

The background of the entire cover is a dark blue grid. Overlaid on this grid are several complex, three-dimensional wireframe structures. These structures are composed of many thin, intersecting lines that form a mesh. The lines are colored in a gradient of cyan, blue, and magenta, giving them a glowing appearance. The shapes are irregular and resemble crumpled sheets of paper or complex mathematical surfaces. They are scattered across the cover, with some appearing larger and more prominent than others.

# On Quantum Corrections in String Compactifications

*Effective Actions and Black Holes*

**Kilian Mayer**



# On Quantum Corrections in String Compactifications

*Effective Actions and Black Holes*

PhD thesis, Utrecht University, 2020

**About the cover:** The cover of this booklet is a schematic depiction of a higher-dimensional spacetime, which is predicted by string theory. The grid represents the four observable dimensions in our universe; they are the dimensions in which we live. The shapes placed on the grid are illustrations of Calabi–Yau manifolds, which represent the ‘hidden’ extra dimensions.



# On Quantum Corrections in String Compactifications

*Effective Actions and Black Holes*

Kwantum Correcties in Snaartheorie  
Compactificaties

*Effectieve Acties en Zwarte Gaten*

(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof. dr. H.R.B.M. Kummeling, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op woensdag 9 september 2020 des middags te 12.45 uur

door

Kilian Mayer

geboren op 31 juli 1991

te Landshut, Duitsland

Promotor: Prof. dr. S. J. G. Vandoren

Copromotor: Dr. T.W. Grimm

*To my parents.*



# List of publications

**Part I** of this thesis is about effective actions from string compactifications with higher-derivative corrections taken into account. Part I of this thesis is based on the publications

- [1] T.W. Grimm, K. Mayer, M. Weissenbacher: *Higher derivatives in type II and M-theory on Calabi-Yau threefolds*, JHEP 1802 (2018) 127, [1702.08404],
- [2] T.W. Grimm, K. Mayer, M. Weissenbacher: *One-modulus Calabi-Yau fourfold reductions with higher-derivative terms*, JHEP 1804 (2018) 021, [1712.07074].

**Part II** of this thesis is about the study of supersymmetric black holes in string theory and their microscopic and macroscopic entropy. It is based on the publications

- [3] C. Couzens, H. het Lam, K. Mayer, S. Vandoren: *Black Holes and (0,4) SCFTs from type IIB on K3*, JHEP 1908 (2019) 043, [1904.05361]<sup>1</sup>,
- [4] T.W. Grimm, H. het Lam, K. Mayer, S. Vandoren: *Four-dimensional black hole entropy from F-theory*, JHEP 1901 (2019) 037, [1808.05228]<sup>1</sup>,
- [5] C. Couzens, H. het Lam, K. Mayer, S. Vandoren: *Anomalies of (0,4) SCFTs from F-theory*, in preparation<sup>1</sup>,

as well as some unpublished results. Another paper to which the author contributed during the course of the PhD, which is however not included in this thesis, is

- [6] C. Couzens, H. het Lam, K. Mayer: *Twisted  $\mathcal{N} = 1$  SCFTs and their  $AdS_3$  duals*, JHEP 03 (2020) 032, [1912.07605].

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<sup>1</sup> This publication is also part of the thesis of H. het Lam.

Before we continue with the main text, we comment on part II of this thesis which has overlap with the thesis of H. het Lam.

- [3] Chapter 6 is based on this publication. Most of the microscopic computations in section 4 of this paper were carried out by the author, whereas most of the macroscopic computations in section 3 were carried out by H. het Lam.
- [4] Chapter 7 is partly based on this publication. The author joined this project after section 2 and in section 3 (two-derivative) of this paper were completed. The computations of the quantum corrections relevant for section 4 were jointly performed by the author and H. het Lam.
- [5] Chapter 7 is partly based on this yet to be published work. In this work, the author performed most of the computations involving the non-abelian flavour symmetries, which are in section 7.5 of this thesis. H. het Lam did most of the computations in section 7.4 which involve the ADE spaces and will also be contained in this paper.

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## Chapter 1

# Introduction and preliminaries

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The purpose of this introductory chapter is to set the stage for the remainder of this thesis. We will start with a more heuristic and historically oriented roadmap of major developments in theoretical high energy physics, gradually becoming more technical in the subsequent sections. The more technical sections will explain the fundamental concepts and techniques used in the main body of this thesis. This will include an overview of ten-dimensional supersymmetric string theories and the duality web through which they are connected. We will also introduce the notion of a low-energy effective theory and how they can be obtained from string theory via compactification of higher-dimensional theories. This process of dimensional reduction will be exemplified for the simple case of a compactification on a circle. These concepts will be particularly important for part I of this thesis. The background material for part II will include a lightning introduction to F-theory, and how black holes can be formed within this framework using D3-branes.

## 1.1 A roadmap towards string theory

The idea of unifying physical theories has in the past led to many deep insights in theoretical physics, and has significantly driven the rapid progress which was made in this field over the last century. The discovery of quantum mechanics and special relativity triggered a true paradigm shift with far-reaching impact on nearly every branch of modern physics.

The quantum physics revolution began in the year 1900 with Max Planck's presentation of his formula for the radiation spectrum of a black body. It was later understood that his formula could only be derived if one makes the crucial assumption that the radiation is emitted in discrete energy packages with elementary unit  $\varepsilon = \hbar\omega$ .<sup>1</sup> This formula perfectly reproduced the experimental data and interpolated between Wien's law, which is valid at high frequencies, and the Rayleigh-Jeans law, which is accurate at low frequencies. Most importantly, Planck's radiation law evades the *ultraviolet catastrophe*, which the Rayleigh-Jeans law famously suffers from. From this point onwards, the theory of quantum mechanics was successfully formulated and formalized. Even today, this remarkable framework is further developed and its frontiers are pushed further out. To date there is overwhelming experimental evidence proving the correctness of quantum mechanics.

The second revolution in the early 1900s was the development of the theory of special relativity and general relativity. Special relativity abandons the concept of an absolute time, which in classical Galilean physics serves as a parameter. Furthermore, it assigns to the speed of light an exceptional role, since in special relativity it is universal in every reference frame. Again, there is countless evidence for this theory up e.g. the Kennedy-Thorndike experiment, which proved that the propagation velocity of light is indeed independent of the inertial frame. A short time after the inception of special relativity, Einstein formulated his theory of general relativity, a geometric theory of gravity based on general covariance. General relativity predicted the existence of gravitational waves (disturbances of the gravitational field, or 'ripples' in spacetime) and black holes nearly a century ago. These two predictions were recently experimentally confirmed in two remarkable discoveries: the measurement of gravitational waves by the LIGO collaboration in 2016 [1], and the first image of a black hole taken by the Event Horizon Telescope in 2019 [2], which furnishes the first *direct* evidence of black holes.

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<sup>1</sup>Planck's constant has the value  $\hbar = 6.58 \times 10^{-16} \text{eV} \cdot \text{s}$  and  $\omega$  is the frequency.

The unification of special relativity and quantum mechanics lead to the framework of Quantum Field Theory (QFT), which is to date one of the most powerful and broad toolkits available to physicists. The prime example for the success story of QFT is Quantum Electrodynamics (QED), which unifies Electrodynamics and quantum mechanics. The unmatched accuracy to which the theoretical prediction of the Sommerfeld fine structure constant in QED agrees with its experimentally determined value, is a more than impressive testimony of QED's success. The tremendous success of QFT heralded a golden era of particle physics, with theoretical advances being driven forward by experimental discoveries, and vice versa. The invention of the Standard Model of particle physics (SM) in the 1970s unified three of the four known forces in nature: the electromagnetic force, which we observe and use in our everyday lives, the strong interaction which is the force holding the nuclei in atoms together, and the weak interaction, which mediates the  $\beta$ -decay of certain radioactive elements. The SM contains all the elementary particles known to date and reproduces experimental data very well. Indeed, the last missing piece of the puzzle, the Higgs boson, was discovered in 2012 at the Large Hadron Collider (LHC) in Switzerland and its mass was found to be  $m_{H^0} = (125.10 \pm 0.14) \text{ GeV}/c^2$  [3]. The Higgs field, whose excitations are the Higgs particles, occupies a special role in the SM. It is the Higgs field, which gives the standard model particles a mass by means of the Brout-Englert-Higgs mechanism.

Despite its great achievements and beauty, the SM also has its limitations. One of the most obvious shortcomings is that neutrinos remain massless in the SM, even after spontaneous symmetry breaking via the Higgs mechanism. However, it is known from experiments which observed neutrino oscillations, i.e. processes which transform a neutrino flavour into a different one, that neutrinos must have a small but non-vanishing mass. Another missing corner in the SM is dark matter, which is known to exist from observations about galaxies and their formation. In the standard model of cosmology, called  $\Lambda\text{CDM}^2$ , dark matter is responsible for 26.4% of the total energy in the universe. Furthermore, the *cosmological constant problem* refers to the spectacular failure of the SM (by a factor of  $10^{120}$ ) to reproduce the experimentally observed dark energy. The situation can be slightly improved by introducing supersymmetry, but the puzzle remains. In fact the origin and nature of dark energy is a currently heavily debated issue in the scientific community.

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<sup>2</sup>Here  $\Lambda$  is the symbol for the cosmological constant, which drives the expansion of the universe. CDM stands for ‘cold dark matter’.

Up to now we have avoided the elephant in the room: the SM only describes three out of the four known forces of nature. The problem of quantizing gravity is a long standing and notoriously difficult problem in theoretical physics. This is because gravity behaves at high energies qualitatively different from the other three forces, which are covered by the SM. Quantization of gravity with standard field theory techniques leads to the introduction of an infinite number of so-called *counterterms*. These counterterms are necessary for scattering amplitudes to remain finite and to avoid unphysical results. An infinite number of counterterms would require measurements for an infinite number of coupling constants in the theory. Theories which require such an infinite amount of counterterms are called *non-renormalizable*. It is often stated in the literature that these theories lose their predictive power due to the necessity to perform an infinite amount of measurements. This need not be the case from the modern Effective Field Theory (EFT) point of view. An EFT is thought to only be valid up to some ultraviolet (UV) scale  $\Lambda_{\text{UV}}$ , after which a more fundamental theory must take over. The top-down approach to effective field theories is to start with the more fundamental UV theory, from which the high-energy degrees of freedom with energies  $E \gtrsim \Lambda_{\text{UV}}$  are integrated out. This produces an effective theory in the Wilsonian sense for excitations with energies well below  $\Lambda_{\text{UV}}$ . The information about the high-energy theory is then contained in the couplings of the EFT. However, the details of the UV theory are washed out. Depending on the UV theory, this process of integrating high-energy degrees of freedom out can be challenging or even practically impossible. In contrast to the top-down approach to EFTs, there exists also the converse bottom-up approach. In this approach, the most general theory, which is compatible with the symmetries that the problem under investigation enjoys, is written down up to a certain satisfactory order. The coupling constants of this theory are then fixed by experimental data. The bottom-up approach does not refer to any UV completion of the EFT, not even the existence of such a completion.<sup>3</sup> However, this bottom-up EFT may be used to describe phenomena up to the energy scale  $\Lambda_{\text{UV}}$ . In this thesis we will (almost) exclusively take the top-down point of view.

The modern view on Einstein gravity is to interpret it as an EFT describing gravity, which is valid at low energies. A natural scale where gravity becomes strong, and therefore quantum gravity effects become important, is the Planck

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<sup>3</sup>Potential criteria to determine whether or not an EFT has a quantum gravity UV-completion are subject of a currently very active area of research, called the ‘Swampland program’, see e.g [4, 5].

scale  $\Lambda_{\text{Planck}} \sim 10^{19}$  GeV. Such a theory of quantum gravity is provided by string theory, the framework on which this thesis relies. The rise of string theory as a theory of quantum gravity was to some extent a lucky coincidence. The origins of string theory are in the late 1960s, where it was designed to explain the spectrum of hadrons and the strong force. Obvious flaws such as the existence of a critical dimension different from four, and the presence of an unknown spin two excitation made it a theory of mediocre success. It was soon realized that if the perspective is slightly changed, string theory has the potential to indeed become a success. The previously troublesome spin two excitation, when interpreted as a graviton, opened an exciting window into quantum gravity. In fact, on very general assumptions, there exists an argument due to Richard P. Feynman and S. Weinberg that the presence of a massless excitation of spin two inevitably leads to Einstein gravity at low energies. From this point onwards the five supersymmetric string theories were established and lead to many far-reaching insights, way beyond string theory itself. It is therefore fair to say, that string theory has succeeded in the long sought-for step of unifying quantum mechanics and gravity. In the next few sections we will introduce the basic concepts of string theory and explain some of its remarkable features. Along the way we will discuss some milestones which this field has passed, and on which many of the results contained in this thesis are based.

## 1.2 Type II superstrings and M-theory

String theory is a candidate for a ‘theory of everything’. The term ‘theory of everything’ refers to a theory which can – at least in principle – accommodate all ingredients which are necessary to describe our world. String theory gives a candidate for quantum gravity, and has also proven to be very successful in engineering gauge theories with different gauge groups and light matter content. The research branch which aims to make contact with the observable universe is called *string phenomenology*. The ultimate goal is to obtain the Standard Model of particle physics (and beyond) and cosmological models, which describe our very own universe accurately, from a top down construction in string theory. The fundamental idea behind string theory is to use extended strings as the elementary objects of the theory, as opposed to the point-like particles of QFT. These strings can either be open or closed strings. The worldlines of zero-dimensional particles are therefore replaced with two-dimensional worldsheets with different topologies. These worldsheets with different topologies represent processes in which strings can split and join, and resemble the familiar Feynman diagrams of QFT. Some

examples of open and closed string worldsheets are depicted in figure 1.1. While

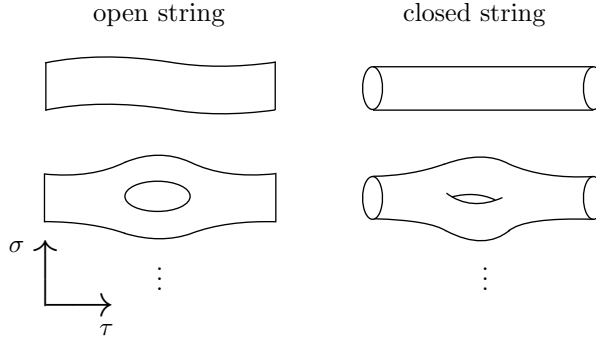


Figure 1.1: Examples of worldsheet topologies of the open and closed string.

string theories consisting solely of closed strings are consistent (and do exist, such as the heterotic string), the same can not be said about purely open string theories. It turns out that there is always a process which joins two open strings to form a closed string, such that consistency of the theory requires a closed string sector to exist. Remarkably, this simple fact forces quantum gravity upon us. We will see in this chapter that the state in the string spectrum which is interpreted as the graviton, the mediator of the gravitational force, is an excitation of the closed string. Therefore, since self-consistency of string interactions require closed strings, string theory requires gravity. It was noticed by Joseph Polchinski in 1995 [6] that surfaces on which open strings can end – the so-called *D-branes* – are in fact stable objects and therefore objects intrinsic to string theory. As the worldvolume theory of D-branes support gauge theories and can be used to construct and study black holes in string theories, the mere existence of these states represented a revolution of string theory.

Before diving into increasingly more technicalities, we will give a short overview of the five existing supersymmetric string theories in ten dimensions and a mysterious yet powerful eleven-dimensional theory called M-theory.

**Type IIA/IIB string.** There are two inequivalent types of string theories with 32 real supercharges in ten dimensions. Type IIA string theory has  $\mathcal{N} = (1, 1)$  supersymmetry and is therefore non-chiral. Type IIB on the other hand has chiral  $\mathcal{N} = (2, 0)$  supersymmetry. Both type IIA and type IIB contain open and closed strings. Furthermore, type IIA (type IIB) contains solitonic D-branes of odd (even) dimension on which the open strings can end. D-branes allow the



construction of gauge theories and are therefore an essential ingredient of type II string model building.

**Heterotic string.** The heterotic string is a hybrid of the bosonic string and the superstring. In contrast to the type II strings it has minimal  $\mathcal{N} = 1$  supersymmetry in ten spacetime dimensions. It is a theory of closed strings only and exists in two variants, namely, the heterotic string theories with gauge groups  $\text{SO}(32)$  and  $E_8 \times E_8$  in ten spacetime dimensions. The reduced amount of supersymmetry as well as the existence of a gauge theory in ten dimensions made the heterotic string a promising starting point for phenomenological model building. Indeed, the heterotic string seems tailored towards model building of grand unified theories (GUTs) since it contains all relevant GUT groups, namely  $\text{SU}(5)$ ,  $\text{SO}(10)$  and  $E_6$  as subgroups. The discovery in 1985 of the heterotic string and its potential for phenomenological applications was an essential driving force of the *first superstring revolution*.

**Type I string.** There also exists the type I string which has  $\mathcal{N} = 1$  supersymmetry in ten spacetime dimensions and  $\text{SO}(32)$  gauge group. It is nowadays viewed as an orientifold of type IIB string theory with 32 spacetime filling D9-branes subject to an orientifold projection.

**M-theory.** Lastly, the hypothetical eleven-dimensional theory called M-theory was introduced by E. Witten in 1995 and sparked the *second superstring revolution*. In its low energy regime it is believed to be described by the unique eleven-dimensional supergravity theory. However, it is not a theory of strings but rather a theory of membranes. There is to date strong evidence that M-theory is related to all supersymmetric string theories in ten dimensions via a web of dualities. That is, all string theories are thought to emerge from special limits of M-theory. In this sense, M-theory incorporates all known string theories and provides a powerful handle on previously inaccessible corners of string theory.

Since the bulk of this thesis is based on type II strings and M-theory we will in the following preliminary sections focus on these theories. The most essential features of the construction of type II strings will be introduced in sections 1.2.1 – 1.2.3. Aspects on how the low-energy effective actions of type II strings may be extracted from string scattering amplitudes are provided in section 1.2.4 and section 1.2.5. Dualities between type II strings and their relation to M-theory will be discussed in the subsequent section. This discussion culminates in the duality web relating M-theory to all five known string theories and is the content of section 1.2.6.

### 1.2.1 Type II superstring worldsheet

The first encounter with string theory is usually bosonic string theory in flat space  $\mathbb{R}^{1,d-1}$ , described by the *Nambu-Goto* action. The string is mathematically described by the string embeddings  $X^\mu$

$$\begin{aligned} X^\mu : \quad \Sigma &\rightarrow \mathbb{R}^{1,d-1} \\ (\sigma, \tau) &\mapsto X^\mu(\sigma, \tau), \end{aligned} \quad (1.1)$$

which are interpreted as coordinates in  $\mathbb{R}^{1,d-1}$ , in which the string is embedded. Furthermore the worldsheet coordinates  $\sigma^\alpha = (\sigma, \tau)$  take values in the domain  $\Sigma = [0, \ell) \times (\tau_i, \tau_f)$ . The Nambu-Goto action describing the dynamics of the string is simply given by the area swept out by the string with embedding  $X^\mu$

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'} \int_{\Sigma} dA = -\frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-\det_{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}}, \quad (1.2)$$

with  $\eta_{\mu\nu}$  the Minkowski metric on  $\mathbb{R}^{1,d-1}$  and  $T = 1/2\pi\alpha'$  the string tension. The square root in the action (1.2) makes a quantization of the theory unnecessarily difficult. It is therefore common to use a more quantization-friendly action to describe the string, the *Polyakov action*

$$S_{\text{P}} = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \quad (1.3)$$

This is nothing but the action of  $d$  free bosons in two dimensions coupled to two-dimensional gravity.<sup>4</sup> Note that the Nambu-Goto action (1.2) and the Polyakov action (1.3) are classically equivalent, with the Nambu-Goto action recovered by integrating out the metric  $h_{\alpha\beta}$ . However bosonic string theory does not stand a chance to be a realistic quantum theory of gravity. This is due to the absence of fermionic states in the Hilbert space, and also due to the tachyonic nature of the vacuum state in the quantum theory. Clearly, since e.g. electrons are fermions, any serious attempt to build a theory which describes our world must contain fermions in its spectrum.

As we shall see, the supersymmetrized version of (1.3) overcomes these deficiencies. This is the RNS (Ramond-Neveu-Schwarz) formulation of the superstring,

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<sup>4</sup>Adding an Einstein-Hilbert action to the 2d action does not change the equations of motion, since the Ricci scalar is the Euler density in 2d. Once the worldsheet theory is coupled to background fields, a coupling of the dilaton to the 2d Ricci scalar leads to the interpretation of the dilaton vacuum expectation as the string coupling. The string coupling controls the perturbative expansion of the string theory partition function in worldsheet genera.

which makes worldsheet supersymmetry manifest. In order to obtain a supersymmetric worldsheet theory we need to add the action of the superpartners of the fields to the Polyakov action. On shell, the supersymmetry multiplets are the matter multiplets  $(X^\mu, \psi^\mu)$  which contain  $d$  Majorana fermions as superpartners of the worldsheet scalars, and the gravity multiplet  $(e_\alpha^a, \chi_\alpha)$  which contains the zweibein  $e_\alpha^a$  and the worldsheet gravitino  $\chi_\alpha$ , which is subject to a Majorana condition. The resulting supersymmetric worldsheet theory enjoys various symmetries which may be used to gauge-fix the action. Local supersymmetry transformations, diffeomorphisms and local Lorentz transformations can be used to locally go to *superconformal gauge*

$$e_\alpha^a = e^\phi \delta_\alpha^a, \quad \chi_\alpha = \rho_\alpha \lambda = \frac{1}{2} \rho_\alpha \rho^\beta \chi_\beta, \quad (1.4)$$

where  $\rho_\alpha$  are the Dirac matrices in two dimensions. In the classical theory, one can make use of the (super)-Weyl transformations to reach the flat gauge  $e_\alpha^a = \delta_\alpha^a$  and  $\chi_\alpha = 0$ . In the quantum theory these symmetries are anomalous except in the critical dimension of the superstring  $d = 10$ . Let us note that the existence of conformal Killing vectors and spinors prevent us from reaching the gauge (1.4) globally on a genus  $g$  worldsheet  $\Sigma_g$ . These conformal Killing vectors and spinors determine the number of (super)-moduli on a given Riemann surface  $\Sigma_g$ , over which one needs to integrate in the computation of string amplitudes. We will not dive into the details of the treatment of these any further but note that an excellent exposition can be found in [7].

In superconformal gauge one obtains the worldsheet action

$$S_{\text{susy}} = -\frac{1}{8\pi} \int_\Sigma d^2\sigma \left( \frac{2}{\alpha'} \partial_\alpha X^\mu \partial^\alpha X_\mu + 2i \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right), \quad (1.5)$$

where the indices are contracted with the flat Minkowski metric  $\eta_{\alpha\beta}$  on the worldsheet. Even though we have used local supersymmetry transformations to gauge fix the worldsheet gravitino, we are still left with residual supersymmetry transformations which do not transform the fields out of the chosen gauge. One finds that the action (1.5) is invariant under the supersymmetry transformations

$$\delta_\epsilon X^\mu = \sqrt{\frac{\alpha'}{2}} i \bar{\epsilon} \psi^\mu, \quad \delta_\epsilon \psi^\mu = \sqrt{\frac{2}{\alpha'}} \rho^\alpha \partial_\alpha X^\mu \epsilon, \quad (1.6)$$

provided the supersymmetry parameter satisfies  $\rho^\beta \rho_\alpha \partial_\beta \epsilon = 0$  which ensures that choice of gauge for the gravitino is preserved. The equations of motions following from (1.5) are

$$\partial_\alpha \partial^\alpha X^\mu = 0, \quad \rho^\alpha \partial_\alpha \psi^\mu = 0. \quad (1.7)$$

Since we have gauge fixed the zweibein  $e_\alpha^a$  and the gravitino  $\chi_\alpha$  these fields do not show up in the action (1.5) anymore. For consistency, we therefore have to impose the equations of motion of the zweibein and the gravitino derived from the action before gauge fixing as constraints on every solution to the equations of motion. These are just the constraints that the energy-momentum tensor  $T_{\alpha\beta}$  and the supercurrent  $T_{F\alpha}$  must vanish, i.e.

$$T_{\alpha\beta} = -\frac{1}{\alpha'} \left( \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \eta_{\alpha\beta} \partial_\gamma X^\mu \partial^\gamma X_\mu \right) \quad (1.8)$$

$$-\frac{i}{4} \left( \bar{\psi}^\mu \rho_\alpha \partial_\beta \psi_\mu + \bar{\psi}^\mu \rho_\beta \partial_\alpha \psi_\mu \right) = 0,$$

$$T_{F\alpha} = -\frac{1}{4} \sqrt{\frac{2}{\alpha'}} \rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu = 0. \quad (1.9)$$

In the quantized theory, these constraints are imposed on states in the Hilbert space. Only states  $|\Psi\rangle$  which satisfy  $T_{\alpha\beta} |\Psi\rangle = T_{F\alpha} |\Psi\rangle = 0$  are part of the physical Hilbert space.

### 1.2.2 Closed and open type II strings

We will make use of the notation

$$\psi^\mu = \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix} \quad (1.10)$$

for the real two-component spinors  $\psi^\mu$ . When deriving the equations of motion of  $X^\mu, \psi^\mu$  from the action (1.5) while imposing  $\delta X^\mu(\sigma, \tau_{i,f}) = \delta \psi^\mu(\sigma, \tau_{i,f}) = 0$ , we pick up a boundary term

$$\delta S_{\text{susy}} \supset -\frac{1}{2\pi\alpha'} \int_{\tau_i}^{\tau_f} d\tau \partial_\sigma X_\mu \delta X^\mu \Big|_{\sigma=0}^{\sigma=\ell} + \frac{1}{2\pi} \int_{\tau_i}^{\tau_f} d\tau (\psi_+ \cdot \delta \psi_+ - \psi_- \cdot \delta \psi_-) \Big|_{\sigma=0}^{\sigma=\ell}, \quad (1.11)$$

which must vanish in order to have a well defined variational principle. We will therefore have to make choices of boundary conditions which we impose on  $X^\mu, \psi^\mu$ . In the following we shall discuss admissible boundary conditions for the fields  $X^\mu, \psi^\mu$  for the closed and open string separately.

**Closed strings.** In the case of the closed string we can impose

$$X^\mu(\sigma, \tau) = X^\mu(\sigma + \ell, \tau) \quad (1.12)$$

as boundary condition on the string embedding, which is intuitively clear. In the case of the worldsheet fermions we can set (1.11) to zero by choosing

$$\psi_+^\mu(\sigma, \tau) = \pm \psi_+^\mu(\sigma + \ell, \tau), \quad \psi_-^\mu(\sigma, \tau) = \pm \psi_-^\mu(\sigma + \ell, \tau). \quad (1.13)$$

In this case, we can make independent choices for the components  $\psi_\pm^\mu$ . Periodic fermions are said to be in the Ramond (R) sector, whereas anti-periodic fermions are said to be in the Neveu-Schwarz (NS) sector. We are therefore left with four possible sets of boundary conditions: (R,R), (R,NS), (NS,R), (NS,NS). Introducing the coordinates  $\sigma^\pm = \tau \pm \sigma$ , such that the equations of motion take the form

$$\partial_+ \partial_- X^\mu = 0, \quad \partial_- \psi_+^\mu = \partial_+ \psi_-^\mu = 0, \quad (1.14)$$

we can see from (1.14) that the string embeddings split into the sum of a left-moving and right-moving part, and  $\psi_\pm$  are purely left- and rightmoving, i.e.

$$X^\mu(\sigma^\pm) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-), \quad \psi_+ = \psi_+(\sigma^+), \quad \psi_- = \psi_-(\sigma^-). \quad (1.15)$$

It is not hard to show that

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{\ell}p^\mu\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n \neq 0}\frac{1}{n}\bar{\alpha}_n^\mu e^{-\frac{2\pi}{\ell}in\sigma^+} \quad (1.16)$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{\ell}p^\mu\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n \neq 0}\frac{1}{n}\alpha_n^\mu e^{-\frac{2\pi}{\ell}in\sigma^-} \quad (1.17)$$

$$\left. \begin{aligned} \psi_+^\mu(\sigma^+) &= \sqrt{\frac{2\pi}{\ell}} \sum_r \bar{b}_r^\mu e^{-\frac{2\pi}{\ell}r\sigma^+} \\ \psi_-^\mu(\sigma^-) &= \sqrt{\frac{2\pi}{\ell}} \sum_r b_r^\mu e^{-\frac{2\pi}{\ell}r\sigma^-} \end{aligned} \right\} \text{ with } \begin{array}{ll} r \in \mathbb{Z} & \text{for (R)} \\ r \in \mathbb{Z} + \frac{1}{2} & \text{for (NS)} \end{array} \quad (1.18)$$

are the most general solution to the equations of motion (1.14) with the given boundary conditions. Since the fields  $X^\mu, \psi^\mu$  are real, we have the relations  $(\alpha_n^\mu)^* = \alpha_{-n}^\mu, (b_n^\mu)^* = b_{-n}^\mu$  and likewise for the left-moving modes. We note that in order to preserve Poincaré invariance in ten dimensions (or more generally, in  $d$  dimensions) we must impose the same boundary conditions for each index  $\mu$ .

**Open strings.** In the case of open strings the boundary terms in (1.11) have to vanish at each boundary point  $\sigma = 0, \ell$  individually. As we already noted, an open string ends on surfaces which are called D-branes. The presence of these D-branes breaks Poincaré invariance which allows us to impose different

boundary conditions for different spacetime indices  $\mu$ . We may even choose different boundary conditions for the individual endpoints of an open string for a fixed spacetime index  $\mu$ . Figure 1.2 shows the situation where an open string has both endpoints attached to the same stack of D-branes (blue), and a string stretching between two D-brane stacks with different worldvolume dimensions (green). For the case of the bosonic worldsheet embeddings  $X^\mu$  we find that (1.11)

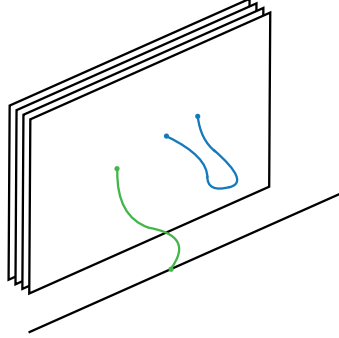


Figure 1.2: An open string attached with both ends on the same stack of D-branes, and an open string stretching between two D-branes of different dimensions.

vanishes at an endpoint  $\sigma = \sigma^* \in \{0, \ell\}$  if one imposes

$$\begin{aligned} \partial_\sigma X^\mu|_{\sigma=\sigma^*} &= 0, & \text{Neumann (N),} \\ \partial_\tau X^\mu|_{\sigma=\sigma^*} &= 0 \quad \Leftrightarrow \quad \delta X^\mu|_{\sigma=\sigma^*} = 0, & \text{Dirichlet (D).} \end{aligned} \quad (1.19)$$

The open string embedding  $X^\mu$  for a fixed index  $\mu$  is therefore fully described by boundary conditions for its two endpoints  $\sigma^* \in \{0, \ell\}$ . This leaves us with four possibilities: (NN), (DD), (ND), (DN). Using e.g. the doubling trick one can show that the mode expansions for the open string embeddings are given by

$$\begin{aligned} \text{(NN)} \quad X^\mu(\sigma, \tau) &= x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-i\frac{\pi}{\ell} n \tau} \cos\left(\frac{n\pi\sigma}{\ell}\right) \\ \text{(DD)} \quad X^\mu(\sigma, \tau) &= x_0^\mu + \frac{1}{\ell} (x_1^\mu - x_0^\mu) \sigma + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-i\frac{\pi}{\ell} n \tau} \sin\left(\frac{\pi n \sigma}{\ell}\right) \\ \text{(ND)} \quad X^\mu(\sigma, \tau) &= x^\mu + i\sqrt{2\alpha'} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{1}{r} \alpha_r^\mu e^{-i\frac{\pi}{\ell} r \tau} \cos\left(\frac{\pi r \sigma}{\ell}\right) \\ \text{(DN)} \quad X^\mu(\sigma, \tau) &= x^\mu + \sqrt{2\alpha'} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \frac{1}{r} \alpha_r^\mu e^{-i\frac{\pi}{\ell} r \tau} \sin\left(\frac{\pi r \sigma}{\ell}\right) \end{aligned} \quad (1.20)$$

with the same reality relations among the modes  $\alpha_n^\mu$  as before. For the worldsheet fermions we can set the boundary terms to zero by imposing

$$\psi_+^\mu|_{\sigma=0} = \pm \psi_-^\mu|_{\sigma=0}, \quad \psi_+^\mu|_{\sigma=\ell} = \pm \psi_-^\mu|_{\sigma=\ell}. \quad (1.21)$$

We now define the phase  $\eta$  which takes values  $\eta = 1$  in the (R) sector and  $\eta = -1$  in the (NS) sector. With the aid of this phase, the boundary conditions for the worldsheet fermions are

$$\begin{aligned} (\text{NN}) \quad \psi_+^\mu|_{\sigma=0} &= +\psi_-^\mu|_{\sigma=0} & \psi_+^\mu|_{\sigma=\ell} &= +\eta\psi_-^\mu|_{\sigma=\ell}, \\ (\text{DD}) \quad \psi_+^\mu|_{\sigma=0} &= -\psi_-^\mu|_{\sigma=0} & \psi_+^\mu|_{\sigma=\ell} &= -\eta\psi_-^\mu|_{\sigma=\ell}, \\ (\text{ND}) \quad \psi_+^\mu|_{\sigma=0} &= +\psi_-^\mu|_{\sigma=0} & \psi_+^\mu|_{\sigma=\ell} &= -\eta\psi_-^\mu|_{\sigma=\ell}, \\ (\text{DN}) \quad \psi_+^\mu|_{\sigma=0} &= -\psi_-^\mu|_{\sigma=0} & \psi_+^\mu|_{\sigma=\ell} &= +\eta\psi_-^\mu|_{\sigma=\ell}. \end{aligned} \quad (1.22)$$

For completeness, we also give the mode expansions of the worldsheet fermions for all possible boundary conditions in the (R) and (NS) sector

$$\begin{aligned} (\text{NN}) \quad \psi_\pm^\mu(\sigma^\pm) &= \sqrt{\frac{\pi}{\ell}} \sum_r b_r^\mu e^{-i\frac{\pi}{\ell} r \sigma^\pm} & \text{with} \quad & \begin{array}{ll} r \in \mathbb{Z} + \frac{1}{2} & \text{for (NS)} \\ r \in \mathbb{Z} & \text{for (R)} \end{array} \\ (\text{DD}) \quad \psi_\pm^\mu(\sigma^\pm) &= \pm \sqrt{\frac{\pi}{\ell}} \sum_r b_r^\mu e^{-i\frac{\pi}{\ell} r \sigma^\pm} & \text{with} \quad & \begin{array}{ll} r \in \mathbb{Z} + \frac{1}{2} & \text{for (NS)} \\ r \in \mathbb{Z} & \text{for (R)} \end{array} \\ (\text{ND}) \quad \psi_\pm^\mu(\sigma^\pm) &= \sqrt{\frac{\pi}{\ell}} \sum_r b_r^\mu e^{-i\frac{\pi}{\ell} r \sigma^\pm} & \text{with} \quad & \begin{array}{ll} r \in \mathbb{Z} & \text{for (NS)} \\ r \in \mathbb{Z} + \frac{1}{2} & \text{for (R)} \end{array} \\ (\text{DN}) \quad \psi_\pm^\mu(\sigma^\pm) &= \pm \sqrt{\frac{\pi}{\ell}} \sum_r b_r^\mu e^{-i\frac{\pi}{\ell} r \sigma^\pm} & \text{with} \quad & \begin{array}{ll} r \in \mathbb{Z} & \text{for (NS)} \\ r \in \mathbb{Z} + \frac{1}{2} & \text{for (R)} \end{array}. \end{aligned} \quad (1.23)$$

We have given all classical mode expansions of the worldsheet bosons and fermions with different boundary conditions. Once the theory is quantized, it is easy to derive the spectrum of states, as well as explicit operator representations of e.g. the mass operator and the Hamiltonian which act on the Hilbert space of the quantized superstring from them.

### 1.2.3 Light-Cone quantization of the superstring

We now proceed and quantize the supersymmetric string theory introduced in the previous section. This is done with standard techniques by imposing canonical



(anti-)commutation relations. These canonical (anti-)commutation relations can be translated into (anti-)commutation relations of the modes  $\alpha_n^\mu, b_r^\mu$ , which are promoted to operators acting on the string theory Hilbert space. One easily derives the (anti-)commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad (1.24)$$

$$\begin{aligned} [\alpha_m^\mu, \alpha_n^\nu] &= [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}, & [\bar{\alpha}_m^\mu, \alpha_n^\nu] &= 0 \\ \{b_r^\mu, b_s^\nu\} &= \{\bar{b}_r^\mu, \bar{b}_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}, & \{\bar{b}_r^\mu, b_s^\nu\} &= 0, \end{aligned} \quad (1.25)$$

which are valid for all mode expansion given in the previous section. Also note that the modes  $b_r^\mu$  and  $\alpha_n^\mu$  commute. We will now proceed by introducing light-cone coordinates in the target space with coordinates  $X^\mu$ . The light-cone coordinates are defined by

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1), \quad X^{i=2,\dots,9}, \quad (1.26)$$

such that the inner product for two vectors  $A^\mu$  and  $B^\mu$  takes the form

$$A \cdot B = \eta_{\mu\nu} A^\mu B^\nu = \delta_{ij} A^i B^i - A^- B^+ - A^+ B^-. \quad (1.27)$$

Recall that after gauge fixing the worldsheet action to superconformal gauge there is still a residual gauge symmetry which we can use to set

$$X^+ = \frac{2\pi\alpha'}{\ell} p^+ \tau, \quad \psi^+ = 0. \quad (1.28)$$

The crux about light-cone quantization is that we can solve the constraints which have to be imposed on physical states  $|\Psi\rangle$ , namely<sup>5</sup>

$$T_{\pm\pm} |\Psi\rangle = T_{F\pm} |\Psi\rangle = 0. \quad (1.29)$$

Indeed, one can show that

$$\partial_\pm X^- = \frac{1}{2p^+} \frac{\ell}{2\pi} \left( \frac{2}{\alpha'} \partial_\pm X^i \partial_\pm X^i + i\psi_\pm^i \partial_\pm \psi_\pm^i \right), \quad (1.30)$$

$$\psi_\pm^- = \frac{2}{\alpha' p^+} \frac{\ell}{2\pi} \psi_\pm^i \partial_\pm X^i, \quad (1.31)$$

solves the constraints

$$T_{\pm\pm} = T_{F\pm} = 0. \quad (1.32)$$

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<sup>5</sup>The worldsheet indices of the tensors are given in terms of the coordinates  $\sigma^\pm$ .

These equations allow us to obtain expressions of the modes  $\alpha_n^-, b_r^-$  in terms of the transverse oscillators  $\alpha_n^i, b_r^i$ , which are the remaining independent operators. We will continue the discussion in the right-moving sector of the closed string. The left-moving part is completely analogous and only requires some bars on the corresponding expressions.

The vacuum states in the (NS) and (R) sector are defined by

$$\begin{aligned} \alpha_m^i |0\rangle_{\text{NS}} &= b_r^i |0\rangle_{\text{NS}} = 0 & \text{for } m \in \mathbb{N}, r \in \mathbb{N}_0 + \frac{1}{2}, \\ \alpha_m^i |A\rangle_{\text{R}} &= b_m^i |A\rangle_{\text{R}} = 0 & \text{for } m \in \mathbb{N}. \end{aligned} \quad (1.33)$$

We can build excited states of the string by acting with oscillators  $\alpha_{-m}^i, b_{-r}^i$  for  $m, r > 0$  on the vacuum states. Furthermore note that if  $|A\rangle_{\text{R}}$  satisfies (1.33), then  $b_0^i |A\rangle_{\text{R}}$  also satisfies (1.33). Since the zero modes  $b_0^i$  satisfy a Clifford algebra

$$\{b_0^i, b_0^j\} = \delta^{ij}. \quad (1.34)$$

One can show that due to this property the (R) ground states form a spinor representation of  $\text{SO}(d-2)$ . In the critical dimension  $d = 10$  there are 16 degrees of freedom in this representation which can be identified with the two Majorana-Weyl ground states of opposite chirality. We denote these with (un)dotted spinor indices

$$|a\rangle \text{ in } \mathbf{8}_s \text{ of } \text{SO}(8), \quad |\dot{a}\rangle \text{ in } \mathbf{8}_c \text{ of } \text{SO}(8). \quad (1.35)$$

Since the little group of massless states in ten dimensions is  $\text{SO}(8)$  and the (R) ground states form representations under this little group, we already conclude that the (R) ground states are massless Majorana-Weyl fermions in ten dimensions. The mass operator for the closed string is defined by  $\alpha' m^2 = 2p^+ p^- - p^i p^i = \alpha' m_L^2 + \alpha' m_R^2$  and can be computed from the mode expansions. One finds

$$\begin{aligned} \alpha' m_L^2 &= 2(N^{(\bar{\alpha})} + N^{(\bar{b})} + \bar{a}) \\ \alpha' m_R^2 &= 2(N^{(\alpha)} + N^{(b)} + a), \end{aligned} \quad (1.36)$$

where the level operators  $N^{(x)}$  count the total level number of oscillators of type  $x$ .<sup>6</sup> The constants  $a, \bar{a}$  arise from a seeming ambiguity in ordering certain operators when passing from the classical to the quantum theory. This ambiguity can be fixed in various ways. The most rigorous way to determine the normal ordering constants  $a, \bar{a}$  is to impose that spacetime Poincaré symmetry still survives at

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<sup>6</sup>That is,  $N^{(x)} x_{-n_1}^{i_1} \cdots x_{-n_M}^{i_M} |0\rangle = \left(\sum_{j=1}^M n_j\right) x_{-n_1}^{i_1} \cdots x_{-n_M}^{i_M} |0\rangle$ .

the quantum level. For the closed string one finds from this analysis the critical dimension  $d = 10$  and

$$a = \bar{a} = \begin{cases} -\frac{1}{16}(d-2) = -\frac{1}{2} & \text{for (NS)} \\ 0 & \text{for (R)} \end{cases}. \quad (1.37)$$

Furthermore, physical states in the closed string theory have to satisfy the level-matching condition

$$\alpha' m_L^2 = \alpha' m_R^2. \quad (1.38)$$

The level-matching condition can be elegantly derived from the fact that on a closed string there is no distinguished point.

We can now proceed and construct the first excited states by acting with the oscillators on the vacuum states. All states up to and including the massless level are shown in table 1.1.

$\alpha' m^2$	state	sector	$((-1)^{\overline{\mathcal{F}}}, (-1)^{\mathcal{F}})$	little group rep.
-2	$ 0\rangle \otimes  0\rangle$	(NS,NS)	$(-1, -1)$	<b>1</b>
0	$\bar{b}_{-\frac{1}{2}}^i  0\rangle \otimes b_{-\frac{1}{2}}^j  0\rangle$	(NS,NS)	$(+1, +1)$	<b>1</b> $\oplus$ <b>28</b> $\oplus$ <b>35<sub>v</sub></b>
0	$ a\rangle \otimes  b\rangle$	(R,R)	$(+1, +1)$	<b>1</b> $\oplus$ <b>28</b> $\oplus$ <b>35<sub>s</sub></b>
0	$ \dot{a}\rangle \otimes  \dot{b}\rangle$	(R,R)	$(-1, -1)$	<b>1</b> $\oplus$ <b>28</b> $\oplus$ <b>35<sub>c</sub></b>
0	$ \dot{a}\rangle \otimes  b\rangle$	(R,R)	$(-1, +1)$	<b>8<sub>v</sub></b> $\oplus$ <b>56<sub>v</sub></b>
0	$ a\rangle \otimes  \dot{b}\rangle$	(R,R)	$(+1, -1)$	<b>8<sub>v</sub></b> $\oplus$ <b>56<sub>v</sub></b>
0	$ a\rangle \otimes b_{-\frac{1}{2}}^i  0\rangle$	(R,NS)	$(+1, +1)$	<b>8<sub>c</sub></b> $\oplus$ <b>56<sub>c</sub></b>
0	$ \dot{a}\rangle \otimes b_{-\frac{1}{2}}^i  0\rangle$	(R,NS)	$(-1, +1)$	<b>8<sub>s</sub></b> $\oplus$ <b>56<sub>s</sub></b>
0	$\bar{b}_{-\frac{1}{2}}^i  0\rangle \otimes  a\rangle$	(NS,R)	$(+1, +1)$	<b>8<sub>c</sub></b> $\oplus$ <b>56<sub>c</sub></b>
0	$\bar{b}_{-\frac{1}{2}}^i  0\rangle \otimes  \dot{a}\rangle$	(NS,R)	$(+1, -1)$	<b>8<sub>s</sub></b> $\oplus$ <b>56<sub>s</sub></b>

Table 1.1: Closed string spectrum up to the massless level before the GSO projection.

However, it turns out that sting theory with all possible states which are listed in table 1.1 is inconsistent at the quantum level, since it fails to be modular invariant. This issue can be cured by projecting out a subset of the spectrum. The

famous GSO projection<sup>7</sup> leads to a consistent string theory with a supersymmetric target space spectrum. The GSO projection involves the right- and left-moving worldsheet fermion number operators  $\mathcal{F}, \bar{\mathcal{F}}$ , whose precise definition does not matter for our purposes. On a state  $|\Psi\rangle_{\text{NS}}$  which is created from the (right-moving) (NS) vacuum with an arbitrary number of  $\alpha_{-m}^i$  oscillators and  $M$  oscillators of the type  $b_{-m}^i$ , the operator  $(-1)^{\mathcal{F}}$  acts as

$$(-1)^{\mathcal{F}} |\Psi\rangle_{\text{NS}} = (-1)^{M+1} |\Psi\rangle_{\text{NS}} . \quad (1.39)$$

A completely analogous relation holds for the operator  $(-1)^{\bar{\mathcal{F}}}$  acting in the left-moving (NS) sector. Likewise a state with an arbitrary number of  $\alpha_{-m}^i$  oscillators and  $M$   $b_{-m}^i$  oscillators built from the right-moving (R) ground state  $|a\rangle$  satisfies

$$(-1)^{\mathcal{F}} |\Psi\rangle_{\text{R}} = (-1)^M |\Psi\rangle_{\text{R}} . \quad (1.40)$$

If the state is created from the ground state with opposite chirality  $|\dot{a}\rangle$  the corresponding relation analogous to (1.40) picks up an additional factor  $(-1)$  on the right hand side. Again, the very same definitions apply to the left-moving sector of the closed string with  $\bar{\mathcal{F}}$ .

We perform the GSO projection for the left- and right-movers separately. The states in the closed string spectrum surviving the GSO projection are then the tensor product states of the left- and right-moving states which survive the left- and right-moving projection and in addition satisfy the level matching condition (1.38). The GSO projection can be performed consistently in two ways:

$$\begin{array}{ll} \text{Type IIA} & \begin{array}{ll} (-1)^{\mathcal{F}} = (-1)^{\bar{\mathcal{F}}} = 1 & \text{for (NS)} \\ (-1)^{\mathcal{F}} = -(-1)^{\bar{\mathcal{F}}} = 1 & \text{for (R)} \end{array} \end{array} \quad (1.41)$$

$$\begin{array}{ll} \text{Type IIB} & \begin{array}{ll} (-1)^{\mathcal{F}} = (-1)^{\bar{\mathcal{F}}} = 1 & \text{for (NS)} \\ (-1)^{\mathcal{F}} = (-1)^{\bar{\mathcal{F}}} = 1 & \text{for (R)} \end{array} . \end{array} \quad (1.42)$$

We can now apply these projection conditions to the spectrum in table 1.1 and we find the massless spectrum of closed type IIA and type IIA string theory which is given in table 1.2.

Type IIA string theory therefore contains at the massless level in the (NS,NS) sector a scalar field  $\phi$ , the dilaton, a two-form  $B$  and the graviton  $h_{\mu\nu}$ . In the (R,R) sector we have a one-form potential  $C_1$  and a three-form potential  $C_3$ . Their fermionic superpartners are two spin-1/2 fermions with opposite chirality  $\lambda_{\frac{1}{2}}, \tilde{\lambda}_{\frac{1}{2}}$

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<sup>7</sup>This projection is named after Gliozzi, Scherk and Olive.

$\alpha' m^2$	state	sector	little group rep.	fields
<b>Type IIA</b>				
0	$\bar{b}_{-\frac{1}{2}}^i  0\rangle \otimes b_{-\frac{1}{2}}^j  0\rangle$	(NS,NS)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v$	$\phi, B_{\mu\nu}, h_{\mu\nu}$
0	$ \dot{a}\rangle \otimes  b\rangle$	(R,R)	$\mathbf{8}_v \oplus \mathbf{56}_v$	$C_{1\mu}, C_{3\mu\nu\rho}$
0	$ \dot{a}\rangle \otimes b_{-\frac{1}{2}}^i  0\rangle$	(R,NS)	$\mathbf{8}_s \oplus \mathbf{56}_s$	$\tilde{\lambda}_{\frac{1}{2}}, \tilde{\Psi}_{\frac{3}{2}}$
0	$\bar{b}_{-\frac{1}{2}}^i  0\rangle \otimes  a\rangle$	(NS,R)	$\mathbf{8}_c \oplus \mathbf{56}_c$	$\lambda_{\frac{1}{2}}, \Psi_{\frac{3}{2}}$
<b>Type IIB</b>				
0	$\bar{b}_{-\frac{1}{2}}^i  0\rangle \otimes b_{-\frac{1}{2}}^j  0\rangle$	(NS,NS)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_v$	$\phi, B_{\mu\nu}, h_{\mu\nu}$
0	$ a\rangle \otimes  b\rangle$	(R,R)	$\mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_s$	$C_0, C_{2\mu\nu}, C_{4\mu\nu\rho\sigma}^{\text{s.d.}}$
0	$ a\rangle \otimes b_{-\frac{1}{2}}^i  0\rangle$	(R,NS)	$\mathbf{8}_c \oplus \mathbf{56}_c$	$\lambda_{\frac{1}{2}}, \Psi_{\frac{3}{2}}$
0	$\bar{b}_{-\frac{1}{2}}^i  0\rangle \otimes  a\rangle$	(NS,R)	$\mathbf{8}_c \oplus \mathbf{56}_c$	$\lambda_{\frac{1}{2}}, \Psi_{\frac{3}{2}}$

Table 1.2: Closed string spectrum up to the massless level in type IIA and type IIB string theory and their corresponding spacetime fields.

and two spin-3/2 gravitinos  $\Psi_{\frac{3}{2}}, \tilde{\Psi}_{\frac{3}{2}}$  of opposite chirality. The appearance of the two gravitinos with different chiralities implies that the theory has  $\mathcal{N} = (1, 1)$  supersymmetry in ten dimensions.

The massless bosonic spacetime fields of type IIB string theory contain the fields in the universal (NS,NS) sector, and a zero-form  $C_0$ , a two-form  $C_2$  and a four-form  $C_4$  with a self-dual field strength in the (R,R) sector. The fermionic spectrum again contains two spin-1/2 fermions  $\lambda_{\frac{1}{2}}^{1,2}$  and gravitinos  $\Psi_{\frac{3}{2}}^{1,2}$ , now with identical chirality. This signals that type IIB string theory has  $\mathcal{N} = (2, 0)$  supersymmetry in ten dimensions.

#### 1.2.4 Low-energy effective actions of type IIA and type IIB

Since in most physical applications energies which are well below the scale set by the string length  $\sqrt{\alpha'} E \ll 1$  are relevant, it is convenient to make use of low-energy effective field theories which can be derived from string theory. In ten spacetime dimensions there exist precisely two supergravity theories with 32 real supercharges: a non-chiral supergravity theory with  $\mathcal{N} = (1, 1)$  supersymmetry and a chiral one with  $\mathcal{N} = (2, 0)$  supersymmetry. The massless field content

agrees with the spacetime field contents of type IIA and type IIB string theory which we listed in table 1.2. These facts alone already strongly suggest that the low-energy dynamics of the massless fields corresponding to the massless superstring excitations is governed by the ten-dimensional supergravity theories.

The approach which we will sketch here is to derive the effective field theory of string theory systematically from string scattering amplitudes. That is, one writes down the most general action for the spacetime fields which is compatible with the symmetries, and matches their couplings with the  $n$ -point amplitudes of string theory in an expansion in  $\alpha'$ . With this matching one ensures that the amplitudes of the effective field theory reproduce the low-energy expansion of string theory amplitudes. The closed string states in table 1.2 are in one-to-one correspondence with certain operators in the worldsheet theory, the so called vertex operators.<sup>8</sup> The insertion of such a vertex operator for a given string state in a correlation function on the string worldsheet theory essentially creates this string state on the insertion point.

In the following we will focus on tree-level amplitudes in the (NS,NS) sector. The general structure of  $n$ -point amplitudes at tree-level in the string coupling  $g_s$  is

$$A^{(n)} \sim g_s^{n-2} \int d^2 z_1 \cdots d^2 z_n \langle V_1(z_1) \cdots V_n(z_n) \rangle_{\Sigma=S^2}, \quad (1.43)$$

where  $V_n$  are vertex operator insertions on the worldsheet. At tree-level the worldsheet is via a conformal map equivalent to a sphere  $S^2$  with  $n$  vertex operator insertions, i.e. a sphere with  $n$  punctures. Specializing to the three-

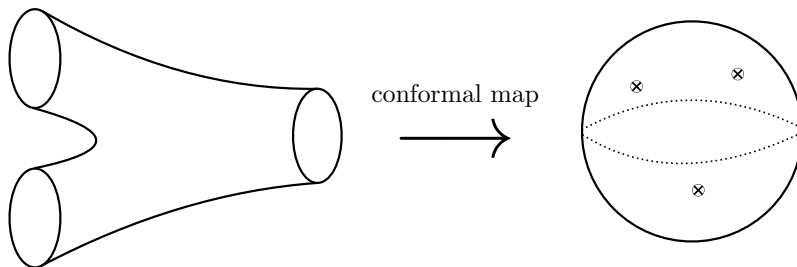


Figure 1.3: The worldsheet of the tree-level three-point function of the closed string is conformal to a three-punctured sphere.

