similarly, the Reeb vector field R_t of α_t is linearly interpolating between the Reeb vector field of α_1 and $\ker \mathrm{d}_\eta\beta$. Hence, by the same argument as before R_n converges to R_∞. This means that $\tilde{J}_n$ converges so that (for n big enough) the contribution of the third property is independent of n. Therefore, the divisor is transverse to the limit foliation $\mathcal{F}$. □

Consider a foliation $\mathcal{F}$ on M that can be type I approximated by contact structures. Combining Theorem 2.2.13 and Theorem 2.4.16 we find an almost CS-submanifold N for $\mathcal{F}$ with the induced CS-structure. In particular N is transverse to $\mathcal{F}$, and we denote the resulting foliation by

$$\mathcal{F}_N := \mathcal{F} \cap TN.$$

Together with Lemma 2.4.5 and Remark 2.4.6, it follows that $\mathcal{F}_N$ admits a type I deformation into contact structures.

On the other hand, observe that if $\mathcal{F}$ can be defined by a closed 1-form, or has no holonomy, then so does $\mathcal{F}_N$. Similarly, if $\mathcal{F}$ has a closed leaf with trivial linear holonomy then so does $\mathcal{F}_N$, provided that N intersects all the leaves of $\mathcal{F}$. We conclude that if $\mathcal{F}$ satisfies any of the conditions in Proposition 2.4.14, but can be type I approximated, then we find a foliation on a 3-manifold contradicting Proposition 2.4.14.

Corollary 2.4.17. *The statement of Proposition 2.4.14 holds in any dimension.*

Example 2.4.18. Recall from Chapter 1 that there exists a SF-deformation (α_t, ω_t), $t \in [0, 1]$, as in Definition 1.8.1, between the standard contact structure on $\mathbb{S}^5$ and the symplectic foliation $(\mathcal{F}, \omega)$ constructed in Theorem 1.9.1. Moreover, the path of 1-forms α_t is of type I as in Definition 2.2.5). However, the compact leaf of $\mathcal{F}$ has trivial linear holonomy since it is obtained by gluing two foliated manifolds with tame boundaries. Therefore, Corollary 2.4.17 that $(\mathcal{F}, \omega)$ cannot be type I approximated by contact structures. △

Using Theorem 2.2.13, the above discussion also implies:

Corollary 2.4.19. *If a foliation $\mathcal{F}$, with modular class $[\nu] \in H^1(\mathcal{F})$, satisfies any of the conditions of Proposition 2.4.14 (in particular M is compact), then it does not admit admit a d_ν-exact leafwise CS-structure.*

Recall that an almost CS-hyperplane field is a "formal CS-foliation" in the sense that it forgets about the "differential conditions" in the Definition 2.2.16. This makes sense more generally for any geometric structure that can be defined as the solution of a (partial) differential equation, see Chapter 3. Clearly, in this situation, the existence of the formal data is a necessary condition for the existence of an honest solution. If the converse holds, i.e. if out of the formal data we can build the structure, we say that the structure satisfies the h-principle. As a consequence of Theorem 2.2.13, the h-principle for CS-foliations does not hold in full generality:

Corollary 2.4.20. *The h-principle for CSfoliations does not hold.*

Proof. Analogously to the h-principle for symplectic structures [43], a formal CS-foliation consists of an almost CS-foliation $(\mathcal{F}, \omega)$ together with a cohomology class $c \in H^2_\nu(\mathcal{F})$. Given this data there should exist a leafwise CS-structure $\widetilde{\omega}$ on $\mathcal{F}$ such that $[\widetilde{\omega}] = c \in H^2_\eta(\mathcal{F})$. However, if $\mathcal{F}$ does not admit a type I deformation into contact structures, then, by Theorem 2.2.13, the class $0 \in H^2(\eta(\mathcal{F}))$ cannot be represented by CS-structures. Hence, any foliation which cannot be type I approximated by contact structures does not satisfy the h-principle. □

2.4.3 Type I and naive approximation

Recall from Corollary 2.3.25 that if a foliation $\mathcal{F}$ admits a type I deformation into contact structures, then it can be conformally approximated (for some carefully chosen 2-form on the leaves). In this section we consider the relationship between naive and type I approximations. We will see that the former implies the latter, but not conversely.

Recall that a foliation $\mathcal{F}$ on M is called **taut**, if there exist an embedded loop $\mathbb{S}^1 \subset M$ transverse to $\mathcal{F}$ and intersecting all the leaves. As shown by Sullivan [102], this is equivalent to the following definition:

Definition 2.4.21. *A foliation $\mathcal{F}$ on M^{2n+1} is **taut** if there exists a globally closed form $\Omega \in \Omega^{2n}(M)$ which restricts to a volume form on the leaves of $\mathcal{F}$.*

Taut foliations are mostly studied in dimension-3 where the above definition says there should exist a globally closed 2-form $\omega \in \Omega^2(M)$ which is symplectic on the leaves of $\mathcal{F}$. Hence, in higher dimensions we also consider the following more restrictive notion.

Definition 2.4.22. *A foliation $\mathcal{F}$ on M^{2n+1} is said to be **strong symplectic** if there exists a globally closed form $\omega \in \Omega^2(M)$ which restricts to a symplectic form on the leaves of $\mathcal{F}$.*

Clearly strong symplectic foliations are taut but the converse need not be true. Furthermore, by Lemma 1.7.23, a strong symplectic foliation together with a globally closed 2-form (F, ω) is a tame symplectic foliation in the sense of Definition 1.7.22.

Lemma 2.4.23. *If an almost CS-foliation $(\mathcal{F}, \omega)$ can be naively approximated by contact structures, as in Definition 2.3.23. Then $\mathcal{F}$ is a strong exact symplectic foliation.*

Proof. Choose a representing pair (α, η) for $(\mathcal{F}, \omega)$, and let α_k be a sequence of contact forms satisfying (as in Definition 2.3.23)

$$\alpha_k \to \alpha, \quad f_k \alpha \wedge \mathrm{d}\alpha_k \to \alpha \wedge \eta,$$

for a sequence of positive functions f_k. Being non-degenerate on $\mathcal{F}$, is an open condition in the space of leafwise 2-forms $\Omega^2(\mathcal{F})$. Hence, the convergence in the above equation implies that, for k sufficiently large, $\mathrm{d}\alpha_k|_{\mathcal{F}}$ defines an exact leafwise symplectic form. □

Clearly, these necessary conditions are extremely restrictive. By the following proposition, they imply that $\mathcal{F}$ admits a type I deformation into contact structures. Hence, together with Example 2.2.12 this means that being naively approximable is strictly stronger condition than being type I approximable.

Proposition 2.4.24. *If a conformal symplectic foliation $(\mathcal{F}, \omega)$ can be naively approximated as in Definition 2.3.23, then it can be type I approximated by contact structures, as in Definition 2.2.5.*

The proof follows directly from the following lemma:

Lemma 2.4.25. *Let α_k be a sequence of contact forms naively approximating a conformal symplectic foliation $(\mathcal{F} = \ker \gamma, \omega)$ as in Definition 2.3.23. Then, for any k sufficiently large,*

$$\alpha_t := \gamma + t\alpha_k,$$

is a type I deformation (Definition 2.2.5) of $\mathcal{F}$ into contact structures.

Proof. We choose a representing pair (γ, η) for $(\mathcal{F}, \omega)$, and denote the associated Reeb vector field by R. Since α_k naively approximates $(\mathcal{F}, \omega)$, it follows that for all k sufficiently large the associated Reeb vector field R_k is transverse to $\mathcal{F}$, and can thus be expressed as:

$$R_k := f_k R + V_k,$$

for positive functions f_k and $V_k \in \ker \alpha_k$. Note that

$$\alpha_k(R_k) = \frac{1}{f_k}.$$

If we multiply α_k by positive constants c_k, the resulting sequence still naively approximates $(\mathcal{F}, \omega)$. Hence, since M is compact, we can assume that $f_k > \epsilon$ for a universal $\epsilon > 0$. Now consider the path

$$\alpha_t := \gamma + t\alpha_k, \quad t \in [0, 1],$$

for which the contact condition equals

$$\alpha_t \wedge \mathrm{d}\alpha_t^n = t^n \left(\gamma \wedge \mathrm{d}\alpha_k^n + n\alpha \wedge \mathrm{d}\gamma \wedge \mathrm{d}\alpha_k^{n-1}\right) + t^{n+1}\alpha_k \wedge \mathrm{d}\alpha_k^n.$$

For t small enough the first term dominates, provided it is nonzero. By assumption $\gamma \wedge \mathrm{d}\alpha_k^n > 0$ for large k. Hence, to show that α_t is contact it suffices to show that $\gamma \wedge \mathrm{d}\alpha_k^n$ dominates $n\alpha_k \wedge \mathrm{d}\gamma \wedge \mathrm{d}\alpha_k^{n-1}$. This condition can be checked pointwise, and at points where $\mathrm{d}\gamma = 0$ it clearly holds. Hence, we can assume $\mathrm{d}\gamma \neq 0$ so that $\operatorname{rank} \mathrm{d}\gamma = 2$. Choose a basis $R, X_{1,k}, Y_{1,k}, \ldots, X_{n,k}, Y_{n,k}$ satisfying

(i) $\gamma(R) = 1$;

(ii) $X_{1,k}, Y_{1,k}, \ldots, X_{n,k}, Y_{n,k} \in \mathcal{F}$ are a sympletic basis for $\mathrm{d}\alpha_k|_{\mathcal{F}}$;

(iii) $\mathrm{d}\gamma(R, X_{1,k}) = 1$ and $\ker \mathrm{d}\gamma = \langle X_{2,k}, \ldots, Y_{n,k}\rangle$.

Furthermore there exist $\varepsilon_{i,k}, \delta_{i,k} \in \mathbb{R}$, $i = 1, \ldots, n$ such that $\widetilde{X}_{i,k} := X_{i,k} + \varepsilon_{i,k} R$ and $\widetilde{Y}_{i,k} := Y_{i,k} + \delta_{i,k} R$ form a basis of ξ_k. Then we have

$$\gamma \wedge \mathrm{d}\alpha_k^n(R, X_{1,k}, \ldots, Y_{n,k}) = 1.$$

On the other hand

$$\begin{aligned}
\alpha_k \wedge \mathrm{d}\gamma \wedge \mathrm{d}\alpha_k^{n-1}(R, X_{1,k}, \ldots, Y_{1,k}) &= \alpha_k \wedge \mathrm{d}\gamma \wedge \mathrm{d}\alpha_k^{n-1}(R, \widetilde{X}_1, \ldots, \widetilde{Y}_n) \\
&= \alpha_k(R)\mathrm{d}\gamma(\widetilde{X}_{1,k}, \widetilde{Y}_{1,k})\mathrm{d}\alpha_k^{n-1}(\widetilde{X}_2, \ldots, \widetilde{Y}_n) \\
&= \frac{\delta_{1,k}}{f_k}.
\end{aligned}$$

Since ξ_k converges to $\mathcal{F}$ in $\mathcal{H}yper(M)$, we must have $\delta_{1,k} \to 0$ for $k \to \infty$. Furthermore, $f_k > 0$ so that for k large enough $\gamma \wedge \mathrm{d}\alpha_k^n \gg \alpha_k \wedge \mathrm{d}\alpha_k^{n-1} \wedge \mathrm{d}\beta$, showing that α_t is contact. □

2.4.3.1 Examples of naive convergence

To construct examples of foliations that can be naively approximated by contact structures, we need to consider manifolds that admit both a taut foliation and a contact structure. A class of manifolds which meet these criteria are those that admit a conformally Anosov flow, see [47, 88], and satisfy an additional symmetry condition.

Definition 2.4.26. *Let (M, g) be a 3-dimensional Riemannian manifold and ϕ_t, $t \in \mathbb{R}$, the flow of a vector field $X \in X(M)$. Then ϕ_t is said to be:*

(i) ***Anosov*** *if there exists a splitting*

$$TM = E_+ \oplus E_- \oplus \langle X \rangle,$$

and a constant $C > 0$ satisfying:

$$||\mathrm{d}\phi_t(v_+)|| \geqslant e^{Ct}||v_+||, \quad ||\mathrm{d}\phi_t(v_-)|| \leqslant e^{-Ct}||v_-||,$$

for all $v_\pm \in E_\pm$.

(ii) ***Conformally Anosov*** *if there exists a splitting as above, and a constant $C > 0$ satisfying:*

$$\frac{||d\phi_t(v_+)||}{||d\phi_t(v_-)||} \geqslant e^{Ct}\frac{||v_+||}{||v_-||}, \tag{2.4.3.1}$$

for any non-zero vectors $v_\pm \in E_\pm$.

The definition of an Anosov flow means that flowing along X contracts the E_- and expands E_+ direction. A conformally Anosov flow generalizes this definition by requiring the flow to contract E_- and expand E_+ only after it has been normalized to have determinant 1 (with respect to the Riemannian metric).

In both cases, the splitting defines two foliations,

$$\mathcal{F}_\pm := \langle X, E_\pm \rangle \subset TM,$$

called the **stable** and **unstable foliations**. The following proposition shows that a conformally Anosov flow is completely determined by these foliations. Hence, such flows can be equivalently described by a pair of differential forms:

Proposition 2.4.27 ([47])**.** *Let X be a vector fields on M^3 whose flow is conformally Anosov, with stable and unstable foliations $\mathcal{F}_\pm$. Then there exist differential forms $\alpha_\pm \in \Omega^1(M)$ such that $\mathcal{F}_\pm = \ker \alpha_\pm$ and*

$$\alpha_+ \wedge \mathrm{d}\alpha_- + \alpha_- \wedge \mathrm{d}\alpha_+ > 0.$$

Conversely, suppose $\alpha_\pm$ satisfy the above condition, and define foliations $\mathcal{F}_\pm :=$ $\ker \alpha_\pm$. Then, any non-vanishing vector field $X \in \mathcal{F}_+ \cap \mathcal{F}_-$ defines a conformally Anosov flow.

For the application we have in mind we are interested in such pairs satisfying an additional symmetry property:

Definition 2.4.28. *A **conformally Anosov pair** on M^3, consists of 1-forms α_+ and α_-, such that $\mathcal{F}_\pm := \ker \alpha_\pm$ are foliations and*

$$\alpha_+ \wedge \mathrm{d}\alpha_- + \alpha_- \wedge \mathrm{d}\alpha_+ > 0. \tag{2.4.3.2}$$

*A conformally Anosov pair (α_+, α_-) is called **symmetric** if (in addition to the above conditions)*

$$\alpha_+ \wedge \mathrm{d}\alpha_- > 0, \quad \alpha_- \wedge \mathrm{d}\alpha_+ > 0.$$

Observe that associated to any symmetric Anosov pair (α_+, α_-) we have a positive function $f \in C^\infty(M)$ defined by

$$\alpha_+ \wedge \mathrm{d}\alpha_- = f\alpha_- \wedge \mathrm{d}\alpha_+.$$

Given a conformally Anosov pair, both the induced foliations $\mathcal{F}_\pm := \ker \alpha_\pm$ admit a type I deformation into contact structures. In fact, they can be connected through a path contact structures:

Proposition 2.4.29 ([47])**.** *If (α, β) is a conformally Anosov pair, then $\mathcal{F}$ and $\mathcal{G}$ are transversal and for all $t \in (0, \pi)$ different from $\pi/2$ the form*

$$\alpha_t := \cos(t)\alpha + \sin(t)\beta,$$

defines a contact structure which is positive for $t \in (0, \pi/2)$ and negative for $t \in$ $(\pi/2, \pi)$.

In general the above family does not produce a naive approximation of $\mathcal{F}_\pm$. For example, consider $\mathbb{T}^3(x, y, z)$ endowed with the conformally Anosov pair

$$\alpha_+ := \mathrm{d}z + \cos(z)\mathrm{d}x, \quad \alpha_- := \mathrm{d}z + \sin(z)\mathrm{d}y.$$

Then, neither $\mathcal{F}_+$ nor $\mathcal{F}_-$ can be naively approximated, by Lemma 2.4.23. Indeed, both foliations contain a leaf $\mathbb{T}^2$ which does not admit an exact symplectic form.

Observe that starting with a symmetric Anosov pair, both foliations $\mathcal{F}_+$ and $\mathcal{F}_-$ are taut and admit leafwise exact symplectic forms. Hence there is no obstruction to naive approximation, in fact we have:

Proposition 2.4.30. *Let (α_+, α_-) be a symmetric Anosov pair, then the symplectic foliations*

$$\left(\mathcal{F}_+ := \ker \alpha_+, \omega_+ := \mathrm{d}\alpha_-|_{\mathcal{F}_+}\right) \quad \textit{and} \quad \left(\mathcal{F}_- := \ker \alpha_-, \omega_- := \mathrm{d}\alpha_+|_{\mathcal{F}_-}\right),$$

can be naively approximated by contact structures (as in Definition 2.3.23).

Proof. We prove the statement for $(\mathcal{F}_+, \omega_+)$. Since (α_+, α_-) is symmetric there exists a positive function $f \in C^\infty(M)$ be such that

$$\alpha_+ \wedge d\alpha_- = f\alpha_- \wedge d\alpha_+.$$

Then, the linear path of contact forms $\alpha_t := \alpha_+ + t\alpha_-$ satisfies

$$\alpha_t \wedge d\alpha_t = t(1 + \frac{1}{f})\alpha_+ \wedge d\alpha_-,$$

showing that Definition 2.3.23 is satisfied. □

As an application of this proposition the following family of foliations can be naively approximated.

Proposition 2.4.31. *Let Σ_g be a Riemann surface of genus $g \geqslant 2$. Then, the unit cotangent bundle $ST^*\Sigma_g$ admits a symmetric Anosov pair.*

Proof. The Lie algebra $(2, \mathbb{R})$ of the projective special linear group $PSL(2, \mathbb{R}) := SL(2, \mathbb{R})/\{\pm I\}$, has an (oriented) basis

$$X_1 := \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad X_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfying the relations $[X_1, X_2] = R, [R, X_1] = -2X_1$ and $[R, X_2] = 2X_2$. Identifying elements of the dual $(2, \mathbb{R})^*$ with left invariant forms on $PSL(2, \mathbb{R})$ the above relations imply that the dual basis $\theta_1, \theta_2, \alpha$, satisfy

$$d\theta_1 = 2\alpha \wedge \theta_1, \quad d\theta_2 = -2\alpha \wedge \theta_2, \quad d\alpha = \theta_1 \wedge \theta_2.$$

Therefore, α is a contact form with Reeb vector field R and θ_1 and θ_2 define foliations. In fact,

$$\theta_1 \wedge d\theta_2 = \theta_2 \wedge d\theta_1 = 2\theta_1 \wedge \theta_2 \wedge \alpha,$$

so (θ_1, θ_2) is a symmetric Anosov pair.

We claim that $PSL(2, \mathbb{R})$ is the total space of the unit cotangent bundle $ST^*\Sigma_g$ for $g \geqslant 2$. Consider the uppper half plane $\mathcal{H} := \{x + iy \mid y >\}$ with the standard hyperbolic metric. The action of $PSL(2, \mathbb{R})$ on $\mathcal{H}$ given by

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

identifies $PSL(2, \mathbb{R})$ with the group of isometries of $\mathcal{H}$. Since, $\pm \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \cdot i = x + iy$, the action is transitive and the stabilizer of i is given by

$$K := \left\{ \pm \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in S^1 \right\}.$$

Hence, $\mathcal{H}$ can be identified with the left coset space $PSL(2,\mathbb{R})/K$ and the action by isometries on $\mathcal{H}$ corresponds to left multiplication on $PSL(2,\mathbb{R})/K$. Note that setting

$$A := \left\{ \pm \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \mid a > 0 \right\},$$

this is just the Iwasawa decomposition $PSL(2,R) \simeq K \times A$, stating that any $g \in PSL(2,\mathbb{R})$ can be uniquely written as $g = ka$ with $k \in K$ and $a \in A$. By the uniformization theorem the universal cover $\widetilde{\Sigma}_g$ is isomorphic to $\mathcal{H}$ and $\Gamma := \pi_1(\Sigma_g)$ act by isometries of $\mathcal{H}$ so can be identified with a subgroup of $PSL(2,\mathbb{R})$. Hence, $\Sigma_g = \Gamma\backslash\mathcal{H}$ and since $\mathcal{H} \simeq PSL(2,R)/K$ we have

Hence, the quotient map

$$\pi : \Gamma\backslash PSL(2,\mathbb{R}) \to \Gamma\backslash PSL(2,\mathbb{R})/K = \Sigma_g,$$

is a circle bundle over Σ_g. Since the forms $\theta_1, \theta_2, \alpha \in \Omega^1(PSL(2,\mathbb{R}))$ are left invariant they descend to a symmetric Anosov pair and a contact form on $\Gamma\backslash PSL(2,\mathbb{R})$. It remains to be shown that this bundle is the unit cotangent bundle $ST^*\Sigma_g$. Note that the Reeb vector field R of α is tangent to K and so the bundle has Legendrian fibers. As shown by Lutz [77] this implies that $\pi : \Gamma\backslash PSL(2,\mathbb{R}) \to \Sigma_g$ is the unit cotangent bundle. □

2.5 An obstruction to conformal approximation

In this section we consider obstructions for a (conformal) symplectic foliation to be approximated by contact structures in ACS$\mathcal{H}yper(M)$. We have seen that in dimension three, by Theorem 2.2.21, only $\mathbb{S}^1 \times \mathbb{S}^2$ with the product foliation cannot be approximated in this sense. As shown in Proposition 2.4.8 and Corollary 2.4.9, this obstruction propagates to higher dimensions using the notion of almost CS-submanifold from Definition 2.4.4. In fact, as illustrated by the examples in Section 2.6.1, in higher dimensions are many (conformal) symplectic foliations containing an almost CS-submanifold isomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$.

In light of these examples we ask if almost CS-submanifolds form the only obstruction to approximation. That is, if a CS-foliation cannot be approximated by contact structures, does it contain an almost CS-submanifold whose induced foliation cannot be approximated? The goal of this section is to proof the following result, showing that the answer to this question is negative:

Theorem. *There exists a CS-foliation (Definition 2.2.16) $(\mathcal{F}, \omega)$ on $\mathbb{S}^3 \times \mathbb{T}^2$ that does not contain any almost CS-submanifolds isomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, and cannot be conformally approximated by contact structures in ACS$\mathcal{H}yper(M)$.*

The proof combines two results about loops of contactomorphisms. In Section 2.5.1 we recall the definition of a contact fibration, i.e. fibrations whose fibers consists of contact manifolds. Analogous to the usual clutching construction for fibrations over spheres, contact fibrations over $\mathbb{S}^2$ correspond to loops in the space of contactomorphisms of the fiber. We will see that the total space of the fibration admits a contact

structure if and only if the corresponding loop is "positive" (Definition 2.5.12). As shown in [46] contact structures which do not admit (contractible) positive loops are special in the sense that the associated group of contactomorphisms amits a partial order.
In Section 2.5.2 we show (closely following the arguments from [29]) that some contact manifolds do not admit any positive loop. Hence, there exist contact fibrations whose total space does not admit any contact structure.
Then, in Section 2.5.3 we construct a CS-foliation $(\mathcal{F}, \omega)$ on $\mathbb{T}^2 \times \mathbb{S}^3$, with the property that any contact structure close to it, in the space of almost CS-hyperplane fields with the compact-open topology, would induce a contact fibration over $\mathbb{S}^2$, whose total space is (automatically) contact. However, we show that the (contact) fiber of this contact fibration does not admit any positive loop, implying that $(\mathcal{F}, \omega)$ cannot be approximated.

2.5.1 Contact fibrations

We recall here the definition and some of the basic properties of contact fibrations, as introduced in [75]. Given a smooth fibration, $\pi : M \to B$ we denote by

$$T^\nu M := \ker d\pi \subset TM,$$

the associated **vertical bundle**. Note that this is just the leafwise tangent bundle $T\mathcal{F}$ of the foliation $\mathcal{F}$ of M by the fibers of π. If each of the leaves is endowed with a contact structure, smoothly varying in the base coordinate, then π is a contact fibration.

Definition 2.5.1. *A* ***contact fibration*** *consists of a locally trivial fibration $\pi : M \to B$ and a (codimension-1) distribution $\xi^\nu \subset T^\nu M$ of the vertical bundle such that, for each $b \in B$,*

$$\xi_b := \xi^\nu|_{M_b} \subset TM_b,$$

defines a contact structure on the fiber M_b.

As usual, we say that $\pi : (M, \xi^\nu) \to B$, is **orientable/oriented** if each ξ_b is orientable on M_b, or equivalently, if the line bundle $T^\nu M/\xi^n u$ is orientable/oriented. Note that if $\pi : M \to B$ is a fibration with oriented total space and base, then there is a canonical orientation induced on the fibers. Indeed, this is the orientation making the local trivializations $\phi_U : M|_U \xrightarrow{\sim} U \times F$ orientation preserving with respect to the product orientation. For an oriented contact fibration we assume that this orientation agrees with the one induced by ξ_b.

The notion of isomorphism for such fibrations is the obvious one:

Definition 2.5.2. *An* ***isomorphism of contact fibrations*** *$\phi : (M, \xi^\nu) \to (N, \zeta^\nu)$ is a diffeomorphism $\phi : M \xrightarrow{\sim} N$, preserving the fibers and sending ξ^ν to ζ^ν. If the contact fibrations are oriented, then we assume ϕ to be orientation preserving.*

It is always possible to extend ξ^ν to a (codimension-1) distribution $\xi \subset TM$. This extension is not unique and does not necessarily define a contact structure on M.

Definition 2.5.3. *A **full contact fibration** $\pi : (M, \xi) \to B$, consists of a fibration $\pi : M \to B$ together with a (codimension-1) distribution $\xi \subset TM$, such that*

$$\xi^\nu := \xi \cap T^\nu M,$$

defines a contact fibration in the sense of Definition 2.5.1. Moreover, we say $\pi : (M, \xi) \to B$ is:

- ***non-negative:*** *if ξ is cooriented, and for any $\alpha \in \Omega^1(M)$ such that $\xi = \ker \alpha$, and inducing the correct orientation on TM/ξ, we have*

$$\alpha \wedge \mathrm{d}\alpha^n \geqslant 0.$$

- ***positive:*** *if ξ is a cooriented contact structure on M. That is, for $\alpha \in \Omega^1(M)$ as above, we have*

$$\alpha \wedge \mathrm{d}\alpha^n > 0.$$

Note that given an extension ξ of ξ^ν, there is a canonical isomorphism of line bundles $TM/\xi \simeq T^\nu M/\xi^\nu$. Hence, the underlying contact fibration of a positive/non-negative contact fibration is always oriented.

Definition 2.5.4. *An **isomorphism of full contact fibrations** $\phi : (M, \xi) \to (N, \zeta)$ is an isomorphism ϕ as in Definition 2.5.2, which additionally sends ξ to ζ, preserving the coorientations.*

If ξ is part of a full contact fibration, then the usual curvature form $c_\xi \in \Omega^2(\xi, TM/\xi)$, defined on sections by

$$c_\xi(X, Y) := [X, Y] \bmod \xi, \quad X, Y \in \Gamma(\xi),$$

is non-degenerate when restricted to ξ^ν. Therefore,

$$\mathcal{H} := (\xi^\nu)^{\perp c_\xi} \subset \xi,$$

is a horizontal distribution, so that it defines a canonical connection on $\pi : M \to B$.

Proposition 2.5.5 ([75, 98])**.** *Let $\pi : (M, \xi) \to B$ be a full contact fibration. Then,*

$$\mathcal{H} = (\xi^\nu)^{\perp_{K_\xi}} \subset \xi,$$

defines an (Ehresmann) connection whose parallel transport is by contactomorphisms. Moreover, if ξ^ν is co-oriented the parallel transport is coorientation preserving.

Proof. Since the construction is local, we can assume without loss of generality that $\xi = \ker \alpha$ for some $\alpha \in \Omega^1(M)$. Since α is unique up to a (positive) conformal factor, so is $\mathrm{d}\alpha|_\xi$ implying that

$$\mathcal{H} := (\xi^\nu)^{\perp \mathrm{d}\alpha} \subset \xi,$$

depends only on ξ. Since ξ^ν defines contact structures on the fibers of π, it follows that $\mathrm{d}\alpha|_{\xi^\nu}$ is non-degenerate, so we have

$$\xi = \xi^\nu \oplus \mathcal{H}.$$

Furthermore, since $\xi^\nu = \xi \cap T^\nu M$, we have that $\mathcal{H}$ is horizontal, and defines a connection.

To see that parallel transport is by contactomorphism let $\mathfrak{X}^\# \in \mathfrak{X}(M)$ denote the horizontal lift of a vector field $X \in \mathfrak{X}(B)$ and $\xi^\nu = \ker \alpha^\nu$, where $\alpha^\nu := \alpha|_{T^\nu M}$. Then for any $Y \in \Gamma(\xi^\nu)$ we have

$$(\mathcal{L}_{X^\#}\alpha^\nu)(Y) = \iota_Y(\iota_{X^\#}d\alpha^\nu + d(\alpha^\nu(X^\#))) = d\alpha^\nu(X^\#, Y) = 0,$$

using that $X^\# \in \mathcal{H}$ and $\mathcal{H} = (\xi^\nu)^{d\alpha}$. This implies that $\mathcal{L}_{X^\#}\alpha^\nu = f\alpha^\nu$ for a nowhere vanishing function $f : M \to \mathbb{R}$. Since parallel transport depends continuously on the endpoint of the path in the base, it follows that it preserves coorientation. □

Let $\pi : (M, \xi) \to B$ be a full contact fibration. The parallel transport of the canonical connection $\mathcal{H}$, allows us to find local trivializations of $\pi : M \to B$ which put ξ in normal form. To define the local model let $(F^{2n-1}, \xi_F := \ker \alpha_F)$ be a (cooriented) contact manifold, playing the role of the fiber. Furthermore, let $H \in C^\infty(F \times \mathbb{D}^2)$ be a smooth function which is of order r^2 at $r = 0$, where (r, θ). denote polar coordinates on $\mathbb{D}^2$. More precisely, this means there exists a smooth function $g \in C^\infty(F \times \mathbb{D}^2)$ such that

$$H = r^2 g.$$

Using this data define the contact manifold

$$\mathcal{M}_{(F,\alpha_F,H)} := \left(F \times \mathbb{D}^2, \xi := \ker(\alpha := \alpha_F + H\mathrm{d}\theta)\right), \tag{2.5.1.1}$$

which together with the obvious projection $\pi : F \times \mathbb{D}^2 \to \mathbb{D}^2$, defines a full contact fibration referred to as the **local model** associated to (F, α_F, H). Observe that the contact condition for α equals:

$$\alpha \wedge \mathrm{d}\alpha^n := \partial_r H\, \alpha_F \wedge \mathrm{d}\alpha_F^{n-1} \wedge \mathrm{d}r \wedge \mathrm{d}\theta.$$

Hence, $\mathcal{M}(F, \alpha_F, H)$ is a positive/non-negative contact fibration, as in Definition 2.5.3, if $\partial_r H$ is positive/non-negative.

Lemma 2.5.6. *Let $\pi : (M, \xi) \to \mathbb{D}^2$ be a full contact fibration. Denote by $(F_0, \xi_0 := \ker \alpha_0)$ the fiber of over $0 \in \mathbb{D}^2$. Then, there exists an isomorphism of full contact fibrations*

$$\phi : (M, \xi) \to \mathcal{M}_{(F_0,\alpha_0,H)},$$

where $H \in C^\infty(\mathbb{D}^2 \times F_0)$ is uniquely determined by ξ. Moreover, if ξ is non-negative or positive, then so is $\partial_r H$.

Proof. Choose a contact form α for ξ. Let $\alpha \in \Omega^1(M)$ be a contact form representing ξ, and let $F_0 := \pi^{-1}(0)$ denote the fiber over $0 \in \mathbb{D}^2$. Using the canonical contact connection $\mathcal{H}$ from Proposition 2.5.5, we obtain a trivialization of F,

Using the parallel transport over radial paths in $\mathbb{D}^2$, of the canonical contact connection $\mathcal{H}$ from Proposition 2.5.5, we obtain a trivialization $\Phi : \mathbb{D}^2 \times F \xrightarrow{\sim} M$. More precisely, let $\phi_{(\theta,r)} : M_0 \xrightarrow{\sim} M_{(r,\theta)}$ be the parallel transport of $\mathcal{H}$, over the path

$$\gamma_\theta : I \to \mathbb{D}^2, \quad t \mapsto (t, \theta).$$

Then, the trivialization is defined by:

$$\Phi : \mathbb{D}^2 \times F \to M, \quad (r, \theta, x) \mapsto \phi_{\theta,r}(x).$$

The parallel transport of $\mathcal{H}$ is coorientation preserving. It follows that in this trivialization we can write:

$$\alpha = e^f \alpha_0 + g dr + hr^2 d\theta,$$

for some functions $f, g, h \in C^\infty(D^2 \times F_0)$. Furthermore, since the velocity vector of γ_θ is equal to ∂_r, the parallel transport $\phi_{(\theta,r)}$ equals the flow of the horizontal lift of ∂_r. Hence, since $\mathcal{H}$ is contained in the kernel of α, in the trivialization ∂_r must be in the kernel of α, implying that $g = 0$. If we define, $H := e^{-f} hr^2$, then we have

$$\alpha = e^f(\alpha_0 + H \mathrm{d}\theta),$$

concluding the proof. □

Recall that given a contact manifold (M, ξ) and a choice of contact form $\alpha \in \Omega^1(M)$ for ξ, there is a one-to-one correspondence between (time dependent) contact vector fields $X_t \in \mathfrak{X}(M)$, and 1-parameter families of function $H_t \in C^\infty(M)$.

Lemma 2.5.7. *[52] Let $\alpha \in \Omega^1(M)$ be a contact form representing a contact structure ξ, and $R \in \mathfrak{X}(M)$ the associated Reeb vector field. Then, for (time dependent) contact vector fields $X_t \in \mathfrak{X}(M)$ and 1-parameter families of functions $H_t \in C^\infty(M)$, the assignments*

- $X_t \mapsto H_t := \alpha(X_t)$;
- $H_t \mapsto X_t$ *uniquely defined by:*

$$\alpha(X_t) = H_t, \quad \iota_{X_t} \mathrm{d}\alpha = \mathrm{d}H_t(R)\alpha - \mathrm{d}H_t,$$

define a one-to-one correspondence. We say that X_t is the ***Hamiltonian vector field*** *of H_t.*

The local model also provides an explicit description of the parallel transport of the contact connection in terms of the triple (F, α_F, H).

Lemma 2.5.8. *Consider $\pi : \mathcal{M}(F, \alpha_F, H) \to D^2$ and the associated contact connection $\mathcal{H}$ from Proposition 2.5.5. Then:*

(i) The parallel transport along radial paths is equal to the identity;

(ii) For a point $(r_0, \theta_0) \in \mathbb{D}^2 \backslash \{0\}$, define a path

$$\gamma : \mathbb{S}^1 \to \mathbb{D}^2, \quad t \mapsto (r_0, \theta_0 - t),$$

and a 1-parameter family of functions

$$H_t := H|_{\pi^{-1}(\gamma(t))} \in C^\infty(F_0).$$

Then, the parallel transport along γ is given by the flow of the Hamiltonian vector field $X_t \in \mathfrak{X}(F)$, of H_t, as in Lemma 2.5.7.

Proof. By definition, the parallel transport over a path γ, equals the flow of the (unique) horizontal lift of the velocity vector of γ. Hence, it suffices to compute the horizontal lifts of ∂_t and $-\partial_\theta$ for the model $\mathcal{M}(F, \alpha_F, H)$ as defined in Equation 2.5.1.1. Since, the extension ξ of α_F is defined as the kernel of

$$\alpha = \alpha_F + H\mathrm{d}\theta,$$

it follows immediately that ∂_r, viewed as a vector field on $F \times \mathbb{D}^2$, is the horizontal lift of ∂_r, proving the first claim. Similarly, observe that

$$\alpha(X_{H_t} - \partial\theta) = H_t - H_t = 0,$$

implying that the horizontal lift of $-\partial_\theta$ is given by $X_{H_t} - \partial_\theta$, proving the second claim. □

Before stating the **clutching construction** for contact fibrations, let us recall the classical construction for smooth fiber bundles. Let F be a smooth manifold playing the role of the fiber, and $\phi : \mathbb{S}^{n-1} \to \mathrm{Diff}(F)$ a family of diffeomorphisms. We use the notation $\phi_x := \phi(x) \in \mathrm{Diff}(F)$, decompose the sphere into the upper and lower hemisphere

$$\mathbb{S}^n := \mathbb{D}^n \cup_{\mathbb{S}^{n-1}} \mathbb{D}^n. \tag{2.5.1.2}$$

Then, we can define a fiber bundle over $\mathbb{S}^n$ by

$$M := \mathbb{D}^n \times F \sqcup \mathbb{D}^n \times F/(x, y) \sim (x, \phi_x(y)) \quad \forall (x, y) \in \partial\mathbb{D}^n \times F,$$

endowed with the obvious projection $\pi : M \to \mathbb{S}^n$. Conversely, given a fibration $\pi : M \to \mathbb{S}^n$, with fiber F, we can restrict it to each of the pieces in the decomposition from Equation 2.5.1.2. Thus, we obtain two fibrations over the disk $\mathbb{D}^n$, which can be trivialized. Therefore, the transition functions yield a family of diffeomoprhisms $\phi : \mathbb{S}^{n-1} \to \mathrm{Diff}(F)$, parametrized by the boundary $\partial\mathbb{D}^n$. It can be shown, see for example [66], that these constructions induce a bijection between $\pi_{n-1}(\mathrm{Diff}(F))$ and isomorphism classes of fiber bundles over $\mathbb{S}^n$ with fiber F.

The same proof works for contact fibrations, replacing F by a contact manifold (F, ξ_F) and the group of diffeomorphisms $\mathrm{Diff}(F)$ by the group of contactomorphisms $\mathrm{Cont}(F, \xi_F)$ of (F, ξ_F). The precise statement is as follows:

Proposition 2.5.9. *Let (F, ξ_F) be a contact manifold, and $\mathrm{Cont}(F, \xi_F)$ its group of contactomorphisms. Then there are one-to-one correspondences*

$$\pi_{n-1}(\mathrm{Cont}(F, \xi_F)) \simeq \pi_0\begin{pmatrix}\textit{contact fibrations over} \\ \mathbb{S}^n \textit{ with fiber } (F, \xi_F)\end{pmatrix} \simeq \left\{\begin{matrix}\textit{contact fibrations over} \\ \mathbb{S}^n \textit{ with fiber } (F, \xi_F)\end{matrix}\right\}\Big/\sim,$$

where the equivalence is up to fiber preserving diffeomorphism.

A similar result holds for positive contact fibrations, as defined in Definition 2.5.3. Observe, that on a cooriented contact manifold (M, ξ) the tangent space at each point, is divided into a positive and negative region seperated by the contact hyperplane. That is, if (locally) $\xi = \ker \alpha$, for a positive contact form α, then

$$T_xM = \{X \in T_xM \mid \alpha(X) > 0\} \cup \xi_x \cup \{X \in T_xM \mid \alpha(X) < 0\}.$$

This allows us to define positive vector fields, and isotopies.

Definition 2.5.10. *Let (M, ξ) be a cooriented contact manifold. Then $X \in \mathfrak{X}(M)$ is said to be* ***positive vector field*** *if*

$$X \bmod \xi \in \Gamma(TM/\xi),$$

is a strictly positive section of the oriented line bundle TM/ξ.

Similarly, we can define non-negative, and negative vector fields. For example Lemma 2.5.7 implies that a contact vector field is positive if and only if its associated Hamiltonian function is. Integrating positive vector fields, we obtain positive diffeomorphisms.

Definition 2.5.11. *Let (M, ξ) be a cooriented contact manifold. Then, $\phi_t \in \mathrm{Diff}(M)$, $t \in [0, 1]$ is said to be a* ***positive isotopy****, if its infinitesimal generator $X_t \in \mathfrak{X}(M)$, defined by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t = X_t \circ \phi_t,$$

is a positive vector field as in Definition 2.5.10.

Again, non-negative, and negative isotopies are defined similarly. Furthermore, this definition also makes sense for loops of diffeomorphisms and contactomorphisms.

Definition 2.5.12. *A* ***positive loop of contactomorphisms*** *$\phi : \mathbb{S}^1 \to \mathrm{Cont}(M, \xi)$ is positive if its infinitesimal generator is a positive contact vector field as in Definition 2.5.10.*

It follows from Lemma 2.5.8 that the parallel transport around the boundary of the local model $\mathcal{M}(F, \alpha_F, H)$ from Equation 2.5.1.1 is a positive loop of contactomorphisms if and only if ξ is a contact structure. As shown in the proof below, the transition functions in the clutching construction can be expressed in terms of this parallel transport. Hence, we obtain the following specialization of Proposition 2.5.9.

Proposition 2.5.13. *Let (F, ξ_F) be a co-oriented contact manifold. Then there are one-to-one correspondences*

$$\pi_0\left(\begin{array}{c}\textit{Positive loops of}\\ \textit{contactomorphisms}\\ \textit{of } (F,\xi_F)\end{array}\right) \simeq \pi_0\left(\begin{array}{c}\textit{Positive contact}\\ \textit{fibrations over } \mathbb{S}^n\\ \textit{with fiber } (F,\xi_F)\end{array}\right) \simeq \left\{\begin{array}{c}\textit{Positive contact}\\ \textit{fibrations over } \mathbb{S}^n\\ \textit{with fiber } (F,\xi_F)\end{array}\right\}\Big/\sim,$$

where the equivalence is up to fiber preserving diffeomorphisms. Moreover, the same result holds when we consider non-negative loops and contact fibrations.

Proof. Let $D_\pm$ denote the disk D^2 with the standard and opposite orientation respectively. Then the identification of their boundaries $S^1 = \partial D_+ \simeq \partial D_-$ is by orientation reversing diffeomorphism and we have

$$S^2 = D_+ \cup_{S^1} D_-,$$

as an oriented manifold. Let $(M, \xi) \to S^2$ be a positive contact fibration with fiber (F, ξ_F). We fix a co-oriented contact form $\alpha_F \in \Omega^1(F)$ for ξ_F and a basepoint $\theta_0 \in S^1$ viewed as the equator of S^2. By choosing identifications $(F_\pm, \xi_\pm) \simeq (F, \xi_F)$, where $F_\pm$ denote the fibers over the north south pole, and applying Lemma 2.5.6 we find (oriented) trivializations

$$\Phi_\pm : M|_{D_\pm} \xrightarrow{\sim} D_\pm \times F.$$

We can identify $\xi = \ker(\alpha_F + H_\pm d\theta)$, with $H_\pm \in C^\infty(D_\pm \times F)$ as in Lemma 2.5.6. Note that $\Phi_\pm$ are orientation preserving diffeomorphisms with respect to the product orientation on $D_\pm \times F$. In particular, by the positivity condition this implies that $\partial_r H_+ > 0$ and $\partial_r H_- < 0$.

The composition $\Phi_- \circ (\Phi_+)^{-1}$ gives a loop $\phi : S^1 \to \mathrm{Cont}(F, \xi_F)$ and we can assume that the identifications $F_\pm \xrightarrow{\sim} F$ are chosen in such a way that $\phi(\theta_0) = id$. We have to show that ϕ defines a positive loop. Let $\gamma_\pm : \mathbb{R} \to \partial D_\pm$ be defined by $t \mapsto (1, t)$. Observe that for $\theta \in S^1$ we have

$$\phi(\theta) = \mathcal{P}^{\theta_0,\theta}_{\gamma_-} \circ (\mathcal{P}^{\theta_0,\theta}_{\gamma_+})^{-1},$$

where $\mathcal{P}^{\theta_0,\theta}_{\gamma_-}$ denotes parallel transport over the path γ_- from time θ_0 to time θ using the contact connection. By Lemma 2.5.8 this is just the composition of the flows of X_{H_+} and $-X_{H_-}$ which are both positive paths. Since the composition of two positive paths is a positive path we conclude that $\phi : S^1 \to \mathrm{Cont}^+(F, \xi_F)$ is a positive loop.

Conversely, let $\phi_\theta \in \mathrm{Cont}(F, \xi_F)$ be a positive loop and denote by ϕ^t_{Reeb} the flow of the Reeb vector field $R \in \mathfrak{X}(F)$. The loop can be written as the composition of two positive paths

$$\phi_\theta = (\phi_\theta \circ \phi^{-\varepsilon\theta}_{Reeb}) \circ \phi^{\varepsilon\theta}_{Reeb}.$$

Indeed, for ε small enough the composition $\theta \mapsto \phi_\theta \circ \phi^{-\varepsilon\theta}_{Reeb})$ is still a positive path. The associated time dependent Hamiltonians are periodic and so define functions $H_\pm : F \times \mathbb{S}^1 \to \mathbb{R}_+$. Gluing the associated local models from Lemma 2.5.6, we obtain a positive contact fibration over $\mathbb{S}^2$.

Both constructions can be done parametrically, giving the first equivalence between positive loops up to homotopy and positive contact fibrations up to homotopy. For the second correspondence note that for a contact fibration $\pi : M \times [0,1] \to S^2 \times [0,1]$ parallel transport gives a fiber preserving diffeomorphism between $M \times \{0\}$ and $M \times \{1\}$. □

Positive loops of contactomorphisms play an important role in the study of contactomorphism groups. In [46] it is shown that there exists a partial order on the group of contactomorphism $\mathrm{Cont}(M, \xi)$ provided that there does not exist a positive loop of contactomorphisms.

Definition 2.5.14. *A contact manifold (M, ξ) is called* ***orderable*** *if there does not exist a contractible positive loop of contactomorphism.*

Note that in the above definition we mean contractible within the space of all contamorphisms $\mathrm{Cont}(M, \xi)$, not contractible within the space of positive loops of contactomorphisms. The existence of a single contractible positive loops implies the existence of many positive loops.

Lemma 2.5.15. *Let (M, ξ) be a closed contact manifold, then the following are true:*

(i) If there exists a contractible positive loop of contactomorphism, then any class in $\pi_1(\mathrm{Cont}(M, \xi))$ can be represented by a positive loop.

(ii) If there exists a class $c \in \pi_1(\mathrm{Cont}(M, \xi))$ for which both c and c^{-1} can be represented by a positive loop, then there exists a contractible positive loop.

Hence, M is orderable if and only if there exists a class in $\pi_1(\mathrm{Cont}(M, \xi))$ which cannot be represented by a positive loop.

Proof. Let $[\phi_t] \in \pi_1(\mathrm{Cont}(M, \xi))$ be a loop of contactomorphism, and $[\psi_t] = [0] \in \pi_1(\mathrm{Cont}(M, \xi))$ a positive contractible loop of contactomorphisms. For $k \in \mathbb{N}$, consider the loop of contactomorphisms

$$\Phi_{t,k} := \psi_t \circ \cdots \circ \psi_t \circ \phi_t \in \mathrm{Cont}(M, \xi), \quad t \in \mathbb{S}^1, \tag{2.5.1.3}$$

where $\circ$ denotes composition of contactomorphisms, not concatenation of loops, and the composition is taken k-times. Recall that a loop of contactomorphisms is positive if and only if the Hamiltonian function of the infinitesimal generator, as in Lemma 2.5.7 is a positive function. Furthermore, given positive loops ϕ_t, and ψ_t, with Hamiltonian functions $H(\phi_t)$ and $H(\psi_t)$, and such that $\phi_t^*\alpha = e^{f_t}\alpha$, for functions $f_t \in C^\infty(M)$, then the composition $\phi_t \circ \psi_t$ is again a positive loop, with Hamiltonian function

$$H(\phi_t \circ \psi_t)(x, t) = H(\phi_t)(x, t) + e^{-f_t} H(\psi_t)(\phi_t(x), t).$$

Hence, if $H(\psi_t)$ is positive, and k is sufficiently large, then $\Phi_{t,k}$, as defined in Equation 2.5.1.3 defines a positive loop, since the associated Hamiltonian is positive. Since, ψ_t is contractible, we have $[\Phi_{t,k}] = [\phi_t]$, which proves the first claim.

For the second claim, observe that the concatenation of two positive loops is again a positive loop. Indeed, on the infinitesimal level this amounts to concatenating the paths of positive vector fields, which yields again a path of positive vector fields. Hence, suppose given a loop of contactomorphism $\phi_t \in \mathrm{Cont}(M, \xi)$ both $[\phi_t]$ and $[\phi_t]^{-1}$, (where the latter denotes the opposite path) can be represented by positive loops. Then, the concatination $[\phi_t] \cdot [\phi_t]^{-1} = [\phi_t \cdot (\phi_t)^{-1}] = [0]$ can be represented by a positive loop which is contractible. ◻

Combining the lemma above with Proposition 2.5.13, we obtain contact fibrations $\pi : M \to \mathbb{S}^2$, whose fiberwise contact structure cannot be extended to a contact structure on M.

Corollary 2.5.16. *Let (F, ξ_F) be an orderable contact manifold, as in Definition 2.5.14. Then, there exists a contact fibration $\pi : (M, \xi^\nu) \to \mathbb{S}^2$ with fiber (F, ξ_F) which cannot be extended to a positive contact fibration as in Definition 2.5.3.*

Proof. By Lemma 2.5.15 there exists a class $[\phi] \in \pi_1(F, \xi_F)$ which cannot be represented by a positive loop of contactomorphisms. Let $\pi : (M, \xi^\nu) \to \mathbb{S}^2$ be the contact fibration associated to $[\pi]$ under the correspondence of Proposition 2.5.9. If ξ^ν can be extended to a contact structure on M, then by Proposition 2.5.13, $[\phi]$ can be represented by a positive loop, giving a contradiction. □

Of course, if a manifold admits no contact structures, then there exist no sequences of contact structures converging to a foliation. Thus, we can use the above idea to construct foliations which cannot be approximated by contact structures. However, observe that Corollary 2.5.16 does not state that M admits no contact structures, only that there exists no contact structures extending ξ^ν. Hence, we need a compatibility condition on the foliation ensuring that any contact structure close to it, is an extension of ξ^ν.

Definition 2.5.17. *Let $\pi : (M, \xi^\nu) \to B$ be a fibration. An almost CS-folation $(\mathcal{F}, \omega)$ on M is said the be **fibered by** π, if for all $b \in B$, the fiber M_b is an almost CS-submanifold as in Definition 2.4.4.*

Furthermore, the condition that there exist no contact structures on M can be weakened. It suffices to require that there exist no contact structures close to $(\mathcal{F}, \omega)$ in ACS$\mathcal{H}yper(M)$. To make this precise, recall that given a contact manifold (M, ξ), the forgetful map $\mathrm{Cont}(M, \xi) \to \mathrm{Diff}(M)$ induces an injection in homotopy

$$\pi_1(\mathrm{Cont}(M, \xi)) \to \pi_1(\mathrm{Diff}(M)).$$

Definition 2.5.18. *Let $(\mathcal{F}, \omega)$ be an almost CS-foliation on M, ξ a contact structure on M, and $[\phi] \in \pi_1(\mathrm{Diff}(M))$. We say that:*

- *$[\phi]$ and ξ are **incompatible**, if $[\phi]$ is not in the image under the above inclusion of a class in $\pi_1(\mathrm{Cont}(M, \xi))$ that can be represented by a positive loop.*
- *$[\phi]$ and $(\mathcal{F}, \omega)$ are **incompatible**, there is a neighborhood $\mathcal{U}$ of $(\mathcal{F}, \omega)$ in the compact-open topology on on ACS$\mathcal{H}yper(M)$, such that ϕ and ξ are incompatible for any contact structure $\xi \in \mathcal{U}$.*

The notions of a contact structure/almost CS-foliation **compatible** with a loop $[\phi] \in \pi_1(\mathrm{Diff}(M))$ are defined analogously.

Remark 2.5.19. The above definitions make use of the compact-open topology on ACS$\mathcal{H}yper(M)$. The reason for this is that the proof of Theorem 2.5.20 depends on Lemma 2.4.5. That is, a contact structure sufficiently close to a fibered almost CS-foliation induces a contact structure on each of the fibers. Recall from Remark 2.4.6, that this result also holds for conformal convergence, and type I/type II deformations. Hence, the above definition and Theorem 2.5.20 can be restated using these the relevant topologies for these type of approximations

Furthermore, here we only consider approximating almost CS-foliations by contact structures. However, in principle, the whole discussion goes through also when considering approximations of general almost CS-hyperplane fields by contact structures. △

With these definitions, we can generalize the observation from Corollary 2.5.16.

Theorem 2.5.20. *Assume we have the following data:*

(i) A fibration $\pi : M \to \mathbb{S}^2$, with fiber F, corresponding to the class $[\phi] \in \pi_1(\mathrm{Diff}(F))$;

(ii) A fibered almost CS-foliation $(\mathcal{F}, \omega)$ on M, as in Definition 2.5.17, with induced almost CS-foliation $(\mathcal{F}_F, \omega_F)$ on the fiber F.

If $[\phi]$ and $(\mathcal{F}_F, \omega_F)$ are incompatible, then $(\mathcal{F}, \omega)$ cannot be (conformally) approximated by contact structures.

Remark 2.5.21. Note that the converse of this theorem is not true. Suppose that we have the same data as in (i) and (ii) of the above theorem. Furthermore, assume that there is a sequence of contact structures $\xi_{F,k}$, $k \in \mathbb{N}$ on F, converging to $(\mathcal{F}_F, \omega_F)$, and such that there exist positive loops $\psi_k : \mathbb{S}^1 \to \mathrm{Cont}(F, \xi_{F,k})$ satisfying

$$[\psi_k] = [\phi] \in \pi_1(\mathrm{Diff}(F)).$$

Then it follows from Proposition 2.5.13, that there is a sequence of contact structures ξ_k on M. By construction the restriction of ξ_k to the fibers of π, converge to $(\mathcal{F}_F, \omega_F)$, but in general the ξ_k do not need to converge to $(\mathcal{F}, \omega)$. △

Proof. UsingLemma 2.4.5 any contact structure sufficiently close to $(\mathcal{F}, \omega)$ inside $\mathrm{ACS}\mathcal{H}yper(M)$, induces a contact structure $\xi_b := \xi \cap TN_b$ on N_b for each $b \in \mathbb{S}^2$. That is, it induces the structure of a positive contact fibration on $\pi : M \to \mathbb{S}^2$, see Definition 2.5.3. This structure is equivalent to a positive loop of contactomorphisms on N by Proposition 2.5.13 and since the underlying fibration must be isomorphic to $\pi : M \to \mathbb{S}^2$ this loop represents the class $[\phi]$. That is, $(\mathcal{F}_F, \omega_F)$ and $[\phi]$ are compatible, so we arrive at a contradiction. □

We emphasize that even though both the proofs of Proposition 2.4.8 and Theorem 2.5.20 make use of Lemma 2.4.5, they are giving different obstructions. The condition that $(\mathcal{F}_F, \omega_F)$ and $[\phi]$ on the fiber of a fibration are incompatible, as in Definition 2.5.18, or even that $(\mathcal{F}_F, \omega_F)$ is completely incompatible as in Definition 2.5.22 below, does not imply that $(\mathcal{F}_F, \omega_F)$ cannot be approximated by contact structures. In fact, in our main example constructed in Section 2.5.3, the foliation on the fiber has many contact structures approximating it.

Of course, if there does not exist any contact structure on N which is sufficiently close to $(\mathcal{F}_N, \omega_N)$ and a positive loop, then the hypothesis of the above theorem are satisfied.

Definition 2.5.22. *An almost CS-hyperplane field $(\mathcal{F}, \omega)$ on a manifold M is said to be* ***completely incompatible****, if there exist a neighborhood $\mathcal{U}$ of $(\mathcal{F}, \omega)$ in the compact-open topology on $ACS\mathcal{H}yper(M)$, such that any contact structure $\xi \in \mathcal{U}$ admits no positive loops.*

Equivalently, this means that any contact structure on M sufficiently close to $(\mathcal{F}, \omega)$ is incompatible with every loop of diffeomorphism $[\phi] \in \mathrm{Diff}(M)$, as in Definition 2.5.18.

Corollary 2.5.23. *Let $\pi : M \to \mathbb{S}^2$ be a fibration, and $(\mathcal{F}, \omega)$ a fibered almost CS-foliation on M, as in Definition 2.5.17. If the induced almost CS-foliation on the fiber $(F_{\mathcal{F}}, \omega_F)$ is completely incompatible, then $(\mathcal{F}, \omega)$ cannot be approximated by contact structures.*

2.5.2 Orderability of $\mathbb{T}^3$

Although Theorem 2.5.20 is very useful from a theoretical perspective, in practice the main difficulty is finding examples where the hypothesis of the theorem are satisfied. The reason for this is twofold. Firstly, as we saw in Lemma 2.5.15, Definition 2.5.18 is closely related to the notion of orderability. It is not known in general when a contact manifold is orderable, so we have a limited number of possible examples to look at. Secondly, both Definition 2.5.18 and Definition 2.5.22, are a condition on all contact structures (in a neighborhood of the foliation). Thus, in order to check this condition is satisfied, we want to consider foliations which only admit a limited number of contact structures close to it.

The rest of this section is devoted to proving the following result, which provides the fiber manifold for the examples of Theorem 2.5.20 constructed in Section 2.5.3.

Theorem 2.5.24. *Any Reebless foliation endowed with any leafwise symplectic form (Definition 2.5.25) on $\mathbb{T}^3$ is completely incompatible (Definition 2.5.22).*

Let us start by recalling the definition of a Reebless foliation. As in Chapter 1, that by a 3-dimensional Reeb component we mean the foliation $\mathcal{F}_{Reeb}$ on $\mathbb{S}^1 \times \mathbb{D}^2$ obtained by turbulizing the product foliation

$$\mathcal{F} = \bigcup_{z \in \mathbb{S}^1} \{z\} \times \mathbb{D}^2,$$

so that the boundary $\mathbb{S}^1 \times \mathbb{S}^2$ becomes a leaf, see also Equation 2.6.4.1.

Definition 2.5.25. *We say that a foliation $\mathcal{F}$ on M is* ***Reebless*** *if it does not contain $(\mathbb{S}^1 \times \mathbb{D}^2, \mathcal{F}_{Reeb})$ as a foliated submanifold.*

Observe that a foliation $\mathcal{F}$ with a Reeb component does not admit any transverse loops. Indeed, the coorientation of the boundary leaf $\mathbb{S}^1 \times \mathbb{S}^1$ implies that a transverse loop has to be contained in the Reeb component, or completely disjoint from it. Thus, the above definition is a generalization of taut foliations as in Definition 2.4.21, since any taut foliation must be Reebless.

Roughly speaking, the reason for requiring the foliation to be Reebless is that such foliations have very few contact structures close to them. To make this more precise, recall that there is a dichotomy of the space of all contact structures into tight and overtwisted ones. In dimension-3 these are defined as follows. On $\mathbb{R}^3$ with Cylindrical coordinates (r, θ, z) the **standard overtwisted contact structure** ξ_{ot} is defined as the kernel of the contact form

$$\alpha_{ot} := \cos(r\pi)\mathrm{d}z + r\sin(r\pi)\mathrm{d}\theta. \tag{2.5.2.1}$$

Then, the disk

$$D_{ot} := \{r \leqslant \pi, z = \sqrt{1 - r^2}\} \subset (\mathbb{R}^3, \xi_{ot})$$

is called the standard **overtwisted disk**.

Definition 2.5.26. *A 3-dimensional contact manifold (M, ξ) is said to be* ***overtwisted*** *if it contains D_{ot} as an almost CS-submanifold, and* ***tight*** *if it does not.*

It is possible, see [83], that given a tight contact manifold (M, ξ) the pullback of ξ to the universal cover $\pi : \widehat{M} \to M$, becomes overtwisted. Hence, we say that a tight contact structure is **universally tight** if its pullback to the universal cover remains tight.

The definitions of overtwisted/tight contact structures and foliations with/without Reeb foliations are very similar, in the sense that they both require the existence/absence of a certain local behaviour. This analogy is made precise by the following result of Bowden, which is a generalization of Proposition 2.7.1 from [47], saying that any contact structure C^0-close to a taut foliation is tight.

Theorem 2.5.27 ([17])**.** *Any contact structure C^0-close to a Reebless foliation is (universally) tight.*

Moreover, tight contact structures on $\mathbb{T}^3$ have been completely classified by Kanda, so that the problem reduces to checking there exists no positive loops of contactomorphisms for any of the following contact structures:

Theorem 2.5.28 ([69])**.** *Any tight contact structure on the 3-torus $T^3(x, y, z)$ is contactomorphic to one of the following*

$$\xi_k := \ker\Big(\alpha_k := \cos(kz)\mathrm{d}x + \sin(kz)\mathrm{d}y\Big), \tag{2.5.2.2}$$

where $k \in \mathbb{N}$ is a positive integer. Moreover, ξ_k and ξ_ℓ are contactomorphic if and only if $k = \ell$.

Observe that $(\mathbb{T}^3, \xi_1)$ is contactomorphic to $(ST^*\mathbb{T}^2, \xi_{st})$, the unit cotangent bundle of the torus. For $k > 1$, define the map

$$\pi_k : \mathbb{T}^3 \to \mathbb{T}^3, \quad (x, y, z) \mapsto (x, y, kz).$$

Then, it follows that any tight contact structure on $\mathbb{T}^3$ is the pullback of $(\mathbb{T}^3, \xi_{st})$ under one of the covering maps π_k.

By the preceding discussion, in order to complete the proof of Theorem 2.5.24 it remains to be shown that the contact manifolds $(\mathbb{T}^3, \xi_k)$, $k \in N$, do not admit positive loops. To see this, we first relate the existence of positive loops of contactomorphisms to the existence of certain families of Legendrian submanifolds. Then, we use results from the theory of generating functions to show the existence of such Legendrian submanifolds is obstructed on $(\mathbb{T}^3, \xi_k)$.

Recall that given a contact manifold (M^{2n+1}, ξ) we say that L is a Legendrian submanifold if

$$\dim L = n, \quad TL \subset \xi|_L.$$

That is, L is an integral submanifold of ξ of the maximal possible dimension.

Definition 2.5.29. *A **path of Legendrian submanifolds** in a contact manifold (M, ξ) is a family $L_t \subset M$, $t \in [0,1]$, such that L_t is an (embedded) Legendrian submanifold of ξ.*

A parametrization of L_t is a map $\phi : L_0 \times [0,1] \to M$ such that $\phi_t(L_0) = L_t$. Any such parametrization defines a section of the normal bundle $X_t \in \Gamma(\nu(L_t))$ by

$$X_t(\phi_t(x)) = \left[\frac{d}{dt}\Big|_{t=0} \phi_t(x) \right], \quad x \in L_0,$$

called the **velocity vector** of L_t. Note that given a fixed path L_t, any two parametrizations differ by an isotopy of L_0, implying that the velocity vector is independent of the choice of parametrization.

If α is a contact form for ξ, and $L \subset (M, \xi)$ a Legendrian, then it follows that $TL \subset \ker \alpha$. Hence, for any section of the normal bundle, $X \in \Gamma(\nu(L))$ the function $\alpha(X) \in C^\infty(M)$ is well-defined.

Definition 2.5.30. *A path of Legendrian submanifolds L_t, $t \in [0,1]$, in a cooriented contact manifold $(M, \xi := \ker \alpha)$ is said to be **positive** if its velocity vector $X_t \in \Gamma(\nu(L_t))$ satisfies*

$$\alpha(X_t) > 0.$$

As claimed above the existence of a positive loop of contactomorphisms, as in Definition 2.5.12 implies that each Legendrian can be displaced from itself by a positive path as in Definition 2.5.30.

Lemma 2.5.31. *Let $\pi : M \to B$ be a fibration, and $\xi := \ker \alpha$ a cooriented contact structure on a compact manifold M such that the fibers of π are Legendrian submanifolds. If $\phi : \mathbb{S}^1 \to \mathrm{Cont}(M, \xi)$ is a positive loop of contactomorphisms, then:*

(i) There exists a positive loop of embedded Legendrians $\psi : L \times \mathbb{S}^1 \to (M, \xi)$;

(ii) There exists a positive path of embedded Legendrians between different fibers of π.

Proof. Assume that there exists a positive loop of contactomorphisms $\phi_t \in \mathrm{Cont}(M, \xi)$, $t \in \mathbb{S}^1$. Since a contactomorphism sends Legendrians to Legendrians, the map

$$\psi : L \times \mathbb{S}^1 \to M, \quad (x,t) \mapsto \phi_t(x),$$

defines a parametrization for the loop of Legendrians $L_t := \phi_t(L)$. Furthermore, it is positive since the velocity vector of ψ is equal to the Hamiltonian vector field of ϕ which is positive.

For the second statement, observe that by the compactness of M we can assume that the Hamiltonian function of ϕ_t satisfies $H_t > \varepsilon$ for a constant $\varepsilon > 0$. Furthermore, we can assume

$$\phi_t^* \alpha = e^{f_t} \alpha,$$

with $e^{f_t} < C$ for a constant $C > 0$.

Now, choose a path of Legendrian submanifolds F_t, $t \in [0,1]$, consisting of fibers of π, between two distinct fibers F_0 and F_1. By the isotopy extension theorem for isotropic submanifolds, see for example [53], there exists a path of contactomorphisms $\psi_t \in \mathrm{Cont}(M, \xi)$ such that $F_t = \psi_t(F_0)$. In general ψ_t is not a positive path of contactomorphisms, however by choosing b_0 and b_1 close enough we can assume that the Hamiltonian function G_t of ψ_t satisfies

$$G_t > \frac{-\varepsilon}{C}.$$

Hence, the composition $\psi_t \circ \phi_t$ is a path of contactomorphisms, and its Hamiltonian function given by

$$G_t + e^{-f_t} H_t \circ \psi_t^{-1},$$

is strictly positive. Thus $\psi_t \circ \phi_t(F_0)$ parametrizes a positive path of Legendrian embeddings connecting two distinct fibers of π. □

It follows directly from Equation 2.5.2.2 that for each tight contact structure ξ_k on $\mathbb{T}^3$, the fibration

$$\pi : \mathbb{T}^3 \to \mathbb{T}^2, \quad (x,y,z) \mapsto (x,y),$$

has Legendrian fibers. Note that under the identification $\mathbb{T}^3 \simeq ST^*\mathbb{T}^2$, this fibration is just the usual projection onto $\mathbb{T}^2$. Thus, the above lemma applies to $(\mathbb{T}^3, \xi_k)$, and in combination with the following result completes the proof of Theorem 2.5.24.

Theorem 2.5.32. *There exists no positive path of Legendrian embeddings between different fibers of* $\pi : (\mathbb{T}^3, \xi_k) \to \mathbb{T}^2$, *for any* $k \in \mathbb{N}$.

For $k = 1$, we have that $(\mathbb{T}^3, \xi_1)$ is isomorphic to $(ST^*\mathbb{T}^2, \xi_{can})$ the unit cotangent bundle of the torus, with the standard contact structure. Thus, in this case the result follows from:

Theorem 2.5.33 ([29]). *There exists no positive path of Legendrian embeddings between two different fibers of* $\pi : ST^*M \to M$ *provided the universal cover of* M *is* $\mathbb{R}^n$.

One of the main ingredients in the proof of this theorem is the so called "hodograph transformation" which is a contactomorphism between the first jet bundle $(J^1(\mathbb{S}^1), \xi_{\text{can}})$ and $(ST^*\mathbb{R}^2, \xi_1)$. For the proof Theorem 2.5.32, we show that this isomorphism can be generalized to obtain contactomorphisms between $(J^1(\mathbb{S}^1), \xi_{\text{can}})$ and $(ST^*\mathbb{R}^2, \xi_k)$. Other than this the proofs are essentially the same.

To define the hodograph transformation, we need explicit coordinates on $J^1(\mathbb{S}^1)$ and $ST^*\mathbb{R}^2$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^2$, and identify the tangent space of the circle with

$$T_q\mathbb{S}^1 = \{v \in \mathbb{R}^2 \mid \langle q, v \rangle = 0\}.$$

Under the identification $J^1(\mathbb{S}^1) \simeq T^*\mathbb{S}^1 \times \mathbb{R}$, the standard contact structure corresponds to

$$\alpha_{st} := \mathrm{d}z - \lambda_{\text{can}},$$

where $\lambda_{\text{can}} \in \Omega^1(T^*\mathbb{S}^1)$ is the tautological 1-form. Thus, in coordinates we have:

$$\left(T^*\mathbb{S}^1 \times \mathbb{R}, \alpha_{st}\right) = \left(\{(q,p,z) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \mid \langle q, q \rangle = 1, \langle q, p \rangle = 0\}, \alpha_{st} = \mathrm{d}z - p\mathrm{d}q\right).$$

That is, the point (q,p) corresponds to the covector $\langle p, \cdot \rangle \in T_q^*\mathbb{S}^1$ explicitly defined by

$$\langle p, \cdot \rangle : T_q\mathbb{S}^1 \to \mathbb{R}, \quad v \mapsto \langle p, v \rangle.$$

Similarly, we have a coordinate description of the unit cotangent bundle

$$\left(ST^*\mathbb{R}^2, \alpha_k\right) = \left(\{(q,p) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \langle p, p \rangle = 1\}, \alpha_k = \rho_k(p)\mathrm{d}q\right),$$

where $\rho_k : \mathbb{R}^2 \to \mathbb{R}^2$ is as in Definition 2.5.34. As before, we identify (q,p) with $\langle p, \cdot \rangle \in ST_q^*\mathbb{R}^2$.

Definition 2.5.34. *For any $k \in \mathbb{N}$ the k-**Hodograph transformation** is defined to be the map*

$$h_k : (J^1(\mathbb{S}^1), \alpha_{st}) \to (ST^*\mathbb{R}^2, \alpha_k), \quad (q,p,z) \mapsto (z\rho_k(q) + \frac{\rho_k(p)}{k}, q),$$

where $\rho_k : \mathbb{R}^2 \to \mathbb{R}^2$ is defined in polar coordinates by $(r, \theta) \mapsto (r, k\theta)$.

For $k = 1$, this is the usual hodograph transform defined by

$$h_1 : J^1(\mathbb{S}^1) \to ST^*\mathbb{R}^2, \quad (q,p,z) \mapsto (zq + p, q). \tag{2.5.2.3}$$

The base of $J^1(\mathbb{S}^1)$ and the fiber over the origin of $ST^*\mathbb{R}^2$ are both Legendrian circles which the hodograph transformation maps to each other. The other fibers of $ST^*\mathbb{R}^2$ get mapped identified with the graphs of the following functions. For $x \in \mathbb{R}^2$, define the function

$$\ell_{x,k} : \mathbb{S}^1 \to \mathbb{R}, \quad q \mapsto \langle q^k, x \rangle. \tag{2.5.2.4}$$

Then, the hodograph transformation h_k has the following properties:

Proposition 2.5.35. *For any $k \in \mathbb{N}$, the map $h_k : (J^1(\mathbb{S}^1), \alpha_{st}) \to (ST^*\mathbb{R}^2, \alpha_k)$, as in Definition 2.5.34, is a contactomorphism sending the graph $\Gamma(j^1 h_{x,k}) \subset J^1(\mathbb{S}^1)$ diffeomorphically to the fiber $\pi^{-1}(x) \subset ST^*\mathbb{R}^2$.*

Proof. It is easily seen that h_k is a diffeomorphism and to check that it preserves the contact forms we parametrize the circle by $q = (\cos(\theta), sin(\theta), \theta \in [0, 2\pi]$. In these coordinates the map becomes:

$$\begin{aligned} h_k(q, p, z) &= h_k(\cos(\theta), \sin(\theta), -p\sin(\theta), p\cos(\theta), z) \\ &= (z\cos(k\theta) - \frac{p}{k}\sin(k\theta), z\sin(k\theta) + \frac{p}{k}\cos(k\theta), \cos(\theta), \sin(\theta)). \end{aligned}$$

We compute that the contact form is preserved:

$$\begin{aligned} h_k^*(\alpha_k) &= \cos(k\theta)d\left(z\cos(k\theta) - \frac{p}{k}\sin(k\theta)\right) + \sin(k\theta)d\left(z\sin(k\theta) + \frac{p}{k}\cos(k\theta)\right) \\ &= dz - pd\theta = \alpha_{st}. \end{aligned}$$

To check that h_k maps the graph of $j^1\ell_{x,k}$ diffeomorphically onto the fiber over x note that in the above coordinates $j^1\ell_{x,k} : \mathbb{S}^1 \to J^1(\mathbb{S}^1)$ is given by

$$\begin{aligned} j^1\ell_{x,k}(\cos(\theta), \sin(\theta)) = (&\cos(\theta), \sin(\theta), kx\sin(k\theta)\sin(\theta) - yk\cos(k\theta)\sin(\theta), \\ &- kx\sin(k\theta)\cos(\theta) + yk\cos(k\theta)\cos(\theta), x\cos(k\theta) + y\sin(k\theta)). \end{aligned}$$

Hence, the composition $h_k \circ j^1 l_{x,k}$ is equal to:

$$\begin{aligned} h_k \circ j^1\ell_{x,k}(q) = \Big(&(x\cos(k\theta) + y\sin(k\theta))\cos(k\theta) + \frac{xk\sin(k\theta)yk\cos(k\theta)\sin(k\theta)}{k}, \\ &(x\cos(k\theta)+y\sin(k\theta))\sin(k\theta) + (-xk\sin(k\theta) + yk\cos(k\theta))\cos(k\theta), \cos(\theta), \sin(\theta)\Big) \\ &= \Big(x\big(\cos^2(k\theta) + \sin^2(k\theta)\big), y\big(\cos^2(k\theta) + \sin^2(k\theta)\big), \cos(\theta), \sin(\theta)\Big) \\ &= (x, y, \cos(\theta), \sin(\theta)). \end{aligned}$$

□

Proof of Theorem 2.5.32. The proof follows exactly the same strategy as that of Theorem 2.5.33 from [29]. The only difference is that for Theorem 2.5.32 the hodograph transform from Equation 2.5.2.3 has to be replaced by the one from Definition 2.5.34, for the case $k > 1$.

We give a sketch of the proof. Assume that the statement is false, and let $L_t \subset ST^*\mathbb{T}^2$, $t \in [0, 1]$, be a path of Legendrian embeddings between distinct fibers. Using the projection of the universal cover $\pi : \mathbb{R}^2 \to \mathbb{T}^2$, this path can be lifted to a path of Legendrian embeddings in $(ST^*\mathbb{R}^2, \alpha_k)$. Since α_k is translation invariant in the base coordinates, we can assume without loss of generality that L_0 is the fiber over $0 \in \mathbb{R}^2$, and L_1 is the fiber over some $x \neq 0 \in \mathbb{R}^2$.

By Proposition 2.5.35, using the hodograph transform, this path corresponds to a path of Legendrian embeddings between the graph of $j^1\ell_{0,k}$, which is just the image of the zero section, and the graph of $j^1\ell_{x,k}$. Note that $l_{x,k}$ is a Morse function with $2k$ critical points with critical values $\pm||x||$, so that at the end of the path all Viterbo numbers must be $\pm||x||$. On the other hand l_0 is constant equal to zero, so that at the start of the path all Viterbo numbers (see [29]) must be 0. This contradicts the fact that along a positive path of Legendrian embeddings (given by a generating family quadratic at infinity) the Viterbo numbers are strictly increasing. □

2.5.3 An example

With the results of the previous two sections, we now return to the question posed at the beginning of the chapter; Is there an obstruction for an almost CS-foliation $(\mathcal{F}, \omega)$ to be approximated by contact structures, different than the one from Proposition 2.4.8? As we have seen in Theorem 2.5.20, there is another obstruction based on the non-existence of positive loops of contactomorphisms. We now show that these obstructions are different, by providing explicit examples.

We first observe that there exist symplectic foliations which are obstructed by Proposition 2.4.8, but not by Theorem 2.5.20.

Proposition 2.5.36. *The product of the Reeb foliation with the sphere,*

$$\left(\mathbb{S}^3 \times \mathbb{S}^2, \mathcal{F} := \mathcal{F}_{Reeb} \times \mathbb{S}^2\right),$$

contains an almost CS-submanifold $\mathbb{S}^1 \times \mathbb{S}^2$, but does not satisfy the conditions of Theorem 2.5.20.

Proof. Since the leaves are products of surfaces, it is clear that $\mathcal{F}$ carries a leafwise symplectic structure, denoted by $\omega \in \Omega^2(\mathcal{F})$. Furthermore, it follows directly from Corollary 2.4.9 that $(\mathcal{F}, \omega)$ cannot be approximated by contact structures.

On the other hand, the obvious projection $\pi : \mathbb{S}^3 \times \mathbb{S}^2 \to \mathbb{S}^2$ is part of a contact fibration, as in Definition 2.5.1, with fiber the standard contact sphere $(\mathbb{S}^3, \xi_{st})$. Clearly, $(\mathcal{F}, \omega)$ is fibred by π as in Definition 2.5.17. Since we are considering the trivial fibration, the associated loop of diffeomorphisms is the trivial loop $[\mathrm{id}] \in \pi_1(\mathrm{Diff}(\mathbb{S}^3))$. By the following lemma, this loop is compatible with $(\mathcal{F}, \omega)$, so that the conditions of Theorem 2.5.20 are not satisfied. ☐

Lemma 2.5.37. *Every open neighborhood of the symplectic foliation $(\mathbb{S}^3, \mathcal{F}_{Reeb}, \omega)$, in the compact-open topology on $\mathcal{ACSHyper}(M)$, contains a contact structure which is compatible with the trivial loop $[\mathrm{id}] \in \pi_1(\mathrm{Diff}(\mathbb{S}^3))$, conform Definition 2.5.18.*

Proof. As shown in [39], the standard contact spheres $(\mathbb{S}^{2n+1}, \xi_{st})$ are not orderable, i.e. they admit a contractible, positive loop of contactomorphisms, as in Definition 2.5.12. Hence, the trivial loop $[\mathrm{id}] \in \pi_1(\mathrm{Diff}(\mathbb{S}^3))$ is compatible with ξ_{st} as in Definition 2.5.18. As shown in Chapter 1, ξ_{st} can be deformed into $\mathcal{F}_{Reeb}$. More precisely, there exists a path of contact structures ξ_t, $t \in (0, 1]$, such that $\xi_0 = \xi_{st}$, and ξ_t converges to $\mathcal{F}_{Reeb}$. By Gray stability, ξ_t is contactomorphic to ξ_{st} for any $t > 0$, and hence non-orderable. Since, the path converges to $\mathcal{F}_{Reeb}$ this means that every open neighborhood of $\mathcal{F}_{Reeb}$ contains a non-orderable contact structure. ☐

The following theorem shows that the converse also holds: there exists a CS-foliation obstructed by Theorem 2.5.20 but not by Proposition 2.4.8.

Theorem 2.5.38. *There exists a CS-foliation, as in Definition 2.2.16, $(\mathcal{F}, \omega)$ on $\mathbb{S}^3 \times \mathbb{T}^2$ that does not contain any almost CS-submanifolds isomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, and cannot be approximated by contact structures.*

The remainder of this section consists of the proof of this theorem. We first construct the CS-foliation. Let $h : \mathbb{S}^3 \to \mathbb{S}^2$ denote the Hopf fibration, and consider the fibration

$$\pi : \mathbb{S}^3 \times \mathbb{S}^2 \to \mathbb{S}^2, \quad \pi(x, y) := h(x),$$

which has fiber $\mathbb{T}^3$. For any (oriented, codimension-1) foliation by lines $\mathcal{L}$ on $\mathbb{T}^2$ consider the product foliation

$$(2.5.3.1) \qquad \left(\mathbb{S}^3 \times \mathbb{T}^2, \mathcal{F}_{\mathcal{L}} := \mathbb{S}^3 \times \mathcal{L}\right),$$

whose basic properties are listed in the following lemma.

Lemma 2.5.39. *Let $\mathcal{F}_{\mathcal{L}}$ on the total space of $\pi : \mathbb{S}^3 \times \mathbb{T}^2 \to \mathbb{S}^2$ be as above then:*

(i) There is a leafwise CS-form defined by

$$\omega := \mathrm{d}_\theta \alpha_{st} = \mathrm{d}\alpha_{st} - \theta \wedge \alpha_{st}.$$

Here, $\theta \in \Omega^1(\mathcal{L})$ is a nowhere vanishing 1-form, and $\alpha_{st} \in \Omega^1(\mathbb{S}^3)$ is the standard contact form.

(ii) Each fiber of π is an almost CS-submanifold for $(\mathcal{F}_{\mathcal{L}}, \omega)$ and the induced CS-foliation on the fiber $\mathbb{T}^3(x, y, z)$ is given by the product foliation

$$(\mathbb{S}^1 \times \mathcal{L}, \mathrm{d}x \wedge \theta).$$

(iii) There does not exist any almost CS-submanifold $\mathbb{S}^1 \times \mathbb{S}^2$ in $(\mathcal{F}_{\mathcal{L}}, \omega)$.

Proof. (i) Since $\mathcal{L}$ is a 1-dimensional oriented foliation on $\mathbb{T}^2$ there exists $\theta \in \Omega^1(\mathbb{T}^2)$ which restricted to the leaves of $\mathcal{L}$ is nowhere vanishing, positive, and closed for dimensional reasons. Note that any two such forms are related by a positive conformal factor. As in Example 2.2.15 it is immediate that $\mathrm{d}_\theta \alpha_{st}$ defines a leafwise CS-structure.

(ii) The fiber of π is equal to $\mathbb{S}^1 \times \mathbb{T}^2$ where $\mathbb{S}^1 \subset \mathbb{S}^3$ is the fiber of the Hopf fibration. Hence, $\mathcal{F}_{\mathcal{L}}$ is transverse to the fibers and the induced foliation on $\mathbb{T}^3$ is the product $\mathbb{S}^1 \times \mathcal{L}$. Lastly, the fibers of of the Hopf fibration are precisely the Reeb orbits of α_{st} so that in our coordinates on $\mathbb{T}^3$ we identify α_{st} with $\mathrm{d}x$.

(iii) Any leaf of $\mathcal{L}$ is diffeomorphic to either $\mathbb{S}^1$ or $\mathbb{R}$, so that any leaf of $\mathcal{F}_{\mathcal{L}}$ is diffeomorphic to either $\mathbb{S}^3 \times \mathbb{S}^1$ or $\mathbb{S}^3 \times \mathbb{R}$. Hence, $\pi_2(L) = 0$ for any leaf L and by the following lemma there does not exist any leaf containing $(\mathbb{S}^2, \omega_{st})$ as a CS-submanifold.

□

Lemma 2.5.40. *Let (M, ω, η) be a conformal symplectic manifold and N a manifold with $H^1(N) = 0$. Then there exist no contractible, conformal symplectic embedding $\phi : N \to M$.*

Proof. Suppose there exists a smooth map $\Phi : N \times [0,1] \to M$, such that $\Phi|_{N\times\{1\}} = \phi$ and $\Phi|_{N\times\{0\}}$ is a constant map. Pulling back the conformal symplectic structure we obtain forms:

$$\omega_N := \Phi^*(\omega), \quad \eta_N := \Phi^*\eta.$$

Note that $\mathrm{d}_{\eta_N}\omega_N = 0$ and since $H^1(N \times I) = 0$ there exists a positive function $f \in C^\infty(N \times I)$ such that $\mathrm{d}(f\omega_N) = 0$. By Stokes theorem this implies

$$0 = \int_{N\times\{0\}} f\omega_N = \int_{N\times\{1\}} f\omega_N.$$

However, since ϕ is a conformal symplectic embedding, the last integral must be strictly positive and we arrive at a contradiction. □

Remark 2.5.41. Let $\mathcal{L}$ on $\mathbb{T}^2$ be as above and consider the product foliation $\mathbb{S}^1 \times \mathcal{L}$ on $\mathbb{T}^3(x, y, z)$. Define a loop of diffeomorphism $\phi_t \in \mathrm{Diff}(\mathbb{T}^3)$ by

$$\phi_t(x, y, z) = (x + t, y, z).$$

The fibration over $\mathbb{S}^2$ resulting from this loop is precisely $\pi : \mathbb{S}^3 \times \mathbb{T}^2 \to \mathbb{S}^2$ and since ϕ_t preserves each leaf of the foliation on $\mathbb{T}^3$ the total space of the fibration carries a foliation, which is just $\mathcal{F}_\mathcal{L}$ as above. In particular, taking multiples of the loop ϕ_t, the same construction allows us to produce non-approximable CS-foliations on the lens spaces $L(p, 1)$ for $p > 1$. △

Lemma 2.5.42. *The CS-foliation $(\mathcal{F}_\mathcal{L}, \omega)$ on $\mathbb{S}^3 \times \mathbb{T}^2$ constructed above cannot be approximated by contact structures. Moreover, the approximation is not obstructed by Proposition 2.4.8.*

Proof. By Theorem 2.2.21 we know that the only CS-foliation in dimension three which cannot be approximated by contact structures is $\mathbb{S}^1 \times \mathbb{S}^2$. Since $\dim \mathbb{S}^3 \times \mathbb{T}^2 = 5$, this means that the only possible almost CS-submanifold that can obstruct $(\mathcal{F}, \omega)$ being approximated is $\mathbb{S}^1 \times \mathbb{S}^2$. By Lemma 2.5.39 we know that such an almost CS-submanifold does not exist.

We show that the conditions in Corollary 2.5.23 are satisfied, i.e. $(\mathcal{F}, \omega)$ is completely incompatible as in Definition 2.5.22. Using Theorem 2.5.24 it suffices to show that the induced foliation on the fiber of π is Reebless as in Definition 2.5.25. By Lemma 2.5.39 the foliation on the fiber equals $\mathbb{S}^1 \times \mathcal{L}$, and hence contains no Reeb components. □

2.6 More examples

2.6.1 Isosymplectic embedding h-principle

Using the h-principle for isosymplectic embeddings, which we recall below, we show that in high dimensions it is common for a symplectic foliation to contain $\mathbb{S}^1 \times \mathbb{S}^2$ as an almost CS-submanifold. Hence, by Proposition 2.4.8, they cannot be approximated by contact structures.

Our strategy is to construct the almost CS-submanifold $\mathbb{S}^1 \times \mathbb{S}^2$ of $(M, \mathcal{F}, \omega)$ out of a suitable $\mathbb{S}^2$ embedded in a single leaf of $\mathcal{F}$. For example, consider a foliation $\mathcal{F}$ induced by a fibration $\pi : M \to \mathbb{S}^1$. Suppose we have an embedded sphere $\iota : \mathbb{S}^2 \hookrightarrow M$ contained in a leaf $M_{t_0} := \pi^{-1}(t_0)$ of $\mathcal{F}$. If we choose a connection on π, then the associate parallel transport $\mathcal{T}_t : M_{t_0} \xrightarrow{\sim} M_t$ defined a family of embedded spheres

$$\sigma_t := \mathbb{S}^2 \to M, \quad \sigma_t := \mathcal{T}_t \circ \iota, \quad t \in [0,1],$$

each contained in a leaf M_t of $\mathcal{F}$. Thus, $\sigma_1(\mathbb{S}^2)$ is the image of $\sigma_0(\mathbb{S}^2)$ under the monodromy map.

Definition 2.6.1. *Let $(M, \mathcal{F})$ be a foliated manifold and $\iota : \mathbb{S}^2 \to M$ an embedded sphere contained in a leaf L of $\mathcal{F}$. By a* ***transverse loop of spheres****, we mean an embedding*

$$\sigma : \mathbb{S}^2 \times [0,1] \to M,$$

defining a family of embedded 2-spheres $\sigma_t : \mathbb{S}^2 \to M$, each inside a leaf L_t of $\mathcal{F}$, with $L_0 = L_1 = L$ and $\sigma_0 = \iota$.

Remark 2.6.2. Although for our purposes it suffices to consider transverse loops of spheres, analogous to the above definition we can define transverse loops of any manifold N. In fact, the following discussion only uses that $\mathbb{S}^2$ is closed and simply connected. △

So, we can think of σ_1 as some kind of "monodromy" map associated to the foliation, for the starting sphere $\iota(\mathbb{S}^2)$. Recall that by a foliated map between foliated manifolds, $f : (M, \mathcal{F}) \to (N, \mathcal{G})$, we mean a (smooth) map sending each leaf of $\mathcal{F}$ into a leaf of $\mathcal{G}$. Hence, viewing $\mathbb{S}^2 \times [0,1]$ as foliated by 2-spheres, a transverse loop of spheres is precisely a foliated embedding mapping the boundary spheres into the same leaf.

For symplectic foliations we consider the following compatibility conditions:

Definition 2.6.3. *Let $(M, \mathcal{F}, \omega)$ be a symplectic foliation. A transverse loop of spheres $\sigma_t : \mathbb{S}^2 \to M$ is said to be*

(i) ***Positive*** *if*

$$\int_{\mathbb{S}^2} \sigma_t^* \omega > 0,$$

for all $t \in [0,1]$.

(ii) ***Symplectic*** *if $\sigma_t^* \omega$ is non-degenerate for each $t \in [0,1]$.*

Consider a symplectic foliation $(\mathcal{F}, \omega)$ on a manifold M of dimension $2n + 1 \geqslant 5$. As shown in Lemma 2.4.11 we can find a closed embedded loop $\mathbb{S}^1 \subset M$ transverse to $\mathcal{F}$. This curve has a tubular neighborhood isomorphic to

$$\left(\mathbb{S}^1 \times \mathbb{D}^{2n}, \bigcup_{z \in \mathbb{S}^1} \{z\} \times \mathbb{D}^{2n} \right).$$

Hence, by choosing an embedded sphere $\mathbb{S}^2 \subset \mathbb{D}^{2n}$, we obtain a transverse loop of spheres with image $\mathbb{S}^1 \times \mathbb{S}^2 \subset M$. However, loops constructed in this way can never be positive. Indeed, each 2-sphere is contractible so by Stokes theorem must have zero area with respect to ω. The following lemma produces loops with non-contractible spheres.

Lemma 2.6.4. *Let $\mathcal{F}$ be a foliation on a (closed) manifold M of dimension $\geqslant 4$. Then, there always exists a transverse loop of spheres. Moreover:*

(i) Any leafwise sphere $\mathbb{S}^2 \subset L$ inside a non-embedded leaf L can be extended to a transverse loop of spheres;

(ii) If the foliation $\mathcal{F}$ is induced by a fibration $\pi : M \to \mathbb{S}^1$, then any leafwise sphere $\mathbb{S}^2 \subset L$, can be extended to a transverse loop of spheres.

Proof. The above discussion shows that, using Lemma 2.4.11, a transverse loop of spheres always exists.

If $\mathcal{F}$ is induced by a fibration $\pi : M \to \mathbb{S}^1$ then, as explained prior to Definition 2.6.1, taking an embedded sphere in any of the fibers, we can use parallel transport to obtain a transverse loop of spheres.

For the other case assume that $L \subset M$ is a non-embedded leaf of M, and $\mathbb{S}^2 \subset M$ an embedded sphere. Let Σ be a transverse section of $\mathcal{F}$, such that $L \cap \Sigma \neq \varnothing$. Then recall that if L is not embedded, $\Sigma \cap L$ is not discrete, see for example [22]. In our case this means that $\mathcal{L}$ must intersect Σ at least twice, and we obtain a transverse loop of spheres as in Definition 2.6.1. ☐

The previous constructions do not necessarily produce positive loops of spheres. However, as shown in the following lemma, it is often enough if the initial sphere is positive.

Lemma 2.6.5. *Let $(M, \mathcal{F}, \omega)$ be a symplectic foliation and $\sigma : \mathbb{S}^2 \times [0,1] \to M$ a transverse loop of spheres starting at a positive sphere $\mathbb{S}^2 \subset L$, i.e.*

$$\int_{\mathbb{S}^2} \omega > 0.$$

Then the following hold:

(i) If L is non-embedded, then there exists a positive transverse loop of spheres starting at $\mathbb{S}^2$;

(ii) If ω is tame (Definition 1.7.22) then σ is positive.

Proof. Observe that the function

$$\mu : [0,1] \to \mathbb{R}, \quad t \mapsto \int_{\mathbb{S}^2} \sigma_t^* \omega$$

is smooth in t. We also have $\mu(0) > 0$ so that there exists $\tilde{t} \in (0,1]$ such that $\mu(t) > 0$ for all $t \in [0, \tilde{t}]$. As in the proof of Lemma 2.6.4 the non-embeddedness of L implies that L intersects the image of $\sigma(\mathbb{S}^2 \times [0, \tilde{t}])$ at least twice. So by restricting σ we obtain a positive transverse loop of spheres.

For the second case, let $\widetilde{\omega}$ be a closed extension of ω. In the coordinates defined by the foliated embedding $\sigma : \mathbb{S}^2 \times [0,1] \to M$ we can write

$$\omega = \omega_t + f_t \mathrm{d}t,$$

for $\omega_t \in \Omega^2(\mathbb{S}^2)$ and $f_t \in C^\infty(\mathbb{S}^2)$. The condition that ω is closed becomes

$$0 = \mathrm{d}\omega = \dot{\omega}_t \wedge \mathrm{d}t + \mathrm{d}f_t \wedge \mathrm{d}t,$$

implying

$$\omega_t := \omega_0 + \mathrm{d}\left(\int_0^t f_t \mathrm{d}t\right).$$

Integrating an exact form over $\mathbb{S}^2$ gives zero, so that $\mu(t) = \mu(0)$ for all t. □

The main result of this section is the following, and states that the existence of a positive transverse loop of spheres implies that the foliation cannot be approximated by contact structures.

Theorem 2.6.6. *Let $(M, \mathcal{F}, \omega)$ be a symplectic foliation with $\dim M \geqslant 7$. If there exists a positive transverse loop of spheres, as in Definition 2.6.3, then there exists an almost CS-submanifold isomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, as in Definition 2.4.4. In particular, $(\mathcal{F}, \omega)$ cannot be approximated by contact structures.*

Combining this theorem with the preceding discussion we conclude that the following classes of symplectic foliations cannot be approximated by contact structures.

Corollary 2.6.7. *Let $(M, \mathcal{F}, \omega)$ be a symplectic foliation with $\dim M \geqslant 7$. If either of the following is satisfied then $(\mathcal{F}, \omega)$ cannot be approximated by contact structures:*

1. *$\mathcal{F}$ is induced by a fibration $\pi : M \to \mathbb{S}^1$, ω is tame (Definition 1.7.22), and $(\mathcal{F}, \omega)$ admits a positive 2-sphere in one of its leaves;*
2. *$(\mathcal{F}, \omega)$ has a non-embedded leaf containing a positive 2-sphere.*

Proof. Combining Lemma 2.6.4 and Lemma 2.6.5 there exists a positive transverse loop of spheres. Hence, Theorem 2.6.6 applies. □

The idea of the proof of Theorem 2.6.6 is to use the h-principle for symplectic embeddings from [43], to homotope the positive transverse loop into a symplectic transverse loop. We also use it to "close up the loop" so that we obtain an almost CS-submanifold isomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, which we know obstructs approximation by contact structures. Let us start by recalling the h-principle for isosymplectic embeddings.

Definition 2.6.8. *Let (M, ω_M) and (N, ω_N) be symplectic manifolds. An* ***isosymplectic embedding*** *$f : (M, \omega_M) \to (N, \omega_N)$ is a smooth embedding satisfying*

$$f^*\omega_N = \omega_M.$$

The word "iso" is meant to emphasize that the symplectic structure on the source manifold is fixed. In other words, we do not only require $f^*\omega_N$ to be symplectic on M, but also to equal ω_M. Following the h-principle philosophy we can forget about the integrability conditions in the definition of an isosymplectic embedding. That is, we consider an "almost isosymplectic embedding", which are usually referred to as isosymplectic homomorphisms.

Definition 2.6.9. *Let (M, ω_M) and (N, ω_N) be two symplectic manifold. A* ***isosymplectic homomorphism*** *from (M, ω_M) to (N, ω_N) consists of fiberwise injective bundle map $F : TM \to TN$ covering a map f, such that $F^*(\omega_N) = \omega_M$ and $f^*[\omega_N] = [\omega_N] \in H^2(M)$.*

Observe that an isosymplectic embedding is the same thing as an isosymplectic homomoprhism satisfying $F = \mathrm{d}f$. As expected, the h-principle states that

Theorem 2.6.10 ([43]). *Let (M, ω_M) and (N, ω_N) be symplectic manifolds such that $\dim M \leqslant \dim N - 4$. Assume there exists an injective bundle map $F : TM \to TN$ covering a map $f : M \to N$ satisfying:*

(i) The map f is an embedding, and satisfies $f^*[\omega_N] = [\omega_M] \in H^2(M)$;

(ii) The map F is an isosymplectic homomorphism, as in Definition 2.6.9, and there exists a homotopy of injective bundle maps $F_t : TM \to TN$, $t \in [0, 1]$ such that $F_0 = \mathrm{d}f$ and $F_1 = F$.

Then, there exists a C^0-small isotopy $f_t : M \to N$, from $f_0 = f$ to an isosymplectic embedding f_1, and the differential $\mathrm{d}f_1$ is homotopic to F_1 through isomosymplectic homomorphisms. Moreover, the same statement holds parametrically.

The first consequence of this theorem is that we can homotope a positive transverse loop of spheres into a symplectic one.

Lemma 2.6.11. *Let $(M, \mathcal{F}, \omega)$ be symplectic foliation with $\dim M \geqslant 7$, and $\sigma : \mathbb{S}^2 \times [0, 1] \to M$ a positive transverse loop of spheres. Then, σ is homotopic to a symplectic transverse loop of spheres, see Definition 2.6.3.*

Proof. We want to apply Theorem 2.6.10 parametrically. In the notation of the theorem we define $f_s = \sigma_s$, interpreting σ as a 1-parameter family of maps $\sigma_s : \mathbb{S}^2 \to M$. The first condition in the theorem is trivially satisfied if we define the symplectic forms on $\mathbb{S}^2$ to be

$$\omega_s := \left(\int_{\mathbb{S}^2} \sigma_s^* \omega \right) \omega_{\mathbb{S}^2}, \tag{2.6.1.1}$$

where $\omega_{\mathbb{S}^2}$ is the standard form on $\mathbb{S}^2$.

Next, we construct the required map $F_{t,s}$ and show it satisfies the conditions in the theorem. The existence of such a map is purely obstruction theoretic. Denote $n := \dim M$. Standard obstruction theory and a careful (but rather straightforward) computation shows that the obstruction to the existence of a one parameter family of isosymplectic homomorphisms $F_{1,s} : T\mathbb{S}^2 \to T\mathcal{F}$, lives in the relative cohomology groups

$$H^i(\mathbb{S}^2 \times [0,1], \pi_{i-1}(\mathbb{S}^{n-2}); \mathbb{R}), \quad i \leqslant 3.$$

Hence, since $n \geqslant 7$ (actually $n \geqslant 5$ suffices here), these obstructions vanish and $F_{1,t}$ always exists. Similarly, the obstruction to the existence of a 1-parameter family of homotopies $F_{s,t} : T\mathbb{S}^2 \to T\mathcal{F}$ through injective bundle maps, connecting $\mathrm{d}f_t$ and F_t, lives in

$$H^i(\mathbb{S}^2 \times I, \pi_i(V_{2,n-1})); \mathbb{R}), \quad \text{and}, \quad \pi_i(V_{2,n-1}) = 0, \quad i \leqslant n-3,$$

where $V_{2,n-1}$ denotes the Stiefel manifold of orthonormal 2-frames in $\mathbb{R}^{n-1}$. Again, since $n \geqslant 7$ these obstructions vanish. Hence, Theorem 2.6.10 applies, giving a foliated isosymplectic embedding

$$f : \Big(\mathbb{S}^2 \times [0,1], \bigcup_{t\in[0,1]} \mathbb{S}^2 \times \{t\}, \omega_t\Big) \to (M, \mathcal{F}, \omega).$$

Finally, since a symplectic form on $\mathbb{S}^2$ is the same as an area form, and the intergrals

$$\int_{\mathbb{S}^2} \sigma_t^* \omega,$$

are strictly positive, it follows from Equation 2.6.1.1 that f is still a foliated isosymplectic embedding if we replace ω_t by $\omega_{\mathbb{S}^2}$. □

The second consequence of Theorem 2.6.10 is that two homotopic, positive 2-spheres in the same leaf, can be connected by an family of symplectic embeddings.

Lemma 2.6.12. *Let (M,ω) be a symplectic manifold with $\dim M \geqslant 6$, and $f_i : (\mathbb{S}^2, \omega_{\mathbb{S}^2}) \to (M,\omega)$, $i = 0,1$ be isosymplectic embedded spheres satisfying $[f_0] = [f_1] \in \pi_2(M)$. Then, there exists an isotopy of isosymplectic embeddings $f_t : (\mathbb{S}^2, \omega_{\mathbb{S}^2}) \to (M,\omega)$, connecting f_0 and f_1.*

Proof. Let $f : \mathbb{S}^2 \times I \to M$ be any homotopy connecting f_0 and f_1. Since $\dim L > 2\dim \mathbb{S}^2 + 1$, we can apply the Whitney embedding theorem to perturb f into a homotopy $\tilde{f} : \mathbb{S}^2 \times I \to M$ satisfying $\tilde{f}_i = f_i$, $i = 0,1$ and each map $\tilde{f} : \mathbb{S}^2 \times \{t\} \to M$ is an embedding. Then, the result follows by applying Lemma 2.6.11. □

Proof of Theorem 2.6.6. By Lemma 2.6.11 we the positive transverse loop of spheres $\sigma : \mathbb{S}^2 \times [0,1] \to M$ can be made symplectic, as in Definition 2.6.3. Then, applying Lemma 2.6.12 to σ_0 and σ_1, we find an isotopy of symplectic embeddings $f_t : \mathbb{S}^2 \to L$ such that $f_0 = \sigma_0$ and $f_1 = \sigma_1$.

To finish the proof we need to slightly modify f_t, so for each t its image is contained in a different leaf of $\mathcal{F}$. Since the image of $f : \mathbb{S}^2 \times [0,1] \to L$ is contained in a simply connected region $V \subset L$, there exists a foliated chart $U \simeq V \times [0,\varepsilon]$, such that $V = V \times \{0\}$. Let $\rho : [0,1] \to [0,1]$ be a smooth bump function satisfying

$$\rho|_{[0,1-\varepsilon]} = 0, \quad \rho|_{[1-\varepsilon/2,1]} = 1.$$

We define

$$\widetilde{\sigma} : \mathbb{S}^2 \times [0,1] \to M, \quad \widetilde{\sigma}_t := f^{-1}_{\rho(t)} \circ \sigma_t.$$

For $\varepsilon > 0$ small enough, this defines a foliated symplectic embedding of $\mathbb{S}^1 \times \mathbb{S}^2$ into M. By Corollary 2.4.9 this means that $(M, \mathcal{F}, \omega)$ cannot be approximated by contact structures. □

2.6.2 Milnor-Wood foliation on $ST^*\Sigma_g$

We give here a family of symplectic foliations which cannot be approximated by contact structures as a consequence of Theorem 2.5.20. These examples are also easily seen to contain $\mathbb{S}^1 \times \mathbb{S}^2$ as an almost CS-submanifold, so that we could apply Proposition 2.4.8 instead. However, they still illustrate the general strategy for finding applications of Theorem 2.5.20.

Given a principal S^1-bundle $\pi : P \to \Sigma_g$, over a closed surface Σ_g of genus g, denote by $e(P) \in H^2(\Sigma; \mathbb{Z})$ the Euler class of P and by $\chi(\Sigma) = 2 - 2g$ the Euler characteristic. The classical Milnor-Wood inequality tells us exactly when P admits a foliation transverse to the fibers:

Theorem 2.6.13 ([114]). *Let $P \to \Sigma$ be a principal S^1-bundle over a Riemann surface of genus $g \geqslant 1$. Then there is a foliation $\mathcal{H}$ on P transverse to the fibers if and only if*

$$|e(P)[\Sigma]| \leqslant -\chi(\Sigma) = 2g - 2.$$

As an immediate consequence we obtain that the unit cotangent bundle of a closed surface Σ_g for $g \geqslant 1$ admits a foliation $\mathcal{H}$ transverse to the fibers. Since all the leaves of $\mathcal{H}$ are oriented surfaces, this foliation carries a leafwise symplectic form.