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<sup>8</sup>This one-to-one correspondence is due to the state-operator map, which is a special property of conformal field theories.

point amplitude of three closed string states, one therefore needs to evaluate a correlation function with three vertex operators inserted. We do not need the explicit form of these vertex operators here but we note instead that the amplitude computation can be universally done for all cases, i.e. graviton, dilaton and two-form vertex operator insertions. We will follow the exposition of [8] here. An (NS,NS) vertex operator is characterized by a transverse polarization tensor  $\epsilon_{\mu\nu}$  which satisfies  $k^\mu \epsilon_{\mu\nu} = \epsilon_{\mu\nu} k^\nu = 0$  when contracted with the on-shell momentum  $k^\mu$  of the associated state. Depending on the choice of  $\epsilon_{\mu\nu}$  we either describe the graviton, the dilaton or the two-form. These choices are

$$\begin{aligned} \text{graviton:} & \quad \epsilon_{\mu\nu} = h_{\mu\nu} = h_{\nu\mu}, \quad \eta^{\mu\nu} \epsilon_{\mu\nu} = 0 \\ \text{two-form:} & \quad \epsilon_{\mu\nu} = B_{\mu\nu} = -B_{\nu\mu} \\ \text{dilaton:} & \quad \epsilon_{\mu\nu} = \frac{1}{\sqrt{8}} (\eta_{\mu\nu} - k_\mu \tilde{k}_\nu - \tilde{k}_\mu k_\nu) \tilde{\phi}, \quad k \cdot \tilde{k} = 1. \end{aligned} \quad (1.44)$$

Once the vertex operators for the closed superstring (NS,NS) states are known, the computation of the three-point amplitude is fairly straightforward. One finds

$$A^{(3)}(\{\epsilon^{(i)}, k_i\}) = \sqrt{4\pi} (2\pi)^3 \alpha' g_s t^{\mu_1 \mu_2 \mu_3} t^{\nu_1 \nu_2 \nu_3} \epsilon_{\mu_1 \nu_1}^{(1)} \epsilon_{\mu_2 \nu_2}^{(2)} \epsilon_{\mu_3 \nu_3}^{(3)} \quad (1.45)$$

where we defined the tensor

$$t^{\mu_1 \mu_2 \mu_3} = \eta^{\mu_1 \mu_2} k_2^{\mu_3} + \eta^{\mu_2 \mu_3} k_3^{\mu_1} + \eta^{\mu_1 \mu_3} k_1^{\mu_2}. \quad (1.46)$$

The derivation of the amplitude (1.45) does not assume a particular form of the polarization tensors except that it satisfies the transversality condition stated above. One can therefore obtain the amplitudes for certain processes by specializing the general expression to polarization tensors in (1.44). The goal is now to find an action in ten dimensions for the spacetime fields, which reproduces the amplitudes described by (1.45). Since we wish to describe the graviton, a scalar field  $\tilde{\phi}$  and a two-form  $\tilde{B}$  with field strength  $\tilde{H} = d\tilde{B}$  we take as an ansatz for the action

$$S_{(\text{NS,NS})} = \int d^{10}x \sqrt{-g} \left[ \frac{1}{2\kappa_{10}^2} R - \frac{1}{6} e^{-2c\tilde{\phi}} \tilde{H}_{\mu\nu\rho} \tilde{H}^{\mu\nu\rho} - \frac{1}{2} g^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} \right]. \quad (1.47)$$

We next expand the action (1.47) around flat Minkowski space  $g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa_{10} h_{\mu\nu}$ , where  $h_{\mu\nu}$  is the on-shell graviton. We can now expand the Einstein-Hilbert action up to third order in the graviton. One finds

$$\int d^{10}x \sqrt{-g} \frac{1}{2\kappa_{10}^2} R|_{3\text{pt}} = - \int d^{10}x \kappa_{10} (h^{\mu\nu} h^{\rho\sigma} \partial_\mu \partial_\nu h_{\rho\sigma} + 2\partial^\sigma h_{\mu\nu} \partial^\mu h^{\nu\rho} h_{\rho\sigma}). \quad (1.48)$$



The three graviton amplitude from string theory is now obtained from (1.45) by choosing  $\epsilon_{\mu\nu}^{(1,2,3)} = h_{\mu\nu}^{(1,2,3)}$  symmetric and traceless. One finds the three graviton amplitude

$$\begin{aligned} A_{hhh}^{(3)} = (2\pi)^4 \alpha'^2 \frac{g_s}{\sqrt{\pi}} & \left[ (h_{\mu\nu}^{(1)} h^{(3)\mu\nu}) (k_1^{\mu_1} h_{\mu_1\nu_1}^{(2)} k_1^{\nu_1}) + 2(k_2^{\mu_1} h_{\mu_1\nu_1}^{(3)} h^{(2)\nu_1\mu_2} h_{\mu_2\nu_2}^{(1)} k_3^{\nu_2}) \right. \\ & + (h_{\mu\nu}^{(1)} h^{(2)\mu\nu}) (k_2^{\mu_1} h_{\mu_1\nu_1}^{(3)} k_2^{\nu_1}) + 2(k_1^{\mu_1} h_{\mu_1\nu_1}^{(2)} h^{(3)\nu_1\mu_2} h_{\mu_2\nu_2}^{(1)} k_3^{\nu_2}) \\ & \left. + (h_{\mu\nu}^{(2)} h^{(3)\mu\nu}) (k_3^{\mu_1} h_{\mu_1\nu_1}^{(1)} k_3^{\nu_1}) + 2(k_1^{\mu_1} h_{\mu_1\nu_1}^{(2)} h^{(1)\nu_1\mu_2} h_{\mu_2\nu_2}^{(3)} k_2^{\nu_2}) \right]. \end{aligned} \quad (1.49)$$

One can show that the expanded Einstein-Hilbert action (1.48) reproduces the string amplitude (1.49) provided one identifies

$$\kappa_{10}^2 = \frac{(2\pi\sqrt{\alpha'})^8}{4\pi} g_s^2. \quad (1.50)$$

Similarly, one can expand the part of the action which contains the two-form in fluctuations around flat Minkowski space

$$\begin{aligned} -\frac{1}{6} \int d^{10}x \sqrt{-g} e^{-2c\tilde{\phi}} \tilde{H}_{\mu\nu\rho} \tilde{H}^{\mu\nu\rho} \Big|_{\text{3pt}} = \int d^{10}x \, c \tilde{\phi} (\partial_\mu \tilde{B}_{\nu\rho} \partial^\mu \tilde{B}^{\nu\rho} + 2\partial_\mu \tilde{B}_{\nu\rho} \partial^\nu \tilde{B}^{\rho\mu}) \\ + \mathcal{O}(\kappa_{10}). \end{aligned} \quad (1.51)$$

The corresponding string amplitude which can fix the coefficient  $c$  in (1.51) is given by a scattering amplitude of a dilaton with two two-forms. Choosing the appropriate polarization tensors<sup>9</sup> one finds

$$A_{BB\phi}^{(3)} = \sqrt{\frac{2}{\pi}} (2\pi\sqrt{\alpha'})^4 g_s k_2^{\mu_1} B_{\mu_1\mu_2}^{(1)} B^{(2)\mu_2\mu_3} k_{1\mu_3} \tilde{\phi} \quad (1.52)$$

which is reproduced by the field theory (1.51) if one chooses

$$c = \frac{(2\pi\sqrt{\alpha'})^4}{\sqrt{8\pi}} g_s. \quad (1.53)$$

The two parameters  $\kappa_{10}^2$  and  $c$  in (1.47) are therefore determined by string theory. We will refrain from doing this here, but one can show that similarly the string amplitudes of two two-forms and a graviton  $A_{hBB}^{(3)}$  and the amplitude of two dilatons and a graviton  $A_{h\phi\phi}^{(3)}$  are reproduced by the field theory (1.47) once the parameters  $\kappa_{10}, c$  are identified according to (1.50) and (1.53).

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<sup>9</sup>The polarization tensors for the two-forms are denoted by  $B_{\mu\nu}^{(1,2)}$ .

We can now furthermore perform the following redefinitions

$$\tilde{B} \rightarrow \frac{1}{2\kappa_{10}} B, \quad \tilde{\phi} \rightarrow \frac{1}{\sqrt{2\kappa_{10}}} (\phi - \phi_0), \quad (1.54)$$

and subsequently perform a Weyl rescaling to string frame, i.e. redefining

$$g_{\mu\nu} \rightarrow e^{-\frac{1}{2}(\phi - \phi_0)} g_{\mu\nu}. \quad (1.55)$$

In this way one obtains the standard string frame action of type II superstrings in the (NS,NS) sector

$$S_{(\text{NS},\text{NS})} = \frac{1}{2\tilde{\kappa}_{10}^2} \int e^{-2\phi} \left( R * 1 - \frac{1}{2} H \wedge *H + 4d\phi \wedge *d\phi \right) \quad (1.56)$$

with  $\tilde{\kappa}_{10} = e^{\phi_0} \kappa_{10} = g_s \kappa_{10}$ . Similar computations can be done for the remaining sectors of the type II strings. These string amplitudes indeed confirm that the dynamics of the massless modes of type II strings at low energies are described by the chiral and non-chiral supergravity theories, which were mentioned in the beginning of this section.

To obtain the full bosonic action of both type IIA and type IIB supergravity one must still include the part of the action which involves the (R,R) fields. As we have seen in section 1.2.3, type IIA string theory contains in the (R,R) sector  $p$ -form potentials of odd degree, whereas type IIB contains potentials of even degree. In fact, the four-form potential with self-dual field strength in type IIB poses an obstruction to construct a proper action of type IIB supergravity. Instead one can construct a pseudo-action for type IIB where the self-duality constraint  $F_5 = *F_5$  is imposed by hand at the level of the equations of motion. The action in the (R,R) sector is given by

$$\begin{aligned} S_{(\text{R},\text{R})}^{\text{IIA}} &= \frac{1}{2\tilde{\kappa}_{10}^2} \int -\frac{1}{2} F_2 \wedge *F_2 - \frac{1}{2} F_4 \wedge *F_4 - \frac{1}{2} B \wedge dC_3 \wedge dC_3, \\ S_{(\text{R},\text{R})}^{\text{IIB}} &= \frac{1}{2\tilde{\kappa}_{10}^2} \int -\frac{1}{2} F_1 \wedge *F_1 - \frac{1}{2} F_3 \wedge *F_3 - \frac{1}{4} F_5 \wedge *F_5 - \frac{1}{2} C_4 \wedge H \wedge F_3, \end{aligned} \quad (1.57)$$

with definitions of the field strengths

$$\begin{aligned} \text{type IIA:} \quad F_2 &= dC_1, & F_4 &= dC_3 + H \wedge C_1, \\ \text{type IIB:} \quad F_1 &= dC_0, & F_3 &= dC_2 - C_0 dB, \\ F_5 &= dC_4 - \frac{1}{2} C_2 \wedge H + \frac{1}{2} B \wedge dC_2. \end{aligned} \quad (1.58)$$

We note that in type IIA supergravity/string theory one has the option to also include a ten-form field strength which is non-dynamical and proportional to the volume form of the ten-dimensional spacetime  $F_{10} = dC_9 = F_0 \text{vol}$ . The quantity  $F_0$  is called *Romans mass* and type IIA string theory with non-vanishing Romans mass is called *massive type IIA*. Massive type IIA includes D8-branes which couple electrically to the nine-form potential  $C_9$  and can be viewed as domain walls which interpolate between domains with different values of the Romans mass.

### 1.2.5 Higher-derivative corrections from string theory

In the last section we have outlined how the low-energy effective action of the massless modes of type II strings can be computed from string amplitudes. The two-derivative action was already fixed by the three-point scattering amplitudes by demanding that the field theory amplitudes reproduce the string theoretic amplitudes. Going beyond the three-point function, we therefore expect to obtain new terms in the effective action. We will address four-graviton scattering at tree-level in type II string theories which induce eight-derivative couplings in the ten-dimensional action. The computation of these amplitudes and their consequences for the effective action of the massless fields were first discussed in [8, 9]. The relevant amplitude for four-graviton scattering can be written as the product of two open string amplitudes by virtue of the KLT relation [10]. The four graviton amplitude is found to be

$$A_{hhhh}^{(4)} \propto \frac{\Gamma(-\frac{1}{2}s)\Gamma(-\frac{1}{2}t)\Gamma(-\frac{1}{2}u)}{\Gamma(1+\frac{1}{8}s)\Gamma(1+\frac{1}{8}t)\Gamma(1+\frac{1}{8}u)} \quad (1.59)$$

$$\times t_8^{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8} k_1^{[\mu_1} k_{1[\nu_1} h^{(1)}{}_{\nu_2]}{}^{\mu_2]} \cdots \left( k_4^{[\mu_7} k_{4[\nu_7} h^{(4)}{}_{\nu_8]}{}^{\mu_8]} \right),$$

where  $s, t, u$  are Mandelstam variables and the tensor  $t_8$  is defined as in (B.4). Next one expands

$$\frac{\Gamma(-\frac{1}{2}s)\Gamma(-\frac{1}{2}t)\Gamma(-\frac{1}{2}u)}{\Gamma(1+\frac{1}{8}s)\Gamma(1+\frac{1}{8}t)\Gamma(1+\frac{1}{8}u)} = -\frac{2^9}{stu} - 2\zeta(3) + \dots \quad (1.60)$$

such that one obtains two contributions in the amplitude (1.59). The first term in the expansion turns out to arise from the effective Lagrangian derived from the three-point amplitude via exchange diagrams. This can heuristically be seen by naive power counting of momenta, i.e. derivatives in position space. Indeed, when the term proportional to  $1/stu$  is combined with the amplitude (1.59) one counts a net amount of two momenta. It should therefore stem from an action with two derivatives. Since the next term in the expansion contains a transcendental factor

$\zeta(3)$  as multiplicative factor, this part of the amplitude corresponds to a new term in the effective action beyond the two-derivative action. We recognize that

$$R^{\mu_1\mu_2}_{\nu_1\nu_2} = k^{[\mu_1} k_{[\nu_1} h_{\nu_2]}^{\mu_2]}. \quad (1.61)$$

is in fact the linearized Riemann tensor. We therefore conclude that the Lagrangian must be supplemented by a term

$$\propto \zeta(3) t_{8\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8} t_8^{\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\nu_7\nu_8} R^{\mu_1\mu_2}_{\nu_1\nu_2} \dots R^{\mu_7\mu_8}_{\nu_7\nu_8} \quad (1.62)$$

to account for the corresponding string amplitude contribution. At the level of the five point function it turns out that yet another coupling quartic in the Riemann tensor appears. The form of the total eight-derivative coupling quartic in the Riemann tensor at tree-level in the string coupling is universal for type IIA and type IIB. Schematically, one obtains a contribution to the ten-dimensional Lagrangian (in string frame)

$$\Delta\mathcal{L}_{\text{tree}} \propto e^{-2\phi} \left( t_8 t_8 + \frac{1}{8} \epsilon_{10} \epsilon_{10} \right) R^4, \quad (1.63)$$

where  $\epsilon_{10}$  is the Levi-Civita tensor.

An analysis at one loop in the string coupling up to the five point function reveals that there are also terms quartic in the Riemann tensor at this order in the expansion of the perturbative string. This time there is a relative sign flip in

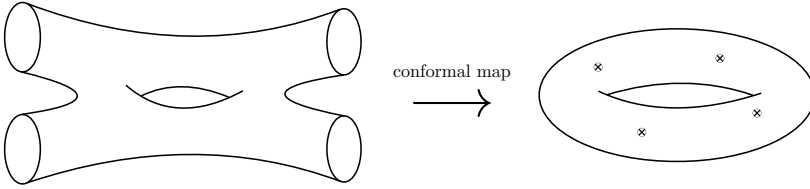


Figure 1.4: The worldsheet of the one-loop four point function of the closed string can be conformally mapped to a torus with four punctures. The punctures correspond to vertex operator insertions.

type IIA compared to type IIB

$$\Delta\mathcal{L}_{\text{loop}} \propto \left( t_8 t_8 \pm \frac{1}{8} \epsilon_{10} \epsilon_{10} \right) R^4, \quad (1.64)$$

where the negative sign corresponds to type IIA and the positive sign to type IIB. Of course, similar computations can be done and have been done for other types

of fields, such as the two-form and the (R,R) fields and the worldvolume fields of D-branes. However, the computational complexity increases dramatically for higher scattering amplitudes and it is non-trivial to identify the new contributions to the effective action. Due to these reasons there is no complete knowledge of the full ten-dimensional effective action at eight derivatives to date.

### 1.2.6 M-theory and the duality web

The discovery of string dualities was without doubt a game changer in string theory. In simplified words, string dualities are potentially very complicated maps between two seemingly very different theories which secretly describe the same physics. Therefore, once a duality and the map translating information from one theory to its dual is understood sufficiently, a seemingly unfeasible task in one duality frame can simplify in a different duality frame dramatically.

*“The principle of duality in physics suggests that the best viewpoint is not an absolute concept and depends on the question being asked.”*

— Cumrun Vafa

The string duality web has enabled tremendous progress in understanding non-perturbative string dynamics in the past decades. At the heart of the string duality web is a still mysterious theory in eleven dimensions, called M-theory. The ten-dimensional superstring theories which were discussed in the previous section are only defined perturbatively, i.e. as a perturbative expansion in terms of string worldsheets with increasing genus. Since a fundamental non-perturbative definition of superstring theories is still lacking, it seems hopeless to obtain insights in the non-perturbative regime of string theory. Even worse, given that only a rigorous perturbative definition is available, this even appears to be a badly posed question. String dualities provide an unexpected window into non-perturbative effects in string theory which will be the topic of this section. We will by no means attempt to give a full list of dualities and describe their maps in detail. Instead we will focus on the dualities which are most relevant for our purposes and give their key features.

**T-duality of type II strings.** Among the oldest known string dualities is T-duality of perturbative type II string theories. When a string theory is compactified on a circle with radius  $R$ , the corresponding compactified string embedding satisfies

$$X^9(\sigma + 2\pi R, \tau) = X^9(\sigma, \tau) + 2\pi R w. \quad (1.65)$$

The integer  $w$  characterizes how often the string winds around the circle on which the theory is compactified. The appearance of these winding modes is a feature which distinguishes string theory from field theory. In addition to the winding number, the string can carry a non-trivial momentum around the circle. Due to the periodicity of the circle, this momentum must be quantized as well. We denote the momentum quantum number with  $n$ . A first observation about the spectrum of states is that it is invariant under

$$R \rightarrow \frac{\alpha'}{R}, \quad n \leftrightarrow w. \quad (1.66)$$

If this map is extended and one performs the transformations

$$(X_L^9, X_R^9) \rightarrow (X_L^9, -X_R^9), \quad (\psi_+^9, \psi_-^9) \rightarrow (\psi_+^9, -\psi_-^9), \quad (1.67)$$

which realized the transformations (1.66), one observes that the original and the transformed worldsheet fields have the same operator product expansions. This shows that the two theories are physically equivalent. One can also show that the transformation (1.67) maps the type IIA GSO projection to the type IIB projection and vice versa. Therefore, T-duality not only implements the map (1.66) but also interchanges type IIA and type IIB string theory. Since type IIA(IIB) string theory contains D-branes extending along an odd(even) number of dimensions, this immediately raises the question how these objects transform under a T-duality transformation. By inspecting the boundary conditions (1.19) and (1.22) we notice that the transformation (1.67) maps (NN) boundary conditions to (DD) boundary conditions. This implies that a  $Dp$ -brane, which wraps the circle  $S^1$ , gets mapped to a  $D(p-1)$ -brane after the T-duality transformation. Likewise, a  $Dp$ -brane which does not wrap the T-duality circle gets mapped to a  $D(p+1)$ -brane in the dual theory.

**M-theory and type IIA.** Type IIA string theory not only contains fundamental strings, but also D-branes in its spectrum. These D-branes are BPS states which are protected from perturbative and non-perturbative quantum corrections. A BPS state forms a short multiplet of the underlying supersymmetry algebra. The reason for the multiplet shortening in the case of a D-brane is a relation between their tension and the central charge of the supersymmetry algebra. The tension and the associated mass scale of a  $Dp$ -brane are

$$T_p = \frac{2\pi}{g_s} \frac{1}{(2\pi\sqrt{\alpha'})^{p+1}}, \quad m_p \sim \frac{1}{g_s^{\frac{1}{p+1}} \sqrt{\alpha'}}. \quad (1.68)$$

Since D $p$ -branes are BPS objects, we expect this relation to be valid at strong coupling. It is clear from (1.68) that in the strong coupling regime  $g_s \gg 1$  the lowest dimensional D-brane is the lightest state. A bound state of  $n$  D0-branes in type IIA string theory at strong coupling therefore has a mass scale

$$m_{\text{D0}} \propto \frac{n}{g_s \sqrt{\alpha'}}. \quad (1.69)$$

These states resemble an infinite tower of Kaluza-Klein modes which one encounters in a circle compactification of a higher-dimensional theory (see section 1.3.1). This suggests that in the strong coupling limit of type IIA string theory an extra dimension opens up and we are effectively dealing with an eleven-dimensional theory, which we call M-theory. Even though we do not know a microscopic formulation of M-theory today, it is believed to be described by the unique eleven dimensional supergravity theory at low energies. Eleven dimensional supergravity has a remarkably simple field content which is the metric  $\hat{g}_{MN}$  and a three-form potential  $\hat{C}_3$  with field strength  $\hat{G}_4 = d\hat{C}_3$  in the bosonic sector and a gravitino in the fermionic sector. The bosonic part of the action of eleven-dimensional supergravity is given by [11]

$$S_{\text{M}} = \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \left[ \hat{R} * 1 - \frac{1}{2} \hat{G}_4 \wedge \hat{*} \hat{G}_4 - \frac{1}{6} \hat{C}_3 \wedge \hat{G}_4 \wedge \hat{G}_4 \right], \quad (1.70)$$

where

$$2\kappa_{11}^2 = (2\pi)^9 \ell_{\text{M}}^8 \quad (1.71)$$

with  $\ell_{\text{M}}$  the Planck length in eleven dimensions. The action (1.70) describes the dynamics of the massless modes of M-theory for energies  $\ell_{\text{M}} E \ll 1$ . Similar to the low energy effective action of type IIA and type IIB, the effective action of M-theory obtains higher-derivative corrections which are suppressed by powers of the eleven dimensional Planck length  $\ell_{\text{M}}$ . These can be obtained from higher-derivative corrections in type IIA string theory by evaluating suitable string amplitudes. For example, the strong coupling limit of (1.64) lifts to a similar  $\hat{R}^4$ -correction in M-theory. The low energy effective action of M-theory at eight derivatives is further discussed in section 2.1.1.

If we compactify the eleventh dimension on a circle with radius  $R_{\text{M}}$  and split the eleven-dimensional index  $M = (\mu, 10)$  we can easily recover the field content of type IIA supergravity

$$\begin{aligned} \hat{g}_{\mu\nu} &= e^{-\frac{2}{3}\phi} g_{\mu\nu} + e^{\frac{4}{3}\phi} C_{1\mu} C_{1\nu}, & \hat{g}_{\mu 10} &= e^{\frac{4}{3}\phi} C_{1\mu}, \\ \hat{C}_{3\mu\nu\rho} &= C_{3\mu\nu\rho}, & \hat{C}_{3\mu\nu 10} &= B_{\mu\nu}, & \hat{g}_{10 10} &= e^{\frac{4}{3}\phi}. \end{aligned} \quad (1.72)$$

Performing the dimensional reduction explicitly one finds the relations

$$\alpha' = \frac{\ell_M^3}{R_M}, \quad g_s = \left( \frac{R_M}{\ell_M} \right)^{\frac{3}{2}} \quad (1.73)$$

between the type IIA and M-theory parameters. The fundamental branes in M-theory are a membrane with two extended spatial dimensions and a brane with five spatial dimensions, called M2- and M5-branes respectively. The relation of these branes to the corresponding type IIA objects is given in table 1.3. The

M-theory object	wrapped on $S^1$	Type IIA object
M2-brane	✓	fundamental string
	✗	D2-brane
M5-brane	✓	D4-brane
	✗	NS5-brane
Kaluza-Klein modes		D0-branes
Taub-NUT space		D6-brane

Table 1.3: M-theory objects and their descendents in type IIA.

D6-brane in type IIA does not have its M-theory origin as an M-theory brane. Instead, it is geometrized by a Taub-NUT space in M-theory. We have illustrated the geometry of Taub-NUT space in figure 1.5. Taub-NUT space is an  $S^1$  fibration

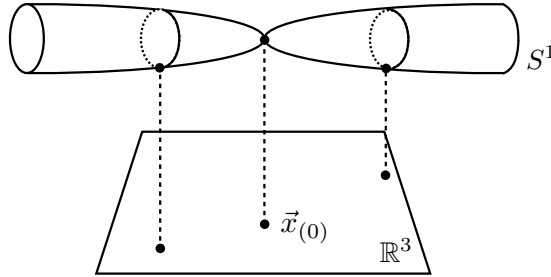


Figure 1.5: Depiction of Taub-NUT space as an  $S^1$  fibration over  $\mathbb{R}^3$ .

over  $\mathbb{R}^3$ , i.e. the radius of the circle may vary over  $\mathbb{R}^3$ . The circle radius shrinks to zero size over a certain point in the base  $\vec{x}_{(0)}$ . One can show that if M-theory is compactified on the  $S^1$  fiber of Taub-NUT space the corresponding type IIA D6-brane extends along the locus  $\vec{x} = \vec{x}_{(0)}$  in the ten-dimensional geometry. One



can similarly construct multi-centered Taub-NUT spaces with degenerating fibers at  $m$  points  $\vec{x}_{(i)}$  in  $\mathbb{R}^3$ . These descend to separated D6-branes in the type IIA picture. In the limit where all degeneration loci of the fiber are taken to be coincident we obtain Taub-NUT space with NUT charge  $m$ . This space develops an orbifold singularity  $\mathbb{R}^4/\mathbb{Z}_m$  close to the centers. This singularity reflects the gauge symmetry enhancement in type IIA when a stack of coincident D6-branes is formed. These Taub NUT spaces with NUT charge  $m$  fit in an infinite family of manifolds, called ALF manifolds which will play a prominent role in part II of this thesis.

**M-theory and type IIB.** To make a connection between M-theory and type IIB string theory we compactify M-theory on a rectangular torus  $T^2 = S^1_M \times S^1_{\text{IIA}}$  with circle radii  $R_M$  and  $R_{\text{IIA}}$ . After compactifying along the M-theory circle  $R_M$  we obtain type IIA string theory with string coupling  $g_s^{\text{IIA}} = (R_M/\ell_M)^{3/2}$  compactified on  $S^1_{\text{IIA}}$ . We now T-dualize to type IIB along  $S^1_{\text{IIA}}$  and we find type IIB compactified on  $S^1_{\text{IIB}}$  with radius  $R_{\text{IIB}} = \alpha'/R_{\text{IIA}}$  and

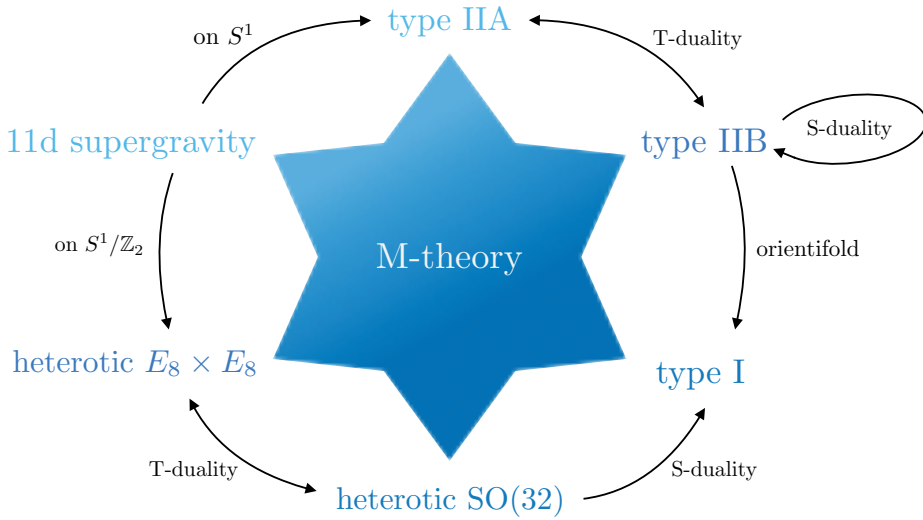
$$g_s^{\text{IIB}} = \sqrt{\frac{R_{\text{IIB}}}{R_{\text{IIA}}}} g_s^{\text{IIA}} = \frac{R_M}{R_{\text{IIA}}} = \frac{1}{\text{Im } \tau}, \quad (1.74)$$

where  $\tau$  is the complex structure parameter of the  $T^2$ . If we wish to recover type IIB string theory in ten dimensions we need to decompactify  $S^1_{\text{IIB}}$  by sending  $R_{\text{IIB}} \rightarrow \infty$ , or equivalently  $R_{\text{IIA}} \rightarrow 0$ . We wish to perform this limit while keeping the string coupling of type IIB (1.74) fixed. This implies we must take  $\text{vol}(T^2) \propto R_M R_{\text{IIA}} \rightarrow 0$ . We have therefore found that

$$\text{M-theory on } T^2 \text{ with } \text{vol}(T^2) \rightarrow 0 \iff \text{type IIB with } g_s = \frac{1}{\text{Im } \tau}. \quad (1.75)$$

This duality holds more generally for tori with complex structure parameters with a real part and torus fibrations. The connection between the complex structure  $\tau$  and S-duality of type IIB string theory will be further elaborated on in section 1.4.

The web of string dualities which relate the known supersymmetric string theories and M-theory with each other can be conveniently visualized in an ‘M-theory star’.



We have discussed only the dualities which are most relevant for this thesis and have skipped fascinating topics such as the relation of M-theory to the heterotic string. To date, there exists overwhelming evidence that these dualities are indeed correct, even though a rigorous proof is still lacking.

### 1.3 Compactifications and effective actions

String theory is naturally defined in ten spacetime dimensions. This poses an apparent contradiction with the observation that our universe is four-dimensional. This looks like bad news for string theory as a fundamental theory of everything which applies to our world. However, there is the possibility that the six extra dimensions are just sufficiently small, such that they have so far escaped detection. More precisely, the energy scales which we are able to reach experimentally today could simply not be high enough to generate a resolution which suffices to detect the extra dimensions. If the characteristic length scale  $\ell_{\text{int}}$  of the compact extra dimensions, henceforth referred to as *internal space*, is small enough such that  $\ell_{\text{int}} E_{\text{exp}} \ll 1$ , experimentalists would not be able to probe these extra dimensions. One can nevertheless determine physical imprints which the extra dimensions leave on four dimensional physics (or more generally, dimensions lower than ten/eleven) behind. This is achieved by compactifying the higher dimensional theories on certain types of manifolds. It is the purpose of this section to introduce the main concepts of Kaluza-Klein reductions and how they lead to a lower dimensional effective action, with information about the internal space stored inside the

couplings of the effective theory. We will first exemplify Kaluza-Klein reductions with the example of five-dimensional Einstein gravity compactified on a circle before turning to a more general discussion. Lastly we give a brief overview over Calabi–Yau manifolds. Choosing these manifolds as internal space is motivated by the desire to preserve some fraction of the original amount of supersymmetry of the higher-dimensional theory.

### 1.3.1 Kaluza-Klein reductions

As a warm up example we consider a real scalar field  $\hat{\phi}(x)$  in five dimensions

$$S_{\text{scalar}}^{(5)} = -\frac{1}{2\kappa_5^2} \int_{\mathcal{M}_5} d^5x \sqrt{-\hat{g}} \hat{g}^{MN} \partial_M \hat{\phi} \partial_N \hat{\phi}, \quad (1.76)$$

on spacetimes of the form  $\mathcal{M}_4 \times S^1$  equipped with a metric

$$\langle d\hat{s}^2 \rangle = \hat{g}_{MN} dx^M dx^N = g_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad y \sim y + 2\pi r, \quad (1.77)$$

where we split the five dimensional spacetime index  $M = (\mu, 5)$ . Since we have a circle with radius  $r$  in the spacetime manifold we can expand the scalar field  $\hat{\phi}$  in Fourier modes while maintaining full generality

$$\hat{\phi}(x^\mu, y) = \sum_{n \in \mathbb{Z}} \phi_n(x^\mu) e^{iny/r}, \quad \phi_n^* = \phi_{-n}. \quad (1.78)$$

Inserting this ansatz into the equations of motion of  $\hat{\phi}$  derived from (1.76) we find

$$\hat{\square} \hat{\phi}(x^\mu, y) = \sum_{n \in \mathbb{Z}} \left( \square - \frac{n^2}{r^2} \right) \phi_n(x^\mu) e^{iny/r} = 0. \quad (1.79)$$

Since the exponential functions form a complete set of functions on  $S^1$  we deduce that the modes  $\phi_n(x^\mu)$  are scalar fields with masses  $m_n^2 = n^2/r^2$  in four dimensions. One observes that for low energies  $rE \ll 1$  the higher Fourier modes cannot be excited and therefore only the massless zero-mode  $\phi_0$  is a dynamical degree of freedom at these energy scales. As a first approximation, one may therefore truncate the Kaluza-Klein tower (1.78) and insert this truncated ansatz into the action (1.76). Integrating over the circle results in the four dimensional effective action

$$S_{\text{scalar}}^{(4)} = -\frac{2\pi r_0}{2\kappa_5^2} \int_{\mathcal{M}_4} d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi_0 \partial_\nu \phi_0. \quad (1.80)$$

The massive Kaluza-Klein states can modify the couplings in the effective action through loop corrections and therefore should be properly integrated out. In

many cases these corrections to the couplings are suppressed by a power of  $\ell_{\text{int}}$  (in the case of the circle  $\ell_{\text{int}} = r$ ) and are therefore negligible at low energies, such that the truncation to the massless modes is justified. In some instances however, quantum corrections which originate from massive Kaluza-Klein modes running in loops can be of crucial importance. We will encounter such situations in the course of this thesis.

We now turn towards compactifying five-dimensional Einstein gravity, which is described by the action

$$S^{(5)} = \frac{1}{2\kappa_5^2} \int_{\mathcal{M}_5} \hat{R} \hat{*} 1, \quad (1.81)$$

on a circle. The most general five-dimensional metric which is compatible with the U(1) isometry of the circle is given by

$$\langle d\hat{s}^2 \rangle = g_{\mu\nu} dx^\mu dx^\nu + \lambda^2(x) (dy + A_\mu(x) dx^\mu)^2, \quad y \sim y + 2\pi r, \quad (1.82)$$

where  $\lambda$  and  $A_\mu$  are from a four-dimensional point of view a scalar field and a vector potential. The gauge field  $A_\mu$  is called the Kaluza-Klein vector and gauges the isometry of the background solution. The background solution is obtained from (1.82) by setting  $A_\mu = 0$  and  $\lambda = 1$ . The scalar field  $\lambda$  may be interpreted as the ‘breathing mode’ which rescales the radius of the circle. We can insert the ansatz (1.82) into the action (1.81) and integrate over the circle. One finds

$$S^{(4)} = \frac{1}{2\kappa_4^2} \int_{\mathcal{M}_4} \left[ \lambda R * 1 - \frac{1}{2} \lambda^3 F \wedge * F \right], \quad \kappa_4^2 = \frac{\kappa_5^2}{2\pi r}. \quad (1.83)$$

Since the action does not have a canonically normalized Einstein-Hilbert term yet, we perform a Weyl rescaling  $g_{\mu\nu} \rightarrow \lambda^{-1} g_{\mu\nu}$ . This Weyl rescaling generates a kinetic term for the scalar field  $\lambda$  and one obtains

$$S^{(4)} = \frac{1}{2\kappa_4^2} \int_{\mathcal{M}_4} \left[ R * 1 - \frac{3}{2} \lambda^{-2} d\lambda \wedge * d\lambda - \frac{1}{2} \lambda^3 F \wedge * F \right]. \quad (1.84)$$

The gauge symmetry of the Kaluza-Klein vector  $A$  can be interpreted as being induced by diffeomorphisms  $\delta y = \xi(x)$ . When the five-dimensional scalar in (1.78) is coupled to five-dimensional Einstein gravity, the parameter  $n$  is identified with the charge of the Kaluza-Klein modes under this U(1) gauge symmetry.

From this simple example one can already draw some lessons which hold more generally. The higher dimensional fields are expanded in a complete basis of eigenfunctions of the corresponding differential operator relevant to the specific type of field. The massless fields correspond to the zero-modes of this operator,

whereas the massive modes belong to modes with non-vanishing eigenvalue. Moreover, isometries in the internal space lead to gauge fields in the Kaluza-Klein reduction; the Kaluza-Klein states transform non-trivially under this gauge symmetry. Increasing the level of complexity slightly, we could perform a Kaluza-Klein reduction on the sphere  $S^2$ , which has an  $SU(2) \simeq SO(3)$  isometry group. The expansion analogous to (1.78) would be an expansion in spherical harmonics. Using this expansion it is then possible to show that the Kaluza-Klein states transform non-trivially under  $SU(2)$  gauge transformations.

We will now briefly discuss the concept of Kaluza-Klein reductions on more general grounds. We assume the  $D$  dimensional theory which we aim to compactify has field content  $\Phi_{M_1 \dots M_I}^I$ , where  $I$  labels the fields and the indices can be tensor or spinor indices. The underlying theory has a background solution in which the spacetime is a product manifold  $\mathcal{M}_D = \mathcal{M}_d \times \mathcal{M}_{\text{int}}$  with coordinates  $(x^\mu, y^m)$ . The fields in the background are denoted by  $\langle \Phi_{M_1 \dots M_I}^I(x, y) \rangle$ . The equations of motion of the field  $\Phi^I$  are given in terms of a differential operator and  $\langle \Phi_{M_1 \dots M_I}^I(x, y) \rangle$  is a solution to this equation. We can split the indices of the fields into arbitrary combinations of internal and external indices  $\Phi_{\mu_1 \dots m_1 \dots}^I$  and perturb the background in all possible ways

$$\Phi_{\mu_1 \dots m_1 \dots}^I = \langle \Phi_{\mu_1 \dots m_1 \dots}^I \rangle + \delta \Phi_{\mu_1 \dots m_1 \dots}^I. \quad (1.85)$$

Inserting this perturbed background into the equations of motion and expanding up to linear order in the perturbations leads to equations of motion of the schematic form<sup>10</sup>

$$(\mathcal{O}_d + \mathcal{O}_{\text{int}}) \delta \Phi_{\mu_1 \dots m_1 \dots}^I = 0, \quad (1.86)$$

where  $\mathcal{O}_{d, \text{int}}$  are differential operators on the external and internal space, respectively. Analogously to the Fourier expansion in the case of the circle reduction, we can now expand the perturbations of the background  $\delta \Phi^I$  in a complete set of eigenfunctions of  $\mathcal{O}_{\text{int}}$  which satisfy  $\mathcal{O}_{\text{int}}(Y_{m_1 \dots}^\alpha) = \lambda^\alpha Y_{m_1 \dots}^\alpha$

$$\delta \Phi_{\mu_1 \dots m_1 \dots}^I(x, y) = \sum_{\alpha} \delta \Phi_{\mu_1 \dots}^{I \alpha}(x) Y_{m_1 \dots}^\alpha(y). \quad (1.87)$$

Inserting this ansatz into the linearized equations of motion (1.86) leads to

$$\sum_{\alpha} Y_{m_1 \dots}^\alpha (\mathcal{O}_d + \lambda^\alpha) \delta \Phi_{\mu_1 \dots}^{I \alpha}(x) = 0, \quad (1.88)$$

---

<sup>10</sup>At this point one should – if necessary – also impose suitable gauge fixing conditions. These conditions distinguish true perturbations of the background from perturbations which can be removed by a gauge transformation.

which by completeness of the  $Y^\alpha$  gives the individual equations of motion for the perturbations. The mass spectrum of the Kaluza-Klein states is therefore determined by the spectrum of eigenvalues of certain differential operators on the internal space  $\mathcal{M}_{\text{int}}$ . In practice it can be very cumbersome to find a basis of tensors  $Y^\alpha$  such that the linearized equations of motion of the independent perturbations decouple.

We can exemplify this strategy with a massless  $p$ -form  $C_p$  with field strength  $F_{p+1} = dC_p$  and action

$$S_p = \int_{\mathcal{M}_D} -\frac{1}{2} F_{p+1} \wedge *F_{p+1}. \quad (1.89)$$

After fixing the gauge to  $d^\dagger C_p = 0$ , the equation of motion become<sup>11</sup>

$$(d d^\dagger + d^\dagger d) C_p = 0. \quad (1.90)$$

The zero-modes of this equation are the harmonic forms on  $\mathcal{M}_{\text{int}}$ . Therefore, the problem of counting massless modes in an expansion of a  $p$ -form can be conveniently reformulated in terms of cohomology groups on  $\mathcal{M}_{\text{int}}$  and their dimensions.

### 1.3.2 Calabi–Yau manifolds and their moduli

Compactifications of string theories on Calabi–Yau manifolds have a long history since its early days. Their phenomenological relevance and the powerful tools provided by algebraic geometry make them an attractive option for studying string compactifications. It follows from supersymmetry that if we wish to compactify string theory while preserving supersymmetry<sup>12</sup> there must exist a covariantly constant, nowhere vanishing and globally defined spinor on the compactification manifold  $X$ . We will focus on the case where the dimension of the internal space  $X$  is (real) six-dimensional.

For a six-dimensional orientable Riemannian manifold the holonomy group must be a subgroup of the structure group  $\text{SO}(6) \simeq \text{SU}(4)$ . A covariantly constant spinor on  $X$  must therefore be a singlet under the holonomy group of  $X$ . A spinor of  $\text{SO}(6)$  can be decomposed into a right-handed Weyl spinor in the  $\mathbf{4}$ , and a left-handed Weyl spinor in the  $\bar{\mathbf{4}}$ . In order to obtain a covariantly constant spinor

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<sup>11</sup>The co-differential  $d^\dagger$  on a  $D$ -dimensional manifold acting on  $p$ -forms is defined by  $d^\dagger = \xi(-1)^{D(p+1)+1} * d *$ , where  $\xi = +1$  for Euclidean signature and  $\xi = -1$  for Lorentzian signature.

<sup>12</sup>We assume the absence of background fluxes here.

on  $X$  the decomposition of these spinors under the holonomy group of  $X$  must contain a singlet representation. It is possible to prove that this condition forces the holonomy of  $X$  to be a subgroup of  $SU(3)$ . The decomposition of the **4** of  $SO(6)$  under  $SU(3) \subset SO(6)$  gives

$$\begin{aligned} SO(6) &\rightarrow SU(3) \\ \mathbf{4} &\rightarrow \mathbf{3} \oplus \mathbf{1}, \end{aligned} \tag{1.91}$$

which indeed confirms that for  $SU(3)$  holonomy there exist two singlets with opposite chirality. The supersymmetry parameter of a ten-dimensional  $\mathcal{N} = 1$  theory is a Majorana-Weyl spinor of  $SO(1, 9)$ . The Weyl representation **16** of  $SO(1, 9)$  decomposes under  $SO(1, 3) \times SU(3)$  as

$$\begin{aligned} SO(1, 9) &\rightarrow SO(1, 3) \times SU(3) \\ \mathbf{16} &\rightarrow (\mathbf{2}_L, \bar{\mathbf{3}}) + (\mathbf{2}_L, \mathbf{1}) + (\mathbf{2}_R, \mathbf{3}) + (\mathbf{2}_R, \mathbf{1}), \end{aligned} \tag{1.92}$$

where  $\mathbf{2}_{L,R}$  are Weyl spinors in four dimensions. Imposing in addition a Majorana condition on the singlets in (1.92) leads to a Majorana spinor in four dimensions, i.e.  $\mathcal{N} = 1$  supersymmetry. If we instead start with 32 supercharges in ten dimensions we end up with  $\mathcal{N} = 2$  supersymmetry in four dimensions. One can build suitable spinor bilinears from the covariantly constant spinors on  $X$  which imply the existence of an integrable complex structure, a Kähler form  $J$  and a unique holomorphic, covariantly constant three-form  $\Omega$  on  $X$ . Putting things together, supersymmetry implies that the internal space on which we compactify needs to be a compact Kähler manifold with holonomy contained in  $SU(3)$ . We will in the following take a compact Kähler manifold with *exact*  $SU(n)$  holonomy as the definition of a Calabi–Yau  $n$ -fold ( $CY_n$ ). By virtue of a famous theorem proven by Yau, these manifolds admit a unique Ricci-flat metric (i.e.  $R_{i\bar{j}} = 0$ ) for each Kähler class.

The cohomology groups on a compact Kähler manifold decompose into subspaces which are refined cohomology groups. These subspaces are formed by  $\bar{\partial}$ -cohomology classes of mixed holomorphic and anti-holomorphic forms, the  $(p, q)$ -forms. More precisely, the cohomology groups split as

$$H^k(X) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{1.93}$$

and we denote the corresponding dimensions of the  $(p, q)$ -cohomology groups by

$$h^{p,q} = \dim_{\mathbb{C}}(H^{p,q}(X)). \tag{1.94}$$

Moreover, it can be shown that each cohomology class has a unique harmonic representative. The Euler number can be computed from the Hodge numbers by

$$\chi(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}. \quad (1.95)$$

The Hodge numbers of a Calabi–Yau manifold are highly restricted. The restrictions on the Hodge numbers of  $CY_n$  are:

- Complex conjugation of  $(p, q)$ -forms provides an isomorphism  $H^{p,q}(CY_n) \simeq H^{q,p}(CY_n)$  and therefore  $h^{p,q} = h^{q,p}$ .
- The Hodge star operator leads on  $CY_n$  to an isomorphism which implies  $h^{p,q} = h^{n-q,n-p} = h^{n-p,n-q}$ .
- The existence of a unique holomorphic  $n$ -form  $\Omega$  immediately implies  $h^{n,0} = h^{0,n} = 1$ .
- The contraction of a harmonic  $(p, 0)$ -form with  $\Omega$  can be shown to result in a harmonic  $(0, n-p)$ -form. Since  $\Omega$  is non-degenerate this is an isomorphism and one obtains  $h^{p,0} = h^{0,p} = h^{0,n-p} = h^{n-p,0}$ .
- For exact  $SU(n)$  holonomy one can show that a harmonic  $(p, 0)$ -form must transform as a singlet under the holonomy group. Since a  $(p, 0)$ -form transforms a priori in the  $p$ -fold antisymmetric tensor representation of  $SU(n)$ , only those  $p$  which lead to singlets are allowed. One therefore finds  $h^{p,0} = h^{0,p} = 0$  for  $0 < p < n$ .

We can now specialize to Calabi–Yau threefolds and arrange the Hodge numbers  $h^{p,q}$  in a Hodge diamond.

$$\begin{array}{ccccccc}
 & & h^{0,0} & & & & \\
 & & & & & & 1 \\
 & h^{1,0} & & h^{0,1} & & 0 & 0 \\
 h^{2,0} & & h^{1,1} & & h^{0,2} & = 1 & h^{1,1} & 0 \\
 h^{3,0} & h^{2,1} & & h^{1,2} & & h^{0,3} & h^{2,1} & 1 \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & 0 & 0 \\
 & & h^{3,2} & & h^{2,3} & & 0 & 0 \\
 & & & & h^{3,3} & & & 1
 \end{array} \quad (1.96)$$

A Calabi–Yau threefold only has two independent Hodge numbers  $h^{1,1}, h^{2,1}$  which count certain types of geometric deformation, as we will see momentarily. The



Euler number of a  $\text{CY}_3$  is given by the simple formula

$$\chi(\text{CY}_3) = 2(h^{1,1}(\text{CY}_3) - h^{2,1}(\text{CY}_3)). \quad (1.97)$$

For Calabi–Yau fourfolds there exists an additional independent Hodge number  $h^{1,3}$ . While the interpretation of  $h^{1,3}$  for  $\text{CY}_4$  in terms of geometric deformations is analogous to  $h^{1,2}$  for  $\text{CY}_3$ , the Hodge number  $h^{1,2}(\text{CY}_4)$  leads to a new sector. The consequences of non-trivial three-form cohomology of Calabi–Yau fourfolds were investigated in [12, 13].

We now turn to the study of moduli of Calabi–Yau manifolds. Moduli are geometric deformations which preserve the Calabi–Yau condition. Upon compactification, these moduli appear as massless fields in the low-energy effective action. Recall that Yau’s theorem guarantees that we can choose a Ricci-flat metric on  $\text{CY}_3$ . We are looking for metric deformations which preserve the Ricci-flatness of the Calabi–Yau metric. Working with real indices  $m, n = 1, \dots, 6$  we therefore need to study solutions to the equation

$$R_{mn}(g + \delta g) = 0. \quad (1.98)$$

In order to identify true deformations of the metric we need to impose the condition  $\nabla^m \delta g_{mn} = 0$  on the perturbations. It is easy to see that this condition ensures that the perturbations are orthogonal to the changes of the metric induced by diffeomorphisms. We expand the condition (1.98) to first order in  $\delta g_{mn}$  and find

$$\nabla^{m_1} \nabla_{m_1} \delta g_{mn} + 2R_m{}^{m_1}{}_{n_1} \delta g_{m_1 n_1} = 0, \quad (1.99)$$

where we used the Ricci-flatness of the unperturbed metric and the condition  $\nabla^m \delta g_{mn} = 0$ . We now utilize complex indices and study the components of (1.99) separately. We start with mixed indices  $(i, \bar{j})$  and we notice, that the corresponding equation is nothing but the Laplace–Beltrami operator acting on  $(1, 1)$ -forms. The corresponding zero-modes are the harmonic two-forms which are counted by the dimension of the cohomology group  $H^{1,1}(\text{CY}_3)$ . we can therefore expand the deformations  $\delta g_{i\bar{j}}$  in a basis of harmonic  $(1, 1)$ -forms  $\omega_a$

$$\delta g_{i\bar{j}} = -i\delta v^a \omega_a{}_{i\bar{j}}, \quad a = 1, \dots, h^{1,1}. \quad (1.100)$$

Clearly, these deformations admit an interpretation as deformations of the cohomology class of the Kähler form.

Equation (1.99) for purely anti-holomorphic indices  $(\bar{i}, \bar{j})$  reads

$$\nabla^m \nabla_m \delta g_{\bar{i}\bar{j}} + 2R_{\bar{i}}{}^{\bar{i}_1}{}_{\bar{j}}{}^{\bar{j}_1} \delta g_{\bar{i}_1 \bar{j}_1} = 0. \quad (1.101)$$

After raising the index  $\bar{i}$ , this equation can be identified with the Laplace-Beltrami operator acting on the one form  $\delta g^i = \delta g^i_{\bar{j}} d\bar{z}^j$ , where  $z^i$  are complex coordinates on  $\text{CY}_3$ . This one-form takes values in the holomorphic tangent bundle and the number of zero-modes is therefore counted by the dimension of  $H^1(\text{CY}_3, T_{\text{CY}_3})$ . One can construct harmonic  $(2, 1)$ -forms from these tangent bundle valued one forms by contracting the tangent bundle index with one of the indices of the holomorphic  $(3, 0)$ -form. We can therefore expand the result of this operation in a basis of harmonic  $(2, 1)$ -forms on  $\text{CY}_3$ , i.e.

$$\Omega_{ijk} \delta g^k_{\bar{l}} = \delta Z^\alpha \chi_{\alpha ij \bar{l}}, \quad \alpha = 1, \dots, h^{2,1}. \quad (1.102)$$

Since  $\Omega$  is non-degenerate we can invert this relation to find

$$\delta g_{i\bar{j}} = \frac{1}{2\|\Omega\|^2} \chi_{\alpha \bar{i}kl} \bar{\Omega}^{kl}_{\bar{j}} \delta Z^\alpha, \quad (1.103)$$

where  $\|\Omega\|^2 = \frac{1}{3!} \Omega_{ijk} \bar{\Omega}^{ijk}$ . The perturbations parametrized by  $\delta Z^\alpha$  introduce purely (anti-)holomorphic components in the metric. This can be understood as coming from an infinitesimal non-holomorphic coordinate transformation parametrized by  $\delta Z^\alpha$ . We therefore conclude that  $\delta Z^\alpha$  are deformations of the complex structure of  $\text{CY}_3$ .

## 1.4 Introducing F-theory

In this section we introduce the basic notions of F-theory, a geometric framework to describe string vacua with minimal supersymmetry in various dimensions. F-theory was formulated in [14] by C. Vafa where he interpreted the axio-dilaton of type IIB string theory as the complex structure parameter of a torus. Today there are two main ways to approach F-theory. We will first discuss F-theory as type IIB with 7-branes and thereafter turn towards its dual formulation in terms of M-theory. Lastly, we will review a recent construction of five-dimensional spinning black holes in F-theory [15]. In part II of this thesis we will build upon and generalize the setting discussed there.

### 1.4.1 F-theory as type IIB with varying axio-dilaton

Type IIB string theory contains (R,R) potentials of even degree in their low-energy spectrum. These (R,R)  $p$ -form potentials couple electrically to a  $D(p-1)$ -brane and magnetically to a  $D(7-p)$ -brane. The D7-branes take a special role among these branes, given that they are co-dimension two objects in type IIB. They

couple electrically to the (R,R) eight-form potential  $C_8$ , which is the magnetic dual of the (R,R) zero-form  $C_0$ . This means that

$$*F_1 = *dC_0 = F_9 = dC_8. \quad (1.104)$$

It is convenient to introduce the complex axio-dilaton

$$\tau = C_0 + ie^{-\phi}, \quad (1.105)$$

which combines the (R,R) axion and the dilaton into a single field. Type IIB supergravity has a remarkable  $\mathrm{SL}(2, \mathbb{R})$  symmetry which acts on the supergravity fields in ten-dimensional Einstein frame by

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} C_2 \\ B \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), \quad (1.106)$$

with all other type IIB fields invariant. It is expected that in the full quantum theory this continuous symmetry group is broken by  $D(-1)$ -instanton effects to the discrete subgroup  $\mathrm{SL}(2, \mathbb{Z})$ . This conjectured duality of type IIB string theory is called S-duality. To highlight the power of this duality consider the case  $C_0 = 0$  and the group element

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}), \quad (1.107)$$

which maps  $g_s \rightarrow 1/g_s$ , i.e. weak coupling to strong coupling. S-duality therefore provides a window to non-perturbative physics of type IIB string theory.

We can now consider a D7-brane whose normal space is given by  $\mathbb{C}$  with complex coordinate  $z$ . We take the D7-brane to be at  $z = 0$ . The equation for motion of  $C_8$  is

$$d * F_9 = \delta^{(2)}(z), \quad (1.108)$$

where the delta function stems from the D7-brane source. Supersymmetry furthermore implies that the axio-dilaton  $\tau$  must be holomorphic in  $z$ . Integrating (1.108) leads to

$$\int_{\mathbb{C}} d * F_9 = \int_{\mathbb{C}} dF_1 = \oint_{S^1} dC_0 = 1. \quad (1.109)$$

A solution close to the D7-brane is provided by the holomorphic axio-dilaton profile

$$\tau(z) = \frac{1}{2\pi i} \log \frac{z}{z_0} + \dots, \quad (1.110)$$

where  $z_0$  is a complex parameter. We note that if we encircle the D7-brane locus  $z = 0$  the  $\tau$ -profile undergoes a monodromy

$$\tau \rightarrow \tau + 1. \quad (1.111)$$

This monodromy can be identified with an  $\mathrm{SL}(2, \mathbb{Z})$  transformation generated by the group element

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (1.112)$$

The analysis of the full D7-brane solution is very subtle but highlights that the presence of D7-branes causes the axio-dilaton to vary non-trivially. We will therefore sketch the main features of the D7-brane supergravity solution. Firstly, the full solution is required to recover (1.110) close to the D7-brane. Secondly,  $\tau(z)$  must be a holomorphic function of  $z$  which is single valued *up to*  $\mathrm{SL}(2, \mathbb{Z})$  transformations. This is done by the Klein  $j$ -function which provides a one-to-one mapping between the fundamental domain of  $\mathrm{SL}(2, \mathbb{Z})$  and the Riemann sphere  $\mathbb{C} \cup \{\infty\}$ . We have depicted the fundamental domain  $\mathcal{F}$  in figure 1.6. The

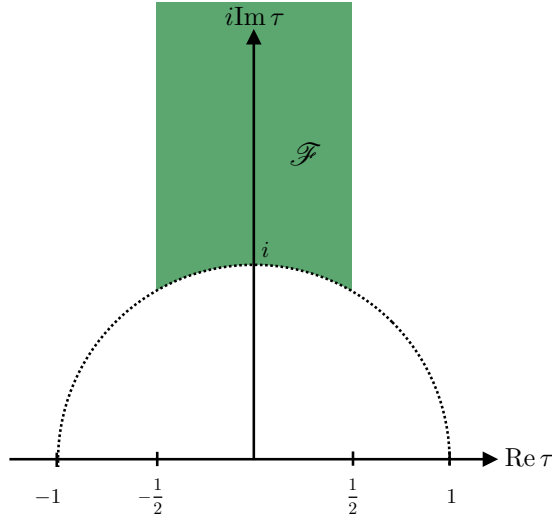


Figure 1.6: The fundamental domain  $\mathcal{F}$  of  $\mathrm{SL}(2, \mathbb{Z})$ .