Proposition 2.6.14. *The product foliation $(ST^*\Sigma_g \times \mathbb{S}^2, \mathcal{H} \times \mathbb{S}^2, \omega_{\mathcal{H}} + \omega_{\mathbb{S}^2})$ cannot be approximated by contact structures.*

Proof. Since $ST^*\Sigma_g$ contains an embedded $\mathbb{S}^1$ transverse to $\mathcal{H}$, the product foliation contains $\mathbb{S}^1 \times \mathbb{S}^2$ as an almost CS-transversal and so cannot be approximated by Proposition 2.4.8.

The following is a (sketch of the) proof using Theorem 2.5.20 instead. The product foliation is taut and so any contact structure approximating it is tight. The tight contact structures on circle bundles are classified, see [63, 64]. The arguments given there show that they are all contactomorphic (through not isotopic to the identity contactomorphisms) to the standard contact structure on the unit cotangent bundle. Therefore, all the tight contact structures on $ST^*\Sigma_g$ are orderable, as in Definition 2.5.14, and the conditions of Corollary 2.5.23 are satisfied. □

2.6.3 Open book decompositions

In this section we show that given a foliated manifold $(M, \mathcal{F})$ and an almost CS-submanifold $(B, \mathcal{F}_B)$, an approximation of $\mathcal{F}_B$ by contact structures can sometimes be extended to an approximation of $\mathcal{F}$. More precisely, if the binding B of a (geometric) open book decomposition of M admits a type II path of contact structures converging to a foliation, then so does M.

Theorem 2.6.15. *Let (B, π) be a (geometric) open book decomposition adapted to (M, α), as in Definition 1.4.21, and $\mathcal{F}_B$ a unimodular foliation on B. Assume*

$$\alpha_{B,t} := \gamma_B + t\alpha_B \in \Omega^1(B) \tag{2.6.3.1}$$

is a type II deformation of $\mathcal{F}_B$ into contact structures, as in Definition 2.2.9. Then there exists a foliation $\mathcal{F}$ on M and a path of contact forms $\alpha_t \in \Omega^1(M)$, $t \in (0, 1]$, such that

(i) $\alpha_1 = \alpha$, and α_t is a type II deformation of $\mathcal{F}$ into contact structures;

(ii) $\mathcal{F}$ has a single closed leaf diffeomorphic to $B \times \mathbb{S}^1$, and coincides with $\ker \pi$ *except on a small neighborhood of the binding;*

(iii) (B, π) is adapted to α_t for all $t > 0$, as in Definition 1.4.21.

Remark 2.6.16. The proof of the theorem shows that if $\alpha_{B,t}$ is a deformation of type I, then α_t can be chosen to be of type I, except on a small neighborhood of the compact leaf $B \times \mathbb{S}^1$. Since the compact leaf has trivial linear holonomy, this is to be expected in light of Corollary 2.4.17. △

Proof. By Example 1.4.11 we can find a tubular neighborhood of the binding isomorphic to $B \times \mathbb{D}^2$, on which α equals

$$\alpha = f\left(\alpha_B + r^2 \mathrm{d}\theta\right),$$

for $f \in C^\infty(B \times \mathbb{D}^2)$ satisfying $f|_{B\times\{0\}} = 1$ and $\partial_r f < 0$ for $r > 0$. We fix such a neighborhood for the rest of the proof.

By assumption $\mathcal{F}_B$ is unimodular so there exists a positive function $g \in C^\infty(B)$ such that $g\gamma_B$ is closed. Hence, multiplying $\alpha_{B,t}$ by g we can assume that γ_B as in Equation 2.6.3.1 is closed. Of course, if we multiply $\alpha_{B,t}$ then we are also required to multiply $\alpha \in \Omega^1(M)$. However, the changed contact form still satisfies the compatibility conditions of Definition 1.4.21. Furthermore, since we will only need γ_B to be closed on the interior of the tubular neighborhood $B \times \mathbb{D}^2$, we can leave α unchanged on $M \backslash (B \times \mathbb{D}^2)$.

If γ_B is closed, the contact condition for $\alpha_{B,t} = \gamma_B + t\alpha_B$ equals:

$$\alpha_{B,t} \wedge \alpha_{B,t}^{n-1} = t^{n-1}\gamma_B \wedge \mathrm{d}\alpha_B^{n-1} + t^n \alpha_B \wedge \mathrm{d}\alpha_B^{n-1}.$$

Thus, $\alpha_{B,t}$ is of type II if $\gamma_B \wedge \mathrm{d}\alpha_B^{n-1} \geqslant 0$ (since α_B is contact), while it is of type I if and only if $\gamma_B \wedge \mathrm{d}\alpha_B^{n-1} > 0$.

Next we construct the foliation $\mathcal{F}$ on M. Choose functions $\rho, \phi, \psi : [0, 1] \to \mathbb{R}_{\geqslant 0}$ satisfying

(i) $\phi(r) > 0$ for $r \in \mathcal{O}p(1/2)$, and $\phi(r) = 0$ for $r \in \mathcal{O}p(0)$ and $r \in \mathcal{O}p(1)$;

(ii) $\rho(r) > 0$ for $r < 1/2$, $\partial_r \rho \leqslant 0$ and $\rho(r) = \begin{cases} 1 & r \in \mathcal{O}p(0) \\ 0 & r \geqslant 1/2 \end{cases}$;

(iii) $\psi(r) > 0$ for $r > 1/2$, $\partial_r \psi \geqslant 0$ and $\psi(r) = \begin{cases} 0 & r \leqslant 1/2 \\ 1 & r \in \mathcal{O}p(1) \end{cases}$,

see Figure 2.1.

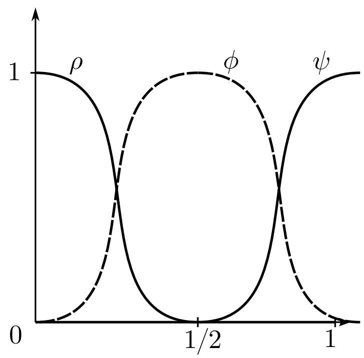

Figure 2.1: Functions ρ, ϕ and ψ, satisfying the properties needed in the proof of Theorem 2.6.15.

We define $\mathcal{F} := \ker \gamma$ where

$$\gamma := \rho(r)\gamma_B + \phi(r)\mathrm{d}r + \psi(r)\mathrm{d}\theta, \in \Omega^1(B \times D^2),$$

which is easily seen to satisfy $\gamma \wedge \mathrm{d}\gamma = 0$. We have $\rho(1/2) = \psi(1/2) = 0$ while $\phi(1/2) > 0$ (and $r = 1/2$ is the only value with these properties), so that $\mathcal{F}$ has a single compact leaf diffeomorphic to $B \times \mathbb{S}^1$. Near the boundary of $B \times D^2$ we have $\gamma = \mathrm{d}\theta$ so γ can smoothly extended to a global form on M by setting $\gamma := \pi^*(\mathrm{d}\theta)$, where by slight abuse of notation we also denote by $\theta \in \mathbb{S}^1$ the angle coordinate.

Hence, both γ and α are global forms on M and we define

$$\alpha_t := \gamma + t\alpha, \quad t \in [0, 1],$$

for which the contact condition equals:

$$\alpha_t \wedge \mathrm{d}\alpha_t^n = t^n \gamma \wedge \mathrm{d}\alpha^n + t^{n+1}\alpha \wedge \mathrm{d}\alpha^n + nt^n \alpha \wedge \mathrm{d}\alpha^{n-1} \wedge \mathrm{d}\gamma. \tag{2.6.3.2}$$

First note that on $M \backslash B \times D^2$ we have $\gamma = \mathrm{d}\theta$, so the above equation becomes:

$$\alpha_t \wedge \mathrm{d}\alpha_t^n = t^n(1-t)\mathrm{d}\theta \wedge \mathrm{d}\alpha^n + t^{n+1}\alpha \wedge \mathrm{d}\alpha^n.$$

The compatibility conditions of the open book imply $\mathrm{d}\theta \wedge \mathrm{d}\alpha^n > 0$, so α_t is of type I on $M \backslash (B \times D^2)$.

On $B \times D^2$ we compute each summand of Equation 2.6.3.2 separately. As observed at the beginning of the proof, we can assume that in this region

$$\alpha = f(\alpha_B + r^2 \mathrm{d}\theta),$$

for a strictly positive function $f \in C^\infty(B \times D^2)$. Furthermore, f can be chosen to satisfy $\partial_r f < 0$ and $2f + \dot{f}r > 0$.

The first summand of Equation 2.6.3.2 equals:

$$\gamma \wedge \mathrm{d}\alpha^n = n f^{n-1} \rho \left(2f + \dot{f}r\right) \gamma_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge (r\mathrm{d}r) \wedge \mathrm{d}\theta - n f^{n-1} \dot{f} \psi \alpha_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}r \wedge \mathrm{d}\theta.$$

The properties of f imply that the first term is non-negative (or strictly positive on $r < 1/2$ if $\alpha_{B,t}$ converges linearly) and the second term is strictly positive. The second summand of 2.6.3.2 is strictly positive since α is a contact form. For the third summand we have:

$$\alpha \wedge \mathrm{d}\alpha^{n-1} \wedge \mathrm{d}\gamma = -f^n \dot{\rho} r^2 \gamma_B \wedge \mathrm{d}\alpha_B^{n-1} \mathrm{d}r \wedge \mathrm{d}\theta + f^n \dot{\psi} \alpha_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}r \wedge \mathrm{d}\theta.$$

Since $\dot{\rho} \leqslant 0$, $\dot{\psi} \geqslant 0$ and $f > 0$ both terms are non-negative (or strictly positive on $r > 1/2$ if $\alpha_{B,t}$ converges linearly).

It follows that α_t is a deformation of type II. Moreover, if $\alpha_{B,t}$ is of type I then so is α_t away from $r = 1/2$.

Lastly, we compute the compatibility condition for α_t. We have:

$$\mathrm{d}\alpha_t^n|_{\Sigma_\theta} = -nt^{n-1}(1-t) f^{n-1} \dot{\rho} \gamma_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}r - nt^n f^{n-1} \dot{f} \alpha_B \wedge \mathrm{d}\alpha_B^{n-1} \wedge \mathrm{d}r. \tag{2.6.3.3}$$

Since $\dot{\rho} \leqslant 0$ and $\dot{g} < 0$ for $r > 0$, it follows that $\mathrm{d}\alpha_t|_{\Sigma_\theta}$ is an exact symplectic form for all $t > 0$. □

In dimension-3 the binding of an open book decomposition is a closed 1-dimensional manifold, i.e. a union of circles. Hence, in this case the hypotheses of the above theorem are automatically satisfied.

Corollary 2.6.17. *Let ξ be a contact structure on a closed 3-manifold M. Then there exists a foliation $\mathcal{F}$ on M admitting a type II deformation into contact structures (Definition 2.2.9).*

2.6.4 Products with $\mathbb{T}^2$

As shown by Bourgeois, see [16], the product of a contact manifold (M, ξ) with a genus-g surface Σ_g, is again a contact manifold. In this section we use Bourgeois' proof to show that if M admits a foliation which can be deformed into a contact structures then so does the product $M \times \Sigma_g$. We first consider the example $\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{T}^2$ in Proposition 2.6.21, making the construction as explicit as possible. Then, we consider the general case in Theorem 2.6.25.

2.6.4.1 An explicit example

To describe the path of contact structures on $\mathbb{S}^1 \times \mathbb{S}^2$, we fix coordinates $(z, r, \theta) \in \mathbb{S}^1 \times \mathbb{S}^2$ which are defined as follows. Let $D^2 := D^2(\pi) \subset \mathbb{R}^2$ be the disk of radius π, endowed with polar coordinates (r, θ). We view the sphere $\mathbb{S}^2$ as the quotient $D^2/\partial D^2$ with induced coordinates (r, θ) in which $(0, \theta)$ corresponds to the northpole and (π, θ) corresponds to the southpole. Furthermore let $z \in S^1 := \mathbb{R}/2\pi\mathbb{Z}$ denote the standard coordinate on the circle.

The Reeb component $(\mathbb{S}^1 \times \mathbb{D}^2, \mathcal{F}_{Reeb})$ (as in Example 1.5.5) can be explicitly described as the kernel of

$$\alpha := \cos(r)\mathrm{d}z + r\sin(r)\mathrm{d}r \in \Omega^1\left(\mathbb{S}^1 \times \mathbb{D}^2(\pi/2)\right).$$

Indeed, not that at $r = \pi/2$ we have $\alpha = \pi/2\mathrm{d}r$ so that the boundary is a compact leaf. Similarly for $r < \pi/2$ we have $\cos(r) > 0$, so all the leaves on the interior are open and diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$. Recall that by gluing two Reeb components "with a twist" one obtains the Reeb foliation on $\mathbb{S}^3$. Similarly, gluing two Reeb components using the identity we obtain a foliation $\mathcal{F}$ on $\mathbb{S}^1 \times \mathbb{S}^2$, explicitely described as the kernel of:

$$\gamma := \cos(r)\mathrm{d}z + r(\pi - r)\sin(r)\mathrm{d}r \in \Omega^1(\mathbb{S}^1 \times \mathbb{S}^2). \tag{2.6.4.1}$$

Furthermore, the standard overtwisted contact structure from Equation 2.5.2.1, induces an (overtwisted) contact structure on $\mathbb{S}^1 \times \mathbb{S}^2$ defined by the contact form:

$$\alpha := \cos(r)\mathrm{d}z + r(\pi - r)\sin(r)\mathrm{d}\theta \in \Omega^1(\mathbb{S}^1 \times \mathbb{S}^2). \tag{2.6.4.2}$$

As shown in the following lemma these forms produce a linear path of contact forms on $\mathbb{S}^1 \times \mathbb{S}^2$ converging to $\mathcal{F}$.

Lemma 2.6.18. *The foliation $\mathcal{F}$ on $\mathbb{S}^1 \times \mathbb{S}^2$ from Equation 2.6.4.1, admits a type I deformation (Definition 2.2.5) into contact structures. More precisely, the path of contact forms defined by:*

$$\alpha_t := \cos(r)\mathrm{d}z + r(\pi - r)\sin(r)\left(\mathrm{d}r + t\mathrm{d}\theta\right), \quad t \in [0, 1],$$

defines a type I deformation of $\mathcal{F}$.

Remark 2.6.19. Observe that even though $\mathcal{F}$ has a compact leaf diffeomorphic $\mathbb{T}^2$, the linear holonomy is non-zero so that the above result does not contradict Proposition 2.4.14. △

Proof. First observe that around $r = 0$ and $r = \pi$ we have that $r(\pi - r)\sin(r)$ is of order r^2 and $(\pi - r)^2$ respectively, so that α_t is well-defined. Furthermore, α_t is essentially of the form $\gamma + t\alpha$, for which the contact condition equals:

$$(\gamma + t\alpha) \wedge (\mathrm{d}\gamma + t\mathrm{d}\alpha) = t(\gamma \wedge \mathrm{d}\alpha + \alpha \wedge \mathrm{d}\gamma) + t^2\alpha \wedge \mathrm{d}\alpha.$$

Hence, to show α_t it is of type I it suffices to show that $\gamma \wedge \mathrm{d}\alpha + \alpha \wedge \mathrm{d}\gamma > 0$. A straight forward computation gives:

$$\gamma \wedge \mathrm{d}\alpha + \alpha \wedge \mathrm{d}\gamma = \big((\pi - 2r)\sin(r)\cos(r) + r(\pi - r)\big)\mathrm{d}z \wedge \mathrm{d}r \wedge \mathrm{d}\theta > 0.$$

□

The contact structure on the product $M \times \Sigma_g$ constructed in [16], depends on the choice of an open book decomposition of M adapted to the contact structure. Hence, to extend the path $\alpha_t \in \Omega^1(M)$ defined above to the product, we need a single open book decomposition of M, adapted to α_t for each $t > 0$.

Lemma 2.6.20. *The open book decomposition of* $\mathbb{S}^1 \times \mathbb{S}^2$ *defined by*

$$B := \mathbb{S}^1 \times \{r = 0, \pi\}, \quad \pi(z, r, \theta) := \theta,$$

is adapted to α_t *for all* $t > 0$*, as in Definition 1.4.21.*

Proof. The Reeb vector field of α_t is given by:

$$R_t = \frac{-t\big(\pi - 2r)\sin(r) + r(\pi - r)\cos(r)\big)\partial_z - \sin(r)\partial\theta)}{t\big((\pi - 2r)\sin(r)\cos(r) + r(\pi - r)\big)}$$

Which is seen to be tangent to B and satisfy $\mathrm{d}\theta(R_t) > 0$. □

Since the open book decomposition is the same for all t we can apply Bourgeois construction [16], to obtain a 1–parameter family of contact structures on $\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{T}^2$. As above denote by $\mathcal{F}$ the Reeb foliation on $\mathbb{S}^1 \times \mathbb{S}^2$ from Equation 2.6.4.1.

Proposition 2.6.21. *The product foliation* $\mathcal{F} \times \mathbb{T}^2$ *on* $\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{T}^2$ *admits a type I deformation into contact structures. More precisely, for any constant* $c \neq 0$ *the family of* 1*-forms*

$$\beta_t := \alpha_t + ctr(\pi - r)\left(\sin(\theta)\mathrm{d}x + \cos(\theta)\mathrm{d}y\right), \quad t \in (0, 1],$$

defines a type I deformation of $\mathcal{F} \times \mathbb{T}^2$*.*

Proof. The proof is the same as that in [16], using the open book decomposition given by Lemma 2.6.20. The contact condition equals

$$\beta_t \wedge (\mathrm{d}\beta_t)^2 = 2c^2t^2\left(r^2(\pi - r)^2\sin(r) + r(\pi - r)(\pi - 2r)\cos(r)\right)\mathrm{d}z \wedge \mathrm{d}r \wedge \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y,$$

which is positive for all $t > 0$. □

Observe that the product foliation has a leaf $\mathbb{T}^2 \times \mathbb{T}^2$ which does not admit an exact symplectic structure. Hence, by Lemma 2.4.23 this foliation cannot be naively approximated by contact structures.

Recall from Lemma 2.4.1, that a foliation which can be approximated by contact structures (in $\mathcal{H}yper(M)$) must admit a leafwise almost CS-structure. The following example, based on the previous lemma, shows that there are sequences of contact structures converging in $\mathcal{H}yper(M)$, but not converging in ACS$\mathcal{H}yper(M)$ to any almost CS-structure on the limit foliation.

Proposition 2.6.22. *On the manifold* $\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{T}^2$ *there exists an almost CS-foliation* $(\mathcal{F}, \omega)$ *together with two sequences of contact structures* ξ_k, ζ_k*,* $k \in \mathbb{N}$ *satisfying:*

(i) (ξ_k, c_{ξ_k}) *conformally converges to* $(\mathcal{F}, \omega)$*;*

(ii) ζ_k *converges to* $\mathcal{F}$ *in* $\mathcal{H}yper(M)$ *but does not (conformally) converge to any almost CS-structure on* $\mathcal{F}$*;*

(iii) ζ_k *can be chosen arbitrary close* ξ_k *as elements in* $\mathcal{H}yper(M)$*.*

Note that even though this example contains an almost CS-submanifold $\mathbb{S}^1 \times \mathbb{S}^2$, the proof does not depend on Corollary 2.4.9.

Proof. Consider the foliation $\mathcal{F}$ from Proposition 2.6.21 which for $c = 1$ gives a type I deformation of $\mathcal{F}$ into contact structures. Hence, by Corollary 2.3.25 $\mathcal{F}$ admits an (exact) leafwise CS-structure ω which can be conformally approximated by contact structures.

Observe that if the constant c is chosen to be a function depending to t which is C^0-close to 1, then $\zeta_t := \ker \beta_t$ is C^0-close to ξ_t and converges to $\mathcal{F}$ in $\mathcal{H}yper(M)$. However, if $c(t)$ does not converge for $t \to 0$, then neither does $\beta_t \wedge \mathrm{d}\beta_t$. In fact, observe that

$$\mathrm{d}\beta_t = \mathrm{d}\alpha_0 + tc(t)\mathrm{d}\big(r(\pi - r)\,(\sin(\theta)\mathrm{d}x + \cos(\theta)\mathrm{d}y\big)\,,$$

and since only one of the summands depend on $c(t)$, the conditions of Lemma 2.3.20 cannot be satisfied if $c(t)$ is chosen correctly. □

Remark 2.6.23. Observe that the proof above shows there exist $\ell_i := \langle X_i, Y_i \rangle$, $i = 1, 2$ tangent to $\mathcal{F}$, such that

$$\frac{\mathrm{d}\beta_k(X_1, Y_1)}{\mathrm{d}\beta_k(X_2, Y_2)},$$

does not converge. Indeed, we can select the vector fields X_i, Y_i such that

$$\mathrm{d}\beta_t(X_1, Y_1) > 0, \quad \mathrm{d}\beta_t(X_2, Y_2) > 0,$$

so that ℓ_1 and ℓ_2 can be thought of as "symplectic lines" for $\mathrm{d}\beta_t$. Furthermore we can choose these lines such that ℓ_1 is tangent to $\mathbb{S}^1 \times \mathbb{S}^2$ and ℓ_2 is tangent to $\mathbb{T}^2$. Then their ratio depends on c. As before, the proof follows by choosing $c(t)$ non-converging. △

2.6.4.2 The general case

To state the general version of Proposition 2.6.21 we need the notion of a path of contact forms adapted to an open book decomposition. The following definition looks rather technical, since we need to require different properties depending on the type of the path of contact forms. However, the conditions are completely analogous to those in Definition 2.2.5 and Definition 2.2.9.

Definition 2.6.24. *Let* (B, π) *be a (geometric) open book decomposition of* M^{2n+1} *and denote by* $\mathrm{d}\theta \in \Omega^1(M \backslash B)$ *the pullback under* π *of the angular form on* $\mathbb{S}^1$*. A path of contact forms* $\alpha_t \in \Omega^1(M)$*,* $t \in (0, 1]$ *is said to be* ***adapted*** *to the open book if* *for all t, we have that B is an almost CS-submanifold of $(\xi_t := \ker\alpha_t, \mathrm{d}\alpha_t|_{\xi_t})$, as in Definition 2.4.4, and*

$$\mathrm{d}\alpha_t^n \wedge \mathrm{d}\theta > 0,$$

on $M \backslash B$. Moreover, if α_t is:

(i) type I, as in Definition 2.2.5, then we additionally require

$$\mathrm{d}\alpha_t^n \wedge \mathrm{d}\theta = t^{n-1} f_t \Omega,$$

for a volume form Ω on $M \backslash B$ and a path of functions $f_t \in C^\infty(M\backslash B)$ such that f_0 is strictly positive.

(ii) type II, as in Definition 2.2.9, then we additionally require

$$\mathrm{d}\alpha_t^n \wedge \mathrm{d}\theta = t^{n-1} f\Omega + t^n g_t \Omega,$$

for a volume form Ω, a non-negative function $f \in C^\infty(M\backslash B)$, and a path of functions $g_t \in C^\infty(M\backslash B)$ such that g_0 is strictly positive.

Observe that if $\ker\alpha_t$ converges (in $\mathcal{H}yper(M)$) to a foliation $\mathcal{F}$ on M, then the above conditions imply that $\mathcal{F}$ is transverse to B and hence induces a foliation $\mathcal{F}_B := \mathcal{F} \cap TB$ on the binding.

Theorem 2.6.25. *Let (B, π) be an open book decomposition of M and α_t, $t \in (0, 1]$ an adapted path of contact forms of type I (resp. type II), as in Definition 2.6.24. Then $M \times \Sigma_g$, $g \geqslant 1$, admits a path of contact forms of type I (resp. type II) converging to the product foliation*

$$\mathcal{F} \times \Sigma_g.$$

Proof. We first prove the case $\Sigma_g = \mathbb{T}^2$. Denote by $\phi = (\phi_1, \phi_2) : M \to \mathbb{R}^2$ the smooth map constructed out of (B, π) as in [16]. Recall that this map has the property that

$$\phi_2 \mathrm{d}\phi_1 - \phi_1 \mathrm{d}\phi_2 = \tilde{r}^2 \mathrm{d}\theta, \quad \mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2 = \rho \wedge \mathrm{d}\tilde{r} \wedge \mathrm{d}\theta,$$

where $\tilde{r} : \mathbb{R}^2 \to \mathbb{R}$ is the radial coordinate multiplied with a suitable bump function, such that $\tilde{r} = r$ near 0 and $\tilde{r} = 0$ for $r \geqslant \epsilon$ for some small $\epsilon > 0$. Furthermore, $\theta : \mathbb{R}^2 \to \mathbb{S}^1$ is the usual angle coordinate. If α_t is type I and adapted as in Definition 2.6.24, then we find:

(2.6.4.3)

$$\beta_t \wedge \mathrm{d}\beta_t^n = c^2 t^n f_t \left(n\tilde{r}^2 \Omega \wedge \mathrm{d}x \wedge \mathrm{d}y + n(n-1)\Omega_B \wedge (\tilde{r}\mathrm{d}\tilde{r}) \wedge \mathrm{d}\theta \wedge \mathrm{d}x \wedge \mathrm{d}y \right),$$

for a volume form Ω on $M \backslash B$, a volume form Ω_B on B and a path $f_t \in C^\infty(M)$ such that f_0 is strictly positive. Hence, it suffices to show that

$$n\tilde{r}^2 \Omega \wedge \mathrm{d}x \wedge \mathrm{d}y + n(n-1)\Omega_B \omega \wedge (\tilde{r}\mathrm{d}\tilde{r}) \wedge \mathrm{d}\theta \wedge \mathrm{d}x \wedge \mathrm{d}y,$$

defines a positive volume form on M. To see this, observe that the first summand is non-negative and vanishes only at points in B, while the second summand is positive at points in B and vanishes away from B. The computation for the case that α_t is type II is completely analogous.

Note that, the contact forms β_t are $\mathbb{T}^2$-invariant and each $M \times \{(x_0, y_0)\}$, $(x_0, y_0) \in \mathbb{T}^2)$ is an almost CS-submanifold, as in Definition 2.4.4. The general case follows from Proposition 2.6.28 below, since Σ_g for $g > 1$ can be expressed as a branched cover of $\mathbb{T}^2$ with downstairs branching locus a finite set of points. □

Corollary 2.6.26. *Let (B, π) be an open book decomposition of M, and α an adapted contact form, as in Definition 1.4.21, and $\mathcal{F}_B := \ker \gamma_B$ a unimodular foliation on B. Assume that*

$$\alpha_{B,t} := \gamma_B + t\alpha_B \in \Omega^1(B),$$

is a type II deformation of $\mathcal{F}_B$ into contact structures, as in Definition 2.2.9. Then, the product foliation

$$\mathcal{F} \times \Sigma_g,$$

on $M \times \Sigma_g$ (where $\mathcal{F}$ is the foliation on M from Theorem 2.6.15) admits a type II deformation into contact structures.

Proof. The proof is the same as that of Theorem 2.6.25, using the family of contact forms α_t, $t \in [0,1]$ on M, as constructed in Theorem 2.6.15. In this case, Equation 2.6.4.3 becomes:

$$\beta_t \wedge \mathrm{d}\beta_t^n = nc^2t^2 \mathrm{d}\alpha_t^{n-1} \wedge (\tilde{r}^2 \mathrm{d}\theta) \wedge \mathrm{d}x \wedge \mathrm{d}y + n(n-1)c^2t^2 \alpha_t \wedge \mathrm{d}\alpha_t^{n-2} \wedge (\tilde{r}\mathrm{d}\tilde{r}) \wedge \mathrm{d}\theta \wedge \mathrm{d}x \wedge \mathrm{d}y.$$

We compute both summands separately. The first summand equals:

$$-n(n-1)c^2 f^{n-2} \tilde{r}^2 \left(t^n \dot{\rho}\gamma_B + t^{n+1} \dot{f}\alpha_B\right) \wedge \mathrm{d}\alpha_B^{n-2} \wedge \mathrm{d}r \wedge \mathrm{d}\theta \wedge \mathrm{d}x \wedge \mathrm{d}y.$$

As in the proof of Theorem 2.6.15, the path $\alpha_{B,t}$ being of type II implies that $\gamma_B \wedge \mathrm{d}\alpha_B^{n-2} \geqslant 0$. Furthermore, we have $\dot{\rho} \leqslant 0$ and $\dot{f} \leqslant 0$ by definition. Hence, β_t is of type II away from $B \times \mathbb{T}^2$ and is of type I precisely when $\alpha_{B,t}$ is.

The second summand equals:

$$n(n-1)c^2t^n \left((1-t)f^{n-2}\rho\gamma_B + tf^{n-1}\alpha_B\right) \wedge \mathrm{d}\alpha_B^{n-2} \wedge \tilde{r}\mathrm{d}\tilde{r} \wedge \mathrm{d}\theta \wedge \mathrm{d}x \wedge \mathrm{d}y.$$

Reasoning as before, we see that if $\gamma_B \wedge \mathrm{d}\alpha_B^{n-2} > 0$ then β_t defines a path of contact forms of type I on $B \times \mathbb{T}^2$. If we only have $\gamma_B \wedge \mathrm{d}\alpha_B^{n-2} \geqslant 0$ then the path is of type II provided that α_B dominates γ_B, which can easily be arranged by rescaling by a constant. □

2.6.5 Branched covers

To complete the proof of Theorem 2.6.25, we show that a type I/type II deformation can be lifted along branched covers. Recall that a **branched cover** is a smooth map $f : M \to N$ between manifolds of the same dimension which is locally equivalent to the map

(2.6.5.1) $$p_k : D^2 \times [-1,1]^{m-2} \to D^2 \times [-1,1]^{m-2}, \quad (z,t) \mapsto (z^k, t),$$

for some $k \in \mathbb{N}$. More precisely, each $y \in N$ has a neighborhood V such that for each connected component $U \subset f^{-1}(V)$ there exists $k \in \mathbb{N}$ and a commutative diagram:

$$\begin{array}{ccc} D^2 \times [-1,1]^{m-2} & \xleftarrow{\sim} & U \\ \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle f|_U} \\ D^2 \times [-1,1]^{m-2} & \xleftarrow{\sim} & V \end{array}$$

It follows that the sets $N_0 := \{y \in N \mid k(y) > 1\}$ and $M_0 := f^{-1}(N_0)$ are codimension-2 submanifolds which we call the **downstairs** and the **upstairs branching set** respectively.

Theorem 2.6.27 ([52]). *Let (N, ξ) be a contact manifold and $f : M \to N$ a branched cover for which the downstairs branching set $N_0 \subset N$ is a contact submanifold. Then M admits a contact form α_M whose kernel agrees with $f^*\xi$ outside a neighborhood of the upstairs branching set M_0.*

We recall from [52] the construction of the contact form on M. If $\xi = \ker \alpha_N$ then $\alpha := f^*\alpha_N$ is a contact form on $M \backslash M_0$ and $\alpha|_{TM_0}$ is a contact form on M_0. Let r be a fiberwise radial coordinate on $\nu(M_0)$ and identify the disk-bundle $\nu(\delta) := \{r \leqslant \delta\} \subset \nu(M_0)$ with a tubular neighborhood of M_0 of radius δ. Let γ be a connection 1–form on the $\mathbb{S}^1$–bundle associated to the normal bundle $\nu(M_0) \to M_0$ and $\rho(r)$ a smooth bump function satisfying $\rho(r) = 1$ near 0 and $\rho(r) = 0$ for $r \geqslant \delta$. Then

$$\alpha_M := C\alpha + \rho(r)r^2\gamma,$$

for $C > 0$ large enough, defines a contact form on M. From this description the proof of the following is almost immediate.

Proposition 2.6.28. *Let $\mathcal{F} := \ker \beta_N$ be a foliation on N with modular form $\mu_N \in \Omega^1(N)$, and*

$$\alpha_{N,t} := \beta_N + t\alpha_N, \quad t \in [0,1],$$

a type I deformation of $\mathcal{F}$ into contact forms. If $f : M \to N$ is a branched cover for which the downstairs branching set is an almost CS-submanifold of $(\mathcal{F}, \mathrm{d}_{\mu_N}\alpha_N)$, as in Theorem 2.2.13, then the pullback foliation $\mathcal{F}_M := f^*\mathcal{F}$ admits a type I deformation into contact structures. Moreover, $\alpha_t = f^*(\alpha_{N,t})$ outside a neighborhood of M_0.

Proof. Note that if the downstairs branching set N_0 is an almost CS-submanifold of $(\mathcal{F}, \mathrm{d}_{\mu_N}\alpha_N)$, then it is in particular a foliated submanifold of $(N, \mathcal{F})$. Hence, from Equation 2.6.5.1 it is clear that $f \pitchfork \mathcal{F}$ and so the pullback foliation $\mathcal{F}_M$ is well-defined. Denote $\beta := f^*\beta_N$, $\mu := f^*\mu_N$, $\alpha := f^*\alpha_N$ and let ρ and γ be as above. For a large constant $C > 0$ define

$$\alpha_t := \beta + tC\alpha + t\rho(r)r^2\gamma, \quad t \in [0,1]. \tag{2.6.5.2}$$

A direct computation shows that

$$\alpha_t \wedge \mathrm{d}\alpha_t^n = t^nC^n\beta \wedge \mathrm{d}_\mu\alpha + t^nC^{n-1}(r\dot{\rho} + 2\rho)\beta \wedge \mathrm{d}_\mu\alpha^{n-1} \wedge r\mathrm{d}r \wedge \gamma + \mathcal{O}(r^2, t^{n+1}), \tag{2.6.5.3}$$

where $\mathcal{O}(r^2, t^{n+1})$ consists of all terms containing either r^2 or t^{n+1}. Observe that $\beta \wedge \mathrm{d}_\mu\alpha^n$ is a positive volume form on $M\backslash M_0$ and non-negative on M_0. Furthermore, $\beta \wedge \mathrm{d}_\mu\alpha^{n-1} \wedge r\mathrm{d}r \wedge \gamma$ is a positive volume form on $TM|_{M_0}$ and zero outside a neighborhood of M_0. Hence, for $C > 0$ large enough $\alpha_t \wedge \mathrm{d}\alpha_t^n = t^n \operatorname{vol} + \mathcal{O}(t^{n+1})$ showing α_t is of type I. □

Remark 2.6.29. Although the above result is stated for type I deformations, the same proof also works for paths of type II with the following changes.

If $\alpha_{N,t}$ is a type II path of contact forms on N then there is no induced almost CS-structure on $\mathcal{F}_N$. Instead of requiring that N_0 is an almost CS-submanifold of $(\mathcal{F}, \mathrm{d}_{\mu_N}\alpha_N)$, we require that N_0 is an almost CS-submanifold of $\alpha_{N,t}$ for all $t > 0$. Then, a similar computation as in the above proof shows that α_t, defined as in Equation 2.6.5.2 is a type II deformation of $\mathcal{F} := f^*\mathcal{F}_N$ into contact structures.

△

2.6.6 Mapping tori

In this section we give a construction that produces type I deformations in any dimension. The construction is based on [54] where it is shown that any product $M \times \mathbb{S}^1$ with $\dim M = 4$ admits a contact structure. In fact, we use exactly the same contact structure constructed there and show it is part of a type I deformation.

Recall from Section 1.4.4 and Section 1.7.1, that the outside component of an adapted (geometric) open book decomposition (B, π) of a manifold M, comes with a fibration $\pi : M\backslash B \to \mathbb{S}^1$, and admits both a contact structure and a symplectic foliation. As in Section 1.8 we can try to extend these structures to the whole of M and deform them into each other.

Here, we consider two outside components and glue them along their boundaries, forgetting about the binding of the open book. This produces a fibration (or mapping torus) over $\mathbb{S}^1$, whose total space admits a CS-foliation together with a type I deformation into contact structures.

Definition 2.6.30. *A **Liouville domain** (W, λ) consists of an exact symplectic manifold such that $\lambda|_{\partial W}$ defines a contact form on ∂W.*

The Liouville manifold plays the role of the page of an (abstract) open book decomposition as in Definition 1.4.25.

Theorem 2.6.31. *Let (W_i, λ_i), $i = 1, 2$ be Liouville domains with the same contact boundary $(B := \partial W, \beta := \lambda_i|_{\partial W_i})$. Furthermore, let $f_i : W_i \to W_i$ be exact symplectomorphisms which are the identity near the boundary, and define*

$$M := W_1 \cup_B B \times [-1, 1] \cup_B \overline{W_2}, \quad f := f_1 \cup \mathrm{id} \cup f_2 : M \xrightarrow{\sim} M.$$

Then the resulting mapping torus

$$X := M \times [0, 1]/(x, 1) \sim (f(x), 0)$$

admits a foliation $\mathcal{F}$ together with a type I deformation into contact structures. Moreover, $\mathcal{F}$ has one compact leaf diffeomorphic to $B \times \mathbb{S}^1$.

There are plenty manifolds which admit a decomposition as in the theorem. In particular, examples exist in any dimension, and the argument in [54] shows that any manifold $M \times \mathbb{S}^1$ with $\dim M = 4$ is of this form.

Proof. The mapping torus $X_1 := W_1 \times [0,1]/\sim$ with return map f_i, admits a contact structure using the standard construction for open books. That is, we let $\rho : [0,1] \to [0,1]$ be a smooth bump function such that $\rho|_{[0,\varepsilon]} = 1$ and $\rho|_{[1-\varepsilon,1]} = 0$. Then,

$$\lambda_{1,\theta} := \rho(\theta) f_i^*(\lambda_1) + (1-\rho(\theta))\lambda_1,$$

makes each fiber of $\pi_1 : X_1 \to \mathbb{S}^1$ into a Liouville domain with boundary (B, β). For $C > 0$ big enough,

$$\alpha_1 := \lambda_{1,\theta} + C\mathrm{d}\theta,$$

defines a positive contact form on X_1. Similarly, we use $\overline{W}_2$ to define a mapping torus X_2 with the contact form

$$\alpha_2 := \lambda_{2,\theta} - C\mathrm{d}\theta.$$

Since f_i is assumed to be the identity near the boundary, we can find collar neighborhoods $(1-\varepsilon, -1] \times B$ of ∂W_i on which $\lambda_i = e^{r+1}\beta$. Choose smooth functions $f, g, h, \ell : (-1-\varepsilon, 1+\varepsilon] \to \mathbb{R}$ satisfying:

(i) f is an even function with $f(r) = e^{r+1}$ near $(1-\varepsilon, -1]$;

(ii) h, g is are odd functions with $h(r) = 1$ near $(-1-\varepsilon, -1]$;

(iii) $f\dot{h} - \dot{f}h > 0$;

(iv) $\ell(0) > 0$ when $h = 0$,

see Figure 2.2. Note that there is a lot of freedom in choosing g. Indeed, since for linear convergence, the choice of g does not matter, and is only needed to ensure that the formula for α given below, is smooth However, it is still useful to keep precise track of g as it in turn allows us freedom in choosing h, see Corollary 2.6.32 below.

We now define a contact structure by

$$\alpha := \begin{cases} \lambda_{1,\theta} + C\mathrm{d}\theta & \text{on } X_1 \\ f\beta + gC\mathrm{d}\theta & \text{on } (-1-\varepsilon, 1+\varepsilon) \times B \times \mathbb{S}^1 \\ \lambda_{2,\theta} - C\mathrm{d}\theta & \text{on } X_2 \end{cases}$$

We also define a foliation by

$$\tau := \begin{cases} \mathrm{d}\theta & \text{on } X_1 \\ h(r)\mathrm{d}\theta + \ell(r)\mathrm{d}r & \text{on } (-1-\varepsilon, 1+\varepsilon) \times B \times \mathbb{S}^1 \\ -\mathrm{d}\theta & \text{on } X_2 \end{cases}$$

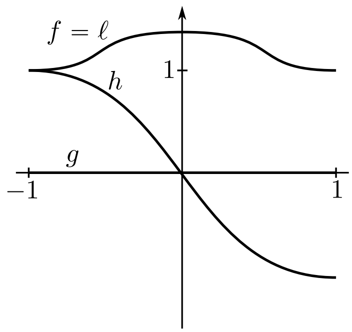

Figure 2.2: Functions f,g,h, and ℓ satisfying the properties needed in the proof of Theorem 2.6.31.

and consider $\alpha_t := \tau + t\alpha$. Computing the contact condition for α_t we find:

$$\alpha_t \wedge \mathrm{d}\alpha_t^n := \begin{cases} t^n(t + h - \dot{h})\mathrm{d}\theta \wedge \mathrm{d}\lambda_{1,\theta}^n & \text{on } X_1 \\ t^n n f^{n-1}\Big(f(t\dot{g} + \dot{h}) - \dot{f}(tg + h)\Big)\,\mathrm{d}r \wedge \beta \wedge \mathrm{d}\beta^{n-1} \wedge \mathrm{d}\theta & \text{on } \widetilde{X} \\ t^n(-t + h - \dot{h})\mathrm{d}\theta \wedge \mathrm{d}\lambda_{2,\theta}^n & \text{on } X_2 \end{cases}$$

where $\widetilde{X} := (-1-\varepsilon, 1+\varepsilon) \times B \times \mathbb{S}^1$. To check that α_t is of type I only the terms containing t^n matter, and with the functions f, g, h satisfying the above conditions each such term is strictly positive for $t > 0$. □

Observe that the condition $f\dot{h} - \dot{f}h > 0$ in the proof of the above theorem implies that h has a single zero at the origin. This means that the foliation $\mathcal{F} := \ker \tau$ has a single compact leaf given by $B \times \mathbb{S}^1$. Choose g to be an odd function with $g(r) = 1$ near $(-1-\varepsilon, -1]$ and also satisfying $f\dot{g} - \dot{f}g > 0$. Then, $\alpha_t = \tau + t\alpha$ is still contact for all $t > 0$ even if $f\dot{h} - \dot{f}h = 0$. In particular, we can choose $h|_{[-\delta,\delta]} = 0$,for some small $\delta > 0$. This implies that on $[-\delta, \delta] \times B \times \mathbb{S}^1 \subset X$ we have

$$\mathcal{F} = \bigcup_{r \in [-\delta,\delta]} \{r\} \times B \times \mathbb{S}^1.$$

These leaves are compact and have no linear holonomy so that Corollary 2.4.17 implies that the deformation cannot be of type I. This can also be seen directly from the formulas in the above proof since $f\dot{h} - \dot{f}h = 0$ implies that only terms with t^{n+1} survive.

On the other hand, it can be checked that $(\xi_t := \ker \alpha_t, \mathrm{d}\alpha_t|_{\xi_t})$ still conformally converges. Define $\lambda \in \Omega^1(X)$ by $\lambda_{X_i} = \lambda_{i,\theta}$, $i = 1, 2$, and $\lambda|_{(1-\varepsilon,1+\varepsilon)\times B \times \mathbb{S}^1} = f\beta$. It is clear that $\mathrm{d}_{\mathrm{d}\theta}\lambda$ defines a leafwise CS-structure on X_i and on $(1-\varepsilon, 1+\varepsilon) \times B \times \mathbb{S}^1$

this follows from:

$$\tau \wedge \mathrm{d}_{\mathrm{d}\theta}(f\beta)^n = nf^{n-1}(\dot{f}h + f\ell)\mathrm{d}r\beta \wedge \mathrm{d}\beta^{n-1} \wedge \mathrm{d}\theta > 0.$$

A straightforward computation shows that

$$\tau \wedge \mathrm{d}_{\mathrm{d}\theta}\alpha_t = t\tau \wedge \mathrm{d}_{\mathrm{d}\theta}\lambda, \quad t \in [0,1]$$

showing that Definition 2.3.23 is satisfied. Although, $\mathrm{d}_{\mathrm{d}\theta}\lambda$ is an exact CS-form, $\mathrm{d}\theta$ does not represent the modular class of $\mathcal{F}$ so that Theorem 2.2.13 does not apply.

Corollary 2.6.32. *In every dimension there exist exact CS-foliations which can be conformally approximated but whose underlying foliation does not admit a type I deformation into contact structures.*

2.7 Preservation of structures in the limit

2.7.1 Taut versus tight.

The goal of this section is to show that with the right definitions the proof of the following 3–dimensional statement goes through in higher dimensions:

Proposition 2.7.1 ([47])**.** *In dimension 3, any contact structure C^0-close to a taut foliation, see Definition 2.4.21, is tight, as in Definition 2.5.26.*

Recall from Definition 2.4.22 that a foliation $\mathcal{F}$ on M^{2n+1} is called strong symplectic if there exists a globally closed form $\omega \in \Omega^2(M)$ which is symplectic on the leaves of $\mathcal{F}$. This generalizes the notion of a taut foliation in dimension 3. As we have seen in Chapter 1, contact structures and (symplectic) foliations often show up as boundaries of symplectic manifolds. For confoliations we have the following:

Definition 2.7.2. *A confoliation ξ on a 3-dimensional manifold M is said to be:*

(i) ***Weakly fillable*** *if there exist a compact symplectic manifold (W^4, ω) such that $M = \partial W$, and the restriction $\omega_\partial := \omega|_M$ satisfies:*

$$\omega_\partial|_\xi > 0.$$

(ii) ***Weakly semi-fillable*** *if it is the connected component of fillable confoliated manifold.*

In particular, if ξ is a contact structure then these definitions coincide with the usual notions of weakly (semi-)fillability.

Remark 2.7.3. To avoid confusion about the terminology, recall that for a (3-dimensional) contact manifold one usually considers two types of fillings by a symplectic manifold. The notion of a weak symplectic filling is as in the above definition, while we say that (W, ω) is a **strong symplectic filling** if $M = \partial W$, and there exists a Liouville vectorfield $X \in \mathfrak{X}(W)$, transverse to the boundary (and pointing outwards) such that

$$\xi = \ker \iota_X \omega|_M.$$

This also implies that ξ is **dominated** by ω by which we mean that

$$\omega_\partial|_\xi > 0.$$

Note that by Lemma 1.2.14 this just means that (M, ω) has contact type boundary as in Definition 1.2.9, and the induced contact structure is ξ. △

The notion of weak fillability of contact structures was generalized to higher dimensions in [85] and also makes sense for almost CS distributions whose coefficient line bundle is trivial. That is, for the rest of this section we work with ACS$\mathcal{H}yper(M, \mathbb{R})$ which denotes ACS$\mathcal{H}yper(M, L)$ with $L = M \times \mathbb{R}$. We also assume, for the rest of this section, that M is compact.

Definition 2.7.4. *An almost CS-hyperplane field $(\xi, \mu) \in ACS\mathcal{H}yper(M, \mathbb{R})$ on a manifold M^{2n-1} is said to be:*

(i) ***Weakly fillable*** *if there exists a symplectic manifold (W^{2n}, ω) such that $M = \partial W$ and the restriction $\omega_\partial := \omega|_M$ satisfies*

$$(\omega_\partial|_\xi + t\mu)^{n-1} > 0, \tag{2.7.1.1}$$

for all $t > 0$;

(ii) ***Weakly semi-fillable*** *when it is a connected component of a weakly fillable almost CS-manifold.*

Theorem 2.7.5. *Any contact structure sufficiently close to a strong symplectic foliation, in the compact-open topology on $ACS\mathcal{H}yper(M, \mathbb{R})$, is tight (as in Definition 2.5.26).*

Proof. By Lemma 2.7.6 below, any strong symplectic foliation is weakly semi-fillable. Observe that this is an open condition in the space of almost CS-hyperplane fields. That is if Equation 2.7.1.1 is satisfied for some (ξ, μ) and ω, then it is also satisfied for $(\widetilde{\xi}, \widetilde{\mu})$ and ω, provided that the almost CS-hyperplane fields are sufficiently close in the compact-open topology.

Thus, if a contact structure is sufficiently close to a strong symplectic foliation then it is weakly semi-fillable. It is shown in [85] that, analogous to the 3-dimensional case, any weakly fillable contact manifold is tight. Hence, the same holds for weakly semi-fillable contact manifolds, completing the proof. □

Lemma 2.7.6. *Let $(\mathcal{F}, \omega)$ be a strong symplectic foliation on M, then it is weakly semi-fillable.*

Proof. Let $\mathcal{F} = \ker \alpha$ be such that $\alpha \wedge \omega^n > 0$ on M. Then consider $W := M \times [0, 1]$ endowed with the closed 2–form $\widetilde{\omega} := \mathrm{d}(t\alpha) + \varepsilon\omega$. For $\varepsilon > 0$ small enough this is a symplectic manifold. We have $\partial W = M \times \{1\} \cup \overline{M} \times \{0\}$ as oriented boundaries. We extend $\mathcal{F}$ by $\mathcal{F} = \ker -\alpha$ on $M \times \{0\}$. Then, since $\mathrm{d}\alpha|_\mathcal{F} = 0$ we have $\widetilde{\omega}|_\mathcal{F} = \omega$ and the condition of weak semi-fillability is satisfied. □

From the above observations the following is immediate:

Proposition 2.7.7. *Any contact structure sufficiently close to a strong symplectic foliation (in the compact-open topology on $ACS\mathcal{H}yper(M, \mathbb{R})$) is tight.*

Proof. The weak filling condition is open in $\mathrm{ACS}\mathcal{H}yper(M)$ (with the compact-open topology). Hence, by Lemma 2.7.6, any contact structure sufficiently close to a strong sympletic foliation is weakly semi-fillable. As remarked above this implies the contact structure is tight. □

2.7.2 Non weakly fillable CS-foliations

Consider a CS-foliation $(\mathcal{F}, \mu)$ on M^{2n-1}, and (W, ω) a symplectic manifold. Analogous to the 3-dimensional case from Remark 2.7.3, if (W, ω) is a strong filling of $(\mathcal{F}, \mu)$ then it follows that ω dominates $\mathcal{F}$, meaning that

$$\omega^{n-1}|_{\mathcal{F}} > 0.$$

Hence, in this case $(\mathcal{F}, \omega)$ is a tame symplectic foliation as in Definition 2.4.22. Such foliations are quite rare, so that there are many CS-foliations which are not strongly fillable. For example, the product foliation of the Reeb foliation with $\mathbb{T}^2$,

$$\left(\mathbb{S}^3 \times \mathbb{T}^2, \mathcal{F} := \mathcal{F}_{Reeb} \times \mathbb{T}^2\right),$$

is not taut and hence not strong symplectically fillable.

If we consider instead weak symplectic fillability, as in Definition 2.7.4, then these obvious obstructions vanish. Nevertheless we have the following:

Theorem 2.7.8. *There exist almost CS-foliations (in dimension $\geqslant 5$) which are not weakly fillable.*

The idea is to use the result from [85] saying that weakly fillable contact structures are tight as in Definition 2.5.26. As observed in the previous section, weak fillability is an open condition in $\mathrm{ACS}\mathcal{H}yper(M, \mathbb{R})$. Hence, it suffices to construct a almost CS-foliation which can be approximated by overtwisted contact structures. To construct the required contact structures, we use that for a branched cover with high enough branching degree the total space admits an overtwisted contact structure. The precise proof is as follows:

Proof. Consider the type I deformation $\alpha_{N,t}$, $t \in (0, 1]$, on $N := \mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{T}^2$ from Proposition 2.6.21. As shown there α_t converges to the product foliation $\mathcal{F}_N := \mathcal{F} \times \mathbb{T}^2$ with $\mathcal{F}$ as in Equation 2.6.4.1. Furthermore, for each $x \in \mathbb{T}^2$, we have that

$$N_x := \mathbb{S}^1 \times \mathbb{S}^2 \times \{x\}$$

is an almost CS-submanifold of $\alpha_{N,t}$. Using Gray stability we see that the induced contact forms, denoted by $\alpha_{N_x,t}$, are contactomorphic to the one in Equation 2.6.4.2 and hence overtwisted.

Let $f : M \to N \times \mathbb{T}^2$ be a branched cover with branching locus N_x, for a fixed $x \in \mathbb{T}^2$, and branching degree k. By Proposition 2.6.28, the pullback foliation $f^*(\mathcal{F} \times \mathbb{T}^2)$ on M admits a type I deformation α_t.

The normal bundle $\nu(M_0)$ of the upstairs branching set is a fiberwise k–fold covering of the trivial normal bundle of $M_x := f^{-1}(N_x) \subset M \times \mathbb{T}^2$. Hence, in fiber coordinates $(r, \theta) \in \mathbb{R}^2$, the form $\gamma := k\mathrm{d}\theta$ is a connection 1-form on $\nu(M_0)$.