$j$ -function enjoys a Laurent-series expansion

$$j(\tau) = e^{-2\pi i \tau} + 744 + \dots, \quad (1.113)$$

which allows us to give an implicit definition of the  $\tau$ -profile via the relation

$$j(\tau) = \frac{z_0}{z}. \quad (1.114)$$

We can see from the expansion (1.113) that we recover the result which is valid close to the brane (1.110). We can now consider the two limiting cases  $z \rightarrow 0$  and  $z \rightarrow \infty$ . In the case  $z \rightarrow 0$  we find from (1.110) that the string coupling  $g_s = e^\phi \rightarrow 0$  close to the D7-brane. When  $z \rightarrow \infty$  the axio-dilaton value is given by

$$\tau(z \rightarrow \infty) = j^{-1}(0) \approx -0.50 + 0.87i \quad \Rightarrow \quad g_s \approx 1.15. \quad (1.115)$$

We notice that the string coupling interpolates from a region of weak string coupling, where string perturbation theory is reliable, to an area which is intrinsically strongly coupled due to the presence of the 7-brane. This emphasizes that D7-branes do not in general lead to a picture where backreaction is negligible.

Now we take the  $\text{SL}(2, \mathbb{Z})$  duality group at face value and recall, that the two-forms of type IIB string theory transform as doublets under  $\text{SL}(2, \mathbb{Z})$ . The (NS,NS) two-form  $B$  couples electrically to the fundamental string and the (R,R) two-form  $C_2$  to the D1-brane. This suggests that there should be more exotic solitonic objects in the spectrum of type IIB string theory. These are the  $(p, q)$ -strings which can end on  $(p, q)$ -7-branes. A  $(p, q)$ -string couples electrically to the combination  $pB + qC_2$  of (NS,NS) two-form and (R,R) two-form. In this notation, the fundamental string is a  $(1, 0)$ -string, whereas the D1-brane is a  $(0, 1)$ -string. We see from the transformation (1.106) that the charge vector of a  $(p, q)$  string transforms as<sup>13</sup>

$$(q', p') = (q, p) \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (1.116)$$

In particular, we can generate a  $(p, q)$  string from a  $(1, 0)$ -string via

$$(q, p) = (0, 1) \begin{pmatrix} r & s \\ q & p \end{pmatrix}, \quad (1.117)$$

provided that  $rp - qs = 1$ . From this and the known expression for the monodromy of a  $(1, 0)$ -7-brane (1.112) we determine the monodromy associated with a general  $(p, q)$ -7-brane

$$M_{(p,q)} = \begin{pmatrix} r & s \\ q & p \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & s \\ q & p \end{pmatrix} = \begin{pmatrix} 1 + pq & p^2 \\ -q^2 & 1 - pq \end{pmatrix}. \quad (1.118)$$

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<sup>13</sup>The charge vector of a  $(p, q)$ -string under the  $\text{SL}(2, \mathbb{Z})$  doublet  $(C_2, B)^T$  is here denoted by  $(q, p)$ .

Now we can in principle, whenever we find a certain monodromy of  $\tau$  in a type IIB setting, determine which kind of  $(p, q)$ -7-brane we circled around. It is easy to see that a  $(p, q)$ -string is invariant under the action of  $M_{(p, q)}$ . This is intuitively clear, since  $(p, q)$ -strings are the objects which end on  $(p, q)$ -7-branes. It is in general possible to find a suitable  $\mathrm{SL}(2, \mathbb{Z})$  transformation which transforms a  $(p, q)$ -7-brane to an ordinary  $(1, 0)$ -7-brane, i.e. a familiar D7-brane from perturbative string theory. However, in the presence of several different types of  $(p, q)$ -7-branes it is in general not possible to find an  $\mathrm{SL}(2, \mathbb{Z})$  frame in which all 7-branes are D7-branes. 7-branes, which cannot be simultaneously transformed into D7-branes are said to be mutually nonlocal.

The transformation law of  $\tau$  in (1.106) is reminiscent of how the complex structure of a torus transforms under a modular transformation. It is tempting to interpret the axio-dilaton of type IIB string theory as the complex structure of a torus and claim that type IIB supergravity emerges from a dimensional reduction of a fundamental twelve-dimensional theory on a torus. This interpretation comes with two main problems. Since type IIB supergravity is invariant under 32 real supercharges the twelve-dimensional parent theory should also be supersymmetric. This is since torus compactifications do not break any supercharges. The low energy limit of this theory should be a suitable supergravity theory in twelve dimensions. However, it can be shown that the highest dimension in which a supergravity theory with Lorentzian signature exists is eleven. A second issue of the twelve dimensional interpretation is the absence of a volume modulus in type IIB supergravity. If type IIB were to descend via dimensional reduction of a twelve-dimensional parent theory on a torus, such a field would clearly exist in ten-dimensions.

The modern point of view is to see the type IIB axio-dilaton as the complex structure of an auxiliary torus which is however not part of the physical spacetime. We will review the duality of type IIB with varying axio-dilaton with M-theory in the following section. This duality uncovers how the physics of type IIB string theory with varying axio-dilaton and  $(p, q)$ -7-branes is encoded in the geometry of elliptic fibrations.

### 1.4.2 F-theory via M-theory

The duality of F-theory with M-theory provides a powerful tool to address questions in type IIB compactifications with varying axio-dilaton and  $(p, q)$ -7-branes. In fact, this duality is so powerful that it may even be viewed as a definition of F-theory. We have seen in section 1.2.6 that M-theory compactified

on a torus  $T^2$  is dual to type IIB compactified on a circle with the axio-dilaton identified with the complex structure of the torus. In the limit of vanishing torus volume  $\text{vol}(T^2) \rightarrow 0$  the type IIB circle decompactifies and full ten-dimensional Lorentz-invariance gets restored.

We now wish to study compactifications of type IIB with axio-dilaton varying on a (complex)  $n$ -dimensional Kähler manifold  $B_n$  and preserve minimal supersymmetry in the corresponding lower dimension. The Einstein equations of type IIB supergravity dictate that the Kähler manifold can not be Calabi–Yau once a non-trivial axio-dilaton profile is included. To get a clearer picture of the conditions implied by supersymmetry, we dualize this system to M-theory. It turns out, that we can still perform the duality chain from section 1.2.6 fiber-wise, once we fiber the  $T^2$  over the space  $B_n$ . Again, the type IIB axio-dilaton is identified with the complex structure  $\tau$  of the  $T^2$  fiber. The non-trivial fibration structure induces the non-trivial axio-dilaton profile on the type IIB side of the duality. We therefore obtain a duality of type IIB with varying axio-dilaton compactified on  $B_n$  and a circle with M-theory compactified on the total space  $X_{n+1}$  with the fibration structure  $T^2 \hookrightarrow X_{n+1} \rightarrow B_n$ . Note that the torus, which is only auxiliary in type IIB is now part of the physical spacetime in M-theory. The supersymmetry conditions in M-theory imply for compactifications to Minkowski space that the total space of the fibration needs to be Calabi–Yau. We therefore find that compactifications of F-theory are described by torus fibered Calabi–Yau  $(n+1)$ -folds

$$T^2 \hookrightarrow \text{CY}_{n+1} \xrightarrow{\pi} B_n, \quad (1.119)$$

where  $\pi : \text{CY}_{n+1} \rightarrow B_n$  is a map which projects a point in the total space to the base. If we wish to decompactify the circle on the type IIB side of the duality, we again need to shrink the volume of the torus fiber  $\text{vol}(T^2) \rightarrow 0$ . We will in the following assume that the fibers have a marked zero-point, i.e. we consider elliptic fibrations with fibers  $\mathbb{E}_\tau$ .

An elliptic curve  $\mathbb{E}_\tau$  can be constructed as a hypersurface in weighted projective space  $\mathbb{P}_{231}$ . The ambient weighted projective space  $\mathbb{P}_{231}$  is defined by  $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$  with complex coordinates  $(x, y, z)$  modulo the equivalence relation

$$(x, y, z) \sim (\lambda^2 x, \lambda^3 y, \lambda z), \quad \lambda \in \mathbb{C}^*. \quad (1.120)$$

A point in  $\mathbb{P}_{231}$ , i.e. an equivalence class, is denoted by  $[x : y : z]$ . An elliptic curve can be constructed in  $\mathbb{P}_{231}$  as the vanishing locus of a polynomial. In Weierstrass form the elliptic curve is defined by the equation

$$\mathbb{E}_\tau : P_W = y^2 - x^3 - fxz^4 - gz^6 = 0 \subset \mathbb{P}_{231}, \quad (1.121)$$

where  $f, g \in \mathbb{C}$  are constant parameters of the Weierstrass equation. The elliptic curve degenerates if

$$P_W = 0, \quad dP_W = 0. \quad (1.122)$$

We first consider the patch  $z = 0$ . The Weierstrass equation reduces to  $y^2 = x^3$ . We can now use the  $\mathbb{C}^*$  equivalence relation (1.120) to set  $x = 1$  which fixes  $y = \pm 1$ . Fixing the residual choice for the sign of  $\lambda$ , we obtain the equivalence class  $[1 : 1 : 0]$ . The point  $[1 : 1 : 0]$  is for all choices of  $f, g$  on the hypersurface which is defined by  $P_W = 0$ . This is the marked point on the elliptic curve which was mentioned earlier. In particular, there are no singularities in the patch  $z = 0$  since  $(0, 0, 0) \notin \mathbb{C}^3 \setminus \{(0, 0, 0)\}$ .

In the patch  $z \neq 0$  we can use the scaling equivalence to set  $z = 1$ . We find the equation

$$y^2 = x^3 + fx + g = (x - x_1)(x - x_2)(x - x_3), \quad (1.123)$$

where  $x_{1,2,3}$  are the three roots of the cubic polynomial in  $x$ . We find a degenerating elliptic curve if

$$\begin{aligned} dP_W = 0 : \quad & 0 = y, \\ & 0 = (x - x_1)(x - x_2) + (x - x_1)(x - x_3) + (x - x_2)(x - x_3), \\ P_W = 0 : \quad & 0 = (x - x_1)(x - x_2)(x - x_3). \end{aligned} \quad (1.124)$$

These conditions are satisfied if  $y = 0$  and at least two of the three roots of the cubic polynomial in  $x$  coincide. This happens when the discriminant of the cubic, given by

$$\Delta = 27g^2 + 4f^3, \quad (1.125)$$

vanishes. Thus, the complex parameters  $f, g$  determine when an elliptic curve degenerates. To find the complex structure  $\tau$  associated with the elliptic curve described by (1.121) we may use a classical result in the theory of elliptic curves. This result states that the complex structure is encoded in the  $j$ -function which we already encountered in section (1.4.1). The complex structure is implicitly given by

$$j(\tau) = \frac{4(24f)^3}{\Delta}. \quad (1.126)$$

Up to now we have only considered elliptic curves without any fibration structure. If we want to construct an elliptic fibration over the base  $B_n$  with coordinates  $u_i$ , we promote the constants  $f, g$  which enter the Weierstrass model to polynomials  $f(u_i), g(u_i)$  of a certain degree on the base. One can see from (1.125) and (1.126)



that the discriminant  $\Delta$  and the complex structure  $\tau$  now depend on the base. In particular, note that this maps on the F-theory/type IIB side of the duality to a varying axio-dilaton. The locus defined by

$$\Delta = 0 \subset B_n \quad (1.127)$$

specifies where in the base the elliptic fiber degenerates. One can show that whenever a  $(p, q)$ -one-cycle in the fiber shrinks over a component of the vanishing locus of  $\Delta$ , that the complex structure of the elliptic fibration undergoes a monodromy transformation associated with a  $(p, q)$ -7-brane. This implies the existence of a  $(p, q)$ -7-brane wrapped on the corresponding vanishing locus. A cartoon of an elliptic fibration is shown in figure 1.7. We have stated before, that the

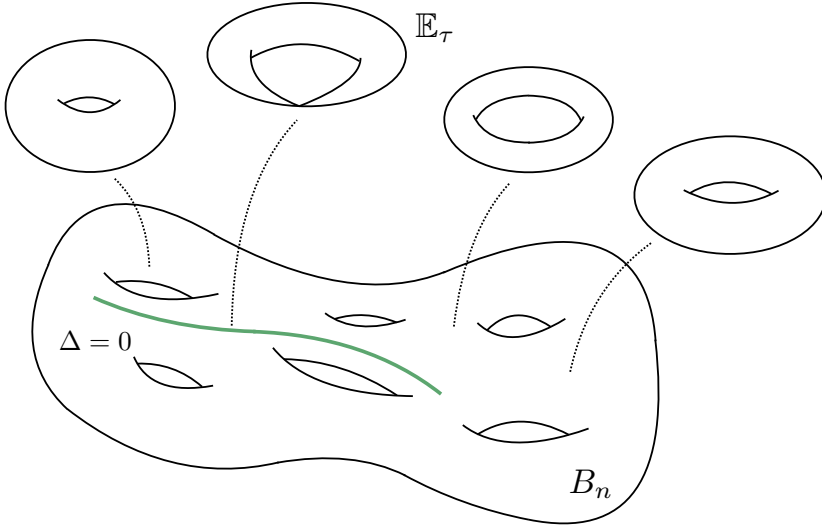


Figure 1.7: Schematic picture of an elliptic fibration. The shape of the elliptic curve  $\mathbb{E}_\tau$  varies as we move over the base  $B_n$ . Over loci where  $\Delta = 0$  (green), a one-cycle of the elliptic fiber collapses. This signals the presence of a 7-brane wrapped on this locus.

total space of the fibration must be Calabi–Yau for Minkowski compactifications. Imposing this property on the fibration leads to the condition

$$c_1(\text{CY}_{n+1}) = c_1(B_n) - \frac{1}{12}[\Delta] = 0, \quad (1.128)$$

since the first Chern-class of  $\text{CY}_{n+1}$  vanishes. Here  $[\Delta]$  denotes the two-form which is Poincaré dual to the vanishing locus  $\Delta = 0$  in the base. This condition

implies that the functions  $f(u_i), g(u_i)$  and coordinates  $x, y, z$  are actually sections of suitable line bundles on  $B_n$ .

The total space of the elliptic fibration  $CY_{n+1}$  can both be smooth and singular. Note that if the total space of the fibration is smooth, this does not imply that the fibration is trivial. There can still be degenerations of the fiber which lead to a smooth total space. If the Calabi–Yau space is singular, this poses a problem for a direct application in M-theory. Recall that there is no microscopic formulation of M-theory to date. We can therefore only rely on its low-energy limit, eleven dimensional supergravity coupled to M2 and M5-branes. In order to justify the use of supergravity, we need to make sure that the compactification manifold is smooth. In many cases one can resolve the singularities of the total space and obtain a smooth Calabi–Yau manifold. Resolution essentially means that a set of  $\mathbb{P}^1$ s are pasted into the singular fibers such that a smooth manifold is obtained. The original singular Calabi–Yau can be recovered by shrinking the  $\mathbb{P}^1$ s to zero size. The resolution  $\mathbb{P}^1$ s in the fiber intersect in the same way, in which nodes of affine Dynkin diagrams intersect. The Dynkin diagram associated with the intersection pattern of the  $\mathbb{P}^1$ s determines the gauge algebra on the 7-branes to which the singularity corresponds. A visualization of this resolution procedure is shown in figure 1.8. We have the following types of divisors in the resolved

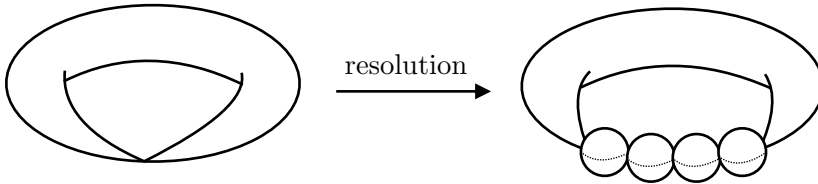


Figure 1.8: The resolution of a singular fiber introduces  $\mathbb{P}^1$ s in the resolved fiber. In this example, the resolution  $\mathbb{P}^1$ s intersect like the nodes of the affine  $A_4 \simeq \mathfrak{su}(5)$  Dynkin diagram.

Calabi–Yau manifold:

- The divisor homologous to the base  $B_n$  with Poincaré dual two-form  $\omega_0$ .
- The  $h^{1,1}(B_n)$  vertical divisors  $D_\alpha = \pi^{-1}(D_\alpha^b)$ , which are obtained as preimages of divisors of the base  $D_\alpha^b$  under the projection map. Their dual two-forms are denoted by  $\omega_\alpha$ .
- The exceptional divisors  $D_i$  which are obtained by fibering the resolution  $\mathbb{P}^1$ s over the discriminant locus. The dual two-forms are  $\omega_i$ .

In principle, there can be additional divisors from rational sections, which we will omit from the discussion. All of the divisors listed above are dual to a harmonic two-form. Expanding the M-theory three-form  $C_3$  along these harmonic forms

$$C_3 = A^0 \wedge \omega_0 + A^\alpha \wedge \omega_\alpha + A^i \wedge \omega_i \quad (1.129)$$

one obtains three sets of vector potentials in the uncompactified dimensions. Recall that M-theory on an elliptically fibered Calabi–Yau manifold with finite fiber volume is dual to F-theory compactified on the base of the fibration times a circle. The vector field  $A^0$  can be shown to be the Kaluza–Klein vector in the circle reduction of F-theory.<sup>14</sup> Therefore, upon decompactifying the circle,  $A^0$  does not lead to a gauge symmetry in F-theory but becomes a part of the metric. The vector fields from the expansion along the two-forms dual to the vertical divisors  $A^\alpha$  can be understood by dualizing to type IIB. There, they correspond to an expansion of the (R,R) four-form potential  $C_4 = B^\alpha \wedge \omega_\alpha$  and therefore also do not lead to a gauge symmetry in F-theory. Lastly, there are the abelian gauge fields  $A^i$  stemming from the exceptional divisors. One can show that they indeed are the abelian gauge fields in the Cartan subalgebra of the 7-brane gauge algebra. It turns out, that the states which deliver the off-diagonal gauge bosons are massive in the resolved phase of the Calabi–Yau manifold and are therefore integrated out. These states are M2-branes wrapping the resolution  $\mathbb{P}^1$ s and their mass is proportional to  $\text{vol}(\mathbb{P}^1)$ . In the singular limit of the Calabi–Yau these states become massless, since the  $\mathbb{P}^1$ s are shrunk to zero size. It is therefore expected that these M2-brane states provide the missing degrees of freedom for a non-abelian enhancement of the gauge symmetry.

The power and beauty of F-theory is that seemingly intractable physical problems are geometrized and can be studied via elliptic fibrations. F-theory has in the past lead to many insights about non-perturbative physics in type IIB compactifications with 7-branes, which can not be reproduced from perturbative type IIB string theory. Among the F-theory milestones are certainly the construction of GUTs and the conjectured classification of six-dimensional superconformal field theories. There are many interesting and beautiful topics in F-theory which have not been covered in this lightning review. These include the physics of abelian gauge groups, discrete symmetries in the effective action, the computation of chiral matter spectra in F-theory compactifications and Yukawa couplings. We refer the interested reader to the excellent reviews [16–20].

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<sup>14</sup>This is true up to a shift in  $\omega_0$ .

### 1.4.3 Black holes from F-theory

The existence of black holes was suggested by general relativity since the discovery of the Schwarzschild solution. One can derive a set of laws of black hole mechanics, which remarkably seem to mimic the laws of thermodynamics, up to the identification of certain quantities. This is particularly striking, given that there is a priori no reason why black holes should behave in any way like a thermodynamical system. The zeroth law suggests that one can associate a temperature  $T$  to black

Law of thremodynamics	Law of black hole mechanics
A body in thermal equilibrium has constant temperature $T$ everywhere	The surface gravity $\kappa$ is constant at the horizon.
$dE = TdS + \mu dQ + \Omega dJ$	$dM = \frac{\kappa}{8\pi} dA + \mu dQ + \Omega dJ$
The entropy never decreases: $dS \geq 0$	The area of the event horizon never decreases: $dA \geq 0$

Table 1.4: The laws thermodynamics vs. the laws of black hole mechanics.

holes, which should be defined in terms of the surface gravity at the black hole horizon. If a black hole can have a non-vanishing temperature, it should radiate as a consequence. This is however in sharp contrast with the naive expectation that nothing can escape from a black hole. This naive expectation turns out to not be true anymore once quantum effects are taken into account. Indeed, a calculation of Hawking showed that a black hole emits thermal radiation with temperature<sup>15</sup>

$$T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}. \quad (1.130)$$

This thermal radiation leads to the so called black hole information paradox. The information paradox is the seeming loss of information in the black hole, which is in contradiction with unitarity. There are many potential resolutions and a vast literature on this topic. We will however not dive deeper into this interesting subject.

The analogy of the first law of thermodynamics with the first law of black hole mechanics suggests that one can assign an entropy to black holes. The form of the first law furthermore suggest that the entropy of a black hole should scale as  $S \sim A$ , i.e. proportional to the area of the event horizon. Using (1.130) one

<sup>15</sup>This formula is shown in natural units  $\hbar = G = k_B = c = 1$ .

finds the Bekenstein-Hawking entropy

$$S = \frac{k_B c^3}{\hbar G} \frac{A}{4} \equiv \frac{A}{4} . \quad (1.131)$$

Another argument why black holes should carry entropy goes as follows. Imagine a box of gas being thrown into a black hole. This box of gas carries a certain amount of entropy which gets lost in the black hole. This seemingly decreases the entropy of the world outside of the black hole which contradicts the second law of thermodynamics. This contradiction may be resolved by postulating that the black hole itself carries entropy which increases once the gas has fallen into the black hole.

Once we take the analogy of black hole mechanics and thermodynamics at face value, we can take it a step further and recall the connection between thermodynamics and statistical mechanics. According to statistical mechanics, the statistical entropy of a system is given by the logarithm of the number microstates  $\Omega_{\text{micro}}$  which can realize the system microscopically

$$S = \log \Omega_{\text{micro}} . \quad (1.132)$$

Viewing this connection in the light of black holes and their entropy, it is natural to ask what these microstates of black holes are and if we are able to identify and count them. String theory provides a promising framework to address this question, given that it is a well-defined quantum theory of gravity. Furthermore, string theory includes heavy extended objects, the D-branes, with which black holes can be constructed. With the aid of these D-brane constructions we have a handle on the microscopic side of the black holes which are formed by them. On the other hand, string theories and their compactifications are at low energies well described by certain supergravity theories. We can therefore use supergravity techniques to compute the black hole entropy; this is the so-called macroscopic side of the black hole. This can be achieved by a direct computation of the area of the event horizon of the corresponding black hole solution, or its generalization to theories with higher derivatives, the Wald entropy. We will however utilize a different technique in part II of this thesis, where we determine the coefficients of certain Chern-Simons terms in the effective action.

The first instance where the microscopic entropy was successfully computed and compared to the macroscopic entropy was in the ground-breaking paper by A. Strominger and C. Vafa [21]. The authors considered type II strings compactified on  $K3 \times S^1$  and the resultant five-dimensional black hole was microscopically realized by D1 and D5-branes. The construction in this paper heavily relied on the

fact, that the black hole in five-dimensions can be obtained by an effective string in six dimensions wrapped around a circle. This allowed the use of conformal field theory techniques to extract the microscopic entropy associated with this D1-D5 system. The microscopic entropy was then shown to match with the entropy computed from the corresponding black hole in five-dimensional supergravity. This spectacular success launched a whole line of research which aims to understand the microscopic and macroscopic entropy of black holes in string theory. As already noted, the Strominger-Vafa black hole can be viewed as descending from strings in type IIB compactified on K3 wrapped on an additional circle. The six-dimensional theory after compactification on K3 is a supergravity theory with chiral  $\mathcal{N} = (2, 0)$  supersymmetry. We will in the following focus on five-dimensional black holes which arise from strings in  $\mathcal{N} = (1, 0)$  supergravity wrapped on an additional circle studied in [15]. The string theory embedding of this  $\mathcal{N} = (1, 0)$  supergravity theory is provided by F-theory compactified on an elliptically fibered Calabi–Yau threefold. In part II of this thesis we will make progress in expanding on this topic.

Consider F-theory compactified on a Calabi–Yau threefold or in type IIB language, type IIB compactified on a compact Kähler base  $B$ . The degenerations of the axio-dilaton on the base signal the presence of spacetime filling 7-branes which wrap the locus  $\Delta = 0$  in the base. We further compactify one dimension on a circle  $S^1$  such that we have the ten-dimensional type IIB spacetime

$$\mathbb{R}_t \times S^1 \times \mathbb{R}^4 \times B, \quad (1.133)$$

where  $\mathbb{R}_t$  denotes time. We are interested in a particular sub-sector of F-theory, which is obtained by wrapping D3-branes over a genus  $g$  curve  $C$  in the Kähler base and  $S^1$ . The D3-brane is therefore effectively a string in six-dimensions which is further compactified on a circle. We also assume that the string has  $n$  units of Kaluza-Klein momentum around the circle. We have depicted the setting in figure 1.9.

We are interested in the worldvolume theory to which the D3-brane flows in the infrared. The theory on the string is expected to flow to a two-dimensional CFT with  $\mathcal{N} = (0, 4)$  supersymmetry. The string in six dimensions probes a transverse  $\mathbb{R}^4$  space with  $\text{SO}(4)$  rotation symmetry. This symmetry acts on the degrees of freedom of the string. At the level of the algebra one has  $\text{SO}(4) \simeq \text{SU}(2)_L \times \text{SU}(2)_R$ . The  $\text{SU}(2)_{L,R}$  symmetries are realized on the string as current algebras with levels  $k_{L,R}$ .  $\text{SU}(2)_R$  is identified with the R-symmetry in the supersymmetric right-moving sector, and  $\text{SU}(2)_L$  is an additional current algebra acting on the left-movers.

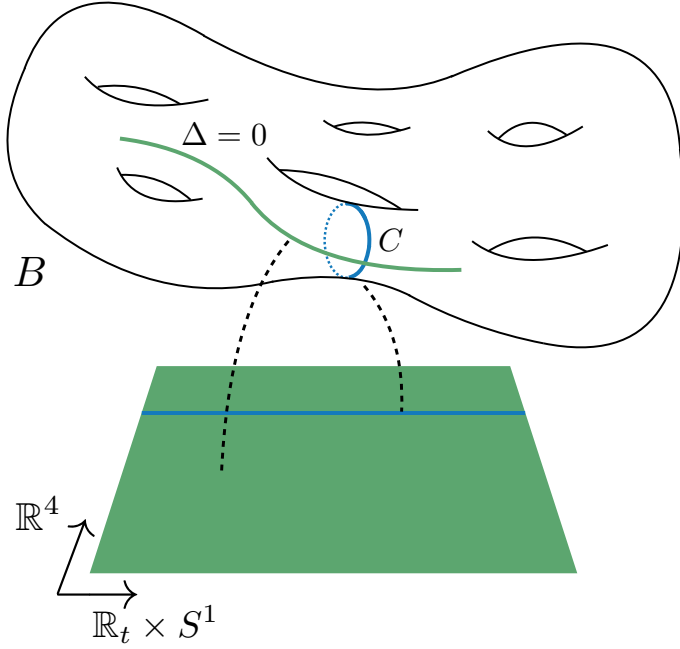


Figure 1.9: The 7-branes (green) wrap the discriminant locus  $\Delta = 0 \subset B$  and extend along the remaining dimensions  $\mathbb{R}_t \times S^1 \times \mathbb{R}^4$ . The D3-brane (blue) wraps the curve  $C \subset B$  and extends along  $\mathbb{R}_t \times S^1$ .

Let us now explore what these current algebras mean for the black hole in five dimensions. A black hole in five dimensions can rotate in two planes with a priori independent angular momenta  $J_{1,2}$ . These angular momenta are eigenvalues of the black hole under the  $\text{SO}(2)_1 \times \text{SO}(2)_2$  subgroup of  $\text{SO}(4)$ . The angular momenta  $J_{1,2}$  can be expressed in terms of the eigenvalues  $J_{L,R}$  under  $\text{U}(1)_L \times \text{U}(1)_R \subset \text{SU}(2)_L \times \text{SU}(2)_R$  via the relations

$$J_1 = J_L + J_R, \quad J_2 = J_L - J_R. \quad (1.134)$$

It turns out that supersymmetry imposes that the angular momenta satisfy  $J_1 = J_2 \equiv J$  which immediately leads to  $J_L = 2J$  and  $J_R = 0$ . In order to obtain the entropy of the five-dimensional spinning BPS black hole we therefore have to count BPS excitations of the string with  $\text{U}(1)_L$  eigenvalue  $J_L = 2J$  and momentum eigenvalue  $n$ . The BPS property on the string forces the right-moving sector in the ground state, while the left-moving sector remains unconstrained.

One obtains for large momentum eigenvalues  $n$  the Cardy formula

$$S = 2\pi \sqrt{\frac{c_L}{6} \left( n - \frac{J_L^2}{4k_L} \right)} = 2\pi \sqrt{\frac{c_L}{6} \left( n - \frac{J^2}{k_L} \right)}. \quad (1.135)$$

Therefore one needs to compute the left-moving central charge and level  $k_L$  of the current algebra in order to obtain the entropy.

On the microscopic side one needs to perform the dimensional reduction of the D3-brane action on the curve  $C \subset B$  and count the bosonic and fermionic degrees of freedom in two dimensions. This counting leads to the microscopic central charges. The action on the D3-brane is four-dimensional  $\mathcal{N} = 4$  super-Yang-Mills with gauge group  $U(1)$ . The complexified gauge coupling of this theory is given by the type IIB axio-dilaton  $\tau$  which now varies over the base, and subsequently also over  $C$ . In particular, the 7-branes located at  $\Delta = 0$  intersect the curve  $C$  in certain points (see figure 1.9), which causes  $\tau$  to undergo  $SL(2, \mathbb{Z})$  monodromies around these intersection loci. These monodromy transformations generate a  $U(1)_D$  bonus symmetry of abelian super-Yang-Mills. In particular, the supercharges on the worldvolume of the D3-brane also transform non-trivially under this  $U(1)_D$  symmetry. In order to make the supercharges on the D3-brane action on  $C$  well defined one needs to perform a *topological duality twist* [22]. From a field theory point of view this means that the theory is coupled to background gauge fields of a suitable subgroup of the R-symmetry and  $U(1)_D$ . This coupling can be done in a way such that  $\mathcal{N} = (0, 4)$  supersymmetry is preserved. Once this is achieved, one can proceed and reduce the field content of the four-dimensional  $U(1)$  gauge theory on  $C$ . The multiplicities of the corresponding fields in two dimensions are counted by suitable cohomology groups on  $C$ . After a careful reduction one finds the left- and right-moving central charges

$$\begin{aligned} c_L &= 3C \cdot C + 9c_1(B) + 2, \\ c_R &= 6k_R = 3C \cdot C + 3c_1(B) \cdot C, \\ k_L &= \frac{1}{2}C \cdot C - \frac{1}{2}c_1(B) \cdot C. \end{aligned} \quad (1.136)$$

Here we have used in slight abuse of notation the same symbol for the curve  $C$  and its Poincaré dual two-form. The quantity  $C \cdot C$  is defined by

$$C \cdot C = \int_B C \wedge C, \quad (1.137)$$

and likewise for  $c_1(B) \cdot C$ .



On the macroscopic side there is a string solution of six-dimensional supergravity with an  $\text{AdS}_3 \times S^3$  near horizon geometry. The sphere is supported by fluxes of the (anti-)self-dual three-form field strengths  $H^\alpha = dB^\alpha$ . These two-forms  $B^\alpha$  arise from the type IIB (R,R) four-form potential by expanding it in a basis of harmonic  $(1,1)$ -forms on the base  $B$

$$C_4 = B^\alpha \wedge \omega_\alpha. \quad (1.138)$$

The sphere has an  $\text{SO}(4) = \text{SU}(2)_L \times \text{SU}(2)_R$  isometry which realizes the corresponding current algebras geometrically. Upon performing a dimensional reduction of the 6d  $\mathcal{N} = (1,0)$  effective action of F-theory on  $S^3$  one produces  $\text{SU}(2)_L \times \text{SU}(2)_R$  gauge fields  $A_L, A_R$  corresponding to the isometries. One may then use that the levels  $k_{L,R}$  and  $c_L - c_R$  are encoded in certain coefficients of Chern-Simons terms in three-dimensions after the reduction on  $S^3$  is performed. More precisely, after compactification of the 6d action on  $S^3$  one finds Chern-Simons terms  $\omega^{\text{CS}}(A_{L,R})$  and a gravitational Chern-Simons term  $\omega_{\text{grav}}^{\text{CS}}$

$$S^{(3d)} \supset \frac{k_R}{4\pi} \int \omega^{\text{CS}}(A_R) - \frac{k_L}{4\pi} \int \omega^{\text{CS}}(A_L) + \frac{c_L - c_R}{96\pi} \int \omega_{\text{grav}}^{\text{CS}} \quad (1.139)$$

from which  $k_{L,R}$  and  $c_L - c_R$  can be read off. Remarkably, one finds (1.136) up to terms independent of  $C$ .

In part II of this thesis we study black holes from D3-branes, which are wrapped on a curve  $C \subset X$ , where  $X = \text{K3}$  in chapter 6 and  $X = B$  in chapter 7. In contrast to the setting discussed in this section, we replace the  $\mathbb{R}^4$  transverse to the string with a non-trivial transverse space  $M_4$ , see figure 1.9. This way we are able to construct four- and five dimensional black holes from type IIB on K3 and F-theory on  $\text{CY}_3$ . Many of the transverse spaces  $M_4$  lead to an additional topological charge on which the entropy may depend. An example of a family of transverse spaces which we consider is the case  $M_4 = \text{TN}_m$ .  $\text{TN}_m$  is the Taub-NUT space with NUT charge  $m$ , which we already encountered in section 1.2.6. We find, that quantum corrections to the Chern-Simons terms (1.139) are crucial in order to obtain the expected results.



## Part I

# Effective actions

In Part I of this thesis we focus on Calabi–Yau compactifications of the type II supergravities and eleven-dimensional supergravity, the low energy limit of M-theory, including certain eight-derivative corrections. In perturbative type II string theories, the relevant couplings arise both at tree-level and one-loop in the string coupling  $g_s$ , whereas in M-theory there is only one type of eight-derivative correction.<sup>16</sup> Upon compactification, these additional terms can have crucial impact on the lower-dimensional effective action. Namely they can generate higher-derivative corrections in the compactified effective action itself, or lead to corrections to the couplings and canonical field variables in the two-derivative effective action after compactification of the higher-dimensional parent theory.

These corrections have in the past played a prominent role in numerous applications. Among these are the (macroscopic) computation of subleading contributions to the entropy of supersymmetric black holes, the computation of subleading terms in the context of the celebrated AdS/CFT correspondence, and applications in scenarios of moduli stabilization. In the latter context, corrections stemming from higher-derivative terms can be used to generate masses for a priori massless fields. The desire of a better understanding of higher-derivative corrections in string theory must however be confronted with two major challenges. Firstly, there is only *partial information* about higher-derivative corrections in ten and eleven dimensions available in the literature. Secondly, the *computational complexity* in compactifications increases dramatically once higher derivatives are taken into account. In this part of the thesis we will make progress in both directions, as we explain in the following.

Most information about higher-derivative corrections in supergravity theories, which describe the low-energy dynamics of string theories, stems from explicit string amplitude computations as explained in section 1.2.4. It is then possible to extract and check higher-derivative couplings from these technically challenging computations of  $n$ -point functions. We will perform in part I several consistency checks of higher-derivative corrections which are available in the literature. We will do these checks by computing (parts of) the effective actions arising from Calabi–Yau threefold compactifications. Compatibility with supersymmetry in lower dimensions imposes stringent constraints on the form of the effective action. The corrections induced by the higher-derivatives then have to obey the rules dictated by supersymmetry, which therefore provides a good and effective way to conduct consistency checks. In addition, we compute new corrections to the

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<sup>16</sup>These correspond to taking the strong coupling, i.e. decompactification limit of the type IIA eight-derivative corrections.

two- and four-derivative effective actions from Calabi–Yau three- and fourfold compactifications. The geometric building blocks which govern the couplings in the effective actions are identified.

We will focus on certain geometric deformations of the Calabi–Yau backgrounds, the Kähler moduli. These moduli appear as scalar fields in the low-energy effective action arising from compactification. In order to perform the relevant computations, we developed a Mathematica code, which enables us to overcome the technical difficulties. Its main ingredients are the ability to perform fast and efficient computations with complex indices as well as the complete, *computer-based generation of total-derivative and Schouten identities*<sup>17</sup> on the Calabi–Yau background. The latter identities are needed in order to simplify the unwieldy expressions, which one obtains after brute force dimensional reduction. This also allows us to identify a minimal set of building blocks, which govern the couplings in the effective actions.

We give a compendium of notations used in the main text.

## Indices.

- external spacetime:  $\mu, \nu, \rho, \dots = 1, \dots, 4, (5)$ , greek letters are used for the non-compact spacetime indices  $\mathcal{M}_{3,4,5}$  in type II and M-theory compactifications.
- Calabi–Yau: lowercase letters  $m, n, \dots = 1, \dots, 6, (7, 8)$  are used as *real* indices of the internal Calabi–Yau manifolds, whereas  $i, j, k, \bar{i}, \bar{j}, \bar{k} \dots = 1, 2, 3, (4)$  are used as *complex* (anti-)holomorphic indices of the Calabi–Yau three- and fourfolds.
- Total spacetime:  $M, N, \dots = 1, \dots, 10, (11)$  are used as spacetime indices of the ten- and eleven-dimensional spacetimes  $\mathcal{M}_{10,11}$ .
- Cohomology and fields:  $a, b, c, \dots = 1, \dots, h^{1,1}$  are used as indices labelling the cohomology groups  $H^{1,1}(\text{CY}_3)$ . In some general discussions of 3d and 5d supersymmetric actions, these indices are also used to label vector- and chiral multiplets. We do this, anticipating that in the string theory realization which we study, their number will be identified with the range given above.

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<sup>17</sup>These identities are sometimes also called ‘dimensionally dependent identities’.

**Further notation.**

- Quantities in ten or eleven dimensions, such as differential forms, Riemann tensors etc. will be denoted with a hat  $\hat{\phantom{x}}$  on top.
- Hodge star: The Hodge star operator in ten and eleven dimensions is denoted by  $\hat{\star}$ , the Hodge star on the external spacetime is denoted by  $\star$ , the Hodge star on the Calabi–Yau manifolds are denoted by  $\star_{6,8}$ .
- Quantities with a superscript  $^{(0)}$  are evaluated using the zeroth order Calabi–Yau background metric  $g_{i\bar{j}}^{(0)}$ .
- Quantities with a superscript  $^{(1)}$  or  $^{(2)}$  indicate the order in  $\alpha$  (type II) or  $\hat{\alpha}$  (M-theory), at which they appear in the corrected Calabi–Yau background solution.

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## Chapter 2

# Calabi–Yau solutions with higher-curvature corrections

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The purpose of this section is to set the stage for the dimensional reductions including higher-derivative terms, which will follow in the subsequent three chapters. At two-derivative level, the dimensional reductions we are interested in are the intensely studied compactifications of type II supergravities and M-theory on Calabi–Yau (CY) manifolds. The dimensional reductions are performed by expanding around a supersymmetric vacuum of the form  $\mathbb{R}^{1,d-2n-1} \times \text{CY}_n$ , where  $d$  is the dimension of the corresponding higher-dimensional theory and  $n$  is the complex dimension of the Calabi–Yau manifold. Naively, one is tempted to use the very same supersymmetric background and perform the dimensional reduction including higher-derivative corrections of the higher-dimensional theory. This would however only lead to an incomplete effective action. The reason for that is, that the supersymmetric background *itself* may be corrected, once higher-derivative corrections are included. The corrected background solution then contributes to the effective action through the dimensional reduction of the higher-dimensional two-derivative action on the corrected background. In this chapter we will derive and collect solutions to the corrected equations of motion, which are at lowest order in the derivative expansion the well known Calabi–Yau solutions. Generally, the solutions under investigation obtain non-trivial corrections which turn out to be of crucial importance for the effective actions.

## 2.1 Higher-derivatives in M-theory and type IIA

In this section we collect various higher-derivative corrections which are available in the literature and which are relevant for our discussion. The detailed index structures of the various pieces in the actions can be found in appendix B.

### 2.1.1 Higher derivatives in M-theory

Higher-derivative corrections to the low-energy effective action of type IIA, which will be presented in section 2.1.2, can be lifted to eleven dimensions. This way, they lead to corrections to the low-energy effective action of M-theory. At two-derivative level the effective action of M-theory is eleven-dimensional  $\mathcal{N} = 1$  supergravity [11].<sup>18</sup> Its bosonic part is given by

$$S_{\text{M}}^{\text{class}} = \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \hat{R} \hat{*} 1 - \frac{1}{2} \hat{G}_4 \wedge \hat{*} \hat{G}_4 - \frac{1}{6} \hat{C}_3 \wedge \hat{G}_4 \wedge \hat{G}_4. \quad (2.1)$$

The dynamical degrees of freedom are the eleven-dimensional metric  $\hat{g}_{MN}$  and the M-theory three-form potential  $\hat{C}_3$  with field strength  $\hat{G}_4 = d\hat{C}_3$ . We introduce an expansion parameter

$$\hat{\alpha}^2 = \frac{(4\pi\kappa_{11}^2)^{2/3}}{(2\pi)^4 3^2 2^{13}}, \quad (2.2)$$

where  $\kappa_{11}$  is related to the eleven-dimensional Planck length  $\ell_{\text{M}}$  as  $\kappa_{11}^2 = \frac{1}{2}(2\pi)^8 \ell_{\text{M}}^9$ . The eight-derivative action of M-theory up to terms quadratic in  $\hat{G}_4$  takes the schematic form

$$S_{\text{M}} = S_{\text{M}}^{\text{class}} + \hat{\alpha}^2 S_{\hat{R}^4} + \hat{\alpha}^2 S_{\hat{C}\hat{X}_8} + \hat{\alpha}^2 S_{\hat{G}^2\hat{R}^3} + \hat{\alpha}^2 S_{(\hat{\nabla}\hat{G})^2\hat{R}^2} + \hat{\alpha}^2 S_{\hat{Z}\hat{G}^2}. \quad (2.3)$$

Let us proceed by introducing the various pieces contributing to the eight-derivative action (2.3). The most prominent part is the well known  $\hat{R}^4$  combination, which is given by [23–28]

$$S_{\hat{R}^4} = \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \left( \hat{t}_8 \hat{t}_8 - \frac{1}{24} \epsilon_{11} \epsilon_{11} \right) \hat{R}^4 \hat{*} 1. \quad (2.4)$$

In addition, there is an  $\hat{R}^4$  coupling to the M-theory three-form  $\hat{C}_3$  via an eight-form curvature polynomial  $\hat{X}_8$ , namely

$$S_{\hat{C}\hat{X}_8} = -\frac{3^2 2^{13}}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \hat{C}_3 \wedge \hat{X}_8. \quad (2.5)$$

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<sup>18</sup>We will in the following often refer to eleven-dimensional supergravity as M-theory, having in mind that, depending on the context, we mean its low energy limit.



The sector involving the four-form field strength  $\hat{G}_4$  is obtained by lifting a set of terms involving the (NS,NS) two-form in type IIA to eleven dimensions. These terms were shown to match with one-loop (in  $g_s$ ) string amplitude computations at the level of five-point functions and partially at the level of six-point functions [29]. The terms relevant to us are the ones quadratic in  $\hat{G}_4$  and the corresponding actions in (2.3) are schematically given by [29–36]

$$S_{\hat{G}^2 \hat{R}^3} = -\frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \left( \hat{t}_8 \hat{t}_8 + \frac{1}{96} \epsilon_{11} \epsilon_{11} \right) \hat{G}_4^2 \hat{R}^3 \hat{*} 1, \quad (2.6)$$

$$S_{(\hat{\nabla} \hat{G})^2 \hat{R}^2} = \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \hat{s}_{18} (\hat{\nabla} \hat{G}_4)^2 \hat{R}^2 \hat{*} 1, \quad (2.7)$$

$$S_{\hat{Z} \hat{G}^2} = \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} 256 \hat{Z} \hat{G}_4 \wedge \hat{*} \hat{G}_4. \quad (2.8)$$

For the detailed index structures of all the terms in (2.3) we refer the reader to appendix B. Note however that the tensorial structure  $\hat{s}_{18}$  is not fully known but contains six unfixed coefficients  $a_i$  [30]. In order to fix these coefficients, an analysis of higher  $n$ -point function needs to be conducted. In the following we will work with generic coefficients  $a_i$  and state, whenever we need to make an assumption on them.

### 2.1.2 Higher derivatives in type IIA

We next turn to type IIA supergravity and their corresponding higher-derivative corrections. We will in the following only be concerned with fields from the (NS,NS) sector and therefore disregard (R,R) fields. The bosonic field content in the (NS,NS) sector of ten-dimensional type IIA supergravity is given by the ten-dimensional metric  $\hat{g}_{MN}$ , the dilaton in ten dimensions  $\hat{\phi}$  and the two-form  $\hat{B}_2$ . In the landscape of ten-dimensional supergravity theories, type IIA is the maximally supersymmetric, non-chiral one, i.e. it has  $\mathcal{N} = (1, 1)$  supersymmetry.

The part of the  $\mathcal{N} = (1, 1)$  low energy effective action of the type IIA superstring we consider takes the form

$$S_{\text{IIA}} = S_{\text{IIA}}^{\text{class}} + \alpha S_{\hat{R}^4}^{\text{tree}} + \alpha S_{\hat{R}^4}^{\text{loop}} + \alpha S_{\hat{H}^2}, \quad (2.9)$$

where we introduced the expansion parameter

$$\alpha = \frac{\alpha'^3}{3 \times 2^{11}}. \quad (2.10)$$

The classical two-derivative action has for the (NS,NS) fields the form

$$2\kappa_{10}^2 S_{\text{IIA}}^{\text{class}} = \int_{\mathcal{M}_{10}} e^{-2\hat{\phi}} \left( \hat{R} \hat{*} 1 + 4d\hat{\phi} \wedge \hat{*} d\hat{\phi} - \frac{1}{2} \hat{H}_3 \wedge \hat{*} \hat{H}_3 \right), \quad (2.11)$$

where  $\mathcal{M}_{10}$  is the spacetime manifold,  $\hat{R}$  is the Ricci scalar in ten spacetime dimensions and  $\hat{H}_3 = d\hat{B}_2$  is the field strength of the (NS,NS) two-form. The gravitational coupling in ten dimensions, denoted by  $\kappa_{10}$ , can be expressed in terms of the string length  $\ell_s^2 = \alpha'$  via the relation

$$2\kappa_{10}^2 = (2\pi)^7 \ell_s^8. \quad (2.12)$$

At eight derivatives the action gets supplemented by additional terms quartic in the Riemann tensor at both tree-level and one-loop in the string coupling  $g_s$ . These pieces of the action take the schematic form [8, 9]

$$2\kappa_{10}^2 S_{\hat{R}^4}^{\text{tree}} = \zeta(3) \int_{\mathcal{M}_{10}} e^{-2\hat{\phi}} \left( \hat{t}_8 \hat{t}_8 + \frac{1}{8} \epsilon_{10} \epsilon_{10} \right) \hat{R}^4 \hat{*} 1, \quad (2.13)$$

$$2\kappa_{10}^2 S_{\hat{R}^4}^{\text{loop}} = \frac{\pi^2}{3} \int_{\mathcal{M}_{10}} \left( \hat{t}_8 \hat{t}_8 - \frac{1}{8} \epsilon_{10} \epsilon_{10} \right) \hat{R}^4 \hat{*} 1. \quad (2.14)$$

For the detailed form of these terms and the definition of the tensor  $\hat{t}_8$  in ten and eleven dimensions we again refer the reader to appendix B. Note that the relative sign flip of the one-loop contribution compared to the tree-level piece is characteristic for type IIA and does not appear in type IIB. Additionally, there are several partial results on higher-derivative terms in ten dimensions involving the (NS,NS) two-form and the dilaton [37–41].

It was conjectured and tested in [29], that the completion of the eight-derivative terms with respect to the (NS,NS) two-form is almost completely captured by introducing a connection with torsion. It was claimed that higher-derivative terms involving  $\hat{B}_2$  can be obtained from the  $\hat{R}^4$ -terms by computing the latter with respect to the aforementioned torsionful connection. We will not collect all the structures emerging from this procedure but outline the strategy how to get the eight-derivative terms we need for our discussion.

Both the tree-level and one-loop contributions to the  $\hat{R}^4$  action can be expressed in terms of the two ‘superinvariants’

$$\mathcal{J}_0 = \left( \hat{t}_8 \hat{t}_8 + \frac{1}{8} \epsilon_{10} \epsilon_{10} \right) \hat{R}^4, \quad (2.15)$$

$$\mathcal{J}_1 = \hat{t}_8 \hat{t}_8 \hat{R}^4 - \frac{1}{4} \hat{t}_8 \epsilon_{10} \hat{B}_2 \hat{R}^4, \quad (2.16)$$

such that the  $\hat{R}^4$ -terms read

$$2\kappa_{10}^2 S_{\hat{R}^4}^{\text{tree}} = \zeta(3) \int_{\mathcal{M}_{10}} e^{-2\hat{\phi}} \mathcal{J}_0 \hat{*} 1, \quad (2.17)$$

$$2\kappa_{10}^2 S_{\hat{R}^4}^{\text{loop}} = \frac{\pi^2}{3} \int_{\mathcal{M}_{10}} (2\mathcal{J}_1 - \mathcal{J}_0) \hat{*} 1. \quad (2.18)$$

The last term in (2.16) is the type IIA analogue of (2.5) coupling  $\hat{B}_2$  to the curvature polynomial  $\hat{X}_8$ , which will not play a role in the rest of this thesis. The  $\hat{B}_2$  field completion at eight derivatives is according to [29] then given by the replacements

$$\begin{aligned} \mathcal{J}_0 &\rightarrow \mathcal{J}_0(\Omega_+) + \Delta\mathcal{J}_0(\Omega_+, \hat{H}_3) = \\ &= \left( \hat{t}_8 \hat{t}_8 + \frac{1}{8} \epsilon_{10} \epsilon_{10} \right) \hat{R}^4(\Omega_+) + \frac{1}{3} \epsilon_{10} \epsilon_{10} \hat{H}_3^2 \hat{R}^3(\Omega_+) \end{aligned} \quad (2.19)$$

$$\mathcal{J}_1 \rightarrow \mathcal{J}_1(\Omega_{\pm}) = \hat{t}_8 \hat{t}_8 \hat{R}^4(\Omega_+) - \frac{1}{8} \epsilon_{10} \hat{t}_8 \hat{B}_2 \left( \hat{R}^4(\Omega_+) + \hat{R}^4(\Omega_-) \right), \quad (2.20)$$

where the Riemann tensor with respect to the connection with torsion  $\Omega_{\pm}$  is in components given by

$$\hat{R}(\Omega_{\pm})_{M_1 M_2}^{N_1 N_2} = \hat{R}_{M_1 M_2}^{N_1 N_2} \pm \hat{\nabla}_{[M_1} \hat{H}_{3 M_2]}^{N_1 N_2} + \frac{1}{2} \hat{H}_{3 [M_1}^{N_1 N_3} \hat{H}_{3 M_2] N_3}^{N_2}. \quad (2.21)$$

For the detailed structure of the coupling  $\epsilon_{10} \epsilon_{10} \hat{H}_3^2 \hat{R}^3$  we again refer the reader to appendix B. The terms generated by this replacement up to quadratic order in the (NS,NS) three-form field strength are in (2.9) denoted by  $S_{\hat{H}^2}$ .

However, as we will explain in section 3.2, the replacements (2.19) and (2.20) applied to the tree-level terms appear not fully consistent with four-dimensional supersymmetry. We therefore propose a corrected replacement given by

$$\begin{aligned} \mathcal{J}_0 &\rightarrow \mathcal{J}_0(\Omega_+) + \Delta\mathcal{J}_0(\Omega_+, \hat{H}_3) + \delta\mathcal{J} \\ \mathcal{J}_1 &\rightarrow \mathcal{J}_1(\Omega_+) + \frac{1}{2} \delta\mathcal{J}, \quad \text{where} \\ \delta\mathcal{J} &= -2 \int_{\mathcal{M}_{10}} \hat{t}_8 \hat{t}_8 \hat{H}_3^2 \hat{R}^3 \hat{*} 1. \end{aligned} \quad (2.22)$$

The explicit index expression of (2.22) is given by

$$\hat{t}_8 \hat{t}_8 \hat{H}_3^2 \hat{R}^3 = \hat{t}_8^{M_1 \dots M_2} \hat{t}_8^{N_1 \dots N_8} \hat{H}_{3 M_1 M_2 M} \hat{H}_3^{N_1 N_2 M} \hat{R}^{N_3 N_4}_{M_3 M_4} \dots \hat{R}^{N_7 N_8}_{M_7 M_8}. \quad (2.23)$$

Notice, that the structure of the modified replacement (2.22) is such that the tree-level terms get modified, whereas the one-loop terms remain untouched.

Furthermore, note that the index contraction of (2.23) is such that it cannot be obtained via a replacement (2.21) applied to the  $\hat{R}^4$ -terms.<sup>19</sup>

## 2.2 Calabi–Yau solutions with corrections in M-theory

In the following we will focus on solutions of eleven-dimensional supergravity with eight-derivative terms, which at lowest order in  $\hat{\alpha}$  reduce to the standard Calabi–Yau backgrounds.

### 2.2.1 Calabi–Yau threefold solution

We will now search for a background solution to the equations of motion of 11d supergravity, including the higher derivative corrections (2.3). The solution we are after should have the property that it reduces to the classical direct product solution of five-dimensional Minkowski spacetime and a compact Calabi–Yau threefold. Concretely, we demand that at lowest order in the expansion parameter  $\hat{\alpha}$  the geometry is given by  $\mathcal{M}_{11} = \mathbb{R}^{1,4} \times \text{CY}_3$ , with no fluxes threading any four-cycles in the geometry. Let us note that that similar to the discussion here, corrections from higher derivatives to classical geometries were studied in the case of a product manifold  $\mathcal{M}_{11} = \mathbb{R}^{1,2} \times \mathcal{M}_7$ , where  $\mathcal{M}_7$  is a (fluxed) Calabi–Yau fourfold [43–45], a Spin(7) holonomy manifold [46], and an internal manifold with  $G_2$  holonomy [47].