For $\varepsilon > 0$ we define the ε-neighborhood of the upstairs branching set:

$$\nu_\epsilon(M_0) := \{r \leqslant \epsilon\} \subset \nu(M_0).$$

Furthermore, in the notation of Proposition 2.6.28, assume $\mathcal{F}_N := \ker \beta_N$, and denote $\beta := f^*(\beta_N)$ and $\alpha := f^*(\alpha_{N,1})$. Then, for ϵ small enough, the explicit description of the family of contact forms from Equation 2.6.5.2 becomes:

$$\alpha_t := \beta + tC\alpha + tkr^2\mathrm{d}\theta.$$

Note that the constant $C > 0$ is independent of ϵ and k. Define

$$\alpha_{t,s} = s(\beta + tC\alpha) + (1 - s)(\beta + tC\alpha)|_{M_0} + tkr^2\mathrm{d}\theta, \quad t, s \in [0, 1].$$

The computation of the contact condition in Equation 2.6.5.3 shows that $\alpha_{t,s}$ is contact for all $(t, s) \in (0, 1] \times [0, 1]$ on a neighborhood of M_0. Hence by Gray stability $\alpha_t = \alpha_{t,1}$ is contactomorphic to $\alpha_{t,0}$.

Since $(\beta + tC\alpha)|_{M_0}$ is contactomorphic to $\alpha_{N_x,t}$ on the almost CS-submanifold downstairs branching set N_x, which is overtwisted, there exist an embedding $\phi : (\mathbb{R}^3, \alpha_{ot}) \to (M_0, \alpha_{t,1}|_{M_0})$ whose image we denote by $U := \phi(\mathbb{R}^3)$. Restricting the normal bundle to U we have $\nu_\epsilon(M_0)|_U \simeq U_0 \times D^2(\epsilon)$, with coordinates (x, r, θ). The map

$$\Phi : \mathbb{R}^3 \times D^2(\sqrt{k}\epsilon) \to \nu_\epsilon(M_0)|_U, \quad (x, r, \theta) \mapsto (\phi(x), \frac{r}{\sqrt{k}}, \theta),$$

satisfies $\Phi^*(\alpha_{t,1}) = \alpha_{ot} + r^2\mathrm{d}\theta$. Since, the choice of ϵ is independent of k, we can pick $k \gg \epsilon$ as large as we want. Then, the main theorem in [26] shows that $\alpha_{t,1}$ is overtwisted concluding the proof. □

Remark 2.7.9. Although we used the concrete example $\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{T}^2$ from Proposition 2.6.21, the above proof holds more generally. Indeed, as observed in Remark 2.6.29, the result of Proposition 2.6.28 holds also for type II deformations and paths in $\mathrm{ACS}\mathcal{H}yper(M, \mathbb{R})$. To run the above argument for these types of approximations we need a path of contact structures ξ_t on a manifold N converging to a foliation $\mathcal{F}$, together with an almost CS-submanifold $N_0 \subset N$ of ξ_t, for which the induced contact structure $\xi_{N_0,t} := \xi_t|_{N_0}$ is overtwisted. Then we can use N_0 as the downstairs branching locus of a branched cover $f : M \to N$ with sufficiently high branching degree. Such examples can be constructed for example using Theorem 2.6.31. △

Chapter 3

Wrinkling h-principles for integral submanifolds of jet spaces

Chapter Contents

3.1 Overview

This chapter is based on work in progress with A. del Pino. We study the h-principle technique of wrinkling in the setting of jet spaces. Recall that given a (fiber) bundle over a manifold $\pi : X \to M$, the r-th jet bundle $J^r(X) \to M$ is the space of r-order derivatives of sections of X. For a more concrete description consider a function $f : \mathbb{R} \to \mathbb{R}$ (i.e a section of the trivial bundle $\pi : \mathbb{R}^2 \to \mathbb{R}$). Its r-order jet, denoted by $j^r f$, is the tuple

$$\left(t, f(t), \frac{\mathrm{d}f}{\mathrm{d}t}, \dots, \frac{\mathrm{d}^r f}{\mathrm{d}t^r}\right), \quad t \in \mathbb{R}.$$

The space of all such tuples, where we think of the derivatives as independent variables, is precisely the jet space $J^r(\mathbb{R}^2)$.

In general, given a section σ of $J^r(X)$ there is no $\sigma \in \Gamma(X)$ such that $\sigma = j^r s$. When such an s exists, σ is called holonomic. We can detect if a section is holonomic using the Cartan distribution ξ_{can} on $J^r(X)$. It is uniquely defined by the property that a section is holonomic if and only if it its image is tangent to ξ_{can}.

This provides a natural framework to study (partial) differential equations. Indeed, a differential equation $\mathcal{R}$ is a subset of $J^r(X)$. Its space of solutions $\text{Sol}(R)$ (resp. formal solutions $\text{Sol}^f(\mathcal{R})$) equals the space of holonomic sections (resp. sections) of $J^r(X)$ whose image lies in $\mathcal{R}$. Furthermore, there is a natural inclusion map

$$\iota : \text{Sol}(\mathcal{R}) \hookrightarrow \text{Sol}^f(\mathcal{R})).$$

The theory of h-principles aims to study this map on the level of homotopy. In particular we say that $\mathcal{R}$ satisfies the h-principle if ι induces an isomorphism on homotopy groups.

Proving h-principles on open and closed manifolds requires different techniques. For open manifolds, many h-principle results are based on the holonomic approximation theorem, [60, 43]. On closed manifolds the so called wrinkling technique, introduced by Eliashberg and Mishashev in [40, 42, 41], is particularly useful. Our aim is to apply wrinkling to the study of submanifolds of jet spaces tangent to the Cartan distribution.Note that (images of) sections are examples of such submanifolds.

In the first three sections we discuss the necessary background material on jet spaces and the h-principle. More precisely, in Section 3.2 we recall some basic properties of (bracket generating) distributions. Most notably, we discuss in detail the coordinate description of the Cartan distribution, and the linear algebra necessary to manipulate integral submanifolds. In Section 3.3 we review the Thom-Boardman stratification of jet spaces and the notion of stability for singularities. We then introduce several classes of singularities we want to work with, including wrinkles. In Section 3.4 we give a brief overview of the h-principle. In particular we review the holonomic approximation theorem and the wrinkling method.

In Section 3.5 we introduce the integral Grassmannian of jet spaces $\text{Gr}_{\text{integral}}(\xi_{\text{can}}, l)$. It is the space of l-dimensional subspaces of ξ_{can}. Given an integral immersion of a submanifold $f : N \to J^r(X)$ there is an associated Gauss map:

$$\text{Gr}(f) : N \to \text{Gr}_{\text{integral}}(\xi_{\text{can}}),$$

sending a point $x \in N$ to the integral element $(\mathrm{d}f)_x(T_xN) \subset \xi_{\text{can},f(x)}$. Such maps can be interpeted as "formal integral submanifolds". Hence, understanding $\text{Gr}_{\text{integral}}(\xi_{\text{can}})$ and its homotopy type is key. Note that most of the arguments in this section are somewhat sketchy, and a precise discussion of the material is left for future work.

The last three sections form the central part of the chapter. In Section 3.6 we define multi-sections of jet bundles. Roughly speaking an (r-times differentiable) multi-section is a smooth map $f : N \to J^r(X)$ which is graphical over M on an open dense set, and whose non-graphical part consists of mild singularities. Multi-sections of jet spaces can (in some sense) be integrated and differentiated. Hence they can be recovered from their image under certain projections (which remember only some of the derivatives). This is similar to the way Legendrian knots can be recovered from their Lagrangian projection. We describe the analogue of this projection in the setting of general jet spaces, and show it provides a convenient way of manipulating multi-sections.

In Section 3.7 we discuss the singularities needed for our applications. We give explicit descriptions of the singularities in coordinates, as well as their images under the various projections mentioned before. Next, we recall Givental's theorem (Theorem 3.7.5) and use it to conclude that our singularities are stable.

Lastly, Section 3.8 contains the proof of our main result:

Theorem (3.8.2). *Let $\sigma : M \to J^r(X \to M)$ be an arbitrary section. Then, for any $\varepsilon > 0$, there exists a map $f : M \to J^r(X \to M)$ satisfying:*

- *f is a holonomic multi-section with fold singularities (in zig-zag position);*
- *$|f - \sigma|_{C^0} < \varepsilon$.*

This is the analogue of holonomic approximation for multi-sections on closed manifolds. Since multi-sections are in particular (singular) integral submanifolds of the Cartan distribution, this theorem is a first step towards a general h-principle for integral submanifolds of jet space.

3.2 Overview: Distributions and jet spaces

In this section we review some standard material from the theory of distributions (Subsection 3.2.1), focusing on the particular case of jet spaces (Subsection 3.2.2). A lot of what we do is needed simply to set up notation.

Throughout this chapter, we work in the smooth category. Given a subset K of a topological space M, we denote by $\mathcal{O}p(K)$ an unspecified neighbourhood of K, whose size is not important as long as it is sufficiently small.

3.2.1 Basics of distributions

The main objects of interest in this chapter are distributions. However, unlike the previous chapters, here we also consider distributions which are not necessarily codimension-1. Recall that by a distribution we mean:

Definition 3.2.1. *Let M be a manifold. A (tangent)* ***distribution*** *ξ of rank k is a section of the Grassmann bundle of k-planes* $\mathrm{Gr}(TM, k) \to M$.

We will look at differential invariants of distributions and at the submanifolds tangent to them. The reader may want to further refer to standard references [90, Chapters 2 and 4],[21], and [55].

3.2.1.1 The Lie flag

The vector fields $\Gamma(\xi)$ tangent to ξ are a C^∞-submodule of the space of all vector fields of M. Therefore, it is natural to analyse to what extent this subspace fails to be a Lie subalgebra (with respect to the Lie bracket of vector fields), as we now describe.

Definition 3.2.2. *Define a sequence of C^∞-modules of vector fields using the inductive formula:*

$$\Gamma(\xi^{(i+1)}) := [\Gamma(\xi), \Gamma(\xi^{(i)})].$$

The rightmost expression denotes taking the C^∞-span of all Lie brackets with entries in $\Gamma(\xi)$ and $\Gamma(\xi^{(i)})$.

*The **Lie flag** associated to ξ is the sequence*

$$\xi^{(0)} := \xi \subset \xi^{(1)} \subset \xi^{(2)} \subset \dots$$

where $\xi^{(i)}$ is the pointwise span of $\Gamma(\xi^{(i)})$.

Remark 3.2.3. In general $\Gamma(\xi^{(i+1)})$ as defined above is only a C^∞-module of vector fields, and not necessarily the module of sections of a distribution as the notation suggests. In particular, unlike ξ the rank of $\xi^{(i)}$ may depend on the basepoint, so it may fail to be a distribution. The precise condition for this to happen is given by the Serre-Swan theorem; On a connected manifold M a C^∞-submodule $\Gamma \subset \mathfrak{X}(M)$ is the space of sections of a distribution $\xi \subset TM$ if and only if it is finitely generated and projective. △

Unless explicitly stated otherwise, we will always assume that all the entries in the Lie flag are distributions. Under this assumption the Lie flag stabilizes after finitely many steps since the rank$\xi^{(i)} \leqslant \dim M$. That is, $\xi^{(r+1)} = \xi^{(r)}$ for some smallest r and thus for all subsequent integers. We can then define:

Definition 3.2.4. *The **growth vector** of ξ is the sequence of integers*

$$(\mathrm{rank}(\xi^{(0)}), \dots, \mathrm{rank}(\xi^{(r)})).$$

When writing a growth vector or a Lie flag, we just write the terms until it stabilizes.

Example 3.2.5. On $\mathbb{R}^3$ with coordinates (x, y, z), consider the standard contact structure

$$\xi_{st} := \ker(\mathrm{d}z - y\mathrm{d}x) = \langle \partial_y, y\partial_z + \partial_x \rangle.$$

Since

$$[\partial_y, y\partial_z + \partial_y] = \partial_z,$$

it follows that $\xi^{(1)} = TM$. This is true for any contact structure, since the curvature $c_\xi : \Lambda^2\xi \to TM/\xi$ being non-degenerate implies that $\xi^{(1)} = TM$. Note however that the converse is not true. △

Example 3.2.6. On $\mathbb{R}^3$ with coordinates (x, y, z), the **Martinet distribution** is defined by

$$\xi := \ker(\mathrm{d}z - y^2\mathrm{d}x) = \langle \partial_y, y^2\partial_z + \partial_x \rangle.$$

Observe that

$$[\partial_y, y^2\partial_z + \partial_x] = y\partial_z, \quad [\partial_y, [\partial_y, y^2\partial_z + \partial_x]] = \partial_z.$$

It follows that away from the hypersurface $\{y = 0\}$, this distribution is contact and $\xi^{(1)} = TM$. At points in the hypersurface we need to take two brackets to span

TM, and we growth vector equals $(2, 2, 3)$. Thus, in this case $\xi^{(1)}$ does not define a distribution (of constant rank). Analogously, for any $k \in \mathbb{N}$ we can define

$$\xi_k := \ker(\mathrm{d}z - y^k \mathrm{d}x),$$

which defines a contact structure away from the hypersurface $\{y = 0\}$, and stabilizer after k steps at points in the hypersurface. △

Example 3.2.7. On $\mathbb{R}^4$ with coordinates (x, y, z, w), the standard **Engel structure** is defined as

$$\mathcal{E} := \ker(\mathrm{d}z - y\mathrm{d}x) \cap \ker(\mathrm{d}y - w\mathrm{d}x) = \langle \partial_w, \partial_x + y\partial_z + w\partial_y \rangle.$$

Thus $\mathcal{E}$ is a (smooth) rank-2 distribution, and its Lie flag equals:

$$\xi^{(1)} = \langle \partial_w, \partial_x + y\partial_z, \partial_y \rangle, \quad \xi^{(2)} = \langle \partial_w, \partial_x, \partial_y, \partial_z \rangle = TM,$$

so that the growth vector equals $(2, 3, 4)$. △

3.2.1.2 Involutive vs. bracket-generating

By definition, $\Gamma(\xi)$ is a Lie subalgebra if and only if $\Gamma(\xi^{(1)}) = [\Gamma(\xi), \Gamma(\xi)] = \Gamma(\xi)$. That is, if and only if the associated Lie flag (or, identically, the growth vector) is constant. Such a ξ is said to be **involutive**. A related notion is that of **integrability**: there exists a partition of the ambient manifold M into submanifolds of dimension $k = \mathrm{rank}(\xi)$ all of which are **integral**, that is, everywhere tangent to ξ. Recall that Frobenius theorem states that involutivity of a distribution is equivalent to integrability. The growth vector is, therefore, a measure of the non-integrability of ξ.

For us, the more interesting case is the complete opposite: ξ is said to be **bracket-generating** if, for some integer r, it holds that $\xi^{(r)} = TM$; i.e. $\Gamma(\xi)$ generates, as an algebra, the space of all vector fields. In particular, all the examples given above are bracket-generating. A well-known theorem of Chow states that any two points in M can be connected by a path tangent to ξ if ξ is bracket-generating. This can be regarded as the first result showing that submanifolds tangent to bracket-generating distributions behave flexibly. The present chapter goes also in this direction following an h-principle approach.

3.2.1.3 Curvature and nilpotentisation

We can define additional invariants of ξ that measure its non-integrability in a more refined manner. They are defined as follows: by construction, there is a map between sections

$$\Gamma(\xi^{(i)}) \times \Gamma(\xi^{(j)}) \to \Gamma(\xi^{(i+j+1)}/\xi^{(i+j)})$$

induced by Lie bracket. It can be checked that this map is C^∞-linear, letting us write:

Definition 3.2.8. *The **(i,j)-curvature** of ξ is the tensor:*

$$\Omega_{i,j}(\xi) : \xi^{(i)}/\xi^{(i-1)} \times \xi^{(j)}/\xi^{(j-1)} \to \xi^{(i+j+1)}/\xi^{(i+j)}.$$

Explicitly, the curvature $\Omega_{0,0}(\xi) : \Lambda^2\xi \to TM/\xi$ is defined on section by

$$\Omega_{0,0}(\xi) : \Gamma(\xi) \times \Gamma(x) \to \Gamma(TM/\xi), \quad (X,Y) \mapsto [X,Y] \bmod \xi.$$

If ξ is corank-1, this is just the curvature $c_\xi \in \Omega^2(\xi, TM/\xi)$. By Koszul's formula this map is uniquely defined by the formula:

$$\alpha \circ \Omega_{0,0} = \mathrm{d}\alpha|_\xi, \quad \forall \alpha \in \alpha \in \Gamma((TM/\xi)^*).$$

Equivalently, the dual map is defined by:

$$\Omega^*_{0,0} : \Gamma()(TM/\xi)^*) \to \Gamma(\Lambda^2\xi^*), \quad \alpha \mapsto \mathrm{d}\alpha|_\xi.$$

Again, it can be checked that these maps are C^∞-linear, and induce maps on the level of vector bundles.

To put all the curvatures together in a more algebraic fashion we recall the notion of a bundle of Lie algebras:

Definition 3.2.9. *A* ***bundle of Lie algebras*** *is a pair $(E, [\cdot,\cdot])$ consisting of a vector bundle $\pi : E \to M$ together with a section $[\cdot,\cdot] \in \Gamma(\Lambda^2 E^* \otimes E)$, such that for each $x \in M$ the restriction*

$$[\cdot,\cdot]_x : E_x \times E_x \to E_x,$$

defines a Lie bracket on the vector space E_x.

Remark 3.2.10. The above definition should not be confused with the more restrictive **Lie algebra bundle**. In the latter case on additionally requires that for each $x \in M$ there exists an isomorphism of bundles of Lie algebras

$$\phi : (E, [\cdot,\cdot])|_{\mathcal{O}p(x)} \xrightarrow{\sim} (\mathcal{O}p(x) \times L, [\cdot,\cdot]_L),$$

for a fixed Lie algebra $(L, [\cdot,\cdot]_L)$. Hence, every Lie algebra bundles is a bundle of Lie algebras but not conversely. △

We can endow the graded vector bundle

$$L(\xi) := \oplus_{i=0}^r L(\xi)_i := \oplus_{i=0}^r (\xi^{(i)}/\xi^{(i-1)}) = \xi \oplus (\xi^{(1)}/\xi) \oplus \cdots \oplus (\xi^{(r)}/\xi^{(r-1)})$$

with a fibrewise graded Lie bracket $\Omega(\xi) = \oplus_{i,j}\Omega_{i,j}(\xi)$.

Definition 3.2.11. *The pair $(L(\xi), \Omega(\xi))$ is a bundle of fibrewise graded Lie algebras called the* ***nilpotentisation*** *of ξ.*

Note that the graded Lie algebras on each fibre may not be pairwise isomorphic (but they will be in the cases we care about). The nilpotentisation should be thought as a linearisation of ξ: it packages the infinitesimal behaviour of ξ under Lie bracket at each point of the manifold.

3.2.1.4 Integral elements and submanifolds

Fix a manifold (M, ξ) endowed with a distribution (the precise nature of the distributions ξ we want to consider will be explained in the next Subsection).

We are interested in maps and submanifolds tangent to ξ:

Definition 3.2.12. *Let M be a manifold endowed with a distribution ξ. We say that:*

- *A map $f : N \to M$ is* ***integral*** *if $\mathrm{d}f(TN) \subset \xi$;*
- *A submanifold $N \subset M$ is* ***integral*** *if $TN \subset \xi$.*

It is immediate that the first curvature $\Omega_{0,0}(\xi)$ vanishes when restricted to an integral submanifold. This leads us to consider the subspaces of TM that might potentially be tangent to one of them:

Definition 3.2.13. *An* ***integral element*** *is an l-dimensional linear subspace $W \subset \xi_p \subset T_pM$, for some $p \in M$, such that:*

$$\Omega_{0,0}(\xi)_p|_W = 0.$$

The collection $\mathrm{Gr}_{\mathrm{integral}}(\xi_p, l)$ of all of them, which is a subset of the usual Grassmannian of l-planes $\mathrm{Gr}(\xi_p, l) \subset \mathrm{Gr}(T_pM, l)$, is called the **integral Grassmannian**. The union of all these subsets for every point $p \in M$ is denoted by $\mathrm{Gr}_{\mathrm{integral}}(\xi, l)$; we call it the **integral Grassmannian bundle**. This name might be misleading: the fibres $\mathrm{Gr}_{\mathrm{integral}}(\xi_p, l)$ are algebraic subvarieties of $\mathrm{Gr}(T_pM, l)$ which often are not smooth. We shall study this in depth in Section 3.5.

For our purposes, it will be necessary to see how integral Grassmannians relate to one another. More precisely, given an integral element $W \subset \xi_p$ of dimension l', we may consider the subset of $\mathrm{Gr}_{\mathrm{integral}}(\xi_p, l)$ consisting of those l-dimensional integral elements that contain it. We denote this by $\mathrm{Gr}_{\mathrm{integral}}(\xi_p, l; W)$. Related to this we have:

Definition 3.2.14. *The* ***polar space*** *of an integral element $W \subset \xi_p$ is*

$$W^\xi := \{v \in \xi_p \mid \Omega_{0,0}(\xi)_p(w, v) = 0, \ \forall w \in W\} \subset \xi_p.$$

That is, the linear subspace of all those vectors in ξ_p which yield zero when paired with W using the curvature. Since W is integral, W^ξ contains W. Tautologically, extensions of W to an integral element of dimension $\dim(W)+1$ are in correspondence with lines in the quotient W^ξ/W.

An element is said to be **maximal** if $W = W^\xi$, i.e. if it is not contained in a larger integral element. Similarly, an integral submanifold $N \subset (M, \xi)$ is **(locally) maximal** if the germ of N at any of its points cannot be extended to an integral submanifold of greater dimension. It is immediate that if T_pN is a maximal integral element, N itself is maximal at p. The converse is not necessarily true, as shown by Example 3.2.6

Definition 3.2.15. *A vector w satisfying $\langle w\rangle^\xi = \xi_p$ is called a* ***Cauchy characteristic****.*

The linear subspace spanned by all the Cauchy characteristics is an integral element denoted by $\ker(\xi_p)$.

If the dimension of $\ker(\xi_p)$ does not vary with $p \in M$, then their union is a distribution $\ker(\xi) \subset \xi$ that we call the **characteristic foliation** of ξ. As the name suggests, a simple computation using the Jacobi identity shows that $\ker(\xi)$ is involutive. Its leaves are integral submanifolds.

It is immediate that any local diffeomorphism preserving ξ must preserve $\ker(\xi)$. Similarly, its differential can identify two vectors tangent to ξ only if their polar spaces have the same dimension. We will exploit these facts in the next subsection.

Example 3.2.16. Let (M, ξ) be a contact manifold. Then the curvature $\Omega_{0,0}(\xi)$ is a nondegenerate form on ξ with values on TM/ξ. Indeed, as in Chapter 2, if α represents ξ, then we have $\alpha \circ \Omega_{0,0}(\xi) = -d\alpha$.

For a subspace $W \subset \xi_p$ to be isotropic is the same as being integral i.e. $W \subset W^\xi$. The polar space W^ξ is the usual **symplectic orthogonal**. The integral Grassmannians are thus the same as the Grassmannians of isotropic subspaces. In particular, for maximal integral elements we look at the Lagrangian Grassmannian. △

3.2.2 Basics of jet spaces

We now recall some elementary notions about jet spaces, putting particular emphasis on their tautological distribution, which is bracket-generating. All of the results in this chapter have to do with integral submanifolds of this tautological distribution. A standard reference in the Geometry of PDEs literature is [70, Chapter IV], but we also recommend [109, Section 2]. The two standard h-principle references also treat jet spaces, namely [60, Section 1.1] and [43, Chapter 1]. Lastly, the reader may want to look at [61, Section 4.1], whose ideas have certainly inspired parts of this work.

One of our goals in this subsection is to stress the metasymplectic viewpoint; see subsections 3.2.2.6, 3.2.2.7, and 3.2.2.8. This will be important later on when we study integral elements in Section 3.5.

3.2.2.1 Jet spaces of sections

Let X be an n-dimensional manifold and let $\pi : Y \to X$ be a smooth fibre bundle with k-dimensional fibres.

Definition 3.2.17. *Two sections $f_0, f_1 : X \to Y$ define the same r-**jet** at a point $x \in X$ if, in any trivialisation, their Taylor polynomials of order r at x agree.*

An r-jet is thus an equivalence class of sections. It is a consequence of the chain rule that this is a well-defined notion across different choices of charts for X and Y. Hence, we can write $J^r(Y \to X)$ for the space of all r-jets of sections $X \to Y$, i.e.

the space of equivalence classes for the above equivalence relation. When Y is the trivial $\mathbb{R}^k$-bundle over X we often write $J^r(X, \mathbb{R}^k) := J^r(Y \to X)$.

We can collect the r-order differential information of a section in the following object:

Definition 3.2.18. *Let $f : X \to Y$ be a section. Its **holonomic lift** is the section*

$$j^r f : X \to J^r(Y \to X)$$

*mapping each $x \in X$ to the r-jet of f at x. A section of $J^r(Y \to X)$ is **holonomic** if it is such a holonomic lift.*

It follows from this definition that the spaces of r-jets, for varying r, fit in a tower

$$J^r(Y \to X) \xrightarrow{\pi_{r,r-1}} J^{r-1}(Y \to X) \xrightarrow{\pi_{r-1,r-2}} \dots \xrightarrow{\pi_{1,0}} J^0(Y \to X) = Y \qquad (3.2.2.1)$$

where each projection forgets the differential information of top order. Each projection maps holonomic sections to holonomic sections. For notational convenience, we single out two of them:

Definition 3.2.19. *The **front projection** and the **base projection** are, respectively, the forgetful maps:*

$$\pi_f := \pi_{r,0} : J^r(Y \to X) \to Y, \qquad \pi_b : J^r(Y \to X) \to X.$$

In the literature these maps are sometimes also called the target and source map respectively. We will see below that $J^r(Y \to X)$ is a smooth manifold and that the $\pi_{r,r'}$ are affine bundles.

3.2.2.2 The Cartan distribution

The notion of holonomicity suggests the following construction:

Definition 3.2.20. *The **tautological distribution** ξ_{can} in $J^r(Y \to X)$ is uniquely defined by the following universal property: a section of $J^r(Y \to X)$ is tangent to ξ_{can} if and only if it is holonomic.*

*The subbundle $V_{\text{can}} := \ker(d\pi_{r,r-1}) \subset \xi_{\text{can}}$ is called the **vertical distribution** and its vector subspaces are said to be **vertical**.*

The subbundle ξ_{can} is also called the **Cartan distribution**. The inclusion $V_{\text{can}} \subset \xi_{\text{can}}$ will be immediate once we introduce local coordinates in Section 3.2.2.3. Moreover, these coordinates give a more explicit definition of ξ_{can} as the simultaneous kernel of a collection of differential forms, see Corollary 3.2.28.

Remark 3.2.21. Observe that the notation ξ_{can} does not contain any reference to the specific r-jet space. We have opted to do so to avoid cluttering our notation with indices. △

The very definition of ξ_{can} implies that images of holonomic sections are integral submanifolds that are everywhere transverse to the vertical distribution. We define:

Definition 3.2.22. *Integral manifolds and integral elements in* $(J^r(Y \to X), \xi_{\mathrm{can}})$ *transverse to* V_{can} *are said to be* ***horizontal****.*

We will show that any integral manifold which is both horizontal and maximal is necessarily the image of a holonomic section (locally).

The collection of all horizontal elements of rank l is denoted by

$$\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, l) \subset \mathrm{Gr}_{\mathrm{integral}}(\xi_{\mathrm{can}}, l),$$

and we call it the **horizontal Grassmannian bundle**. The subscript Σ^0 is inspired by the Thom-Boardman notation introduced in Subsection 3.3.1. If we restrict to a point $p \in J^r(Y \to X)$, we write $\mathrm{Gr}_{\Sigma^0}((\xi_{\mathrm{can}})_p, l)$ for the corresponding horizontal Grassmannian at p.

Example 3.2.23. If the fibres Y_x are 1-dimensional, then $(J^1(Y \to X), \xi_{\mathrm{can}})$ is a contact manifold and its maximal integral submanifolds are legendrians. Similarly, if Y_x and X are 1-dimensional, $(J^2(Y \to X), \xi_{\mathrm{can}})$ is an Engel manifold. Holonomic sections are then curves tangent to the canonical Engel structure and it can be shown that no higher dimensional integral manifolds exist △

3.2.2.3 Local coordinates

We now provide a more explicit description of ξ_{can}, by in terms of local coordinates. By working locally we may assume that X is a n-dimensional vector space, denoted by B, and that the fibre of Y is a k-dimensional vector space, denoted by F. In this local setting the jet space $J^r(Y \to X)$ can be identified with $J^r(B, F)$. To be explicit we choose coordinates $x := (x_1, \cdots, x_n)$ in B and coordinates $y := (y_1, \cdots, y_k)$ in F. We may then use (x, y) to endow $J^r(B, F)$ with coordinates, as we now explain.

Two maps $f_0, f_1 : B \to K$ have the same r-jet at $x \in B$ if and only if their Taylor expansions at x are the same. Equivalently: a point $p \in J^r(B, F)$ is uniquely represented by an r-order Taylor polynomial based at $\pi_b(p) \in X$. Now, the r-order Taylor polynomial of a map $f : B \to F$ at x reads:

$$f(x+h) \cong \sum_{0 \leqslant |I| \leqslant r} (\partial^I f(x)) \frac{dx^{\odot I}}{I!}(h, \ldots, h),$$

where $I = (i_1, \ldots, i_n)$ ranges over all multi-indices of length at most r. Here $\odot$ denotes the symmetric tensor product and we use the notation

$$\mathrm{d}x^{\odot I} := \mathrm{d}x_{i_1} \odot \cdots \odot \mathrm{d}x_{i_n}, \quad I = (i_1, \ldots, i_n).$$

This tells us that $J^r(B, F) \to B$ is a vector bundle and that, formally, we can use the monomials

$$\frac{dx^{\odot I}}{I!} \otimes e_j, \qquad 0 \leqslant |I| \leqslant r' \quad 1 \leqslant j \leqslant k$$

as a framing; here $\{e_j\}_{1 \leqslant j \leqslant k}$ is the standard basis of F in the (y)-coordinates.

The monomials above with $|I| = r'$ form a basis of $\mathrm{Sym}^{r'}(B^*, F)$, the space of a symmetric tensors with r' entries in B and values in F. This leads us to write, in more conceptual terms:

Lemma 3.2.24. $J^r(B,F) = B \times F \times \mathrm{Hom}(B,F) \times \mathrm{Sym}^2(B^*,F) \times \cdots \times \mathrm{Sym}^r(B^*,F)$.

Corollary 3.2.25. *$\pi_{r,r-1} : J^r(Y \to X) \to J^{r-1}(Y \to X)$ is an affine bundle with fibres modelled on $\mathrm{Sym}^r(B^*,F)$.*

Proof. We cannot speak, intrinsically, of a section having vanishing derivatives of order exactly r. I.e. there is no natural choice of zero section in $\pi_{r,r-1}$. However, once a reference section $f : X \to Y$ is chosen, the space of r-jets $\{j^r g \mid j^{r-1} g = j^{r-1} f\}$ is isomorphic to $\mathrm{Sym}^r(B^*,F)$.

Once we fix a trivialisation $B \times F \to B$ of $Y \to X$ we can take this a step further. The zero section of $J^r(B,F)$ over $J^{r-1}(B,F)$ is the space of polynomials of degree $r-1$. □

In particular, the fibres of $\pi_{r,r-1}$ are $k\binom{n+r-1}{n-1}$-dimensional.

We can now write $z_j^{(I)}$ for the coordinate dual to the vector $\frac{dx^{\odot I}}{I!} \otimes e_j \in \mathrm{Sym}^{|I|}(B^*,F)$. This definition depends only on the choice of coordinates $(x,y) : Y \to B \times F$. We give these coordinates a name:

Definition 3.2.26. *The coordinates*

$$(x,y,z) := (x, y = z^0, z^1, \ldots, z^r), \qquad z^{r'} := \{z_j^{(I)} \mid |I| = r', 1 \leqslant j \leqslant k\},$$

in $J^r(Y \to X)$ are said to be ***holonomic****.*

3.2.2.4 The Cartan distribution in coordinates

We continue using the notation from the previous subsection. It is immediate from the expression of the Taylor expansion shown above that:

Lemma 3.2.27. *In terms of the holonomic coordinates $(x,y,z) \in J^r(B,F)$, the holonomic lift of a map $f : B \to F$ reads:*

$$\begin{aligned} j^r f : B &\to J^r(B,F) = && B \times F \times \mathrm{Hom}(B,F) \times \mathrm{Sym}^2(B^*,F) \times \cdots \times \mathrm{Sym}^r(B^*,F), \\ x &\to j^r f(x) = && (x, y = f(x), z^1 = (\partial f)(x), z^2 = (\partial^2 f)(x), \ldots, z^r = (\partial^r f)(x)). \end{aligned}$$

That is, a holonomic section satisfies the relations

$$z_j^{(I)}(x) = (\partial^I y_j)(x), \qquad I = (i_1, \ldots, i_n),\ 0 \leqslant |I| \leqslant r,\ 1 \leqslant j \leqslant k.$$

In other words:

Corollary 3.2.28. *The tautological distribution ξ_{can} is the simultaneous kernel of the* ***Cartan*** ***1-forms****:*
(3.2.2.2)

$$\alpha_j^I = dz_j^{(I)} - \sum_{a=1}^{n} z_j^{(i_1,\cdots,i_a+1,\cdots,i_n)} dx_a, \qquad I = (i_1, \ldots, i_n),\ 0 \leqslant |I| < r,\ 1 \leqslant j \leqslant k.$$

Looking at the Cartan 1-forms we deduce that n-dimensional horizontal elements are maximal (because all the directions in V_{can} pair non-trivially, through the curvature, with them). The same argument shows that V_{can} is maximal as well. Note that these are, in general, integral elements of different dimensions.

Using the dual viewpoint allows us to write ξ_{can} as the span of $n + k\binom{n+r-1}{n-1}$ linearly independent vector fields:

(3.2.2.3)

$$\xi_{\text{can}}(x,y,z) = \bigoplus_{1\leqslant a\leqslant n} \langle \partial_{x_a} + \sum_{0\leqslant |I|<r} z_j^{(i_1,\cdots,i_a+1,\cdots,i_n)} \partial_{z_j^{(I)}} \rangle \quad \oplus \bigoplus_{|I'|=r, 1\leqslant j'\leqslant k} \langle \partial_{z_{j'}^{(I')}} \rangle,$$

where the first n vectors lift TX and the others span V_{can}. The distribution determined by the first n vectors is not canonically defined and depends on the framing chosen (and thus on our choice of coordinates).

3.2.2.5 The bracket-generating property

From Equation 3.2.2.3 we readily compute the Lie flag:

Corollary 3.2.29. *The Lie flag associated to $(J^r(Y \to X), \xi_{\text{can}})$ is given by the expression:*

$$\xi_{\text{can}}^{(i)} = d\pi_{r,r-i}^{-1}(\xi_{\text{can}}),$$

where the right hand side is the preimage of the Cartan distribution on $J^{r-i}(Y \to X)$.

In particular, ξ_{can} bracket-generates in r steps, i.e. $\xi_{\text{can}}^{(r)} = TJ^r(Y \to X)$.

We can additionally observe that $[\xi_{\text{can}}^{(1)}, V_{\text{can}}] \subset \xi_{\text{can}}^{(1)}$. I.e. the curvature $\Omega_{0,0}(\xi_{\text{can}}^{(1)})$ pairs trivially with the vertical space of ξ_{can}. This is true for trivial reasons if $r = 1$ (because then $\xi_{\text{can}}^{(1)}$ is the whole tangent space) but, in general:

Corollary 3.2.30. *The following statements hold for $r > 1$:*

- *The vertical distribution V_{can} is the characteristic foliation $\ker(\xi_{\text{can}}^{(1)})$ of $\xi_{\text{can}}^{(1)}$ on $J^r(Y \to X)$.*
- *Inductively, $\ker(\xi_{\text{can}}^{(i)}) = \ker(d\pi_{r,r-i})$ for every $0 < i < r$.*

We will say that $\ker(\xi_{\text{can}}^{(i)})$ is the i**th characteristic foliation**.

If we regard $(J^r(Y \to X), \xi_{\text{can}})$ as an abstract manifold endowed with a distribution, i.e. forgetting that projections $\pi_{r,r'}$, the Corollary tells us that we can recover the fibres of $\pi_{r,r'}$ intrinsically, as long as $r' > 0$. In fact, even the front projection π_f (Definition 3.2.19) can be recovered as long as $k = \dim(Y_x) > 1$. This follows by observing that the polar space of a horizontal vector is always smaller in dimension than the polar space of a vertical one. In this case, we say that the fibre of π_f is the **polar foliation** associated to ξ_{can}.

The one case in which the fibres are not defined intrinsically is when we look at the front projection with $k = 1$. This is related to the fact that $(J^1(X, \mathbb{R}), \xi_{\text{can}})$, as a contact manifold, has many more symmetries than those appearing as lifts of symmetries of the front; see below.

3.2.2.6 Standard metasymplectic space

Let us revisit subsection 3.2.2.4, particularly the Cartan 1-forms (as defined in Equation 3.2.2.3). We work locally in jet space, so we may write $J^r(B,F)$ with holonomic coordinates (x,y,z).

Consider the collection of Cartan 1-forms of the form:

$$\alpha_j^I = dz_j^{(I)} - \sum_{a=1}^{n} z_j^{(i_1,\cdots,i_a+1,\cdots,i_n)} dx_a, \qquad I=(i_1,\dots,i_n),\ |I|=r-1,\ 1\leqslant j\leqslant k,$$

which only depend on the coordinates z^r. Their differentials are the 2-forms:

$$\Omega_j^I = \sum_{a=1}^{n} dx_a \wedge z_j^{(i_1,\cdots,i_a+1,\cdots,i_n)}, \qquad I=(i_1,\dots,i_n),\ |I|=r-1,\ 1\leqslant j\leqslant k,$$

which, by construction, are pullbacks of forms in the product $B\oplus\mathrm{Sym}^r(B^*,F)$ (which have the same coordinate expression, so we abuse notation and denote them the same). We can package them in the following intrinsic manner:

Definition 3.2.31. *The* ***canonical metasymplectic structure*** *in* $B\oplus\mathrm{Sym}^r(B^*,F)$ *is the* 2*-form:*

$$\Omega_{\mathrm{can}} := (\Omega_j^I)_{|I|=r-1,\,1\leqslant j\leqslant k} : \quad \wedge^2(B\oplus\mathrm{Sym}^r(B^*,F)) \quad\to\quad \mathrm{Sym}^{r-1}(B^*,F).$$

The pair $(B\oplus\mathrm{Sym}^r(B^*,F),\Omega_{\mathrm{can}})$ *is called* ***standard metasymplectic space.***

We remark that we can regard standard metasymplectic space as a vector space endowed with a linear 2-form, or as a manifold endowed with a smooth 2-form. The tangent fibres of the latter are, of course, isomorphic to the former.

A more manageable way of expressing Ω_{can} is provided by the following tautological lemma:

Lemma 3.2.32. *Given a point* $p\in B\oplus\mathrm{Sym}^r(B^*,F)$ *and vectors* $v_i+A_i\in T_p(B\oplus\mathrm{Sym}^r(B^*,F))\cong B\oplus\mathrm{Sym}^r(B^*,F)$*:*

$$\Omega_{\mathrm{can}}(v_0+A_0,v_1+A_1)=\iota_{v_0}A_1-\iota_{v_1}A_0.$$

I.e. the canonical metasymplectic structure is precisely the contraction map of tensors with vectors.

Example 3.2.33. When $r=k=1$, the standard metasymplectic space $(B\oplus B^*,\Omega_{\mathrm{can}})$ is the standard $2n$-dimensional symplectic space. △

3.2.2.7 The metasymplectic projection

Let us consider the map:

Definition 3.2.34. *The **metasymplectic projection** is the map*

$$\begin{aligned}\pi_L : J^r(B,F) &\rightarrow B \oplus \mathrm{Sym}^r(B^*,F)\\ (x,y,z) &\rightarrow \pi_L(x,y,z) := (x,z^r).\end{aligned}$$

In the contact setting this is usually called the **Lagrangian projection**, because it maps legendrians to lagrangians. A similar situation holds in general, as we now explain.

Observe first that, by construction, the differential

$$d_p\pi_L : T_pJ^r(B,F) \quad\rightarrow\quad T_{\pi_L(p)}(B \oplus \mathrm{Sym}^r(B^*,F))$$

is an epimorphism that restricts to an isomorphism $(\xi_{\mathrm{can}})_p \rightarrow T_{\pi_L(p)}(B \oplus \mathrm{Sym}^r(B^*,F))$. Furthermore, using the duality between distributions and their annihilators it readily follows that:

Lemma 3.2.35. *The differential is an isomorphism of metasymplectic linear spaces:*

$$d_p\pi_L : ((\xi_{\mathrm{can}})_p, \Omega(\xi_{\mathrm{can}})) \rightarrow (T_{\pi_L(p)}(B \oplus \mathrm{Sym}^r(B^*,F)), \Omega_{\mathrm{can}}),$$

where $\Omega(\xi_{\mathrm{can}})$ is the curvature of ξ_{can}.

We can use this to study integral submanifolds of ξ_{can}.

Definition 3.2.36. *A vector subspace V of the standard metasymplectic (linear) space is said to be an **isotropic element** if*

$$V^{\Omega_{can}} \subset V.$$

An isotropic element is maximal if it is not contained in a larger isotropic subspace.

*A submanifold of standard metasymplectic space is **isotropic** if all its tangent subspaces are isotropic elements.*

It immediately follows from the previous Lemma (and the comments preceding it) that:

Corollary 3.2.37. *Let $f : N \rightarrow J^r(B,F)$ be a map. Then:*

- *f is integral if and only if $\pi_L \circ f$ is isotropic.*
- *If f is integral then f is an immersion if and only if $\pi_L \circ f$ is an immersion.*

In subsection 3.6.4.1 we will prove a converse: any isotropic map has a unique integral lift up to choice of basepoint.

3.2.2.8 The nilpotentisation

According to the computations in the previous subsections, the nilpotentisation of ξ_{can} at any point is isomorphic to the following graded Lie algebra:

Definition 3.2.38. *Let B and F be real vector spaces of dimensions n and k, respectively.*

*The **jet Lie algebra** (depending on n, r, and k) is:*

- *The graded vector space $\mathfrak{g} := \bigoplus_{i=0}^{r} \mathfrak{g}_i$ with*

$$\mathfrak{g}_0 := B \oplus \mathrm{Sym}^r(B^*, F), \qquad \mathfrak{g}_i := \mathrm{Sym}^{r-i}(B^*, F).$$

- *Endowed with the Lie bracket defined by the contraction of vectors with tensors*

$$[v, \beta] = \iota_v \beta, \qquad v \in B, \quad \beta \in \mathrm{Sym}^j(B^*, F).$$

  All other brackets are either defined by the antisymmetry or zero.

We will often abuse notation and use $\mathfrak{g}$ to denote the graded Lie algebra as a whole (instead of just the underlying vector space). Note that the bracket and the grading are compatible, making the jet Lie algebra a *graded Lie algebra*. This Lie algebra is *nilpotent*; in particular, the composition of $r+1$ brackets is zero. Disclaimer: the grading we use is shifted by one with respect to the usual conventions for graded Lie algebras found in the literature.

The degree zero part $\mathfrak{g}_0$ is the direct sum $B \oplus \mathrm{Sym}^r(B^*, F)$. When identified with ξ_{can} at a point p, the first part corresponds to a lift of T_pX (in a canonical manner once we choose local coordinates). The second term corresponds to the vertical distribution. We will henceforth say that B is the **horizontal component** and $\mathrm{Sym}^r(B^*, F)$ is the **vertical component**. We write $\pi_b : \mathfrak{g}_0 \to B$ for the projection to the horizontal factor.

As claimed:

Proposition 3.2.39. *The nilpotentisation of $(J^r(Y \to X), \xi_{\mathrm{can}})$, at any point, is isomorphic to the jet Lie algebra $\mathfrak{g}$ with parameters $n = \dim(X)$, r, and $k = \dim(Y_x)$. In particular, the zeroeth order part $\mathfrak{g}_0$ corresponds to $(\xi_{\mathrm{can}})_p$.*

Proof. The last claim is simply a consequence of the definition of nilpotentisation. That the underlying vector space is $\mathfrak{g}$ follows from the discussion in subsection 3.2.2.1. The form of the Lie bracket is immediate from the local presentation of ξ_{can} given in subsection 3.2.2.4, together with Lemmas 3.2.32 and 3.2.35. □

It readily follows from the Proposition that:

Corollary 3.2.40. *Integral elements of ξ_{can} correspond to vector subspaces $W \subset \mathfrak{g}_0$ which are, additionally, Lie subalgebras.*

Horizontal elements of ξ_{can} correspond to Lie subalgebras $W \subset \mathfrak{g}_0$ transverse to the vertical component.

Hence, a Lie subalgebra contained in $\mathfrak{g}_0$ is said to be an **integral element**. Similarly, an integral element transverse to $\mathrm{Sym}^r(B^*, F)$ is called a **horizontal element**.

For fixed l, the collection $\mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g}, l)$ of all integral elements is called the **integral Grassmannian**. It is isomorphic to $\mathrm{Gr}_{\mathrm{integral}}((\xi_{\mathrm{can}})_p, l)$, for any $p \in J^r(B, F)$. The collection of all horizontal elements of dimension l is the **horizontal Grassmannian**, which we denote by $\mathrm{Gr}_{\Sigma_0}(\mathfrak{g}, l) \subset \mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g}, l)$.

Corollary 3.2.41. *The space* $\mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g}, l)$ *is an algebraic subvariety of the standard Grassmannian of* l*-planes* $\mathrm{Gr}(\mathfrak{g}, l)$*.*

Proof. Being a Lie subalgebra and lying in zero degree are both algebraic conditions. □

We will study the integral Grassmannian in more detail in Section 3.5.

3.2.2.9 Closing remarks

We have observed that the fibre of $J^r(Y \to X) \to J^{r-1}(Y \to X)$ is an exceptional integral submanifold: its dimension $k\binom{r+n-1}{n-1}$ is larger than n (i.e. the dimension of a maximal horizontal submanifold) unless $k = 1$ and $r = 1$ (i.e the contact setting) or if $k = 1$ and $n = 1$ (the Goursat setting).

In the contact case, the fibre is locally isomorphic to any other legendrian, according to Weinstein's legendrian neighbourhood theorem. In particular, up to a C^∞-small perturbation, it can be assumed to be in generic position with respect to the front projection. This is related to the fact that the front projection

$$\pi_f : J^1(\mathbb{R}^n, \mathbb{R}) \cong T^*\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$$

is not canonical.That is, if a legendrian is not horizontal, we can find some other front projection (locally) in which it is.

In the Goursat case with $r > 1$, the fibre has a special local model. It can be put in generic position (through integral manifolds) if we allow the ends not to be constrained, but it was proven in [20] that it admits no compactly-supported deformations. This phenomenon is called **rigidity**. A related notion is that of **singularity**: an integral manifold is singular if it has less compactly-supported deformations than expected. Many works [65, 73, 1] study these ideas for the case of curves but, to our knowledge, this has not been studied in depth for higher dimensional integral submanifolds in general bracket-generating distributions.

The upshot is that, for general jet spaces, the fibre is rather exceptional. In particular, there are marked differences between horizontal submanifolds and general integral submanifolds. In particular, one cannot expect a full analogue of the Legendre transform to hold. Despite of this, we will introduce some tools in Section 3.6 to deal with the general case.

3.2.3 Distributions modelled on jet spaces

Much like contact manifolds look locally like the 1-jet space of functions, we can, more generally, consider manifolds with distributions locally modelled on some other jet space.

Definition 3.2.42. *Let (M, ξ) be a manifold endowed with a distribution. We say that ξ is* ***modelled*** *on the tautological distribution $(J^r(B,F), \xi_{\text{can}})$ if, for each $p \in M$, there are local coordinates (x, y, z) around p, with domain a subset of $J^r(B,F)$, so that $\xi = \xi_{\text{can}}$.*

In particular, the numbers $n = \dim(B)$, $k = \dim(F)$, and r are invariants of ξ.

According to Corollary 3.2.29 from subsection 3.2.2.5, ξ bracket-generates in r steps. Furthermore, according to Corollary 3.2.30, M is endowed with a flag $\{\ker(\xi^{(i)})\}_{i=1,\ldots,r-1}$ of **characteristic foliations**. Indeed: in each local model these are simply the successive fibres of the projections $\pi_{r,r'}$, $r' > 0$. When $k > 1$, we also have a well-defined **polar foliation**, defined locally to the fibres of π_f.

Our constructions of integral submanifolds will be (semi-)local in nature. Therefore, they will apply too to integral submanifolds of distributions modelled on jet spaces. Note that one can still talk about horizontal submanifolds in this setting (as long as $r > 1$ or $k > 1$), as those that are transverse to the first characteristic/polar foliation.

3.2.3.1 Automorphisms

In Section 3.7 we will study local models of integral submanifolds of ξ_{can}. Understanding their locus of tangency with V_{can} will take us into Singularity Theory (see Section 3.3 for an overview of the concepts we need). One of our goals then will be to show that certain models of tangency are *stable* (defined in Subsection 3.3.2). However, in order to discuss stability, we must fix the allowed space of automorphisms (i.e. the left-right equivalences). To this end, we look at the *symmetries* of ξ_{can}.

Definition 3.2.43. *Let (M, ξ) be a distribution modelled on a jet space. A* ***(contact) transformation*** *of (M, ξ) is a ξ-preserving diffeomorphism.*

A more restrictive notion of symmetry (which only makes sense for jet spaces) is the following:

Definition 3.2.44. *Let $Y \to X$ be a fibre bundle. Let $\Psi : Y \to Y$ be a fibre-preserving diffeomorphism lifting a diffeomorphism $\psi : X \to X$.*

The ***point symmetry*** *lifting Ψ is defined as:*

$$\begin{aligned} j^r\Psi : (J^r(Y \to X), \xi_{\text{can}}) &\to (J^r(Y \to X), \xi_{\text{can}}) \\ j^r f(x) &\to (j^r\Psi)(j^r f(x)) := j^r(\Psi \circ f \circ \psi^{-1})(\psi(x)). \end{aligned}$$

Point symmetries form a subgroup of the group of contact transformations. It is well-known in Contact Geometry that the space of contact transformations of $J^1(X, \mathbb{R})$ is strictly larger than the space of point symmetries. However:

Lemma 3.2.45. *Assume $r > 1$ or $\dim(Y_x) > 1$. Any contact transformation of $J^r(Y \to X)$ is the lift of a contact transformation of $J^{r-1}(Y \to X)$.*

Proof. Suppose that $r > 1$ and let Ψ be a contact transformation of $J^r(Y \to X)$. Then Ψ preserves the vertical distribution. Therefore, Ψ induces a well-defined map $\tilde{\Psi}$ in the quotient

$$J^r(Y \to X)/V_{\text{can}} \cong J^{r-1}(Y \to X).$$

Furthermore, since Ψ preserves ξ_{can}, it preserves $\xi_{\text{can}}^{(1)}$. From this we deduce that $\tilde{\Psi}$ preserves the Cartan distribution in $J^{r-1}(Y \to X)$. Hence Ψ is a lift of $\tilde{\Psi}$ (we will not explain that this lift is, in fact, unique).

The same argument applies to the polar foliation if $r = 1$ and $\dim(Y_x) > 1$. □

3.2.3.2 Jet spaces of submanifolds

Let us provide an example of distribution (M, ξ) locally modelled on a jet space.

Definition 3.2.46. *Let Y be a smooth manifold and fix an integer $n < \dim(Y)$. We say that two n-submanifolds have the same r-**jet** at $p \in Y$ if they are tangent at p with multiplicity r.*

More precisely, two submanifolds $N_1, N_2 \subset Y$ have the same r-jet at $x \in N_1 \cap N_2$ if, N_2 is graphical in a neighborhood of x, and the induced section vanishes up to order r at x.

An r-jet is therefore an equivalence class of (germs of) embedded submanifolds. We denote the space of r-jets of n-submanifolds as $J^r(Y, n)$. We have, just like in the case of sections, a sequence of forgetful projections

$$\pi_{r,r'} : J^r(Y, n) \to J^{r'}(Y, n),$$

with $\pi_f := \pi_{r,0}$ being called the **front projection**.