The ansatz for the fourfold solution in [44, 45] involves a warp factor, fluxes and an overall Weyl rescaling. The necessity for warping and fluxes can be traced back to the fact, that the eight-form curvature polynomial  $\hat{X}_8$  does not vanish on the internal Calabi–Yau manifold and one therefore has to include fluxes in order to ensure, that the  $\hat{C}_3$  equation of motion is satisfied. In our case the situation is different, since the eight-form  $\hat{X}_8$  trivially vanishes on the Calabi–Yau threefold  $\text{CY}_3$ . Therefore, we take the following ansatz for our background solution

$$\begin{aligned} \langle d\hat{s}^2 \rangle &= e^{\hat{\alpha}^2 \langle \Phi^{(2)}(y^m) \rangle} \left( \eta_{\mu\nu} dx^\mu dx^\nu + \left( g_{mn}^{(0)} + \hat{\alpha}^2 g_{mn}^{(2)} \right) dy^m dy^n \right), \\ \langle \hat{G}_4 \rangle &= 0, \end{aligned} \quad (2.24)$$

where  $g_{mn}^{(0)}$  is the Ricci-flat Calabi–Yau metric. The variation of the quantum corrected action (2.3) then gives rise to the following conditions on the corrections  $\langle \Phi^{(2)}(y^m) \rangle$  and  $g_{mn}^{(2)}(y^m)$ .

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<sup>19</sup>For a state of the art overview of tree-level eight-derivative terms in the (NS,NS) sector we refer the reader to [42].

**external Einstein equation:**

$$g^{(0)mn} R_{mn}^{(2)} - 9g^{(0)mn} \nabla_m^{(0)} \nabla_n^{(0)} \langle \Phi^{(2)} \rangle = 0 \quad (2.25)$$

**internal Einstein equations:**

$$\begin{aligned} 0 = R_{mn}^{(2)} - \frac{1}{2} g_{mn}^{(0)} g^{(0)kl} R_{kl}^{(2)} - \frac{9}{2} \nabla_m^{(0)} \nabla_n^{(0)} \langle \Phi^{(2)} \rangle + \frac{9}{2} g_{mn}^{(0)} \nabla_k^{(0)} \nabla^{(0)k} \langle \Phi^{(2)} \rangle \\ + 768(2\pi)^3 J_m^{(0)l} J_n^{(0)k} \nabla_l^{(0)} \nabla_k^{(0)} (*_6^{(0)} c_3^{(0)}) . \end{aligned} \quad (2.26)$$

We have defined  $R_{mn}(g^{(0)} + \hat{\alpha}^2 g^{(2)}) \equiv R_{mn}^{(0)} + \hat{\alpha}^2 R_{mn}^{(2)}$  and  $J_m^{(0)n}$  is the lowest order complex structure. The mixed internal/external Einstein equations are trivially satisfied on our ansatz. In order to fix the overall Weyl factor  $\langle \Phi^{(2)} \rangle$  one takes the trace over the internal Einstein equation and eliminates all expressions involving  $R_{mn}^{(2)}$  by making use of the external Einstein equation resulting in

$$\langle \Phi^{(2)} \rangle = -\frac{512}{3} (2\pi)^3 *_6^{(0)} c_3^{(0)}, \quad (2.27)$$

$$R_{mn}^{(2)} = -768(2\pi)^3 (\nabla_m^{(0)} \nabla_n^{(0)} + J_m^{(0)l} J_n^{(0)k} \nabla_l^{(0)} \nabla_k^{(0)}) *_6^{(0)} c_3^{(0)}. \quad (2.28)$$

It is important to notice that the correction to the internal Ricci tensor is governed by an expression which is twice a covariant derivative of the Hodge dual of the third Chern form. The strategy to solve this equation is to split  $*_6^{(0)} c_3^{(0)}$  into a part which is constant on the internal space and therefore drops out of the equation of motion and a part which varies non trivially over  $\text{CY}_3$ . To make this separation one uses the fact that the third Chern form satisfies  $\text{dc}_3^{(0)} = 0$  but  $\text{d}^\dagger c_3^{(0)} \neq 0$  and can therefore be expanded as

$$c_3^{(0)} = \Pi_H c_3^{(0)} + i\partial\bar{\partial}\xi \quad (2.29)$$

by applying the  $\partial\bar{\partial}$ -Lemma, where  $\Pi_H c_3^{(0)}$  is the harmonic part of  $c_3^{(0)}$  which is unique by virtue of the Hodge decomposition theorem and  $\xi$  is a  $(2,2)$ -form satisfying  $\partial^\dagger \xi = \bar{\partial}^\dagger \xi = 0$ . The unspecified  $(2,2)$ -form parametrizes the non-harmonicity of the third Chern form. Since  $\Pi_H c_3^{(0)}$  is harmonic by definition and  $[\Delta_d, *_6] = 0$  it follows immediately that the scalar function  $h(y^m) \equiv *_6^{(0)} \Pi_H c_3^{(0)}$  is constant on the compact  $\text{CY}_3$  and therefore drops out of the equations of

motion. Furthermore, using that on a Kähler manifold the Laplacian satisfies  $\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}$  one shows that

$$i *^{(0)}_6 \partial \bar\partial \xi = -\frac{1}{2} \Delta^{(0)} *^{(0)}_6 (J^{(0)} \wedge \xi), \quad (2.30)$$

where  $J^{(0)}$  is the Kähler form of  $\text{CY}_3$  and  $\Delta^{(0)} = \nabla_k^{(0)} \nabla^{(0)k}$  is the Laplace-Beltrami operator. Having determined the non-trivial part of the correction to the internal Ricci tensor the equation determining  $g^{(2)}$  reads

$$R_{mn}^{(2)} = 384(2\pi)^3 \left( \nabla_m^{(0)} \nabla_n^{(0)} \nabla_k^{(0)} \nabla^{(0)k} + J_m^{(0)r} J_n^{(0)s} \nabla_r^{(0)} \nabla_s^{(0)} \nabla_k^{(0)} \nabla^{(0)k} \right) *^{(0)}_6 (J^{(0)} \wedge \xi) \quad (2.31)$$

whose solution is given by

$$g_{mn}^{(2)} = -768(2\pi)^3 \left( J_m^{(0)k} J_n^{(0)l} \nabla_k^{(0)} \nabla_l^{(0)} + \nabla_m^{(0)} \nabla_n^{(0)} \right) *^{(0)}_6 (J^{(0)} \wedge \xi). \quad (2.32)$$

One observes that the correction to the metric (2.32) is twice the derivative of a scalar function. In fact the metric is Kähler. We stress that it is crucial for the derivation of a consistent two-derivative effective action to perform the dimensional reduction on the fully backreacted background solution, as we will see in section 3.1. In all subsequent M-theory  $\text{CY}_3$  compactifications we work in conventions such that

$$\int_{\text{CY}_3} d^6 y \sqrt{g^{(0)}} = \frac{1}{3!} \int_{\text{CY}_3} J^{(0)3} = (2\pi\ell_M)^6 \mathcal{V}^{(0)} = \mathcal{V}^{(0)}, \quad (2.33)$$

where  $J^{(0)}$  is the Kähler form of the zeroth order Calabi-Yau background and  $\mathcal{V}^{(0)}$  is its corresponding volume.<sup>20</sup> We furthermore choose units  $(2\pi\ell_M) = 1$ .

### 2.2.2 Calabi–Yau fourfold solution

In this section we review the fourfold solutions including eight-derivative terms studied in [43–45]. The background solution is again taken to be an expansion in terms of the dimensionful parameter  $\hat{\alpha}$ , defined in (2.2), which reduces to the ordinary direct product solution  $\mathbb{R}^{1,2} \times \text{CY}_4$  without fluxes and warping to lowest order in  $\hat{\alpha}$ . All terms up to and including  $\mathcal{O}(\hat{\alpha}^2)$  are taken into account, while

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<sup>20</sup>Due to (2.33) we also have that  $\kappa_{11}^{-2} = \kappa_5^{-2} = 4\pi$ .

higher powers are neglected. At second order both a warp-factor  $\langle A^{(2)} \rangle$  and fluxes are induced. The solution then takes the form

$$\langle d\hat{s}^2 \rangle = e^{\hat{\alpha}^2 \langle \Phi^{(2)} \rangle} \left( e^{-2\hat{\alpha}^2 \langle A^{(2)} \rangle} \eta_{\mu\nu} dx^\mu dx^\nu + 2e^{\hat{\alpha}^2 \langle A^{(2)} \rangle} g_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} \right), \quad (2.34)$$

$$\langle \hat{G} \rangle = \hat{\alpha} G^{(1)} + \text{dvol}_{\mathbb{R}^{1,2}} \wedge d(e^{-3\hat{\alpha}^2 \langle A^{(2)} \rangle}), \quad (2.35)$$

where  $\text{dvol}_{\mathbb{R}^{1,2}}$  denotes the volume form on the external spacetime. Using this ansatz one can work out the constraints following from the equations of motion. It turns out that the metric  $g_{i\bar{j}}$  is, similar to the threefold case in section 2.2.1, given by an expansion

$$g_{i\bar{j}} = g_{i\bar{j}}^{(0)} + \hat{\alpha}^2 g_{i\bar{j}}^{(2)}, \quad g_{i\bar{j}}^{(2)} \sim \partial_i \bar{\partial}_{\bar{j}} *_8^{(0)} (J^{(0)} \wedge J^{(0)} \wedge F_4), \quad (2.36)$$

where  $g^{(0)}$  is the lowest order Ricci-flat Calabi–Yau metric and  $J^{(0)}$  is its associated Kähler form. We furthermore denote with  $F_4$  the non-harmonic part of the third Chern form, which will however be irrelevant for the following discussion, as it drops out of all expressions in the effective action. The metric solution also includes an overall Weyl factor  $\langle \Phi^{(2)} \rangle = -\frac{512}{3}(2\pi)^3 *_8^{(0)} (c_3^{(0)} \wedge J^{(0)})$  and a warp-factor  $A^{(2)}(z, \bar{z})$  satisfying the warp-factor equation

$$\Delta^{(0)} e^{3\hat{\alpha}^2 \langle A^{(2)} \rangle} \text{dvol}_{\text{CY}_4}^{(0)} + \frac{1}{2} \hat{\alpha}^2 G^{(1)} \wedge G^{(1)} - 3^2 2^{13} \hat{\alpha}^2 X_8^{(0)} = 0. \quad (2.37)$$

The background value of the four-form field strength is parametrized by the internal flux  $G^{(1)} \in H^4(\text{CY}_4)$ , which is self-dual with respect to the lowest order Calabi–Yau metric, and a piece proportional to the volume form  $\text{dvol}_{\mathbb{R}^{1,2}}$  on  $\mathbb{R}^{1,2}$ . In order to obtain a supersymmetric background, the internal flux must be chosen to be of  $(2, 2)$  type with respect to the lowest order complex structure. The corrected supersymmetric fourfold background is not Calabi–Yau anymore, but still (conformally) Kähler. In fact, the corrected manifolds are still very closely related to Calabi–Yau manifolds. In the language of  $\text{SU}(4)$  structure manifolds, there is only one non-vanishing but exact torsion form for the corrected geometry.

## 2.3 Calabi–Yau threefold solutions with corrections in type IIA

We next aim to find a background solution to the equations of motion of type IIA supergravity including higher derivatives (2.9). At lowest order in  $\alpha'$  preserving four-dimensional  $\mathcal{N} = 2$  supersymmetry implies that the ten-dimensional space-time is a product manifold  $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times \text{CY}_3$  with a constant dilaton and no

background fluxes. In the presence of eight-derivative couplings at order  $\alpha^3$ , we expect the classical Calabi–Yau solution to obtain non-trivial corrections. We make an ansatz for the corrected solution, which at lowest order in the expansion parameter  $\alpha$  reduces to the classical CY<sub>3</sub> solution. Therefore, we make the ansatz<sup>21</sup>

$$\begin{aligned}\langle d\hat{s}^2 \rangle &= \eta_{\mu\nu} dx^\mu dx^\nu + (g_{mn}^{(0)} + \alpha g_{mn}^{(1)}) dy^m dy^n, \\ \langle \hat{\phi} \rangle &= \phi_0 + \alpha \langle \phi^{(1)}(y) \rangle, \\ \langle \hat{H}_3 \rangle &= 0.\end{aligned}\tag{2.38}$$

The goal is now to fix the correction to the dilaton  $\langle \phi^{(1)} \rangle$  as well as the correction to the metric  $g_{mn}^{(1)}$ . Varying the corrected action (2.9) with respect to the fields and evaluating the resulting equations of motion on the ansatz (2.38) yields the following equations.

**external Einstein equation:**

$$g^{(0)mn} R_{mn}^{(1)} + 4 \nabla_n^{(0)} \nabla^{(0)n} \langle \phi^{(1)}(y) \rangle = 0 \tag{2.39}$$

**internal Einstein equations:**

$$\begin{aligned}R_{mn}^{(1)} - \frac{1}{2} g_{mn}^{(0)} g^{(0)kl} R_{kl}^{(1)} - 2 g_{mn}^{(0)} \nabla_n^{(0)} \nabla^{(0)k} \langle \phi^{(1)}(y) \rangle + 2 \nabla_m^{(0)} \nabla_n^{(0)} \langle \phi^{(1)}(y) \rangle \\ + 768 (2\pi)^3 \left( \zeta(3) + \frac{\pi^2}{3} e^{2\phi_0} \right) J_{m \quad n}^{(0) \quad r} J_n^{(0) \quad s} \nabla_r^{(0)} \nabla_s^{(0)} (*_6^{(0)} c_3^{(0)}) = 0.\end{aligned}\tag{2.40}$$

In (2.39) and (2.40) we have again employed the definition  $R_{mn}(g^{(0)} + \alpha g^{(1)}) \equiv R_{mn}^{(0)} + \alpha R_{mn}^{(1)}$ . The equation of motion for the (NS,NS) two-form is trivially satisfied and the equation of motion of the dilaton is implied by the external Einstein equation. Compatibility of the external and internal Einstein equation is imposed by taking the trace of the internal Einstein equation and comparing it to the external Einstein equation, which in turn fixes the correction of the dilaton to

$$\langle \phi^{(1)}(y) \rangle = 384 (2\pi)^3 \left( \zeta(3) + \frac{\pi^2}{3} e^{2\phi_0} \right) *_6^{(0)} c_3^{(0)}.\tag{2.41}$$

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<sup>21</sup>We did not include a Weyl factor in the ansatz here, since it turns out to be trivial in this case.



Inserting (2.41) into the internal Einstein equation results in

$$R_{mn}^{(1)} = -768 (2\pi)^3 \left( \zeta(3) + \frac{\pi^2}{3} e^{2\phi_0} \right) \left( \nabla_m^{(0)} \nabla_n^{(0)} + J_m^{(0) r} J_n^{(0) s} \nabla_r^{(0)} \nabla_s^{(0)} \right) *_6^{(0)} c_3^{(0)}. \quad (2.42)$$

Going to complex indices in (2.42) shows that  $R_{i\bar{j}}^{(1)} \propto \partial_i \bar{\partial}_{\bar{j}} (*_6^{(0)} c_3^{(0)})$ , which in turn has a solution  $g_{i\bar{j}}^{(1)} \sim \partial_i \bar{\partial}_{\bar{j}} f(y)$  for some specific function  $f$  which depends on the compact manifold and serves of as a Kähler potential. The precise form of  $f$  can be computed explicitly, as for the case of M-theory in section 2.2.1. Since the procedure is the same and the precise form of  $f$  seems to be of no physical importance, as the metric correction drops out of the effective action, we will refrain from giving the explicit expression here. We will again use conventions for type IIA compactifications such that the volume of the lowest order background metric is normalized as

$$\int_{\text{CY}_3} d^6 y \sqrt{g^{(0)}} = \frac{1}{3!} \int_{\text{CY}_3} J^{(0)3} = (2\pi\ell_s)^6 \mathcal{V}^{(0)} = \mathcal{V}^{(0)}, \quad (2.43)$$

where we chose units  $(2\pi\ell_s) = 1$ .



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## Chapter 3

# Dimensional reduction with Kähler moduli: general case

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The next step we would like to take is to perform a dimensional reduction on the corrected background solutions discussed in chapter 2. This includes a dimensional reduction of the effective action of M-theory at eight derivatives as well as type II supergravity with eight-derivative terms. In order to perform the dimensional reduction, we will need to specify the reduction ansatz and the dynamical fields which we consider. Starting from this reduction ansatz, we then obtain the lower-dimensional low-energy effective action by integrating over the compact internal space.

The main complication in dimensional reductions with higher derivatives lies in the unwieldy and seemingly unstructured expressions one obtains after inserting the reduction ansatz into the higher-dimensional action. In order to recognize any structure in the effective action, it is therefore of crucial importance to identify a minimal set of building blocks, of which the couplings entering the lower-dimensional effective action are composed. To this end, numerous integrations by parts and Schouten identities are needed to simplify the integrals which have to be evaluated in the compactification.

To make sure that all possibilities to simplify the expressions are exhausted, we developed a computer algorithm, which systematically generates all Schouten

identities and total derivatives of expressions with a given structure.<sup>22</sup> Armed with this toolkit, we are then able to show that at four-derivative level all couplings we compute descend from a single building block.

In section 3.1 we consider M-theory compactified on Calabi–Yau threefolds to five dimensions in the presence of higher-derivative corrections. We first review the general formulation of five dimensional two-derivative supergravity in section 3.1.1. Corrections to the two-derivative effective action of the Kähler moduli and the vector modes descending from the M-theory three-form potential are then computed in section 3.1.2. We find that a scalar field residing in the universal hypermultiplet, which is interpreted as the overall volume of the Calabi–Yau threefold, gets renormalized by a constant shift proportional to the Euler characteristic. In section 3.1.3 we compute four-derivative terms of the Kähler deformations and show, that they are governed by a single object. Section 3.2 is devoted to Calabi–Yau threefold compactifications of type IIA with higher derivatives. We study corrections to the two-derivative effective action for the Kähler moduli and the axion from the (NS,NS) two-form, and their compatibility with  $\mathcal{N} = 2$  supersymmetry in four dimensions. We find tension between a tree-level (in  $g_s$ ) correction and supersymmetry, which we resolve by proposing an additional eight-derivative tree-level coupling in ten dimensions.<sup>23</sup> We close this chapter by computing four-derivative couplings for the Kähler moduli in type IIA in section 3.2.2. Appendix 3.A contains an extension of section 3.1.2 to include complex structure moduli.

### 3.1 M-theory on $CY_3$ with higher-derivative corrections

We are now in a position to study the dimensional reduction of M-theory on the background solution found in section 2.2.1. The effective action in five dimensions is expected to preserve  $\mathcal{N} = 2$  supersymmetry. We perturb the background solution and derive the two-derivative effective action as well as four-derivative operators up to second order in the Kähler class deformations.

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<sup>22</sup>Such a structure could for example schematically be of the form  $(R^{(0)})^2(\nabla^{(0)}\omega)^2$ . This notation stands for any possible index contraction of two Riemann tensors on  $CY_3$  and two covariant derivatives of harmonic  $(1, 1)$ -forms.

<sup>23</sup>This additional tree-level coupling was later confirmed by a string amplitude computation in [42].

### 3.1.1 $\mathcal{N} = 2$ supergravity in five dimensions

For later reference the basic ingredients of five-dimensional  $\mathcal{N} = 2$  ungauged supergravity are collected. In the dimensional reduction of M-theory we focus entirely on the massless sector. For this reason, the relevant massless multiplets are given in table 3.1.1. Note that the entire tensor multiplet can be dualized into

multiplet	bosonic field content	# of multiplets
gravity multiplet	metric $g_{\mu\nu}$ , graviphoton $A_\mu^0$	1
tensor multiplet	tensor $B_{\mu\nu}$ , real scalar $\phi$	$n_T$
vector multiplet	vector $A_\mu^a$ , real scalar $L^a$	$n_V$
hypermultiplet	four real scalars $q^{u=1,\dots,4}$	$n_H$

Table 3.1: Multiplets of five-dimensional  $\mathcal{N} = 2$  supergravity and their field content.

a vector multiplet since in five dimensions a two-form is dual to a vector. Let us now turn to the geometry of the scalar field space in the various multiplets. Since the main focus will lie on the vector- and gravity multiplet, we will only briefly mention the hypermultiplet sector. The scalar field space  $\mathcal{M}_{\text{scalar}}$  is locally given by a direct product [48]

$$\mathcal{M}_{\text{scalar}} = \mathcal{M}_{\text{real sp.}} \times \mathcal{M}_{\text{quat. K\"ah.}}, \quad (3.1)$$

where  $\mathcal{M}_{\text{quat. K\"ah.}}$  is a quaternionic K\"ahler manifold parametrized by the hypermultiplet scalars. The vector multiplet scalar geometry is encoded in a real very special manifold  $\mathcal{M}_{\text{real sp.}}$  whose metric is determined by a cubic potential. This sector is highly restricted and allows for precise tests of corrections induced by higher-derivative couplings, based on supersymmetry in five dimensions. The vector multiplet scalar geometry is described by the  $(n_V + 1)$  very special coordinates  $L^a$ , where  $a = 0, \dots, n_V$  exceeds the number counting the physical vector multiplet scalars by one. However in the end, the scalars  $L^a$  parametrize only  $n_V$  degrees of freedom. This can be understood in a geometric way as follows. The scalar sector of the vector multiplet can be interpreted as a  $n_V$  dimensional submanifold embedded in an  $(n_V + 1)$ -dimensional ambient space with coordinates  $L^a$ . The hypersurface spanned by the vector multiplet scalars is defined by

a cubic polynomial, which in general takes the form

$$\mathcal{N}(L) = \frac{1}{3!} C_{abc} L^a L^b L^c, \quad (3.2)$$

where  $C_{abc}$  is constant and symmetric in its indices. The hypersurface constraint that has to be satisfied by the very special coordinates  $L^a$  is then simply given by

$$\mathcal{N}(L) = \frac{1}{3!} C_{abc} L^a L^b L^c = 1. \quad (3.3)$$

The canonical  $\mathcal{N} = 2$  supergravity action in the bosonic sector can then be written as [49, 50]

$$\begin{aligned} \kappa_5^2 S_{\text{can.}}^{(5)} = \int_{\mathcal{M}_5} & \frac{1}{2} R \star 1 - \frac{1}{2} G_{ab} dL^a \wedge \star dL^b - h_{uv} dq^u \wedge \star dq^v \\ & - \frac{1}{2} G_{ab} F^a \wedge \star F^b - \frac{1}{6} C_{abc} A^a \wedge F^b \wedge F^c. \end{aligned} \quad (3.4)$$

The notation indicates that the vector in the gravity multiplet  $A^0$  was combined with the vectors in the vector multiplets in a collective notation such that the index  $a$  takes values  $a = 0, \dots, n_V$ . The hypermultiplet metric  $h_{uv}$  does not play a major role in the following and will therefore not be explained in more detail. The restrictive nature of  $\mathcal{N} = 2$  supergravity in five dimensions follows from the constraint (3.3) and the fact that the metric for the vector multiplet scalars is determined from the cubic polynomial  $\mathcal{N}$  by

$$G_{ab} = -\frac{1}{2} \partial_{L^a} \partial_{L^b} \log \mathcal{N} \Big|_{\mathcal{N}=1} = -\frac{1}{2} \mathcal{N}_{ab} + \frac{1}{2} \mathcal{N}_a \mathcal{N}_b \Big|_{\mathcal{N}=1}, \quad (3.5)$$

where the notation  $\mathcal{N}_a \equiv \partial_{L^a} \mathcal{N}$  was introduced. Since the cubic potential  $\mathcal{N}(L)$  is fully determined by the set of constants  $C_{abc}$ , the geometry of the vector multiplet moduli space is fully determined by the Chern-Simons couplings of the vectors in (3.4).

### 3.1.2 The two-derivative effective action

The first step in our analysis will be the derivation of the two-derivative effective action for the gravity- and vector multiplet fields as well as a hypermultiplet scalar, which in the classical case is the volume modulus of  $\text{CY}_3$ . To perform the dimensional reduction of M-theory we perturb the background solution found in section 2.2.1. As already mentioned, a crucial observation is the fact, that the correction to the Calabi-Yau metric  $g_{i\bar{j}}^{(2)}$  drops out of the effective action [51].

Effectively, the dimensional reduction of (2.3) can therefore be performed with the reduction ansatz

$$d\hat{s}^2 = e^{\hat{\alpha}^2 \Phi^{(2)}} \left( g_{\mu\nu} dx^\mu dx^\nu + 2(g_{i\bar{j}}^{(0)} - i\delta v^a \omega_{a i\bar{j}}) dz^i d\bar{z}^{\bar{j}} \right), \quad (3.6)$$

$$\Phi^{(2)} = -\frac{512}{3}(2\pi)^3 *_6 c_3 = \langle \Phi^{(2)} \rangle + \langle \partial_a \Phi^{(2)} \rangle \delta v^a + \frac{1}{2} \langle \partial_a \partial_b \Phi^{(2)} \rangle \delta v^a \delta v^b + \mathcal{O}(\delta v^3),$$

where the deformations of the Kähler class of  $CY_3$  parametrized by  $\delta v^a$  are expanded in a real basis  $\omega_a \in H^{1,1}(CY_3)$ ,  $a = 1, \dots, h^{1,1}$  as

$$\delta g_{i\bar{j}} = -i\delta v^a \omega_{a i\bar{j}}, \quad (3.7)$$

and we have used the notation  $\partial_a \equiv \partial_{v^a}$ . In the zero-mode expansion of the M-theory three-form we only keep terms contributing to the vector and gravity multiplet in five dimensions. More precisely, we only take into account modes giving rise to vectors  $A^a$  in five dimensions. The expansion is thus

$$\hat{C}_3 = A^a \wedge \omega_a, \quad \hat{G}_4 = d\hat{C}_3 = F^a \wedge \omega_a, \quad (3.8)$$

i.e. along the  $H^{1,1}(CY_3)$  cohomology. In principle, the massless modes in the effective theory with higher derivatives do not have to coincide with the ones from the classical reduction. The reason for this is that the linearized equations of motion, which are solved by the massless modes, can receive non-trivial corrections. Along the lines of [51], using that the massless deformations of the corrected background in the compactification ansatz should preserve the Kähler condition and the Bianchi identity for the four-form field strength, it is possible to show on general grounds that the corrections to the massless fields at most contribute as total derivatives to the effective action. Thus, we may ignore these corrections in the following and treat the perturbations as the ones of the classical M-theory reduction on a Calabi–Yau threefold.

We will proceed by recording the results of the contributions of the classical and the eight-derivative action to the kinetic terms separately. We will also consider all contributions at quadratic order without any derivatives on the external spacetime, i.e. terms contributing to a scalar potential.

### Reduction of the two-derivative action.

First let us perform the dimensional reduction of the classical Einstein-Hilbert term on the perturbed and  $\ell_M$ -corrected background (3.6). Focusing on terms carrying two derivatives in five dimensions up to second order in the fluctuations

we obtain

$$\begin{aligned} \int_{\mathcal{M}_{11}} \hat{R} \hat{*} 1 \Big|_{\text{kin.}} &= \int_{\mathcal{M}_5} (\mathcal{V}_M - 768 (2\pi)^3 \hat{\alpha}^2 \chi(\text{CY}_3)) R \star 1 \\ &+ \int_{\mathcal{M}_5} d\delta v^a \wedge \star d\delta v^b \int_{\text{CY}_3} \left( \frac{1}{2} \omega_{a i \bar{j}} \omega_b^{\bar{j} i} - \omega_{a i}^i \omega_{b j}^j \right) *_6^{(0)} 1 \\ &- \int_{\mathcal{M}_5} 768 \hat{\alpha}^2 \left( \frac{1}{2} \mathcal{R}_{ab} + \mathcal{T}_{ab} \right) d\delta v^a \wedge \star d\delta v^b, \end{aligned} \quad (3.9)$$

where we made use of the shorthand notation

$$\mathcal{V}_M = \int_{\text{CY}_3} \left[ 1 - i\delta v^a \omega_{a i}^i + \frac{1}{2} \left( \omega_{a i \bar{j}} \omega_b^{\bar{j} i} - \omega_{a i}^i \omega_{b j}^j \right) \delta v^a \delta v^b \right] *_6^{(0)} 1, \quad (3.10)$$

$$\mathcal{R}_{ab} = (2\pi)^3 \int_{\text{CY}_3} \omega_{a i \bar{j}} \omega_b^{\bar{j} i} c_3^{(0)}, \quad (3.11)$$

$$\mathcal{T}_{ab} = (2\pi)^3 \int_{\text{CY}_3} \omega_{a i}^i \omega_{b j}^j c_3^{(0)}. \quad (3.12)$$

From the classical action we furthermore pick up a correction to the kinetic terms of the vectors and a Chern-Simons term in five dimensions

$$\begin{aligned} \int_{\mathcal{M}_{11}} -\frac{1}{2} \hat{G} \wedge \hat{*} \hat{G}_4 - \frac{1}{6} \hat{C}_3 \wedge \hat{G}_4 \wedge \hat{G}_4 \Big|_{\text{kin.} + \text{C.S.}} &= \int_{\mathcal{M}_5} \frac{1}{2} F^a \wedge \star F^b \int_{\text{CY}_3} \omega_{a i \bar{j}} \omega_b^{\bar{j} i} *_6^{(0)} 1 \\ &- \int_{\mathcal{M}_5} 128 \hat{\alpha}^2 \mathcal{R}_{ab} F^a \wedge \star F^b \\ &- \int_{\mathcal{M}_5} \frac{1}{6} \mathcal{K}_{abc} A^a \wedge F^b \wedge F^c, \end{aligned} \quad (3.13)$$

where we introduced the triple intersection numbers

$$\mathcal{K}_{abc} = \int_{\text{CY}_3} \omega_a \wedge \omega_b \wedge \omega_c \quad (3.14)$$

on  $\text{CY}_3$ , which appear as the coefficients of the Chern-Simons term.

### Reduction of the eight-derivative action.

We obtain further contributions to the kinetic terms of the five-dimensional theory by reducing the eight-derivative terms in the action (2.3) on the lowest order Calabi-Yau background. The  $\hat{R}^4$ -terms (2.4) lead to a correction to the kinetic term of the Kähler class deformations and a correction to the Ricci scalar

$$2\kappa_5^2 S_{\hat{R}^4} \Big|_{\text{kin.}} = \int_{\mathcal{M}_5} 768 (2\pi)^3 \chi(\text{CY}_3) R \star 1 + 384 \mathcal{R}_{ab} d\delta v^a \wedge \star d\delta v^b, \quad (3.15)$$



and from (2.6) and (2.7) we obtain the corrections to the kinetic terms of the vectors

$$2\kappa_5^2(S_{\hat{G}^2\hat{R}^3} + S_{(\hat{\nabla}\hat{G})^2R^2})|_{\text{kin.}} = \int_{\mathcal{M}_5} 384 \mathcal{R}_{ab} F^a \wedge \star F^b. \quad (3.16)$$

Note that in order to obtain the result (3.16) we had to fix  $a_1 = a_2$  in  $\hat{s}_{18}$ . This is necessary to arrive at an expression, which is solely built of internal space Riemann tensors and harmonic (1,1)-forms without explicit derivatives after applying internal space total derivative identities. Then the final result can be shown to be independent of the unfixed coefficients  $a_n$  by applying Schouten identities. We furthermore perform the reduction of (2.8) yielding

$$2\kappa_5^2 S_{\hat{Z}\hat{G}^2}|_{\text{kin.}} = - \int_{\mathcal{M}_5} 256 \hat{\alpha}^2 \mathcal{R}_{ab} F^a \wedge \star F^b. \quad (3.17)$$

Before putting the results obtained in this section together to obtain the five-dimensional two-derivative effective action we discuss the scalar potential.

### The scalar potential.

Let us consider the higher curvature terms proportional to  $\hat{R}^4$  once again. The dimensional reduction of these, focusing on the contributions with no external derivatives, gives

$$2\kappa_5^2 S_{\hat{R}^4}|_{\text{sc.p.}} = -768(2\pi)^3 \int_{\mathcal{M}_5} \delta v^a \delta v^b \star 1 \int_{CY_3} \nabla_k^{(0)} \nabla^{(0)k} (*_6^{(0)} c_3^{(0)}) \omega_{a i \bar{j}} \omega_b^{\bar{j} i} *_6^{(0)} 1. \quad (3.18)$$

This looks like a mass term for the fluctuations  $\delta v^a$  is induced by the  $\hat{R}^4$  corrections. Another potential source for a scalar potential is the classical Einstein-Hilbert action, since it is possible to pick up a mass term if one performs the dimensional reduction on the  $\hat{\alpha}^2$ -corrected background solution (3.6). One finds from the reduction of the Einstein-Hilbert action

$$\int_{\mathcal{M}_{11}} \hat{R} \hat{*} 1 \Big|_{\text{sc.p.}} = 768(2\pi)^3 \hat{\alpha}^2 \int_{\mathcal{M}_5} \delta v^a \delta v^b \star 1 \int_{CY_3} \nabla_k^{(0)} \nabla^{(0)k} (*_6^{(0)} c_3^{(0)}) \omega_{a i \bar{j}} \omega_b^{\bar{j} i} *_6^{(0)} 1, \quad (3.19)$$

which is exactly the contribution needed to cancel the one coming from the higher curvature terms (3.18). From the reduction results (3.18) and (3.19) it is also possible to see that their contribution entirely stems from the non-harmonicity of the third Chern form given by  $i\partial\bar{\partial}\xi$ . Moreover this cancellation shows that taking into account the backreaction and expanding the perturbations around a consistent background solution is crucial for the consistency of the five-dimensional effective action.

We are now in a position to collect the various pieces (3.9) and (3.13) – (3.17) and merge them into the five-dimensional two-derivative effective action. The result is

$$\begin{aligned}
2\kappa_5^2 S^{(5)} = & \int_{\mathcal{M}_5} \mathcal{V}_M R \star 1 + \int_{\mathcal{M}_5} d\delta v^a \wedge \star d\delta v^b \int_{\text{CY}_3} \left( \frac{1}{2} \omega_{ai\bar{j}} \omega_b^{\bar{j}i} - \omega_{ai}{}^i \omega_{bj}{}^j \right) \star_6^{(0)} 1 \\
& - \int_{\mathcal{M}_5} 768 \hat{\alpha}^2 \mathcal{T}_{ab} d\delta v^a \wedge \star d\delta v^b - \frac{1}{2} \int_{\mathcal{M}_5} F^a \wedge \star F^b \int_{\text{CY}_3} \omega_a \wedge \star_6^{(0)} \omega_b \\
& - \int_{\mathcal{M}_5} \frac{1}{6} \mathcal{K}_{abc} A^a \wedge F^b \wedge F^c .
\end{aligned} \tag{3.20}$$

Note that all terms involving the coupling  $\mathcal{R}_{ab}$  dropped out of the action (3.20). These cancellations are in fact crucial for compatibility with  $\mathcal{N} = 2$  supergravity in five dimensions, as the coupling  $\mathcal{R}_{ab}$  is not proportional to the Euler characteristic  $\chi(\text{CY}_3)$ . The non-harmonic part of the third Chern class prevents us from splitting the integral to obtain an expression proportional to  $\chi(\text{CY}_3)$ . The surviving coupling  $\mathcal{T}_{ab}$  however satisfies

$$\mathcal{T}_{ab} = -(2\pi)^3 \frac{\chi(\text{CY}_3)}{\mathcal{V}^{(0)}} \mathcal{K}_a^{(0)} \mathcal{K}_b^{(0)} , \tag{3.21}$$

since the traces of the harmonic  $(1,1)$ -forms  $\omega_{ai}{}^i$  can easily be shown to be constant on  $\text{CY}_3$ . The quantity  $\mathcal{V}^{(0)}$  in (3.21) denotes the volume of the zeroth order Calabi-Yau manifold and  $\mathcal{K}_a^{(0)}$ ,  $\mathcal{K}_{ab}^{(0)}$  are contractions of the intersection numbers with the Kähler moduli evaluated in the background, whose precise form can be found in appendix A in equations (A.19)–(A.21).

We now perform a Weyl rescaling of (3.20) according to  $g_{\mu\nu} \rightarrow \mathcal{V}_M^{-2/3} g_{\mu\nu}$  and the uplift<sup>24</sup> from infinitesimal Kähler class deformations to finite fields  $v^a$  leading to the action in Einstein frame

$$\begin{aligned}
2\kappa_5^2 S^{(5)} = & \int_{\mathcal{M}_5} \left[ R \star 1 + \frac{1}{2\mathcal{V}} \left( \mathcal{K}_{ab} - \frac{5}{3\mathcal{V}} \mathcal{K}_a \mathcal{K}_b \right) dv^a \wedge \star dv^b \right. \\
& + \frac{1}{2\mathcal{V}^{\frac{1}{3}}} \left( \mathcal{K}_{ab} - \frac{1}{2\mathcal{V}} \mathcal{K}_a \mathcal{K}_b \right) F^a \wedge \star F^b - \frac{1}{6} \mathcal{K}_{abc} A^a \wedge F^b \wedge F^c \\
& \left. + 768(2\pi)^3 \hat{\alpha}^2 \frac{\chi(\text{CY}_3)}{\mathcal{V}^3} \mathcal{K}_a \mathcal{K}_b dv^a \wedge \star dv^b \right] .
\end{aligned} \tag{3.22}$$

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<sup>24</sup>The term ‘uplift’ here refers to the procedure of going from infinitesimal fluctuations  $\delta v^a$  around a reference point in moduli space  $v_0^a$  to finite fields  $v^a$ . Whenever couplings in the effective action are topological in nature, this procedure can be trivial. In non-topological cases this is significantly harder and can require an educated guess.

We used in (3.22) the definition of the Calabi-Yau volume  $\mathcal{V} = \frac{1}{3!}\mathcal{K}_{abc}v^av^bv^c$ . The goal is now to make contact with  $\mathcal{N} = 2$  supergravity outlined in section 3.1.1. It is already clear from (3.4) that we make the identification  $C_{abc} = \mathcal{K}_{abc}$ , since there are no corrections to the Chern-Simons term in five dimensions, leading to the cubic constraint

$$\mathcal{N}(L) = \frac{1}{3!}\mathcal{K}_{abc}L^aL^bL^c, \quad (3.23)$$

which the physical scalars in the vector multiplets  $L^a$  have to satisfy. This constraint is the same as in the classical case and does not get corrected. The cubic constraint is therefore solved by  $L^a = \mathcal{V}^{-\frac{1}{3}}v^a$ . Due to the relation (3.5) this data is enough to completely fix the geometry of the vectormultiplet moduli space and shows, that there are no corrections present in this sector. The correction  $\propto \chi(\text{CY}_3)$  in (3.22) must therefore belong to a hypermultiplet. Using the explicit form of the physical scalars  $L^a$  one can show that (3.22) is equivalent to

$$\begin{aligned} \kappa_5^2 S^{(5)} = & \int_{\mathcal{M}_5} \frac{1}{2} R \star 1 - \frac{1}{2} G_{ab}(L) dL^a \wedge \star dL^b - \frac{1}{2} G_{ab}(L) F^a \wedge \star F^b \\ & - \int_{\mathcal{M}_5} \frac{1}{4} d \log \mathcal{V} \wedge \star d \log \mathcal{V} - 384 \hat{\alpha}^2 \frac{\chi(\text{CY}_3)}{\mathcal{V}} d \log \mathcal{V} \wedge \star d \log \mathcal{V} \quad (3.24) \\ & - \int_{\mathcal{M}_5} \frac{1}{6} \mathcal{K}_{abc} A^a \wedge F^b \wedge F^c, \end{aligned}$$

where we used the metric  $G_{ab}$  derived from the cubic potential (3.23) given by

$$G_{ab} = -\frac{1}{2} \partial_a \partial_b \log \mathcal{N}(L) \Big|_{\mathcal{N}=1} = -\frac{1}{2} \mathcal{K}_{abc} L^c + \frac{1}{8} \mathcal{K}_{acd} \mathcal{K}_{bef} L^c L^d L^e L^f. \quad (3.25)$$

Classically, one identifies one of the hypermultiplet scalars  $D$  with  $D = -\frac{1}{2} \log \mathcal{V}$  [50], whereas when taking corrections from higher-derivative terms into account it is possible to define the corrected scalar field

$$D = -\frac{1}{2} \log (\mathcal{V} + 768 \hat{\alpha}^2 \chi(\text{CY}_3)), \quad (3.26)$$

such that the final action is

$$\begin{aligned} \kappa_5^2 S^{(5)} = & \int_{\mathcal{M}_5} \frac{1}{2} R \star 1 - \frac{1}{2} G_{ab}(L) dL^a \wedge \star dL^b - \frac{1}{2} G_{ab}(L) F^a \wedge \star F^b - dD \wedge \star dD \\ & - \int_{\mathcal{M}_5} \frac{1}{6} \mathcal{K}_{abc} A^a \wedge F^b \wedge F^c. \quad (3.27) \end{aligned}$$

We have thus shown that our dimensional reduction of M-theory at two-derivative level is compatible with  $\mathcal{N} = 2$  supergravity. The metric of the vectormultiplet coincides with the one of the classical reduction, such that the only net effect at two derivatives is a corrected field identification of one hypermultiplet scalar. This can in turn be interpreted as a renormalization of the volume of  $\text{CY}_3$  at order  $\hat{\alpha}^2$ , see (3.26). In appendix 3.A we furthermore collect some unpublished results including complex structure moduli, which reside in hypermultiplets in five dimensions. These results show that the kinetic terms of the complex structure moduli remain unmodified at  $\mathcal{O}(\hat{\alpha}^2)$  due to a non-trivial cancellation.

### 3.1.3 Four-derivative terms of the Kähler moduli

We now aim to compute four-derivative terms in the effective action including at most two fluctuations  $\delta v^a$ . Obviously, this truncation misses e.g. the four-derivative interaction of the form  $(\partial v^a)^4$  and we shall not attempt to compute them here. Performing the computation of this  $(\partial v^a)^4$  coupling would require at least an analysis up to order  $\delta v^4$ , which increases the computational complexity by an enormous amount.<sup>25</sup> We now introduce the main building block which captures the four-derivative couplings we find for the Kähler deformations. It is a non-topological, co-closed  $(2, 2)$ -form  $Z$  whose components are given by

$$Z_{i\bar{j}k\bar{l}} = \varepsilon_{i\bar{j}i_1\bar{j}_1i_2\bar{j}_2} \varepsilon_{k\bar{l}k_1\bar{l}_1k_2\bar{l}_2} R^{\bar{j}_1i_1\bar{l}_1k_1} R^{\bar{j}_2i_2\bar{l}_2k_2} \quad (3.28)$$

satisfying the relations

$$Z_{i\bar{j}k}{}^k = -2i(2\pi)^2 (*_6 c_2)_{i\bar{j}}, \quad (3.29)$$

$$Z_l{}^l{}_k{}^k = 2(2\pi)^2 *_6 (c_2 \wedge J), \quad (3.30)$$

$$Z_{i\bar{j}k}{}^k \omega_a{}^{\bar{j}i} = 2i(2\pi)^2 *_6 (c_2 \wedge \omega_a). \quad (3.31)$$

This object was already recognized to play a role in the context of  $\mathcal{N} = 2$  four-derivative couplings arising from string compactifications in [52, 53]. The four-derivative couplings from the  $\hat{R}^4$ -terms obtained by reducing them to five dimensions are

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<sup>25</sup>We will compute this coupling in the for a single Kähler modulus in chapter 4.

$$\begin{aligned}
2\kappa_5^2 S_{\hat{R}^4}|_{\text{four der.}} &= \int_{\mathcal{M}_5} 192 (c_2^{(0)} \cdot J) [R^2 \star 1 - 4R_{\mu\nu} R^{\mu\nu} \star 1 - 16 \text{tr } \mathcal{R} \wedge \star \mathcal{R}] \\
&\quad - \int_{\mathcal{M}_5} 96 c_{ab}^{(0)} [R \text{d}\delta v^a \wedge \star \text{d}\delta v^b - 4R^{\mu\nu} \partial_\mu \delta v^a \partial_\nu \delta v^b \star 1 \\
&\quad + 2(\square \delta v^a)(\square \delta v^b) \star 1], \tag{3.32}
\end{aligned}$$

where we performed external spacetime integrations by parts and defined  $\square \equiv \nabla_\mu \nabla^\mu$ . Additionally, we introduced the five-dimensional curvature two-form  $\mathcal{R}$  satisfying  $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \star 1 = -8 \text{tr } \mathcal{R} \wedge \star \mathcal{R}$  and the shorthand notation

$$\mathcal{Z}_{ab}^{(0)} = \int_{CY_3} Z_{ij\bar{k}l}^{(0)} \omega_a^{\bar{j}i} \omega_b^{\bar{l}k} *_6^{(0)} 1, \tag{3.33}$$

$$c_2^{(0)} \cdot \omega_a = (2\pi)^2 \int_{CY_3} c_2^{(0)} \wedge \omega_a, \tag{3.34}$$

$$c_2^{(0)} \cdot J = (2\pi)^2 \int_{CY_3} c_2^{(0)} \wedge (J^{(0)} + \delta v^a \omega_a), \tag{3.35}$$

which we will use frequently in the following.<sup>26</sup> Note that the coupling  $(c_2^{(0)} \cdot J)$  in the first line of (3.32) contains the perturbed Kähler form  $J = J^{(0)} + \delta v^a \omega_a$ , and the contribution at second order in the fluctuations to the same terms cancel in a non-trivial way.

In the following we will discuss the Riemann squared terms in (3.32) when moving to the five-dimensional Einstein frame. Note that the second and third line in (3.32) is already second order in the fluctuations  $\delta v^a$  such that the Weyl rescaling simply leads to an overall factor proportional to the zeroth order Calabi–Yau volume  $\mathcal{V}^{(0)}$ . The terms involving two five-dimensional Riemann tensors however come with at most linear terms in  $\delta v^a$  and get therefore a less trivial modification from the Weyl rescaling considering terms up to order  $\mathcal{O}(\delta v^2)$ . The explicit form of the action after the Weyl rescaling in terms of the fluctuations is very involved and we will not display it here. Since we are in the end interested in the corresponding action of the finite fields we will give an action, which precisely reproduces the Weyl rescaled action when expanding it in infinitesimal fluctuations. We therefore introduce the scalar

$$u = \log \mathcal{V} \quad \text{with} \quad \mathcal{V} = \frac{1}{3!} \mathcal{K}_{abc} v^a v^b v^c. \tag{3.36}$$

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<sup>26</sup>Recall that we always denote evaluation on the background with a zero superscript.

The Riemann squared action in Einstein frame then reads

$$\begin{aligned}
2\kappa_5^2 S_{R^2}^{\text{Einst.}} = & \int_{\mathcal{M}_5} 192 \hat{\alpha}^2 (c_2 \cdot J) e^{-u/3} \left[ R^2 \star 1 - 4 R_{\mu\nu} R^{\mu\nu} \star 1 - 16 \text{tr } \mathcal{R} \wedge \star \mathcal{R} \right] \\
& + \int_{\mathcal{M}_5} \frac{256}{3} \hat{\alpha}^2 (c_2 \cdot J) e^{-u/3} \left[ 6 R (\Box u) \star 1 - 6 R^{\mu\nu} \nabla_\mu \nabla_\nu u \star 1 \right. \\
& \quad \left. - 2 R du \wedge \star du + R^{\mu\nu} \partial_\mu u \partial_\nu u \star 1 + 4 (\Box u)^2 \star 1 \right], \tag{3.37}
\end{aligned}$$

where we now have the moduli dependent coupling

$$c_2 \cdot J = (2\pi)^2 \int_{\text{CY}_3} c_2 \wedge J, \quad \text{with} \quad J = v^a \omega_a. \tag{3.38}$$

In contrast to the uplift of  $(c_2^{(0)} \cdot J)$ , the uplift of the terms in the second and third line of (3.32) is significantly more tricky. This is due to the non-topological nature of the coupling  $\mathcal{Z}_{ab}^{(0)}$ . The naive guess would be to promote  $\mathcal{Z}_{ab}^{(0)} \rightarrow \mathcal{Z}_{ab} \equiv \mathcal{Z}_{ab}(v^a)$  to its moduli dependent counterpart and to lift  $\delta v^a \rightarrow v^a$ . A higher order analysis in the fluctuations  $\delta v^a$  might provide further evidence for this claim.

### 3.2 Type IIA on CY<sub>3</sub> with higher-derivative corrections

In this section we derive the four-dimensional two-derivative effective action of type IIA supergravity including both tree-level and one-loop eight-derivative terms in ten dimensions. We stress that higher-derivative terms of the ten-dimensional dilaton and the (R,R)-fields are not fully known and therefore not taken into account. We therefore focus on the  $\hat{R}^4$ -terms in ten dimensions and their conjectured  $\hat{H}_3$  completion from [29] reviewed in section 2.1.2. Since these conjectured terms are based on introducing a connection with torsion, with which the superinvariants  $\mathcal{J}_{0,1}$  given in (2.19) and (2.20) are evaluated, we will apply the same strategy to the tree-level terms in this section. One consistency check of our computations will be to obtain a known correction to the prepotential  $\mathcal{F}$ , from which the metric of the complexified Kähler moduli  $t^a = b^a + iv^a$  derives. The prepotential takes the form

$$\mathcal{F}(t) = \mathcal{F}_{\text{class}}(t) - i \frac{\zeta(3)}{2(2\pi)^3} \chi(\text{CY}_3) = \frac{1}{3!} \mathcal{K}_{abc} t^a t^b t^c - i \frac{\zeta(3)}{2(2\pi)^3} \chi(\text{CY}_3) + \dots, \tag{3.39}$$

where the ellipses denote terms  $\sim k_{ab} t^a t^b + c_a t^a$ , which do not contribute to the Kähler potential, as well as non-perturbative contributions stemming from

worldsheet instantons. This correction is well known [54–56], we however show a first derivation which is solely based on a dimensional reduction of type IIA supergravity.

### 3.2.1 The two-derivative effective action

The purpose of this section is to dimensionally reduce the ten-dimensional type IIA supergravity action including the eight-derivative corrections introduced in section 2.1.2 to four dimensions. We are considering the modified  $CY_3$  solution (2.38) including deformations of the Kähler class parametrized by the fluctuations  $\delta v^a$  as before, see (3.7). In addition we take scalars  $b^a$  from the zero-mode expansion of the (NS,NS) two-form

$$\hat{B}_2 = b^a \omega_a \quad (3.40)$$

in harmonic  $(1,1)$ -forms into account. Our focus will lie on the kinetic terms of these scalar fields. These fields combine into the complexified Kähler moduli  $t^a = b^a + i v^a$ . The geometry on the space of complexified Kähler moduli is specified by a prepotential  $\mathcal{F}(t^a)$  which can be determined from the two-derivative effective action. We will again split the reduction into two separate parts: the reduction of the classical two-derivative type IIA action on the corrected background (2.38) and the reduction of the eight-derivative terms in ten dimensions on the lowest order Calabi-Yau background.

#### Reduction of the two-derivative action.

The reduction of the classical action gives rise to the following contribution to the kinetic terms in four dimensions

$$\begin{aligned} 2\kappa_4^2 S_{\text{IIA}}^{\text{class}}|_{\text{kin.}} = & \int_{\mathcal{M}_4} [\mathcal{V}_M - 1536 (2\pi)^3 \alpha \chi(CY_3) (\ell_0 + \ell_1)] R \star 1 \\ & + \int_{\mathcal{M}_4} e^{-2\phi_0} d\delta v^a \wedge \star d\delta v^b \int_{CY_3} \left( \frac{1}{2} \omega_{a i \bar{j}} \omega_b^{\bar{j} i} - \omega_{a i}^i \omega_{b j}^j \right) \star_6^{(0)} 1 \\ & - \int_{\mathcal{M}_4} 768 \alpha (\ell_0 + \ell_1) \left( \mathcal{T}_{ab} + \frac{1}{2} \mathcal{R}_{ab} \right) d\delta v^a \wedge \star d\delta v^b \\ & + \int_{\mathcal{M}_4} \frac{1}{2} e^{-2\phi_0} db^a \wedge \star db^b \int_{CY_3} \omega_{a i \bar{j}} \omega_b^{\bar{j} i} \star_6^{(0)} 1 \\ & - \int_{\mathcal{M}_4} 384 \alpha (\ell_0 + \ell_1) \mathcal{R}_{ab} db^a \wedge \star db^b, \end{aligned} \quad (3.41)$$

where we defined the constants

$$\ell_0 = \zeta(3) e^{-2\phi_0}, \quad \ell_1 = \frac{\pi^2}{3} \quad (3.42)$$

and made use of the definition (3.10).

### Reduction of the eight-derivative action.

The reduction of the eight-derivative terms in ten dimensions yields the contribution to the action

$$\begin{aligned}
2\kappa_4^2(S_{\tilde{R}^4}^{\text{tree}} + S_{\tilde{R}^4}^{\text{loop}} + S_{\tilde{H}^2})|_{\text{kin.}} &= \int_{\mathcal{M}_4} -768(2\pi)^2(\ell_0 - \ell_1)\chi(\text{CY}_3)R \star 1 \quad (3.43) \\
&+ \int_{\mathcal{M}_4} 384(\ell_0 + \ell_1)\mathcal{R}_{ab}db^a \wedge \star db^b \\
&+ \int_{\mathcal{M}_4} 384(\ell_0 + \ell_1)\mathcal{R}_{ab}d\delta v^a \wedge \star d\delta v^b,
\end{aligned}$$

which again has the property that, combining it with the reduction results from the classical action (3.41), all couplings involving  $\mathcal{R}_{ab}$  cancel. Summing the two pieces (3.41) and (3.43) of the action in four dimensions up leads to

$$\begin{aligned}
2\kappa_4^2 S^{(4)} &= \int_{\mathcal{M}_4} \mathcal{V}_{\text{IIA}} R \star 1 + \frac{1}{2}e^{-2\phi_0} \left( \mathcal{K}_{ab}^{(0)} - \frac{1}{\mathcal{V}^{(0)}} \mathcal{K}_a^{(0)} \mathcal{K}_b^{(0)} \right) db^a \wedge \star db^b \quad (3.44) \\
&+ \int_{\mathcal{M}_4} \frac{1}{2}e^{-2\phi_0} \left( \mathcal{K}_{ab}^{(0)} + \frac{1}{\mathcal{V}^{(0)}} \mathcal{K}_a^{(0)} \mathcal{K}_b^{(0)} \right) d\delta v^a \wedge \star d\delta v^b \\
&+ \int_{\mathcal{M}_4} 768(2\pi)^3(\ell_0 + \ell_1)\alpha \frac{\chi(\text{CY}_3)}{\mathcal{V}^{(0)^2}} \mathcal{K}_a^{(0)} \mathcal{K}_b^{(0)} d\delta v^a \wedge \star d\delta v^b.
\end{aligned}$$

We furthermore introduced the notation for the prefactor of the four-dimensional Ricci scalar

$$\mathcal{V}_{\text{IIA}} = \mathcal{V}_{\text{M}} e^{-2\phi_0} - 1536(2\pi)^3 \alpha \ell_0 \chi(\text{CY}_3), \quad (3.45)$$

where  $\mathcal{V}_{\text{M}}$  is given in (3.10). This factor  $\mathcal{V}_{\text{IIA}}$  can be removed by a Weyl rescaling of the metric  $g_{\mu\nu} \rightarrow \mathcal{V}_{\text{IIA}}^{-1} g_{\mu\nu}$ . Performing this rescaling as well as the uplift from infinitesimal fluctuations to finite fields  $v^a$  leads to the effective action in Einstein frame

$$S^{(4)} = S_{\alpha'^0}^{(4)} + \alpha'^3 S_{\alpha'^3}^{(4)}, \quad (3.46)$$

where we have restored the explicit  $\alpha'$ -dependence by using (2.10). We split the action (3.46) into a classical and correction (tree-level and one-loop combined) part which are explicitly given by

$$\begin{aligned}
2\kappa_4^2 S_{\alpha'^0}^{(4)} &= \int_{\mathcal{M}_4} R \star 1 + \frac{1}{2\mathcal{V}} \left( \mathcal{K}_{ab} - \frac{1}{\mathcal{V}} \mathcal{K}_a \mathcal{K}_b \right) db^a \wedge \star db^b \quad (3.47) \\
&+ \int_{\mathcal{M}_4} \frac{1}{\mathcal{V}} \left( \frac{1}{2} \mathcal{K}_{ab} - \frac{1}{\mathcal{V}} \mathcal{K}_a \mathcal{K}_b \right) dv^a \wedge \star dv^b
\end{aligned}$$



and

$$\begin{aligned}
2\kappa_4^2 S_{\alpha'^3}^{(4)} = & \int_{\mathcal{M}_4} \frac{\chi(CY_3) \zeta(3) \pi^3}{\mathcal{V}^2} \left( \mathcal{K}_{ab} - \frac{1}{\mathcal{V}} \mathcal{K}_a \mathcal{K}_b \right) db^a \wedge \star db^b \\
& + \int_{\mathcal{M}_4} \frac{\chi(CY_3) \zeta(3) \pi^3}{\mathcal{V}^2} \left( \mathcal{K}_{ab} - \frac{4}{\mathcal{V}} \mathcal{K}_a \mathcal{K}_b \right) dv^a \wedge \star dv^b \\
& + \int_{\mathcal{M}_4} \frac{\chi(CY_3) \pi^5}{3\mathcal{V}^3} e^{2\phi_0} \mathcal{K}_a \mathcal{K}_b dv^a \wedge \star dv^b.
\end{aligned} \tag{3.48}$$

Let us now comment on the origin of the various corrections. It is evident in (3.44) that the corrections to the kinetic term of the scalars  $b^a$  cancel and can thus only come from the Weyl rescaling. This in turn implies, that the correction is at tree-level in  $g_s$ , since the contributions to the Ricci scalar for the one-loop terms precisely cancel as well. This can be traced back to the fact, that the  $\hat{R}^4$ -terms in type IIA suffer from a relative sign flip when comparing tree-level to one-loop. The Kähler moduli receive both tree-level and one-loop corrections. The one-loop corrections in (3.48) are proportional to the string coupling.

Now let us discuss the possible corrections of the vector multiplets containing the complexified Kähler moduli  $t^a$ . The vector multiplet metric can not depend on a scalar in a hypermultiplet; in particular, this is true for the dilaton, which identifies the term  $\propto e^{2\phi_0}$  in (3.48) as a correction to a hypermultiplet. It is furthermore well known, that the prepotential in type IIA obtains a tree-level (in  $g_s$ ) correction to the prepotential of the form

$$\mathcal{F}(t) = \frac{1}{3!} \mathcal{K}_{abc} t^a t^b t^c - i a \chi(CY_3), \quad a \in \mathbb{R}. \tag{3.49}$$

The constant  $a$  is already completely fixed by the terms  $\propto \mathcal{K}_{ab}$  in (3.48), while terms  $\propto \mathcal{K}_a \mathcal{K}_b dv^a \wedge \star dv^b$  can in principle be absorbed in the definition of a hypermultiplet scalar. One can then compute the metric following from the prepotential (3.49) and compare the coefficient with the one of the  $\propto \mathcal{K}_{ab}$  terms in (3.48) which leads to

$$a = (2\pi\alpha')^3 \frac{\zeta(3)}{2}. \tag{3.50}$$

Setting explicitly  $2\pi\ell_s = 1$  one obtains the desired result for the corrected prepotential

$$\mathcal{F}(t) = \frac{1}{3!} \mathcal{K}_{abc} t^a t^b t^c - i \frac{\zeta(3)}{2(2\pi)^3} \chi(CY_3). \tag{3.51}$$

Note however, that a correction  $\propto \mathcal{K}_a \mathcal{K}_b db^a \wedge \star db^b$  cannot be absorbed in the definition of a hypermultiplet scalar  $\phi_4$  with e.g. a contribution  $\phi_4 = \hat{\phi} + \mathcal{K}_a b^a + \dots$ , since it would spoil the shift symmetry of the axion  $b^a$ . In order to match the  $b^a$  sector with the metric derived from (3.51) we therefore need the additional contribution

$$2\kappa_4^2 \Delta S_b^{(4)} = - \int_{\mathcal{M}_4} \frac{\zeta(3)}{8(2\pi)^3} \chi(\text{CY}_3) \mathcal{V}^{-3} \mathcal{K}_a \mathcal{K}_b db^a \wedge \star db^b \quad (3.52)$$

from the reduction of the tree-level eight-derivative terms involving  $\hat{H}_3$ . This missing structure strongly indicates, that the logic of obtaining the  $\hat{B}_2$  field completion of the eight-derivative terms employed in [29] cannot be applied to the tree-level terms. However, if one considers the modified replacement (2.22) the new structure  $\hat{t}_8 \hat{t}_8 \hat{H}_3^2 \hat{R}^3$  gives precisely the lacking contribution (3.52).

The presence of the additional terms which we proposed in (2.22) was later confirmed in [42] by an explicit string amplitude computation. This highlights that it is possible to ‘bootstrap’ certain information about higher derivative terms in higher dimension by demanding consistency with supersymmetry in lower dimensions.

### 3.2.2 Four-derivative terms of the Kähler moduli

We proceed by determining the four-derivative terms of type IIA supergravity on a Calabi–Yau threefold in a similar way to section 3.1.3. Due to the lack of a complete eight-derivative action of the dilaton and  $\hat{B}_2$  in ten dimensions we do not take the latter into account. We thus consider the action

$$S_{\text{IIA}, \hat{R}^4} = S_{\hat{R}^4}^{\text{tree}} + S_{\hat{R}^4}^{\text{loop}}, \quad (3.53)$$

since the classical action does not lead to any four-derivative coupling in five dimensions. However, let us add a comment on the impact of the classical action on the four-derivative couplings. After the reduction, the four-derivative terms at order  $\alpha'^3$  are not in Einstein frame. Performing a Weyl rescaling of the Einstein–Hilbert term, which is after the reduction given by  $\propto e^{-2\phi} \mathcal{V} R$ , one obtains the Einstein frame four-derivative couplings. Any order  $\alpha'^3$  contribution to the Ricci scalar does not alter the form of the four-derivative terms at order  $\alpha'^3$ .

**Reduction of the tree-level eight-derivative terms.**

The tree-level terms give rise to the four-derivative couplings

$$\begin{aligned}
2\kappa_4^2 \ell_0^{-1} S_{\hat{R}^4}^{\text{tree}}|_{4 \text{ der.}} &= \int_{\mathcal{M}_4} 768 (c_2^{(0)} \cdot \omega_a) [R(\square \delta v^a) - 2R^{\mu\nu} \nabla_\mu \nabla_\nu \delta v^a] \star 1 \\
&+ \int_{\mathcal{M}_4} 768 (c_2^{(0)} \cdot J) [R_{\mu\nu} R^{\mu\nu} - 4R^2] \star 1 \\
&+ \int_{\mathcal{M}_4} 96 \mathcal{Z}_{ab}^{(0)} [R d\delta v^a \wedge \star d\delta v^b - 4R^{\mu\nu} \partial_\mu \delta v^a \partial_\nu \delta v^b \star 1] \\
&+ \int_{\mathcal{M}_4} 192 \mathcal{Z}_{ab}^{(0)} [(\square \delta v^a)(\square \delta v^b) \star 1 - 2\nabla_\mu \nabla_\nu \delta v^a \nabla^\mu \nabla^\nu \delta v^b] \star 1,
\end{aligned} \tag{3.54}$$

where again  $J = J^{(0)} + \delta v^a \omega_a$  in the coupling  $(c_2^{(0)} \cdot J)$  in the second line of (3.54). Notice that the terms in the first line in (3.54) proportional to  $(c_2^{(0)} \cdot \omega_a)$  vanish up to dilaton terms by using a contracted version of the second Bianchi identity. Performing the Weyl rescaling  $g_{\mu\nu} \rightarrow e^{2\phi_0} \mathcal{V}^{-1} g_{\mu\nu}$  one obtains the effective action in Einstein frame. Due to the appearance of the  $R^2$  terms in four dimensions the Weyl rescaling is rather involved. We therefore refrain from spelling the effective action in terms of fluctuations out but rather give an action in terms of the fields after the uplift, which precisely reproduces the action obtained in terms of the fluctuations  $\delta v^a$ . By defining  $\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu}$ , in close analogy to the Einstein tensor<sup>27</sup> one finds

$$\begin{aligned}
2\kappa_4^2 \ell_0^{-1} S_{\hat{R}^4}^{\text{tree}}|_{4 \text{ der.}} &= \int_{\mathcal{M}_4} 768 \alpha \left[ \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu} (c_2 \cdot J) + \frac{2}{\mathcal{V}} \mathcal{K}_a (c_2 \cdot J) \mathcal{G}_{\mu\nu} \nabla^\mu \nabla^\nu v^a \right] \star 1 \\
&- \int_{\mathcal{M}_4} 768 \alpha \left( \frac{1}{2} \mathcal{Z}_{ab} - \frac{2}{\mathcal{V}} \mathcal{K}_{ab} (c_2 \cdot J) \right. \\
&\quad \left. + \frac{1}{\mathcal{V}^2} \mathcal{K}_a \mathcal{K}_b (c_2 \cdot J) \right) \mathcal{G}_{\mu\nu} \nabla^\mu v^a \nabla^\nu v^b \\
&+ \int_{\mathcal{M}_4} 192 \alpha \left( \mathcal{Z}_{ab} - \frac{1}{\mathcal{V}^2} \mathcal{K}_a \mathcal{K}_b (c_2 \cdot J) \right) (\square v^a) (\square v^b) \star 1 \\
&- \int_{\mathcal{M}_4} 384 \alpha \left( \mathcal{Z}_{ab} - \frac{2}{\mathcal{V}^2} \mathcal{K}_a \mathcal{K}_b (c_2 \cdot J) \right) \nabla_\mu \nabla_\nu v^a \nabla^\mu \nabla^\nu v^b \star 1 \\
&+ \int_{\mathcal{M}_4} 768 \alpha (c_2 \cdot \omega_a) [R(\square v^a) - 2R^{\mu\nu} \nabla_\mu \nabla_\nu v^a] \star 1.
\end{aligned} \tag{3.55}$$

The last line in (3.55) involving the coupling  $c_2 \cdot \omega_a$  again vanishes up to dilaton terms upon integrating by parts. We once more stress, that the trivial uplift  $\mathcal{Z}_{ab}^{(0)} \rightarrow \mathcal{Z}_{ab}$  is not necessarily the complete answer and the role the tensor  $Z_{ijk\bar{l}}^{(0)}$

<sup>27</sup> The Einstein tensor is given by  $\mathcal{G}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ .

plays from a physical point of view is yet to be explored.

### Reduction of one-loop eight-derivative terms.

The one-loop terms in ten dimensions reduce to the four-derivative terms

$$\begin{aligned}
2\kappa_4^2 \ell_1^{-1} S_{\hat{R}^4}^{\text{loop}}|_{4 \text{ der.}} &= \int_{\mathcal{M}_4} 768 (c_2^{(0)} \cdot \omega_a) [2R^{\mu\nu} \nabla_\mu \nabla_\nu \delta v^a - R(\square \delta v^a)] \star 1 \\
&+ \int_{\mathcal{M}_4} 192 (c_2^{(0)} \cdot J) [2R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2] \star 1 \\
&- \int_{\mathcal{M}_4} 96 \mathcal{Z}_{ab}^{(0)} [R d\delta v^a \wedge \star d\delta v^b - 4R^{\mu\nu} \partial_\mu \delta v^a \partial_\nu \delta v^b \star 1 \\
&\quad + 2(\square \delta v^a)(\square \delta v^b)] \star 1. \quad (3.56)
\end{aligned}$$

In the following we will again consider the Weyl rescaling of the action (3.56). To do this we redefine the metric as  $g_{\mu\nu} \rightarrow e^{2\phi_0} \mathcal{V}^{-1} g_{\mu\nu}$  leading to a canonically normalized Einstein-Hilbert term. Performing the rescaling and the uplift to finite fields one obtains

$$\begin{aligned}
2\kappa_4^2 \ell_1^{-1} S_{\hat{R}^4}^{\text{loop}}|_{4 \text{ der.}} &= \int_{\mathcal{M}_4} 384 (c_2 \cdot J) \left[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \frac{1}{\mathcal{V}} R \mathcal{K}_a (\square v^a) - 2 \mathcal{G}_{\mu\nu} \mathcal{G}^{\mu\nu} \right] \star 1 \\
&+ \int_{\mathcal{M}_4} 384 \left[ \mathcal{Z}_{ab} \mathcal{G}_{\mu\nu} + \frac{1}{\mathcal{V}} (c_2 \cdot J) R \mathcal{K}_{ab} g_{\mu\nu} \right. \\
&\quad \left. - \frac{3}{2\mathcal{V}^2} (c_2 \cdot J) R \mathcal{K}_a \mathcal{K}_b g_{\mu\nu} \right] \nabla^\mu v^a \nabla^\nu v^b \star 1 \\
&- \int_{\mathcal{M}_4} 192 \left[ \mathcal{Z}_{ab} - \frac{3}{\mathcal{V}^2} (c_2 \cdot J) \mathcal{K}_a \mathcal{K}_b \right] (\square v^a) (\square v^b) \star 1. \quad (3.57)
\end{aligned}$$

## 3.A Complex structure deformations in M-theory on $\text{CY}_3$

In this short appendix we study another class of geometric deformations admitted by Calabi–Yau manifolds, the complex structure deformations. As the main focus in this thesis lies on Kähler deformations, we will only present the results of the dimensional reduction of complex structure moduli for  $\text{CY}_3$  compactifications of M-theory. The results contained in this appendix furnish unpublished work. These unpublished results are however included in this thesis to demonstrate that the Mathematica code developed to carry out the computations in chapter 3 can be used to include other types of fluctuations. Complex structure deformations are the zero modes of the Lichnerowicz equation with purely (anti-)holomorphic index structure, see section 1.3.2. They are counted by the cohomology groups  $H^{2,1}(\text{CY}_3)$  and  $H^{1,2}(\text{CY}_3)$  and can be expanded in contractions of harmonic  $(2, 1)$

and  $(1, 2)$ -forms with the holomorphic  $(3, 0)$  and the antiholomorphic  $(0, 3)$ -form on the  $CY_3$ . The latter are denoted by  $\Omega$  and  $\bar{\Omega}$ , respectively. The expansion then reads

$$\delta g_{ij} = \bar{b}_{\bar{\alpha} i j} \delta \bar{Z}^{\bar{\alpha}}, \quad \delta g_{\bar{i} \bar{j}} = b_{\alpha \bar{i} \bar{j}} \delta Z^{\alpha}, \quad (3.58)$$

where the quantities  $b$  and  $\bar{b}$  are defined by

$$\bar{b}_{\bar{\alpha} i}{}^{\bar{j}} = \frac{1}{2\|\Omega\|^2} \bar{\chi}_{\bar{\alpha} i \bar{k} l} \Omega^{\bar{k} l \bar{j}}, \quad b_{\alpha \bar{i}}{}^j = \frac{1}{2\|\Omega\|^2} \chi_{\alpha \bar{i} k l} \bar{\Omega}^{k l j}, \quad (3.59)$$

and we furthermore have  $\bar{\chi}_{\bar{\alpha}} \in H^{1,2}(CY_3)$  and  $\chi_{\alpha} \in H^{2,1}(CY_3)$ . Since for Calabi-Yau threefolds we have  $H^{2,1}(CY_3) \cong H^{1,2}(CY_3)$  it follows that  $\alpha, \bar{\alpha} = 1, \dots, h^{2,1}(CY_3)$ . In (3.59) we have furthermore made use of the definition

$$\|\Omega\|^2 = \frac{1}{3!} \Omega_{ijk} \bar{\Omega}^{ijk}. \quad (3.60)$$

Upon performing the dimensional reduction of (2.3) on the background (2.24) including complex structure deformations we again have to take into account two contributions.<sup>28</sup> The first source of corrections to the kinetic terms comes from the dimensional reduction of the  $\hat{R}^4$ -terms (2.4) in eleven-dimensional supergravity. Applying numerous Schouten identities and integrations by parts in the integrals over  $CY_3$  one can show that these terms give rise to a contribution

$$\int_{\mathcal{M}_{11}} \left( \hat{t}_8 \hat{t}_8 - \frac{1}{24} \epsilon_{11} \epsilon_{11} \right) \hat{R}^4 \hat{*} 1 \Big|_{\text{kin.}} = 384 (2\pi)^3 \int_{\mathcal{M}_5} d\delta Z^{\alpha} \wedge \star d\delta \bar{Z}^{\bar{\beta}} \int_{CY_3} b_{\alpha}^{ij} \bar{b}_{\bar{\beta} ij} c_3^{(0)}. \quad (3.61)$$

Note that mixing terms between Kähler- and complex structure deformations, i.e. terms like  $dv^{\alpha} \wedge \star dZ^{\alpha}$  and their complex conjugate counterparts, drop out of the final expression (3.61) in a non-trivial way using Schouten identities. It remains to reduce the two-derivative action of eleven-dimensional supergravity on the corrected background (2.3), now including complex structure deformations. The only term giving rise to kinetic terms for these geometric deformations is the Einstein-Hilbert action. One finds in this case

$$\int_{\mathcal{M}_{11}} \hat{R} \hat{*} 1 \Big|_{\text{kin.}} = -384 (2\pi)^3 \hat{\alpha}^2 \int_{\mathcal{M}_5} d\delta Z^{\alpha} \wedge \star d\delta \bar{Z}^{\bar{\beta}} \int_{CY_3} b_{\alpha}^{ij} \bar{b}_{\bar{\beta} ij} c_3^{(0)}, \quad (3.62)$$

which precisely cancels the contribution (3.61) from the  $\hat{R}^4$ -terms. We have therefore shown that the kinetic terms of the complex structure moduli remain uncorrected at  $\mathcal{O}(\hat{\alpha}^2)$ .

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<sup>28</sup>We will exclusively focus on the terms which give corrections to the kinetic terms of the complex structure deformations here.



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## Chapter 4

# Dimensional reduction: one Kähler modulus

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As discussed in chapter 3, the dimensional reduction of higher dimensional supergravity theories with higher-derivative terms can lead to insights from various perspectives. Among these are top-down derivations of corrections to effective actions of string compactifications and bottom-up ‘bootstrapping’ of higher-derivative operators in the higher-dimensional theory based on consistency with supersymmetry. Since we performed the dimensional reduction in the case of generic  $h^{1,1}$  by including fluctuations around a certain background solution, the effective action is a power series expansion in the fluctuations. For non-topological couplings such as (3.33), the uplift from fluctuations to finite fields may however not be unique and require computationally intense checks.

Chapter 4 is devoted to the study of the specific case where  $h^{1,1} = 1$ . The only Kähler modulus in this setting is the overall volume of the internal manifold. By restricting ourselves to this specific case, we are able to overcome the difficulties mentioned above. We will be able to perform the dimensional reduction with the finite volume modulus from the very beginning. Therefore, any uplifting procedure becomes redundant. In addition to this advantage we shall also see, that all couplings in the case of a single modulus are topological in nature. In section 4.1–4.3 we perform the dimensional reduction of M-theory, type IIA and type IIB for the overall volume modulus. We compute all four-derivative couplings

of the volume and curvature terms for each case. In section 4.4 we examine a recently proposed scenario of moduli stabilization in the framework of type IIB orientifolds. We compute the coupling of a certain higher-derivative operator, which is of crucial importance in the scenario, explicitly. The chapter is closed by critical remarks and potential caveats to the aforementioned scenario of moduli stabilization.

## 4.1 M-theory on $CY_3$ : four-derivative action

We now specialize to compactifications of eleven-dimensional supergravity on Calabi–Yau threefolds with  $h^{1,1}(CY_3) = 1$ . We focus on the gravitational terms in eleven dimensions in the following discussion. Compared to the case with a general number of Kähler moduli, this case is computationally less involved and allows for a complete treatment of four-derivative couplings of the Kähler modulus. The corresponding ansatz for the dimensional reduction with one modulus is

$$ds_{11}^2 = e^{\hat{\alpha}^2 \Phi^{(2)}} \left( g_{\mu\nu} dx^\mu dx^\nu + 2e^{u/3} (-i\omega_{i\bar{j}}) dz^i d\bar{z}^{\bar{j}} \right), \quad (4.1)$$

$$\Phi^{(2)} = -\frac{512}{3} (2\pi)^3 * c_3, \quad (4.2)$$

where in (4.1) the Weyl factor  $\Phi^{(2)}$  is computed using the metric  $g_{i\bar{j}} = e^{u/3} (-i\omega_{i\bar{j}})$ . Here and in the following,  $\omega$  is the harmonic  $(1,1)$ -form on the Calabi–Yau threefold. We normalize the self-intersection number of the latter to unity, i.e.

$$\frac{1}{3!} \int_{CY_3} \omega^3 = (2\pi\ell_M)^6 = 1. \quad (4.3)$$

The form  $\omega$  can be identified with the Kähler form of the unit volume Calabi–Yau threefold. The scalar field  $u(x)$  parametrizes the volume of the Calabi–Yau threefold, as can be seen from

$$\frac{1}{3!} \int_{CY_3} J^3 = e^u \frac{1}{3!} \int_{CY_3} \omega^3 = e^u. \quad (4.4)$$

As a first step we dimensionally reduce the action on the ansatz (4.1) including the  $\hat{R}^4$ -terms in eleven dimensions up to four external spacetime derivatives. One obtains the following action in five dimensions

$$\begin{aligned} 2\kappa_5^2 S^{(5)} = & \int_{\mathcal{M}_5} e^u R \star 1 + \frac{5}{6} e^u du \wedge \star du + 768 (2\pi)^3 \hat{\alpha}^2 \chi(CY_3) du \wedge \star du \\ & + \int_{\mathcal{M}_5} \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) e^{u/3} [384 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 768 R_{\mu\nu} R^{\mu\nu} + 192 R^2] \star 1 \end{aligned} \quad (4.5)$$



$$\begin{aligned}
& + \int_{\mathcal{M}_5} \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) e^{u/3} \left[ \frac{256}{3} R^{\mu\nu} \partial_\mu u \partial_\nu u \star 1 + 512 R^{\mu\nu} \nabla_\mu \nabla_\nu u \star 1 \right. \\
& \quad \left. - 256 R (\square u) \star 1 - 64 R du \wedge \star du \right] \\
& + \int_{\mathcal{M}_5} \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) e^{u/3} \left[ \frac{128}{3} (\square u)^2 \star 1 + \frac{128}{9} (\square u) du \wedge \star du + \frac{80}{9} (\partial u)^4 \star 1 \right].
\end{aligned}$$

The coupling  $c_2^{(0)} \cdot \omega$  is defined as

$$c_2^{(0)} \cdot \omega = (2\pi)^2 \int_{\text{CY}_3} c_2^{(0)} \wedge \omega, \quad (4.6)$$

where  $c_2^{(0)}$  is computed using the unit-volume metric  $g_{i\bar{j}}^{(0)} = -i\omega_{i\bar{j}}$ . We furthermore made use of the schematic notation  $(\partial u)^4 \equiv \partial_\mu u \partial^\mu u \partial_\nu u \partial^\nu u$ . For a canonical normalization of the five-dimensional Einstein-Hilbert term we perform a Weyl rescaling

$$g_{\mu\nu} \rightarrow e^{-\frac{2}{3}u} g_{\mu\nu},$$

which results in the action

$$\begin{aligned}
2\kappa_5^2 S^{(5)} = & \int_{\mathcal{M}_5} R \star 1 - \frac{1}{2} du \wedge \star du + 768 (2\pi)^3 \hat{\alpha}^2 \chi(\text{CY}_3) e^{-u} du \wedge \star du \\
& + \int_{\mathcal{M}_5} \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) \left[ 384 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 768 R_{\mu\nu} R^{\mu\nu} + 192 R^2 \right] \star 1 \\
& + \int_{\mathcal{M}_5} \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) \left[ 384 (\square u)^2 \star 1 - \frac{1280}{3} (\square u) du \wedge \star du + \frac{368}{3} (\partial u)^4 \star 1 \right] \\
& + \int_{\mathcal{M}_5} \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) \left[ 256 R (\square u) \star 1 - \frac{448}{3} R du \wedge \star du \right]
\end{aligned} \quad (4.7)$$

in Einstein frame, where we have furthermore performed integrations by parts. The form of higher-derivative actions can be altered by perturbative higher-derivative field redefinitions. We shall now propose new field variables such that the action (4.7) takes a rather simple form. If one redefines the five-dimensional metric  $g_{\mu\nu}$  and the scalars  $u$  as

$$\begin{aligned}
g_{\mu\nu} \rightarrow & g_{\mu\nu} + m_1 \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) (\square u) g_{\mu\nu} + m_2 \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) (\partial u)^2 g_{\mu\nu} \\
& + m_3 \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) R g_{\mu\nu} + m_4 \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) R_{\mu\nu} + m_5 \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) \partial_\mu u \partial_\nu u \\
& + m_6 \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) \nabla_\mu \nabla_\nu u,
\end{aligned} \quad (4.8)$$

$$u \rightarrow u + m_7 \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) (\square u) + m_8 \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) (\partial u)^2 + m_9 \hat{\alpha}^2 (c_2^{(0)} \cdot \omega) R,$$

where the nine coefficients  $m_i$  take the values

$$\begin{aligned}
m_1 &= -\frac{512}{3} & m_2 &= \frac{320}{9} & m_3 &= -128 \\
m_4 &= 768 & m_5 &= 384 & m_6 &= 0 \\
m_7 &= -384 & m_8 &= \frac{896}{3} & m_9 &= 0,
\end{aligned} \tag{4.9}$$

the five-dimensional action boils down to

$$\begin{aligned}
2\kappa_5^2 S^{(5)} &= \int_{\mathcal{M}_5} R \star 1 - \frac{1}{2} du \wedge \star du + 768(2\pi)^3 \hat{\alpha} \chi(\text{CY}_3) e^{-u} du \wedge \star du \\
&+ \int_{\mathcal{M}_5} 384 \hat{\alpha} (c_2^{(0)} \cdot \omega) [R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2] \star 1 \\
&+ \int_{\mathcal{M}_5} 192 \hat{\alpha} (c_2^{(0)} \cdot \omega) (\partial u)^4 \star 1.
\end{aligned} \tag{4.10}$$

When performing the field redefinition, terms at higher order in the expansion parameter  $\hat{\alpha}^2$  are neglected. The only four-derivative couplings in (4.10) surviving this field redefinition are the squared Riemann tensor terms in the Gauss-Bonnet combination and the  $(\partial u)^4$  interaction which leads to a particularly simple form of the effective action. Effectively, one obtains a massless scalar field  $u$  coupled to Gauss-Bonnet (super)gravity and a  $(\partial u)^4$ -like interaction term.

## 4.2 Type IIA on $\text{CY}_3$ : four-derivative action

In the dimensional reduction of type IIA supergravity for the case of  $h^{1,1} = 1$  we will take both types of  $\hat{R}^4$ -couplings into account. These are given by eight-derivative couplings induced at tree-level and one-loop in string perturbation theory, see (2.13) and (2.14). Starting from the background solution (2.38) we take the following reduction ansatz for the case of a single Kähler modulus

$$\begin{aligned}
d\hat{s}_{10}^2 &= g_{\mu\nu} dx^\mu dx^\nu + 2e^{u/3} (-i\omega_{i\bar{j}}) dz^i d\bar{z}^{\bar{j}}, \\
\hat{\phi} &= \phi_0 + \alpha \phi^{(1)}.
\end{aligned} \tag{4.11}$$

The form of the correction to the dilaton  $\phi^{(1)}$  is given in (2.41), and is computed with the metric  $g_{i\bar{j}} = e^{u/3} (-i\omega_{i\bar{j}})$ . The form  $\omega$  is again the harmonic  $(1,1)$ -form on the Calabi–Yau threefold. We normalize the self-intersection number of the latter to unity (in string units), i.e.

$$\frac{1}{3!} \int_{\text{CY}_3} \omega^3 = (2\pi\ell_s)^6 = 1. \tag{4.12}$$

Performing the dimensional reduction of the type IIA action including tree-level and one-loop eight-derivative corrections results in a four-dimensional effective action of the form

$$S^{(4)} = S_{2\text{ der.}}^{(4)} + \alpha S_{R^2}^{(4)} + \alpha S_{R\partial u}^{(4)} + \alpha S_u^{(4)}. \quad (4.13)$$

The various contributions to (4.13) are the two-derivative action  $S_{2\text{ der.}}^{(4)}$ , the piece containing the quadratic Riemann tensor terms  $S_{R^2}^{(4)}$ , the part of the action containing the terms mixing Riemann tensors and derivatives of  $u$   $S_{R\partial u}^{(4)}$ , and finally the contribution where all four derivatives act on the scalar  $u$ , denoted by  $S_u^{(4)}$ . The various pieces are

$$2\kappa_4^2 S_{2\text{ der.}}^{(4)} = \int_{\mathcal{M}_4} [e^{-2\phi_0} e^u - 1536 \alpha (2\pi)^3 \chi(CY_3) \ell_0] R \star 1 + \frac{5}{6} e^{-2\phi_0} e^u du \wedge \star du + 768 \alpha \ell_+ \chi(CY_3) (2\pi)^3 du \wedge \star du, \quad (4.14)$$

$$2\kappa_4^2 S_{R^2}^{(4)} = \int_{\mathcal{M}_4} (c_2 \cdot J) [384 \ell_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \ell_- R_{\mu\nu} R^{\mu\nu} - 192 \ell_- R^2] \star 1, \quad (4.15)$$

$$2\kappa_4^2 S_{R\partial u}^{(4)} = \int_{\mathcal{M}_4} (c_2 \cdot J) \left[ -\frac{256}{3} \ell_- R^{\mu\nu} \partial_\mu u \partial_\nu u \star 1 - 512 \ell_- R^{\mu\nu} \nabla_\mu \nabla_\nu u + 256 \ell_- R (\square u) \star 1 + 64 \ell_- R du \wedge \star du \right], \quad (4.16)$$

$$2\kappa_4^2 S_u^{(4)} = \int_{\mathcal{M}_4} (c_2 \cdot J) \left[ -\frac{128}{3} \ell_- (\square u)^2 \star 1 - \frac{128}{9} \ell_- (\square u) du \wedge \star du + \frac{80}{9} \ell_+ (\partial u)^4 \star 1 + \frac{256}{9} \ell_0 \partial^\mu u \nabla_\mu \nabla_\nu u \partial^\nu u \star 1 + \frac{256}{3} \ell_0 \nabla_\mu \nabla_\nu u \nabla^\mu \nabla^\nu u \star 1 \right], \quad (4.17)$$

where we made use of the shorthand notation (3.42) and abbreviated  $\ell_\pm = \ell_0 \pm \ell_1$ . The coupling  $c_2 \cdot J$  of the four-derivative interactions is the modulus-dependent analogue of  $c_2^{(0)} \cdot \omega$  defined in (4.6), i.e.

$$c_2 \cdot J = (2\pi)^2 \int_{CY_3} c_2 \wedge J = e^{u/3} (2\pi)^2 \int_{CY_3} c_2^{(0)} \wedge \omega = e^{u/3} (c_2^{(0)} \cdot \omega). \quad (4.18)$$

In order to go to Einstein frame, we perform a Weyl rescaling of the four-dimensional metric

$$g_{\mu\nu} \rightarrow [e^{-2\phi_0} e^u - 1536 \alpha (2\pi)^3 \chi(CY_3) \ell_0]^{-1} g_{\mu\nu}, \quad (4.19)$$

which now involves a shift in the volume proportional to the Euler characteristic. The action in Einstein frame takes the form

$$S_{\text{IIA}}^{(4)} = S_{\text{class}}^{(4)} + \alpha \ell_0 S_{\text{tree}}^{(4)} + \alpha \ell_1 S_{\text{loop}}^{(4)}, \quad (4.20)$$

where we split the action (4.20) in a classical two-derivative piece  $S_{\text{class}}^{(4)}$  at lowest order in  $\alpha$ , a tree-level (in  $g_s$ ) part  $S_{\text{tree}}^{(4)}$  and a one-loop correction  $S_{\text{loop}}^{(4)}$ . Using external space total derivative identities one can show that the relations

$$\int_{\mathcal{M}_4} d^4x \sqrt{-g} e^{u/3} \nabla^\mu u \nabla_\mu \nabla_\nu u \nabla^\nu u = -\frac{1}{2} \int_{\mathcal{M}_4} d^4x \sqrt{-g} e^{u/3} \left[ \frac{1}{3} (\partial u)^4 + (\square u) (\partial u)^2 \right] \quad (4.21)$$

and

$$\begin{aligned} \int_{\mathcal{M}_4} d^4x \sqrt{-g} e^{u/3} \nabla_\mu \nabla_\nu u \nabla^\mu \nabla^\nu u &= \int_{\mathcal{M}_4} d^4x \sqrt{-g} e^{u/3} \left[ (\square u)^2 + \frac{1}{2} (\square u) (\partial u)^2 \right. \\ &\quad \left. + \frac{1}{18} (\partial u)^4 - R^{\mu\nu} \partial_\mu u \partial_\nu u \right] \end{aligned} \quad (4.22)$$

hold. The individual parts of (4.20) take the form

$$2\kappa_4^2 S_{\text{class}}^{(4)} = \int_{\mathcal{M}_4} R \star 1 - \frac{2}{3} du \wedge \star du, \quad (4.23)$$

$$\begin{aligned} 2\kappa_4^2 S_{\text{tree}}^{(4)} &= \int_{\mathcal{M}_4} -2560 (2\pi)^3 \chi(\text{CY}_3) e^{2\phi_0} e^{-u} du \wedge \star du \\ &\quad + \int_{\mathcal{M}_4} (c_2 \cdot J) [768 R_{\mu\nu} R^{\mu\nu} - 192 R^2] \star 1 \\ &\quad + \int_{\mathcal{M}_4} (c_2 \cdot J) \left[ \frac{6592}{9} (\square u)^2 \star 1 - 128 (\square u) du \wedge \star du + \frac{2048}{81} (\partial u)^4 \star 1 \right] \\ &\quad + \int_{\mathcal{M}_4} (c_2 \cdot J) \left[ \frac{2048}{9} R^{\mu\nu} \partial_\mu u \partial_\nu u \star 1 + 1024 R^{\mu\nu} \nabla_\mu \nabla_\nu u \star 1 \right. \\ &\quad \left. - 128 R (\square u) \star 1 - 128 R du \wedge \star du \right], \end{aligned} \quad (4.24)$$

$$\begin{aligned} 2\kappa_4^2 S_{\text{loop}}^{(4)} &= \int_{\mathcal{M}_4} (c_2 \cdot J) [384 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 768 R_{\mu\nu} R^{\mu\nu} + 192 R^2] \star 1 \\ &\quad + \int_{\mathcal{M}_4} 768 (2\pi)^3 e^{2\phi_0} \chi(\text{CY}_3) e^{-u} du \wedge \star du \\ &\quad + \int_{\mathcal{M}_4} (c_2 \cdot J) \left[ 448 (\square u)^2 \star 1 - \frac{6016}{9} (\square u) du \wedge \star du + \frac{4928}{27} (\partial u)^4 \star 1 \right] \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathcal{M}_4} (c_2 \cdot J) \left[ -256 R^{\mu\nu} \partial_\mu u \partial_\nu u \star 1 + 512 R^{\mu\nu} \nabla_\mu \nabla_\nu u \star 1 \right. \\
& \quad \left. + 128 R (\square u) \star 1 - 256 R du \wedge \star du \right]. \tag{4.25}
\end{aligned}$$

Similar to the M-theory discussion in section 4.1 we can again consider higher-derivative field redefinitions. The general ansatz for these redefinitions reads

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + a_1 \alpha (c_2 \cdot J) (\square u) g_{\mu\nu} + a_2 \hat{\alpha} (c_2 \cdot J) (\partial u)^2 g_{\mu\nu} \tag{4.26}$$

$$+ a_3 \alpha (c_2 \cdot J) R g_{\mu\nu} + a_4 \alpha (c_2 \cdot J) R_{\mu\nu} + a_5 \alpha (c_2 \cdot J) \partial_\mu u \partial_\nu u,$$

$$+ a_6 \alpha (c_2 \cdot J) \nabla_\mu \nabla_\nu u,$$

$$u \rightarrow u + a_7 \alpha (c_2 \cdot J) (\square u) + a_8 \alpha (c_2 \cdot J) (\partial u)^2 + a_9 \alpha (c_2 \cdot J) R. \tag{4.27}$$

The coefficients  $a_i$  can then have tree-level and one-loop contributions proportional to  $\zeta(3)e^{-2\phi_0}$  and  $\pi^2/3$  respectively which we again choose in a way, such that the action takes a particularly simple form. The factors multiplying these tree-level and one-loop coefficients in the various parameters  $a_i \equiv \alpha_i \ell_0 + \beta_i \ell_1$  are listed in table 4.1.

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$	$i = 9$
$\alpha_i$	-384	$\frac{2048}{9}$	-192	768	$\frac{512}{9}$	1024	$-\frac{3184}{3}$	-96	0
$\beta_i$	-384	$\frac{1280}{3}$	-192	768	$-\frac{256}{3}$	512	-592	$\frac{928}{3}$	0

Table 4.1: Our choice for the coefficients in the field redefinitions.

The individual pieces of the four-dimensional action which we obtain after these redefinitions are then

$$2\kappa_4^2 S_{\text{class}}^{(4)} = \int_{\mathcal{M}_4} R \star 1 - \frac{2}{3} du \wedge \star du, \tag{4.28}$$

$$\begin{aligned}
2\kappa_4^2 S_{\text{tree}}^{(4)} = & - \int_{\mathcal{M}_4} 2560 (2\pi)^3 \chi(CY_3) e^{2\phi_0} e^{-u} du \wedge \star du + \frac{5632}{81} (c_2 \cdot J) (\partial u)^4 \star 1, \\
& \tag{4.29}
\end{aligned}$$

$$\begin{aligned}
2\kappa_4^2 S_{\text{loop}}^{(4)} = & \int_{\mathcal{M}_4} 384 (c_2 \cdot J) \left[ R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right] \star 1 \\
& + \int_{\mathcal{M}_4} 768 (2\pi)^3 e^{2\phi_0} \chi(CY_3) e^{-u} du \wedge \star du - \frac{3008}{27} (c_2 \cdot J) (\partial u)^4 \star 1. \\
& \tag{4.30}
\end{aligned}$$

We note that we find again Gauss-Bonnet-like gravity coupled to a scalar with a  $(\partial u)^4$  interaction. The only imprints of the Calabi–Yau threefold in the four-dimensional effective action are the topological Euler characteristic, and the second Chern-class.

### 4.3 Type IIB on CY<sub>3</sub>: four-derivative action

We will now make a detour to compactifications of type IIB supergravity with higher derivative terms. We will take the classical Einstein-Hilbert action, which is included in type IIB supergravity, as well as the tree-level and one-loop  $\hat{R}^4$ -corrections in ten dimensions into account. Since the higher-derivative terms of the dilaton and the (NS,NS) two-form are not fully known in type IIB, we do not include them as dynamical fields. The action we are considering thus takes the form

$$S^{\text{IIB}} = S_{\text{class}}^{\text{IIB}} + \alpha S_{\hat{R}^4}^{\text{IIB}} , \quad (4.31)$$

with the classical action

$$2\kappa_{10}^2 S_{\text{class}}^{\text{IIB}} = \int_{\mathcal{M}_{10}} \hat{R} \hat{*} 1 - \frac{1}{2(\text{Im}\tau)^2} d\tau \wedge \hat{*} d\bar{\tau} + \dots , \quad (4.32)$$

where

$$\tau = \tau_1 + i\tau_2 = \hat{C}_0 + ie^{-\hat{\phi}} \quad (4.33)$$

is the complex axio-dilaton. The ellipses in (4.32) stand for the two-derivative terms of the (NS,NS) two-form and the (R,R) potentials of even degree, which we do not display here. In addition to the two-derivative action, we again have eight-derivative terms quartic in the Riemann tensor in type IIB. They take the form

$$2\kappa_{10}^2 S_{\hat{R}^4}^{\text{IIB}} = \frac{1}{2} \int_{\mathcal{M}_{10}} E_{\frac{3}{2}}(\tau, \bar{\tau}) \left( \hat{t}_8 \hat{t}_8 + \frac{1}{8} \epsilon_{10} \epsilon_{10} \right) \hat{R}^4 \hat{*} 1 , \quad (4.34)$$

where  $E_{\frac{3}{2}}(\tau, \bar{\tau})$  is the  $SL(2, \mathbb{Z})$ -invariant Eisenstein-series defined by

$$E_{\frac{3}{2}}(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^{3/2}}{|m + n\tau|^3} = 2\zeta(3) \tau_2^{3/2} + \frac{2\pi^2}{3} \tau_2^{-1/2} + \mathcal{O}(e^{-2\pi\tau_2}) . \quad (4.35)$$

The first two terms in the expansion in (4.35) represent the tree-level and one-loop contributions to the  $\hat{R}^4$ -action, whereas the  $\mathcal{O}(e^{-2\pi\tau_2})$  terms represent non-perturbative corrections.

The corrected background solution to the equations of motion of the action (4.31), which at lowest order in  $\alpha$  reduces to the ordinary Calabi–Yau solution

without background fluxes, was found in [57]. Based on this background solution, we make the following reduction ansatz for the dimensional reduction of the volume modulus:

$$\begin{aligned} d\hat{s}^2 &= e^{\alpha\Phi^{(1)}} \left( g_{\mu\nu} dx^\mu dx^\nu + 2e^{u/3} (-i\omega_{i\bar{j}}) dz^i d\bar{z}^{\bar{j}} \right) \\ \Phi^{(1)} &= -192 (2\pi)^3 \mathcal{Y} (*_6 c_3) \\ \hat{\phi} &= \phi_0 + \mathcal{O}(\alpha). \end{aligned} \quad (4.36)$$

We have defined

$$\mathcal{Y} \equiv \zeta(3) e^{-3\phi_0/2} + \frac{\pi^2}{3} e^{\phi_0/2} \quad (4.37)$$

as a shorthand notation for the coefficients of the perturbative tree-level and one-loop corrections. As in the type IIA reduction, we introduced the modulus  $u$ , and normalized  $\omega$  to unity, see (4.12). The overall Weyl factor  $\Phi^{(1)}$  is again computed using  $g_{i\bar{j}} = e^{u/3} (-i\omega_{i\bar{j}})$ . There is furthermore a correction to the dilaton  $\hat{\phi}^{(1)} \propto *_6 c_3$ , which however only contributes at order  $\alpha^2$  to the part of the action we consider and can therefore be ignored.

The results of the dimensional reduction are very similar to the tree-level terms of type IIA presented in section 4.2. We therefore refrain from giving the results before the Weyl rescaling and simply state the result in Einstein frame. The four dimension effective action is then

$$\begin{aligned} 2\kappa_4^2 S_{\text{IIB}}^{(4)} &= \int_{\mathcal{M}_4} R \star 1 - \frac{2}{3} du \wedge \star du - 2560 (2\pi)^3 \alpha \chi(CY_3) \mathcal{Y} e^{-u} du \wedge \star du \\ &+ \int_{\mathcal{M}_4} \alpha (c_2 \cdot J) \mathcal{Y} [768 R_{\mu\nu} R^{\mu\nu} - 192 R^2] \star 1 \\ &+ \int_{\mathcal{M}_4} \alpha (c_2 \cdot J) \mathcal{Y} \left[ \frac{2048}{9} R^{\mu\nu} \partial_\mu u \partial_\nu u \star 1 + 1024 R^{\mu\nu} \nabla_\mu \nabla_\nu u \star 1 \right. \\ &\quad \left. - 128 R (\Box u) \star 1 - 128 R du \wedge \star du \right] \\ &+ \int_{\mathcal{M}_4} \alpha (c_2 \cdot J) \mathcal{Y} \left[ \frac{6592}{9} (\Box u)^2 \star 1 - 128 (\Box u) du \wedge \star du \right. \\ &\quad \left. + \frac{2048}{81} (\partial u)^4 \star 1 \right] \end{aligned} \quad (4.38)$$

We can furthermore use the field redefinitions (4.26) and (4.27) and the coefficients in the first column of table 4.1 to simplify (4.38). This finally leads to the action

$$\begin{aligned} 2\kappa_4^2 S_{\text{IIB}}^{(4)} &= \int_{\mathcal{M}_4} R \star 1 - \frac{2}{3} du \wedge \star du - 2560 (2\pi)^3 \alpha \chi(CY_3) \mathcal{Y} e^{-u} du \wedge \star du \\ &- \int_{\mathcal{M}_4} \alpha \frac{5632}{81} (c_2 \cdot J) \mathcal{Y} (\partial u)^4 \star 1. \end{aligned} \quad (4.39)$$

Note that compared to (4.13) the couplings in the action (4.38) involve different powers of the string coupling  $g_s$ . This is due to the fact, that we started on the one hand in the ten-dimensional string frame for type IIA, whereas on the other hand our starting point for type IIB was the Einstein frame action in ten dimensions. We can obtain the same powers of the dilaton as in (4.13) if we shift  $u \rightarrow u - \frac{3}{2}\phi_0$ .

#### 4.4 Comments on $\mathcal{N} = 1$ type IIB orientifolds

In the final section of this chapter we will discuss the orientifold truncation of the  $\mathcal{N} = 2$  compactification performed in the previous section. An orientifold truncation means that an  $\mathcal{N} = 2$  reduction is effectively truncated to the sector, which is not projected out by an orientifold projection. This procedure allows us to infer parts of the couplings coming from the bulk of type IIB supergravity. The resulting theory is expected to be an  $\mathcal{N} = 1$  supergravity theory and we will focus on the subsector of the theory involving the Kähler structure deformations. In addition to this restricted focus we will not try to complete the compactification to a fully consistent  $\mathcal{N} = 1$  setup. In fact, the presence of orientifold planes would require a more complete treatment involving D-branes, which can themselves eventually contribute higher-derivative terms to the four-dimensional theory. With these caveats in mind we can nevertheless try to push our  $\mathcal{N} = 2$  results and comment on the recent proposal of [58] to stabilize moduli. This procedure of direct truncation from  $\mathcal{N} = 2$  to  $\mathcal{N} = 1$  was successfully used before in [57, 59] to determine corrections to the  $\mathcal{N} = 1$  Kähler potential.

In the  $\mathcal{N} = 1$  settings one faces similar difficulties as for in the discussion of the  $\mathcal{N} = 2$  Calabi-Yau threefold compactifications of the previous sections. In particular, a general  $\mathcal{N} = 1$  four-derivative on-shell action to which one can compare the effective action after performing a reduction is currently not available. There are, however, partial results obtained by expanding certain four-derivative terms in superspace [60]. Therefore, it is tempting to compare the  $\mathcal{N} = 1$  truncated one modulus reduction of section 4.3 to the action of [60] as suggested in [58]. We will therefore briefly review the required results. The main idea is to take a four-derivative  $\mathcal{N} = 1$  Lagrangian and infer a contribution to the scalar potential from a particular four-derivative coupling. The relevant



four-derivative Lagrangian is [58, 60]

$$\begin{aligned} \frac{\mathcal{L}_{\text{bos.}}}{\sqrt{-g}} = & \frac{1}{2}R - G_{a\bar{b}}(A, \bar{A}) \partial_\mu A^a \partial^\mu \bar{A}^{\bar{b}} - 2e^K T^{\bar{d}}_a{}^c{}_{\bar{b}} D_c W \overline{D_{\bar{d}} W} \partial_\mu A^a \partial^\mu \bar{A}^{\bar{b}} \\ & + T_{a\bar{b}\bar{c}\bar{d}} \partial_\mu A^a \partial^\mu A^b \partial_\nu \bar{A}^{\bar{c}} \partial^\nu \bar{A}^{\bar{d}} - V(A, \bar{A}). \end{aligned} \quad (4.40)$$

In (4.40)  $A^a$  are complex scalars in chiral multiplets,  $W$  is the holomorphic superpotential,  $T_{a\bar{b}\bar{c}\bar{d}}$  is the coupling tensor of the four-derivative interaction and  $D_a$  is the Kähler covariant derivative with respect to  $A^a$ . Additionally, the Kähler metric  $G_{a\bar{b}}$  is given in terms of a Kähler potential  $G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K(A, \bar{A})$ . The scalar potential in (4.40) consists of two terms  $V(A, \bar{A}) = V_{(0)} + V_{(1)}$ , where

$$\begin{aligned} V_{(0)} &= e^K \left( G^{a\bar{b}} D_a W \overline{D_{\bar{b}} W} - 3|W|^2 \right), \\ V_{(1)} &= -e^{2K} T^{\bar{a}\bar{b}cd} \overline{D_{\bar{a}} W} \overline{D_{\bar{b}} W} D_c W D_d W. \end{aligned} \quad (4.41)$$

In a simplified setup with only a single Kähler modulus sitting in a chiral multiplet after the  $\mathcal{N} = 1$  truncation we have to compare (4.40) with the reduction result (4.39). In order to obtain the correct coupling tensor  $T_{a\bar{b}\bar{c}\bar{d}}$ , we have to rewrite the four-derivative interaction  $(\partial u)^4$  in the canonical  $\mathcal{N} = 1$  field variables. Once the coupling tensor  $T$  and the Kähler potential  $K$  are known, one can compute the induced potential (4.41) for a given superpotential  $W$ .

In order to perform the suggested comparison we therefore first have to determine the correct  $\mathcal{N} = 1$  coordinates. It is well-known [59, 61, 62] that at leading order one has to introduce a complex field  $A \equiv \rho + i e^{\frac{2}{3}u}$ , where  $\rho$  is the appropriately rescaled scalar arising from the (R,R) four-form and we recall  $\mathcal{V} = e^u$  is the volume of  $\text{CY}_3$ . Taking into account the order  $\alpha'^3$  corrections at the two-derivative level, the  $\mathcal{N} = 1$  coordinates may potentially receive corrections. However, it was argued in [57, 59] that the above  $A$  and the complex axio-dilaton  $\tau = C_0 + i e^{-\phi}$  are still the correct  $\mathcal{N} = 1$  coordinates, once  $\alpha'^3$ -corrections are taken into account. In the following we will freeze  $\tau$  and only consider the dynamics of the field  $A$ .

In order to match the action (4.40) with the reduction result (4.39) we now have to assume, that the complexified coordinates  $A$ , especially the scalars  $\rho$ , arrange themselves in a way, such that only the contribution  $(\partial A)^2 (\partial \bar{A})^2$  enters the four dimensional action. Comparing the action in the correct field variables

with (4.40) leads to<sup>29</sup>

$$T_{AA\bar{A}\bar{A}} = -\frac{11}{384} \frac{(2\pi)^{-4}}{\mathcal{V}^{8/3}} \zeta(3) (\text{Im}\tau)^{3/2} (c_2 \cdot J). \quad (4.42)$$

This fixes, at least under the stated assumptions, the numerical factor discussed in [58]. Let us stress two points. First, we have used a non-trivial coordinate redefinition to obtain (4.39) and it would be desirable to study its significance in this  $\mathcal{N} = 1$  truncated scenario. Second, it would be desirable to compute the terms involving  $\rho$  in order to justify the crucial assumption about the dependence on the complex moduli  $A$ .

Let us close this section by some further comments. So far we have not generalized the derivation of  $T_{ab\bar{c}\bar{d}}$  to more than one modulus, since the computational complexity increases significantly. Also the direct derivation of the actual scalar potential seems currently difficult, since many of the required higher-derivative terms in ten dimensions are not known. While (4.42) seems to suggest that a potential  $V_{(1)}$  given in (4.40) is induced if a non-trivial superpotential is included, we are not able to give any further evidence for this claim. It has been shown, for example in [63], that it can happen in string reductions that the potential is significantly altered if one finds other structures for the kinetic terms not appearing in (4.40), such as contributions from D-branes. It would be interesting to explore this further.

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<sup>29</sup>In this expression we have explicitly set  $2\pi\sqrt{\alpha'} = 1$ .

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## Chapter 5

# M-theory on $CY_4$ : the corrected 3d Kähler potential

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We now turn to the study of M-theory compactifications on Calabi–Yau fourfolds. These compactifications have led to many advances in building top-down models of particle physics with phenomenological relevance. Phenomenological applications of M-theory compactifications are mostly within the framework of F-theory [14]. The physics of F-theory is most conveniently studied in the dual formulation given by M-theory on an elliptically fibered Calabi–Yau manifold, see e.g. [16–20] and chapter 1.4. Corrections to the effective action of M-theory compactified on elliptically fibered Calabi–Yau manifolds may then lift to corrections in F-theory, provided they survive the M-theory to F-theory limit.<sup>30</sup> Among these corrections there could possibly be a correction to the three-dimensional Kähler potential stemming from the  $\hat{R}^4$ -terms in eleven dimensions. Even though much effort was made in the past few years to identify such a correction, a definite answer remained elusive, see [64–66, 45, 51, 41, 67]. Due to the lack of a fundamental formulation of F-theory we take its dual formulation in terms of M-theory as a definition of F-theory. It is obvious from this point of view, that before attempting to lift certain corrections to a four-dimensional F-theory model, one first has to

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<sup>30</sup>The M-theory to F-theory limit is the limit, in which the volume of the elliptic fiber vanishes.

understand the dual three-dimensional setting sufficiently well.

In this chapter we aim to elaborate on the existence of a correction to the three-dimensional  $\mathcal{N} = 2$  Kähler potential by considering the most simple setup. We consider M-theory including  $\ell_{\text{M}}^6$  corrections and reduce them on a Calabi–Yau fourfold with only one Kähler modulus, namely the overall volume of the compact manifold. We use the corrected fourfold solution involving fluxes and warping found in [43–45] and reviewed in section 2.2.2 to derive a three-dimensional effective action of the gravity multiplet and one vector multiplet. We show that the resulting action is compatible with three-dimensional  $\mathcal{N} = 2$  supersymmetry. Upon dualizing the vector multiplet into a chiral multiplet we derive the corrected Kähler potential and associated complex coordinate. We show that the result breaks the no-scale condition in three dimensions which leads to a non-vanishing scalar potential for the overall volume. Our results allow us to make some first observations about the M-theory to F-theory limit. We point out that the corrections we find are reminiscent of one-loop corrections found in three-dimensional effective theories. These may be interpreted as one-loop corrections induced by massive Kaluza-Klein modes in a circle reduction from four to three dimensions [68–73].