Definition 3.2.47. *The **holonomic lift** of an n-submanifold $N \subset Y$ is the submanifold $j^r N \subset J^r(Y, n)$ consisting of all the r-jets of N at each of its points.*

*The **Cartan distribution** ξ_{can} in $J^r(Y, n)$ is the smallest distribution which is tangent to every holonomic lift.*

$J^r(Y \to X)$ was defined (when Y fibres over some base n-manifold X) using exactly the same equivalence relation as $J^r(Y, n)$: two sections have the same r-order Taylor polynomial at a point if and only if their images have an r-tangency. Therefore:

Lemma 3.2.48. *There is a distribution preserving embedding with open and dense image:*

$$(J^r(Y \to X), \xi_{\text{can}}) \quad \to \quad (J^r(Y, n), \xi_{\text{can}}).$$

The jets of submanifolds having non-trivial tangencies with the fibres of $Y \to X$ are not in the image of this inclusion.

Remark 3.2.49. If $n = \dim(Y) - 1$ and $r = 1$, the structure we just constructed is precisely the **space of contact elements**. In general, if $r = 1$, the space $J^1(Y, n)$ is precisely the Grassmannian of n-planes $\mathrm{Gr}(TY, n)$.

Given an immersion $f : N \to Y$, we often talk of its **Gauss map**

$$\mathrm{Gr}(f) := j^1 f : N \to \mathrm{Gr}(TY, n)$$

which at every point assigns the corresponding tangent plane. △

3.2.4 The foliated setting

Due to the parametric nature of the statements we want to prove, we will need to phrase our constructions in a foliated setting. An alternate (seemingly weaker but ultimately equivalent way) would be to use the fibered setting [43, 6.2.E].

Let $Y \to (M, \mathcal{F})$ be a smooth fiber bundle over a foliated manifold. We write k for the dimension of the fibres and n for the dimension of the leaves. We define the bundle of **foliated r-jets** $J^r(Y \to (M, \mathcal{F}))$ to be the space of equivalence classes of leafwise sections that are r-tangent to one another. The fibres of $J^r(Y \to (M, \mathcal{F})) \to M$ are again modelled on r-order Taylor polynomials of k functions in n variables. Given a global section $f : M \to Y$, we can consider its corresponding leafwise r-jet $j^r_{\mathcal{F}} f : M \to J^r(Y \to (M, \mathcal{F}))$. Such a section of the space of foliated jets is said to be **holonomic**. Note that $j^r_{\mathcal{F}} f$ encodes no information about the derivatives of f along the normal bundle of $\mathcal{F}$.

Given manifolds X and K, where the latter is thought of as a parameter space, we may consider the foliated manifold

$$(M = X \times K, \mathcal{F} = \coprod_{a \in K} X \times \{a\}).$$

If $Y \to X$ is a fibre bundle, we can pull it back to $X \times K$ using the obvious projection. The corresponding space of foliated r-jets $J^r(Y \to (M, \mathcal{F}))$ is the natural place to carry out parametric arguments for K-families of sections of $Y \to X$.

3.3 Overview: Singularity theory

We are interested in integral submanifolds of jet spaces. Often, we will look at them using their front projections (see Definition 3.2.19), which we would like to regard as "multiply-valued sections". We will define precisely what this means in Subsection 3.6.1, but it is clear that for any approach to work one should assume that the singularities of the front are manageable. At the very least, they should form a set of positive codimension, so the submanifold is graphical over the zero section in an open dense set.

As we mentioned in subsection 3.2.2.9, it is not always possible to study integral submanifolds through their fronts. For instance, some integral submanifolds tangent to the vertical distribution have no homotopies making them somewhere transverse to it. That is, no homotopy allows us to assume that the front is not a point, which is a seemingly very degenerate situation. This tells us that the integrality condition constrains the submanifolds heavily, and certain behaviours that are generic for unconstrained manifolds (like being able to move them to put them in general position with respect to a fibration) cannot be achieved. The general case contrasts with the contact case, where one may always assume that the front singularities are generic.

Even when one can obtain generic singularities, these might be terribly complicated. In this Section we will review the Thom-Boardman hierarchy (Subsection 3.3.1) and the notion of stability for singularities (Subsection 3.3.2), both of which quantify how complicated a singularity is. Then, in Subsections 3.3.4, 3.3.5, and 3.3.6, we will describe some of the singularities appearing later, called *wrinkles* and *double folds*.

3.3.1 The Thom-Boardman stratification theorem

Our goal in this Subsection is to state the Thom-Boardman Theorem 3.3.6. For this we need to set up some notation first. We refer the reader to the original papers [103, 14] and to the more modern reference [43, Chapter 2].

3.3.1.1 Types of singularities

We will look at two different notions of singularity. The first one being:

Definition 3.3.1. *Let M and N be smooth manifolds of dimensions m and n. Fix a map $f : N \to M$.*

A point $p \in N$ is said to be a ***singularity of mapping*** *of f if*

$$\mathrm{rank}(d_p f) < \min(m, n).$$

Suppose instead that f has no singularities of mapping:

Definition 3.3.2. *Let M be endowed with a foliation $\mathcal{F}$ of rank k, and $f : N \to M$ a smooth map.*

A point $p \in N$ is a ***singularity of tangency*** *with $\mathcal{F}$ if $d_p f(TN)$ and $\mathcal{F}_{f(p)}$ are not transverse to one another.*

In particular, for inclusions $\iota : N \to M$ of submanifolds, a point $p \in N$ is a singularity of tangency if T_pN and $\mathcal{F}_p$ are not transverse.

We can be more precise about the structure of the loci of singular points, as we now explain.

3.3.1.2 The stratification I

Let us focus on singularities of tangency for $N \subset (M, \mathcal{F})$.

Definition 3.3.3. *The locus of singularities of corank j is denoted by:*

$$\Sigma^j(N, \mathcal{F}) := \{p \in N \mid \dim(T_pN \cap \mathcal{F}_p) - \max(k + n - m, 0) \geqslant j\}.$$

That is, the set of points where the dimension of the intersection $T_pN \cap \mathcal{F}_p$ surpasses the transverse case by j.

Assuming that the set $\Sigma^j(N; \mathcal{F})$ is a submanifold, and the restriction $\mathcal{F}|_{\Sigma^j(N;\mathcal{F})}$ is a foliation, one can recursively define:

Definition 3.3.4. *The* ***higher tangency locus*** *of corank $J = j_0, \ldots, j_l$ is:*

$$\Sigma^{j_0 \cdots j_l}(N, \mathcal{F}) := \Sigma^{j_l}(\Sigma^{j_0 j_1 \cdots j_{l-1}}, \mathcal{F}).$$

Remarkable work of Thom [103] and Boardman [14] (see below) shows that one may perturb N so that all the $\Sigma^J(N, \mathcal{F})$ are well-defined smooth submanifolds. It can further be assumed that they form a **stratification**: i.e. $\Sigma^{j_0 j_1 \cdots j_l j_{l+1}}(N, \mathcal{F})$ lies in the closure of $\Sigma^{j_0 j_1, \cdots j_l (j_{l+1}-1)}(N, \mathcal{F})$.

One can phrase this in more homotopical terms. Consider the Grassmannian fibrations $\mathrm{Gr}(TM, n) \to M$, for varying n. The presence of $\mathcal{F}$ defines:

Definition 3.3.5. *The* Schubert decomposition *of* $\mathrm{Gr}(T_pM, n)$ *is the partition into smooth algebraic submanifolds:*

$$\Sigma^j = \{H \in \mathrm{Gr}(T_pM, n) \mid \dim(H \cap \mathcal{F}_p) - \max(k + n - m, 0) = j\}.$$

The stratification of the Grassmannian they provide varies smoothly with the point, defining submanifolds $\Sigma^j(n, \mathcal{F}) \subset \mathrm{Gr}(TM, n)$. The Poincaré duals of the $\Sigma^j(n, \mathcal{F})$ define cohomology classes in $\mathrm{Gr}(TM, n)$ that may be pulled back to N using the tangent map of the inclusion. These classes are dual to the intersections $TN \cap \Sigma^j(n, \mathcal{F})$: they represent obstructions to removing the singularities of N with $\mathcal{F}$ by homotoping N.

3.3.1.3 The stratification II

Instead of looking at singularities of tangency, one may look at singularities of mapping. This works in a completely analogous way: given a map $f : N \to M$ we define the loci of singularity:

$$\Sigma^j(f) = \{p \in N \mid \min(m, n) - \mathrm{rank}(Tf) \geqslant j\}.$$

And higher loci may be defined by iterating the process:

$$\Sigma^{j_0 j_1 \cdots j_l j_{l+1}}(f) = \Sigma^{j_{l+1}}(f|_{\Sigma^{j_0 j_1 \cdots j_l}(f)}).$$

3.3.1.4 The Thom-Boardman theorem

The two cases we have explained are examples of the same phenomenon: we look at the 1-jet of a submanifold/map and we study how it interacts with a certain subspace of 1-jet space. The Thom-Boardman theorem tells us, in much more generality, that the r-jet of a submanifold/map can be assumed to intersect a stratified submanifold of r-jet space in a transverse manner.

Theorem 3.3.6. *Let A be a stratified subset of $J^r(M,n)$ and let $f : N \to M$ be a n-submanifold with holonomic lift $j^r f : N \to J^r(M,n)$. After a C^∞-small perturbation of f, it may be assumed that $j^r f$ is transverse to each stratum of A.*

In particular, the stratification of A induces a stratification of N by pullback $(j^r f)^*$. The codimension of a stratum $j^r f^*(A^j) \subset N$ is the codimension of $A^j \subset A$.

3.3.2 Stability

We will introduce later the singularities we want to work with. We will be able to define many of them by simply saying that they are given by a particular local model, up to reparametrisation in domain and target. In the smooth setting, this reparametrisation process has a name:

Definition 3.3.7. *Two maps $f, g : N \to M$ are* ***(left-right) equivalent*** *if there exists a pair $(\psi, \phi) \in \mathrm{Diff}(N) \times \mathrm{Diff}(M)$ such that*

$$f = \phi \circ g \circ \psi.$$

That is, we define a natural action of the group $\mathrm{Diff}(N) \times \mathrm{Diff}(M)$ on the space $C^\infty(N,M)$ and two maps are equivalent if they lie in the same orbit.

One can consider the same definition where instead of maps and diffeomorphisms we look at germs of both. Furthermore, one can restrict the notion of equivalence by restricting the allowed diffeomorphisms in the domain or the target. This is important when the manifolds N and M are endowed with geometric structures that should be preserved by these automorphisms. See below.

As we deform a map, its equivalence class may change. If this is not the case we say that:

Definition 3.3.8. *A map $f : N \to M$ is* ***stable*** *if its orbit under the action of $\mathrm{Diff}(N) \times \mathrm{Diff}(M)$ is open. Equivalently, if it has no non-trivial deformations.*

A property $\mathcal{P}$ for maps in $C^\infty(N,M)$ is called **generic** if it is natural (i.e it is preserved by the $\mathrm{Diff}(N) \times \mathrm{Diff}(M)$ action) and the set

$$\{f \in C^\infty(N,M) \mid f \text{ has property } \mathcal{P}\} \subset C^\infty(N,M),$$

is dense. Therefore, stable maps are generic in the sense that they satisfy every generic property.

3.3.2.1 Stability under constraints

One can study families of maps, defined as subsets $A \subset C^\infty(N, M)$. In the presence of a geometric structure, one might consider classes A of maps that interact with the structure in a meaningful manner (preserving it most likely).

It then makes sense to restrict the groups by which we reparametrise so that they preserve A. In our setting, where we look at integral maps (Definition 3.2.12) of a manifold N into the r-jet space $M = J^r(Y \to X)$, the transformations from the right will still be $\mathrm{Diff}(N)$, but the transformations on the left will be the contact transformations of the Cartan distribution (or, more restrictively, the point symmetries).

When we look at the orbit $\mathcal{O}(a)$ of an element $a \in A$ using this restricted group of symmetries, we have that $\mathcal{O}(a) \cap A$ might be open in A, even if a was not stable as an element of $C^\infty(N, M)$; such an a is then *stable* as an element of A. This tells us that generic phenomena for maps in A might be quite different than for general maps.

For instance: any contact transformation of $J^r(Y \to X)$ preserves the characteristic and polar foliations. Hence, singularities of tangency of N with any of them cannot be removed by applying a symmetry. In the contact case, where none of these foliations exist, a contact transformation is instead capable of removing all the singularities of tangency of a legendrian with a given front projection.

3.3.2.2 Unfoldings

Even if a map $f \in C^\infty(N, M)$ is not stable, it can be part of a finite dimensional family of maps which is stable. This happens precisely when orbit of f has finite codimension in $C^\infty(N, M)$. Let us elaborate.

Definition 3.3.9. *Fix a map $f : N \to M$. A d-parametric **unfolding** of f is a map $F : N \times \mathbb{R}^d \to M \times \mathbb{R}^d$ fibered over $\mathbb{R}^d$ and satisfying:*

$$F(x, 0) = (f(x), 0), \quad x \in N$$

Additionally:

- *Two unfoldings are said to be **equivalent** if they are left-right equivalent as smooth fibered maps.*
- *An unfolding F is **trivial** if it is equivalent to the trivial unfolding $F(x, t) = f(x)$.*

In this language, f is stable if and only if every unfolding is trivial.

Given a fibered map $F : N \times \mathbb{R}^d \to M \times \mathbb{R}^d$ such that

$$F(x, u) = (\tilde{f}(x, u), u),$$

and a map $\phi : \mathbb{R}^e \to \mathbb{R}^d$, we define the pullback of F along ϕ by:

$$\phi^* F(x, u) := (\tilde{f}(x, \phi(u)), u),$$

yielding a new unfolding.

Definition 3.3.10. *F is said to be a* ***versal*** *unfolding of f if any other unfolding is fibered left-right equivalent to a pullback of F.*

This means that F contains all the possible deformations of f.

We can look at the infinitesimal action of $\mathrm{Diff}(N) \times \mathrm{Diff}(M)$ on $C^\infty(N, M)$. Given a map f, we may compute the quotient

$$T^1(f) := \frac{\{\frac{d}{dt}|_{t=0} f_t \mid f_0 = f\}}{\{\frac{\mathrm{d}}{\mathrm{d}t}(\phi_t \circ f \circ \psi_t) \mid \phi_0 = \mathrm{id}, \psi_0 = \mathrm{id}\}}.$$

Here $(f_t)_{t\in\mathbb{R}}$ is a 1-parametric deformation of f and $(\psi_t, \phi_t)_{t\in\mathbb{R}} \in \mathrm{Diff}(N) \times \mathrm{Diff}(M)$ is a 1-parametric deformation of the identity diffeomorphisms. This quotient computes the difference between the deformations of f and the deformations arising from reparametrisation.

In general, $T^1(f)$ is a vector space. It corresponds to the normal bundle of the $\mathrm{Diff}(N) \times \mathrm{Diff}(M)$-orbit of f. Its dimension measures the failure of f to be stable. However, if it is finite-dimensional, we can integrate representatives of the elements $T^1(f)$ to yield a $\dim(T^1(f))$-parameter versal unfolding F of f.

3.3.3 Whitney singularities

The Thom-Boardman invariants (Definition 3.3.4) are not sufficient to classify singularities of maps between manifolds of arbitrary dimension. In most cases, such a classification is not possible due to the existence of moduli (i.e. a singularity may not be stable even after we fix its Thom-Boardman class).

For maps between manifolds of equal dimension there is a particular countable family of singularities, called the *Whitney singularities*, which are completely classified in terms of the Thom-Boardman stratification. Let us describe them in a slightly roundabout way.

3.3.3.1 Spaces of polynomials

Endow $\mathbb{R}^{n+1}$ with coordinates $(q_1, \dots, q_{n-1}, q_n, x) = (\tilde{q}, q_n, x) = (q, x)$. Consider the function

$$\begin{aligned} F_n : \mathbb{R}^{n+1} &\to \mathbb{R} \\ (q, x) &\to x^{n+1} + q_1 x^{n-1} + \dots + q_{n-1}x + q_n. \end{aligned}$$

Here the q–variables function as parameters. As they vary, they parametrise the space of all polynomials of degree $n+1$ in one variable x. That is, F_n is an unfolding of the map $x \to x^{n+1}$ with n parameters. This unfolding is, in fact, versal.

The roots of the family of polynomials $(x \to F_n(q, x))_{q\in\mathbb{R}^n}$ can be obtained by solving for q_n:

Lemma 3.3.11. *The locus of roots*

$$\Gamma_n := \{(q,x) \in \mathbb{R}^{n+1} \mid F_n(q,x) = 0\},$$

can be explicitly parametrised as the graph of:

$$\begin{array}{rcl} s_n : \mathbb{R}^n & \to & \Gamma_n \subset \mathbb{R}^{n+1} \\ (\tilde{q}, x) & \to & (\tilde{q}, Q_n(\tilde{q}, x) = -x^{n+1} - q_1 x^{n-1} - \cdots - q_{n-1}x, x). \end{array}$$

We regard Γ_n as the graph of a multiply-valued function over the q coordinates. Indeed, for each q, the corresponding polynomial $x \to F_n(q,x)$ has at most $n+1$ real roots, so Γ_n may be thought of as a function of q with finitely many values.

The locus of roots Γ_n has a stratification given by the multiplicity of the root. By definition, the locus $\Gamma_n^j \subset \Gamma_n$ of roots of multiplicity at least j is the common zero of the functions

$$F_n, \frac{\partial F}{\partial x}, \ldots, \frac{\partial^{j-1} F}{\partial^{j-1} x}.$$

3.3.3.2 The definition

We can project down the locus of roots Γ_n to the q-coordinates, yielding:

Definition 3.3.12. *The n**-th Whitney singularity** is the germ at the origin of the map:*

$$\begin{array}{c} \mathrm{Whit}_n : \mathbb{R}^n \to \mathbb{R}^n \\ (\tilde{q}, x) \to (\tilde{q}, Q_n(\tilde{q}, x) = -x^{n+1} - q_1 x^{n-1} - \cdots - q_{n-1}x). \end{array} \tag{3.3.3.1}$$

*The number n is called the **index** of* Whit_n*. For $n = 1, 2$, these maps are referred to as the **fold** and the **pleat**, respectively.*

The maps Whit_n, just like their parametrizations s_n, are fibered over $\mathbb{R}^{n-1}$. I.e. we think of Whit_n as an $\mathbb{R}^{n-1}$-family of maps $\mathbb{R} \to \mathbb{R}$. Note that the higher singularity locus $\Sigma^2(\mathrm{Whit}_n)$ are empty.

We also need to introduce:

Definition 3.3.13. *The **i-fold stabilisation** of* Whit_n *is the map:*

$$\begin{array}{rcl} \mathbb{R}^{n+i} & \to & \mathbb{R}^{n+i} \\ (q_0, \tilde{q}, x) & \to & (q_0, \mathrm{Whit}_n(\tilde{q}, x)),. \end{array}$$

H. Whitney proved in [113] that Whit_n is a stable map, which corresponds to the fact that F_n is a versal unfolding. In [93], Morin proved a converse:

Theorem 3.3.14. *The germ at $p \in \Sigma^1(f)$ of a stable map $f : M \to N$, between manifolds of the same dimension, is left-right equivalent to the germ at the origin of a stabilisation of a Whitney map.*

Let us remark that the left and right actions on germs are allowed not to fix the origin; otherwise, the orbit of the Whitney map of index l would have codimension l in the space of all germs.

3.3.3.3 Some properties

We claim that the singularity locus

$$\Sigma(\mathrm{Whit}_n) = \{(\tilde{q}, x) \mid \frac{\partial Q_n}{\partial x}(\tilde{q}, x) = 0\}$$

coincides with the locus of roots:

Lemma 3.3.15. *For each positive integer j:*

$$\Sigma^{1^j}(\mathrm{Whit}_n) = \Sigma^{1^j}(s_n, \langle\partial_x\rangle) = \Gamma_n^{j+1} \subset \mathbb{R}^{n+1}.$$

Proof. The first identity follows by definition, so we focus on the second one. We work inductively on j. The induction hypothesis is that the claim holds. I.e. $\Sigma^{1^j}(s_n, \langle\partial_x\rangle)$ is the locus of common zeroes of

$$F_n, \frac{\partial F}{\partial x}, \dots, \frac{\partial^{j-1} F}{\partial^{j-1} x}.$$

$\Sigma^{1^{j+1}}(s_n, \langle\partial_x\rangle)$ is the locus of tangency of $\Sigma^{1^j}(s_n, \langle\partial_x\rangle)$ with the x-directions. Therefore, the 1-forms

$$dF_n, d\frac{\partial F}{\partial x}, \dots, d\frac{\partial^{j-1} F}{\partial^{j-1} x}$$

vanish precisely when restricted to $\Sigma^{1^{j+1}}(s_n, \langle\partial_x\rangle)$ and evaluated on ∂_x. But precisely:

$$d\frac{\partial^l F}{\partial^l x}(\partial_x) = \frac{\partial^{l+1} F}{\partial^{l+1} x}.$$

□

Invoking Theorem 3.3.14 we see that the Whitney singularities are adjacent to one another:

Lemma 3.3.16. *At each point in $\Sigma^{1^j 0}(\mathrm{Whit}_n)$, the map Whit_n is equivalent to the $(n-l)$-fold stabilisation of Whit_j.*

For instance, a pleat consists of two folds coming together in a birth/death phenomenon.

3.3.4 The equidimensional wrinkle

The fold and its stabilisations are the simplest (non-trivial) singularities of equidimensional maps. Ideally, we would work in the category of folded maps. However, this is not possible when we consider families of maps: we must, at the very least, allow folds to appear and disappear in birth/death events, i.e. pleats. We will want the pleat locus itself to be a closed submanifold. This leads to the definition:

Definition 3.3.17. *The (equidimensional)* ***wrinkle*** *is the map*

$$\begin{aligned} \mathrm{Wrin}_n : \mathcal{O}p(\mathbb{D}^n) &\to \mathbb{R}^n \\ (q,x) &\mapsto \left(q, w(q,x) = \frac{x^3}{3} + (|q|^2-1)x\right). \end{aligned} \tag{3.3.4.1}$$

The region bounded by the singular locus, i.e. the interior of the disc $\mathbb{D}^n$ in the domain, is called the ***membrane*** *of the wrinkle.*

In Subsection 3.4.4 we will introduce wrinkled submersions, which will be maps locally modelled on the wrinkle. All these notions were introduced by Y. Eliashberg and N. Mishachev in [40] to prove Theorem 3.4.18, which computes the homotopy type of the space of wrinkled submersions.

3.3.4.1 Singularity locus

We see that Wrin_n is a map fibered over $\mathbb{R}^{n-1}$. Its singularities (which are of corank 1) correspond to the vanishing of $\frac{\partial w}{\partial x} = x^2+|q|^2-1$, i.e. the unit sphere $\Sigma(\mathrm{Wrin}_n) = \mathbb{S}^{n-1}$ bounding the membrane. If we further restrict Wrin_n to $\Sigma(\mathrm{Wrin}_n)$ we observe that its singularities live in $\{x = 0\}$, i.e. the equator $\Sigma^{11}(\mathrm{Wrin}_n) = \mathbb{S}^{n-2}$. The map $\mathrm{Wrin}_n|_{\Sigma^{11}(\mathrm{Wrin}_n)}$ is non-singular so

$$\Sigma(\mathrm{Wrin}_n) = \Sigma^{10}(\mathrm{Wrin}_n) \cup \Sigma^{11}(\mathrm{Wrin}_n).$$

Thus, the equator is a codimension-2 sphere of pleats and the two open hemispheres consist of folds. Each two points in $\Sigma^{10}(\mathrm{Wrin}_n)$ sharing the same q–coordinate are a local maximum and a local minimum of the corresponding function $x \to \frac{x^3}{3} + (|q|^2 - 1)x$. As we move in q towards the equator $\Sigma^{11}(\mathrm{Wrin}_n)$, these two points collapse in a birth/death event. Hence, the singularities of the wrinkle are seemingly in cancelling position, but not really: the domain of definition of Wrin_n is not the whole of $\mathbb{R}^n$ (in which the cancellation is possible) but a small neighbourhood of the unit ball $\mathbb{D}^n$.

3.3.4.2 Formal desingularisation

Nonetheless, the singularities of the wrinkle are homotopically inessential from the point of view of obstruction theory: consider the homotopy of functions

$$W_s(q,x) = (x^2 + |q|^2 - 1) + s\rho(q,x), \qquad s \in [0,1],$$

where $\rho : \mathcal{O}p(\mathbb{D}^n) \to [0,\infty)$ is a non-negative function which is greater than 1 over $\mathbb{D}^n$ and identically zero in a neighbourhood of the boundary of its domain. It provides a compactly-supported homotopy between $W_0 = \frac{\partial w}{\partial x}$ and a strictly positive function. We can use W_s to construct a compactly-supported homotopy between the differential $T\mathrm{Wrin}_n$ and a bundle monomorphism. Indeed, we keep the formal derivatives of Wrin_n with respect to the q-coordinates fixed, and we homotope the formal derivative with respect to x using W_s. We call this the **formal desingularisation**. Its existence implies that the wrinkle, as a singularity, represents a trivial class (relative to the boundary of the model).

3.3.4.3 The fibered nature of a wrinkle

Let us regard the wrinkle Wrin_{n+k} as a fibered over $\mathbb{R}^k$ map. Explicitly, we write (q_0) for the coordinates in $\mathbb{R}^k$ and (q_1, x) for the coordinates in $\mathbb{R}^n$. The restriction of Wrin_{n+k} to the fibre over a fixed q_0 is left-right equivalent to Wrin_n if $|q_0| < 1$, non-singular if $|q_0| > 1$, and left-right equivalent to:

$$\begin{aligned} \mathcal{O}p(\{0\}) &\to \mathbb{R}^n \\ (q_1, x) &\mapsto (q_1, \frac{x^3}{3} + |q_1|^2 x), \end{aligned} \tag{3.3.4.2}$$

if $|q_0| = 1$. This singularity is called the **embryo**. It is precisely the event in which a wrinkle Wrin_n is born. It follows from the previous subsection that the embryo can be formally desingularised in a unique manner up to homotopy.

3.3.5 Double folds, wrinkles, and surgery

A wrinkle has non-empty Σ^{11}-locus. Sometimes, it is useful to work with maps whose singularity locus is just Σ^{10}; we call such maps, *folded.* A key idea in wrinkling is that one may produce a folded map out of a wrinkled map using *surgery of singularities* [35, 44]. Conversely, one can pass from a map having *double folds*, defined below, to a wrinkled map by a procedure called *wrinkle chopping* (but we will not explore this). Hence, wrinkles and double folds are essentially equivalent.

3.3.5.1 The definition

Definition 3.3.18. *We define the **double fold** to be the map:*

$$\begin{aligned} f : \mathcal{O}p(\mathbb{S}^{n-1} \times [-1, 1]) &\to \mathbb{R}^n \\ (q, x) &\mapsto (q, \frac{x^3}{3} - x). \end{aligned} \tag{3.3.5.1}$$

*The region bounded by the singular locus, i.e. the open annulus $\mathbb{S}^{m-1} \times (-1, 1)$ in the domain, is called the **membrane** of f.*

The singularity locus $\Sigma(f) = \Sigma^{10}(f)$ is the union of the spheres bounding the membrane

$$\left\{ \frac{\partial f}{\partial x} = x^2 - 1 \right\} = (\mathbb{S}^{n-1} \times \{-1\}) \cup (\mathbb{S}^{n-1} \times \{+1\}).$$

At each sphere the singularity is modelled on (a stabilisation of) the usual fold. Like the wrinkle, the two fold points sharing the same q-coordinate seem to be in cancelling position, but they are not due to the size of the domain.

We often speak of the spheres $\mathbb{S}^{n-1} \times \{\pm 1\}$ as being the double fold, leaving the existence of the membrane bounding them implicit. We could also define the folds to take place along hypersurfaces other than spheres, but for our purposes this is unnecessary.

3.3.5.2 Embryos

Just like wrinkles are born in an embryo event, we may define the analogous birth/death singularity for double folds. It is given by the following expression:

$$
\begin{aligned}
f : \mathcal{O}p(\mathbb{S}^{n-1} \times \{0\}) &\to \mathbb{R}^n \\
(q, x) &\mapsto (q, x^3),
\end{aligned}
\tag{3.3.5.2}
$$

which we call the (double fold) **embryo**. It is simply a parametric version of the 1-dimensional birth/death critical point.

3.3.5.3 Surgery: opening a wrinkle

Consider the wrinkle Wrin_n, whose domain of definition is $\mathcal{O}p(\mathbb{D}^n)$. We may find an $(n-1)$-disc $D \subset \mathcal{O}p(\mathbb{D}^n)$ satisfying:

- ∂D is the equator $\mathbb{S}^{n-2}$ (consisting of pleats),
- D intersects the unit sphere $\mathbb{S}^{n-1}$ transversally at its boundary,
- the interior of D is disjoint from the unit ball.

One may picture D running very closely along the northern/southern hemisphere.

We want to modify the map Wrin_n in $\mathcal{O}p(D)$ to yield instead a double fold. Indeed, we can find coordinates (q, x) in a neighbourhood of $\mathcal{O}p(D)$, with values in $\mathbb{D}^{n-1}_{1+\delta} \times [-\delta, \delta]$, such that:

- $\mathbb{D}^{n-1}_1 \times \{0\} = D$,
- the map Wrin_n reads

$$(q, x) \to (q, \frac{x^3}{3} + (1 - |q|^2)x)$$

I.e. at the boundary of D, which was the equator of the wrinkle, a pair of folds appears in a birth-death event.

Consider the piecewise smooth family of functions

$$
\begin{aligned}
x &\to \frac{x^3}{3} + (1 - |q|^2)x, && \text{if } |q|^2 > 1 + \delta^2/2 \\
x &\to \frac{x^3}{3} - \delta^2 x/2, && \text{otherwise.}
\end{aligned}
$$

It may be smooothed at $|q|^2 = 1 + \delta^2/2$ to yield a smooth family f_q with a double fold. Replacing Wrin_n in $\mathcal{O}p(D)$ by the map $(q, x) \to (q, f_q(x))$ yields a map with a double fold. Each fold is a smoothing of the union of a slight push-off of D and one of the hemispheres of the original wrinkle.

3.3.6 The (first order) wrinkle in positive codimension

The main result in this chapter (see Section 3.8) says that one can control the r-jet of an embedding/submanifold as long as one is allowed to introduce (simple) singularities. The case of 1-jets was studied by Y. Eliashberg and N. Mishachev in [44], and it relies on a particular model of singularity, which we now review.

3.3.6.1 The definition

Definition 3.3.19. *We define the* ***wrinkle*** *(of dimension m into $n > m$, and of order 1) to be the map*

$$\mathrm{Wrin}_{m,n} : \mathcal{O}p(\mathbb{S}^{m-1}) \to \mathbb{R}^n$$

$$(q,x) \to (q, x^3 + 3(|q|^2 - 1)x, \int_0^x (s^2 + |q|^2 - 1)^2 ds, 0, \dots, 0).$$

Its projection to $\mathbb{R}^m$ is (the germ along the unit sphere of) the wrinkle Wrin_m between equidimensional manifolds.

Observe that $\mathrm{Wrin}_{m,n}$ is not defined in the interior of the disc $\mathbb{D}^m$. The reason for this is that, in our constructions, wrinkles will be **nested** inside one another. We will assume that the wrinkles bound a disc (which we still call the **membrane**) in whatever manifold we are working in, but the membrane might contain further wrinkles.

In Subsection 3.4.4 we will introduce (first order) wrinkled embeddings, which will be locally modelled on $\mathrm{Wrin}_{m,n}$.

3.3.6.2 Singularity locus

The $(m+1)$th coordinate of $\mathrm{Wrin}_{m,n}$ is a function that has exactly the same singularity locus as Wrin_m. Therefore, the singularity locus $\Sigma(\mathrm{Wrin}_{m,n})$ is the unit sphere $\Sigma^1(\mathrm{Wrin}_{m,n}) = \mathbb{S}^{m-1}$. It is the union of the equator $\Sigma^{11}(\mathrm{Wrin}_{m,n}) = \mathbb{S}^{m-2}$ and its complement $\Sigma^{10}(\mathrm{Wrin}_{m,n})$. The singularity along $\Sigma^{10}(\mathrm{Wrin}_{m,n})$ is a stabilisation of the usual planar semicubic cusp. The families of cusps in each hemisphere approach each other at the equator $\Sigma^{11}(\mathrm{Wrin}_{m,n})$, cancelling in a sphere of open semicubic swallowtails.

3.3.6.3 Regularisation

Unlike Wrin_n, the wrinkle $\mathrm{Wrin}_{m,n}$ is not stable as soon as $m < n$. Indeed, the small perturbation:

$$(q,x) \to (q, x^3 + 3(|q|^2 - 1)x, \varepsilon x + \int_0^x (s^2 + |q|^2 - 1)^2 ds, 0, \dots, 0)$$

is a smooth embedding. A cut-off may be applied to make this perturbation compactly supported. This smoothing process is unique up to isotopy (which may also be assumed to be compactly supported); we call it the **regularisation**.

3.3.6.4 The Gauss map

Despite being singular, $\mathrm{Wrin}_{m,n}$ has a well-defined **Gauss map** $\mathrm{Gr}(\mathrm{Wrin}_{m,n})$, i.e. a lift to the space of 1-jets of submanifolds; see subsection 3.2.3.2. This is clear along the cusp locus $\Sigma^{10}(\mathrm{Wrin}_{m,n})$, because the planar cusp has a well-defined tangent line at every point. We claim that the same is true along the swallowtail region $\Sigma^{11}(\mathrm{Wrin}_{m,n})$. This is a simple computation, but we will justify it, in the setting of integral submanifolds of general jet spaces, in Subsection 3.7.1.

3.3.6.5 Embryos

Just as in the equidimensional setting, we may think of the wrinkle $\mathrm{Wrin}_{k+m,k+n}$ as a fibered over $\mathbb{R}^k$ map. We write (q_0) for the coordinates in $\mathbb{R}^k$ and (q_1, x) for those in $\mathbb{R}^m$. For $|q_0| < 1$ given, the restriction of $\mathrm{Wrin}_{k+m,k+n}$ to the fibre over q_0 is left-right equivalent to $\mathrm{Wrin}_{m,n}$. For $|q_0| > 1$, it has no singularities. Lastly, for $|q_0| = 1$ the map is equivalent to:

$$(q_1, x) \to (q_1, x^3 + 3|q_1|^2 x, \int_0^x (s^2 + |q_1|^2)^2 ds, 0, \ldots, 0).$$

whose only singularity is the origin. This is exactly the birth/death phenomenon for $\mathrm{Wrin}_{m,n}$, which we also call **embryo**. It can be regularised as above.

3.4 Overview: The h-principle

The h-principle is a collection of techniques and heuristic approaches whose purpose is to describe the space of solutions of a partial differential relation/equation. The main results of this chapter (see Section 3.8) are of this type.

In this Section we provide a quick overview of some of the h-principle techniques that we will need. We first review some of the necessary language (Subsection 3.4.1). Then we go over some classic techniques: *holonomic approximation* in Subsection 3.4.2, *triangulations in general position* in Subsection 3.4.3, and *wrinkling* in Subsection 3.4.4.

For a panoramic view of h-principles we refer the reader to the two standard texts [43] and [60] (which we do suggest to check in that order). Wrinkling techniques were introduced first in the *wrinkling saga* [40, 42, 41, 45, 44].

3.4.1 Differential relations

Having looked at jet spaces in Section 3.2, let us explain how they fit in the geometric formalism of PDEs: given smooth bundles $Y \to X$ and $Y' \to X$, a **local partial differential operator** is a map taking sections $\Gamma(Y)$ to sections $\Gamma(Y')$ which, in local coordinates, may be written as a function of the section and its derivatives up to a given order. Given a section $g \in \Gamma(Y')$, we can define a PDE whose solutions are the sections $f \in \Gamma(Y)$ that are mapped to g.

Now, there exists a local differential operator of order r which is universal. Namely, the map

$$j^r : \Gamma(Y) \to \Gamma(J^r(Y \to X))$$

collects all the derivatives up to order r, so any other local order-r differential operator $\Gamma(Y) \to \Gamma(Y')$ can be decomposed as $A \circ j^r$; here A is the lift of a bundle map $J^r(Y \to X) \to Y'$, which we also denote by A. Given g as above, we can simply look at the subset $A^{-1}(g) \subset J^r(Y \to X)$. A section f of Y such that $j^r f$ is contained in $A^{-1}(g)$ is a **solution** of our PDE.

In more formal terms:

Definition 3.4.1. *Let $Y \to X$ be a smooth fibre bundle.*

- *A **partial differential relation** (PDR) of order r is a subset $\mathcal{R} \subset J^r(Y \to X)$.*
- *The PDR $\mathcal{R}$ is said to be **open** if it is open as a subset.*
- *A section $F : X \to \mathcal{R}$ is said to be a **formal solution** of $\mathcal{R}$.*
- *A section $f : X \to Y$ is a **solution** if $j^r f$ is a formal solution.*

We have presented the framework of PDRs of sections, but we could do the same for PDRs of n-submanifolds by looking at subsets of $J^r(Y, n)$, with Y a manifold and $n < \dim(Y)$.

3.4.1.1 The h-principle

The goal of the h-principle is to determine the homotopy type of the space of solutions of a given PDR $\mathcal{R}$. For this, we need to fix topologies on the spaces $\Gamma(J^r(Y \to X))$ and $\Gamma(Y)$: on $\Gamma(J^r(Y \to X))$ we consider the C^0-topology. Using the map j^r, this induces the Whitney C^r-topology on $\Gamma(Y)$. This choice makes j^r continuous.

We write $\mathrm{Sol}^f(\mathcal{R})$ for the subspace of sections in $\Gamma(J^r(Y \to X))$ whose image lies in $\mathcal{R}$, i.e. the space of formal solutions. Similarly, we write $\mathrm{Sol}(\mathcal{R})$ for the space of solutions, which is a subspace of $\Gamma(Y)$. Then, we may look at the forgetful map:

$$\begin{array}{rcl} \iota_{\mathcal{R}} : \mathrm{Sol}(\mathcal{R}) & \to & \mathrm{Sol}^f(\mathcal{R}) \\ f & \to & \iota_{\mathcal{R}}(f) := j^r f. \end{array}$$

Definition 3.4.2. *We say that the **(complete) h-principle** holds for $\mathcal{R}$ if $\iota_{\mathcal{R}}$ is a weak homotopy equivalence.*

In particular, if $\mathrm{Sol}^f(\mathcal{R})$ is non-empty and the h-principle holds, solutions of $\mathcal{R}$ do exist.

3.4.1.2 The space of solutions

By construction PDRs are local so, given open sets $U \subset V \subset X$, we have restriction maps $\mathrm{Sol}(\mathcal{R}) \to \mathrm{Sol}(\mathcal{R}|_V) \to \mathrm{Sol}(\mathcal{R}|_U)$. Hence, $\mathrm{Sol}(\mathcal{R})$ is the space of global sections of the **sheaf of topological spaces** which assigns to each open subset in X the space of solutions $\mathrm{Sol}(\mathcal{R}|_U)$.

We may then pass to the **étale space** viewpoint, regarding $\mathrm{Sol}(\mathcal{R})$ as the space of germs of solutions of $\mathcal{R}$. It has the structure of a (non-second-countable, non-Hausdorff) manifold such that the projection $\pi : \mathrm{Sol}(\mathcal{R}) \to X$ is a local diffeomorphism. Local sections of $\mathrm{Sol}(\mathcal{R}) \to X$ correspond to local solutions of $\mathcal{R}$. Using this point of view, the Whitney topology on the space of solutions is forgotten.

Instead, we can endow the étale space $\mathrm{Sol}(\mathcal{R})$ (as a set) with the structure of a **C^r-diffeological space** [68]. That is, given any smooth manifold K, we have a distinguished subset $C^r(K, \mathrm{Sol}(\mathcal{R}))$ of $\mathrm{Maps}_{\mathrm{Set}}(K, \mathrm{Sol}(\mathcal{R}))$; a map in this subset is said to be a (C^r-)*plot*. In this case, a map is a plot if it can be extended to a K-family of local solutions of $\mathcal{R}$ which is C^r in the parameter. Observe that precomposing a C^r-plot by an actual C^r-map is still a C^r-plot. In this manner we encode the C^r-topology, but we forget the étale one.

This discussion applies as well to the étale space associated to $\mathrm{Sol}^f(\mathcal{R})$. It is endowed with the C^0-diffeology and the étale topology.

Remark 3.4.3. The subtlety here is that, even though $\mathrm{Sol}(\mathcal{R})$ and $\mathrm{Sol}^f(\mathcal{R})$ are sheaves of topological spaces, their stalks are not topological spaces themselves because we are taking a direct limit. We are forced to look at diffeological or quasi-topological spaces to work with germs of sections [60, Sections 1.4 and 2.2]. △

When we think of $\mathrm{Sol}(\mathcal{R})$ and $\mathrm{Sol}^f(\mathcal{R})$ as sheaves of topological spaces, the map $\iota_{\mathcal{R}}$ becomes a sheaf morphism. Hence, it is a continuous map, for the étale topology, between the corresponding étale spaces. It also takes plots to plots, so we can say that it is **continuous** with respect to the diffeological structures.

In this text we will work with all these different structures on $\mathrm{Sol}(\mathcal{R})$ and $\mathrm{Sol}^f(\mathcal{R})$. Usually, unless stated otherwise, they just denote the spaces of (global) sections and not their sheaf or étale space structures.

3.4.1.3 Flavours of h-principle

Often, one is unable to prove that $\iota_{\mathcal{R}}$ is a weak homotopy equivalence, but partial results hold. For instance, if $\iota_{\mathcal{R}}$ is surjective at the level of connected components, we say that the **existence h-principle** holds. Similarly, if $\iota_{\mathcal{R}}$ is a bijection of connected components, we may say that the h-principle holds in π_0; analogous statements hold for higher homotopy groups.

Furthermore, we may ask whether the h-principle holds over each open set $U \subset X$ in a way that is coherent with respect to the sheaf structure. This can be phrased as follows. The h-principle is **relative in the domain** when: any family of formal solutions of $\mathcal{R}|_U$, which are already honest solutions in a neighbourhood of a closed set

$A \subset U$, can be homotoped to become solutions over the whole of U while remaining unchanged over $\mathcal{O}p(A)$. I.e. the homotopy equivalences between solutions and formal solutions can be assumed to respect closed subsets where they already hold.

Similarly, the h-principle is **relative in the parameter** when: any family of formal solutions $\{F_k\}_{k \in K}$, parametrised by a closed manifold K, and with $F_{k'}$ holonomic for every k' in an open neighbourhood of a fixed closed subset $K' \subset K$, can be homotoped to be holonomic relative to $\mathcal{O}p(K')$.

Lastly, we say that the h-principle is **C^0-close** if the zeroeth order part of any formal solution can be approximated by a genuine solution.

3.4.1.4 Local integrability

We momentarily forget the étale topology on the étale space $\mathrm{Sol}(\mathcal{R})$; we are only interested in the diffeological structure of the germs. We look at the forgetful map $\mathrm{Sol}(\mathcal{R}) \to J^r(Y \to X)$ which evaluates a germ to the corresponding r-jet at the point in which it is defined. This map then necessarily takes values in $\mathcal{R}$.

Definition 3.4.4. *A PDR $\mathcal{R}$ is **locally integrable** if $\mathrm{Sol}(\mathcal{R}) \to \mathcal{R}$ is a Serre fibration with weakly contractible fibres.*

In particular, $\mathrm{Sol}(\mathcal{R})$ is weak homotopy equivalent to $\mathcal{R}$ itself.

One can equivalently rephrase this as follows: any finite dimensional family of pointwise formal solutions can be extended, relative in the parameter, to a family of solution germs. This extension is unique up to homotopy.

3.4.1.5 Flexibility, microflexiblity

Local integrability takes into account only the stalks of the sheaf $\mathrm{Sol}(\mathcal{R})$, i.e. germs at individual points. One would like to look at germs along higher dimensional submanifolds of X. We denote Θ_l for the pair

$$(A = [-1,1]^n, B = \partial(A) \cup ([-1,1]^l \times \{0\})).$$

Definition 3.4.5. *A PDR $\mathcal{R}$ is **microflexible** if, for any:*

- *ball U and integer m,*
- *embeddings $(h_p)_{p \in [0,1]^m} : \Theta_l \to U$,*
- *holonomic sections $(F_p)_{p \in [0,1]^m} : \mathcal{O}p(h_p(A)) \to \mathcal{R}$*
- *homotopy of holonomic sections*

$$(\tilde{F}_{p,s})_{p \in [0,1]^m, s \in [0,1]} : \mathcal{O}p(h_p(B)) \to \mathcal{R}$$

satisfying $\tilde{F}_{p,s} = F_p$ for $s = 0$ or $p \in \mathcal{O}p([0,1]^m)$.

There exists $s_0 \in (0,1]$ *and a family of holonomic sections*

$$(F_{p,s})_{p\in[0,1]^m, s\in[0,s_0]} : \mathcal{O}p(h_p(A)) \to \mathcal{R}$$

satisfying:

- $F_{p,s} = \tilde{F}_{p,s}$ *in* $\mathcal{O}p(B)$.
- $F_{p,s} = F_p$ *for every* $p \in \mathcal{O}p([0,1]^m)$.

We say that $\mathcal{R}$ *is* ***flexible*** *if* s_0 *can taken to be* 1.

That is, a PDR is microflexible if any local deformation of a solution can be extended to a global deformation, at least for small times. Note that PDRs that are open are immediately microflexible and locally integrable.

3.4.1.6 Natural bundles and PDRs

Some PDRs can be intrinsically formulated, without referring to the particular manifold in which they live. Identically, they are invariant under diffeomorphisms, so the corresponding spaces of solutions are naturally endowed with an action by the diffeomorphism group. PDRs of this type are ubiquitous in geometry and they will play a role later on. Let us formalise this idea; we refer to [97].

Definition 3.4.6. *A* ***natural fibre bundle*** *is a functor* F *from the category of* n*-manifolds (where we take morphisms to be embeddings) to the category of fibre bundles (with morphisms being fibrewise diffeomorphisms lifting embeddings), satisfying:*

- $F(X)$ *is a fibre bundle over* X.
- $F(f : X \to X')$ *covers* f.

We sometimes abuse notation and say that a particular $F(X)$ is a natural fibre bundle (but we implicitly remember the rest of the data). Observe then that the pseudogroup $\mathrm{Diff}_{\mathrm{loc}}(X)$ acts on $F(X)$. The bundles associated to the tangent bundle (frame bundles, the cotangent bundle, wedge products, symmetric products) are all examples.

Given F, we can take r-jets, yielding a new functor j^rF which is still natural. The map $F \to j^rF$ is itself a natural transformation; a fibrewise diffeomorphism f is mapped to the corresponding point transformation j^rf.

Definition 3.4.7. *A* Diff***-invariant*** *PDR (of order* r *and for* n*-manifolds) is a natural fibre bundle* $\mathcal{R}$ *together with a natural transformation* $\mathcal{R} \to j^rF$*, for some* F*, which realises* $\mathcal{R}(X)$ *as a subbundle of* $j^rF(X)$*, for any* n*-manifold* X.

In particular, $\mathcal{R}(X)$ is preserved by the action of $\mathrm{Diff}_{\mathrm{loc}}(X)$. This naturality allows us to abstract the relation from the particular manifold in which it lives.

3.4.2 Holonomic approximation

One of the cornerstones of the classical theory of h-principles is the holonomic approximation theorem. It states that any formal section of a jet bundle can be approximated by a holonomic one in a neighbourhood of a perturbed CW-complex of codimension at least 1. The precise statement reads as follows:

Theorem 3.4.8 ([43]). *Let $Y \to X$ be a fiber bundle, K a compact manifold, $A \subset M$ a polyhedron of positive codimension, and $(F_{k,0})_{k\in K} : X \to J^r(Y \to X)$ a family of formal sections. Then, for any $\varepsilon > 0$ there exists*

- *a family of isotopies $(\phi_{k,t})_{t\in[0,1]} : X \to X$,*
- *a homotopy of formal sections $(F_{k,t})_{k\in K, t\in[0,1]} : X \to Y$,*

satisfying:

- *$F_{k,1}$ is holonomic in $\mathcal{O}p(\phi_{k,1}(A))$,*
- *$|\phi_{k,t} - \mathrm{id}|_{C^0} < \varepsilon$ and is supported in a ε-neighbourhood of A,*
- *$|F_{k,t} - F_{k,0}|_{C^0} < \varepsilon$.*

Moreover the following hold:

- *If $V \in \mathfrak{X}(\mathcal{O}p(A))$ is a vector field transverse to A, then we can arrange that $\phi_{k,t}(A)$ is transverse to V for all t and k.*
- *If the $F_{k,t}$ are already holonomic in a neighborhood of a subcomplex $B \subset A$, then we can take $F_{k,t} = F_{k,0}$ and $\phi_{k,t} = \mathrm{id}$ on $\mathcal{O}p(B)$, for all k.*
- *If $F_{k,t}$ is everywhere holonomic for every k in a neighbourhood of a CW-complex $K' \subset K$, then we can take $F_{k,t} = F_{k,0}$ and $\phi_{k,t} = \mathrm{id}$ for $k \in \mathcal{O}p(K')$.*

Remark 3.4.9. Note that in the above statement, the equations

$$|\phi_{k,t} - \mathrm{id}|_{C^0} < \varepsilon, \quad |F_{k,t} - F_{k,0}|_{C^0} < \varepsilon,$$

depend on a choice of Riemannian metric on X and Y. Alternatively, these conditions can be phrased using the C^0-Whitney topology. △

For the proof and a much longer account of its history, we refer the reader to [43]. Essentially, this theorem recasts the method of flexible sheaves due to M. Gromov (itself a generalisation of the methods used by S. Smale in his proof of the sphere eversion and the general h-principle for immersions) in a different light. Let us go over the statement.

The starting point is the family of formal sections $F_{k,0}$, which we want to homotope until they become holonomic. This is not possible, but the theorem tells us that at least we can achieve holonomicity in a neighbourhood of a set of positive codimension.

We are not allowed to fix this set. Instead, we begin with a polyhedron A, which we deform in a C^0 small way to yield an isotopic polyhedron $\phi_{k,1}(A)$. This isotopy occurs in the normal directions of A (which we may prefix by taking a transverse vector field V), and essentially produces a copy $\phi_{k,1}(A)$ of A of greater length. This process is called, descriptively, **wiggling**. The space we gain by wiggling is what allows us to achieve holonomicity: the main idea is that, at each point $p \in A$, we approximate $F_{k,0}$ by the corresponding Taylor polynomial $F_{k,0}(p)$ and then we use the directions normal to A to interpolate between these polynomials keeping control of the derivatives. Hence, we can take the $F_{k,t}$ to be arbitrarily close to our initial data, and the wiggling to be C^0-small. However, if we desire better C^0-bounds, we will be forced to wiggle more aggressively, i.e. the isotopies $\phi_{k,t}$ will become C^1-large.

It is additionally possible to achieve a relative statement both in parameter and domain: if all the formal sections $F_{k,t}$ are already holonomic over some region $\mathcal{O}p(B)$, we do not have to perturb them nor wiggle there. Similarly, we can leave the $F_{k,t}$ untouched close to a subset K' of the parameter space as long as the $\{F_{k,t}\}_{\mathcal{O}p(K')}$ are holonomic (everywhere in M).

3.4.3 Thurston's triangulations

An important step in the application of many h-principles (including ours), is the reduction of the global statement (global in the manifold M), to a local statement taking place in a small ball. These reductions allow us not to worry about (global) topological considerations, making the geometric nature of the arguments involved more transparent. Working on small balls (i.e. "zooming-in") usually has the added advantage of making the geometric structures we consider seem "almost constant"; this will play a role later on.

A possible approach to achieve this is to triangulate the ambient manifold M and then work locally simplex by simplex. A small neighbourhood of a simplex is a smooth ball which can be assumed to be arbitrarily small if the subdivision is sufficiently fine; thus, this achieves our intended goal. When we deal with parametric results (phrased using the foliated setup, see subsection 3.2.4), we want to zoom-in in the parameter space too. This requires us to triangulate in parameter directions as well. For us, this means that we must triangulate a foliated manifold in a manner that is nicely adapted to the foliation.