## 5.1 Three-dimensional gauged $\mathcal{N} = 2$ supergravity

In this section we briefly review  $\mathcal{N} = 2$  gauged supergravity in three dimensions. Three-dimensional maximal and non-maximal supergravities and their gaugings were exhaustively discussed in [74]. The case which is relevant for our setting is an  $\mathcal{N} = 2$  supergravity theory with a gauged shift symmetry and was studied, for example, in [75, 76]. This shift symmetry corresponds to an isometry of the geometry describing the scalar field space of the  $\mathcal{N} = 2$  theory. We consider three-dimensional  $\mathcal{N} = 2$  supergravity coupled to chiral multiplets whose complex scalars are denoted by  $N^\Lambda$ . The gaugings are realized along certain isometries  $\tilde{X}^{\Lambda\Sigma}$ . We also use a constant embedding tensor denoted by  $\Theta_{\Lambda\Sigma}$ . The  $\mathcal{N} = 2$  action then reads

$$\kappa_3^2 S_{\mathcal{N}=2}^{(3)} = \int_{\mathcal{M}_3} \frac{1}{2} R \star 1 - K_{\Lambda\bar{\Sigma}} \mathcal{D}N^\Lambda \wedge \star \mathcal{D}\bar{N}^{\bar{\Sigma}} - \frac{1}{2} \Theta_{\Lambda\Sigma} A^\Lambda \wedge F^\Sigma - (V_{\mathcal{T}} + V_F) \star 1, \quad (5.1)$$

where  $K_{\Lambda\bar{\Sigma}} = \partial_{N^\Lambda} \partial_{\bar{N}^{\bar{\Sigma}}} K$  is a Kähler metric with Kähler potential  $K$ . The gauge covariant derivative  $\mathcal{D}N^\Lambda$  is defined by

$$\mathcal{D}N^\Lambda = dN^\Lambda + \tilde{X}^{\Lambda\Sigma} \Theta_{\Sigma\Xi} A^\Xi. \quad (5.2)$$

The individual contributions to the scalar potential in (5.1) are given by

$$V_{\mathcal{T}} = K^{\Lambda\bar{\Sigma}} \partial_{\Lambda} \mathcal{T} \partial_{\bar{\Sigma}} \mathcal{T} - \mathcal{T}^2, \quad (5.3)$$

$$V_F = e^K (K^{\Lambda\bar{\Sigma}} D_{\Lambda} W \overline{D_{\Sigma} W} - 4|W|^2),$$

where  $\mathcal{T}$  is a real function of the chiral fields  $N^{\Lambda}$  which will be given more explicitly below. The Kähler-covariant derivative is defined by  $D_{\Lambda} W = (\partial_{\Lambda} + (\partial_{\Lambda} K)W)$ . Note that the vectors entering (5.1) via a Chern-Simons term are non-dynamical. We introduced the hermitian matrix  $K^{\Lambda\bar{\Sigma}}$ , which is the inverse of  $K_{\Lambda\bar{\Sigma}}$ , and a holomorphic superpotential  $W$  in (5.3).

**Dualizing the action.** We now split the chiral fields as  $N^{\Lambda} = (M^I, T_a)$  to obtain a dual description of the action (5.1) by dualizing the chiral multiplets with bosonic components  $T_a$  into vector multiplets. The detailed procedure can be found in [75, 76], we will therefore just quote the result. Since the dualization is in general not possible, one has to assume that the action (5.1) is invariant under shifts of  $\text{Im } T_a$ . The relevant gauging is achieved by choosing a constant embedding tensor and

$$\tilde{X}^{ab} = -2i\delta^{ab}, \quad \tilde{X}^{IJ} = \tilde{X}^{I\bar{J}} = 0, \quad \tilde{X}^{aI} = 0, \quad \Theta_{IJ} = 0. \quad (5.4)$$

The dual action is then given by

$$\begin{aligned} S_{\mathcal{N}=2, \text{dual}}^{(3)} = & \int_{\mathcal{M}_3} \frac{1}{2} R \star 1 - \tilde{K}_{M^I \bar{M}^{\bar{J}}} \mathcal{D} M^I \wedge \star \mathcal{D} \bar{M}^{\bar{J}} + \frac{1}{4} \tilde{K}_{L^a L^b} dL^a \wedge \star dL^b \\ & + \int_{\mathcal{M}_3} \frac{1}{4} \tilde{K}_{L^a L^b} F^a \wedge \star F^b + \frac{1}{2} \Theta_{ab} A^a \wedge F^b + F^a \wedge \text{Im} [\tilde{K}_{L^a M^I} \mathcal{D} M^I] \\ & - \int_{\mathcal{M}_3} (V_{\mathcal{T}} + V_F) \star 1. \end{aligned} \quad (5.5)$$

The physical couplings in (5.5) such as  $\tilde{K}_{L^a L^b} = \partial_{L^a} \partial_{L^b} \tilde{K}$  are now derived from a kinetic potential  $\tilde{K}$ , which is obtained by performing a Legendre transformation of the Kähler potential  $K$

$$K(M, T) = \tilde{K}(M, L) - \text{Re } T_a L^a, \quad (5.6)$$

where the real coordinates conjugate to  $\text{Re } T_a$  are defined by  $L^a = -2K_{T_a} = -2\partial_{T_a} K$ . The scalars  $L^a$  are now scalars in (propagating) vector multiplets. Using the Legendre transformation (5.6) one can derive many expressions relating

derivatives of  $K$  with derivatives of  $\tilde{K}$ , among which the most important ones are

$$K_{T_a \bar{T}_b} = -\frac{1}{4} \tilde{K}^{L^a L^b}, \quad \text{Re } T_a = \tilde{K}_{L^a}, \quad \frac{\partial L^a}{\partial T_b} = \frac{1}{2} \tilde{K}^{L^a L^b}. \quad (5.7)$$

These relations are extensively used in the explicit dualization procedure. The scalar potential  $V_{\mathcal{T}}$  reads in the vector multiplet variables

$$\begin{aligned} V_{\mathcal{T}} &= \tilde{K}^{M^I \bar{M}^{\bar{J}}} \partial_{M^I} \mathcal{T} \partial_{\bar{M}^{\bar{J}}} \mathcal{T} - \tilde{K}^{L^a L^b} \partial_{L^a} \mathcal{T} \partial_{L^b} \mathcal{T} - \mathcal{T}^2, \\ \mathcal{T} &= -\frac{1}{2} L^a \Theta_{ab} L^b. \end{aligned} \quad (5.8)$$

The scalar potential  $V_F$  is in the vector multiplet language given by

$$V_F = e^K \left[ \tilde{K}^{M^I \bar{M}^{\bar{J}}} D_{M^I} W \overline{D_{\bar{M}^{\bar{J}}} W} - (4 + L^a \tilde{K}_{L^a L^b} L^b) |W|^2 \right]. \quad (5.9)$$

The only relevant part to compute the scalar potential in the chiral multiplet formulation (5.3) will be the function  $\mathcal{T}$  given in (5.8) and the Kähler potential  $K$  with coordinates  $T_a$ , since we will assume a constant superpotential later.

## 5.2 Computation of the quantum-corrected Kähler potential

We now perform the dimensional reduction of eleven-dimensional supergravity on the background reviewed in section 2.2.2. We will do this for the simplified case  $h^{1,1} = \dim H^{1,1}(\text{CY}_4) = 1$ . The single Kähler modulus is then given by the volume of the Calabi–Yau fourfold  $\mathcal{V}$ . In other words, we expand the Kähler form in a single harmonic  $(1,1)$ -form  $\omega$  as

$$J = \mathcal{V}^{\frac{1}{4}} \omega, \quad \frac{1}{4!} \int_{\text{CY}_4} \omega^4 = (2\pi \ell_{\text{M}})^8 = 1, \quad (5.10)$$

where we have normalized  $\omega$  such that it is identified with the Kähler form of the unit volume Calabi–Yau fourfold. The volume  $\mathcal{V}$  is the the Calabi–Yau fourfold volume measured in units of the eleven-dimensional Planck length.

The simplified analysis with  $h^{1,1} = 1$  comes with two main advantages. Firstly, one can deduce the dependence of the warp-factor on the volume from the warp-factor equation (2.37). Secondly, the couplings in the three-dimensional effective action are all topological as opposed to the case for general  $h^{1,1}$  considered in [51, 67]. Upon a rescaling of the metric  $g_{i\bar{j}}^{(0)} \rightarrow \mathcal{V}^{\frac{1}{4}} g_{i\bar{j}}^{(0)}$ , the warp-factor equation should scale homogeneously, mapping a solution of the equation to another solution. This implies that  $\tilde{A}(z, \bar{z}, \mathcal{V}) = \mathcal{V}^{-1} \langle A^{(2)} \rangle$ , where  $\langle A^{(2)} \rangle$  is a solution

of the warp-factor equation (2.37) with respect to the metric  $g_{i\bar{j}}^{(0)} = -i\omega_{i\bar{j}}$ . We already noted that the correction to the metric  $g_{i\bar{j}}^{(2)}$  decouples from the effective action since it only contributes with total derivatives. In the following we therefore drop the metric correction from all expressions. The reduction ansatz for the metric and the M-theory four form field strength is therefore

$$d\hat{s}^2 = e^{\hat{\alpha}^2\Phi} \left( e^{-2\hat{\alpha}^2\tilde{A}} g_{\mu\nu} dx^\mu dx^\nu + 2e^{\hat{\alpha}^2\tilde{A}} \mathcal{V}^{\frac{1}{4}} (-i\omega_{i\bar{j}}) dz^i d\bar{z}^{\bar{j}} \right), \quad (5.11)$$

$$\hat{G}_4 = \hat{\alpha} G^{(1)} + \text{dvol}_{\mathbb{R}^{1,2}} \wedge \text{d}_{\text{CY}_4} (e^{-3\hat{\alpha}^2\tilde{A}}) + F \wedge \omega, \quad (5.12)$$

where  $F = dA$  is the field strength of a three-dimensional vector from the expansion of  $\tilde{C}_3$  along the harmonic  $(1, 1)$ -form  $\omega$ ,  $\Phi = \mathcal{V}^{-\frac{3}{4}} \langle \Phi^{(2)} \rangle$  with

$$\langle \Phi^{(2)} \rangle = -\frac{512}{3} (2\pi)^3 *^{(0)} (c_3^{(0)} \wedge J^{(0)}), \quad (5.13)$$

and  $\tilde{A}$  as given above. Before we continue with the reduction, let us introduce the notation

$$\begin{aligned} c_3^{(0)} \cdot \omega &= (2\pi)^3 \int_{\text{CY}_4} c_3^{(0)} \wedge \omega, \\ c_3 \cdot J &= (2\pi)^3 \int_{\text{CY}_4} c_3 \wedge J, \\ \mathcal{A} &= \frac{1}{4!} \int_{\text{CY}_4} \tilde{A} J^4 = \frac{1}{4!} \int_{\text{CY}_4} A^{(2)} \omega^4, \end{aligned} \quad (5.14)$$

where  $(c_3^{(0)} \cdot \omega)$  is a constant depending on the topology of  $\text{CY}_4$ , and  $\mathcal{A}$  is a constant depending on the warp-factor profile and background metric of  $\text{CY}_4$ .

We are now in a position to perform the dimensional reduction of the eleven dimensional action including the eight-derivative couplings of interest. The resulting theory has  $\mathcal{N} = 2$  supersymmetry in three dimensions and contains the gravity multiplet, whose bosonic field is the three-dimensional metric  $g_{\mu\nu}$ , and a vector multiplet formed by the 3d vector and the volume  $\mathcal{V}$ . Focusing only on the bosonic part of the action, we first use the reduction results in appendix 5.A and then perform a Weyl rescaling to Einstein frame in three dimensions. The resulting action for the kinetic terms takes the form

$$\begin{aligned} \kappa_3^2 S_{\text{kin}}^{(3)} &= \int_{\mathcal{M}_3} \left[ \frac{1}{2} R \star 1 - \frac{9}{16} d \log \mathcal{V} \wedge \star d \log \mathcal{V} - \mathcal{V}^{\frac{3}{2}} F \wedge \star F \right. \\ &\quad + \frac{9}{2} \hat{\alpha}^2 \mathcal{V}^{-1} \mathcal{A} d \log \mathcal{V} \wedge \star d \log \mathcal{V} + 216 \hat{\alpha}^2 \mathcal{V}^{-\frac{3}{4}} (c_3^{(0)} \cdot \omega) d \log \mathcal{V} \wedge \star d \log \mathcal{V} \\ &\quad \left. - 6 \hat{\alpha}^2 \mathcal{V}^{\frac{1}{2}} \mathcal{A} F \wedge \star F + 384 \hat{\alpha}^2 \mathcal{V}^{\frac{3}{4}} (c_3^{(0)} \cdot \omega) F \wedge \star F \right]. \end{aligned} \quad (5.15)$$

To compare this result with the general action (5.5) we first define  $L = \mathcal{V}^{-\frac{3}{4}} - 3\hat{\alpha}^2 \mathcal{A} \mathcal{V}^{-\frac{7}{4}}$  to rewrite (5.15) as

$$\begin{aligned} \kappa_3^2 S_{\text{kin}}^{(3)} = \int_{\mathcal{M}_3} & \left[ \frac{1}{2} R \star 1 - \frac{1}{L^2} dL \wedge \star dL + 384 \hat{\alpha}^2 (c_3^{(0)} \cdot \omega) \frac{1}{L} dL \wedge \star dL \right. \\ & \left. - \frac{1}{L^2} F \wedge \star F + 384 \hat{\alpha}^2 (c_3^{(0)} \cdot \omega) \frac{1}{L} F \wedge \star F \right]. \end{aligned} \quad (5.16)$$

Comparing this action with the standard  $\mathcal{N} = 2$  action (5.5) we find that

$$\kappa_3^2 S_{\text{kin}}^{(3)} = \int_{\mathcal{M}_3} \left[ \frac{1}{2} R \star 1 + \frac{1}{4} \tilde{K}_{LL}(L) dL \wedge \star dL + \frac{1}{4} \tilde{K}_{LL}(L) F \wedge \star F \right], \quad (5.17)$$

with

$$\tilde{K}_{LL}(L) = -\frac{4}{L^2} \left( 1 - 384 \hat{\alpha}^2 (c_3^{(0)} \cdot \omega) L \right) = -\frac{4}{L^2} + 1536 \hat{\alpha}^2 (c_3^{(0)} \cdot \omega) \frac{1}{L}. \quad (5.18)$$

We can integrate the metric  $\tilde{K}_{LL}$  to obtain the kinetic potential  $\tilde{K}(L)$  and coordinate

$$\tilde{K} = 4 \log L + 1536 \hat{\alpha}^2 (c_3^{(0)} \cdot \omega) L (\log(L) - 1) + 4, \quad (5.19)$$

$$L = \mathcal{V}^{-\frac{3}{4}} - 3\hat{\alpha}^2 \mathcal{A} \mathcal{V}^{-\frac{7}{4}}, \quad (5.20)$$

where we have chosen the integration constants in a convenient way.

**Determining the Kähler potential.** We will now dualize the vector multiplet to a chiral multiplet, whose metric derives from a Kähler potential. This is achieved by a Legendre transformation of the kinetic potential as outlined in section 5.1

$$K = \tilde{K} - L \operatorname{Re} T, \quad \operatorname{Re} T = \partial_L \tilde{K}. \quad (5.21)$$

One finds the Kähler potential  $K$

$$\begin{aligned} K &= 4 \log L - 1536 \hat{\alpha}^2 (c_3^{(0)} \cdot \omega) L \\ &= -3 \log \left( \frac{1}{4!} \int_{CY_4} e^{4\hat{\alpha}^2 \tilde{A}} J^4 + 512 \hat{\alpha}^2 c_3 \cdot J \right), \end{aligned} \quad (5.22)$$



with corresponding coordinate

$$\begin{aligned} \operatorname{Re} T &= \frac{4}{L} + 1536 \hat{\alpha}^2 (c_3^{(0)} \cdot \omega) \log L \\ &= 4 \mathcal{V}^{\frac{3}{4}} + 12 \hat{\alpha}^2 \mathcal{V}^{-\frac{1}{4}} \mathcal{A} - 1152 \hat{\alpha}^2 (c_3 \cdot J) \mathcal{V}^{-\frac{1}{4}} \log \mathcal{V} . \end{aligned} \quad (5.23)$$

All quantities in the Kähler potential (5.22) now depend on the modulus  $\mathcal{V}$ . On a first sight one might wonder about the unusual correction to the Kähler coordinate  $\propto \log \mathcal{V}$ . We will comment on the physical interpretation of this correction in section 5.3.

Let us stress that the analysis of [67] also lead to a Kähler potential of the form (5.22). However, in the analysis performed there, it was not possible to fix all the coefficients in  $K$  and  $\operatorname{Re} T$  unambiguously. The discussion of the Kähler coordinates was incomplete, due to the presence of many non-topological terms in the case of generic  $h^{1,1}$  and an ansatz for the Kähler coordinates, which was not general enough. In this one modulus analysis we were able to avoid these problems. Nevertheless, it is important to point out that the pure warping result of [67], which was inspired by [77], agrees with our findings here.

**The no-scale condition and the scalar potential.** One of the essential key results is that the no-scale condition in three dimensions is broken once  $\ell_M^6$ -suppressed corrections to the Kähler potential in (5.22) are taken into account. We can straightforwardly compute

$$K_T K^{T\bar{T}} K_{\bar{T}} = 4 - 1536 \hat{\alpha}^2 \mathcal{V}^{-1} c_3 \cdot J , \quad (5.24)$$

which indeed shows that the no-scale condition is broken.<sup>31</sup> We can now also use this result to determine the scalar potential. We first evaluate

$$\begin{aligned} V_{\mathcal{T}} &= K^{T\bar{T}} \partial_T \mathcal{T} \partial_{\bar{T}} \mathcal{T} - \mathcal{T}^2 \\ &= (16 \mathcal{V}^{\frac{3}{2}} + \dots)^{-1} \left[ \frac{1}{2} \left( \frac{\partial \operatorname{Re} T}{\partial \mathcal{V}} \right)^{-1} \frac{\partial \mathcal{T}}{\partial \mathcal{V}} \right]^2 - \mathcal{T}^2 = 0 + \mathcal{O}(\hat{\alpha}^3) , \end{aligned}$$

where we used

$$\mathcal{T}(\mathcal{V}) = -\frac{1}{2} \hat{\alpha} \Theta \mathcal{V}^{-\frac{3}{2}} + \dots , \quad \Theta = -\frac{1}{2} \int_{CY_4} \omega \wedge \omega \wedge G^{(1)} . \quad (5.25)$$

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<sup>31</sup>The no-scale condition is  $K_T K^{T\bar{T}} K_{\bar{T}} = 4$ .

This means that the only contribution to the scalar potential comes from the breaking of the no-scale condition. It enters the effective action via the F-term scalar potential

$$V_F = e^K \left( K^{T\bar{T}} D_T W_0 \overline{D_T W_0} - 4|W_0|^2 \right) = -1536 \hat{\alpha}^2 \frac{|W_0|^2}{\mathcal{V}^4} c_3 \cdot J, \quad (5.26)$$

which has a runaway direction for  $\mathcal{V} \rightarrow \infty$  if  $c_3 \cdot J < 0$ . We assumed a constant superpotential  $W_0$  in (5.26) which may arise from stabilizing complex structure moduli and inserting their fixed values in the GVW superpotential [78]

$$W = \frac{1}{\ell_M^3} \int_{CY_4} G^{(1)} \wedge \Omega, \quad \Omega \in H^{4,0}(CY_4). \quad (5.27)$$

The runaway behavior of (5.26) for large volume  $\mathcal{V}$  signals an instability of the solution in case the complex structure moduli are stabilized in a non-supersymmetric vacuum with a non-vanishing  $W_0$ . This raises doubts about the validity of the reduction for such a non-vanishing, supersymmetry breaking  $W_0$  as recently stressed in [79].

### 5.3 Comments on loop corrections and M/F-theory duality

One of the main motivations to study M-theory compactifications on Calabi–Yau fourfolds is its duality to 4d F-theory models with minimal supersymmetry. Upon compactifying the 4d F-theory action on a circle and taking the F-theory limit of vanishing torus-fiber volume  $\text{vol}(T^2) \rightarrow 0$ , one can infer F-theory data from the matching with the 3d M-theory compactification. Since F-theory requires an elliptic fibration with  $h^{1,1} > 1$  our analysis does not immediately apply to this case. Nevertheless we will try to give some first interpretation of the result (5.22) in the context of this duality.

Let us assume for a moment that our result is valid beyond the one modulus case. While this is seemingly straightforward for the Kähler potential  $K$  it is less obvious how to generalize the complex coordinates  $T_a$ . A reasonable assumption appears to be that the  $T_a$  contain a correction proportional to

$$c_3 \cdot \omega_a = (2\pi)^3 \int_{CY_4} c_3 \wedge \omega_a, \quad a = 1, \dots, \dim H^{1,1}(CY_4), \quad (5.28)$$

multiplied with the volume of some submanifold of  $CY_4$ . The 3d  $\mathcal{N} = 2$  theory with this Kähler potential  $K$  and coordinates  $T_a$  can then be thought of as a circle-compactified 4d  $\mathcal{N} = 1$  theory with an infinite tower of (massive) Kaluza-Klein

states. The 3d Wilsonian effective action which is valid below some energy scale  $\Lambda_{\text{cutoff}}$  is then calculated by integrating out the massive fields. These massive fields can, when integrated out, in general modify the couplings in the effective action through diagrams in which the massive fields run in loops. These loop-correction can appear up to an arbitrary order in the diagrammatic loop expansion, unless some non-renormalization theorem comes to the rescue. Loop effects of this sort are certainly important when employing M/F-theory duality. In particular, the duality may mix classical contributions on one side of the duality with quantum corrections to the effective action on the other side of the duality. The inclusion of such one-loop corrections turned out to be crucial in tracking Chern-Simons terms through M-theory/F-theory duality in the case of compactifications to three dimensions [80–82] and five dimensions [50, 83, 81].

The simplest 4d setting to start with is a supergravity theory with only the  $\mathcal{N} = 1$  gravity multiplet, i.e. a pure supergravity theory. Compactifying this theory on the background  $\mathbb{R}^{1,2} \times S^1$  leads to a 3d  $\mathcal{N} = 2$  supergravity theory coupled to a chiral multiplet.<sup>32</sup> The leading perturbative correction to the corresponding 3d Kähler coordinate was determined in [73]. It was inferred from one-loop determinants of fluctuations of the graviton and the gravitino (plus their ghosts) around the aforementioned background. The correction was found to be

$$\text{Re } T_0^{1\text{-loop}} = 2\pi^2 M_{\text{pl}}^2 R^2 + \frac{7}{48} \log(M_{\text{pl}}^2 R^2), \quad (5.29)$$

where  $R$  is the radius of the  $S^1$  and  $M_{\text{pl}}$  is the 4d Planck's mass. This logarithmic correction to the lowest order complex structure is reminiscent of the  $\log \mathcal{V}$  correction to the Kähler coordinate from the M-theory reduction given in (5.22). However, it is well-known that an F-theory setting will not only lead to pure  $\mathcal{N} = 1$  supergravity, but include some moduli fields. These are counted by the Hodge numbers  $h^{1,1}$ ,  $h^{3,1}$  and  $h^{2,1}$  of  $\text{CY}_4$ . Hence, one would need to generalize the analysis of [73] to include further complex fields in 4d.

We are not aware of a study of such a more general setting. However, perturbative and non-perturbative quantum corrections to  $\mathcal{N} = 2$  supersymmetric gauge theories without gravity were studied intensively from various point of views, for example, in [68–72]. We review here the case of having a  $\text{U}(1)^{N_c}$  gauge theory with  $N_f$  flavours labelled by  $I = 1, \dots, N_f$  with Coulomb branch masses  $q_a^I L^a$ , where  $q_a^I$  are the charges under the  $a$ -th  $\text{U}(1)$  factor and  $L^a$  is the real scalar in the  $a$ -th  $\text{U}(1)$  vector multiplet. For simplicity we set any further real and complex

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<sup>32</sup>The chiral multiplet emerges after dualizing the KK vector to a scalar.

masses to zero. The one-loop corrected Kähler coordinates are then determined to be (see e.g. [70])

$$\text{Re } T_a^{1\text{-loop}} = \frac{1}{e^2} \mathcal{C}_{ab} L^b + \sum_I q_a^I \log |q_b^I L^b|. \quad (5.30)$$

The corresponding kinetic potential was shown to be of the form

$$\tilde{K}(L^a) = \frac{1}{2e^2} \mathcal{C}_{ab} L^a L^b + \sum_I q_a^I L^a (\log |q_b^I L^b| - 1). \quad (5.31)$$

The corrected coordinates are those of a 3d one-loop Wilsonian effective action with all the massive flavours integrated out. If one now thinks about these coordinates as coming from a 4d  $\mathcal{N} = 1$  supersymmetric F-theory model compactified on a circle, one is led to identify the massive KK-modes on the circle with the massive modes that have been integrated out in order to obtain (5.30). From the M-theory side of the duality the massive states responsible for the loop corrections may admit an interpretation in terms of M2-brane states wrapping curves  $\mathcal{C}_I \in H_2(CY_4)$ . These states have a mass  $q_a^I L^a$  proportional to

$$\text{vol}(\mathcal{C}_I) = \int_{\mathcal{C}_I} J, \quad q_a^I = \int_{\mathcal{C}_I} \omega_a. \quad (5.32)$$

Remarkably, the  $L$  found in (5.20) has the elegant more-moduli generalization

$$L^a = \frac{v^a}{\mathcal{V}_W}, \quad \mathcal{V}_W = \frac{1}{4!} \int e^{3\hat{\alpha}^2 \tilde{A}} J^4, \quad (5.33)$$

where  $v^a$  are the expansion coefficients in  $J = v^a \omega_a$  and  $\mathcal{V}_W$  is the standard warped volume. As suggested by our result (5.20) this  $L^a$  does not include any higher-derivate corrections.

Note that the results (5.30), (5.31) are obtained in a theory without gravity. Coupling to gravity will lead to a logarithmic term such that the coupled result is expected to be of the form<sup>33</sup>

$$\tilde{K}(L^a) = -\log(e^2 - \mathcal{C}_{ab} L^a L^b) + \sum_I q_a^I L^a (\log |q_b^I L^b| - 1), \quad (5.34)$$

which expands to (5.31) for small  $\mathcal{C}_{ab}$ . Appropriately combining (5.29) with (5.34) one indeed finds an immediate resemblance with our reduction result (5.19). Even though this is not a proof that the correction found in (5.22) is a loop correction in the effective action, this reasoning supports this interpretation.

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<sup>33</sup>Strictly speaking one has to include more vector multiplets  $L^0$ ,  $L^\alpha$  with  $e^2$  and  $\mathcal{C}_{ij}$  depending on these fields.  $L^0$  is the dual variable to  $T_0^{1\text{-loop}}$ .

## 5.A Reduction results of section 5

In this appendix we collect the intermediate results of the reduction performed in section 5, especially of the higher-derivative terms.<sup>34</sup> The results are always shown up to at most two derivatives in three dimensions.

- Kinetic and potential terms from  $\hat{t}_8 \hat{t}_8 \hat{R}^4$ .

$$\int_{\mathcal{M}_{11}} \hat{t}_8 \hat{t}_8 \hat{R}^4 \hat{*} 1 = \int_{\mathcal{M}_3} 1536 \chi(\text{CY}_4) \star 1 - 72 \mathcal{V}^{\frac{1}{4}} (c_3^{(0)} \cdot \omega) d \log \mathcal{V} \wedge \star d \log \mathcal{V} \quad (5.35)$$

- Kinetic and potential terms from  $\epsilon_{11} \epsilon_{11} \hat{R}^4$ .

$$\begin{aligned} -\frac{1}{24} \int_{\mathcal{M}_{11}} \epsilon_{11} \epsilon_{11} \hat{R}^4 \hat{*} 1 &= \int_{\mathcal{M}_3} 768 \mathcal{V}^{\frac{1}{4}} (c_3^{(0)} \cdot \omega) R \star 1 \\ &+ \int_{\mathcal{M}_3} 24 \mathcal{V}^{\frac{1}{4}} (c_3^{(0)} \cdot \omega) d \log \mathcal{V} \wedge \star d \log \mathcal{V} + 1536 \chi(\text{CY}_4) \star 1 \end{aligned} \quad (5.36)$$

- Kinetic terms from  $\hat{t}_8 \hat{t}_8 \hat{R}^3 \hat{G}_4^2$ .

$$\int_{\mathcal{M}_{11}} \hat{t}_8 \hat{t}_8 \hat{R}^3 \hat{G}_4^2 \hat{*} 1 = 1152 \int_{\mathcal{M}_3} \mathcal{V}^{-\frac{1}{4}} (c_3^{(0)} \cdot \omega) F \wedge \star F \quad (5.37)$$

- Kinetic terms from  $\epsilon_{11} \epsilon_{11} \hat{R}^3 \hat{G}_4^2$ .

$$\frac{1}{96} \int_{\mathcal{M}_{11}} \epsilon_{11} \epsilon_{11} \hat{R}^3 \hat{G}_4^2 \hat{*} 1 = -384 \int_{\mathcal{M}_3} \mathcal{V}^{-\frac{1}{4}} (c_3^{(0)} \cdot \omega) F \wedge \star F \quad (5.38)$$

- Kinetic terms from  $\hat{Z} \hat{G}_4^2$ .

$$\int_{\mathcal{M}_{11}} 256 \hat{Z} \hat{G}_4 \wedge \hat{*} \hat{G}_4 = 1024 \int_{\mathcal{M}_3} \mathcal{V}^{-\frac{1}{4}} (c_3^{(0)} \cdot \omega) F \wedge \star F, \quad (5.39)$$

- Einstein Hilbert and kinetic term for  $\hat{G}_4$ .

$$\begin{aligned} \int_{\mathcal{M}_{11}} \frac{1}{2} \hat{R} \hat{*} 1 - \frac{1}{4} \hat{G}_4 \wedge \hat{*} \hat{G}_4 &= \int_{\mathcal{M}_3} \frac{1}{2} \mathcal{V} e^{3\hat{\alpha}^2 \mathcal{A}} (1 - 768 \hat{\alpha}^2 \mathcal{V}^{-\frac{3}{4}} (c_3^{(0)} \cdot \omega)) R \star 1 \\ &+ \int_{\mathcal{M}_3} \frac{7}{16} \mathcal{V} d \log \mathcal{V} \wedge \star d \log \mathcal{V} - \frac{3}{16} \hat{\alpha}^2 \mathcal{A} d \log \mathcal{V} \wedge \star d \log \mathcal{V} \\ &+ \int_{\mathcal{M}_3} 240 \hat{\alpha}^2 \mathcal{V}^{\frac{1}{4}} (c_3^{(0)} \cdot \omega) d \log \mathcal{V} \wedge \star d \log \mathcal{V} \\ &+ \int_{\mathcal{M}_3} \mathcal{V}^{\frac{1}{2}} F \wedge \star F - 3 \hat{\alpha}^2 \mathcal{V}^{-\frac{1}{2}} \mathcal{A} F \wedge \star F + 128 \hat{\alpha}^2 \mathcal{V}^{-\frac{1}{4}} (c_3^{(0)} \cdot \omega) F \wedge \star F \end{aligned}$$

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<sup>34</sup>The dimensional reduction was performed using the xAct bundle [84–86].

$$- \int_{\mathcal{M}_3} \star 1 \int_{CY_4} \hat{\alpha}^{2\frac{1}{4}} G^{(1)} \wedge \star^{(0)} G^{(1)}$$

- Chern-Simons term.

$$2\kappa_{11}^2 S_{CS} = \int_{\mathcal{M}_{11}} -\frac{1}{6} \hat{C}_3 \wedge \hat{G}_4 \wedge \hat{G}_4 = \int_{\mathcal{M}_3} \hat{\alpha} \Theta \mathcal{A} \wedge F, \quad \Theta = -\frac{1}{2} \int_{CY_4} \omega \wedge \omega \wedge G^{(1)}.$$

- Kinetic terms from  $(\hat{\nabla} \hat{G})^2 \hat{R}^2$ .

$$\int_{\mathcal{M}_{11}} \hat{s}_{18} (\hat{\nabla} \hat{G})^2 \hat{R}^2 \star 1 = 0 \tag{5.40}$$

## Part II

# Black Holes from D3-branes

In part II of this thesis we study black holes constructed from D3-branes in type IIB and F-theory. The goal is to compute certain anomaly coefficients which are relevant for the black hole entropy from both a microscopic and a macroscopic point of view. The setups we consider are D3-branes wrapped on a compact genus  $g$  Riemann surface  $C$  inside a complex Kähler two-fold  $X$ . This will include the cases where  $X = K3$  in chapter 6 and  $X = B$  in chapter 7, where  $B$  is the Kähler base of an elliptically fibered Calabi–Yau manifold. Wrapping the D3-brane over the curve  $C$ , we obtain a string in six dimensions which is expected to flow to a two-dimensional superconformal field theory. Macroscopically these strings may be described within the corresponding six-dimensional low-energy supergravity theory. For  $X = K3$  one obtains  $\mathcal{N} = (2, 0)$  supersymmetry in six dimensions, whilst for  $X = B$  the theory has a minimal amount of  $\mathcal{N} = (1, 0)$  supersymmetry. The distinguished feature in our setup is that we take the spaces transverse to these strings not to be  $\mathbb{R}^4$ . Instead, we consider certain families of hyperkähler manifolds as transverse spaces, such that the strings have an  $\text{AdS}_3 \times S^3/\Gamma$  near horizon geometry labelled by a discrete group  $\Gamma$ . Upon wrapping the strings on an additional circle, one obtains black holes in four and five dimensions.

For the microscopic description of the black holes we make use of convenient duality frames, which enable us to perform the relevant computations. For instance, for certain types of transverse spaces it turns out to be beneficial to dualize the type IIB/F-theory setting to M-theory. On the M-theory side the system can be reformulated in terms of M5-branes, where standard techniques apply. This enables us to extract the relevant data in a straightforward way. In some other cases, it turns out to be possible to perform the microscopic computations directly in type IIB. The microscopic theories on the D3-branes are  $\mathcal{N} = 2$  quiver gauge theories wrapped on  $C$  with a suitable topological twist to preserve supersymmetry. Utilizing this microscopic description one can then compute the relevant anomaly coefficients from the spectrum of the compactified theory.

The anomaly coefficients we are interested in are on the macroscopic side linked to Chern-Simons coefficients after compactifying the six-dimensional supergravity theory on a space which is topologically  $S^3/\Gamma$ . In certain cases, it turns out to be necessary to include one-loop contributions to these Chern-Simons coefficients which are induced by massive Kaluza-Klein modes. The inclusion of loop contributions is one of the main results contained in part II. The computation of these one-loop Chern-Simons terms can be formulated as a three-step-procedure, which we will outline in the following.



1. Determine the Kaluza-Klein spectrum on  $S^3$ , i.e. masses and representations of the modes under the  $\text{SO}(4) \simeq \text{SU}(2)_L \times \text{SU}(2)_R$  isometries
2. Compute the contribution of a single field for each type (e.g. spin-1/2, spin-3/2, etc.) in the Kaluza-Klein spectrum to the Chern-Simons coefficients
3. Sum the single field contributions over the whole Kaluza-Klein tower and project on the  $\Gamma$ -invariant states

With this recipe we are able to compute corrections to the Chern-Simons coefficients which, when combined with the classical contribution from the reduction of the supergravity action, determine the black hole entropy.



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## Chapter 6

# Wrapped D3-branes in Type IIB on K3

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In this chapter we study the central charges and levels of 2d  $\mathcal{N} = (0, 4)$  superconformal field theories that are dual to four- and five-dimensional BPS black holes in compactifications of type IIB string theory on a K3 surface. These black holes arise from wrapping a D3-brane on a genus  $g$  curve  $C$  inside K3 and have transverse space either an ALE or ALF space. The D3-branes have an  $\text{AdS}_3 \times S^3/\Gamma$  near horizon geometry where  $\Gamma$  is a discrete subgroup of  $\text{SU}(2)$ . These subgroups of  $\text{SU}(2)$  have, according to the McKay correspondence, an ADE classification. We compute the central charges and levels of the 2d SCFTs both in the microscopic picture and from six-dimensional  $\mathcal{N} = (2, 0)$  supergravity.<sup>35</sup> These quantities determine the black hole entropy via Cardy's formula. Agreement between the microscopic and macroscopic computations are found. Contributions from one-loop quantum corrections to the macroscopic result are crucial for this matching.

We are interested in compactifications of type IIB string theory on K3 and the black strings emerging in this theory. The goal is to match macroscopic data with a complimentary microscopic description. The literature on the topic is vast, in particular for black holes arising from the D1-D5-(p) system [21, 87]. We shall

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<sup>35</sup>To be more precise, we compute the 't Hooft anomaly coefficients.

instead concern ourselves with black strings arising from wrapping D3-branes on a curve inside K3, which, in comparison to the D1-D5-(p) system, has been far less studied. Explicitly we shall consider D3-branes in an asymptotic geometry  $\mathbb{R} \times S^1 \times M \times \text{K3}$  where the D3-branes are wrapped on  $S^1 \times C$  with  $C$  a curve in K3 and probe the transverse space  $M$ . The equations of motion and supersymmetry conditions imply that the four-manifold  $M$  is a Ricci-flat hyper-Kähler manifold which we take to be non-compact. We consider two families of such manifolds  $M = M_\Gamma$ : asymptotically locally Euclidean (ALE) and asymptotically locally flat (ALF) spaces, both of which may be defined by a choice of discrete subgroup  $\Gamma \subset \text{SU}(2)$  as we review in section 6.1. Different choices of  $\Gamma$  lead to black strings with different charges, some of which have not been considered before in the literature. In particular, black strings arising from D3-branes probing  $M_\Gamma$  corresponding to the D- and E-series within the ADE-classification of discrete subgroups of  $\text{SU}(2)$  have not been studied previously. Microscopically the strings are dual to  $(0, 4)$  superconformal field theories (SCFTs) which admit both left- and right-moving central charges  $c_{L,R}$ . In addition they always admit a right-moving  $\text{SU}(2)_R$  current algebra and in the case of the A-series also a left-moving  $\text{U}(1)_L$  current algebra, each with an associated level. From this data the entropy of the black string follows via Cardy's formula. In this chapter we compute the central charges and levels corresponding to the settings described above both from a macroscopic and microscopic viewpoint.

Macroscopically we shall work exclusively in six-dimensional  $\mathcal{N} = (2, 0)$  supergravity [88, 89] which is obtained by compactifying type IIB on K3 [90]. The techniques for computing the various central charges and current algebra levels corresponding to the black string solutions of this theory were developed in [91–94, 15, 95]. Concretely, we reduce the classical six-dimensional action to three dimensions in the black string background. The central charges and levels correspond to coefficients of Chern–Simons terms in the three-dimensional effective action. When  $M_\Gamma$  is ALF one also has to include one-loop Chern–Simons terms that arise from integrating out massive Kaluza-Klein modes [95]. Other settings, where one-loop Chern-Simons terms lead to corrections to the black hole entropy were studied in [96, 97].

Alternatively, we could have chosen to look at these solutions directly in type IIB supergravity and compute the central charges there. The geometries of wrapped D3-brane solutions giving rise to  $\text{AdS}_3$  near horizons were classified in [98]. This was further extended in [99, 100] to include seven-branes in addition to the D3-branes. The analysis performed in [99] shows that for wrapped D3-brane solutions preserving  $\mathcal{N} = (0, 4)$  supersymmetry the geometry is essentially unique

and takes the form  $\text{AdS}_3 \times S^3/\Gamma \times \text{K3}$ .

Microscopically the central charges and levels follow from the construction of new 2d  $\mathcal{N} = (0, 4)$  SCFTs. When  $M_\Gamma$  is ALE these arise from wrapping known 4d  $\mathcal{N} = 2$  quiver gauge theories on a Riemann surface. Wrapping 4d  $\mathcal{N} = 2$  quiver theories on Riemann surfaces are in themselves not new, see for example [101]. However, in the context of black hole microstate counting this is the first instance of such a 2d construction. The 4d parent quiver gauge theories are given by projections of non-abelian  $\mathcal{N} = 4$  supersymmetric Yang–Mills preserving  $\mathcal{N} = 2$  supersymmetry [102, 103]. One then wishes to place the theory on a curve  $C$  inside K3. In order to preserve  $\mathcal{N} = (0, 4)$  in 2d this requires a particular topological twist to be performed. We then compute the central charges and levels and find a perfect matching to the macroscopic computations. When  $M_\Gamma$  is ALF and  $\Gamma = \mathbb{Z}_m$  corresponding to the A-series, we can compute the central charges and levels by considering a dual M-theory setting. In this case  $M_\Gamma$  is the Taub-NUT space and we can T-dualize along the NUT-circle to obtain a type IIA setting. Lifting this to M-theory results in M-theory on  $\text{K3} \times T^2$  with an M5-brane that wraps  $C \times T^2$  and  $m$  M5-branes wrapping K3. These M5-branes combine into a single M5-brane wrapping  $C \times T^2 + m\text{K3}$  when the corresponding class is very ample. From this duality frame it is possible to determine the central charges [104, 105, 94]. For the D-series such a clean description is not available, we shall comment on this case further later.

In section 6.1 we begin by outlining the various setups which we consider in this chapter, in the process reviewing ALE and ALF spaces. Section 6.2 is devoted to the computation of the macroscopic central charges and levels of the 6d strings considered here. Beginning with a discussion on six-dimensional  $\mathcal{N} = (2, 0)$  supergravity and its relevant black string solutions, section 6.2 proceeds with the calculation of the classical and quantum contributions to the central charges and levels before a summary section collating the macroscopic results. The complimentary microscopic calculation of the central charges and levels, split between ALE and ALF spaces, is performed in section 6.5. We find perfect agreement with the macroscopic results of section 6.2. The appendices to this chapter contain some additional technical material.

## 6.1 Wrapped D3-branes, black holes, and ALE/ALF spaces

In this section we shall give an overview of the various setups that we consider in the following. All the cases considered here have the same underlying theory arising from compactifying type IIB string theory on a K3 surface. In the low energy limit this leads to an effective six-dimensional supergravity theory with chiral  $\mathcal{N} = (2, 0)$  supersymmetry, see section 6.2.1 for further details.

We shall consider D3-brane states giving rise to strings in the six-dimensional effective theory. As such the (R,R) and (NS,NS) fields that couple to D1, F1, D5 and NS5-branes will be switched off as we explain in section 6.2.1. To obtain strings in six dimensions we wrap the D3-branes on a genus  $g$  Riemann surface  $C \subset K3$ . When the non-compact transverse space to the string in 6d is taken to be  $\mathbb{R}^4$  the effective two-dimensional worldvolume theory living on the D3-branes is known to flow in the IR to a 2d  $\mathcal{N} = (4, 4)$  SCFT [106]. The  $SO(4) = SU(2)_L \times SU(2)_R$  rotation group of the transverse  $\mathbb{R}^4$  realizes the left- and right-moving  $SU(2)_{L,R}$  current algebras with associated levels  $k_{L,R}$  holographically. One should contrast this with the setups which form the basis of this chapter and to which we now turn.

We replace the transverse  $\mathbb{R}^4$  with a different non-compact, Ricci-flat hyper-Kähler four-manifold. Such spaces have been classified into four categories depending on their asymptotic volume growth: ALE, ALF, ALG, ALH. At infinity the metrics approach a quotient of the flat metric on  $\mathbb{R}^{4-k} \times T^k$  where the fibration at infinity is trivial except in the  $k = 1$  case where it may be fibered. The metric is ALE when  $k = 0$ , ALF when  $k = 1$ , ALG when  $k = 2$  and finally ALH when  $k = 3$ . We shall only consider two of these classes in the following: *asymptotically locally Euclidean* (ALE) and *asymptotically locally flat* (ALF) spaces<sup>36</sup>. For the benefit of the reader not familiar with these manifolds, or in need of a refresher, we summarize the salient points here.

A manifold is said to admit an ALE metric if it is diffeomorphic to  $\mathbb{R}^+ \times S^3/\Gamma$ , with  $\Gamma$  a freely acting discrete subgroup of  $SU(2)$ , and has a metric that asymptotically approaches a quotient of the Euclidean flat space metric. The possible subgroups of  $SU(2)$  admit an ADE-classification and are summarized in table 6.1. The metric is only explicitly known for  $\Gamma = \mathbb{Z}_m$  in which case it is given by the Gibbons-Hawking metric [107] which is a generalization of the

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<sup>36</sup>Note the the ‘G’ and ‘H’ do not have a meaning like in the ‘E’ and ‘F’ cases but are named as such by induction.

Eguchi-Hanson metric [108]. For the D- and E-series, the metric is known to exist [109] as a hyper-Kähler resolution of a quotient singularity  $\mathbb{C}^2/\Gamma$ , but its explicit form has yet to be established.

Likewise, ALF metrics are also diffeomorphic to  $\mathbb{R}^+ \times S^3/\Gamma$ . However, their metric instead asymptotically approaches the metric on  $(\mathbb{R}^3 \times S^1)/\Gamma$ . In fact they are only known to exist for the A- and D-series and not for the E-series of the ADE-classification [110]. In the case of the A-series they are the Taub-NUT spaces with NUT charge  $m$  [111] constructed as a circle fibration over a base  $\mathbb{R}^3$ . At the location of the centers the circle fiber shrinks to zero size whereas at infinity the radius becomes constant. In the case of the D-series the metric is only known explicitly asymptotically but an implicit construction of the full metric may be found in [112], see also [113] for the metric with quotient  $D_1$ . Sufficiently far from the center, the metric can be approximated by a Taub-NUT space with  $2m$  centers, plus a contribution to the centers with a negative mass parameter, subject to a quotient by  $\mathbb{Z}_2$  which acts by reflecting the coordinates on both  $\mathbb{R}^3$  and  $S^1$  [114]. In the black string solutions we always take the ALE and ALF spaces in their singular limit, i.e. with a  $\mathbb{C}^2/\Gamma$  singularity at the center. The black string metric however, has a smooth near horizon limit. Note that ALF spaces admit a circle at infinity whilst ALE spaces lack such a circle. This will play a crucial role in the calculation of one-loop corrections to the current levels in section 6.4.

$\Gamma \subset \text{SU}(2)$	$ \Gamma $	singularity type
cyclic group $\mathbb{Z}_m$	$m$	$A_{m-1}$
binary dihedral $\mathbb{D}_m^*$	$4m$	$D_{m+2}$
binary tetrahedral $\mathbb{T}^*$	24	$E_6$
binary octahedral $\mathbb{O}^*$	48	$E_7$
binary icosahedral $\mathbb{I}^*$	120	$E_8$

Table 6.1: The freely acting discrete subgroups of  $\text{SU}(2)$ .

Both of the setups outlined above lead to six-dimensional black strings. It is known [99], that the near horizon geometry of such a black string with  $\mathcal{N} = (0, 4)$  supersymmetry must be

$$\text{AdS}_3 \times S^3/\Gamma. \quad (6.1)$$

For the A-series these black strings have a  $\text{U}(1)_L \times \text{SU}(2)_R$  isometry group of the transverse space and may be spinning. For the D- and E-series the isometry group of the transverse space is  $\text{SU}(2)_R$  and the strings are static. We may reinterpret

these black string solutions as five-dimensional black holes by compactifying on the  $S^1$  wrapped by the D3 common to both setups. These 5d black holes have a near horizon geometry  $\text{AdS}_2 \times S^3/\Gamma$ . For ALF spaces one may perform a further compactification on the asymptotic circle to obtain four-dimensional black holes. The entropy of these black holes follows from the left-moving central charge  $c_L$  and level (for the A-series)  $k_L$  via Cardy's formula

$$S = 2\pi \times \begin{cases} \sqrt{\frac{c_L}{6} \left( n - \frac{J_L^2}{k_L} \right)} & \text{if } \Gamma = \mathbb{Z}_m, \\ \sqrt{\frac{c_L}{6} n} & \text{if } \Gamma \neq \mathbb{Z}_m. \end{cases} \quad (6.2)$$

In contrast to the  $\mathbb{R}^4$  case, where the SCFT has  $\mathcal{N} = (4, 4)$  supersymmetry, the black string solutions with either ALE or ALF transverse spaces are dual to chiral  $\mathcal{N} = (0, 4)$  SCFTs. This is manifest in the different current algebras admitted in the latter cases compared to the  $\mathbb{R}^4$  one. The current algebras of the dual 2d SCFTs are realized holographically by the isometries of the solution. Therefore the A-series SCFT has a  $U(1)_L \times SU(2)_R$  current algebra. In the D and E cases the  $U(1)_L$  is broken, implying the SCFT has only an  $SU(2)_R$  current algebra. To each current there is an associated level, denoted by  $k_{L,R}$ , and in the case of the R-symmetry this uniquely determines the right-moving central charge. In all cases the  $SU(2)_R$  is identified with the  $SU(2)_r$  R-symmetry of the small superconformal algebra of a 2d  $\mathcal{N} = (0, 4)$  SCFT and supersymmetry implies that the right-moving central charge and right-moving level are related via  $c_R = 6k_R$ .<sup>37</sup>

## 6.2 Macroscopics of 6d strings

In this section we compute the central charges and levels corresponding to the setups described in the previous section using six-dimensional  $\mathcal{N} = (2, 0)$  supergravity [90, 88, 89]. As we shall review below, compactifying type IIB supergravity on K3 results in a gravity multiplet coupled to 21 tensor multiplets [90, 88]. The macroscopic configuration is that of a black string solution in 6d with near horizon geometry  $\text{AdS}_3 \times S^3/\Gamma$  which asymptotically approaches  $\mathbb{R}^{1,1} \times M_{\Gamma\infty}$ , with  $M_{\Gamma\infty}$  denoting the asymptotic geometry of the ALE/ALF space  $M_\Gamma$ . In principle

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<sup>37</sup>This is a subtle point and technically only applies when one decouples the contributions of the center of mass modes, see [94]. However as we only compute the levels and  $c_L - c_R$  on both sides this will not matter for the matching between the microscopics and the macroscopics. We discuss this issue in more detail in section 6.5.1.



one could compute the central charges and levels by dimensionally reducing the six-dimensional action on the compact spherical part of the near horizon geometry in order to obtain an effective action on  $\text{AdS}_3$  [91–93]. Using the AdS/CFT dictionary the central charges and levels then correspond to the coefficients of Chern–Simons terms in the dimensionally reduced action. Concretely,  $k_{L,R}$  correspond (when the former is present) to the coefficients of the  $\text{U}(1)_L$  (when present) and  $\text{SU}(2)_R$  Chern–Simons terms respectively, whilst  $c_L - c_R$  gives the coefficient of the gravitational Chern–Simons term.

**Reduction at asymptotic infinity.** The Bekenstein–Hawking entropy of a black hole scales with the area of its event horizon and since this entropy can be calculated from the central charges and levels, one would expect that one has to do the reduction to three dimensions in the near horizon geometry. However, black holes can have hair, in other words degrees of freedom living outside of the horizon and contributing to the microscopic degeneracy [115, 116]. A well studied example is provided by considering the BMPV black hole [117], which is microscopically described by a D1–D5 system of type IIB on  $\text{K3} \times S^1$  carrying momentum along  $S^1$  and having equal angular momentum in two planes transverse to the D5-brane. Macroscopically this is a five-dimensional rotating black hole. This BMPV black hole can be placed at the center of a Taub–NUT space to get a four-dimensional black hole, since Taub–NUT space with  $m = 1$  looks like  $\mathbb{R}^4$  close to its center. While the five-dimensional near horizon geometries of the BMPV black hole and its Taub–NUT generalization are the same, the microscopic degeneracies were shown to be different [118]. The difference can be explained by invoking the aforementioned hair. For example the center of mass degrees of freedom of the brane system are not captured by the near horizon geometry. Since our setting also includes, among other non-trivial transverse spaces, a Taub–NUT space we expect non-vanishing contributions from hair. This hair has to be taken into account to match the microscopic results we find for the four- and five-dimensional black holes. However, instead of explicitly constructing the hair modes as done in [115] for the BMPV black hole, we use the approach suggested in [94]. More precisely, we perform the reduction to three dimensions at asymptotic infinity. Concretely, this means that the reduction is done on the spherical part at large radial coordinate  $r$ . According to [94] the macroscopic levels and central charges, which we will compare with their microscopic counterparts, are then in terms of the asymptotic quantities given by

$$k_L = k_L^{\text{asympt}} + \delta_L, \quad k_R = k_R^{\text{asympt}} + \delta_R,$$

$$c_L = c_L^{\text{asympt}} + \Delta, \quad c_R = 6k_R. \quad (6.3)$$

The quantities  $\delta_L$ ,  $\delta_R$ ,  $\Delta$  are further  $\mathcal{O}(1)$  contributions. Since the main focus of this part of the thesis is on the terms that are proportional to the charges of the four- and five-dimensional black holes, we will not compute these contributions explicitly. In [94] the difference of the asymptotic and near horizon reduction manifested itself at the level of  $\mathcal{O}(1)$  contributions, which we do not consider in the following.

In addition to performing the reduction of the supergravity action at infinity one is moreover required to include contributions from one-loop Chern–Simons terms arising from integrating out massive Kaluza-Klein (KK) modes [95]. For the setups considered here, the classical contributions turn out to be the same whether they are computed in the near horizon geometry or at asymptotic infinity. The quantum corrections, on the other hand do depend on this. For this reason we shall perform computations at asymptotic infinity, reducing the action using the asymptotic solution. Having said this, it is sometimes more convenient to perform the computation in the near horizon geometry when it is known that the asymptotic computation will agree with the near horizon one, i.e. when the contributions of the hair vanish.

We begin this section with a review of six-dimensional  $\mathcal{N} = (2, 0)$  supergravity arising from type IIB supergravity compactified on K3. We then compute the classical and quantum corrections to the central charges and levels by first reducing the six-dimensional action to three dimensions and then including the one-loop contributions. We conclude the section with a summary of the results obtained therein in anticipation of comparing with the microscopic computations in the subsequent section.

### 6.2.1 Six-dimensional supergravity from type IIB on K3

The massless bosonic field content of type IIB supergravity includes a metric, a complex scalar, two real two-forms and a four-form with self-dual field strength. The cohomology of the K3 surface admits one scalar, three self-dual two-forms, nineteen anti-self-dual two-forms and one four-form. As usual the reduction to the massless 6d sector follows by expanding the various type IIB supergravity fields in terms of the generators of the cohomology of K3. Reducing the four-form leads to one scalar, three self-dual rank two tensors and nineteen anti-self-dual rank two tensors. The two 10d two-forms each lead to twenty-two scalars, one self-dual rank two tensor and one anti-self-dual rank two tensor in 6d. The complex scalar

trivially reduces to two real scalars, whilst the metric leads to the graviton in 6d and fifty-eight scalars. In total there are 105 scalars, 21 anti-self-dual rank two tensors, five self-dual rank two tensors and the graviton.

Having determined the massless bosonic fields in 6d we may now rearrange them into 6d  $\mathcal{N} = (2, 0)$  multiplets. The various representations, labeled by the spins  $(j_1, j_2)$  of the little group  $\text{SO}(4) \simeq \text{SU}(2)_1 \times \text{SU}(2)_2$ , are:

- one gravity multiplet:  $(1, 1) \oplus 4(\frac{1}{2}, 1) \oplus 5(1, 0)$ , containing one graviton, 2 left-handed gravitinos and five self-dual rank two tensors,
- $n_T$  tensor multiplets:  $(0, 1) \oplus 4(0, \frac{1}{2}) \oplus 5(0, 0)$ , containing one anti-self-dual rank two tensor, two right-handed tensorinos and 5 real scalars.

As sketched above, the compactification on K3 results in a theory coupled to  $n_T = 21$  tensor multiplets. This is the exact number of tensors necessary for the theory to be anomaly free and is in fact the unique anomaly-free six-dimensional  $\mathcal{N} = (2, 0)$  theory [90]. Uniqueness follows from the fact that the only possible matter multiplet in chiral  $\mathcal{N} = (2, 0)$  supergravity is the tensor multiplet and that the cancellation of gravitational anomalies requires  $n_T = 21$ .

To perform computations it is advantageous to collectively denote the tensors in the gravity- and tensor multiplets by  $\hat{B}^\alpha$ , with  $\alpha = 1, \dots, 26$ , and their corresponding (self-dual) field strengths as  $\hat{G}^\alpha = d\hat{B}^\alpha$ . The 105 scalars in the tensor multiplets parametrize the coset

$$\mathcal{M}_{\text{tensor}} = \frac{\text{SO}(5, 21)}{\text{SO}(5) \times \text{SO}(21)}. \quad (6.4)$$

It turns out that in order to parametrize the scalar manifold it is more convenient to use the 130 scalars  $\hat{j}_k^\alpha$ ,  $k = 1, \dots, 5$ , satisfying the constraints

$$\Omega_{\alpha\beta} \hat{j}_k^\alpha \hat{j}_l^\beta = \delta_{kl}. \quad (6.5)$$

Here  $(\Omega_{\alpha\beta})$  is the  $\text{SO}(5, 21)$  invariant constant metric with mostly minus signature. It may be used to define a positive definite metric via

$$g_{\alpha\beta} = 2\hat{j}_{k\alpha}\hat{j}_\beta^k - \Omega_{\alpha\beta}, \quad \hat{j}_{k\alpha} \equiv \Omega_{\alpha\beta}\hat{j}_k^\beta, \quad \hat{j}^{k\alpha} \equiv \delta^{kl}\hat{j}_l^\alpha. \quad (6.6)$$

With this formalism the bosonic part of the pseudo-action is given by [89]<sup>38</sup>

$$S^{(6)} = \frac{1}{(2\pi)^3} \int_{\mathcal{M}_6} \left[ \frac{1}{2} \hat{R} \hat{*} 1 - \frac{1}{4} g_{\alpha\beta} \hat{G}^\alpha \wedge \hat{*} \hat{G}^\beta - \frac{1}{2} (\hat{j}_{k\alpha} \hat{j}_\beta^k - \Omega_{\alpha\beta}) d\hat{j}_l^\alpha \wedge \hat{*} d\hat{j}^{l\beta} \right], \quad (6.7)$$

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<sup>38</sup>Note that our conventions differ with respect to [89], i.e.  $\hat{G}_{\text{there}}^\alpha = \frac{1}{2} \hat{G}_{\text{here}}^\alpha$ .

where we chose conventions such that  $\kappa_6^2 = (2\pi)^3$ . This is a pseudo-action since the self-duality constraints for the tensors,

$$g_{\alpha\beta} \hat{*} \hat{G}^\beta = \Omega_{\alpha\beta} \hat{G}^\beta, \quad (6.8)$$

do not follow from the action and must be imposed by hand at the level of the equations of motion.

### 6.2.2 Black string solutions

Recall that we are interested in black strings arising from wrapping a D3-brane on a curve inside K3. As such we cannot turn on tensor fields arising from the reduction of the two 10d two-forms, each providing a self-dual and anti-self-dual tensor field in 6d. Moreover, preservation of supersymmetry implies that the Poincaré dual of the wrapped curve is in  $H^{1,1}(K3)$  [119, 120, 99]. This implies that the six-dimensional string is only charged under tensors arising from the reduction of the four-form along two-forms  $\omega_\alpha$  in  $H^{1,1}(K3)$ . Consequently we may only have non-vanishing values for one self-dual tensor and 19 anti-self-dual tensors. Note that these are exactly the tensors we find in  $\mathcal{N} = (1, 0)$  supergravity coupled to 19 tensor multiplets. This theory has black string solutions with a transverse hyper-Kähler space [121, 122]. These solutions may be embedded into the  $\mathcal{N} = (2, 0)$  theory by setting the scalars  $\hat{j}_k^\alpha = \delta_k^{27-\alpha}$  for  $k = 2, 3, 4, 5$  and  $\hat{j}_1^\alpha = 0$  for  $\alpha = 21, \dots, 26$ .<sup>39</sup> Therefore, from now on,  $\alpha = 1, \dots, 20$ , and the scalars  $\hat{j}^\alpha \equiv \hat{j}_1^\alpha$  satisfy

$$\Omega_{\alpha\beta} \hat{j}^\alpha \hat{j}^\beta = 1, \quad (6.9)$$

where  $(\Omega_{\alpha\beta})$  is the (canonical)  $\text{SO}(1, 19)$  invariant constant metric with mostly minus signature. It can be identified with the metric on  $H^{1,1}(K3)$

$$\Omega_{\alpha\beta} = \eta_{\alpha\beta} = \int_{K3} \omega_\alpha \wedge \omega_\beta. \quad (6.10)$$

The black string solutions in question have a metric of the form [121, 122]

$$ds_6^2 = 2H^{-1} (du + \beta) \left( dv + \omega + \frac{1}{2} \mathcal{F} (du + \beta) \right) + H ds^2(M_\Gamma), \quad (6.11)$$

where  $ds^2(M_\Gamma)$  is either the ALE or ALF hyper-Kähler metric corresponding to the finite group  $\Gamma$ . The one-forms  $\omega, \beta$  are defined on  $M_\Gamma$  and, like the functions

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<sup>39</sup>For later simplicity we have taken a non-canonical choice for  $\Omega$ . We take the first diagonal entry to be +, the next 21 – and the last four entries to be +.

$H$  and  $\mathcal{F}$ , are independent of  $u$  and  $v$ . They satisfy certain equations [121] that in principle can be solved once an explicit expression for the hyper-Kähler metric is given.

Note that  $\partial_u$  is a Killing vector of the metric (6.11) and taking it to be spacelike we find a metric for a black string wound in the  $u$ -direction. The near horizon geometry of this string is  $\text{AdS}_3 \times S^3/\Gamma$  [99] with asymptotics given by (6.11) up to replacing  $ds^2(M_\Gamma)$  by its asymptotic metric  $ds^2(M_{\Gamma\infty})$ . The asymptotic space has a finite covering and its metric approaches a  $\Gamma$ -quotient of the following metric sufficiently fast:

$$ds^2(M_\infty) = \begin{cases} dr^2 + r^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) & M_\Gamma \text{ is ALE}, \\ dr^2 + r^2(\sigma_1^2 + \sigma_2^2) + \sigma_3^2 & M_\Gamma \text{ is ALF}. \end{cases} \quad (6.12)$$

Here  $\sigma_i$  are the left-invariant one-forms on  $S^3$ . We use the covering space to treat all quotients simultaneously. The charges of the black string follow from integrating the three-forms  $\hat{G}_\Gamma^\alpha$  in the quotient space over the spherical part of the metric  $M^{\text{sph}}$  (given by the  $\sigma_i$  part). We choose the normalization

$$\int_{M_\Gamma^{\text{sph}}} \hat{G}_\Gamma^\alpha = \frac{1}{|\Gamma|} \int_{M^{\text{sph}}} \hat{G}^\alpha = -(2\pi)^2 Q^\alpha, \quad (6.13)$$

by using the quotient map  $\pi : M \rightarrow M_\Gamma$ , and the pullback to the covering space  $\hat{G}^\alpha = \pi^* \hat{G}_\Gamma^\alpha$ . These charges are related to the microscopic charges  $q^\alpha$  via

$$Q^\alpha = q^\alpha, \quad (6.14)$$

where  $C = q^\alpha \omega_\alpha$  is the Poincaré dual of the curve  $C$  wrapped by the D3-brane in the microscopic picture. This identification may be proven in the same manner as in section 7.2.2, where we demonstrate this identification in more detail in the F-theory setting. In F-theory, we will find a correction to the relation (6.14), which is due to the appearance of a certain higher-derivative coupling. This coupling is not present in the type IIB setup with D3-branes only, since this higher-derivative coupling has its origin on D7-branes and O7-planes.

### 6.3 Classical contribution for ALE and ALF transverse spaces

In this section we determine the classical contribution to the central charges and levels. Here ‘classical’ refers to the contribution that is obtained by reducing the

six-dimensional  $\mathcal{N} = (2, 0)$  pseudo-action to three dimensions along the compact space. The contributions to the central charges and levels are given by the coefficients of 3d Chern–Simons terms. We therefore only need to consider the part of the pseudo-action that can give rise to terms of this form, specifically the term

$$-\frac{1}{32\pi^3} \int_{\mathcal{M}_6} g_{\alpha\beta} \hat{G}^\alpha \wedge \hat{*}\hat{G}^\beta \subset S^{(6)}. \quad (6.15)$$

The reduction for the black strings with either ALE or ALF transverse space will be the same and therefore we treat them concurrently.

To perform the reduction one gauges the isometries of  $S^3/\Gamma$  by introducing gauge fields for these symmetries. This requires a modification of the three-form flux and we take the ansatz for  $\hat{G}^\alpha$  to be [93, 94]<sup>40</sup>

$$\hat{G}^\alpha = -Q^\alpha [(2\pi)^2 |\Gamma| (e_3 - \chi_3) + \omega(\mathcal{M}_3)]. \quad (6.16)$$

The three-dimensional space  $\mathcal{M}_3$  is the non-spherical part of the covering space, to which we reduce the action, and  $\omega(\mathcal{M}_3)$  is a three-form on  $\mathcal{M}_3$  whose form we do not specify. The three-form  $e_3$  appearing in (6.16) is invariant under the isometries of the spacetime, integrates to unity over the spherical part

$$\int_{M^{\text{sph}}} e_3 = 1, \quad (6.17)$$

and has exterior derivative

$$de_3 = \begin{cases} \frac{1}{16\pi^2} F_L \wedge F_L + \frac{1}{8\pi^2} \text{tr } F_R \wedge F_R & \Gamma = \mathbb{Z}_m, \\ \frac{1}{8\pi^2} \text{tr } F_R \wedge F_R & \Gamma \neq \mathbb{Z}_m. \end{cases} \quad (6.18)$$

An explicit form for this three-form can be obtained from [93]. We will refrain from giving this explicit expression here, as the general properties mentioned above will be sufficient for our purposes. The two cases, namely  $\Gamma = \mathbb{Z}_m$  and  $\Gamma \neq \mathbb{Z}_m$ , are qualitatively different. In the case where  $\Gamma = \mathbb{Z}_m$  we have gauge fields for  $U(1)_L \times SU(2)_R$ , whereas for  $\Gamma \neq \mathbb{Z}_m$  the group  $U(1)_L$  is broken, and we only have the remaining  $SU(2)_R$  gauge field. Therefore, the  $U(1)_L$  gauge field

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<sup>40</sup>Strictly the expression (6.16) only satisfies the self-duality constraint after setting the gauge fields to zero. However this is sufficient as the gauge fields are treated as fluctuations around the background.

has to be set to zero in this case. The three-form  $\chi_3$  in (6.16) is given by

$$\chi_3 = \begin{cases} \frac{1}{16\pi^2} A_L \wedge F_L + \frac{1}{8\pi^2} \text{tr} \left( A_R \wedge dA_R + \frac{2}{3} A_R^3 \right) & \Gamma = \mathbb{Z}_m, \\ \frac{1}{8\pi^2} \text{tr} \left( A_R \wedge dA_R + \frac{2}{3} A_R^3 \right) & \Gamma \neq \mathbb{Z}_m, \end{cases} \quad (6.19)$$

and is included to ensure that  $\hat{G}^\alpha$  satisfies its Bianchi identity.

We are now ready to determine the contribution of (6.15) to the central charges and levels. We calculate it by determining the gauge variation of the reduced action under either  $U(1)_L \times SU(2)_R$  gauge transformations for  $\Gamma = \mathbb{Z}_m$  or  $SU(2)_R$  gauge transformations for  $\Gamma \neq \mathbb{Z}_m$ . We parametrize the gauge transformations by  $\Lambda$  and determine the variation of the three-dimensional action by reducing the variation of the six-dimensional action on  $M_\Gamma^{\text{sph}}$ . By construction  $e_3$  is invariant under all gauge transformations, consequently only the variation of  $\chi_3$  contributes to the variation of the six-dimensional action. We find<sup>41</sup>

$$\begin{aligned} \delta_\Lambda \mathcal{L}_{\text{CS}} *_3 1 &= -\frac{1}{16\pi^3} \int_{M_\Gamma^{\text{sph}}} g_{\alpha\beta} \delta_\Lambda \hat{G}_\Gamma^\alpha \wedge \hat{*} \hat{G}_\Gamma^\beta = -\frac{1}{16\pi^3 |\Gamma|} \int_{M^{\text{sph}}} g_{\alpha\beta} \delta_\Lambda \hat{G}^\alpha \wedge \hat{*} \hat{G}^\beta \\ &= \pi |\Gamma| \eta_{\alpha\beta} Q^\alpha Q^\beta \int_{M^{\text{sph}}} \delta_\Lambda \chi_3 \wedge e_3 = \pi |\Gamma| \eta_{\alpha\beta} Q^\alpha Q^\beta \delta_\Lambda \chi_3, \end{aligned} \quad (6.20)$$

where the third equality follows by using the (anti-)self-duality condition (6.8). This gives the gauge variation of the three-dimensional action given by

$$\begin{aligned} S_{\text{CS}} &= \pi |\Gamma| \eta_{\alpha\beta} Q^\alpha Q^\beta \int_{\mathcal{M}_{\Gamma 3}} \chi_3 \\ &= \begin{cases} \frac{k_L^{\text{class}}}{8\pi} \int_{\mathcal{M}_{\Gamma 3}} A_L \wedge F_L + \frac{k_R^{\text{class}}}{4\pi} \int_{\mathcal{M}_{\Gamma 3}} \text{tr} \left( A_R \wedge dA_R + \frac{2}{3} A_R^3 \right) & \Gamma = \mathbb{Z}_m, \\ \frac{k_R^{\text{class}}}{4\pi} \int_{\mathcal{M}_{\Gamma 3}} \text{tr} \left( A_R \wedge dA_R + \frac{2}{3} A_R^3 \right) & \Gamma \neq \mathbb{Z}_m. \end{cases} \end{aligned} \quad (6.21)$$

Comparing the two expressions we thus find for the central charges and levels

$$\begin{aligned} k_L^{\text{class}} &= \frac{1}{2} |\Gamma| \eta_{\alpha\beta} Q^\alpha Q^\beta = \frac{1}{2} |\Gamma| C \cdot C & \text{only for } \Gamma = \mathbb{Z}_m, \\ k_R^{\text{class}} &= \frac{1}{2} |\Gamma| C \cdot C, \end{aligned} \quad (6.22)$$

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<sup>41</sup>In our conventions  $\int_{\mathcal{M}_6} = \int_{\mathcal{M}_{\Gamma 3}} \cdot \int_{M_\Gamma^{\text{sph}}}$ , where  $\mathcal{M}_{\Gamma 3}$  is the three-dimensional non-spherical part of the quotient space.

$$c_L^{\text{class}} = c_R^{\text{class}},$$

where we have introduced the notation

$$C \cdot C = \int_{K3} C \wedge C = q^\alpha \eta_{\alpha\beta} q^\beta. \quad (6.23)$$

In writing (6.22) we have used that  $c_L^{\text{class}} - c_R^{\text{class}} = 0$  due to the absence of a 3d gravitational Chern–Simons term. It is tempting to set  $c_R = 6k_R$  using the field theory result, however as we will explain in section 6.5.1 this is only true modulo center of mass modes.

We note that if we had performed the reduction in the near horizon geometry we would have obtained exactly the same results as in (6.22). This is a quirk of the current setup, as in general a difference in the two reductions is possible when the supergravity action contains a certain higher derivative term, but as such a term is missing here, the results agree. We will encounter such a difference in the F-theory setup in section 7.2.3. When  $M_T$  is ALE (6.22) gives the full answer. On the other hand for ALF transverse space one must also include one-loop contributions originating from integrating out massive Kaluza-Klein modes, as we have an asymptotic circle in the geometry. The content of the next section is devoted to the calculation of these one-loop contributions and showing that they only contribute in the ALF case and not the ALE one.

## 6.4 Quantum contributions for ALF transverse spaces

We will see here and in chapter 7 that including contributions from one-loop Chern–Simons terms is essential for reproducing the correct central charges and levels. These terms arise from integrating out massive Kaluza-Klein (KK) modes running in the loops of the relevant two-point functions. The relevant KK modes come from the chiral fields in six dimensions, i.e. the six-dimensional gravitinos, the spin- $\frac{1}{2}$  fermions in the tensor multiplets and the (anti-)self-dual two-forms. After dimensional reduction to three dimensions these fields lead to massive spin- $\frac{3}{2}$ , spin- $\frac{1}{2}$  and chiral vector fields. For simplicity we shall perform the calculation of the one-loop Chern–Simons terms in the  $\text{AdS}_3 \times S^3/\Gamma$  near horizon geometry. Given our previous comments this may seem contradictory. However, we shall argue that for ALF transverse spaces the near horizon computation matches the analogous computation at infinity. Further we will explain why for ALE transverse spaces the quantum contributions vanish when dimensionally reducing in the asymptotic geometry.



Let us end this introduction by outlining the strategy we will follow in the remainder of the section to compute the quantum corrections. We begin by giving the relevant KK spectrum for  $\mathcal{N} = (2, 0)$  supergravity on  $\text{AdS}_3 \times S^3$  as determined in [123, 124]. In particular, for each mode we give the three-dimensional Lorentz representation, their  $\text{SO}(4) = \text{SU}(2)_L \times \text{SU}(2)_R$  representation following from the isometries of the three-sphere, and the sign of the mass. We shall then determine which modes are projected out under the action of  $\Gamma = \mathbb{Z}_m$  and  $\Gamma = \mathbb{D}_m^*$ , thereby determining the KK-spectrum after the reduction on  $\text{AdS}_3 \times S^3/\Gamma$ . Having determined the spectrum we then compute the contribution of each mode to the three-dimensional  $\text{U}(1)_L$  (only when  $\Gamma = \mathbb{Z}_m$ ),  $\text{SU}(2)_R$  and gravitational Chern-Simons terms using the Atiyah-Patodi-Singer index theorem. Finally we sum the individual contributions over all modes. Adding these to the classical part determined in section 6.3 we obtain the macroscopic data up to order  $\mathcal{O}(1)$  in the charges.

### 6.4.1 Kaluza-Klein spectrum

The six-dimensional fields giving rise to massive three-dimensional chiral modes are the 2 gravitinos,  $2n_T = 42$  tensorinos, 5 self-dual rank two tensors and  $n_T = 21$  anti-self-dual rank two tensors. The gravitinos and tensorinos are given by two Weyl fermions obeying a symplectic-Majorana condition and the tensors are subject to a reality condition. The gravitinos give rise to three-dimensional spin- $\frac{3}{2}$  and spin- $\frac{1}{2}$  particles, the tensorinos to spin- $\frac{1}{2}$  particles and the (anti)-self-dual tensors to ‘chiral’ vector fields. These somewhat exotic chiral, (anti)-self-dual vector fields in three dimensions were first discussed in [125]. One loop corrections due to massive chiral vectors and higher rank tensors were studied in [126].

The KK spectrum on  $\text{AdS}_3 \times S^3$  was computed in [123, 124]. Initially we will not take into account the symplectic-Majorana nor reality condition, instead imposing these later when the sums over all states are performed. The representations are labeled in terms of  $\text{SU}(2)_{L,R}$  spins  $j_{L,R}$  and the sign of the mass,  $\text{sgn}(M)$ , as  $(j_L, j_R)^{\text{sgn}(M)}$ <sup>42</sup>. The representations are:

- Spin- $\frac{3}{2}$ :

$$4 \bigoplus_{j_L=\frac{1}{2}}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\mp}.$$

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<sup>42</sup>We use notation similar to [124]. The same spectrum in [123] is listed in terms of the highest weight vector  $(l_1, l_2)$  of  $\mathfrak{so}(4)$ . The relation with our notation is given by  $l_1 = j_L + j_R$ ,  $l_2 = j_L - j_R$ .

- Spin- $\frac{1}{2}$ :

$$\begin{aligned}
& 4 \bigoplus_{j_L=\frac{3}{2}}^{\infty} (j_L, j_L \pm \frac{3}{2})^{\mp} \oplus 4 \bigoplus_{j_L=0}^1 (j_L, j_L + \frac{3}{2})^{-} \oplus 4 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \quad (6.24) \\
& \oplus 4(\frac{1}{2}, 1)^{+} \oplus 4(0, \frac{1}{2})^{+} \oplus 84 \bigoplus_{j_L=\frac{1}{2}}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \oplus 84(0, \frac{1}{2})^{+}.
\end{aligned}$$

- Chiral vectors:

$$\begin{aligned}
& 5 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm 1)^{\mp} \oplus 5(\frac{1}{2}, \frac{3}{2})^{-} \oplus 5(0, 1)^{-} \oplus 21 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm 1)^{\pm} \\
& \oplus 21(\frac{1}{2}, \frac{3}{2})^{+} \oplus 21(0, 1)^{+}.
\end{aligned}$$

The notation

$$(j_L, j_L \pm \frac{1}{2})^{\mp} = (j_L, j_L + \frac{1}{2})^{-} \oplus (j_L, j_L - \frac{1}{2})^{+} \quad (6.25)$$

symbolizes two towers of KK modes. Furthermore, the three-dimensional fermions in the above spectrum are Dirac spinors and the chiral vectors are complex. We also must apply the six-dimensional symplectic-Majorana and reality conditions. Denoting the eigenvalues of the generators of  $U(1)_L \subset SU(2)_L$  and  $U(1)_R \subset SU(2)_R$  by  $j_L^{(3)}$  and  $j_R^{(3)}$  respectively, these conditions imply that modes with  $j_L^{(3)}$  and  $j_R^{(3)}$  get mapped to modes with  $-j_L^{(3)}$  and  $-j_R^{(3)}$  respectively [127]. As a result we only need to sum over modes with  $j_R^{(3)} \geq 0$  or  $j_L^{(3)} \geq 0$ . To determine the KK spectrum on  $S^3/\Gamma$  we determine which states are invariant under the action of  $\Gamma$  for  $\Gamma = \mathbb{Z}_m$  and  $\Gamma = \mathbb{D}_m^*$ . The full details are presented in appendix C. For  $\Gamma = \mathbb{Z}_m$  the invariant states satisfy  $j_L^{(3)} = \frac{1}{2}mk$  with  $k \in \mathbb{Z}$ , the symplectic-Majorana and reality conditions further refine this to  $k \in \mathbb{Z}_{\geq 0}$ . For  $\mathbb{D}_m^*$  the invariant representations have  $j_L^{(3)} = mk$  with  $k \in \mathbb{Z}_{\geq 0}$ . The reality conditions imply in addition that we only keep states with  $j_R^{(3)} \geq 0$ .

At asymptotic infinity the reduction is not performed on  $S^3/\Gamma$  but on the compact part of the quotient of (6.12). It is therefore not a priori clear that the KK spectrum is the same as the one given above. For  $\Gamma = \mathbb{Z}_m$  the reduction is on a squashed Lens space, where the radius of the two-sphere is taken to infinity. The Hopf circle still has finite radius and therefore the expectation is that the representation content of the KK spectrum will be the same as for  $S^3/\mathbb{Z}_m$ . We shall assume that this expectation is met and moreover that the signs of the masses do not change. For  $\Gamma = \mathbb{D}_m^*$  there is still a two-sphere in the asymptotic

geometry whose radius tends to infinity. However in contrast to the A-series there is no longer a circle, instead there is a segment with finite length, see for example [114]. Again the expectation is that we can use the previously determined KK spectrum.

This discussion sheds light on why we do not have to include one-loop Chern–Simons terms for a transverse ALE space. From (6.12) it is easy to see that there is no compact space at asymptotic infinity on which one can perform a KK reduction, ergo there are no quantum corrections in the ALE case and the classical result of section 6.3 is the final result.

#### 6.4.2 One-loop Chern–Simons terms from massive Kaluza-Klein modes

The last step which we need to take in order to be able to compute the one loop contributions to the Chern–Simons terms is to determine the contribution of a single field to the Chern–Simons terms. We need to determine the contributions for each type of field in the Kaluza-Klein spectrum (6.24). Once we have succeeded in this last step, we are able to sum over the whole spectrum using the data given in section 6.4.1.

Quantum corrections to Chern-Simons terms can be interpreted as compensations for the parity violation introduced by families of massive fields, after they are integrated out [83]. The fields that contribute in our case to the three-dimensional parity anomaly are massive spin- $\frac{1}{2}$  fermions, spin- $\frac{3}{2}$  fermions and massive vectors in three dimensions. We can thus calculate these corrections by calculation of the parity-violating piece of the effective action which can be expressed using the Atiyah-Patodi-Singer  $\eta$ -invariant [128] corresponding to the relevant Dirac operator. This  $\eta$ -invariant can be expressed in Chern-Simons terms by extending the Dirac operator to one dimension higher and using the Atiyah-Patodi-Singer index theorem [128]. This calculation is valid for three-dimensional Riemannian manifolds of the form  $\mathcal{M}_3 = \mathbb{R} \times \mathcal{M}_2$ , where  $\mathcal{M}_2$  is a compact manifold without boundary. Since we are doing the reduction at infinity, where the three-dimensional manifold (after Wick rotation) is of the form  $\mathbb{R}^2 \times S^1$ , the index theorem is indeed applicable by treating this manifold as  $\mathbb{R} \times S_R^1 \times S^1$ , where we take the radius of the  $S_R^1$  circle to be very large.

We now first treat the spin- $\frac{1}{2}$  fermions, the spin- $\frac{3}{2}$  fermions and the massive vectors separately. The loop corrections induced by these three types of fields are summarized in table 6.2. After these corrections are determined, we sum the

latter over the spectrum determined in the previous section to compute the full one-loop correction to the central charges and levels.

**Spin- $\frac{1}{2}$  fermions.** We consider a massive spin- $\frac{1}{2}$  fermion coupled to the gauge fields  $A = (A_L, A_R)$  taking values in the Lie algebra  $\mathfrak{u}(1)_L \oplus \mathfrak{su}(2)_R$  and to an external gravitational field denoted by the vielbein  $e$  with spin connection  $\omega$ . The parity anomaly resulting from this particle can be canceled by a term [128]

$$-i\pi \operatorname{sgn}(M) \int_{\mathcal{M}_3} Q_{\frac{1}{2}}(A, \omega), \quad (6.26)$$

with

$$dQ_{\frac{1}{2}}(A, \omega) = \hat{A}(\mathcal{M}_3) \wedge \operatorname{ch}(F_L) \wedge \operatorname{ch}(F_R)|_{4\text{-form}}, \quad (6.27)$$

where the vertical dash denotes that we pick out the four-form contribution of the whole expansion on the right hand side of (6.27). The form on the right-hand side is the index density of the Dirac operator for spin- $\frac{1}{2}$  particles in four dimensions. It is expressed in terms of the Dirac genus  $\hat{A}$  and Chern character, which have an expansion

$$\hat{A}(\mathcal{M}_3) = 1 + \frac{1}{(4\pi)^2} \frac{1}{12} \operatorname{tr} \mathcal{R} \wedge \mathcal{R} + \dots, \quad (6.28)$$

$$\operatorname{ch}(F) = r + \frac{i}{2\pi} \operatorname{tr} F - \frac{1}{2} \frac{1}{(2\pi)^2} \operatorname{tr} F \wedge F - \frac{i}{6} \frac{1}{(2\pi)^3} \operatorname{tr} F \wedge F \wedge F + \dots,$$

where  $r$  is the dimension of the representation of the gauge group, under which the spin- $\frac{1}{2}$  fermion transforms. We use that

$$\hat{A}(\mathcal{M}_3) \wedge \operatorname{ch}(F_L) \wedge \operatorname{ch}(F_R)|_{4\text{-form}} = \frac{1}{(2\pi)^2} \left( \frac{r}{4} F_L \wedge F_L - \frac{1}{2} \operatorname{tr} F_R \wedge F_R + \frac{r}{48} \operatorname{tr} \mathcal{R} \wedge \mathcal{R} \right), \quad (6.29)$$

where now  $r$  is the dimension of the  $\mathfrak{su}(2)_R$  representation of the spin- $\frac{1}{2}$  fermion and we used that the generator for the  $\mathfrak{u}(1)_L$  is in terms of the Pauli matrices given by  $-\frac{i}{2}\sigma_3$ . We find that the counterterm to cancel the parity anomaly is then given by

$$\operatorname{sign}(M) \left( -\frac{ir}{16\pi} A_L \wedge F_L + \frac{i}{8\pi} \omega_{\text{CS}}(A_R) - \frac{ir}{192\pi} \omega_{\text{grav}}^{\text{CS}} \right). \quad (6.30)$$

Note that these are the corrections to the action on the Riemannian manifold. We still have to Wick rotate to Lorentzian signature by multiplying with a factor  $i$ , which yields the counter terms

$$\operatorname{sign}(M) \left( \frac{r}{16\pi} A_L \wedge F_L - \frac{1}{8\pi} \omega_{\text{CS}}(A_R) + \frac{r}{192\pi} \omega_{\text{grav}}^{\text{CS}} \right). \quad (6.31)$$

**Spin- $\frac{3}{2}$  fermions.** For spin- $\frac{3}{2}$  fermions the counterterm is given by (6.26) but now with [128, 129]

$$dQ_{\frac{3}{2}}(A, \omega) = \hat{A}(\mathcal{M}_3) \wedge \left[ \text{tr} \exp \left( \frac{i\mathcal{R}}{2(2\pi)^2} \right) - 1 \right] \wedge \text{ch}(F_L) \wedge \text{ch}(F_R) \Big|_{4\text{-form}}. \quad (6.32)$$

Using that  $\text{tr} \exp \left( \frac{i\mathcal{R}}{2\pi} \right) - 1 = 3 - \frac{1}{2(2\pi)^2} \text{tr} \mathcal{R} \wedge \mathcal{R} + \dots$ , we find that

$$dQ_{\frac{3}{2}}(A, \omega) = \frac{3r}{4(2\pi)^2} F_L \wedge F_L - \frac{3}{2(2\pi)^2} \text{tr} F_R \wedge F_R - \frac{7r}{16(2\pi)^2} \text{tr} \mathcal{R} \wedge \mathcal{R}. \quad (6.33)$$

The counterterm to the Lorentzian action then becomes

$$\text{sign}(M) \left( \frac{3r}{16\pi} A_L \wedge F_L - \frac{3}{8\pi} \omega_{\text{CS}}(A_R) - \frac{7r}{64\pi} \omega_{\text{grav}}^{\text{CS}} \right). \quad (6.34)$$

**Chiral vectors.** In this case we were unaware of the existence of an appropriate index theorem in the literature. Without coupling to gauge fields one has [129]

$$\text{ind } iD_A = \frac{1}{2} \int_M L(M) \Big|_{4\text{-form}}, \quad (6.35)$$

where the Hirzebruch  $L$ -polynomial is given by

$$L(M) = 1 - \frac{1}{(2\pi)^2} \frac{1}{6} \text{tr} \mathcal{R} \wedge \mathcal{R} + \dots. \quad (6.36)$$

The equality in (6.35) only holds for the four-form and we multiplied the right hand side by two with respect to the result in [129] since we consider complex instead of real vector fields. However, we now use that the  $L$ -polynomial according to the Hirzebruch signature theorem also determines the Hirzebruch signature

$$\tau = \int_M L(M) \Big|_{4\text{-form}}. \quad (6.37)$$

If one instead considers a tensor product with another vector bundle with connection  $F$ , the Hirzebruch theorem becomes [130]

$$\tau = \int_M L(M) \wedge \text{ch}(2F) \Big|_{4\text{-form}}. \quad (6.38)$$

Based on these considerations, we now postulate that

$$\text{ind } iD_A = \frac{1}{2} \int_M L(M) \wedge \text{ch}(2F) \Big|_{4\text{-form}}. \quad (6.39)$$

In [83] one-loop corrections to Chern-Simons terms in 5d, which are induced by integrating out massive chiral Kaluza-Klein modes after a circle reduction from 6d, are computed. The authors do this by explicit calculation of the corresponding Feynman diagrams. In appendix D we reproduce these results using the index theorems in which we also use the index (6.39). This is non-trivial evidence that (6.39) is indeed the right quantity.

Using the index (6.39), the counterterm is now given by

$$i\pi \operatorname{sign}(M) \int_{\mathcal{M}_3} Q_{\text{vec}}(A, \omega) \quad (6.40)$$

where

$$dQ_{\text{vec}}(A, \omega) = \frac{1}{2} L(M) \wedge \operatorname{ch}(2F_L) \wedge \operatorname{ch}(2F_R) \big|_{4\text{-form}}. \quad (6.41)$$

Notice that (6.40) has an extra minus-sign compared to (6.26), which is due to the fact that vectors are bosons [129]. We then find

$$dQ_{\text{vec}}(A, \omega) = \frac{r}{2(2\pi)^2} F_L \wedge F_L - \frac{1}{(2\pi)^2} \operatorname{tr} F_R \wedge F_R - \frac{1}{12} \frac{r}{(2\pi)^2} \operatorname{tr} \mathcal{R} \wedge \mathcal{R}. \quad (6.42)$$

This implies that the counterterms to the Lorentzian action are given by

$$\operatorname{sign}(M) \left( -\frac{r}{8\pi} A_L \wedge F_L + \frac{1}{4\pi} \omega_{\text{CS}}(A_R) + \frac{r}{48\pi} \omega_{\text{grav}}^{\text{CS}} \right). \quad (6.43)$$

**Corrections to the levels and central charges.** Note that all the corrections above were derived for an arbitrary representation under  $\mathfrak{u}(1)_L \oplus \mathfrak{su}(2)_R$  specified by the quantum numbers  $j_L^{(3)}$  and  $j_R$ . Expressing the left Chern-Simons term in the representation we used in the classical part gives a factor  $2(j_L^{(3)})^2$  and expressing the right Chern-Simons terms in the fundamental representation gives a factor  $\frac{2}{3}j_R(j_R+1)(2j_R+1)$ . We also use that the dimension of the representation under  $\mathfrak{su}(2)_R$  specified by  $j_R$  is given by  $\dim \mathbf{R}_{j_R} = 2j_R + 1$ . The constants  $\alpha_L$ ,  $\alpha_R$ ,  $\alpha_{\text{grav}}$  multiplying the Chern-Simons terms  $\omega_{\text{CS}}(A_L)$ ,  $\omega_{\text{CS}}(A_R)$  and  $\omega_{\text{grav}}^{\text{CS}}$  are then given in table 6.2.

With these results we can now sum the contributions of table 6.2 over the complete spectrum determined in section 6.4.1. As we are summing over an infinite number of states the result will diverge and needs to be regularized. Following [95] we use zeta-function regularization. The regularized summations that we need are

	spin- $\frac{1}{2}$	spin- $\frac{3}{2}$	chiral vectors
$\alpha_L$	$\frac{1}{2}(j_L^{(3)})^2 \dim \mathbf{R}_{j_R}$	$\frac{3}{2}(j_L^{(3)})^2 \dim \mathbf{R}_{j_R}$	$-(j_L^{(3)})^2 \dim \mathbf{R}_{j_R}$
$\alpha_R$	$-\frac{j_R}{3}(j_R + 1) \dim \mathbf{R}_{j_R}$	$-j_R(j_R + 1) \dim \mathbf{R}_{j_R}$	$\frac{2j_R}{3}(j_R + 1) \dim \mathbf{R}_{j_R}$
$\alpha_{\text{grav}}$	$\frac{1}{48} \dim \mathbf{R}_{j_R}$	$-\frac{7}{16} \dim \mathbf{R}_{j_R}$	$\frac{1}{12} \dim \mathbf{R}_{j_R}$

Table 6.2: Contributions of a single field to the left-, right- and gravitational Chern-Simons terms. The table should be read as  $\alpha_I = \frac{\text{sgn}(M)}{4\pi} \times (\text{entry of table})$ .

$$\begin{aligned}
\sum_{n=1}^{\infty} 1 &= -\frac{1}{2}, & \sum_{n=1}^{\infty} n &= -\frac{1}{12}, \\
\sum_{n=1}^{\infty} n^2 &= 0, & \sum_{n=1}^{\infty} n^3 &= \frac{1}{120}.
\end{aligned} \tag{6.44}$$

The relation between the coefficients  $\alpha_L$ ,  $\alpha_R$ ,  $\alpha_{\text{grav}}$  of the Chern-Simons terms, after summing over all of the modes, to the central charges and levels are

$$k_L = 8\pi\alpha_L, \quad k_R = 4\pi\alpha_R, \quad c_L - c_R = 96\pi\alpha_{\text{grav}}. \tag{6.45}$$

Before proceeding, a few comments regarding the regularization of the contributions from the massive KK spectrum are in order. Firstly, in the three-dimensional theory one expects that there is a UV cut-off determined by the scale at which gravity becomes strongly coupled [131, 132]. Regularization with this cut-off will agree with the zeta-function regularization performed here. Secondly it is only possible to apply zeta-function regularization because the higher dimensional theory is anomaly free [82].

Let us proceed with the calculation of the quantum corrections for the A- and D-series. We shall only present the final results here, leaving a more detailed exposition to appendix 6.A.

**A-series.** We must sum over the representations listed in section 6.4.1 with  $j_L^{(3)} = \frac{1}{2}mk$ ,  $k \in \mathbb{Z}_{\geq 0}$ . We implement this by first summing over all representations that include a state with  $j_L^{(3)} = \frac{1}{2}mk$ , that is over  $j_L = \frac{1}{2}mk, \frac{1}{2}mk + 1, \dots$  and then performing the sum over  $k \in \mathbb{Z}_{\geq 0}$ , keeping in mind that we must regularize the summations using zeta-function regularization. This is performed explicitly in appendix 6.A, with the final results

$$k_L^{\text{loop}} = 0, \quad k_R^{\text{loop}} = 2m, \quad (c_L - c_R)^{\text{loop}} = 12m, \tag{6.46}$$

up to terms of  $\mathcal{O}(1)$ , i.e. terms independent of the charges. Adding these results to the classical contributions (6.22), and using  $|\mathbb{Z}_m| = m$ , the central charges and levels up to order  $\mathcal{O}(1)$  in the charges are

$$k_L = \frac{1}{2}mC \cdot C, \quad k_R = \frac{1}{2}mC \cdot C + 2m, \quad c_L = c_R + 12m. \quad (6.47)$$

**D-series.** As there is no left-moving current in this case there is no left level  $k_L$  to compute. As in the A-series case we sum the contributions of the individual representations over the KK spectrum of section 6.4.1, this time with  $j_L^{(3)} = mk$ ,  $k \in \mathbb{Z}_{\geq 0}$  and  $j_R^{(3)} \geq 0$ . This is most easily implemented by first summing over all representations with  $j_L^{(3)} = mk$ ,  $k \in \mathbb{Z}_{\geq 0}$  and then imposing  $j_R^{(3)} \geq 0$ . Comparing with the condition in the A-series it is clear that the first step may be achieved by using the A-series results (6.46) and replacing  $m \rightarrow 2m$ . We must still impose the second condition, which may be imposed *effectively* by dividing the result of the first step by two<sup>43</sup>. The one-loop contributions up to order  $\mathcal{O}(1)$  in the charges are

$$k_R^{\text{loop}} = 2m, \quad (c_L - c_R)^{\text{loop}} = 12m. \quad (6.48)$$

Adding these to (6.22) and using  $|\mathbb{D}_m^*| = 4m$ , the final results for the central charges and right-moving level up to and including linear terms in the charges are

$$k_R = 2mC \cdot C + 2m, \quad c_L = c_R + 12m. \quad (6.49)$$

### 6.4.3 Summary

For the ease of the reader we conclude by collating the results of this section. In the subsequent section we shall compare the results obtained here with the microscopic computations conducted there.

For the case of ALE transverse space we have shown that the only contribution is from the classical reduction of the action performed in section 6.3. The results

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<sup>43</sup>We use the word effectively as the contribution from the  $j_R^{(3)} = 0$  terms is half of the actual result using this method. The difference, however, is  $\mathcal{O}(1)$  in the charges and therefore for our purposes may be neglected.



for the central charges and levels are

$$\begin{aligned} k_L &= \frac{1}{2} |\Gamma| \eta_{\alpha\beta} Q^\alpha Q^\beta = \frac{1}{2} |\Gamma| C \cdot C && (\text{exists only for } \Gamma = \mathbb{Z}_m), \\ k_R &= \frac{1}{2} |\Gamma| C \cdot C, \\ c_L &= c_R, \end{aligned} \tag{6.50}$$

where  $|\Gamma|$  may be read off of table 6.1.

In contrast, for the case of ALF transverse space we must also include, in addition to the classical contribution of section 6.3, the quantum corrections obtained in the preceding section. For the A-series we find<sup>44</sup>

$$k_L = \frac{1}{2} m C \cdot C, \quad k_R = \frac{1}{2} m C \cdot C + 2m, \quad c_L = c_R + 12m, \tag{6.51}$$

whilst for the D-series we find

$$k_R = 2m C \cdot C + 2m, \quad c_L = c_R + 12m. \tag{6.52}$$

All results presented here are up to terms linear in the charges, we do not include any  $\mathcal{O}(1)$  terms.

## 6.5 Microscopics of 6d strings

In the previous section we have computed the central charges and levels macroscopically; the content of this section is to reproduce these results from a microscopic computation. Concretely, we shall compute the central charges and levels of the 2d  $\mathcal{N} = (0, 4)$  SCFTs living on the strings considered previously. For the case of transverse ALE space we utilize the results of [102, 103] to determine the worldvolume theory on D3-branes probing an ADE singularity. By placing the resultant four-dimensional SCFTs on  $\mathbb{R}^{1,1} \times C$  and performing a suitable

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<sup>44</sup>Note that the central charges and levels do not follow from the results of [95] immediately by specializing to  $\text{CY}_3 = \text{K3} \times T^2$ . This is due to the presence of non-trivial one-cycles that are not present in Calabi–Yau threefolds with  $\text{SU}(3)$  holonomy.

topological twist [106, 133, 101, 134] we obtain 2d  $\mathcal{N} = (0, 4)$  SCFTs, whose central charges and levels we may compute via a spectrum computation and anomaly arguments. In the ALF case it is convenient to use a dual M-theory description for the A-series. However, for the ALF D-series case such a clean microscopic setup is lacking and the matching of the microscopic results to the macroscopic ones remains an open problem.

### 6.5.1 Transverse ALE spaces: ADE-series

We first consider the microscopic theory describing a D3-brane wrapping a curve  $C$  inside K3 and probing the singular limit of an ALE space  $\mathbb{C}^2/\Gamma$  transverse to the brane. The worldvolume theory of a D3-brane on  $\mathbb{R}^{1,3}$  probing a transverse ADE singularity is obtained by performing a projection of  $\mathcal{N} = 4$   $U(|\Gamma|)$  super Yang–Mills. The resulting theories are  $\mathcal{N} = 2$  supersymmetric gauge theories with known Lagrangian description [102, 103]. Their gauge and hypermultiplet field content is summarized in table 6.3. From the table we can read off that the 4d  $\mathcal{N} = 2$  theories all have  $n_H = n_V = |\Gamma|$  hyper- and vector multiplets. Our ultimate interest is the 2d IR theory arising from wrapping the D3-brane on a curve  $C$  inside K3, preserving  $\mathcal{N} = (0, 4)$  supersymmetry.<sup>45</sup> In order to preserve supersymmetry in 2d it is necessary to perform a topological twist. Before proceeding with performing the twist, counting fields in the 2d theory and evaluating the central charges and levels, we first discuss the representations of the field content in the 4d theory.

A 4d  $\mathcal{N} = 2$  SCFT admits an  $SU(2)_R \times U(1)_r$  R-symmetry. The supercharges transform under the total symmetry group  $SO(1, 3)_\ell \times SU(2)_R \times U(1)_r$  in the representations

$$Q_{\alpha I} \in (\mathbf{2}, \mathbf{1}, \mathbf{2})_1, \quad \tilde{Q}_{\dot{\alpha}}^I \in (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-1}. \quad (6.53)$$

Let us consider a generic 4d  $\mathcal{N} = 2$  SCFT with  $n_H \mathcal{N} = 2$  hypermultiplets and  $n_V \mathcal{N} = 2$  vector multiplets. An  $\mathcal{N} = 2$  hypermultiplet has field content

$$q^I \in (\mathbf{1}, \mathbf{1}, \mathbf{2})_0, \quad \psi_\alpha^{i=1,2} \in (\mathbf{2}, \mathbf{1}, \mathbf{1})_{-1}, \quad \tilde{\psi}_{\dot{\alpha}}^{i=1,2} \in (\mathbf{1}, \mathbf{2}, \mathbf{1})_1, \quad (6.54)$$

whilst an  $\mathcal{N} = 2$  vector multiplet contains the fields

$$A_\mu \in (\mathbf{2}, \mathbf{2}, \mathbf{1})_0, \quad \lambda_\alpha^I \in (\mathbf{2}, \mathbf{1}, \mathbf{2})_1, \quad \tilde{\lambda}_{\dot{\alpha}}^I \in (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-1}, \quad \Phi \in (\mathbf{1}, \mathbf{1}, \mathbf{1})_2. \quad (6.55)$$

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<sup>45</sup>Interesting proposals for the IR sigma model of the D1-D5 system probing the A-type ALE and ALF spaces were studied in [135].

$\Gamma$	Gauge multiplets	Hypermultiplets
$\mathbb{Z}_m$	$U(1)^m$	$m \times (\mathbf{1}, \mathbf{1})$
$\mathbb{D}_m^*$	$U(1)^4 \times U(2)^{m-1}$	$4 \times (\mathbf{1}, \mathbf{2}) + (m-2) \times (\mathbf{2}, \mathbf{2})$
$\mathbb{T}^*$	$U(1)^3 \times U(2)^3 \times U(3)$	$3 \times (\mathbf{1}, \mathbf{2}) + 3 \times (\mathbf{2}, \mathbf{3})$
$\mathbb{O}^*$	$U(1)^2 \times U(2)^3 \times U(3)^2 \times U(4)$	$2 \times [(\mathbf{1}, \mathbf{2}) + (\mathbf{2}, \mathbf{3}) + (\mathbf{3}, \mathbf{4})]$ $+ (\mathbf{2}, \mathbf{4})$
$\mathbb{I}^*$	$U(1) \times U(2)^2 \times U(3)^2 \times U(4)^2 \times U(5) \times U(6)$	$(\mathbf{1}, \mathbf{2}) + (\mathbf{2}, \mathbf{3}) + (\mathbf{3}, \mathbf{4})$ $+ (\mathbf{4}, \mathbf{5}) + (\mathbf{5}, \mathbf{6}) + (\mathbf{2}, \mathbf{4})$ $+ (\mathbf{4}, \mathbf{6}) + (\mathbf{3}, \mathbf{6})$

Table 6.3:  $\mathcal{N} = 2$  multiplets for the different quivers associated with the groups  $\Gamma$ .

Note that in the case of the A-series we expect the 4d  $\mathcal{N} = 2$  SCFT to enjoy an additional  $U(1)_L$  global symmetry. This flavor symmetry manifests itself in the D-brane picture as a  $U(1)_L$  isometry of the transverse ALE space under which the supercharges are invariant, and is absent for the D- and E-series. The discussion of the assignment of  $U(1)_L$  charges for the  $\mathcal{N} = 2$  fields is relegated to appendix 6.B. They will appear later in this section when we give the final 2d spectrum in table 6.4.

Placing the 4d theories on  $\mathbb{R}^{1,1} \times C$  breaks Lorentz symmetry with the group decomposing as  $SO(1, 3)_\ell \rightarrow SO(1, 1) \times U(1)_C$ . This splitting leads to the following decompositions of the  $SO(1, 3)_\ell$  representations:

$$\begin{aligned}
SO(1, 3)_\ell &\rightarrow SO(1, 1) \times U(1)_C \\
(\mathbf{2}, \mathbf{2}) &\rightarrow \mathbf{1}_{2,0} \oplus \mathbf{1}_{-2,0} \oplus \mathbf{1}_{0,2} \oplus \mathbf{1}_{0,-2} \\
(\mathbf{2}, \mathbf{1}) &\rightarrow \mathbf{1}_{1,1} \oplus \mathbf{1}_{-1,-1} \\
(\mathbf{1}, \mathbf{2}) &\rightarrow \mathbf{1}_{1,-1} \oplus \mathbf{1}_{-1,1} .
\end{aligned} \tag{6.56}$$

The decomposition of the supercharges under  $SU(2)_R \times SO(1, 1) \times U(1)_C \times U(1)_r$  is:

$$\begin{aligned}
SO(1, 3)_\ell \times SU(2)_R \times U(1)_r &\rightarrow SU(2)_R \times SO(1, 1) \times U(1)_C \times U(1)_r \\
Q_{\alpha I} \in (\mathbf{2}, \mathbf{1}, \mathbf{2})_1 &\rightarrow \mathbf{2}_{1,1,1} \oplus \mathbf{2}_{-1,-1,1} \\
\tilde{Q}_{\dot{\alpha}}^I \in (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-1} &\rightarrow \mathbf{2}_{1,-1,-1} \oplus \mathbf{2}_{-1,1,-1} .
\end{aligned} \tag{6.57}$$

In order to preserve  $\mathcal{N} = (0, 4)$  in the 2d theory we must perform a partial topological twist with respect to  $U(1)_r$  [106, 133, 101, 134]. The unique twist preserving  $\mathcal{N} = (0, 4)$  in 2d, up to redefinition, is implemented by defining the generator of  $U(1)'_C$  to be

$$T'_C = \frac{1}{2}(T_C + T_r). \quad (6.58)$$

In this way we obtain two singlet supercharges, both with negative chirality thereby giving  $\mathcal{N} = (0, 4)$  supersymmetry in 2d. Next consider the decomposition of the various matter fields after the twist. The fields of an  $\mathcal{N} = 2$  hypermultiplet decompose as:

$$\begin{aligned} \text{SO}(1, 3)_\ell \times \text{SU}(2)_R \times U(1)_r &\rightarrow \text{SU}(2)_R \times \text{SO}(1, 1) \times U(1)'_C \times U(1)_r \\ q^I &\in (\mathbf{1}, \mathbf{1}, \mathbf{2})_0 \rightarrow \mathbf{2}_{0,0,0} \equiv q^I \\ \psi_\alpha^{i=1,2} &\in (\mathbf{2}, \mathbf{1}, \mathbf{1})_{-1} \rightarrow \mathbf{1}_{1,0,-1} \oplus \mathbf{1}_{-1,-1,-1} \equiv \psi_+^{i=1,2} \oplus \psi_-^{i=1,2} \\ \tilde{\psi}_{\dot{\alpha}}^{i=1,2} &\in (\mathbf{1}, \mathbf{2}, \mathbf{1})_1 \rightarrow \mathbf{1}_{1,0,1} \oplus \mathbf{1}_{-1,1,1} \equiv \tilde{\psi}_+^{i=1,2} \oplus \tilde{\psi}_-^{i=1,2}. \end{aligned} \quad (6.59)$$

Likewise a 4d  $\mathcal{N} = 2$  vector multiplet decomposes as:

$$\begin{aligned} \text{SO}(1, 3)_\ell \times \text{SU}(2)_R \times U(1)_r &\rightarrow \text{SU}(2)_R \times \text{SO}(1, 1) \times U(1)'_C \times U(1)_r \\ A_\mu &\in (\mathbf{2}, \mathbf{2}, \mathbf{1})_0 \rightarrow \mathbf{1}_{2,0,0} \oplus \mathbf{1}_{-2,0,0} \oplus \mathbf{1}_{0,1,0} \oplus \mathbf{1}_{0,-1,0} \\ &\equiv v_+ \oplus v_- \oplus a \oplus \tilde{a} \\ \lambda_\alpha^I &\in (\mathbf{2}, \mathbf{1}, \mathbf{2})_1 \rightarrow \mathbf{2}_{1,1,1} \oplus \mathbf{2}_{-1,0,1} \equiv \lambda_+^I \oplus \lambda_-^I \\ \tilde{\lambda}_{\dot{\alpha}}^I &\in (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-1} \rightarrow \mathbf{2}_{1,-1,-1} \oplus \mathbf{2}_{-1,0,-1} \equiv \tilde{\lambda}_+^I \oplus \tilde{\lambda}_-^I \\ \Phi &\in (\mathbf{1}, \mathbf{1}, \mathbf{1})_2 \rightarrow \mathbf{1}_{0,1,2} \equiv \Phi. \end{aligned} \quad (6.60)$$

Having determined the 2d fields we now turn our attention to evaluating their multiplicities. The 2d fields with  $U(1)'_C$  charges  $Q'_C = \pm 1$  have multiplicities

$$\dim H^0(C, K_C) = \dim H^0(C, K_C^{-1}) = \dim H^1(C, \mathcal{O}_C) = g, \quad (6.61)$$

whereas the states with charge  $Q'_C = 0$  have multiplicity  $\dim H^0(C, \mathcal{O}_C) = 1$ . The latter identity follows since the only holomorphic functions on a compact Riemann surface are constants. The full 2d massless spectrum follows from the decompositions (6.59) and (6.60) coupled with the above discussion of the multiplicities, and is given in table 6.4. We have included in table 6.4 the  $U(1)_L$  flavor charge which is only present for the A-series case. We remind the reader that the details of the  $U(1)_L$  charge assignment may be found in appendix 6.B.

Bosons	$SU(2)_R \times U(1)_L$	Multiplicity
$a, \tilde{a}, \Phi$	$2 \times \mathbf{1}_0, 2 \times \mathbf{1}_0$	$g$
$q^I$	$\mathbf{2}_1, \mathbf{2}_{-1}$	1
$v_+, v_-$	$2 \times \mathbf{1}_0$	1
Fermions		
$\lambda_+^I, \tilde{\lambda}_+^I$	$2 \times \mathbf{2}_0$	$g$
$\psi_+^1, \psi_+^2, \tilde{\psi}_+^1, \tilde{\psi}_+^2$	$2 \times \mathbf{1}_1, 2 \times \mathbf{1}_{-1}$	1
$\psi_-^1, \psi_-^2, \tilde{\psi}_-^1, \tilde{\psi}_-^2$	$2 \times \mathbf{1}_1, 2 \times \mathbf{1}_{-1}$	$g$
$\lambda_-^I, \tilde{\lambda}_-^I$	$2 \times \mathbf{2}_0$	1

Table 6.4: Spectrum of the 2d  $\mathcal{N} = (0, 4)$  theory. The  $U(1)_L$  charges are only relevant for the A-series.

With the full 2d spectrum in hand we may compute 't Hooft anomaly coefficients and use their relation to the central charges and levels, see the recent paper [136] in the context of  $\mathcal{N} = (0, 2)$  theories as an example. The gravitational anomaly is easily computed via

$$c_L - c_R = \text{tr}_{\text{Weyl}} \gamma_3 = 0, \quad (6.62)$$

where  $\gamma_3$  is the chirality matrix in 2d with eigenvalues  $\pm 1$  and the sum is over all Weyl fermions.

The right level is (recall  $n_V = n_H = |\Gamma|$ )

$$\begin{aligned} k_R &= \text{tr}_{\text{Weyl}}(\gamma_3 Q_R^2) = |\Gamma| \left[ 2g \times \left[ \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 \right] - 2 \times \left[ \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 \right] \right] \\ &= |\Gamma|(g - 1) = \frac{1}{2} |\Gamma| C \cdot C, \end{aligned} \quad (6.63)$$

where  $Q_R$  is the eigenvalue under the Cartan generator of  $SU(2)_R$ . In the final line we have used adjunction which implies  $g = \frac{1}{2} C \cdot C + 1$  in order to express the level in terms of  $C \cdot C$ .

Proceeding in the same way to compute the 't Hooft anomaly associated to  $U(1)_L$  we find<sup>46</sup>

$$k_L = -\text{tr}_{\text{Weyl}}(\gamma_3 Q_L^2) = |\Gamma|(g - 1) = \frac{1}{2} |\Gamma| C \cdot C. \quad (6.64)$$

<sup>46</sup>We compute the coefficient with half-integer  $U(1)_L$  charges, with the embedding  $U(1)_L \subset SU(2)_L$  in mind.