Let $(M, \mathcal{F})$ be a manifold of dimension n endowed with a foliated of rank k . Given a triangulation $\mathcal{T}$, we write $\mathcal{T}^{(i)}$ for the collection of i-simplices, where $i = 0, \ldots, \dim(M) = n$. We think of each i-simplex $\sigma \in \mathcal{T}^{(i)}$ as being parametrised $\sigma : \Delta^i \to M$, where the domain is the standard simplex in $\mathbb{R}^i$. The parametrisation σ allows us to pull-back data from M to Δ^i. In particular, if σ is a top-dimensional simplex, it is a diffeomorphism with its image and we may assume that σ extends to an embedding $\mathcal{O}p(\Delta^n) \to M$ of a ball.

If the image of σ is sufficiently small, we would expect that the parametrisation σ can be chosen to be reasonable enough so that $\sigma^*\mathcal{F}$ is almost constant. This can be phrased as follows:

Definition 3.4.10. *A top-dimensional simplex σ is in **general position** with respect to the foliation $\mathcal{F}$ if the linear projection (identifying $T_p\mathbb{R}^n = \mathbb{R}^n$)*

$$\Delta^n/(\sigma^*\mathcal{F})_p \to \mathbb{R}^{n-k}$$

restricts to a map of maximal rank over each subsimplex of σ. In particular, $\sigma^*\mathcal{F}$ is transverse to each subsimplex.

*The triangulation $\mathcal{T}$ is in **general position** with respect to $\mathcal{F}$ if all of its top-simplices are in general position.*

We may then state:

Theorem 3.4.11. *Let $(M, \mathcal{F})$ be a foliated manifold. Then, there exists a triangulation $\mathcal{T}$ of M which is in general position with respect to $\mathcal{F}$.*

This statement was first stated and proven by W. Thurston in [104, 105], playing a central role in his h-principles for foliations. His statement is slightly more general and works for general distributions, but this is not needed in our setting.

Given a submanifold $N \subset (M, \mathcal{F})$, we cannot expect it to be transverse to $\mathcal{F}$. However, after a C^∞-perturbation, we can assume that the singularities of tangency between N and $\mathcal{F}$ are generic: i.e. they form a smooth manifold whose singularities of tangency are themselves generic. Recursively, this provides a stratification of N in the sense of Thom-Boardman, as described in Subsection 3.3.1. The singular strata may represent non-trivial homology classes in N, which are invariants of N and $\mathcal{F}$ up to homotopy. Non-triviality of these invariants tells us that we cannot homotope N to make it transverse to $\mathcal{F}$.

Suppose now that N is just a piecewise-smooth submanifold. Then, its tangent bundle is not defined everywhere: it has discontinuities and the singularities of tangency are not defined there. One is able, by introducing additional discontinuities, to remove all singularities of tangency. In homotopical terms, there is a classifying map associated to the locus of tangency and one can make the map into classifying space discontinuous by passing to the piecewise category. Discontinuities allow us to "jump" over the homology classes of the classifying space. This is called an *h-principle without homotopical assumptions* [4].

Theorem 3.4.11 was one of the first statements along these lines, where instead of a submanifold we have a triangulation. The argument goes roughly as follows: we start with a triangulation $\mathcal{T}'$. We then subdivide it (in a controlled fashion called *crystalline subdivision*, which ensures that angles remain controlled and that the cardinality of the star of a vertex is uniformly bounded). As we subdivide, the foliation seems progressively flatter from the perspective of each simplex. In particular, the measure of the set of planes that intersect the foliation non-transversally goes to zero. This allows us to apply *Thurston's jiggling*: we tilt slightly the vertices, yielding simplices that are transverse to $\mathcal{F}$.

3.4.4 Wrinkling

Wrinkling is an h-principle method to construct mildly singular solutions of partial differential relations. It has been used by Y. Eliashberg and M. Mishachev to prove flexibility results for submersions [40], equidimensional immersions with prescribed folds [45], foliations [41], and fibrations [38]. It entered the world of Contact Topology with [44], which would then lead to the works of E. Murphy on loose legendrians [94] and D. Álvarez-Gavela on the simplification of front singularities of legendrians [3, 2]. It is also one of the central ingredients in the construction and classification of overtwisted contact structures in all dimensions [15] due to M.S. Borman, Y. Eliashberg, and E. Murphy. More recently, it has been used in Engel Geometry to classify overtwisted Engel structures [31] and integral knots in Engel manifolds [25].

We want to phrase some of these results in a coherent light. In Section 3.8 we will reprove and improve on some of the aforementioned works (namely [44]) to produce and classify embedded integral manifolds in higher jet spaces.

Our approach is more general and, hopefully, more transparent/streamlined than previous iterations in the literature. Despite having a different flavour in implementation, we present no fundamentally new geometric ingredients. Our work owes a lot to the papers cited above.

For the reader to have a somewhat complete picture, let us provide a list of sample theorems on wrinkling. We will refer back to them later on.

3.4.4.1 Wrinkled submersions

Let M and N be n-dimensional manifolds (we assume equidimensionality for simplicity).

Definition 3.4.12. *A **formal submersion** is a bundle map $F : TM \to TN$ which is a fibrewise epimorphism (no assumptions on the underlying map $M \to N$).*

It is well-known that the space of submersions $M \to N$ is not homotopy equivalent to the space of formal submersions if M is closed. Indeed, any map $M \to \mathbb{R}^n$ with M closed must have critical points, so it cannot be submersive.

The first wrinkling result of Y. Eliashberg and M. Mishachev [40] says that one may salvage the h-principle by relaxing the submersion condition. They do so by allowing mild singularities, as introduced in Subsection 3.3.4:

Definition 3.4.13. *A **wrinkled submersion** is:*

- *a map $f : N \to M$ between n-dimensional manifolds,*
- *a finite collection of disjoint open balls $\{B_i\}$,*

such that:

- *f is a submersion in the complement of the B_i.*

- $f|_{B_i}$ *is left-right equivalent to* Wrin_n *(Definition 3.3.17).*

*A **wrinkled submersion with embryos** has an additional collection of balls in which f is modelled by the embryo (Equation 3.3.4.2).*

As explained in subsection 3.3.4.2, the wrinkle and the embryo possess a formal desingularisation: i.e. a homotopy of the formal derivative to a monomorphism. This implies that there is a (well-defined up to homotopy) map from the space of wrinkled submersions with embryos to the space of formal submersions. Then:

Theorem 3.4.14 (Eliashberg and Mishachev [40])**.** *The space of wrinkled submersions with embryos is homotopy equivalent to the space of formal submersions. This h-principle is, additionally, C^0-close.*

We can similarly define **submersions with double folds** to be maps which are submersions in the complement of a finite collection of disjoint annuli in which they are modelled by a double fold. They may additionally have finitely many spheres in which they are modelled by a double fold embryo. Then, using the surgery of singularities from subsection 3.3.5.3 and the previous theorem, one can deduce:

Corollary 3.4.15. *The space of submersions with double folds and embryos is homotopy equivalent to the space of formal submersions. This h-principle is, additionally, C^0-close.*

3.4.4.2 Wrinkled embeddings

Let $M \subset N$ be smooth manifolds with $\dim(M) < \dim(N)$.

Definition 3.4.16. *A **tangential homotopy** $M \to N$ is a family of bundle monomorphisms*

$$(G_s)_{s\in[0,1]} : TM \to TN|_M, \qquad G_0 = \mathrm{id}.$$

In [44], Y. Eliashberg and M. Mishachev study the problem of isotoping M, as an embedded submanifold of N, to approximate a given tangential homotopy (in a holonomic manner). This problem is, in general, obstructed. However, it is solvable if we relax the embedding condition to allow for wrinkle singularities.

We take the local model of a first order, positive codimension wrinkle (from Subsection 3.3.6) and we globalise it as follows:

Definition 3.4.17. *A smooth map $f : M \to N$ is a **wrinkled embedding** if:*

- *it is a topological embedding,*
- *it is a smooth embedding away from a collection of disjoint embedded codimension-1 spheres S_i,*
- $f|_{\mathcal{O}p(S_i)}$ *is left-right equivalent to* $\mathrm{Wrin}_{\dim(M),\dim(N)}$.

A map $f : M \to N$ is a ***wrinkled embedding with embryos*** *if it is a wrinkled embedding in the complement of a finite collection $\{p_i\}$ of points and it is left-right equivalent to an embryo in each neighbourhood $\mathcal{O}p(p_i)$.*

Wrinkles provide enough flexibility to yield the following approximation result, which is both parametric and relative in the parameter and in the domain:

Theorem 3.4.18 (Eliashberg and Mishachev [44]). *Let N and K be smooth manifolds. Let $(M_k)_{k\in K} \subset N$ be a K-family of submanifolds of N. Assume that there is a family of tangential homotopies $(\nu_{k,s})_{k\in K,s\in[0,1]}$ starting at $\nu_{k,0} = TM_k$.*

Then, there is a $K\times[0,1]$-family of wrinkled submanifolds with embryos $(M_{k,s})_{k\in K,s\in[0,1]}$, starting at $M_{k,0} = M_k$, such that $TM_{k,s}$ is C^0-close to $\nu_{k,s}$.

Furthermore:

- *Assume there is a closed submanifold $K' \subset K$ such that $\nu_{k,s} = TM_k$ for every $k \in K'$. Then, we may assume that $M_{k,s} = M_k$ for all $k \in K'$.*
- *Assume there are closed submanifolds $M'_k \subset M_k$ such that $\nu_{k,s}(x) = T_xM_k$ for all $x \in M'_k$. Then we may assume that $M_{k,s}$ agrees with M_k in $\mathcal{O}p(M'_k)$.*

This should be understood as an analogue of the holonomic approximation Theorem 3.4.8. A minor difference is that it is stated for submanifolds as opposed to sections. More importantly, it applies to closed manifolds, and the price paid is that singularities must be introduced. Lastly, it applies only for 1-jets. We extend it to general r-jets; see Section 3.8.

3.5 The integral Grassmannian

Let B and F be vector spaces of dimensions $n = \dim(B)$ and $k = \dim(F)$. We are interested in l-dimensional integral submanifolds of $(J^r(B,F), \xi_{\text{can}})$. Our goal in this Section is to understand their linear counterpart, i.e. the corresponding integral elements.

We will do this step by step, looking first at the horizontal elements (Subsection 3.5.2), then at the elements that intersect the vertical distribution in a given dimension (Subsection 3.5.3), and finally at how these different pieces glue together (Subsections 3.5.4 and 3.5.5).

Let us provide some context about integral manifolds and integral elements: the first to regard general integral submanifolds of jet space as "generalised solutions" seems to have been R. Thom in [24], where he sketched the proof of his famous "homological h-principle". Later, A.M. Vinogradov brought attention to them, in the context of Geometry of PDEs, in [108]. Several works have followed in this direction [9, 10, 109].

It is within the Geometry of PDEs literature [70, 71] that the integral Grassmannian has been studied. As far as the we are aware, the majority of what is currently known can be found in the works of V. Lychagin [79, 78, 80, 81]. Despite containing beautiful results, these articles follow an announcement format and proofs are often missing or just outlined. One of our goals in this Section is to provide a detailed account of Lychagin's work.

We note that our homotopy type computations for the integral Grassmannian in Subsection 3.5.5 seem to be new.

3.5.1 Decomposing the integral Grassmannian

Following subsection 3.2.2.3, we identify the tangent space of $J^r(B,F)$ at any point with the vector space

$$\mathfrak{g} = B \oplus F \oplus \mathrm{Hom}(B,F) \oplus \mathrm{Sym}^2(B^*,F) \oplus \cdots \oplus \mathrm{Sym}^r(B^*,F).$$

In Definition 3.2.38 we endowed $\mathfrak{g}$ with a natural graded Lie algebra structure given by the contraction of vectors with tensors. We called this the jet Lie algebra with parameters n, k, and r. It was then proven in Proposition 3.2.39 that $\mathfrak{g}$ models the nilpotentisation of ξ_{can}. Under this isomorphism, integral elements (of a given dimension l) correspond to Lie subalgebras lying in the zero degree part

$$\mathfrak{g}_0 = B \oplus \mathrm{Sym}^r(B^*,F).$$

The space of integral elements is denoted by $\mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g},l)$. It decomposes into several pieces, depending on how integral elements intersect the vertical component. We define:

$$\mathrm{Gr}_{\Sigma^i}(\mathfrak{g},l) := \{W \in \mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g},l) \mid \dim(W \cap \mathrm{Sym}^r(B^*,F)) = i\},$$

where the subscript Σ^i is inspired by the Thom-Boardman notation.

The piece $\mathrm{Gr}_{\Sigma_0}(\mathfrak{g},l)$ is precisely the horizontal Grassmannian, as introduced in subsection 3.2.2.8. We also call it the **regular cell** even though it is, in general, not dense in $\mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g},l)$. This is shown in subsection 3.5.3.3 below. We will describe the spaces $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g},l)$ in Subsections 3.5.2 and 3.5.3.

3.5.1.1 The grassmannian of multi-sections

In Section 3.6 we will introduce *multi-sections*, i.e. integral submanifolds that are horizontal in a dense set. These are submanifolds that one can manipulate through their front projection. Any integral element tangent to a multi-section must be in the closure $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g},n)}$ of the horizontal elements; we call this space the **Grassmannian of multi-section elements**.

Furthermore, we are interested in multi-sections with mild singularities of tangency, which will be, in particular, of corank 1. Therefore, we content ourselves with describing how the two strata $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g},n)$ and $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g},n)$ glue together.

Definition 3.5.1. *The* Σ^2***-free integral Grassmannian****, is the union*

$$\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g},n) := \mathrm{Gr}_{\Sigma^0}(\mathfrak{g},n) \cup \mathrm{Gr}_{\Sigma^1}(\mathfrak{g},n).$$

We will study its topology in Subsection 3.5.5.

We will study $\mathrm{Gr}_{\mathrm{integral}}(\mathfrak{g}, l)$ as a whole in the future. In particular, in the present work we do not look at the closures $\overline{\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, n)}$ with $i > 1$.

3.5.2 Horizontal elements

We now prove Lemma 3.5.3: the Grassmannians of horizontal elements are vector bundles with (standard) Grassmannian base. This description appeared already in the recent work [10].

3.5.2.1 Maximal horizontal elements

A maximal horizontal element W is graphical over B. We can represent it (uniquely) as the graph of a homomorphism $A \in \mathrm{Hom}(B, \mathrm{Sym}^r(B^*, F))$. Then:

Lemma 3.5.2. *Let $W = \mathrm{graph}(A)$ be a n-dimensional subspace of $\mathfrak{g}_0$ graphical over B. Then, W is integral if and only if $A \in \mathrm{Sym}^{r+1}(B^*, F)$.*

Proof. The Lie subalgebra condition for W means that for any pair $w_0 + A(w_0), w_1 + A(w_1) \in W$ we have:

$$0 = [w_0 + A(w_0), w_1 + A(w_1)] = \iota_{w_0} A(w_1) - \iota_{w_1} A(w_0)$$

which implies that A is symmetric with respect to the first variable as well. The claim follows. ☐

This Lemma realises the correspondence between horizontal elements at a point $p \in J^r(Y \to X)$ and points in the fibre of $J^{r+1}(Y \to X)$ over p.

3.5.2.2 General dimension

More generally, if W is horizontal and of dimension $l \leqslant n$, it projects down to some l-dimensional subspace $H \subset B$, defining a map

$$\pi_b : \mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l) \to \mathrm{Gr}(B, l)$$

to the l-Grassmannian of the base. We claim that this is a vector bundle which can be explicitly described in terms of the tautological bundle γ over $\mathrm{Gr}(B, l)$.

Lemma 3.5.3. *There is a canonical isomorphism of vector bundles over $\mathrm{Gr}(B, l)$:*

$$\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l) \cong \frac{\mathrm{Sym}^{r+1}(B^*, F)}{\mathrm{Sym}^{r+1}(\gamma^\perp, F)},$$

where $\gamma^\perp$ is the annihilator of the tautological bundle γ.

Proof. We look at all the graphical l-subspaces in $\mathfrak{g}_0$, not necessarily integral: given $H \subset B$, its possible lifts correspond to the elements of $\mathrm{Hom}(H, \mathrm{Sym}^r(B^*, F))$. Packaged all together, for varying H, they are elements of the total space of the vector bundle:

$$\mathrm{Hom}(\gamma, \mathrm{Sym}^r(B^*, F)) \to \mathrm{Gr}(B, l).$$

We want to determine which of these are horizontal.

To do so, we use the auxiliary trivial vector bundle $\underline{\mathrm{Sym}^{r+1}(B^*, F)} \to \mathrm{Gr}(B, l)$. We look at the bundle map given by evaluation on each l-subspace:

$$\mathrm{ev}_\gamma : \underline{\mathrm{Sym}^{r+1}(B^*, F)} \subset \underline{\mathrm{Hom}(V, \mathrm{Sym}^r(B^*, F))} \mapsto \mathrm{Hom}(\gamma, \mathrm{Sym}^r(B^*, F)).$$

The image of this map is necessarily contained in $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l)$. We claim that the map is an epimorphism: this follows from the fact that any horizontal W, projecting to $H \subset B$, may be extended to a maximal horizontal element by direct summing with the complement of H in B.

The kernel of ev_γ is, by definition, the subspace of those elements of $\mathrm{Sym}^{r+1}(B^*, F)$ which vanish when a vector in γ is plugged in. By symmetry, we deduce that there is a exact sequence

$$0 \to \mathrm{Sym}^{r+1}(\gamma^\perp, F) \to \underline{\mathrm{Sym}^{r+1}(B^*, F)} \to \mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l) \to 0$$

of vector bundles, proving the claim. □

3.5.2.3 The subspace filtration

Let $H \subset B$ be a linear subspace. In the proof above we looked at those elements in $\mathrm{Sym}^{r+1}(B^*, F)$ which vanish when an element of H is plugged in. One can, more generally, consider those tensors that vanish when a collection of elements in H is used. This leads us to define the following filtration:

$$\mathrm{Sym}^{r+1}(B^*, F)^{(H,j)} := \{A \in \mathrm{Sym}^{r+1}(B^*, F) \mid \iota_{v_j} \cdots \iota_{v_1} A = 0, \text{ for any } v_i \in H\},$$

$$\cdots \subset \mathrm{Sym}^{r+1}(B^*, F)^{(H,j)} \subset \mathrm{Sym}^{r+1}(B^*, F)^{(H,j+1)} \subset \ldots$$

By the discussion in the previous subsection, we have that

$$\mathrm{Sym}^{r+1}(B^*, F)^{(H,1)} = \mathrm{Sym}^{r+1}(H^\perp, F).$$

In general, by choosing a direct summand of H, we can identify:

$$\frac{\mathrm{Sym}^{r+1}(B^*, F)^{(H,j)}}{\mathrm{Sym}^{r+1}(B^*, F)^{(H,j-1)}} \cong \mathrm{Sym}^{j-1}(H^*, F) \otimes \mathrm{Sym}^{r+2-j}(H^\perp, F).$$

yielding the dimension formula:

$$\dim\left(\frac{\mathrm{Sym}^{r+1}(B^*, F)^{(H,j)}}{\mathrm{Sym}^{r+1}(B^*, F)^{(H,j-1)}}\right) = k\binom{n+j-2}{n-1}\binom{n+r+1-j}{n-1}.$$

In Subsection 3.5.4 we will study the *principal cone* in $\mathrm{Sym}^{r+1}(B^*, F)$, i.e. the space of tensors A of the form $A \in \mathrm{Sym}^{r+1}(H^\perp, F)$, for some $H \subset B$.

3.5.2.4 Aside: the conormal

We finish this Subsection presenting the *conormal construction*. Given a horizontal submanifold of $J^r(Y \to X)$, it produces a maximal integral submanifold containing it. This will not be needed later on, but it helps us emphasise that maximal integral submanifolds are often exotic looking (compared to those integral submanifolds that are almost everywhere horizontal).

We first present the linear analogue of this phenomenon:

Definition 3.5.4. *Let $W \subset \mathfrak{g}_0$ be l-dimensional and horizontal. Denoting its projection to B by H, we define the* ***conormal*** *of W to be the subspace:*

$$\text{conormal}(W) := W \oplus \text{Sym}^r(H^\perp, F) \subset \mathfrak{g}_0.$$

The space $\text{Sym}^r(H^\perp, F)$ is the intersection of the polar space of H with the vertical component. Hence, the conormal is a maximal integral element.

In the contact case, conormal(W) is middle-dimensional and therefore a lagrangian subspace of ξ_{can}. In the general case, $\text{Sym}^r(H^\perp, F)$ has dimension $k\binom{(n-l)+r-1}{n-l-1}$ which is often (much) larger than $n - l$. For instance:

- If $l = n - 1$, we have $\dim(\text{Sym}^r(H^\perp, F)) = k$.
- If $l = n - 2$, we have $\dim(\text{Sym}^r(H^\perp, F)) = k(r + 1)$.
- If $l = n - 3$, we have $\dim(\text{Sym}^r(H^\perp, F)) = k\frac{(r+2)(r+1)}{2}$.

Therefore, the conormal construction produces integral elements which are tangent to the fibre along a large subspace, and whose dimension is often much larger than n.

Now for the manifold version:

Definition 3.5.5. *Let $N \subset J^r(Y \to X)$ be a l-dimensional, integral submanifold with immersed projection $\pi_b(N) \subset X$. We define its* ***conormal*** *to be the manifold:*

$$\text{conormal}(N) := \{p \in J^r(Y \to X) \mid \pi_{r,r-1}(p) \in \pi_{r,r-1}(N),\ p \supset T_{\pi_{(r,r-1}(p)}\pi_{r,r-1}(N)\}.$$

In the last inclusion we think of $p \in J^r(Y \to X)$ as a maximal horizontal element in $\pi_{r,r-1}(p) \in J^{r-1}(Y \to X)$.

To see how this corresponds to the linear version, we choose a trivialisation so we may work with $J^r(B, F)$, where B and F are vector spaces. Then the conormal is precisely the space

$$\{p \in J^r(B, F) \mid \pi_{r,r-1}(p) \in \pi_{r,r-1}(N),\ p \in \text{conormal}(T_{\pi_{r,r-1}(p)}\pi_{r,r-1}(N))\}.$$

Here we use the fact that both the base B and the fibre F are vector spaces to canonically identify the fibre of r-jet space with $\text{Sym}^r(B^*, F)$ and therefore invoke the linear definition.

3.5.3 Integral elements of given corank

Having understood the horizontal case (which we will have to invoke repeatedly), we may look now at more general integral elements. Namely, those intersecting the vertical component in a subspace of dimension i.

3.5.3.1 The setup

The space $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$ is endowed with two canonical maps. The first is simply the restriction of the base projection; we denote it by:

$$\pi_b : \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l) \mapsto \mathrm{Gr}(B, l - i).$$

The second one intersects an integral element with the vertical component. We write:

$$\cap \mathrm{Sym}^r(B^*, F) : \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l) \to \mathrm{Gr}(\mathrm{Sym}^r(B^*, F), i).$$

Given $W \in \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$, the subspaces $H = \pi_b(W)$ and $W_v = W \cap \mathrm{Sym}^r(B^*, F)$ must be orthogonal with respect to the curvature/Lie bracket. This means that W_v must be, in fact, an element of $\mathrm{Gr}(\mathrm{Sym}^r(H^\perp, F), i)$. Reasoning in this fashion for all W simultaneously leads us to look at the total space of the bundle $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i) \to \mathrm{Gr}(B, l - i)$. We write ν for the tautological bundle over it.

The two canonical maps defined above yield a projection $\pi : \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l) \to \mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i)$. It is immediate that π is a vector bundle in which a natural choice of zero section is:

$$(H, W_v) \to H \oplus W_v, \tag{3.5.3.1}$$

where $H \in \mathrm{Gr}(B, l - i)$ and $W_v \in \mathrm{Gr}(\mathrm{Sym}^r(H^\perp, F), i)$.

3.5.3.2 The result

We may describe $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$ explicitly:

Lemma 3.5.6. *There is a canonical isomorphism of vector bundles:*

$$\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l) \cong \frac{\mathrm{Sym}^{r+1}(B^*, F)}{\mathrm{Sym}^{r+1}(\gamma^\perp, F) \oplus \mathrm{Hom}(\gamma, \nu)}$$

over the total space of $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i) \to \mathrm{Gr}(B, l - i)$.

Proof. As before denote by $\underline{\mathrm{Sym}^{r+1}(B^*, F)} \to \mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i)$ the trivial vector-bundle, with fiber $\mathrm{Sym}^{r+1}(B^*, F)$. We define a vector bundle epimorphism

$$\oplus : \underline{\mathrm{Sym}^{r+1}(B^*, F)} \mapsto \mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$$

which, at a point $W_v \in \mathrm{Gr}(\mathrm{Sym}^r(H^\perp, F), i)$, is given by

$$A \quad \mapsto \quad \oplus_{H, W_v}(A) := \mathrm{graph}(A|_H) \oplus W_v.$$

The tensor A is in the kernel of $\oplus_{H,W_v}$ (i.e. gets mapped to the zero section from Equation 3.5.3.1) if and only if the associated quotient map

$$\tilde{A} : H \mapsto \mathrm{Sym}^r(B^*, F)/W_v$$

is zero. I.e. $\iota_v A \in W_v$ for every $v \in H$. Therefore, after choosing a direct summand for H, we can identify:

$$\ker(\oplus_{H,W_v}) \cong \mathrm{Sym}^{r+1}(H^\perp, F) \oplus \mathrm{Hom}(H, W_v),$$

which is a vector subspace of $\mathrm{Sym}^{r+1}(B^*, F)^{(H,2)} \cong \mathrm{Sym}^{r+1}(H^\perp, F) \oplus \mathrm{Hom}(H, \mathrm{Sym}^r(H^\perp, F))$. □

3.5.3.3 Dimension counting

From the previous proof, we deduce that:

Corollary 3.5.7. *The fibre of* $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$*, as a vector bundle over* $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i)$*, has dimension*

$$\left[\binom{n+r}{n-1} - \binom{n-l+i+r}{n-l+i-1}\right] k - i(l-i).$$

Similarly, we deduce:

Corollary 3.5.8. *The manifold* $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$ *has dimension*

$$\begin{aligned} \dim(\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)) = \quad & (l-i)(n-l+i) + \\ & \left[\binom{r+(n-l+i)-1}{n-l+i-1} k - i\right] i + \\ & \left[\binom{n+r}{n-1} - \binom{n-l+i+r}{n-l+i-1}\right] k - i(l-i). \end{aligned}$$

Proof. The space $\mathrm{Gr}(B, l-i)$ has dimension $(l-i)(n-l+i)$. The fibre of $\mathrm{Sym}^r(\gamma^\perp, F)$ has dimension $\binom{r+(n-l+i)-1}{n-l+i-1}$, so it follows that the fibre of $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), i)$ has dimension:

$$\left[\binom{r+(n-l+i)-1}{n-l+i-1} k - i\right] i.$$

Putting all these computations together, we deduce the claim. □

We are particularly interested in comparing $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)$ with the regular cell $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l)$, which we want to regard as the "generic" ones. To do so with define a number, which we call the **codimension**, as follows:

$$\mathrm{codim}(r, n, k, l, i) := \dim(\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, l)) - \dim(\mathrm{Gr}_{\Sigma^i}(\mathfrak{g}, l)).$$

We particularise to the case $n = l$ and we compute:

$$\begin{aligned} \mathrm{codim}(r,n,k,n,i) &= i^2 + kr\frac{1-i}{1+r}\binom{r+i-1}{i-1}, \\ \mathrm{codim}(r,n,k,n,1) &= 1, \\ \mathrm{codim}(r,n,k,n,2) &= 4-kr, \\ \mathrm{codim}(r,n,k,n,3) &= 9-kr(r+2). \end{aligned}$$

So we deduce:

Corollary 3.5.9. *The space* $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g},n)$ *has codimension* 1 *in* $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g},n)$.

In the contact setting $k = r = 1$, *the space* $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g},n)$ *has codimension* $\frac{i(i+1)}{2}$ *in* $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g},n)$.

That is: with the exception of a few cases in which r and k are small, the strata $\mathrm{Gr}_{\Sigma^i}(\mathfrak{g},n)$, $i > 1$, are often larger than the regular cell.

The most interesting component, from a PDE perspective, is the closure $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g},l)}$ of the horizontal cell. We will not attempt to look at it in depth. As pointed out in the introduction, it is enough that we understand how $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g},l)$ sits inside; we will do so in Subsection 3.5.5.

3.5.4 Principal subspaces

It is convenient that we introduce some auxiliary concepts before we look at $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g},l) \subset \overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g},l)}$. The main definition of interest in this Subsection is:

Definition 3.5.10. *A horizontal element* $A \in \mathrm{Gr}_{\Sigma^0}(\mathfrak{g},n) \cong \mathrm{Sym}^{r+1}(B^*,F)$ *is* ***principal*** *if*

$$A = f^{r+1} \otimes \alpha,$$

for some (unique) $f \in B^*$ *and* $\alpha \in F$. *The span of a principal element is said to be a* ***principal subspace***.

Any non-zero principal element defines a kernel subspace $\ker(A) := \ker(f) \subset B$ which is of codimension 1, and an image subspace $\mathrm{Image}(A) \subset \mathrm{Sym}^r(B^*,F)$ which is by definition the 1-dimensional space spanned by $f^r \otimes \alpha$.

Remark 3.5.11. As points in $(r+1)$-jet space, principal elements correspond precisely to *pure derivatives* (i.e. derivatives of order $r+1$ along a single direction in the base). △

3.5.4.1 The principal cone

We claim that the set of all principal subspaces in $\mathrm{Sym}^{r+1}(B^*,F)$ is the cone of an algebraic subvariety in the projectivisation. Let us recall two constructions from classic algebraic geometry.

Let V and W be vector spaces. We define the **Veronese mapping**:

$$\begin{array}{rcl} \mathbb{P}(V) & \mapsto & \mathbb{P}(\mathrm{Sym}^{r+1}(V)), \\ [v] & \mapsto & [v^{r+1}]. \end{array}$$

Similarly, the **Segre mapping** is defined by the expression:

$$\begin{array}{rcl} \mathbb{P}(V) \times \mathbb{P}(W) & \mapsto & \mathbb{P}(V \otimes W), \\ ([v],[w]) & \mapsto & [v \otimes w]. \end{array}$$

Both of them are algebraic maps.

In our setting, we can put them together to define the **principal mapping**:

$$\begin{array}{rcl} \mathbb{P}(B^*) \times \mathbb{P}(F) & \mapsto & \mathbb{P}(\mathrm{Sym}^{r+1}(B^*, F)), \\ ([f],[\alpha]) & \mapsto & [f^{r+1} \otimes \alpha]. \end{array}$$

We are interested in the cone it defines. It is given by the image of the map:

$$\begin{array}{rcl} B^* \times F & \mapsto & \mathrm{Sym}^{r+1}(B^*, F), \\ (f,\alpha) & \mapsto & f^{r+1} \otimes \alpha. \end{array}$$

We will abuse notation and still call this map the *principal mapping*, as long as no confusion may arise. Its image, which we denote by $\mathcal{V}_0$ and we call the **principal cone**, is an algebraic subvariety. By construction, a horizontal element is principal if and only if it is contained in $\mathcal{V}_0$.

3.5.4.2 The closure of the principal cone

Fix $A_0, A_1 \in \mathrm{Sym}^{r+1}(B^*, F)$, with A_1 principal, and consider the linear combinations $(A_0 + sA_1)_{s\in\mathbb{R}}$. We can see that

$$(A_0 + sA_1)|_{\ker(A_1)} = A_0|_{\ker(A_1)},$$

i.e. the graph over $\ker(A_1)$ does not depend on s. However, $A_0 + sA_1$ explodes in the complement of $\ker(A_1)$ as s goes to infinity. This implies that the sequence of horizontal elements $(A_0 + sA_1)_{s\in\mathbb{R}}$ has well-defined limit in $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$: the integral element

$$\mathrm{graph}(A_0|_{\ker(A_1)}) \oplus \mathrm{Image}(A_1).$$

In terms of r-jet space, this phenomenon corresponds to an explosion of a pure derivative of order $r+1$. Any element in $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ may be written as such a limit, so we deduce:

Lemma 3.5.12. $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ *is contained in the closure of* $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$*.*

Applying this reasoning with $A_0 = 0$, we are effectively looking at the closure $\mathcal{V} := \overline{\mathcal{V}_0}$ in $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)}$ of the principal cone:

Lemma 3.5.13. *The* ***principal subvariety*** $\mathcal{V}$ *is the union of two pieces* $\mathcal{V}_0$ *and* $\mathcal{V}_1$*. The latter piece is the zero section of* $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ *as a bundle over* $\mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), 1) \to \mathrm{Gr}(B, n-1)$*.*

Proof. Any element in the closure of $\mathcal{V}_0$ can be realised as the limit of a path $(sA)_{s\in\mathbb{R}}$, with A principal. As reasoned above, its limit is then the direct sum $\ker(A) \oplus \mathrm{Image}(A)$, where the first term is a hyperplane in B and the second one is a line in $\mathrm{Sym}^r(\ker(A)^\perp, F)$. This concludes the claim. □

Lastly, we remark that $\mathcal{V}_1 = \mathrm{Gr}(\mathrm{Sym}^r(\gamma^\perp, F), 1)$, as a bundle over $\mathrm{Gr}(B, n-1)$, is trivial. Indeed, an element in the fibre is a line in $\mathrm{Sym}^r(\gamma^\perp, F)$, which can be uniquely identified with its image in F, which is again a line. This shows that:

Corollary 3.5.14. *There is an identification*

$$\mathcal{V}_1 = \mathrm{Gr}(B, n-1) \times \mathrm{Gr}(F, 1) = \mathbb{P}(B^*) \times \mathbb{P}(F).$$

3.5.4.3 The topology of the principal subvariety

We want to determine the homotopy type of $\mathcal{V}$ by putting its pieces together. This is relevant because, as we will see in Subsection 3.5.5.2, $\mathcal{V}$ is homotopy equivalent to $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$.

Let us make a preliminary remark. We write $\widetilde{\mathcal{V}}$ for be the blow-up of $\mathcal{V}$ at the origin. We denote the tautological bundles over $\mathbb{P}(B^*)$ and $\mathbb{P}(F)$ by γ_{B^*} and γ_F, respectively. We then look at the forgetful map

$$\widetilde{\mathcal{V}} \to \mathbb{P}(B^*) \times \mathbb{P}(F).$$

One can check that it is a fibration with $\mathbb{RP}^1$ fibres and, in fact, it is the fibrewise compactification of the real line bundle $\gamma_{B^*}^{\otimes r+1} \otimes \gamma_F$. From this expression we see that there is a certain asymmetry depending on the parity of r, so we must tackle each case separately.

Write $\widehat{B^*} \cong \mathbb{RP}^n$ for the compactification of B^* by adding $\mathbb{P}(B^*)$ at infinity. Denote by $\mathbb{S}(F)$ the unit sphere (with respect to some scalar product). Then:

Lemma 3.5.15. *Let r be even. Then, there is a fibration*

$$\mathbb{Z}_2 \to \widehat{B^*} \times \mathbb{S}(F) \to \mathcal{V}.$$

In particular, if $k = \dim(F) = 1$, we have that $\mathcal{V}$ is homotopy equivalent to $\widehat{B^*} \cong \mathbb{RP}^n$.

Proof. We define maps

$$\begin{array}{rcl} B^* \times \mathbb{S}(F) & \mapsto & \mathcal{V}_0, \\ (f, \alpha) & \mapsto & f^{r+1} \otimes \alpha; \\ \mathbb{P}(B^*) \times \mathbb{S}(F) & \mapsto & \mathcal{V}_1, \\ ([f], \alpha) & \mapsto & ([f], [\alpha]). \end{array}$$

Their composition defines a continuous map $\widehat{B^*} \times \mathbb{S}(F) \mapsto \mathcal{V}$, as claimed. For the second claim we note that the bundle is trivial because $\mathbb{S}(F) = \mathbb{Z}_2$. □

Similarly:

Lemma 3.5.16. *Let r be odd. Then $\mathcal{V}$ is homotopy equivalent to the quotient*

$$\frac{\mathbb{P}(B^*) \times \widehat{F}}{\mathbb{P}(B^*) \times 0}.$$

Proof. Regard $\mathbb{P}(B^*)$ as the quotient of the unit sphere (for some scalar product) under the antipodal map. Consider the map:

$$\begin{aligned} \mathbb{P}(B^*) \times F &\mapsto \mathcal{V}_0, \\ ([f], \alpha) &\mapsto f^{r+1} \otimes \alpha, \end{aligned}$$

which is well-defined because r is odd. Together with the identity map $\mathbb{P}(B^*) \times \mathbb{P}(F) \mapsto \mathcal{V}_1$, this defines a mapping

$$\mathbb{P}(B^*) \times \widehat{F} \mapsto \mathcal{V}$$

which is surjective, maps $\mathbb{P}(B^*) \times \{0\}$ to the origin in $\mathcal{V}$, and is a homeomorphism in the complement; quotienting we deduce the claim. □

3.5.4.4 The tangent variety of the principal cone

Lastly, being a subvariety of a vector space, we can look at the tangent variety $T\mathcal{V}_0 \subset \mathrm{Sym}^{r+1}(B^*, F)$ associated to $\mathcal{V}_0$.

To determine $T\mathcal{V}_0$, we look at the map $\psi(f, \alpha) = f^{r+1} \otimes \alpha$. Its differential at a covector $f \in B^*$ and a vector $\alpha \in F$ is readily computed:

$$d_{f,\alpha}\psi : B^* \times F \to \mathrm{Sym}^{r+1}(B^*, F),$$

$$d_{f,\alpha}\psi(g, \beta) = f^{r+1} \otimes (\alpha + \beta) + (r+1) g \cdot f^r \otimes \alpha.$$

Equivalently, if we set $H = \ker(f) \subset B$, we see that the tangent space to $\mathcal{V}_0$ at $f^{r+1} \otimes \alpha \neq 0$ is the subspace:

$$\mathrm{Sym}^{r+1}(H^\perp, F) \oplus H^* \otimes \mathrm{Sym}^r(H^\perp, \langle\alpha\rangle) \subset \mathrm{Sym}^{r+1}(B^*, F)^{(H,2)}.$$

This identifies the normal fibre to $\mathcal{V}_0$ at (f, α) with the quotient

$$\frac{\mathrm{Sym}^{r+1}(B^*, F)}{\mathrm{Sym}^{r+1}(H^\perp, F) \oplus H^* \otimes \mathrm{Sym}^r(H^\perp, \langle\alpha\rangle)},$$

as we would expect from our description of $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ as a bundle over $\mathcal{V}_1$.

3.5.5 The Σ^2-free integral Grassmannian

In this last subsection we state some structural results about $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ and we provide sketches of proofs. A more comprehensive account will appear in future work.

3.5.5.1 Smoothness

According to Subsections 3.5.2 and 3.5.3, the pieces $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$ and $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ are smooth manifolds. The first is a vector space. The second one is a vector bundle over a smooth bundle with grassmannian base and fibre. The computations in subsection 3.5.3.3 show that the later has dimension one less than the former. One can put together these facts to show:

Proposition 3.5.17. $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ *is a smooth open manifold, embedded in* $\mathrm{Gr}(\mathfrak{g}_0, n)$*. Furthermore,* $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ *sits inside as a smooth hypersurface.*

Proof. It is sufficient to describe, at each point $W \in \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$, a chart that is simultaneously a submanifold chart of $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ inside of $\mathrm{Gr}(\mathfrak{g}_0, n)$ and a submanifold chart of $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ inside $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$. We will just provide the latter.

Let W be presented as $\lim_{s\to\pm\infty} \mathrm{graph}(A_0 + sA_1)$, with $A_0, A_1 \in \mathrm{Sym}^{r+1}(B^*, F)$ and A_1 principal. We write L for a neighbourhood of A_0 within the normal fibre to the principal cone at A_0. Additionally, we fix a $(n+k-1)$-dimensional family U of rank-1 maps whose projectivisations are a neighbourhood of $[A_1]$ in the space of principal subspaces.

Then, the map

$$\Phi : L \times U \times (-\delta, \delta) \to \mathrm{Gr}(\mathfrak{g}, n)$$
$$(A, A', s) \to A + \frac{1}{s}A'$$

is a smooth embedding with image a neighbourhood of W in $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$. Further, the map $\Phi|_{U\times L\times\{0\}}$ parametrises the hypersurface $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$. ☐

We remark that we do not know whether $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)}$ is smooth in general. In the contact case it is known that it is.

3.5.5.2 Homotopy type

We can put together Proposition 3.5.17 with the work we did in the previous Subsection about the principal subvariety to show that:

Proposition 3.5.18. *The* Σ^2*-free Grassmannian* $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ *is homotopy equivalent to the principal subvariety* $\mathcal{V}$*.*

Proof. We just provide a sketch of proof.

Let us fix a metric in $\mathfrak{g}_0$ making the horizontal and vertical components orthogonal. This immediately defines a distance function $\measuredangle$ between lines in $\mathfrak{g}_0$, given as the sine squared of the angle they make. We can readily extend this function to $\mathrm{Gr}(\mathfrak{g}_0, n)$ as follows:

$$\measuredangle(A, A') := \max_{L \subset A, L' \subset A'} \measuredangle(L, L').$$

We restrict $\measuredangle$ to $\overline{\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)}$.

Note that the horizontal cell $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$ is the set of points at distance strictly less that 1 from the zero map. Similarly, $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ is the set at distance strictly less than 1 from $\mathcal{V}$. We may then define the distance function

$$d : \mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n) \to [0, 1)$$

$$d(A) := \inf_{B \in \mathcal{V}} \measuredangle(A, B),$$

whose zero set is $\mathcal{V}$.

The function d is smooth. It can be seen that its restriction to $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ is Morse-Bott and its critical set is precisely $\mathcal{V}_1$. The situation in $\mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n)$ is more delicate because d is not Morse-Bott: its zero locus is the principal cone, which is singular, and the additional critical points (corresponding to the cut locus of d) form a conical algebraic subvariety S.

We may then proceed as follows: we modify d by adding a perturbation $h(A) = |A|^2\rho$; here $\rho : \mathrm{Gr}_{\Sigma^0}(\mathfrak{g}, n) \to \mathbb{R}$ is a bump function supported in the intersection of a neighbourhood of S and the complement of a ball around zero. In particular, this perturbation is zero in the hypersurface $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$. The effect of this is that minus the gradient flow of $d + h$ retracts everything to a neighbourhood of $\mathcal{V}$, which itself retracts onto $\mathcal{V}$. □

3.5.5.3 The Maslov hypersurface

In the Lagrangian Grassmannian, the complement of the regular cell is usually called the *Maslov cycle*. As studied by V. Maslov and V. Arnol'd [84, 5], it is a two-sided (i.e. cooriented) and non-separating hypersurface and, it defines a first homology class through the intersection pairing. Let us study this phenomenon in general jet spaces. We will henceforth denote:

Definition 3.5.19. $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n) \subset \mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ *is called the* ***Maslov hypersurface****.*

The Maslov hypersurface is non-separating in general. Furthermore:

Proposition 3.5.20. *The Maslov hypersurface is two-sided if and only if one of the following conditions holds:*

- $\dim(F) = 1$ *and* r *is odd, or*
- $\dim(B) = \dim(F) = 1$.

These are not mutually exclusive.

Proof. According to Proposition 3.5.18, it is sufficient that we prove that $\mathcal{V}_1$ is coorientable within $\mathcal{V}$. Then, we refer back to subsection 3.5.4.3, where it was explained that $\widetilde{\mathcal{V}}$ (the blow-up at the origin of $\mathcal{V}$) is the fibrewise compactification of the tautological bundle $\gamma_{B*}^{\otimes r+1} \otimes \gamma_F$ over $\mathbb{P}(B^*) \times \mathbb{P}(F)$. Here the zero section corresponds to the blow-up of the origin and the infinity section is precisely $\mathcal{V}_1$, but their roles are symmetric.

Now we observe that $\gamma_{B*}^{\otimes r+1} \otimes \gamma_F$ is isomorphic to the normal bundle of $\mathcal{V}_1$ in $\widetilde{\mathcal{V}}$, and therefore isomorphic to the normal bundle of $\mathcal{V}_1$ in $\mathcal{V}$. Furthermore, this bundle is trivial if and only if the terms γ_F and $\gamma_{B*}^{\otimes r+1}$ are individually trivial. This proves the claim. ☐

Furthermore:

Corollary 3.5.21. *Let $\dim(F) = 1$ and r be odd. Then a choice of orientation for F determines a coorientation for the Maslov hypersurface.*

Proof. Indeed, as computed in the proof of Proposition 3.5.20, the normal bundle to $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ is precisely γ_F, which is canonically identified with F. ☐

Similarly:

Corollary 3.5.22. *Let $\dim(B) = \dim(F) = 1$ with r even. Then, a choice of orientation for $B^* \oplus F$ determines a coorientation for the Maslov hypersurface.*

Proof. The normal bundle to $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ is $\gamma_{B*} \otimes \gamma_F$, which is identified with $\det(B^* \oplus F)$. ☐

In both cases, once we have oriented either F or $B^* \oplus F$, we will call the resulting coorientation the **Maslov coorientation**.

3.5.5.4 The Maslov class

Under the assumptions of Proposition 3.5.20, the Maslov hypersurface is non-separating, cooriented, and closed as a subset. This is enough to have a well-defined cohomology class using the intersection pairing:

Definition 3.5.23. *Suppose one of the following conditions holds:*

- $\dim(F) = 1$ *and r is odd, or*
- $\dim(B) = \dim(F) = 1$*,*

and that a Maslov coorientation has been fixed.

*Then, the **Maslov index** or **Maslov class** is the non-zero, non-torsion element*

$$\mathrm{Ind} \in H^1(\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n), \mathbb{Z})$$

defined by:

$$\mathrm{Ind}([\gamma]) := |\gamma \cap \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)| \in \mathbb{Z} \quad ,$$

where γ is a curve representative intersecting the Maslov hypersurface transversally. The count of intersection points takes into account signs, comparing the orientation of γ with the Maslov coorientation.

3.5.5.5 The local Maslov class

Even if the Maslov hypersurface is not two-sided, it still makes sense to talk about a **local Maslov coorientation**: indeed, let $W \in \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ and consider a ball $U \subset \mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\mathfrak{g}, n)$ containing W. In U, the intersection $U \cap \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ is two-sided, so a coorientation can be chosen.

Given a local Maslov coorientation for $U \cap \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$, we can reason as before to define a **local Maslov class** for oriented curves

$$([0,1], \{0,1\}) \to (U, (\partial U)\backslash(U \cap \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)))$$

using the intersection pairing. It can only take the values $\{0, 1, -1\}$.

This will play a role in Subsection 3.7.2.

3.6 Multi-sections: Definition and elementary properties

Having looked at the linear situation in Section 3.5, we turn our attention back to integral submanifolds of r-jet space $J^r(Y \to X)$. We always denote $\dim(X) = n$ and $\dim(Y) = k$. When we pass to local coordinates we replace X by a vector space B and the fibres of Y by a vector space F.

In Subsection 3.6.1 we set up the language of *multi-sections*, i.e. integral submanifolds that are horizontal almost everywhere. In Subsection 3.6.2 we focus on Σ^2-free multi-sections and we explain what it means for their singularities to be in Thom-Boardman form. We then introduce two techniques that will allow us to manipulate Σ^2-free multi-sections: *generating functions* (Subsection 3.6.3) and *metasymplectic lifts* (Subsection 3.6.4). Using these ideas we prove some structure results (Subsection 3.6.5).

3.6.1 Multi-sections

Multi-sections are defined in subsection 3.6.1.3. Before we get there, we need to introduce some notation.

3.6.1.1 Grassmannian bundles

In Subsection 3.5.1 we singled out several subsets of the integral Grassmannian of $\mathfrak{g}$. According to Proposition 3.2.39 we can identify $\mathfrak{g}$ with any tangent fibre $T_pJ^r(Y \to X)$ (uniquely up to point symmetries). In doing so we define bundle analogues of these subsets.

Namely: the **bundle of horizontal elements** will be denoted by

$$\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, l) \subset \mathrm{Gr}_{\mathrm{integral}}(\xi_{\mathrm{can}}, l).$$

Similarly, we use the notation

$$\mathrm{Gr}_{\Sigma^i}(\xi_{\mathrm{can}}, l) := \{W \in \mathrm{Gr}_{\mathrm{integral}}(\xi_{\mathrm{can}}, l) \mid \dim(W \cap V_{\mathrm{can}}) = i\}.$$

For us it is of particular importance the union

$$\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\xi_{\mathrm{can}}, l) := \mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, l) \cup \mathrm{Gr}_{\Sigma^1}(\xi_{\mathrm{can}}, l),$$

which we call the **Σ^2-free Grassmannian bundle**. It is a submanifold of the **bundle of multi-section elements** $\overline{\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, l)}$.

3.6.1.2 Singularities of integral maps

Let N be a n-dimensional manifold, and consider an integral map $f : N \to J^r(Y \to X)$ (Definition 3.2.12), possibly with singularities. We will denote: $\Sigma(f)$ for its locus of singularities of mapping, $\Sigma(\pi_f \circ f)$ for the locus of singularities (Definition 3.3.1) of its front projection, and $\Sigma(f, V_{\mathrm{can}})$ for the locus of singularities of tangency (Definition 3.3.2) with the vertical distribution V_{can}.

Lemma 3.6.1. *The following statements hold:*

- *The sets $\Sigma(f, V_{\mathrm{can}})$ and $\Sigma(f)$ are not necessarily disjoint.*
- $\Sigma(\pi_f \circ f) = \Sigma(f) \cup \Sigma(f, V_{\mathrm{can}})$.
- *In particular, if f is an immersion, then $\Sigma(\pi_f \circ f) = \Sigma(f, V_{\mathrm{can}})$.*

3.6.1.3 Multi-sections

At a point in the complement of $\Sigma(\pi_f \circ f)$, $f(N)$ is a submanifold graphical over X; i.e. it can be expressed as the graph of the r-jet of a locally defined section $X \to Y$. Motivated by this, we introduce the following definition:

Definition 3.6.2. *Let N be an n-dimensional manifold.*

A (parametrised, r-times differentiable) ***multi-section*** *is a smooth map $f : N \to J^r(Y \to X)$ which:*

a. is tangent to ξ_{can}, *b.* *is transverse to* V_{can} *on an open dense set,*

c. *has a well-defined Gauss map* $\mathrm{Gr}(f) : N \to \mathrm{Gr}_{\text{integral}}(\xi_{\text{can}}, n)$ *with* $\mathrm{Image}(df) \subset \mathrm{Image}\mathrm{Gr}(f)$*.*

Each component of the complement of $\Sigma(\pi_f \circ f)$ *is called a* ***branch****.*

Property (b) implies that f is an immersion in an open dense set, and its Gauss map $\mathrm{Gr}(f)$ takes values in the Grassmannian of multi-section elements $\overline{\mathrm{Gr}_{\Sigma^0}(\xi_{\text{can}}, n)}$, justifying its name.

The key property of multi-sections is that they are uniquely recovered from their front projection: indeed, this holds for each of the branches and the global claim follows by density. Thus, we can go back and forth between the multi-section f and its front projection $\pi_f \circ f$. For this reason, we will sometimes be sloppy about this back and forth and we will say that $\pi_f \circ f$ *is a multi-section* (and, in fact, this agrees with the usual picture of what a multi-section should be).

Sometimes the parametrisation itself is not important so:

Definition 3.6.3. *An* ***unparametrised multi-section*** *is a subset of* $J^r(Y \to X)$ *which is the image of a parametrised multi-section.*

3.6.1.4 The space of multi-sections

The space of all parametrised, r-times differentiable multi-sections with domain N is denoted by $C^r_{\text{multi}}(N, Y)$. We can then observe:

Lemma 3.6.4. *The projection* $\pi_{r,r'} : J^r(Y \to X) \to J^{r-1}(Y \to X)$ *maps* $C^r_{\text{multi}}(N, Y)$ *to* $C^{r-1}_{\text{multi}}(N, Y)$*.*

Proof. The projection $\pi_{r,r'}$ preserves Properties (a) and (b) in Definition 3.6.2. It is enough that we prove that Property (c) holds as well.

Let $f \in C^r_{\text{multi}}(N, Y)$. We then observe that, tautologically, $\pi_{r,r'} \circ f$ must be tangent, at a point p, to the horizontal element represented by $f(p)$. Hence, under the correspondence between horizontal elements and points in the fibre in one jet space higher, f itself is the Gauss map of $\pi_{r,r'} \circ f$. In particular, the Gauss map of a projection always takes values in the horizontal Grassmannian. □

In our definition of multi-sections we allow maps not to be immersed. In view of its proof, dropping the immersion condition is necessary for the Lemma to hold. Furthermore, even though we are mostly interested in embedded multi-sections, we will need to consider multi-sections with mild singularities of mapping in the course of our arguments.

3.6.1.5 Scaling

We finish this Subsection with a trivial remark:

Lemma 3.6.5. *Let B and F be vector spaces and let N be a smooth manifold. Fix an integral mapping $f : N \to J^r(B, F)$ (Definition 3.2.12) and a constant $\delta > 0$. The following statements hold:*

- *$J^r(B, F)$ is a vector bundle over B.*
- *The Cartan 1-forms are invariant under scaling of the fibres of $J^r(B, F) \to B$.*
- *In particular, δf is integral.*

Proof. This is immediate from the description of the Cartan 1-forms in holonomic coordinates given in subsection 3.2.2.4, Equation 3.2.2.2. □

Despite being trivial, this Lemma plays an important role in the construction of multi-sections with arbitrarily small derivatives that we shall need in Section 3.8.

3.6.2 Σ^2-free multi-sections

We further narrow the scope of our work to:

Definition 3.6.6. *A multi-section f is Σ^2**-free** if $\Sigma^i(\pi_f \circ f) = \varnothing$ for every $i \geqslant 2$.*

Hence, the Gauss map of a Σ^2-free multi-section takes values in $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\xi_{\mathrm{can}}, n)$.

Despite having no singularities of higher rank, the locus $\Sigma^1(\pi_f \circ f)$ may be complicated, involving both singularities of tangency and of mapping. We would like to say that, if f is generic, the singularities are nicely stratified. However, the Thom transversality Theorem 3.3.6 does not apply because the class of integral maps is not generic. Still, in subsection 3.6.2.2 we discuss how the singularities of an integral Σ^2-free immersion would look if they resembled those arising from Thom transversality. Before we do so, let us introduce a couple of auxiliary definitions.

3.6.2.1 Kernel and image

Let $f : N \to J^r(Y \to X)$ be an integral Σ^2-free map (not necessarily a multi-section). At every singular point we can single out the direction in which the rank is dropping:

Definition 3.6.7. *The **kernel line field** of f is defined as:*

$$\ker(f) := \{\ker(d_q(\pi_f \circ f)) \subset T_qN \mid q \in N\} \subset TN.$$

Note that it is not defined over the whole of N, only over $\Sigma^1(\pi_f \circ f)$.

Conversely, we have a partially defined (and possibly multiply-defined) hyperplane distribution along the image of the singularities:

Definition 3.6.8. *The* ***singular hyperplanes*** *of f are defined as:*

$$\{df(T_qN) \subset T_{f(q)}J^r(Y \to X) \mid q \in \Sigma^1(\pi_f \circ f)\}.$$

It is immediate that:

Lemma 3.6.9. *The singular hyperplanes of f provide a well-defined Gauss map for the restriction $f|_{\Sigma^1(\pi_f \circ f)}$.*

Proof. Indeed, at a point $q \in \Sigma^1(\pi_f \circ f)$ we set

$$\mathrm{Gr}(f|_{\Sigma^1(\pi_f \circ f)})(q) = df(T_qN).$$

This is well-defined even if the set $\Sigma^1(\pi_f \circ f)$ is not smooth. □

3.6.2.2 Σ^2-free singularities in Thom-Boardman form

In Subsection 3.3.3, Theorem 3.3.14 we recalled Morin's result: a Σ^2-free generic mapping between equidimensional manifolds has only singularities of Whitney-type. This leads us to the following definition:

Definition 3.6.10. *Let N be a smooth n-dimensional manifold. An integral Σ^2-free immersion $f : N \to J^r(Y \to X)$ is in* ***Thom-Boardman form*** *if the singularities of the base projection $\pi_b \circ f$ are of Whitney-type.*

In particular, a Σ^2-free immersed multi-section F in Thom-Boardman form contains a sequence of smooth submanifolds

$$\Sigma^{1^l}(f, V_{\mathrm{can}}) = \Sigma^1(f|_{\Sigma^{1^{l-1}}}, V_{\mathrm{can}}),$$

each of which is a hypersurface in the previous one. The locus $\Sigma^{1^l}(f, V_{\mathrm{can}})$ is precisely the tangency locus of $\Sigma^{1^{l-1}}(f, V_{\mathrm{can}})$ with the kernel line field of f. Similarly, df maps $T\Sigma^1(f, V_{\mathrm{can}})$ to the singular hyperplanes, as in Definition 3.6.8, of f.

Our h-principle statements are parametric in nature. For this we must generalise Definition 3.6.10 to families:

Definition 3.6.11. *Let N be a smooth n-dimensional manifold. Fix a smooth manifold K, which we regard as a parameter space. Write $\mathcal{F}$ for the foliation in $K \times N$ by fibres of $K \times N \to K$.*

A K-family of Σ^2-free immersions $(f_s)_{s \in K}$ is in ***Thom-Boardman form*** *if:*

- *The singularities $\Sigma(f_s, V_{\mathrm{can}})$ of the fibered map $(s, p) \to (s, \pi_b \circ f_s(p))$ are of Whitney type.*
- *The stratified locus $\Sigma(f_s, V_{\mathrm{can}})$ has generic tangencies with respect to $\mathcal{F}$.*

The second item deserves some comment. The projection map $\Sigma^{1^l}(f_s, V_{\text{can}}) \to K$ of each stratum is endowed with a Thom-Boardman stratification in terms of the rank. The dimensions of these new strata can be computed using the Thom transversality Theorem 3.3.6, depending only on l and $\dim(K)$. For our purposes it is not important what these numbers are.

Remark 3.6.12. For most jet spaces, the dimension computations from subsection 3.5.3.3 show that most vertical curves are not tangent to principal directions and, therefore, their only deformations are other vertical curves. Additionally, in subsection 3.2.2.9 we stated the following result of R. Bryant and L. Hsu [20]: under fairly weak hypotheses on a bracket-generating distribution, there exist integral curves that do not admit any compactly supported deformations. For jet spaces $J^r(\mathbb{R}, \mathbb{R})$, $r > 1$, these are the curves tangent to the vertical distribution.

That is to say, any transversality statement must bypass these two issues. The first one is avoided by requiring integral submanifolds to have tangent spaces in $\text{Gr}_{\Sigma^2-\text{free}}(\xi_{\text{can}}, n)$. The second one can be ignored for curves as long as our curves are somewhere not tangent to the vertical. The case of higher dimensional manifolds will be discussed in subsection 3.6.5.3. △

3.6.3 Generating functions

V. Arnold proved in [6, 7] that front singularities of embedded legendrians/lagrangians can always be (locally) described by generating functions. This is not true for arbitrary integral submanifolds of jet spaces [101, p. 14] [115], but it nonetheless holds that front singularities are rather special compared to the singularities of a general map. This was first explored by V. Lychagin [78] for 1-jet spaces in more than one variable, and later by A. Givental [58] for general jet spaces.

Our goal in this Section is to define what a generating function is for a general jet space. We do this in a possibly novel way: the key ingredient is the concept of reduction, which we introduce in subsection 3.6.3.1. This allows us, in subsection 3.6.3.4, to provide a recipe for corank-1 front singularities admitting a generating function description. We will see in subsection 3.7.1.2 that this recipe can be particularised to recover Givental's description of integral submanifolds that have Whitney type front singularities.

3.6.3.1 Reduction

The main idea behind generating functions is that we can follow a two step process when constructing non-horizontal integral submanifolds: first, we produce a horizontal submanifold over a base of greater dimension. Then, we use a "reduction" procedure to go down to the actual jet space we want to work in. It is in this latter step in which the horizontality condition is lost.

The "enlarged base" will be the foliated manifold $(X, \mathcal{F})$. The actual base manifold will be the quotient space $X/\mathcal{F}$, which we assume is smooth (even though part of the construction goes through without this assumption). We denote the quotient map by $\pi : X \to X/\mathcal{F}$.

Let Y be another smooth manifold. We denote by $C^\infty(X, Y)$ the space of smooth functions $X \to Y$. Using the pullback of the quotient map π, we have a natural inclusion $\pi^* : C^\infty(X/\mathcal{F}, Y) \to C^\infty(X, Y)$, whose image we denote by $C^\infty_{\mathcal{F}}(X, Y)$. A function in $C^\infty_{\mathcal{F}}(X, Y)$ is said to be **basic**. We collect all the r-jets of basic functions to yield:

Definition 3.6.13. *The space of **basic** r**-jets** is defined as:*

$$J^r_{\mathcal{F}}(X, Y) := \{j^r_x f \in J^r(X, Y) \mid x \in X, f \in C^\infty_{\mathcal{F}}(X, Y)\}.$$

The canonical projection map

$$\begin{array}{rcl} \tilde{\pi} : J^r_{\mathcal{F}}(X, Y) & \mapsto & J^r(X/\mathcal{F}, Y) \\ j^r_x(f \circ \pi) & \mapsto & j^r_{\pi(x)} f, \end{array}$$

*is called the **reduction map**.*

In this general setting, the familiar properties of the contact reduction process still hold. We leave the proof to the reader:

Lemma 3.6.14. *The following statements hold:*

- *$J^r_{\mathcal{F}}(X, Y)$ is a smooth submanifold of $J^r(X, Y)$.*
- *The restriction*
$$\xi^{\mathcal{F}}_{\text{can}} := \xi_{\text{can}} \cap TJ^r_{\mathcal{F}}(X, Y)$$
has a $\text{rank}(\mathcal{F})$*-dimensional characteristic foliation* $\ker(\xi^{\mathcal{F}}_{\text{can}})$ *which is a lift of* $\mathcal{F}$*.*
- *The reduction map $\tilde{\pi}$ preserves the Cartan distribution.*
- *Leaves of the characteristic foliation* $\ker(\xi^{\mathcal{F}}_{\text{can}})$ *correspond to fibres of* $\tilde{\pi}$*.*

So we can legitimately say that $J^r(X/\mathcal{F}, Y)$ is the **reduction** of $J^r(X, Y)$ with respect to $\ker(\xi^{\mathcal{F}}_{\text{can}})$.

We may study next how integral submanifolds interact with the reduction process:

Definition 3.6.15. *Let $L \subset J^r(X, Y)$ be an integral submanifold. Its **reduction** is the set*

$$L/\mathcal{F} := \tilde{\pi}(L \cap J^r_{\mathcal{F}}(X, Y)) \subset J^r(X/\mathcal{F}, Y).$$

*We say that $f : X \to Y$ is the **generating function** of*

$$L_f := \text{Image}(j^r f)/\mathcal{F}.$$

As suggested by the definition, even if the intersection $L \cap J^r_{\mathcal{F}}(X, Y)$ is a smooth submanifold, it may have singularities of tangency with $\ker(\xi^{\mathcal{F}}_{\text{can}})$. Therefore, the reduction $L/\mathcal{F}$ is often not smooth. However, it is integral (in the sense that it is the image of an integral map).

3.6.3.2 Reduction in concrete terms

Take local coordinates (q, x) in X with values in $\mathbb{R}^n \times \mathbb{R}^m$; the x–coordinates denote the foliation directions. Take also local coordinates in Y with values in $\mathbb{R}^k$. In this manner, the reduced space is $J^r(\mathbb{R}^n, \mathbb{R}^k)$.

Lemma 3.6.16. *A function $G : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ generates the subset:*
(3.6.3.1)
$$L_G = \{(q, G(q,x), \partial_q G(q,x), \cdots, \partial_q^r G(q,x)) \mid \forall (q,x) \ s.t.\ \partial_q^a \partial_x^b G(x,q) = 0 \quad \forall b \neq 0, a\}.$$

Proof. The lift of G is given by the expression:

$$j^r G(q,x) = (q, x, G(q,x), \partial_q G(q,x), \partial_x G(q,x), \partial_q^2 G(q,x), \partial_q \partial_x G(q,x), \ldots, \partial_x^r G(q,x)).$$

The intersection of $j^r G$ with the space of basic r-jets is the subset of $j^r G$ in which all derivatives of G involving x at least once are zero. I.e. the set of points in which the derivatives of G take place purely in the q-directions. In particular, this set is contained in the locus of leafwise critical points of G and the two agree if $r = 1$. □

3.6.3.3 Remark: dimension counting

In the contact case (i.e. $r = 1$ and $m = 1$) the collection of leafwise critical points on a given leaf $q \times \mathbb{R}^m$ is, generically, a finite collection of points and, for most leaves, the points will be Morse. In particular, the reduction L_G is an n-dimensional submanifold (a legendrian), which can be regarded as the 1-jet of a multiply-valued function $\mathbb{R}^n \to \mathbb{R}$.

For $mr > 1$, having derivative purely in the q-directions is an overdetermined condition. The expected dimension of L_G may be computed to be:

$$(n+m) - k \sum_{l=1}^{r} \left(\binom{n+m+l-1}{l} - \binom{n+l-1}{l} \right).$$

The expected dimension is n only in the contact setting, and it is non-negative only if $r = 1$ and $n \geqslant (k-1)m$. Otherwise, and in particular for all higher jet spaces, the expected dimension is negative.

This tells us that any generating function theory for higher jet spaces would not rely on generic functions, but rather on a subclass of functions (of positive codimension given by the formula above) with prescribed singularities. We will look at one particularly manageable example next. Developing a general theory is left as an open question. It is worth remarking that this has been explored already in connection with the theory of legendrian/lagrangian singularities [101], which deals with the *local* existence of generating functions. In light of the extreme *global* flexibility results, it is unclear whether generating functions in higher jet spaces may be useful from a *topological* perspective.

3.6.3.4 Integral expressions

Let M be an n-dimensional manifold. We set $X = M \times \mathbb{R}$ and we endow it with the foliation $\mathcal{F}$ by fibres of $X \to M$. We restrict to the case in which the target space is $Y = \mathbb{R}^k$. In this Subsection we explain how to use generating functions on X to obtain integral manifolds in the reduction $J^r(M, \mathbb{R}^k)$. Since X is a rank-1 bundle over M, any integral manifold we produce will have front tangencies of rank at most 1.

Let $F : X \to \mathbb{R}$ be submersion whose zero set has generic singularities of tangency (in the sense of Thom-Boardman; see Subsection 3.3.1) with respect to $\mathcal{F}$. For dimensional reasons, this singularity locus is thus Σ^2-free. We then define a function:

$$\begin{aligned} G : M \times \mathbb{R} &\to \mathbb{R}^k \\ (q, x) &\to \left(G_1(q, x) := \int_0^x F(q, s)^r ds, 0, \ldots, 0 \right). \end{aligned}$$

The only relevant entry is G_1, since the other $(k-1)$ entries are zero and therefore singular everywhere.

We see that $\partial_x G_1(q, x) = F(q, x)^r$. Furthermore, using induction we can prove:

Lemma 3.6.17. *Let $a \geqslant 0$ and $b > 0$ be integers. Then, there are functions $\Psi_l : M \times \mathbb{R} \to \mathbb{R}$ such that*

$$\partial_q^a \partial_x^b G_1(q, x) = \sum_{l=0}^{a+b-1} F^{r-l}(q, x) \Psi_l(q, x).$$

That is, all the derivatives (up to order r) involving x at least once vanish at the fibrewise critical points of G. Therefore, according to subsection 3.6.3.2:

Corollary 3.6.18. *The reduction L_G is parametrised by the locus of zeroes of F:*

$$L_G = \{(q, G(q, x), \partial_q G(q, x), \cdots, \partial_q^r G(q, x)) \mid \forall (q, x) \ s.t. \ F(q, x) = 0\}.$$

According to the Corollary, L_G is parametrised by a smooth manifold. Furthermore:

Lemma 3.6.19. *L_G is an embedded Σ^2-free multi-section. Its singularities of tangency with the vertical distribution correspond to the singularities of tangency of $F^{-1}(0)$ with $\mathcal{F}$.*

Proof. The locus of zeroes $F^{-1}(0)$ is a smooth hypersurface in X which is in general position with respect to $\mathcal{F}$. In particular, its locus of tangencies $\Sigma(F^{-1}(0), \mathcal{F})$ with $\mathcal{F}$ is of codimension 1. In each branch of $F^{-1}(0)$, the variable x can be regarded as a function of q. Hence, branches of $F^{-1}(0)$ are mapped to branches of L_G simply by taking the graph $j^r(G(q, x(q))$. Conversely, the locus $\Sigma(F^{-1}(0), \mathcal{F})$ maps to the singularity locus of L_G. The singularities of L_G may be front tangencies or singularities of mapping. We claim that they are always tangencies.

Fix $(\tilde{q}, \tilde{x}) \in \Sigma(F^{-1}(0), \mathcal{F})$. Since F is a submersion, we have that $\partial_{q_i} F(\tilde{q}, \tilde{x}) \neq 0$, for some i. We may then compute:

$$\partial_x \partial^r_{q_i} G_1(\tilde{q}, \tilde{x}) = r![\partial_{q_i} F(\tilde{q}, \tilde{x})]^r \neq 0$$

because all other terms involve F and are zero. Therefore, the map $x \to \partial^r_{q_i} G_1(q, x)$ is a local diffeomorphism of $\mathbb{R}$ to itself. This implies that $(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n, x)$ locally parametrises L_G as a smooth embedded manifold, concluding the proof. □

We can define additional Σ^2-free multi-sections, for every $0 \leqslant l < r$, as follows:

$$\pi_{r,l}(L_G) = \{(q, G(q,x), \partial_q G(q,x), \cdots, \partial^l_q G(q,x)) \mid \forall (q,x) \text{ s.t. } \partial_x G(x,q) = 0\} \subset J^l(M, \mathbb{R}^k),$$

which are none other than the usual projections of L_G to lower jet spaces. All of them are generated by G and have a well-defined Gauss map into the horizontal Grassmannian. They have singularities of mapping corresponding to the front tangencies of L_G.

3.6.4 Metasymplectic projections and lifts

In Contact Topology it is fruitful to manipulate legendrian knots using their lagrangian projection. In this Subsection we present the analogue of this process for general jet spaces. We work locally in $J^r(B, F)$, with B and F vector spaces. We fix holonomic coordinates (x, y, z).

In subsections 3.2.2.6 and 3.2.2.7 we introduced the notion of *metasymplectic space*

$$(B \oplus \mathrm{Sym}^r(B^*, F), \Omega_{\mathrm{can}}).$$

The terms B and $\mathrm{Sym}^r(B^*, F)$ were called the horizontal and vertical component, respectively. The projection onto B was denoted by π_b. Furthermore, we defined the *metasymplectic projection*:

$$\pi_L : J^r(B, F) \to B \oplus \mathrm{Sym}^r(B^*, F),$$

which is the natural generalisation of the lagrangian projection. The coordinates of $J^r(B, F)$ induce coordinates (x, z^r) in $B \oplus \mathrm{Sym}^r(B^*, F)$.

In subsection 3.6.4.1 we prove Theorem 3.6.22: isotropic submanifolds of $B \oplus \mathrm{Sym}^r(B^*, F)$ can always be lifted to $J^r(B, F)$. This is sufficient to manipulate multi-sections when B is 1-dimensional: the theory of general integral curves is then very similar to the theory of immersed legendrian curves; see subsection 3.6.4.2.

For higher-dimensional integral submanifolds the story is more complicated, because it is non-trivial to manipulate their metasymplectic projections directly. To address this, we work "*one direction at a time*", effectively thinking about them as parametric families of curves. This is done in subsection 3.6.4.3.

3.6.4.1 Integral lift of an isotropic

We want to prove that any isotropic submanifold can be lifted to an integral one. First we need an auxiliary concept:

Definition 3.6.20. *The* ***Liouville form***

$$\lambda_{\mathrm{can}} \in \Omega^1(B \oplus \mathrm{Sym}^r(B^*, F); \mathrm{Sym}^{r-1}(B^*, F))$$

is defined, at a point (v, A)*, by the following tautological expression:*

$$\lambda_{\mathrm{can}}(v, A)(w, B) := -\iota_w A.$$

The computations in subsection 3.2.2.6 imply that:

Lemma 3.6.21. *Then following statements hold:*

- *The Liouville form can be explicitly written as:*

$$\lambda_{\mathrm{can}}(x, z^r) = \left(-\sum_{a=1}^{n} z_j^{(i_1,\cdots,i_a+1,\cdots,i_n)} dx_a \right)_{|(i_1,\cdots,i_a,\cdots,i_n)|=r-1}.$$

- *The Cartan* 1*-forms* $\alpha^r \in \Omega^1(J^r(B, F); \mathrm{Sym}^{r-1}(B^*, F))$ *are given by the expression*

$$\alpha_r(x, y, z) = dz_{r-1} + \lambda_{\mathrm{can}}(x, z^r).$$

- *In particular,* $d\lambda_{\mathrm{can}} = \Omega_{\mathrm{can}}$.

That is, the familiar properties for the Liouville form in the symplectic/contact setting hold as well in more general jet spaces. Then:

Theorem 3.6.22. *Let* N *be a smooth, connected, contractible manifold. Given an isotropic map*

$$g : N \to (B \oplus \mathrm{Sym}^r(B^*, F), \Omega_{\mathrm{can}})$$

there exists an integral map (Definition 3.2.12)

$$\mathrm{Lift}(g) : N \to J^r(B, F)$$

satisfying $\pi_L \circ \mathrm{Lift}(g) = g$.

The lift $\mathrm{Lift}(g)$ *is unique once we fix* $\mathrm{Lift}(g)(x)$ *for some* $x \in N$.

Proof. Write $g(p) = (x(p), z^r(p))$. By construction, $g^*\Omega_{\mathrm{can}} = 0$. Hence, $g^*\lambda_{\mathrm{can}}$ is closed. Using the contractibility of N, we deduce that each component of $g^*\lambda_{\mathrm{can}}$ is exact. We choose primitives, which we denote suggestively by $z^{r-1} : N \to \mathrm{Sym}^{r-1}(B^*, F)$. These functions are unique up to a shift by an element of $\mathrm{Sym}^{r-1}(B^*, F)$.

We put together g with the chosen primitives to produce a map

$$h := (x, z^r, z^{r-1}) : N \to B \oplus \mathrm{Sym}^r(B^*, F) \oplus \mathrm{Sym}^{r-1}(B^*, F).$$

We can readily check, using Lemma 3.6.21, that

$$h^*\alpha^r = dz^{r-1} + g^*\lambda_{\text{can}} = 0.$$

Furthermore, consider the 2-form with values in $\text{Sym}^{r-2}(B^*, F)$:

$$\Omega^{r-1}_{\text{can}} = \left(\sum_{a=1}^{n} dx_a \wedge dz_j^{(i_1,\cdots,i_a+1,\cdots,i_n)} \right)_{|(i_1,\cdots,i_a,\cdots,i_n)|=r-2}.$$

It corresponds to the curvature of $\xi^{(1)}_{\text{can}}$, which depends only on the coordinates (x, z^{r-1}). We can compute:

$$h^*\Omega^{r-1}_{\text{can}} = h^* \left(-\sum_{a,b=1}^{n} z_j^{(i_1,\cdots,i_a+1,\cdots,i_b+1,\cdots,i_n)} dx_a \wedge dx_b \right) = (0).$$

In the last step we get zero because cross derivatives agree. This computation tells us that the map

$$(x, z^{r-1}) : N \to B \oplus \text{Sym}^{r-1}(B^*, F)$$

is isotropic. Therefore, the argument can be iterated for decreasing r to produce a lift. □

From the proof we see that the contractibility assumption on N is used to ensure that the restriction of the Liouville form at each step is exact. More in general, we could define:

Definition 3.6.23. *An isotropic submanifold N of $(B \oplus \text{Sym}^r(B^*, F), \Omega_{\text{can}})$ is said to be **exact** if it admits an integral lift to $(J^r(B, F), \xi_{\text{can}})$.*

Corollary 3.6.24. *N is exact if and only if:*

- *$\lambda_{\text{can}}|_N$ is exact.*
- *Its isotropic lift to $(B \oplus \text{Sym}^{r-1}(B^*, F), \Omega_{\text{can}})$ is exact.*

Observe that by recursion N is exact if and only if all of its lifts are exact.

3.6.4.2 Lifting curves

Let us particularise now to the case $\dim(B) = 1$. Then, in holonomic coordinates $(x, y = z^0, z)$ the Cartan 1-forms read

$$\alpha^l = dz^l - z^{l+1}dx, \qquad l = 0, \ldots, r-1.$$

The particular flexibility of curves (compared to higher dimensional integral submanifolds) stems from the fact that any

$$g(t) = (x(t), z_r(t)) : [0, 1] \to B \oplus \text{Sym}^r(B^*, F)$$

is automatically isotropic. Then, following the recipe outlined in the proof of Theorem 3.6.22, we solve for the z^{r-1} coordinates using α^r:

$$g^*\alpha^r = z_{r-1}(t)dt - z_r(t)x'(t)dt$$

leading to the integral expression

$$z_{r-1}(t) = z_{r-1}(0) + \int_0^t z_r(s)x'(s)ds$$

which uniquely recovers z_{r-1} up to the choice of lift $z_{r-1}(0)$. Proceeding decreasingly in l we can solve for all the $z^l(t)$, effectively lifting g to an integral curve $\text{Lift}(g) : [0,1] \to J^r(B,F)$.

According to Lemma 3.2.37, the lift $\text{Lift}(g)$ is immersed if and only if g was immersed. Assuming g is immersed, the front tangencies $\Sigma(\text{Lift}(g), \pi_f)$ correspond precisely to the singularities of tangency $\Sigma(g, \pi_b)$. This implies that to control the singularities of an integral curve it is sufficient to control the singularities of its metasymplectic projection, which is a smooth curve with no constraints.

3.6.4.3 Restricted metasymplectic projection

Unlike curves, higher-dimensional isotropic/integral submanifolds cannot be deformed freely. To get rid of differential constraints we consider instead:

Definition 3.6.25. *The **principal metasymplectic projection** with respect to the principal direction determined by the coordinate x_n is the map:*

$$\begin{aligned} \pi_L^n : J^r(B,F) &\to B \oplus \text{Sym}^r(\mathbb{R}^*, F) \\ (x,y,z) &\to (x, z^{(0,\dots,0,r)}). \end{aligned}$$

That is, we only remember the pure r-order derivatives associated to x_n. We then work with Σ^2-free maps whose rank drops along the x_n-directions. We think of them as $(n-1)$-families of curves, allowing us to prove:

Lemma 3.6.26. *Given a smooth map:*

$$\begin{aligned} g : B &\to B \oplus \text{Sym}^r(\mathbb{R}^*, F) \\ (t) = (\tilde{t}, t_n) = (t_1, \dots, t_n) &\to (\tilde{t}, x_n(t), z^{(0,\dots,0,r)}(t)), \end{aligned}$$

there exists an integral map $\text{Lift}(g) : B \to J^r(B,F)$ satisfying $\pi_L^n \circ \text{Lift}(g) = g$.

The map $\text{Lift}(g)$ is unique up to the choice of $\text{Lift}(g)|_{\{t_n=0\}}$.

Proof. The integral lift Lift(g) is given by the formula:

$$\begin{aligned}(t) \quad \mapsto \quad & (\tilde{t}, x_n; \\ & y = z^{(0,\ldots,0,0)}; \\ & \partial_{\tilde{t}} y, z^{(0,\ldots,0,1)}; \\ & \partial_{\tilde{t}}^2 y, \partial_{\tilde{t}} z^{(0,\ldots,0,1)}, z^{(0,\ldots,0,2)}; \\ & \ldots; \\ & \partial_{\tilde{t}}^r y, \ldots, \partial_{\tilde{t}} z^{(0,\ldots,0,r-1)}, z^{(0,\ldots,0,r)}).\end{aligned}$$

All the terms on the right hand side depend only on t. Let us explain how the other functions are obtained from t, x_n and $z^{(0,\ldots,0,r)}$.

The term $z^{(0,\ldots,0,l)}$ is the (formal) pure derivative of order l in the direction of x_n and it is defined (for decreasing l) by the integral expression:

$$z^{(0,\ldots,0,l)}(t) := z^{(0,\ldots,0,l)}(\tilde{t}, 0) + \int_0^{t_n} z^{(0,\ldots,0,l+1)}(\tilde{t}, s) x_n'(\tilde{y}, s) ds,$$

following what we did in the previous subsection for curves. In particular, the coordinate $y = z^{(0,\ldots,0,0)}$ is recovered by integrating r times. At every step we can choose the value of $z^{(0,\ldots,0,l)}(\tilde{t}, 0)$.

All other functions are derivatives of the form $\partial_{\tilde{t}}^i z^{(0,\ldots,0,j)}$, for some integers i and j. Hence, we obtain them, uniquely, by differentiation. □

Recall from Section 3.5 that the polar space of a $(n-1)$-dimensional horizontal element is $(n+k)$-dimensional and intersects the vertical fibre exactly in the associated principal direction. This implies that any construction of Σ^2-free integral maps by fixing $n-1$ base directions and a (formal) derivative must necessarily use a pure derivative. This indicates that the method presented is general.

Most of the key properties of the lift can be read from the original map:

Corollary 3.6.27. *Let g be a map into a principal metasymplectic projection. Then:*

- *The map* Lift(g) *is well-defined, smooth, integral and* Σ^2*-free.*
- $\Sigma(\mathrm{Lift}(g))$ *is in correspondence with* $\Sigma(g)$*.*
- $\Sigma(\mathrm{Lift}(g), V_{\mathrm{can}})$ *is in correspondence with* $\Sigma(g, V_{\mathrm{can}})$*. In the second term,* V_{can} *denotes the component* $\mathrm{Sym}^r(\mathbb{R}^*, F)$ *of* $B \oplus \mathrm{Sym}^r(\mathbb{R}^*, F)$*.*

3.6.5 Structure results about Σ^2-free multi-sections

In this Subsection we exploit the ideas introduced in Subsection 3.6.4 to prove three results about Σ^2-free integral maps. Under some mild assumptions, they are lifts of maps into a principal metasymplectic projection (subsection 3.6.5.1), they admit a generating function description (subsection 3.6.5.2), and they can be put in Thom-Boardman form (subsection 3.6.5.3).

3.6.5.1 Σ^2-free multi-sections are lifts

The statement we want to prove is:

Proposition 3.6.28. *Let N be a smooth n-manifold. Let $f : N \to J^r(Y \to X)$ be a Σ^2-free integral map.*

Then, given $p \in N$, there exists g mapping into a principal metasymplectic projection such that $f|_{\mathcal{O}p(p)} = \mathrm{Lift}(g)$.

Proof. If p is non singular the claim is immediate. Suppose then that p is a singular point. Choose a locally defined hyperplane $L \ni p$ such that $f|_L$ has maximal rank. Its image $f(L)$ is then an $(n-1)$-dimensional horizontal submanifold of $J^r(Y \to X)$. Hence, there is a trivialisation $J^r(B, F)$ of $\mathcal{O}p(f(p)) \subset J^r(Y \to X)$ with holonomic coordinates (x, y, z) such that: $f(L)$ is contained in the zero section B and, further, is spanned by the first $n-1$ coordinates $(\tilde{x})$.

Consider the foliation $\mathcal{F}$ in $\mathcal{O}p(\pi_b \circ f(p))$ by lines parallel to the x_n-axis in X. The foliation by lines $(\pi_b \circ f)^*\mathcal{F}$ extends the kernel line field of f (Definition 3.6.7) to a smooth line field defined everywhere in $\mathcal{O}p(p) \subset N$. We pullback the $(\tilde{x})$–coordinates on $f(L)$ to L, where we denote them by $(\tilde{t})$. Then we use the flow of a vector field spanning $(\pi_b \circ f)^*\mathcal{F}$ to produce coordinates (t) around p.

Consider the principal metasymplectic projection π associated to x_n (Definition 3.6.25). The composition $g := \pi \circ f$ is fibered over the $\tilde{t} = \tilde{x}$ coordinates and, by construction, its integral lift is f (where the choice of initial values is zero). □

3.6.5.2 Σ^2-free multi-sections admit generating functions

The following is a modest generalisation to higher jets of Arnold's result stating that any embedded legendrian can locally be given by a generating function [6]. Note that our result does not apply to singularities of rank greater than 1. However, we do allow arbitrary Σ^2-free singularities, generalising the case of Whitney singularities studied by Givental [58].

Proposition 3.6.29. *Let B and F be vector spaces with $\dim(F) = 1$. Then, any germ f of Σ^2-free integral embedding into $J^r(B, F)$ admits a generating function.*

It is unclear whether the assumption $\dim(F) = 1$ can be dropped. That would require studying more general generating functions in Subsection 3.6.3.4.

Proof. Outside of its singularity locus, f is (up to reparametrisation) a holonomic section and it therefore admits a generating function. Consider then a singular point $p \in N$. Since f is embedded, the point p is a singularity of tangency. According to our description of integral elements of corank 1 (Subsection 3.5.3), the integral element $df(T_pN)$ intersects the vertical distribution in a principal direction: a pure r-order derivative associated to the singular hyperplane $d(\pi_b \circ f)(T_pN)$, as in Definition 3.6.8.

We can then use Proposition 3.6.28 to produce:

- holonomic coordinates (x, y, z) in $J^r(B, F)$; we write $(x) = (\tilde{x}, x_n)$,
- coordinates $(t) = (\tilde{t}, t_n)$ in $\mathcal{O}p(p) \subset N$,
- a principal metasymplectic projection π such that: f is a lifting of $\pi \circ f$ and its projection $\pi \circ f$ is graphical over the $z^{(0,\dots,0,r)}$ derivative.

In these coordinates we may write explicitly:

$$(\pi \circ f)(t) = (\tilde{x}(t) = \tilde{t}; x_n(t); z^{(0,\dots,0,r)}(t) = t_n).$$

For clarity of notation we denote $g(t)$ for the function $x_n(t)$.

We follow the method presented in Subsection 3.6.3.4. We claim that f is generated by a function of the form:

$$\begin{aligned} G : B \times \mathbb{R} &\to F \\ G(x, s) &= \left(H(\tilde{x}) + \frac{1}{r!} \int_0^s (x_n - g(\tilde{x}, s))^r ds \right), \end{aligned}$$

where H is a function (to be specified now) which depends only of the $\tilde{x}$–coordinates.

Indeed: let Γ be the fibrewise singularity locus $\{x_n = g(\tilde{x}, s)\}$ of G. It parametrises the integral submanifold L_G. Since Γ is graphical over the $(\tilde{x}, s)$ coordinates, it is a smooth manifold. Applying Lemma 3.6.19 we deduce that so is L_G. From the computation $\partial^r_{x_n} G = s$ it follows that s parametrises the $z_k^{(0,\dots,0,r)}$-coordinate of L_G. That is, both L_G and f are lifts of the same principal metasymplectic projection $\pi \circ f$. Hence, they differ only on the initial conditions as we lift. Choosing H amounts to choosing these initial conditions. Indeed, H specifies the front of G at $t_n = 0$, so we set it to be $H(\tilde{t}) = y \circ f(\tilde{t}, 0)$, concluding the proof. □

3.6.5.3 Thom-Boardman for Σ^2-free multi-sections

Before stating the result let us define:

Definition 3.6.30. *Let $f : N \to J^r(Y \to X)$ be an integral map (Definition 3.2.12). A curve $\gamma : I \to N$ is said to be* ***vertical*** *if $f \circ \gamma$ is tangent to V_{can} (Definition 3.2.20).*

Due to the rigidity phenomenon pointed out in Remark 3.6.12, vertical curves may represent an obstruction to achieving transversality. Constraining them allows us to prove the following genericity statement:

Proposition 3.6.31. *Let N be an n-dimensional manifold. Let $f : N \to J^r(Y \to X)$ be a Σ^2-free integral immersion satisfying:*

- *Every vertical curve in f can be extended to a curve that is somewhere not vertical.*

Then, up to a C^∞-small perturbation, f can be assumed to be in Thom-Boardman form.

Proof. We provide a sketch. For each singular point $p \in N$:

- We take the maximal vertical curve γ containing it.
- We choose a principal metasymplectic projection π (Definition 3.6.25) such that $f(\gamma)$ is contained in the domain of π.
- As in the proof of Proposition 3.6.28 we can extend the kernel linefield of f to a foliation by lines $\mathcal{F}$ everywhere on N. Let U be a neighborhood of γ saturated by integral leaves of $\mathcal{F}$.

According to the last item, U can be identified with the cube $[-1,1]^n$ so that the last coordinate corresponds to $\mathcal{F}$. We write $U^{\pm} \cong [-1,1]^{n-1} \times \{\pm 1\}$ under this identification. By assumption, γ can be slightly enlarged to a curve that is not vertical at its endpoint. This allows us to assume that the points in U^+ are non-singular.

We can now cover N by a finite collection $\{U_i\}$ of such neighbourhoods. We denote the corresponding curve, kernel line field, and projection by γ_i, $\mathcal{F}_i$, and π_i, respectively. We also write $U_i^{\pm}$.

Inductively on i, we regard $\pi_i \circ f|_{U_i}$ as a fibered over $[-1,1]^{n-1}$ family of maps of $[-1,1]$ into $\mathbb{R} \oplus \mathrm{Sym}^r(R^*, F)$. We apply the standard Thom transversality Theorem 3.3.6 to them so that the singularities are in Thom-Boardman form. We then apply Lemma 3.6.26 to lift this to a perturbation of $f|_{U_i}$ itself. In order to work inductively in i:

- We observe that, due to the lack of singularities, $f|_{\mathcal{O}p(U_i^+)}$ is graphical over the zero section. Thus, it is a reparametrisation of a holonomic section. This allows us to interpolate freely between the original value of $f|_{\mathcal{O}p(U_i^+)}$ and whatever perturbation we choose (since the two are very close).
- Along the other end U_i^- we leave $\pi_i \circ f$ untouched. Further, we choose $f|_{U_i^-}$ as the initial value for the lifting Lemma 3.6.26.
- Additionally, we do not perturb $\pi_i \circ f$ close to the rest of the boundary of U_i.

These requirements imply that the perturbation we construct is relative to the boundary of $\mathcal{O}p(U_i)$, allowing us to iterate the argument. □

3.7 Multi-sections: Models of singularities

In this Section we present some simple singularities for Σ^2-free multi-sections. In Subsection 3.7.1 we describe singularities of tangency of Whitney type. In Subsection 3.7.2 we use Whitney singularities to define models of singularities of tangency along submanifolds (as opposed to germs at points). Lastly, in Subsection 3.7.3 we look at singularities of mapping.

Remark 3.7.1. Our naming conventions for singularities reflect the behaviour of the integral maps themselves, not their front projections. In particular, the names we use often refer to their singularities of tangency with the vertical distribution. When singularities of mapping are present, we point it out explicitly. △

3.7.1 Whitney singularities in jet spaces

In Definition 3.3.12 we introduced smooth Whitney singularities. In this Subsection we study their analogues for integral submanifolds in jet space $J^r(B,F)$.

In subsection 3.7.1.2 we use the generating function method presented in subsection 3.6.3.4 to provide explicit models of Whitney singularities. We then state Givental's Theorem 3.7.5: Whitney singularities are stable when $\dim(F) = 1$ [58], generalising the theorem of Morin [93] to jet spaces. In subsection 3.7.1.4 we use Theorem 3.7.5 to prove a global stability result for embedded multi-sections with Whitney singularities. Lastly, in subsections 3.7.1.5 and 3.7.1.6 we describe the fold and the pleat in detail. They will play a role in Subsections 3.7.2 and 3.7.3.

3.7.1.1 The definition

Definition 3.7.2. *Let $f : N \to J^r(B,F)$ be a Σ^2-free integral mapping. The germ of f at a point p is a **Whitney singularity** if:*

- *f is an immersion at p, and*
- *the base map $\pi_b \circ f$ has a Whitney singularity at p.*

*The **index** of f at p is the index of $\pi_b \circ f$. The Whitney singularities of indices 1 and 2 are called the **fold** and the **pleat**, respectively.*

Sometimes, in order to stress that we are referring to Whitney singularities in jet space, we call them *r-times differentiable Whitney singularities.*

Lemma 3.3.16 implies:

Corollary 3.7.3. *A germ of Whitney singularity $f : \mathcal{O}p(\{0\}) \to J^r(B,F)$ has index-j Whitney singularities along $\Sigma^{1^j0}(f, V_{\rm can})$.*

3.7.1.2 Generating functions

Recall the notation from Subsection 3.3.3: endow $\mathbb{R}^{n+1}$ with coordinates $(q_1,\dots,q_n,x)$ and denote $q = (q_1,\dots,q_n)$ and $\tilde{q}_l = (q_1,\dots,\widehat{q}_l,\dots,q_n)$. Consider the fibration $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by $(q,x) \mapsto q$. We set

$$F_l(q,x) = x^{l+1} + q_1 x^{l-1} + \cdots + q_l,$$

and we let $\Gamma_l := F_l^{-1}(0)$ be the locus of roots of $x \to F_l(q,x)$. The coordinates $(\tilde{q}_\ell, x)$ parametrise Γ_l; we denote $s_l : \mathbb{R}^n(\tilde{q}_l, x) \to \Gamma_l$. Finally, $\Gamma_l^j \subset \Gamma_l$ denotes the locus of roots of multiplicity at least j.

Define the generating functions:

$$
\begin{aligned}
G_{r,l} : \mathbb{R}^{n+1} &\to \mathbb{R}^k \\
(q,x) &\mapsto \left(\int_0^x (F_l(q,s))^r ds, 0, \ldots, 0 \right),
\end{aligned}
\tag{3.7.1.1}
$$

where r is the order of the jet space and $l \leqslant n$.

The function F_l is of the form prescribed in the subsection 3.6.3.4: a submersion whose zero locus is smooth with generic tangencies. Therefore, the loci $L_{G_{r,l}}$ are smooth integral manifolds which are parametrised by the locus of roots $\Gamma_l \cong \mathbb{R}^n(\tilde{q}, x)$. This is shown as the dashed diagonal arrow in the following diagram:

$$
\begin{array}{ccccc}
\mathbb{R}^n(\tilde{q},x) & \xrightarrow{s_l} & \Gamma_l \subset \mathbb{R}^{n+1}(q,x) & \xrightarrow{j^r G_{r,l}} & J^r(\mathbb{R}^{n+1}, \mathbb{R}^k) \\
 & & \downarrow \pi & \searrow & \downarrow \\
 & & \mathbb{R}^n(q) & \xleftarrow[\pi_b]{} & L_{G_{r,l}} \subset J^r(\mathbb{R}^n, \mathbb{R}^k)
\end{array}
$$

The composition of the parametrisation with the base projection $\pi_b : J^r(\mathbb{R}^n, \mathbb{R}^k) \to \mathbb{R}^n$ is precisely the $(n-l)$-fold stabilisation of the l-th Whitney map. It follows that:

Lemma 3.7.4. *The germ at the origin of $L_{G_{r,l}}$ is a $(n-l)$-fold stabilization of Whitney singularity of index l.*

The main result in [58] says that the models just constructed are in fact unique up to equivalence if $k = 1$. The left equivalences we consider are point symmetries (as defined in subsection 3.2.3.1).

Theorem 3.7.5 (A. Givental)**.** *Any r-times differentiable Whitney singularity of index l in $J^r(B, \mathbb{R})$ is equivalent to $L_{G_{r,l}}$.*

For $k > 1$ the same result holds as long as $l \leqslant k$, due to general position arguments. However, the general case seems not to be addressed in [81, 58] and we do not know whether the uniqueness statement fails. We leave this as an open question.

3.7.1.3 Remark: Whitney singularities as discriminants

In [58], Givental interpreted the r-times differentiable Whitney singularities as discriminants in certain spaces of polynomials. For completeness, let us review this construction. Consider the space:

$$
\mathcal{P}_{r,n} := \left\{ \begin{array}{c} \text{Polynomials of degree } r(n+1)+1 \text{ in one} \\ \text{variable } x, \text{ whose derivative has } n+1 \\ \text{roots of multiplicity } r \end{array} \right\}.
$$

We explicitly parametrise $\mathcal{P}_{r,n}$ as the space of polynomials of the form:

$$f(x) = \int_0^x (s^{n+1} + q_1 s^{n-1} + \cdots + q_n)^r ds - y = G_{r,n}(q,x) - y$$

with coefficients $(q,y) \in \mathbb{R}^{n+1}$.

We let $\Delta_{r,n} \subset \mathcal{P}_{r,n}$ be the subset consisting of polynomials with multiple roots. Now, f has a multiple root at x if and only if $f(x) = 0$ and $f'(x) = 0$. By construction, any multiple root of f has multiplicity at least $r+1$, since all the roots of $f'(x)$ have multiplicity r. Then, the polynomials with multiple roots can be parametrised by

$$\Delta_{r,n} = \{(q,y) \in \mathbb{R}^{n+1} \mid \exists x \in \mathbb{R}, \text{ such that } F_n(q,x) = 0, y = G_{r,n}(q,x)\},$$

which is precisely the front projection of $L_{G_{r,n}}$.

3.7.1.4 Global stability

The main result in this Subsection is the global counterpart of Theorem 3.7.5:

Proposition 3.7.6. *Let $f : N \to J^r(B,\mathbb{R})$ be a multi-section with Whitney singularities and such that $\pi_b \circ f : N \to B$ is an embedding. Then f is stable (Definition 3.3.8), up to point symmetries (Definition 3.2.44), among integral maps (Definition 3.2.12).*

Proof. Let $(f_s)_{s\in[0,1]} : N \to J^r(B,\mathbb{R})$ be a deformation of $f_0 := f$. Since $\pi_b \circ f$ is topologically embedded and has Whitney singularities, it is stable. We can therefore assume that the deformation $(\pi_b \circ f_s)_{s\in[0,1]}$ is trivial, so that all the f_s lift the same base map. In particular, all of them have the same singularity locus.

Now we proceed by induction, decreasingly on the index l of the Whitney singularities of f_s. We assume, by induction hypothesis, that $(f_s)_{s\in\mathcal{O}p(0)}$ is trivial in a neighbourhood of $\Sigma^{1^{l+1}}(f)$. Then, for each point $p \in \Sigma^{1^l0}(f)$ we apply Theorem 3.7.5 to produce a fibrewise isotopy $\psi_s^p : Y \to Y$, supported on a neighbourhood of $\pi_f \circ f(p)$, so that

$$(j^r\psi_s^p) \circ f_s|_{\mathcal{O}p(p)} = f|_{\mathcal{O}p(p)}.$$

We choose a finite collection $p_i \in \Sigma^{1^l0}$ such that the domains of the corresponding $\psi_s^{p_i}$ cover $\Sigma^{1^l0}(f)\backslash\mathcal{O}p(\Sigma^{1^{l+1}}(f))$. Then we define a semi-local isotopy $(\psi_s)_{s\in\mathcal{O}p(0)}$ on $\mathcal{O}p(\Sigma^{1^l}(f))$ by interpolating between the $\psi_s^{p_i}$ using a partition of unity. Note that, indeed, we can simply interpolate linearly between the different diffeomorphisms because, for small s, all of them are graphical over the identity.

The isotopy ψ_s makes all the $(f_s)_{s\in\mathcal{O}p(0)}$ agree in $\mathcal{O}p(\Sigma^{1^l}(f))$. This completes the inductive step. □

Using the Proposition we can prove a (rather weak) version of the Weinstein neighbourhood theorem in jet spaces:

Corollary 3.7.7. *Let $f : N \to J^r(B,\mathbb{R})$ be an embedded multi-section with Whitney singularities. Then f is stable, up to germs of contact transformations, among integral maps.*

Proof. Let $(f_s)_{s\in[0,1]} : N \to J^r(B,\mathbb{R})$ be a deformation of $f_0 := f$. We claim that there is a germ of isotopy $(\psi_s)_{s\in[0,1]}$ in $\mathcal{O}p(\text{Image}(f))$ satisfying:

- $\psi_s \circ f_s = f$,
- $\psi_s^*\xi_{\text{can}} = \xi_{\text{can}}$.

The key claim is the following: since f is embedded, we may assume that its front projection $\pi_f \circ f$ is a topological embedding. To see this first note that the claim is true locally for any germ of Whitney singularity. We then globalise as follows: We consider $\mathcal{O}p(\text{Image}(f))$ and we quotient it by the connected components of the fibres of the base projection. This yields a new front manifold in which the claim holds.

Now we apply the Proposition 3.7.6 to produce the family ψ_s. □

Arguing similarly, one should be able to prove the following stronger version: the germ of ξ_{can} along an integral embedding with Whitney singularities $f : N \to J^r(B,\mathbb{R})$ is fully encoded in its singularity locus, together with Maslov coorientation data (see subsection 3.5.5.3).

3.7.1.5 Folds

We use the same notation as in subsection 3.7.1.2.

Definition 3.7.8. *The A_{2r}-**cusp** is the germ at the origin of the map:*

$$\begin{aligned} A_{2r} : \mathbb{R}^n &\to \mathbb{R}^{n+k} \\ (\tilde{q}, x) &\to (\tilde{q}, x^2, x^{2r+1}, 0, \dots, 0). \end{aligned} \tag{3.7.1.2}$$

We see that its singularity locus is the hyperplane

$$\Sigma(A_{2r}) = \Sigma^{10}(A_{2r}) = \{x = 0\}.$$

In subsection 3.7.1.2 we showed that the A_{2r}-cusp is the front projection of an r-times differentiable fold.

3.7.1.6 Pleats

We continue using the same notation.

Definition 3.7.9. *The A_{2r}-**swallowtail** is the germ at the origin of the mapping:*

$$\begin{aligned} \text{Sw}_{2r} : \mathbb{R}^n &\to \mathbb{R}^{n+k} \\ (\tilde{q}, x) &\to (\tilde{q}, -x^3 - q_1 x, \int_0^x (s^3 + q_1 s - x^3 - q_1 x)^r ds, 0, \dots, 0). \end{aligned} \tag{3.7.1.3}$$

This is the front projection of the (r-times differentiable) pleat. Its singularity locus reads:

$$\Sigma^1(\mathrm{Sw}_{2r}) = \{3x^2 + q_1 = 0\}, \qquad \Sigma^{11}(\mathrm{Sw}_{2r}) = \{x, q_1 = 0\}.$$

The A_{2r}-swallowtail has a fibered nature: We may split the q-coordinates into two groups $q^0 = (q_1, \dots, q_l)$ and $q^1 = (q_{l+1}, \dots, q_{n-1})$. The first we regard as parameters and the second as base variables. We write

$$\mathrm{Sw}_{2r}^{q^0} : \mathbb{R}^{n-l} \quad \to \quad \mathbb{R}^{n-l+k}$$

for the map obtained by fixing the q^0-variables.

If $q_1 > 0$, the map $\mathrm{Sw}_{2r}^{q^0}$ has no singularities and is graphical over the base. If $q_1 < 0$, $\mathrm{Sw}_{2r}^{q^0}$ has a pair of A_{2r}-cusps. At $q_1 = 0$ a birth/death phenomenon takes place:

Definition 3.7.10. *The lift to r-jet space of the front:*

$$(q^1, x) \mapsto (q^1, x^3, x^{3r+1}, 0, \dots, 0)$$

*is called the **first Reidemeister move**.*

It is an embedded integral manifold whose singularity locus is not of Whitney type.

3.7.2 Semi-local singularities of tangency

We now describe several models of singularities of tangency for Σ^2-free integral embeddings. These models rely on the lifting techniques for submanifolds in a principal metasymplectic projection (Definition 3.6.25); see Lemma 3.6.26 in subsection 3.6.4.3. The singularities we present are semi-local in the sense that they are not germs at points but around higher dimensional submanifolds.

The singularities we go through are: the double fold (subsection 3.7.2.3), the regularised wrinkle (subsection 3.7.2.4) and the stabilisation (subsection 3.7.2.6). We also discuss their birth/death phenomena.

3.7.2.1 Semi-local Maslov coorientation

To describe our singularities intrinsically (instead of through a local model), we want to prescribe how they intersect the Maslov hypersurface $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ (Definition 3.5.19). We need a notion of coorientation to do so but, as seen in subsection 3.5.5.3, $\mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$ is not always coorientable. Coorientability was the necessary ingredient for defining a global Maslov class (Definition 3.5.23).

Despite of this, in subsection 3.5.5.5 we pointed out that, in a neighbourhood of an integral element $W \in \mathrm{Gr}_{\Sigma^1}(\mathfrak{g}, n)$, it is always possible to define a local Maslov class. We now explain a global analogue of this; it will allow us to talk about Maslov class and coorientation.

Our singularities will be fibered over some base manifold D, which will be either $\mathbb{S}^{n-1}$ or $\mathbb{R}^{n-1}$. We set $X = D \times \mathbb{R}$ with (x) coordinates and we let x_n be the coordinate in the $\mathbb{R}$ factor. Similarly, we let F be a vector space with coordinates $(y_1, \ldots, y_k)$. We work in $J^r(X, F)$. Inside of $\mathrm{Gr}_{\Sigma^1}(\xi_{\mathrm{can}}, n)$ we define the submanifold:

$$\mathrm{Gr}_D := \left\{ W \in \mathrm{Gr}_{\Sigma^1}(\xi_{\mathrm{can}}, n) \mid \begin{array}{c} d\pi_b(W) \text{ is transverse to } \partial_{x_n} \\ dy_1(W) \neq 0 \end{array} \right\}.$$

I.e. Gr_D is the submanifold of integral planes whose projection to the base is graphical over TD and whose principal direction is graphical over the y_1 coordinate. It follows that:

Lemma 3.7.11. $\mathrm{Gr}_D \subset \mathrm{Gr}_{\Sigma^1}(\xi_{\mathrm{can}}, n)$ *is open, contractible, and dense.*

Contractibility of Gr_D implies coorientability. Then, just like in Proposition 3.5.20:

Corollary 3.7.12. *Fix a Maslov coorientation along* Gr_D*.*

There non-zero, non-torsion element:

$$\mathrm{Ind}([\gamma]) := |\gamma \cap \mathrm{Gr}_D| \in H^1(\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, n) \cup \mathrm{Gr}_D, \mathbb{Z}),$$

where γ *is a curve representative intersecting* Gr_D *transversally. The count of intersection points takes into account signs, comparing the orientation of* γ *with the Maslov coorientation.*

3.7.2.2 Preferred principal metasymplectic projections

We will construct explicit models of our singularites using the lifting Lemma 3.6.26. We single out a preferred principal metasymplectic projection (subsection 3.6.4.3) to do so.

Using the coordinates (x, y) in $X \times F$ we define holonomic coordinates (x, y, z) in $J^r(X, F)$ and we look at the projection:

$$\pi_L^n(x, y, z) := (x, z^{(0,\ldots,r)}).$$

Of particular interest is the term $z_1^{(0,\ldots,r)}$, i.e. the projection onto the pure r-derivative of y_1 with respect to x_n.

3.7.2.3 The double fold

The reader should compare the following definition to its smooth analogue Definition 3.3.18:

Definition 3.7.13. *Set* $D = \mathbb{S}^{n-1}$*. An integral embedding fibered over* D

$$f : D \times \mathcal{O}p([0,1]) \quad \rightarrow \quad J^r(X, F)$$

is a ***double fold*** *if*

$$\Sigma(f, V_{\mathrm{can}}) = D \times \{0\} \cup D \times \{1\}$$

and these are folds of opposite Maslov coorientations. The image $f(D \times (0,1))$ *is called the* ***membrane*** *of the double fold.*

A particular model can be provided using the lifting procedure of subsection 3.6.4.3. We define a map into the domain of the principal metasymplectic projection π_L^n:

$$\begin{aligned} \sigma : D \times \mathcal{O}p([0,1]) &\rightarrow X \oplus \mathrm{Sym}^r(\mathbb{R}^*, F) \\ (\tilde{x}, x_n) &\rightarrow (\tilde{x}, x_n^3/3 - x_n; z_1^{(0,\dots,r)} = x_n, 0, \dots 0). \end{aligned}$$

I.e. all the functions $z_i^{(0,\dots,r)}$ are zero for $i \neq 1$. Using Corollary 3.6.27 we see that $\mathrm{Lift}^r(\sigma)$ is a double fold. Its front projection reads:

$$(3.7.2.1) \quad (x) \rightarrow \left(\tilde{x}, x_n^3 - x_n; \int_0^{x_n} \int_0^{s_1} \dots \int_0^{s_{r-1}} s_r \prod_j (s_j^2 - 1) ds_r \dots ds_1, 0 \dots, 0\right).$$

Its singularity locus is comprised of two spheres $\{|x_n| = 1\}$ of A_{2r}-singularities.

Lemma 3.7.14. *Suppose* $\dim(F) = 1$*. Then any double fold* f *is equivalent to* $\mathrm{Lift}(\sigma)$ *(using point symmetries in the target, and diffeomorphisms in the domain).*

Proof. The first observation is that we may assume $\pi_b \circ f = \pi_b \circ \mathrm{Lift}(\sigma)$ because both maps are the usual smooth double fold. We take the identification to preserve the fibered structure of the maps.

Denote the singular spheres of f by S_0 and S_1. S_0 is horizontal, so we can find a point symmetry identifying $f|_{S_0}$ with $\mathrm{Lift}(\sigma)|_{D\times\{0\}}$; the analogous statement holds for S_1. Since $\dim(F) = 1$, we have that the Maslov coorientation along S_0 induced by f can be extended over the membrane to be compared with the Maslov coorientation over S_1. The same can be done with $\mathrm{Lift}(\sigma)$. By assumption, in both cases we obtain the opposite Maslov coorientation.

We can then reason as in Proposition 3.7.6 (invoking Givental's stability Theorem 3.7.5 pointwise and then glueing), to show that f along S_0 (resp. S_1) is equivalent to $\mathrm{Lift}(\sigma)$ along $D \times \{0\}$ (resp. $D \times \{1\}$). Since any two holonomic sections are equivalent (and the membrane is the graph of a holonomic section) we can extend these identifications to the interior, concluding the claim. Note that we must use the condition on the Maslov coorientation hypothesis when invoking Givental: this allows us to restrict to point symmetries that preserve the coorientation. □

3.7.2.4 The regularised wrinkle

Compare the following notion to the smooth wrinkle (Definition 3.3.17):

Definition 3.7.15. *Set* $D = \mathbb{R}^{n-1}$*. A fibered over* D *integral embedding*

$$f : \mathcal{O}p(\mathbb{S}^{n-1}) \subset X \quad \rightarrow \quad J^r(X, F)$$

*is a **regularised wrinkle** if its singularities are:*

- $\Sigma^{110}(f, V_{\mathrm{can}}) = \mathbb{S}^{n-2}$ *pleat locus,*
- $\Sigma^{10}(f, V_{\mathrm{can}}) = \mathbb{S}^{n-1} \backslash \mathbb{S}^{n-2}$ *fold locus.*

If f extends to a horizontal embedding of the open disc, we say that $f(\mathring{\mathbb{D}}^n)$ is the ***membrane*** *of the wrinkle.*

Observe that, close to the pleat locus, the two hemispheres have opposite Maslov coorientations, so the same is true semi-locally in the whole regularised wrinkle (note that this statement uses the fibered nature of the map).

A particular model can be produced from the following map into the domain of π_L^n:

$$(3.7.2.2) \qquad \begin{aligned} \sigma : \mathcal{O}p(\mathbb{S}^{n-1}) &\to X \oplus \mathrm{Sym}^r(\mathbb{R}^*, F) \\ (\tilde{x}, x_n) &\to (\tilde{x}, x_n^3/3 + (|\tilde{x}|^2 - 1)x_n; z_1^{(0,\dots,r)} = x_n, 0, \dots 0). \end{aligned}$$

Its lift $\mathrm{Lift}^r(\sigma)$ is a regularised wrinkle. Its front projection reads:

$$(\tilde{x}, x_n) \mapsto \left(\tilde{x}, x_n^3/3 + (|\tilde{x}|^2 - 1)x_n; \int_0^{x_n} \int_0^{s_1} \dots \int_0^{s_{r-1}} s_r \prod_j (s_j^2 + |\tilde{x}|^2 + 1) ds_r \dots s_1, 0, \dots, 0\right).$$

Reasoning as in Lemma 3.7.14:

Lemma 3.7.16. *Suppose* $\dim(F) = 1$*. Then any regularised wrinkle f is equivalent to* $\mathrm{Lift}(\sigma)$ *(using point symmetries in the target, and diffeomorphisms in the domain).*

3.7.2.5 Fibered regularised wrinkles

Usual smooth wrinkles are fibered, as explained in subsection 3.3.4.3. The same is true for the regularised wrinkle in r-jet space. We let $D = \mathbb{R}^{m+n-1}$, where the first m-coordinates (q) are regarded as parameters and the last $(n-1)$-coordinates $(\tilde{x})$ are domain coordinates. We fix $X = \mathbb{R}^n$, with coordinates $(x) = (\tilde{x}, x_n)$.

Definition 3.7.17. *A* ***fibered over*** $\mathbb{R}^m$ ***regularised wrinkle*** *is a map*

$$f : \mathcal{O}p(\mathbb{S}^{m+n-1}) \quad \to \quad \mathbb{R}^m \times J^r(X, F)$$

which we regard as a m-parameter family of integral embeddings $f_q(x) = f(q, x)$ whose singularities are:

- $\Sigma^{110}(f_q, V_{\mathrm{can}}) = \mathbb{S}^{m+n-2}$,
- $\Sigma^{10}(f_q, V_{\mathrm{can}}) = \mathbb{S}^{m+n-1} \backslash \mathbb{S}^{m+n-2}$.

We see from the description of the singularities and the embedding and integrality conditions that f_q is a regularised wrinkle for every $|q| < 1$. Similarly, if $|q| > 1$, the map f_q has no singularities. We denote:

Definition 3.7.18. *The integral embeddings*

$$f_q : \mathcal{O}p(0) \quad \to \quad J^r(X, F)$$

with $|q| = 1$ are called (regularised wrinkle) ***embryos****.*

A particular incarnation of the embryo is given by lifting the map:

$$(x) \mapsto (\tilde{x}, x_n^3/3 + |\tilde{x}|^2 x_n, ; z_1^{(0,\dots,r)} = x_n, 0, \dots 0).$$

Lemma 3.7.16 implies that this model is unique if $\dim(F) = 1$.

3.7.2.6 The stabilisation

Definition 3.7.19. *Set $D = \mathbb{S}^{n-1}$. A fibered over D integral embedding*

$$f : D \times \mathcal{O}p([0,1]) \quad \to \quad J^r(X, F)$$

*is a **stabilisation** if*

$$\Sigma(f, V_{\rm can}) = D \times \{0\} \cup D \times \{1\}$$

*and these are folds with the same Maslov coorientation. The image $f(D \times (0,1))$ is called the **membrane** of the stabilisation.*

For a model we may consider the lift $\mathrm{Lift}^r(\sigma)$ of the map:

$$\begin{aligned} \sigma : D \times \mathcal{O}p([0,1]) &\quad \to \quad X \oplus \mathrm{Sym}^r(\mathbb{R}^*, F) \\ (\tilde{x}, x_n) &\quad \to \quad (\tilde{x}, x_n^3/3 - x_n; z_1^{(0,\dots,r)} = x_n^2, 0, \dots 0). \end{aligned} \tag{3.7.2.3}$$

As before:

Lemma 3.7.20. *Suppose* $\dim(F) = 1$. *Then any stabilisation is equivalent to* $\mathrm{Lift}(\sigma)$ *(using point symmetries in the target, and diffeomorphisms in the domain).*

However, unlike previous singularities:

Lemma 3.7.21. *Suppose* $\dim(F) = 1$. *Then there is* no *generating function $G : X \times \mathbb{R} \to F$ such that L_G is a stabilisation.*

Proof. This fact is well-known in the contact case and we will mimick the usual proof.

We assume that a generating function G does exist. Due to the fibered nature of the stabilisation we may assume that we are looking at a curve $\gamma : \mathcal{O}p([0,1]) \to J^r(\mathbb{R}, \mathbb{R})$, with principal metasymplectic projection $x \mapsto (x^3/3 - x, x^2)$. This curve must be parametrised by the fibrewise singularity locus Γ of G.

The singularity locus $\{x = \pm 1\}$ of γ consists of two folds. The corresponding fronts are A_{2r}-swallowtails: i.e. birth-death events of two A_{2r}-singularities. In particular, if r is even, one of them is a maximum and the other is a minimum (these are degenerate if $r > 1$, but it does not matter). The maximum must be the one with greater value, i.e. it lies above in the front projection. Reasoning in this manner at both folding points, it follows that the only possible configuration in the front projection is that $\pi_f \circ \gamma((0,1))$ lies above (or below) both branches of the complement $\pi_f \circ \gamma(\mathcal{O}p([0,1] \backslash (0,1))$. This implies that the fold loci have opposite Maslov coorientations; a contradiction.

For r even the critical points of G meeting at the birth-death are both increasing (or both decreasing). This implies, similarly, that the three consecutive branches $\pi_f \circ \gamma(\{x < 0\})$, $\pi_f \circ \gamma((0,1))$, and $\pi_f \circ \gamma(\{x > 0\})$ have increasing (resp. decreasing) y-coordinates. Yet again this implies that the fold loci have opposite Maslov coorientations. □

In the next subsection 3.7.2.7 we provide some additional details about this proof.

3.7.2.7 Zig-zags

In the proof of Lemma 3.7.21 we see one of the incarnations of a phenomenon we call *open/closed switching*. It was first observed by A. Givental in [58]. Let us explain what it is.

Let us recall Equation 3.7.2.1, which defines the front projection of a double fold:

$$f(\tilde{x},t) = \left(\tilde{x}, x_n = t^3/3 - t; y_1 = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{r-1}} s_r \prod_j (s_j^2 - 1) ds_r \dots ds_1, 0 \dots, 0\right).$$

The term y_1 is defined by an iterated integral, as explained in Lemma 3.6.26. The way in which we obtained it was as follows: let $j^r f(\tilde{x},t)$ be the holonomic lift of f to a multi-section. Consider one of its components, the odd function

$$(z_1^{(0,\dots,0,r)} \circ j^r f)(\tilde{x},t) = t.$$

We then multiply it by $t^2 - 1$, so it remains odd, and then we integrate it once to yield the even function

$$(z_1^{(0,\dots,0,r-1)} \circ j^r f)(\tilde{x},t) = \int_0^t s_r(s_r^2 - 1) ds_r.$$

Inductively we see that:

Lemma 3.7.22. *The function $z_1^{(0,\dots,0,r-l)} \circ j^r f$ is:*

- *odd if l is even,*
- *even if l is odd.*

This alternation between even and odd is precisely what we call open/closed switching. It can be rephrased using Maslov coorientations in each $(r-l)$-jet space, but we leave this for the reader. We can interpret it geometrically:

Lemma 3.7.23. *The following statements hold:*

- *If r is even, the function y_1 increases at a fold point if and only if it increases at the other.*
- *If r is odd, the function y_1 increases at a fold point if and only if decreases at the other.*

Proof. Being critical points, when we say increase/decrease we mean as continuous functions, without considerations on the derivative. Note that the model at each fold point tells us that y_1 must be either increasing or decreasing.

If r is even, the function y_1 is odd. This is equivalent to the first statement. Similarly, if r is odd, the function y_1 is even, so the second statement follows. □

We can reason in exactly the same manner for the stabilisation and prove that the situation is exactly the opposite.

Lemma 3.7.24. *Let g be a stabilisation:*

- *If r is odd, the function $y_1 \circ g$ increases at a fold point if and only if it increases at the other.*
- *If r is even, the function $y_1 \circ g$ increases at a fold point if and only if decreases at the other.*

What this means is that if we want to have two A_{2r}-singularities in the front projection forming a "*zig-zag*" shape, we must use a double fold if r is even and a stabilisation if r is odd. We define:

Definition 3.7.25. *Set $D = \mathbb{S}^{n-1}$. A fibered over D integral embedding*

$$f : D \times \mathcal{O}p([0,1]) \quad \to \quad J^r(X, F)$$

*is a **zig-zag** if:*

- *r is even and f is a double fold,*
- *r is odd and f is a stabilisation.*

The front of the zig-zag is what we would call an *open* shape, and the other two situations (double fold with r odd, stabilisation with r even) we would call them *closed*. The importance of zig-zags is that they can be stacked on top of each other keeping the front projection embedded. This will be central in our h-principle in Section 3.8.

3.7.3 Singularities of mapping

The singularities we have presented so far are all of tangency, i.e. the integral maps themselves are non-singular. We will now look at singularities of mapping having well-defined Gauss map taking values in $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\xi_{\mathrm{can}}, n)$.

The main source of examples of singularities of mapping are projections of singularities of tangency (from a higher jet space). We make some remarks in this direction in subsection 3.7.3.1. We then define several germs: the cusp in its two incarnations (subsections 3.7.3.2 and 3.7.3.3) and the swallowtail (subsection 3.7.3.4). These are the pieces we need to then define some semi-local singularities: the wrinkly stabilisation (subsection 3.7.3.5), the double cusp (subsection 3.7.3.6), and the wrinkle (subsection 3.7.3.7).

We continue using the notation from the previous Subsection 3.7.2.

3.7.3.1 Projecting singularities

Let $f : N \to J^r(B,F)$ be an integral map. Then the projection $\pi_{r,r-1} \circ f : N \to J^{r-1}(B,F)$ is integral as well. In Lemma 3.6.4 we additionally showed that if f is a multi-section then $\pi_{r,r-1} \circ f$ is a multi-section with a well-defined Gauss map $\mathrm{Gr}(\pi_{r,r-1} \circ f) = f$ into the horizontal elements (where we use the identification between horizontal elements and lifts to $J^r(B,F)$). Hence, when we project, singularities of tangency become singularities of mapping.

Some of the singularities we will describe below are obtained by projecting an r-times differentiable Whitney singularity. For instance, in subsections 3.7.1.5 and 3.7.1.6 we already saw that the front projection of the fold and the pleat are the A_{2r} cusp and swallowtail, respectively.

One important observation is:

Lemma 3.7.26. *Assume* $\dim(F) = 1$. *Let* $f : N \to J^r(B,F)$ *be a topologically embedded multi-section of the form* $f = \pi_{r+l,r} \circ g$, *with*

$$g : N \to J^{r+l}(B,F)$$

an embedded multi-section with Whitney singularities.

Then f *is stable among multi-sections lifting to* $J^{r+l}(B,F)$.

Proof. Let $(f_s)_{s\in[0,1]}$ be a deformation of $f_0 := f$ and let $(g_s)_{s\in[0,1]}$ be the corresponding deformation of $g_0 := g$ lifting it. Observe that the lifts, when they exist, are uniquely defined (by lifting on each branch).

According to Corollary 3.7.7, the map g is stable up to contact transformation germs. Higher contact transformations are lifts of contact transformations in $J^r(B,F)$ (Lemma 3.2.45). This implies that the isotopy of contact transformations identifying g_s with g is a lift of an isotopy taking f_s to f, proving the claim. □

Remark 3.7.27. We will encounter below singularities of mapping that have a well-defined Gauss map taking values in $\mathrm{Gr}_{\Sigma^1}(\xi_{\mathrm{can}}, n)$. Therefore, none of those singularities can admit a lift to $J^{r+1}(B,F)$. However, one may instead look the total space of

$$\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\xi_{\mathrm{can}}, n) \to J^r(B,F)$$

and endow it with its tautological distribution. This partially compactifies $J^{r+1}(B,F)$ and, by definition, the singularities we describe admit a lift to $\mathrm{Gr}_{\Sigma^2-\mathrm{free}}(\xi_{\mathrm{can}}, n)$.

For $\dim(B) = \dim(F)$, iterating this construction yields the *Monster tower*, as introduced by R. Montgomery and M. Zhitomirskii in the treatise [91]. They show that there is a correspondence between points in the tower and singularities of fronts. Their results should partly translate to our context of Σ^2-free singularities, but we point out some difficulties in Remark 3.7.34 below.

An intriguing question is whether the whole Grassmannian of multi-section elements $\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, n)$ is smooth. If this were true, the natural next step would be to construct the analogue of the Monster tower. △

3.7.3.2 The horizontal cusp

As we prove below, projecting a fold down one level yields:

Definition 3.7.28. *An integral map (Definition 3.2.12)*

$$f : \mathcal{O}p(\{0\}) \quad \to \quad J^r(X,F)$$

is a ***horizontal cusp*** *if:*

- *The singularities of* $\pi_L^n \circ f$ *form a hypersurface of semicubic cusps.*
- $\mathrm{Gr}(f)$ *takes values in* $\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, n)$.

A explicit fibered model can be obtained by lifting

$$(\tilde{x}, x_n) \mapsto (\tilde{x}, x_n^2; z_1^{(0,\dots,0,r)} = x_n^3, 0, \dots, 0).$$

Lemma 3.7.29. *Let* $\dim(F) = 1$. *Then any horizontal cusp is equivalent to the model (using point symmetries in the target, and diffeomorphisms in the domain).*

Proof. By assumption f can be lifted to an integral map $\mathrm{Gr}(f) : N \to J^{r+1}(X,F)$. Since its metasymplectic projection has semicubic cusps, this lift is an embedding. The singularities of mapping of f correspond to fold singularities of tangency of $\mathrm{Gr}(f)$. The claim follows from Lemma 3.7.26. □

In particular, a horizontal cusp f is a topological embedding, even if it is not an immersion. Its front singularities are A_{2r+2}-cusps.

3.7.3.3 The vertical cusp

We can instead consider:

Definition 3.7.30. *An integral map (Definition 3.2.12)*

$$f : \mathcal{O}p(\{0\}) \quad \to \quad J^r(X,F)$$

is a ***vertical cusp*** *if:*

- *The singularities of* $\pi_L^n \circ f$ *form a hypersurface of semicubic cusps.*
- $\mathrm{Gr}(f)$ *takes values in* $\mathrm{Gr}_{\Sigma^1}(\xi_{\mathrm{can}}, n)$ *along the locus* $\Sigma(f)$.

It is a topological embedding as well. Note that the singularities are both of tangency and mapping.

A model can be obtained by lifting

$$(\tilde{x}, x_n) \mapsto (\tilde{x}, x_n^3; z_1^{(0,\dots,0,r)} = x_n^2, 0, \dots, 0).$$

However, it is unclear whether a vertical cusp has a unique local model even if we assume $\dim(F) = 1$ (but the answer may be in [91]). Additionally, one could define cusp loci that are horizontal almost everywhere but become vertical over a submanifold of the singularity locus.

3.7.3.4 The swallowtail

In subsection 3.3.6.2 we defined the smooth the open semicubic swallowtail within the context of the wrinkle in positive codimension (Subsection 3.3.6). Now we define its jet space analogue:

Definition 3.7.31. *An integral map (Definition 3.2.12)*

$$f : \mathcal{O}p(\{0\}) \quad \to \quad J^r(X, F)$$

*is a **horizontal swallowtail** if:*

- $\pi^n_L \circ f$ *has a open semi-cubic swallowtail at the origin.*
- $\mathrm{Gr}(f)$ *takes values in* $\mathrm{Gr}_{\Sigma^0}(\xi_{\mathrm{can}}, n)$.

It is yet again a topological embedding because that is the case for $\pi^n_L \circ f$.

We can produce a model by lifting the following map into a principal metasymplectic projection:

$$(\tilde{x}, x_n) \mapsto (\tilde{x}, \int_0^{x_n} (s^2 - x_1) ds; z_1^{(0,\dots,0,r)} = \int_0^{x_n} (s^2 - x_1)^2 ds, 0, \dots, 0).$$

Its singularity locus Γ consists of the parabola $\{x_n^2 = x_1\}$, which is tangent to the x_n-lines along the codimension-2 linear subspace $A = \{x_n = x_1 = 0\}$. A is the locus of swallowtails, and its complement in Γ consists of horizontal cusps. Hence, the swallowtail serves as a birth/death of cusps (as is the case in the smooth setting).

Lemma 3.7.32. *Let* $\dim(F) = 1$. *Then any horizontal swallowtail is equivalent (using point symmetries in the target, and diffeomorphisms in the domain) to the model.*

Proof. We lift f to $\mathrm{Gr}(f) : \mathcal{O}p(\{0\}) \to J^{r+1}(X, F)$, which is smooth, embedded, and has a pleat at the origin. Lemma 3.7.26 applies. □

One can also consider *vertical* swallowtails or swallowtails with singularity locus becoming vertical over a submanifold. We will not study this.

3.7.3.5 The wrinkly stabilisation

We explained in subsection 3.3.5.3 that there is a correspondence between smooth wrinkles and double folds by performing surgeries. We will not provide a justification of this, but the same is true in jet spaces. For instance, the double fold (subsection 3.7.2.3) and the regularised wrinkle (subsection 3.7.2.4) are, up to surgery, equivalent. Similarly, there is a "wrinkle" analogue of the stabilisation, and one can pass between them through surgeries. It is defined as follows:

Definition 3.7.33. *Set $D = \mathbb{R}^{n-1}$. An integral map (Definition 3.2.12) fibered over D*

$$f : \mathcal{O}p(\mathbb{S}^{n-1}) \quad \to \quad J^r(X, F)$$

*is a **wrinkly stabilisation** if:*

- $\Sigma^{10}(f) = \mathbb{S}^{n-2}$ *is a locus of vertical cusps,*
- $\Sigma^{10}(f, V_{\text{can}}) = \mathbb{S}^{n-1}$*,*
- *The hemispheres $\mathbb{S}^{n-1} \backslash \mathbb{S}^{n-2}$ are folds with the same Maslov coorientation.*
- *It is a topological embeddeding and has no other singularities.*

Note that along $\mathbb{S}^{n-2}$ there is discontinuity in the Gauss map. Hence, the wrinkly stabilisation is not a multi-section in the sense of Definition 3.6.2.

Remark 3.7.34. This is a continuation of Remark 3.7.27 above. The wrinkly stabilisation shows the first difficulty with the Monster tower approach for higher dimensional manifolds: some singularities do not admit a continuous Gauss map.

If we look at the maps induced by f on each fibre, we see that if $|\tilde{x}| < 1$ then they are curves with two folds, if $|\tilde{x}| > 1$ they are curves graphical over the zero section, and if $|\tilde{x}| = 1$, they are vertical cusps. That is, it corresponds to the standard unfolding of the cusp. Thus, not admitting a continuous Gauss map corresponds to a phenomenon already observed in [91, Section 9.1]: the lifting procedure to the Monster tower is not continuous in the unfolding parameter. This is something to be explored in future work. △

Lemma 3.7.35. *The topological embedding condition is implied, in the vicinity of its cusp locus, from the first three items.*

Proof. For $|\tilde{x}|$ smaller than but close to one, the curve $\pi_L^n \circ f(\{\tilde{x}\} \times \mathbb{R})$ is an unfolding of the cusp. It describes a little loop when projected to $(x_n, z_1^{(0,\dots,0,r)})$. In particular, it has a self-intersection point. However, according to the subsection 3.6.4.2, the two intersection points have different lifts by integration. □

A model we may consider is the lift of

$$(\tilde{x}, x) \mapsto (\tilde{x}, x_n^3/3 + (|\tilde{x}|^2 - 1)x_n; z_1^{(0,\dots,0,r)} = x_n^2, 0, \dots, 0).$$

The principal metasymplectic projection of any wrinkly stabilisation is equivalent, as a smooth map, to this model. However, it is unclear whether the model is unique up to point symmetries.

3.7.3.6 The double (horizontal) cusp

Now we consider two spheres of horizontal cusps bounding an annulus:

Definition 3.7.36. *Set $D = \mathbb{S}^{n-1}$. A fibered over D integral map (Definition 3.2.12)*

$$f : D \times \mathcal{O}p([0,1]) \quad \to \quad J^r(X,F)$$

is a ***double cusp*** *if*

- *f is a topological embedding.*
- *$\mathrm{Gr}(f) : D \times \mathcal{O}p([0,1]) \to J^{r+1}(X,F)$ is a stabilisation.*

The image $f(D \times (0,1))$ is called the ***membrane*** *of f.*

In particular, we are requiring that

$$\Sigma(f) = D \times \{0\} \cup D \times \{1\}$$

are horizontal cusps. If that is the case, the lift $\mathrm{Gr}(f)$ exists and is an immersion with two folds. Hence, it may be a double fold or a stabilisation. We require that it is the latter.

The key property here is:

Lemma 3.7.37. *The front singularities of the double cusp are two A_{2r+2}-cusps in an open configuration (i.e. a zig-zag).*

This follows from the open/closed switching from Lemma 3.7.24, see subsection 3.7.2.7.

3.7.3.7 The wrinkle

The "wrinkly" analogue of the double cusp is precisely:

Definition 3.7.38. *Set $D = \mathbb{R}^{n-1}$. An integral map (Definition 3.2.12), fibered over D,*

$$f : \mathcal{O}p(\mathbb{S}^{n-1}) \quad \to \quad J^r(X,F)$$

is a ***wrinkle*** *if*

- *$\mathrm{Gr}(f) : D \times \mathcal{O}p([0,1]) \to J^{r+1}(X,F)$ is a wrinkly stabilisation (Definition 3.7.33).*
- *f is a topological embedding.*

The image $f(D \times (0,1))$ is called the ***membrane****.*

A possible model is the lift of the wrinkled map of positive codimension (see Subsection 3.3.6):

$$F(\tilde{x}, x_n) = (\tilde{x}, \int_0^{x_n} (s^2 + |\tilde{x}|^2 - 1)ds; z_1^{(0,\dots,0,r)} = \int_0^{x} (s^2 + |\tilde{x}|^2 - 1)^2 ds, 0, \dots, 0).$$

We do not know if $\mathrm{Lift}(F)$ is the only possible model. However, the principal meta-symplectic projection of a wrinkle is equivalent to F if we let left equivalences be diffeomorphisms preserving the base projection. From this we deduce:

Lemma 3.7.39. *Equivalently, a wrinkle is an integral topological embedding*

$$f : \mathcal{O}p(\mathbb{S}^{n-1}) \quad \to \quad J^r(X, F)$$

with singularity locus $\Sigma(f) = \mathbb{S}^{n-1}$ *satisfying:*

- *The equator* $\mathbb{S}^{n-2}$ *consists of semicubic swallowtails.*
- *The hemispheres are horizontal cusps.*

Remark 3.7.40. The wrinkle is unique for smooth maps (i.e. $r = 0$). Uniqueness for $r > 0$, as we stated, is unknown. In the contact case (i.e. $r = 1$ and $\dim(F) = 1$), wrinkles for legendrians were defined by D. Álvarez-Gavela in [2], providing a explicit model. Although not stated explicitly in his paper, it seems like uniqueness follows from the constructions he provides. △

3.7.3.8 Fibered wrinkles

Let us present the fibered version. We fix coordinates (q) in $\mathbb{R}^m$ and (x) in $X = \mathbb{R}^n$.

Definition 3.7.41. *A* ***fibered over*** $\mathbb{R}^m$ ***wrinkle*** *is a map*

$$f : \mathcal{O}p(\mathbb{S}^{m+n-1}) \quad \to \quad \mathbb{R}^m \times J^r(X, F),$$

which we regard as a m-parameter family of integral topological embeddings $f_q(x) = f(q, x)$ *with singularity locus* $\mathbb{S}^{m+n-1}$ *satisfying:*

- $\Sigma^{110}(\pi^n_L \circ f_q) = \mathbb{S}^{m+n-2}$ *are open semicubic swallowtails,*
- $\Sigma^{10}(\pi^n_L \circ f_q) = \mathbb{S}^{m+n-1} \backslash \mathbb{S}^{m+n-2}$ *are horizontal cusps.*

The maps with $|q| = 1$ *are called (wrinkle)* ***embryos****.*

A possible model for the principal metasymplectic projection of an embryo reads:

$$(\tilde{x}, x_n) \to (\tilde{x}, \int_0^{x_n} (s^2 + |\tilde{x}|^2) ds; z_1^{(0,\ldots,0,r)} = \int_0^{x} (s^2 + |\tilde{x}|^2)^2 ds, 0, \ldots, 0).$$

However, we do not know whether this model is unique.

3.8 Holonomic approximation by multi-sections

The main result of this chapter is an h-principle with PDE flavour. It states that the holonomic approximation Theorem 3.4.8 applies to closed manifolds as long as we are willing to be flexible and allow for multi-sections. A particular consequence is that any open partial differential relation admits a solution in the class of multi-sections.

The interesting part of the result is that it is sufficient to work with multi-sections with simple singularities. Namely, they will satisfy that:

- Their only singularities are folds in a zig-zag configuration.
- Their front projection is topologically embedded.

In Subsection 3.8.1 we formulate this formally. In Subsection 3.8.2 we present the key geometric insight needed for our arguments. Lastly, in Subsection 3.8.3 we provide the proof.

As in previous Sections, we fix a smooth fibre bundle $Y \to X$, with X compact. We work on the jet space $J^r(Y \to X)$. In order to quantify how close two sections of $J^r(Y \to X)$ are, we fix a metric.

3.8.1 Statement of the result

Recall the notion of zig-zag from subsection 3.7.2.7. We are interested in multi-sections of the form:

Definition 3.8.1. *A **section with zig-zags** is:*

- *an embedded multi-section $f : X \to J^r(Y \to X)$,*
- *a finite collection of disjoint annuli $\{A_j \subset X\}$,*

satisfying:

- *$\pi_f \circ f$ is a topological embedding,*
- *$f|_{X\setminus(\cup_j A_j)}$ is horizontal,*
- *$f|_{A_j}$ is a zig-zag.*

Our main result is the obvious multi-section version of the holonomic approximation Theorem 3.4.8:

Theorem 3.8.2. *Let $\sigma : X \to J^r(Y \to X)$ an arbitrary section. Then, for any $\varepsilon > 0$, there exists a map $f : X \to J^r(Y \to X)$ satisfying:*

- *f is a section with zig-zags;*
- *$|f - \sigma|_{C^0} < \varepsilon$.*

We want to stress that this statement is a proof of concept: it should be immediate to the reader experienced in h-principles, after inspecting the proof, that a parametric and relative (in the domain and the parameter) version also holds. Furthermore, the theorem is the graphical case of the analogous result about approximating r-jets of submanifolds through submanifolds with zig-zags (that is, the generalisation to higher jets of the wrinkled embeddings Theorem 3.4.18). Lastly, it is the first step towards a general h-principle for Σ^2-free integral submanifolds of distributions modelled on jet spaces.

The additional ingredient needed for these more general statements is a careful description of the birth/death of zig-zags. To avoid additional technical difficulties we have chosen to leave this to future work.

3.8.2 The key ingredient of the proof

We now present the simple observation that constitutes the basis of our work:

Definition 3.8.3. *Let $I = [a, b]$ be an interval. An* ***asymptotically flat sequence of zig-zag bump functions*** *is a sequence of maps*

$$(\rho_N)_{N\in\mathbb{N}} : [a, b] \to J^0([a, b], \mathbb{R})$$

satisfying

- *their holonomic lifts $j^r \rho_N : [a, b] \to J^r([a, b], \mathbb{R})$ are sections with zig-zags,*
- $\rho_N|_{\mathcal{O}p(a)}(t) = (x = t, y = 0)$,
- $\rho_N|_{\mathcal{O}p(b)}(t) = (x = t, y = 1)$,
- $|z^{(r')} \circ \rho_N| < \frac{1}{N}$ *for all $r' > 0$.*

The name follows from the fact that an element ρ_N, with N sufficiently large, allows us to interpolate between two given sections without introducing big derivatives (unlike a normal bump function).

Proposition 3.8.4. *An asymptotically flat sequence of zig-zag bump functions exists on any interval.*

Before we provide a proof, let us explain a Corollary that showcases this.

Corollary 3.8.5. *Let $\varepsilon, \delta > 0$ be given. Consider sections $s_0, s_1 : \mathbb{D}^n \to \mathbb{R}^k$ satisfying $|s_0 - s_1|_{C^r} < \varepsilon$.*

Then, there exists a section with zig-zags $f : \mathbb{D}^n \to J^r(\mathbb{D}^n, \mathbb{R}^k)$ satisfying:

- $(\pi_f \circ f)|_{\mathbb{D}^n_{1-\delta}} = s_0$,
- $(\pi_f \circ f)|_{\mathcal{O}p(\partial\mathbb{D}^n)} = s_1$,
- $|j^r s_0 - f|_{C^0} < 4\varepsilon$.

Proof. We write $(y_1, \ldots, y_k)$ for the coordinates in the fibre $\mathbb{R}^k$ and (x) for the coordinates in the base. We break down the proof into elementary steps.

The pushing trick. Since $|s_0 - s_1|_{C^0} < \varepsilon$, we can shift s_0 by adding a constant in $\mathbb{R}^k$:

$$\tilde{s}_0(x) := s_0(x) + (2\varepsilon, 0, \ldots, 0).$$

Replacing s_0 by $\tilde{s}_0$ guarantees that:

$$\tilde{s}_0(x) \neq s_1(x), \qquad \text{for every } x \in \mathbb{S}^{n-1} \times [1 - \delta, 1],$$

while retaining a bound $|\tilde{s}_0 - s_1|_{C^r} < 3\varepsilon$. We henceforth restrict the domain of $\tilde{s}_0$ and s_1 to the region of interest $\mathbb{S}^{n-1} \times [1 - \delta, 1]$.

First simplification. We can simplify the setup by applying the fibrewise translation:

$$\begin{array}{rcl} J^0(\mathbb{S}^{n-1}\times[1-\delta,1],\mathbb{R}^k) & \to & J^0(\mathbb{S}^{n-1}\times[1-\delta,1],\mathbb{R}^k) \\ p & \to & p-\tilde{s}_0(\pi_b(p)), \end{array}$$

It preserves the C^r–distance and maps $\tilde{s}_0$ to the zero section. The section s_1 is mapped to $s := s_1 - \tilde{s}_0$. Consequently, we just need to explain how to interpolate between the zero section and some arbitrary section s satisfying $|s|_{C^r} < 3\varepsilon$ and $s(x) \neq 0$ for all x.

Second simplification. A second symmetry allows us to put s in normal form. Due to the nature of the shift we performed, we have that

$$\varepsilon < |y_1 \circ s(x)| < 3\varepsilon$$

for all x. This allows us to define a framing

$$\begin{array}{rcl} A:\mathbb{S}^{n-1}\times[1-\delta,1] & \to & \mathrm{GL}(\mathbb{R}^k) \\ A(x) & = & (s,e_2,e_3,\dots,e_k), \end{array}$$

where $\{e_j\}_{j=1,\dots,k}$ is the framing dual to the coordinates y_i in $\mathbb{R}^k$. The framing A defines a fibre-preserving transformation of the $\mathbb{R}^k$-bundle by left multiplication. By construction $Ae_1 = s$.

Main construction. Apply Proposition 3.8.4 to produce an asymptotically flat sequence of zig-zag bump functions

$$(\rho_N)_{N\in\mathbb{N}} : [1-\delta,1] \quad\to\quad J^0([1-\delta,1],\mathbb{R}).$$

We use it to define a sequence of front projections:

$$\begin{array}{rcl} Z_N:\mathbb{S}^{n-1}\times[1-\delta,1] & \to & J^0(\mathbb{S}^{n-1}\times[1-\delta,1],\mathbb{R}^k) \\ (\tilde{x},t) & \to & A[\rho_N(t)e_1]. \end{array}$$

We claim that, for N large enough, the holonomic lift $f_N := j^r Z_N$ satisfies the properties prescribed.

Checking the claimed properties. We first observe that f_N is a section with zigzags. This follows from the fact that $j^r(\rho_N e_1)$ is a section with zigzags and f_N is obtained from it by applying the point symmetry $j^r A$. In particular, the singularities of f_N are codimension-1 spheres of folds, corresponding to the values of t in which ρ_N has an A_{2r}-singularity.

The second and final claim is that $|f_N|_{C^0} < 4\varepsilon$ if N is large enough. Equivalently, we have to bound the C^r-size of:

$$A(\rho_N e_1) = \rho_N s.$$

Note that we can pretend that ρ_N is an actual function, because this is true over a dense set. Therefore, for each multi-index I with $|I| \leqslant r$ we compute:

$$|\partial^I(\rho_N s)|^2 = \left|\sum_{I'+I''=I}(\partial^{I'}\rho_N)(\partial^{I''}s)\right|^2 \leqslant \sum_{I'+I''=I}|\partial^{I'}\rho_N|^2|\partial^{I''}s|^2$$

Now, each derivative $|\partial^{I'}\rho_N|$ is smaller than $1/N$, with the exception of $|\rho_N| = 1$. Similarly, $|\partial^{I''}s| < 3\varepsilon$ for all I''.

Let K_1 be the maximum number of decompositions $I' + I'' = I$ that a multi-index $|I| \leqslant r$ in n variables and k outputs may have. Let K_2 be the number of multi-indices $|I| \leqslant r$. Then:

$$\begin{aligned} |\partial^I(\rho_N s)|^2 < \quad & |\partial^I s|^2 + \frac{9K_1}{N^2}\varepsilon^2 \\ |\rho_N s|^2_{C^r} < \quad & \sum_I \left(|\partial^I s|^2 + \frac{9K_1}{N^2}\varepsilon^2\right) < |s|^2_{C^r} + \frac{9K_1K_2}{N^2}\varepsilon^2. \end{aligned}$$

Therefore, by setting $N^2 > 9K_1K_2$, we conclude:

$$|f_N|_{C^0} = |\rho_N s|_{C^r} < |s|_{C^r} + \varepsilon < 4\varepsilon.$$

□

Remark 3.8.6. An interesting feature of the proof is that the sections with zig-zags we construct are obtained from the "standard" sections with zig-zags $j^r(\rho_N e_1)$ by applying a point symmetry. The same argument would work if instead of $j^r\rho_N$ we used a particular model of wrinkle (subsection 3.7.3.7). Hence, we can bypass the potential uniqueness issues for wrinkles pointed out in Remark 3.7.40. △

Now we construct the zig-zag bump functions:

*Proof of Proposition 3.8.4. Observe that i*t is sufficient to prove the claim for $I = [0,1]$, since any two intervals are diffeomorphic by a scaling and a translation. The scaling dilates the fibres of jet space in a homogeneous manner, so any asymptotically flat sequence is mapped to an asymptotically flat sequence.

Fix N. We will construct ρ_N as the holonomic lift $\rho_N = j^r(\pi_f \circ \rho_N)$ of its front projection $\pi_f \circ \rho_N$.

The infinite zig-zag. We first define:

$$\begin{aligned} Z : \mathbb{R} \quad & \rightarrow \quad J^0([0,1], \mathbb{R}), \\ (t) \quad & \rightarrow \quad \left(x(t) = \frac{1}{2}\int_0^t \sin(s)ds, y(t) = \int_0^t \sin(s)^{2r}ds\right). \end{aligned}$$

We claim that, at each of its critical points $\{t = 0, \pi, 2\pi, \dots\}$, the map Z is modelled on the A_{2r}-singularity. To prove this we compute the Taylor expansion at each of these points:

$$\begin{aligned} \sin(l\pi + h) &= \frac{h}{2} + O(h^3), \qquad & \sin(l\pi + h)^{2r} &= h^{2r} + O(h^{2r+2}), \\ x(l\pi + h) &= \frac{h^2}{4} + O(h^4), \qquad & y(l\pi + h) &= \frac{h^{2r+1}}{2r+1} + O(h^{2r+3}). \end{aligned}$$

Which proves the claim because the A_{2r} singularity is stable.

From this computation we deduce that the lift

$$j^r Z : \mathbb{R} \to J^r([0,1], \mathbb{R})$$

is an integral mapping with fold singularities. Since its front is topologically embedded, $j^r Z$ is embedded. Lastly, according to the definition in Subsubection 3.7.2.7, the germ $j^r Z|_{\mathcal{O}p([(2l-1)\pi, 2l\pi])}$ is a zig-zag. The section with zig-zags $j^r Z$ has infinitely many of them stacked.

A piece of the infinite zig-zag. Next, observe that Z is graphical over $[0,1]$ in the intervals $(2l\pi, (2l+1)\pi)$. In particular, we can flatten Z in $\mathcal{O}p(0)$ so that it is identically 0, without introducing self-intersections of the front. Similarly, for any l, we can flatten Z in the region $\mathcal{O}p((2l+1)\pi)$ so that it is identically $Z((2l+1)\pi)$. Lastly, we can scale this modification of Z, dividing by the constant $Z((2l+1)\pi)$. In this manner we obtain a front that is identically 0 and 1 in $\mathcal{O}p(0)$ and $\mathcal{O}p((2l+1)\pi)$, respectively. We denote it by Z_N.

We claim that, if l is large enough, then $|z^{(a)} \circ j^r Z_N| < \varepsilon$ for all $a > 0$. This follows immediately from the scaling we just did: Z was 2π-periodic, so the quantities $z^{(a)} \circ j^r Z$ were bounded. The quantity $Z((2l+1)\pi)$ goes to infinity as l does, so a sufficiently large choice guarantees that the derivatives of $j^r Z_N$ are smaller than $1/N$.

Lastly, we simply reparametrise

$$\pi_f \circ \rho_N(t) = Z_N \circ \phi(t),$$

where $\phi : [0,1] \to [0, (2l+1)\pi]$ is a suitable diffeomorphism. □

3.8.3 The proof

The proof of Theorem 3.8.2 follows the standard structure of an h-principle.

In subsection 3.8.3.2 we prove the *reduction step*. Its output is a holonomic section g, defined along the codimension-1 skeleton of X and approximating the given formal section σ.

In subsection 3.8.3.3 we provide the *extension argument*: we extend g to the interior of the top dimensional cells. In order to obtain a good approximation of σ, the extension to the interior must be a multi-section, as presented in Corollary 3.8.5.

3.8.3.1 Preliminaries

We must fix some auxiliary data first. Depending on the constant $\varepsilon > 0$ we fix a finite collection of pairs $\{(U_i, f_i)\}$ such that

- $\{U_i\}$ is a covering of X by balls,
- $f_i : U_i \to J^r(Y|_{U_i} \to U_i)$ is a holonomic section satisfying $|f_i - \sigma|_{U_i}| < \varepsilon$.

The existence of such a collection follows from the standard holonomic approximation Theorem 3.4.8 applied to each point in X. By compactness of X we get a finite refinement.

We then triangulate X, yielding a triangulation $\mathcal{T}$. We assume that this triangulation is fine enough to guarantee that each simplex is contained in one of the U_i. Given a top-simplex $\Delta \in \mathcal{T}$, we choose a preferred U_i and we denote the corresponding section f_i by f_Δ.

We remark that $Y|_{U_i}$ is trivial, so we can make the identification $J^r(Y|_{U_i} \to U_i) \cong J^r(\mathbb{D}^n, \mathbb{R}^k)$. We can then relate the C^0-norm in the former with the standard C^0-norm in the latter. By finiteness of the cover there is a constant bounding one in terms of the other. We assume this constant is 1 to avoid cluttering the notation.

3.8.3.2 Reduction

The codimension-1 skeleton of X is a CW-complex of positive codimension. Thus, according to Theorem 3.4.8, there exists:

- a wiggled version $\tilde{\mathcal{T}}$ of $\mathcal{T}$,
- a holonomic section $g : \mathcal{O}p(\tilde{\mathcal{T}}) \to Y$ satisfying $|\sigma - j^r g| < \varepsilon$.

The wiggling can be assumed to be C^0-small, so each top-simplex $\Delta \in \tilde{\mathcal{T}}$ is contained in the same U_i as the original simplex. I.e., we have sections g (defined over $\mathcal{O}p(\partial\Delta)$) and f_Δ (defined over the whole of Δ), both of them approximating σ.

3.8.3.3 Extension

We focus on a single top-simplex $\Delta \in \tilde{\mathcal{T}}$ because the argument is the same for all of them. We simply observe that Corollary 3.8.5 applies to g and f_Δ over the annulus $\mathcal{O}p(\partial\Delta)$, producing the desired multi-section extension f of $j^r g$ to the interior of Δ. The Corollary guarantees that:

$$|f - \sigma| < |f - j^r f_\Delta| + |j^r f_\Delta - \sigma| < 5\varepsilon.$$

This concludes the proof of Theorem 3.8.2. □

We close with an extremely biased remark about the proof: the idea presented (zig-zag bump functions together with the pushing trick) seems simpler than the path followed in [44] (reducing to simple tangential homotopies and approximating them with a model zig-zag). Additionally, it has a more transparent connection with holonomic approximation. Therefore, Theorem 3.8.2 provides a new understanding even in the classic case $r = 1$.

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Samenvatting

Het onderwerp van deze scriptie is de interactie tussen contact structuren en (symplectische) foliaties van codimensie-één. Beiden zijn speciale gevallen van (codimensie-één) distributies. Voor een foliatie eisen we dat deze distributie integreerbaar is, terwijl we voor een contact structuur eisen dat hij "maximaal niet-integreerbaar" is. Deze definities zijn dus in zekere zin tegenovergesteld aan elkaar. Desalniettemin vertonen deze structuren meer overeenkomsten dan hun definitie doet vermoeden. Ze hebben bijvoorbeeld dezelfde onderliggende "algebraische structuur"; een paar differentiaalvormen $(\alpha, \eta) \in \Omega^1(M) \times \Omega^2(M)$ die voldoen aan de vergelijking:

$$\alpha \wedge \eta^n \neq 0. \tag{3.8.3.1}$$

Ook is het veel gevallen mogelijk om beide structuren in elkaar te "vervormen".

In **Hoofdstuk 1** behandelen we constructies van symplectische foliaties en contact structuren. Onze aanpak is als volgt; we splitsen de ruimte op in (simpelere) stukken, daarna construeren we op elk van deze stukken de gewenste structuur, en als laatste lijmen we de stukken weer aan elkaar zodat we de originele ruimte terugkrijgen.

Voor deze aanpak is het belangrijk om te begrijpen hoe symplectische foliaties en contact structuren zich gedragen op wiskundige ruimtes met een rand. We onderscheiden verschillende type randen (analoog aan de definities van contact/cosymplectische randen van symplectische ruimtes) en bewijzen expliciete normaalvormen.

We merken op dat de resultaten (en bewijzen) voor beide structuren zo goed als hetzelfde zijn. Dit stelt ons instaat om de constructies tegelijkertijd uit te voeren en de resulterende structuren in elkaar te vervormen. We bewijzen een algemene stelling die in het bijzonder toepasbaar is voor de 5-dimensionale bol (dit geeft ons een resultaat van Mitsumatsu [89]), en elke (gesloten, georienteerde) 3-dimensionale ruimte.

Hoofdstuk 2 is gebaseerd op gezamelijk werk met F. Presas. Hier bestuderen we de convergentie van contact structuren naar symplectische foliaties in hoger dimensionale ruimten. In dimensie-3 beschouwt men gewoonlijk convergentie als secties van de Grasmanniaanse bundel van codimensie-een distributies, zoals in de theorie van confoliaties. In hogere dimensies merken we op dat de (formele) symplectische vorm (η in Vergelijking 3.8.3.1) een belangrijke rol speelt. We definieren verschillende soorten convergentie en bestuderen hun relatie doormiddel van voorbeelden.

Een belangrijk resultaat in de 3-dimensionale theorie, zie [47], is dat elke foliatie behalve die op $\mathbb{S}^1 \times \mathbb{S}^2$ (doormiddel van bollen) benaderd kan worden door contact structuren. We bewijzen dat er in hogere dimensies veel meer voorbeelden van dit soort foliaties bestaan. Een van onze argumenten is gebaseerd op de theorie van contact fibraties over de 2-dimensionale bol. Dit levert voorbeelden op van foliaties die niet benadert kunnen worden om redenen die essentieel verschillen van het 3-dimensionale geval.

Zoals we opgemerkt hebben in Vergelijking 3.8.3.1 heeft elke differentieerbare structuur een onderliggende algebraische vergelijking. Voor verscheidene structuren is het altijd mogelijk om een oplossing van de algebraische vergelijking te vervormen in een echte oplossing. In dit geval zeggen we dat de structuur voldoet aan het "h-principe". In **Hoofdstuk 3** bestuderen we een specifieke techniek, de zogenoemde "plooi techniek" in de setting van jet bundels. De inhoud van dit hoofdstuk maakt deel uit van een lopend project met A. del Pino. Ons doel is om deze technieken toe te passen op de vraagstukken uit de voorgaande hoofdstukken. Voor nu is ons belangrijkste resultaat een generalizatie van de klassieke holonome benadering stelling uit [43], die stelt dat (onder gepaste voorwaarden) elke formele snede van een jet bundel benaderd kan woorden door een holonome snede. We bewijzen dat als we toestaan dat de snedes milde singulariteiten hebben dit resultaat ook op gesloten ruimtes geldt.

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Curriculum Vitae

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