The minus sign appearing in the definition of  $k_L$ , in contrast to the definition of  $k_R$ , arises as  $U(1)_L$  is a left-moving current whilst  $SU(2)_R$  is a right-moving current. Unitarity imposes that the anomalies are positive semi-definite, and the minus sign for the left-moving 't Hooft anomaly is included to 'cancel off' the minus sign arising from the chirality matrix. Our 't Hooft anomaly considerations thus give

$$k_R = \frac{1}{2}|\Gamma|C \cdot C, \quad k_L = \frac{1}{2}|\Gamma|C \cdot C, \quad c_L - c_R = 0, \quad (6.65)$$

which precisely matches the result from the macroscopic computation (6.50).

We would now like to exploit the relation  $c_R = 6k_R = 3|\Gamma|C \cdot C$  dictated by supersymmetry to determine the central charges. However, this only holds if one can correctly identify the  $SU(2)_R$  with the 2d R-symmetry and as already noted in section 6.1, this does not work for the center of mass modes. This can be seen explicitly from table 6.4. The center of mass modes are given by  $q^I$ ,  $\psi_+^{1,2}$ ,  $\tilde{\psi}_+^{1,2}$ ,  $\lambda_-^I$  and  $\tilde{\lambda}_-^I$ . For example, it is easy to see that  $\lambda_-^I$  and  $\tilde{\lambda}_-^I$  are left-moving but transform under  $SU(2)_R$  which is forbidden if it is the (right-moving) R-symmetry. One should repeat these computations ignoring the center of mass modes in the traces. This yields

$$c_L = c_R = 6k_R = 3|\Gamma|C \cdot C + 6|\Gamma|, \quad (6.66)$$

which are the central charges of the theory without the center of mass modes and is consistent with the results in [96, 137] for  $|\Gamma| = 1$ .

The central charges also follow from the relations  $c_{L,R} = N_B^{L,R} + \frac{1}{2}N_F^{L,R}$ , where  $N_B^{L,R}$  is the number of left- and right-moving bosonic degrees of freedoms and  $N_F^{L,R}$  the number of left- and right-moving fermionic degrees of freedom after extracting the center of mass degrees of freedom.

### 6.5.2 Transverse ALF spaces: AD-series

We now turn to studying the microscopics of the wrapped D3-brane probing a transverse ALF space. As there is strictly speaking no known worldvolume theory for this setting, unlike in the ALE case, we use a convenient duality frame to determine the central charges and levels. We do this for the A-series where the

ALF space is given by Taub-NUT space with NUT-charge  $m$ . Dualizing to an M5-brane picture allows us to compute the microscopic data. Unfortunately, for the D-series this method runs into difficulties which we shall explain in more detail below.

**A-series.** To obtain an M-theory picture we first T-dualize along the NUT-circle, obtaining a type IIA setup with a D4-brane wrapping  $C$  and the NUT-circle (with inverted radius), and  $m$  NS5-branes wrapping the aforementioned circle and the entire K3. Uplifting to M-theory, we obtain M-theory compactified on  $X = K3 \times T^2$  with an M5-brane wrapping  $C \times T^2$  and  $m$  M5-branes wrapping K3. As explained in [138], this system can effectively be described by an M5-brane wrapping  $\mathcal{P} = C \times T^2 + mK3$ . This is the well-studied MSW CFT, where the Calabi-Yau threefold has been specialized to  $K3 \times T^2$ . This leads to the added complication that there now exist one-cycles in the geometry which were assumed not to be present in the original MSW setup, where the Calabi-Yau was taken to have  $SU(3)$  holonomy. In order to determine the microscopic central charges we compute the bosonic and fermionic spectrum in the effective 2d  $\mathcal{N} = (0, 4)$  theory on the M5-brane. This setup was already studied in [139, 105, 94] before, but for convenience and completeness we review the main points here.

The worldvolume theory of the M5-brane is a 6d  $(2, 0)$  theory with a  $(2, 0)$  tensor multiplet containing a chiral two-form and five scalar fields. Three of the five scalars parametrize the center of mass motion of the string in the transverse  $\mathbb{R}^3$ , whereas the remaining two scalars parametrize the position of the surface  $\mathcal{P}$  inside  $X = K3 \times T^2$ . Additionally, one obtains 2d scalar fields from reducing the chiral two-form along two-forms on  $\mathcal{P}$ . Due to the self-duality constraint there will be  $b_2^-(\mathcal{P})$  left-moving, and  $b_2^+(\mathcal{P})$  right-moving scalar fields in two dimensions, where  $b_2^\pm(\mathcal{P})$  denote the number of harmonic (anti-)self-dual two-forms on  $\mathcal{P}$ . In total, we find

$$N_B^L = d_{\mathcal{P}} + 3 + b_2^-(\mathcal{P}), \quad N_B^R = d_{\mathcal{P}} + 3 + b_2^+(\mathcal{P}) \quad (6.67)$$

left- and right-moving bosons. The contribution from the two scalars parametrizing the motion of  $\mathcal{P}$  inside  $K3 \times T^2$  is denoted by  $d_{\mathcal{P}}$  and will be determined momentarily. The number of left- and right-moving fermions can be determined in the standard way by using the formulas

$$\begin{aligned} N_F^L &= 4h^{0,1}(\mathcal{P}) = 2b_1(\mathcal{P}) = 2b_1(X) = 4, \\ N_F^R &= 4(b_0(\mathcal{P}) + h^{0,2}(\mathcal{P})) = 4(b_0(X) + h^{0,2}(\mathcal{P})) = 4 + 4h^{0,2}(\mathcal{P}). \end{aligned} \quad (6.68)$$

Supersymmetry in the right-moving sector implies that the number of right-moving bosons equals the number of right-moving fermions, allowing us to determine  $d_{\mathcal{P}}$  via

$$0 = N_B^R - N_F^R = d_{\mathcal{P}} + b_2^+(\mathcal{P}) - 4h^{0,2}(\mathcal{P}) - 1 = d_{\mathcal{P}} - 2h^{0,2}(\mathcal{P}), \quad (6.69)$$

where we have used that  $b_2^+(\mathcal{P}) = b_2(\mathcal{P}) - b_2^-(\mathcal{P})$  and  $b_2^-(\mathcal{P}) = h^{1,1}(\mathcal{P}) - 1$ . For the left- and right-moving central charges we find,

$$c_L = b_2(\mathcal{P}) + 4, \quad c_R = 3b_2^+(\mathcal{P}) + 3. \quad (6.70)$$

We may evaluate these explicitly by rewriting the topological numbers in terms of integrals on  $X = \text{K3} \times T^2$  as [104, 94]

$$c_L = \int_X (\mathcal{P}^3 + c_2(X) \wedge \mathcal{P}) + 6, \quad c_R = \int_X \left( \mathcal{P}^3 + \frac{1}{2} c_2(X) \wedge \mathcal{P} \right) + 6. \quad (6.71)$$

These are straightforward to evaluate for the curve  $\mathcal{P}$ , giving the results<sup>47</sup>:

$$c_L = 3mC \cdot C + 24m + 6, \quad c_R = 3mC \cdot C + 12m + 6. \quad (6.72)$$

Using  $c_R = 6k_R$  this matches the macroscopic result (6.51) up to and including linear order terms in the charges. We expect that the left level can be computed by adapting the techniques in [138, 95].

**D-series.** We saw above that the microscopics in the A-series' case is relatively clean once we have dualized to an M-theory picture. The D-series' case on the other hand, is not as clean cut. This is because the dual M-theory setup is no longer accessible with the standard MSW techniques. To see why, let us perform the analogous duality chain as in the A-series. Performing a T-duality along the fiber of the D-series ALF space to type IIA we obtain a D4-brane wrapping  $S^1 \times C \times S_D^1$  from the D3-brane,  $m$  NS5-branes wrapping  $\text{K3} \times S^1$  and in addition a so called ON5-plane on top of the NS5-branes [140]. An ON5-plane is the analogue of an ordinary orientifold plane for the NS5-brane, and leads to orthogonal gauge groups when placed on top of the NS5-branes. Lifting the type IIA setup to

<sup>47</sup>When one calculates the central charges using 't Hooft anomalies and the relation  $c_R = 6k_R$  one finds the results (6.72) up to order  $\mathcal{O}(1)$  [94]. The difference is again caused by subtleties in identifying  $\text{SU}(2)_R$  with the R-symmetry.



M-theory the D4 lifts to an M5-brane wrapping  $C \times T^2$ , the NS5s to M5s on K3 whilst the ON5-plane becomes an OM5 plane wrapping the K3. At low energies the system of M5-branes with an OM5-plane on top realize the familiar D-type  $(2, 0)$  SCFT in the ADE classification. This is a very involved setup and does not directly lead itself to the application of MSW to compute the central charges due to the presence of the orientifold.

A second chain of dualities, which can also be applied to all the other cases we consider, is to first T-dualize along the circle wrapped by the D3-brane, leading to a type IIA setup of a D2-brane probing the ALE/ALF space. Uplifting to M-theory we obtain an M2-brane on the curve  $C$  probing the ALE/ALF space. The problem has now been rephrased in terms of M2-brane counting for five-dimensional black holes. Note that as this duality frame exists for all the cases considered here, it is to some extent the most universal setup. In flat space there is a connection between the black hole partition function and the topological string partition function. It is natural to conjecture that a generalization of this connection would be a useful tool in understanding the counting of M2-branes in this setup.

Since the center of the ALF space looks like  $\mathbb{C}^2/\mathbb{D}_m^*$  it is not unreasonable to expect that the leading order contribution to the central charges and level agrees with the corresponding D-series ALE space. The macroscopic results (6.52) and (6.50) confirm this expectation. Clearly it is desirable to have a better understanding of this case and to obtain a first principles derivation of the microscopic central charges and level.

**Summary.** We studied the central charges and levels of 2d  $\mathcal{N} = (0, 4)$  SCFTs dual to black strings that arise from D3-branes in type IIB compactifications on K3. The branes wrap  $S^1 \times C$ , where  $C$  is a curve in K3, and have as transverse space an ALE or ALF space. We computed the central charges and levels both from six-dimensional  $\mathcal{N} = (2, 0)$  supergravity and from the microscopic  $\mathcal{N} = (0, 4)$  SCFTs. We found excellent matching between microscopics and macroscopics for all ALE spaces and for ALF spaces corresponding to the A-series. For ALF spaces corresponding to the D-series we only performed a macroscopic analysis.

An extension of this work is to include 7-branes in the setup. The natural framework for this is F-theory [14]. Whereas in this chapter we have compactified type IIB on K3, wrapping the D3-brane on a curve  $C$  inside K3, we would instead compactify F-theory on an elliptically fibered Calabi–Yau threefold, wrapping the D3-brane on a curve  $C$  inside the base of the threefold. We will turn to these settings in chapter 7.

## 6.A Summation of 3d one-loop corrections

In this appendix we present the full and explicit computation of the one-loop contributions to the three-dimensional  $U(1)_L$ ,  $SU(2)_R$  and gravitational Chern–Simons terms from integrating out the massive KK modes arising in the reduction on  $S^3/\mathbb{Z}_m$ . With these results, and a little further work, we may also obtain the results for the reduction on  $S^3/\mathbb{D}_m^*$ . For this reason in this appendix we will almost exclusively consider the A-series case. The necessary adaptation to the D-series case will be explained in section 6.A.5 of this appendix.

To determine the corrections furnished by the massive KK modes we sum the contributions of individual modes (see table 6.2) over the full KK spectrum derived in section 6.4.1. As explained in the main text we must regularize the summations using zeta-function regularization. In particular we will use the following regularized sums:

$$\begin{aligned} \sum_{j_L=\frac{1}{2}mk}^{\infty} 1 &= \frac{1}{2} - \frac{1}{2}mk, & \sum_{j_L=\frac{1}{2}mk}^{\infty} j_L &= \frac{1}{24}(-2 + 6km - 3k^2m^2), \\ \sum_{j_L=\frac{1}{2}mk}^{\infty} j_L^2 &= \frac{1}{24}(-2km + 3k^2m^2 - k^3m^3). \end{aligned} \quad (6.73)$$

The sums are performed over the integers or half integers when  $\frac{1}{2}mk$  is integer or half integer respectively. In fact the regularized sums we need are

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} 1 &= -\frac{1}{4} + \frac{m}{24}, & \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} j_L &= \frac{1}{24} - \frac{m}{48}, \\ \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} j_L^2 &= -\frac{m^3}{24 \cdot 120} + \frac{m}{144}, & \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} k^2 &= -\frac{m}{240}, \end{aligned} \quad (6.74)$$

where we have used the previous results and (6.44).

We now calculate the corrections  $k_L^{\text{loop}}$ ,  $k_R^{\text{loop}}$  and  $(c_L - c_R)^{\text{loop}}$  separately. As described in the main text we implement the projection condition  $j_L = \frac{1}{2}mk$  for  $k \in \mathbb{Z}_{\geq 0}$  by first summing over the representations  $j_L = \frac{1}{2}mk, \frac{1}{2}mk + 1, \dots$  and subsequently summing over  $k$ . The structure of the KK spectrum is such that we have to restrict to  $m \geq 3$  and do the sums for  $k = 0$  separately. In the calculation of the one-loop contribution to  $k_L$ ,  $k_R$  and  $c_L - c_R$  we therefore first sum over the representations relevant for  $k = 0$  and afterwards we sum the contributions for  $k \neq 0$ . We will not perform the computation for the special cases  $m = 1, 2$  explicitly as they give the same result up to order  $\mathcal{O}(1)$  as the formulas for  $m \geq 3$  evaluated in  $m = 1, 2$ .

### 6.A.1 Relevant spectrum

The part of the spectrum that contributes to  $k = 0$  is given by

- Spin- $\frac{3}{2}$ :

$$4 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\mp},$$

- Spin- $\frac{1}{2}$ :

$$\begin{aligned} & 4 \bigoplus_{j_L=2}^{\infty} (j_L, j_L \pm \frac{3}{2})^{\mp} \oplus 4 \bigoplus_{j_L=0}^1 (j_L, j_L + \frac{3}{2})^{-} \oplus 4 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \\ & \oplus 4(0, \frac{1}{2})^{+} \oplus 84 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \oplus 84(0, \frac{1}{2})^{+}, \end{aligned}$$

- Chiral vectors:

$$5 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm 1)^{\mp} \oplus 5(0, 1)^{-} \oplus 21 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm 1)^{\pm} \oplus 21(0, 1)^{+},$$

where all the sums are over the integers.

The vector representations  $(0, 1)^{-} \oplus 21(0, 1)^{+}$  are mapped to themselves by the reality condition on the six-dimensional tensors. We thus count their contribution with a factor of  $\frac{1}{2}$ .

When  $k > 0$ , we find that  $j_L^{(3)} = \frac{1}{2}mk \geq \frac{m}{2}$  such that for  $m \geq 3$  the following part of the spectrum contributes:

- Spin- $\frac{3}{2}$ :

$$4 \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\mp},$$

- Spin- $\frac{1}{2}$ :

$$4 \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm \frac{3}{2})^{\mp} \oplus 4 \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \oplus 84 \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm},$$

- Chiral vectors:

$$5 \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm 1)^{\mp} \oplus 21 \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm 1)^{\pm},$$

where the sums are again with integer steps.

### 6.A.2 Correction to the left level

Note that the contribution from the  $k = 0$  modes is zero in this case as they come with an overall factor of  $(j_L^{(3)})^2$  which clearly vanishes in the  $k = 0$  case. We next consider, separately at first, the contributions for  $k > 0$  for the different kinds of fields before summing all the results. For the spin- $\frac{3}{2}$  fermions we find

$$\begin{aligned}\alpha_L^{(3/2)} &= 4 \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \frac{3}{8\pi} \left(\frac{1}{2}mk\right)^2 \left[2\left(j_L - \frac{1}{2}\right) + 1 - 2\left(j_L + \frac{1}{2}\right) - 1\right] \\ &= -\frac{3m^2}{4\pi} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} k^2 = \frac{1}{8\pi} \frac{m^3}{40}.\end{aligned}\quad (6.75)$$

Likewise the spin- $\frac{1}{2}$  fermions contribute as

$$\begin{aligned}\alpha_L^{(1/2)} &= 4 \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \frac{1}{8\pi} \left(\frac{1}{2}mk\right)^2 (-6 + 2 + 42) \\ &= \frac{19}{4\pi} m^2 \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} k^2 = -\frac{1}{8\pi} \frac{19}{120} m^3.\end{aligned}\quad (6.76)$$

Finally, the vectors contribute as

$$\alpha_L^{(\text{vect})} = \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \frac{1}{4\pi} \left(\frac{1}{2}mk\right)^2 (20 - 84) = \frac{1}{8\pi} \frac{2}{15} m^3.\quad (6.77)$$

Adding (6.75), (6.76) and (6.77), it follows that the one-loop contribution to the left level vanishes

$$k_L^{\text{loop}} = 8\pi \cdot \left( \alpha_L^{(3/2)} + \alpha_L^{(1/2)} + \alpha_L^{(\text{vect})} \right) = 0.\quad (6.78)$$

### 6.A.3 Correction to the right level

Let us now perform the analogous computation for the right level. We begin by studying the contribution from the  $k = 0$  modes which do not vanish in this case. The spin- $\frac{3}{2}$  fermions contribute as

$$4 \sum_{j_L=1}^{\infty} \frac{1}{4\pi} \left[ \left(j_L + \frac{1}{2}\right) \left(j_L + \frac{3}{2}\right) (2j_L + 2) - \left(j_L - \frac{1}{2}\right) \left(j_L + \frac{1}{2}\right) (2j_L) \right] = -\frac{5}{4\pi}.\quad (6.79)$$

The infinite towers contained within the spin- $\frac{1}{2}$  fermion spectrum give

$$\frac{1}{3\pi} \sum_{j_L=2}^{\infty} \left[ \left(j_L + \frac{3}{2}\right) \left(j_L + \frac{5}{2}\right) (2j_L + 4) - \left(j_L - \frac{3}{2}\right) \left(j_L - \frac{1}{2}\right) (2j_L - 2) \right] = -\frac{83}{4\pi},$$

$$\frac{-1}{3\pi}(1+21)\sum_{j_L=1}^{\infty}\left[(j_L+\frac{1}{2})(j_L+\frac{3}{2})(2j_L+2)-(j_L-\frac{1}{2})(j_L+\frac{1}{2})(2j_L)\right]=\frac{55}{6\pi}, \quad (6.80)$$

whilst the isolated representations, which are not part of an infinite tower, give

$$-\frac{1}{3\pi}\left[-\frac{3}{2}\cdot\frac{5}{2}\cdot 4-\frac{5}{2}\cdot\frac{7}{2}\cdot 6+(1+21)\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot 2\right]=\frac{23}{2\pi}. \quad (6.81)$$

Finally, the infinite towers contained within the vector spectrum contribute

$$(5-21)\sum_{j_L=1}^{\infty}\frac{1}{6\pi}\left[-(j_L+1)(j_L+2)(2j_L+3)+(j_L-1)j_L(2j_L-1)\right]=-\frac{32}{3\pi} \quad (6.82)$$

and the isolated representations  $5(0,1)^- \oplus 21(0,1)^-$  give (note that they come with an extra factor of  $\frac{1}{2}$ )

$$-\frac{1}{2}(5-21)\cdot\frac{1}{6\pi}\cdot 1\cdot 2\cdot 3=\frac{8}{\pi}. \quad (6.83)$$

Enumerating all the contributions from the above  $k=0$  results gives

$$\alpha_R^{k=0}=-\frac{16}{4\pi}. \quad (6.84)$$

We now turn to the evaluation of the  $k>0$  contributions. For the spin- $\frac{3}{2}$  fermions the contribution is

$$\begin{aligned} \alpha_R^{(3/2)} &= 4\sum_{k=1}^{\infty}\sum_{j_L=\frac{1}{2}mk}^{\infty}\frac{1}{4\pi}\left[(j_L+\frac{1}{2})(j_L+\frac{3}{2})(2j_L+2)-(j_L-\frac{1}{2})(j_L+\frac{1}{2})(2j_L)\right] \\ &= \frac{1}{\pi}\sum_{k=1}^{\infty}\sum_{j_L=\frac{1}{2}mk}^{\infty}\left(\frac{3}{2}+6j_L+6j_L^2\right)=\frac{1}{4\pi}\left(-\frac{1}{2}-\frac{m}{12}-\frac{m^3}{120}\right). \end{aligned} \quad (6.85)$$

Analogously, the spin- $\frac{1}{2}$  fermions give

$$\begin{aligned} \alpha_R^{(1/2)} &= 4\sum_{k=1}^{\infty}\sum_{j_L=\frac{1}{2}mk}^{\infty}\frac{1}{12\pi}\left[(j_L+\frac{3}{2})(j_L+\frac{5}{2})(2j_L+4)-(j_L-\frac{3}{2})(j_L-\frac{1}{2})(2j_L-2)\right] \\ &\quad -4(1+21)\sum_{k=1}^{\infty}\sum_{j_L=\frac{1}{2}mk}^{\infty}\frac{1}{12\pi}\left[(j_L+\frac{1}{2})(j_L+\frac{3}{2})(2j_L+2)-(j_L-\frac{1}{2})(j_L+\frac{1}{2})(2j_L)\right] \\ &= \frac{1}{4\pi}\sum_{k=1}^{\infty}\sum_{j_L=\frac{1}{2}mk}^{\infty}(22+24j_L+24j_L^2)-\frac{22}{3\pi}\sum_{k=1}^{\infty}\sum_{j_L=\frac{1}{2}mk}^{\infty}\left(\frac{3}{2}+6j_L+6j_L^2\right) \end{aligned}$$

$$= \frac{1}{4\pi} \left( -\frac{5}{6} + \frac{43}{36}m + \frac{19}{360}m^3 \right). \quad (6.86)$$

Lastly, the vector spectrum provides the contribution

$$\begin{aligned} \alpha_R^{(\text{vect})} &= (21 - 5) \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \frac{1}{6\pi} [(j_L + 1)(j_L + 2)(2j_L + 3) - (j_L - 1)j_L(2j_L - 1)] \\ &= -\frac{16}{\pi} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} (1 + 2j_L + 2j_L^2) = \frac{1}{4\pi} \left( \frac{32}{3} - \frac{8}{9}m + \frac{2}{45}m^3 \right). \end{aligned} \quad (6.87)$$

Adding (6.84), (6.85), (6.86) and (6.87) the one-loop contribution to the right level is

$$k_R^{\text{loop}} = 4\pi \cdot \left( \alpha_R^{k=0} + \alpha_R^{(3/2)} + \alpha_R^{(1/2)} + \alpha_R^{(\text{vect})} \right) = 2m - 28.$$

#### 6.A.4 Correction to $c_L - c_R$

As in the previous section we first calculate the  $k = 0$  contribution before proceeding to calculate the  $k > 0$  contributions. The spin- $\frac{3}{2}$  fermion, spin- $\frac{1}{2}$  fermion and vector representations give

$$\begin{aligned} 4 \sum_{j_L=1}^{\infty} \frac{7}{64\pi} \cdot 2 &= -\frac{7}{16\pi}, \\ \frac{1}{48\pi} \left[ -\sum_{j_L=2}^{\infty} 6 - 4 - 6 + \sum_{j_L=1}^{\infty} 2 + 2 + 21 \sum_{j_L=1}^{\infty} 2 + 42 \right] &= \frac{7}{16\pi}, \\ (21 - 5) \sum_{j_L=1}^{\infty} \frac{1}{48\pi} \cdot 4 + \frac{1}{2}(21 - 5) \frac{1}{48\pi} \cdot 3 &= \frac{1}{6\pi}, \end{aligned} \quad (6.88)$$

respectively. We have again included an extra factor of  $\frac{1}{2}$  for the isolated vector representations. The  $k = 0$  contribution from the above corrections is

$$\alpha_{\text{grav}}^{k=0} = -\frac{16}{96\pi}. \quad (6.89)$$

The  $k > 0$  contribution of the spin- $\frac{3}{2}$  fermions is

$$\alpha_{\text{grav}}^{(3/2)} = 4 \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \frac{7}{64\pi} \cdot 2 = \frac{1}{96\pi} \left( -21 + \frac{7}{2}m \right), \quad (6.90)$$

the spin- $\frac{1}{2}$  fermions contribute

$$\alpha_{\text{grav}}^{(1/2)} = 4 \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \frac{1}{48 \cdot 4\pi} (-6 + 2 + 42) = \frac{1}{96\pi} \left( -19 + \frac{19}{6}m \right), \quad (6.91)$$

and finally the vector modes give

$$\alpha_{\text{grav}}^{(\text{vect})} = (19 - 5) \sum_{k=1}^{\infty} \sum_{j_L = \frac{1}{2}mk}^{\infty} \frac{1}{48\pi} \cdot 4 = \frac{1}{96\pi} \left( 32 - \frac{16}{3}m \right). \quad (6.92)$$

The total one-loop correction for  $c_L - c_R$ , obtained by summing (6.89), (6.90), (6.91) and (6.92) is

$$\begin{aligned} (c_L - c_R)^{\text{loop}} &= 96\pi \cdot \left( \alpha_{\text{grav}}^{k=0} + \alpha_{\text{grav}}^{(3/2)} + \alpha_{\text{grav}}^{(1/2)} + \alpha_{\text{grav}}^{(\text{vect})} \right) \\ &= 12m - 88. \end{aligned} \quad (6.93)$$

### 6.A.5 $\mathbb{D}_m^*$ quantum corrections from $\mathbb{Z}_m$

In the previous sections we have computed the one-loop corrections to the levels and gravitational anomaly for the A-series. In the case of the D-series, we have to compute the right level and gravitational anomaly as well. In principle one should be able to perform the equivalent computations of the previous sections, however in practice this is not the most efficient way to obtain the desired result. Instead we can use the observation that the summations for the D-series are, modulo the  $j_R^{(3)} \geq 0$  condition, equivalent to the summations for the A-series with  $m \rightarrow 2m$ . We may therefore use the previous results and then impose the  $j_R^{(3)} \geq 0$  condition. As the contribution for a given  $j_R^{(3)}$  within a fixed  $j_R$  representation is the same for  $j_R^{(3)}$  and  $-j_R^{(3)}$ , imposing  $j_R^{(3)} \geq 0$  can be performed by halving the above result and adding in  $\frac{1}{2}$  of the contribution from  $j_R^{(3)} = 0$ . The latter contributions can be seen to be of  $\mathcal{O}(1)$  in the charges and therefore, as we are working to linear order, we may neglect this discrepancy and use the above shortcut. Performing the computation as outlined in the A-series section above for the D-series will result in the same linear order results. The final quantum corrections for the D-series are then

$$k_R^{\text{loop}} = 2m, \quad (c_L - c_R)^{\text{loop}} = 12m. \quad (6.94)$$

## 6.B $U(1)_L$ charges of $\mathcal{N} = 2$ fields

In this appendix we discuss the  $U(1)_L$  charges of the various fields in the  $\mathcal{N} = 2$  quivers associated to a transverse ALE spaces in the A-series. Geometrically, this symmetry is realized as a  $U(1)_L$  isometry of the ALE space corresponding to the A-series. This isometry is absent for the D- and E-series such that this  $U(1)_L$  flavor symmetry is not present in the latter field theories.

Recall that the 4d  $\mathcal{N} = 2$  quiver gauge theories we are interested in can be obtained by a quotient of  $\mathcal{N} = 4$  super Yang–Mills. An  $\mathcal{N} = 4$  vector multiplet consists of a gauge field  $A_\mu^{\mathcal{N}=4}$ , six scalars  $\varphi^i$  and fermions  $\Psi_\alpha^{\mathcal{I}}, \tilde{\Psi}_{\dot{\alpha}\mathcal{I}}$ , where  $\mathcal{I} = 1, \dots, 4$  and  $i = 1, \dots, 6$ . They transform under  $\mathrm{SO}(1, 3)_\ell \times \mathrm{SU}(4)_R$  in the representations

$$A_\mu^{\mathcal{N}=4} \in (\mathbf{2}, \mathbf{2}, \mathbf{1}), \quad \varphi^i \in (\mathbf{1}, \mathbf{1}, \mathbf{6}), \quad \Psi_\alpha^{\mathcal{I}} \in (\mathbf{2}, \mathbf{1}, \mathbf{4}), \quad \tilde{\Psi}_{\dot{\alpha}\mathcal{I}} \in (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}}), \quad (6.95)$$

where  $\mathrm{SU}(4)_R$  is the R-symmetry of  $\mathcal{N} = 4$  SYM. An  $\mathcal{N} = 4$  vector multiplet can be decomposed into an  $\mathcal{N} = 2$  vector multiplet and a hypermultiplet. Identifying  $\mathrm{SU}(2)_R \times \mathrm{U}(1)_r \subset \mathrm{SU}(4)_R$  as the  $\mathcal{N} = 2$  R-symmetry, the  $\mathcal{N} = 4$  vector multiplet decomposes as

$$\begin{aligned} \mathrm{SO}(1, 3)_\ell \times \mathrm{SU}(4)_R &\rightarrow \mathrm{SO}(1, 3)_\ell \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_r \times \mathrm{U}(1)_L \\ A_\mu^{\mathcal{N}=4} \in (\mathbf{2}, \mathbf{2}, \mathbf{1}) &\rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{1})_{0,0} \\ \varphi^i \in (\mathbf{1}, \mathbf{1}, \mathbf{6}) &\rightarrow (\mathbf{1}, \mathbf{1}, \mathbf{1})_{2,0} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{1})_{-2,0} \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2})_{0,1} \\ &\quad \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2})_{0,-1} \\ \Psi_{\alpha\mathcal{I}} \in (\mathbf{2}, \mathbf{1}, \mathbf{4}) &\rightarrow (\mathbf{2}, \mathbf{1}, \mathbf{1})_{1,1} \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1})_{1,-1} \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2})_{-1,0} \\ \tilde{\Psi}_{\dot{\alpha}\mathcal{I}} \in (\mathbf{1}, \mathbf{2}, \bar{\mathbf{4}}) &\rightarrow (\mathbf{1}, \mathbf{2}, \mathbf{1})_{-1,1} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1})_{-1,-1} \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_{1,0}. \end{aligned} \quad (6.96)$$

The only states in the decomposition with non-vanishing  $\mathrm{U}(1)_L$  charge are the  $(\mathbf{2}, \mathbf{1}, \mathbf{1})_{1,\pm 1}$  states in the decomposition of  $\Psi_{\alpha\mathcal{I}}$ , the  $(\mathbf{1}, \mathbf{2}, \mathbf{1})_{-1,\pm 1}$  states from  $\tilde{\Psi}_{\dot{\alpha}\mathcal{I}}$ , and finally the  $(\mathbf{1}, \mathbf{1}, \mathbf{2})_{0,\pm 1}$  states from the decomposition of  $\varphi^i$ . Since the fermionic states  $(\mathbf{2}, \mathbf{1}, \mathbf{1})_{1,\pm 1}$  and  $(\mathbf{1}, \mathbf{2}, \mathbf{1})_{-1,\pm 1}$  are singlets under  $\mathrm{SU}(2)_R$ , they are identified with the fermions in the  $\mathcal{N} = 2$  hypermultiplet. Similarly, since the states  $(\mathbf{1}, \mathbf{1}, \mathbf{2})_{0,\pm 1}$  are scalar doublets under  $\mathrm{SU}(2)_R$ , we conclude that they comprise the scalar degrees of freedom in the  $\mathcal{N} = 2$  hypermultiplet. In summary, the two scalars and the two fermions in an  $\mathcal{N} = 2$  hypermultiplet have opposite charges  $Q_L = \pm 1$  under  $\mathrm{U}(1)_L$ . On the other hand the fields in the vector multiplet are uncharged under this  $\mathrm{U}(1)_L$  global symmetry. This then leads to the  $\mathrm{U}(1)_L$  charges shown in table 6.4. Let us stress again that this additional global  $\mathrm{U}(1)_L$  symmetry only exists for the A-series.



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## Chapter 7

# Wrapped D3-branes in F-theory on $CY_3$

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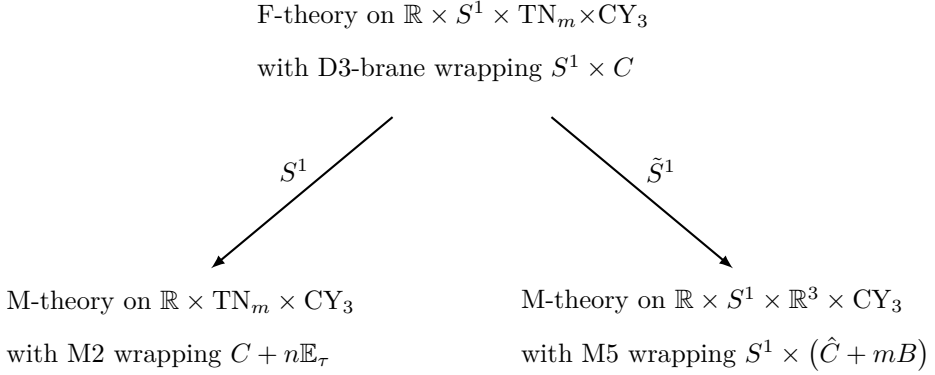
In the early days of F-theory the microscopic entropy of a D3-brane in an asymptotic geometry  $\mathbb{R}^{1,4} \times S^1 \times CY_3$  was computed exploiting its dual M-theory formulation. The D3-brane is wrapped on  $S^1 \times C$ , where  $C$  is a curve in the base  $B$  of an elliptically fibered Calabi-Yau threefold  $CY_3$  and corresponds to a non-spinning black hole in five dimensions. The microscopic entropy was then successfully matched to its macroscopic counterpart [141]. Some years later, this setup was generalized to a background  $\mathbb{R} \times S^1 \times TN_m \times CY_3$  [138] corresponding to macroscopic four-dimensional black holes. The microscopic analysis was carried out by mapping the F-theory setup to the MSW setting [104]. The macroscopic analysis was focused on the leading contribution to the entropy.

Studying gravitational aspects in F-theory attracted renewed interest recently using diverse approaches. In [15] the authors extended the study of [141] to the case of five-dimensional spinning black holes that previously only had been studied in compactifications of type II strings on  $T^5$  and  $K3 \times S^1$  [117]. On the CFT side the main difference between [141] and [15] is the identification of an  $SU(2)_L$  current algebra in spite of the absence of supersymmetry in the left-moving sector. We reviewed this setting in section 1.4.3. Most recently, supersymmetric  $AdS_3$  solutions of type IIB supergravity with varying axio-dilaton profile and five-form flux dual to  $\mathcal{N} = (0, 2)$  and  $\mathcal{N} = (0, 4)$  SCFTs were analyzed in [99, 100]. The

$AdS_3$  solutions dual to  $\mathcal{N} = (0, 4)$  SCFTs can be interpreted as near horizon geometries of six-dimensional strings from wrapped D3-branes, as described above.

This chapter is devoted to the study of strings in six dimensions which are constructed by wrapping D3-branes on  $S^1 \times C$  while probing a transverse ALE or ALF space. These transverse ALE and ALF spaces, which we denote by  $M_\Gamma$ , are labelled by a discrete subgroup  $\Gamma$  of  $SU(2)$ , as described in section 6.1 for type IIB on K3. The main focus of this chapter will be on the case where  $M_\Gamma = TN_m$  since this case admits a clean microscopic description. The extension of the  $TN_m$  case to other transverse spaces is given at the end of this chapter. We derive characteristic data of the SCFT corresponding to the D3-brane placed inside the background geometry  $\mathbb{R} \times S^1 \times TN_m \times CY_3$  [138] from macroscopic considerations. The two-dimensional SCFT has  $\mathcal{N} = (0, 4)$  supersymmetry with left- and right-moving central charges  $c_{L,R}$ , as well as left- and right-moving current algebras  $U(1)_L \times SU(2)_R$  with levels  $k_{L,R}$ . This setting is motivated by the 4d/5d black hole correspondence [142, 143]. Making the radius of the circle  $S^1$  wrapped by the D3-brane very small, we perform a T-duality along  $S^1$  to obtain a type IIA setting that then lifts to an M-theory background  $\mathbb{R} \times TN_m \times CY_3$ . Under this duality the wrapped D3-brane turns into an M2-brane wrapping  $C$ . Including momentum  $n$  along  $S^1$ , the D3-brane maps to bound states of M2-branes wrapping a curve in the class  $C + n\mathbb{E}_\tau$ , where  $\mathbb{E}_\tau$  is the elliptic fiber of the  $CY_3$ . After compactification on  $CY_3$  one obtains a five-dimensional black hole with a transverse  $TN_m$  space. This five-dimensional black hole has an eigenvalue  $J_L$  corresponding to the  $U(1)_L$  symmetry along the NUT-circle. Compactifying the M-theory setting further along the circle fiber of the Taub-NUT space results in a type IIA compactification on  $CY_3$ . The M2-brane configuration gets mapped to a D6-D2-D0 system on the same Calabi-Yau threefold. The D6-brane has multiplicity  $m$  and one has  $2J_L$  units of D0-brane charge. This is the four-dimensional side of the correspondence in [142, 143].

Instead of using the circle  $S^1$  wrapped by the D3-brane to go to five dimensions, we can also reduce along the Taub-NUT circle, which we denote by  $\tilde{S}^1$ . Performing a T-duality along  $\tilde{S}^1$ , and lifting to M-theory, the D3-brane wrapping the curve  $C$  turns into an M5-brane wrapping  $\hat{C} = \pi^{-1}(C) \subset CY_3$ , where  $\pi : CY_3 \rightarrow B$  is the projection to the base. Following the same duality chain the Taub-NUT space gives rise to  $m$  M5-branes wrapping the base  $B$  of the elliptic fibration. The two groups of M5-branes can be combined into a single M5-brane wrapping the curve  $\hat{C} + mB$  in the  $CY_3$  if the corresponding class is very ample. In summary, the two main dualities just introduced can be schematically depicted as:



If we take both the NUT circle and the circle wrapped by the D3-brane to be small, we effectively obtain a four-dimensional black hole. We therefore obtain an F-theory description of a four-dimensional black hole. The central charges and levels which we determine then give the black hole entropy via Cardy's formula.

We use the six-dimensional effective  $\mathcal{N} = (1, 0)$  supergravity action of F-theory compactified on an elliptically fibered Calabi-Yau threefold derived in [144, 50] to determine the contributions from classical six-dimensional supergravity to the central charges and levels using techniques of [91–93]. Concretely, we dimensionally reduce the six-dimensional effective action to three dimensions and read off the sought-after quantities from the coefficients of Chern-Simons terms. It turns out that in order to fully reproduce the microscopic quantities one also has to include one-loop Chern-Simons terms in three dimensions. These one-loop induced terms arise from integrating out massive Kaluza-Klein modes. This interplay between classical and quantum contributions to complete M/F-theory duality in this case is in fact not unexpected. Including one-loop corrections was already crucial for the matching of the five-dimensional M-theory effective action on  $\text{CY}_3$  and its dual six-dimensional F-theory action [50, 83]. Another ingredient we also take into account for the comparison with the microscopic quantities is a shift in the charges. This shift stems from a higher-derivative term in the six-dimensional effective action which does not integrate to zero on Taub-NUT space.

In section 7.1 we start with a more extensive description of the setting with a transverse Taub-NUT space with NUT-charge  $m$ . In this section we will also state the microscopic quantities that we aim to reproduce from supergravity. We then proceed by calculating the classical and quantum contributions to the central charges and levels in sections 7.2 and 7.3 respectively. Section 7.4 includes the macroscopic computations for transverse ALE and ALF spaces, which are labelled

by  $\Gamma \subset SU(2)$ . In section 7.5 we extend the previous sections to include vector multiplets in six-dimensions. This leads to new flavour symmetries acting on the string, whose levels we compute.

## 7.1 Microscopics

As already stated, we consider an F-theory background  $\mathbb{R} \times S^1 \times TN_m \times CY_3$ , where we have a D3-brane wrapping  $S^1 \times C$  with  $C \subset B$  a curve in the elliptically fibered Calabi-Yau threefold  $\pi : CY_3 \rightarrow B$ . For simplicity we only consider threefolds with mild fiber degenerations which render the total elliptic fibration smooth. Using a basis  $\omega_\alpha$  of  $H^{1,1}(B)$  we can expand the Poincaré dual of the curve and the first Chern class of the base such that we have  $C = q^\alpha \omega_\alpha$  and  $c_1(B) = c^\alpha \omega_\alpha$ . The intersection numbers on the base are given by

$$\eta_{\alpha\beta} \equiv \int_B \omega_\alpha \wedge \omega_\beta. \quad (7.1)$$

Furthermore, here and in the following we make use of the notation

$$\begin{aligned} C \cdot C &= \int_B C \wedge C \equiv C^2, \\ c_1(B) \cdot C &= \int_B c_1(B) \wedge C, \\ c_1(B) \cdot c_1(B) &= \int_B c_1(B) \wedge c_1(B) \equiv c_1(B)^2. \end{aligned} \quad (7.2)$$

Microscopically the central charges corresponding to this setting were derived by considering the dual system in M-theory [138]. As already described in the introduction of this section one can start from type IIB, T-dualize along the NUT-circle and then lift the system to M-theory. Performing a T-duality along the NUT circle we end up with a D4-brane wrapping  $S^1 \times \tilde{S}^1 \times C$  and  $m$  NS5-branes wrapping  $B$ . These type IIA objects lift in M-theory to an M5-brane wrapping  $\hat{C}$  and  $m$  M5-branes wrapping  $B$ . If the class of the curve  $\hat{C} + mB$  is very ample the two M5-brane groups can be combined into a single M5-brane wrapping  $\hat{C} + mB$ . We therefore assume that  $q^\alpha > 0$  and  $q^\alpha \gg mc^\alpha$ ,  $\forall \alpha = 1, \dots, h^{1,1}(B)$  in order to ensure this ampleness condition.

This system falls in the class of settings studied by MSW [104] such that the

central charges and right level are given by<sup>48</sup>

$$\begin{aligned} c_L &= 3mC^2 - 3m^2c_1(B) \cdot C + m^3c_1(B)^2 + 12c_1(B) \cdot C + 12m - 2mc_1(B)^2, \\ c_R &= 6k_R = 3mC^2 - 3m^2c_1(B) \cdot C + m^3c_1(B)^2 + 6c_1(B) \cdot C + 6m - mc_1(B)^2, \end{aligned} \quad (7.3)$$

where the relation between the right-moving central charge and level follows from supersymmetry in the right-moving sector of the SCFT.

**Left level.** Although not explicitly calculated, we can extract the left level  $k_L$  from the data provided in [138]. The formula for the entropy of the black string in [138] reads

$$S = 2\pi \sqrt{\frac{c_L}{6} \hat{m}}, \quad (7.4)$$

where

$$\hat{m} = n + \frac{1}{12} (D^{00} \tilde{Q}_0 \tilde{Q}_0 + 2D^{0\alpha} \tilde{Q}_0 \tilde{Q}_\alpha + D^{\alpha\beta} \tilde{Q}_\alpha \tilde{Q}_\beta) \quad (7.5)$$

and the matrix  $D$  is given by

$$D_{00} = \frac{1}{6} c_1(B) \cdot C, \quad D_{0\alpha} = \frac{1}{6} q_\alpha, \quad D_{\alpha\beta} = \frac{1}{6} m \eta_{\alpha\beta}. \quad (7.6)$$

The elements  $D^{AB} = D_{AB}^{-1}$  denote components of the inverse matrix with respect to the full matrix  $D_{AB}$ , in particular, it is not the inverse of a sub-matrix of  $D$ . The charge  $\tilde{Q}_0$  contains a term  $2J_L/m$  [138]. The entropy of BPS states in the CFT with  $U(1)_L$  eigenvalue  $J_L$  and momentum eigenvalue  $n$  is given by the Cardy formula

$$S = 2\pi \sqrt{\frac{c_L}{6} \left( n - \frac{J_L^2}{k_L} \right)}, \quad (7.7)$$

from which we can extract  $k_L$ , as we will show in the following. The level  $k_L$  can not depend on the momentum  $n$ , since the level is part of the data which *defines* the symmetries in the CFT, whereas the momentum  $n$  is a label of states in the CFT. We can then take the limit  $2J_L/m \rightarrow \infty$  in (7.5) and compare the resulting expression with the spectral flow invariant  $n - \frac{J_L^2}{k_L}$ . In particular, we have that

$$\frac{1}{12} D^{00} \left( \frac{2J_L}{m} \right)^2 = -\frac{J_L^2}{k_L}. \quad (7.8)$$

<sup>48</sup>These results follow straightforwardly from the data given in [138] using identities valid for elliptically fibered Calabi-Yau threefolds.

Calculating the inverse of  $D$  yields

$$D^{00} = \frac{6m}{mc_1(B) \cdot C - C^2}, \quad (7.9)$$

such that we find

$$k_L = \frac{1}{2}mC^2 - \frac{1}{2}m^2c_1(B) \cdot C \quad (7.10)$$

for the left level.

It is the main objective of this chapter to reproduce the central charges and levels, given in (7.3) and (7.10), from six-dimensional  $\mathcal{N} = (1, 0)$  supergravity up to  $\mathcal{O}(1)$  terms.

As described in the introduction of this chapter, our setting is motivated by obtaining four-dimensional black holes from F-theory. Using  $c_L$  and  $k_L$ , one can compute the entropy of this black hole in the Cardy limit via the formula (7.7). In the same limit the Wald entropy [145] agrees with this formula, with  $c_L$  and  $k_L$  derived from the supergravity action [146, 91].

Parts of the central charges and levels have been computed in [99] from type IIB supergravity. The authors studied  $AdS_3$  solutions of type IIB supergravity with varying axio-dilaton using the spinorial geometry approach. They studied the constraints on the compact geometry arising from preserving  $\mathcal{N} = (0, 4)$  supersymmetry in the dual two-dimensional SCFT while preserving all  $AdS_3$  isometries. The class of ten-dimensional solutions takes the form  $AdS_3 \times S^3/\mathbb{Z}_m \times (\mathbb{E}_\tau \hookrightarrow CY_3 \xrightarrow{\pi} B)$ , with non-trivial five-form flux and axio-dilaton profile, where  $B$  is the Kähler base of an elliptically fibered Calabi-Yau threefold. The solution can be interpreted as the near horizon limit of  $N$  D3-branes wrapping a curve  $C$  in the Kähler base in the presence of D7-branes and a Taub-NUT space with NUT-charge  $m$  in the four non-compact directions transverse to the D3-branes. The dual  $\mathcal{N} = (0, 4)$  SCFT has again a  $U(1)_L \times SU(2)_R$  current algebra with levels  $k_{L,R}$ . These levels and the central charges of the CFT were computed in the large  $N$  limit and were for general  $m$  found to be

$$\begin{aligned} c_R^{\text{IIB}} &= 6k_R^{\text{IIB}} = 3N^2 m C^2, \\ c_L^{\text{IIB}} &= 3N^2 m C^2, \\ k_L^{\text{IIB}} &= \text{unknown}. \end{aligned} \quad (7.11)$$

A subleading correction  $c_L^{\text{IIB}} - c_R^{\text{IIB}}$  at  $\mathcal{O}(N)$  was also found for general  $m$ , it is however expected from the dual M-theory result (7.3) that there exist additional

$\mathcal{O}(N)$  contributions to the central charges and levels. The full answer for  $c_{L,R}^{\text{IIB}}$  and  $k_R^{\text{IIB}}$  including subleading  $\mathcal{O}(N)$  contributions was given for the distinguished case  $m = 1$ , where the near horizon geometry is  $\text{AdS}_3 \times S^3 \times B$  corresponding to an unbroken  $\text{SU}(2)_L \times \text{SU}(2)_R$  current algebra in the CFT.

## 7.2 Macroscopics in F-theory from 6d: classical contributions

In this section we use six-dimensional  $\mathcal{N} = (1, 0)$  supergravity [88, 147–149] to compute parts of the microscopic central charges and levels (7.3) and (7.10). An F-theory compactification on a smooth elliptically fibered Calabi-Yau threefold results in a gravity multiplet,  $n_T = h^{1,1}(B) - 1$  tensor multiplets and  $n_H = h^{2,1}(\text{CY}_3) + 1$  hypermultiplets, but no vector multiplets [14, 150, 151]. Recall that we restricted ourselves to smooth threefolds for simplicity, e.g. to avoid charged matter.

We reproduce part of the central charges and levels utilizing the approach used in [91–94], which in principle means that one has to reduce the six-dimensional action on the spherical part of the near horizon geometry  $\text{AdS}_3 \times S^3/\mathbb{Z}_m$  of the black string solution. Dimensionally reducing the six-dimensional action on  $S^3/\mathbb{Z}_m$  one can infer the levels and central charges of the dual CFT from coefficients of Chern-Simons terms in three dimensions using the AdS/CFT dictionary, see e.g. [152]. In fact we find, based on [94] and as explained in section 6.2, that one has to do this dimensional reduction at spatial infinity of the solution to get the correct result for central charges and levels and to take into account the effect of the Taub-NUT space transverse to the string.

We first provide a few details about the six-dimensional  $\mathcal{N} = (1, 0)$  supergravity theory arising from F-theory compactified on a Calabi-Yau threefold, which shall be the starting point for our investigation. Thereafter, we perform the dimensional reduction of the supergravity action to three dimensions, pointing out the difference between the reduction in the near horizon geometry and the reduction at asymptotic infinity. In both cases one finds a mismatch with the microscopic prediction. The mismatch in the reduction at asymptotic infinity can be cured using one loop induced Chern-Simons terms in three dimensions. This will be the subject of section 7.3, which is one of the main results of this chapter.

### 7.2.1 Six-dimensional $\mathcal{N} = (1, 0)$ supergravity from F-theory on $CY_3$

We consider the six-dimensional effective action arising from compactifying F-theory on an elliptically fibered Calabi-Yau threefold  $CY_3$ . The characteristic data of the underlying  $\mathcal{N} = (1, 0)$  supergravity theory in six dimensions was determined in [50] by matching a generic circle reduced six-dimensional  $\mathcal{N} = (1, 0)$  theory with the geometric data arising from compactifying M-theory on a smooth Calabi-Yau threefold. The massless spectrum assembles itself in representations of the little group in six dimensions  $SO(4) \simeq SU(2)_1 \times SU(2)_2$ , which are labelled by the spins  $(j_1, j_2)$ . We will focus on a six-dimensional theory with field content

- one gravity multiplet:  $(1, 1) \oplus 2(\frac{1}{2}, 1) \oplus (1, 0)$  i.e. one graviton, one left-handed gravitino and one self-dual rank two tensor
- $n_T$  tensor multiplets:  $(0, 1) \oplus 2(0, \frac{1}{2}) \oplus (0, 0)$  i.e. one anti-self-dual rank two tensor, one right-handed tensorino and one real scalar per multiplet
- $n_H$  uncharged hypermultiplets:  $2(0, \frac{1}{2}) \oplus 4(0, 0)$  i.e. one right-handed hyperino and two complex scalars per multiplet.

We will furthermore assume throughout the chapter that the six-dimensional spectrum satisfies the constraint

$$n_H = 273 - 29n_T, \quad (7.12)$$

which ensures the cancellation of gravitational anomalies in six dimensions.

**Tensor multiplets.** The rank two tensors in the gravity- and tensor multiplets are collectively denoted by  $\hat{B}^\alpha$  with  $\alpha = 1, \dots, n_T + 1$ . The scalars in the tensor multiplets parametrize the manifold

$$\mathcal{M}_{\text{tensor}} = \frac{SO(1, n_T)}{SO(n_T)}. \quad (7.13)$$

The scalar sector of the tensor multiplets is usually described by  $n_T + 1$  scalar fields  $\hat{j}^\alpha$  subject to the constraint

$$\Omega_{\alpha\beta} \hat{j}^\alpha \hat{j}^\beta = 1, \quad (7.14)$$

where  $(\Omega_{\alpha\beta})$  is an  $SO(1, n_T)$  invariant constant metric with mostly minus signature. In the six-dimensional F-theory models we consider here this matrix  $\Omega_{\alpha\beta}$  is



identified with the intersection numbers on the base, i.e.

$$\Omega_{\alpha\beta} = \eta_{\alpha\beta} . \quad (7.15)$$

The constraint (7.14) is the six-dimensional analogue of the cubic constraint in very special geometry (3.23), which governs the vector multiplet sector in five-dimensional  $\mathcal{N} = 2$  supergravity. One furthermore introduces the non-constant, positive-definite metric

$$g_{\alpha\beta} = 2\hat{j}_\alpha \hat{j}_\beta - \eta_{\alpha\beta} , \quad \hat{j}_\alpha \equiv \eta_{\alpha\beta} \hat{j}^\beta . \quad (7.16)$$

The gauge-invariant field-strength  $\hat{G}^\alpha$  is defined by

$$\hat{G}^\alpha = d\hat{B}^\alpha + \frac{1}{8} c^\alpha \hat{\omega}_{\text{grav}}^{\text{CS}} , \quad \hat{\omega}_{\text{grav}}^{\text{CS}} = \text{tr} \left( \hat{\omega} \wedge d\hat{\omega} + \frac{2}{3} \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega} \right) , \quad (7.17)$$

where  $\hat{\omega}$  is the six-dimensional spin connection.

**Hypermultiplets.** Every hypermultiplet contains four real scalars, such that we denote the scalars collectively by  $\hat{q}^U$  ( $U = 1, \dots, 4n_H$ ). The hypermultiplets have a geometric interpretation as coordinates on a quaternionic manifold, whose metric is denoted by  $h_{UV}$ . Since we do not include vector multiplets in our setting and therefore the hypermultiplets are neutral, we will not need any further information about the hypermultiplets.

**Standard form of 6d (1,0) supergravity.** We choose conventions  $\kappa_6^2 = (2\pi)^3$  such that the bosonic part of the standard  $\mathcal{N} = (1,0)$  supergravity theory takes the form [144, 50]

$$\begin{aligned} S^{(6)} = \frac{1}{(2\pi)^3} \int_{M_6} \left[ \frac{1}{2} \hat{R} \hat{*} 1 - \frac{1}{4} g_{\alpha\beta} \hat{G}^\alpha \wedge \hat{*} \hat{G}^\beta - \frac{1}{2} g_{\alpha\beta} d\hat{j}^\alpha \wedge \hat{*} d\hat{j}^\beta \right. \\ \left. - h_{UV} d\hat{q}^U \wedge \hat{*} d\hat{q}^V - \frac{1}{8} \eta_{\alpha\beta} c^\alpha \hat{B}^\beta \wedge \text{tr} \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} \right] . \quad (7.18) \end{aligned}$$

The last term in (7.18) is a Green-Schwarz term which ensures gauge invariance at one-loop level [149, 153] and  $\hat{\mathcal{R}}$  denotes the curvature two-form in six dimensions. This higher curvature term in F-theory can be understood via its counterpart in M-theory [54, 50, 154], as well as from higher-curvature corrections on D7-branes and O7-planes. The latter perspective will be briefly explained in section 7.2.2. The field strengths satisfy non-standard Bianchi-identities

$$d\hat{G}^\alpha = \frac{1}{8} c^\alpha \text{tr} \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} , \quad (7.19)$$

and the (anti-)self-duality constraints for the tensors of the tensor- and gravity multiplets, which are imposed at the level of the equations of motion, are collectively given by

$$g_{\alpha\beta} \hat{*} \hat{G}^\beta = \eta_{\alpha\beta} \hat{G}^\beta . \quad (7.20)$$

This six-dimensional pseudo-action will be the starting point, similar to the approach to the macroscopic description in [15].

### 7.2.2 Black string solution in six-dimensional $\mathcal{N} = (1, 0)$ supergravity

Two-derivative six-dimensional  $\mathcal{N} = (1, 0)$  supergravity coupled to tensor multiplets has a black string solution which has the same asymptotics as  $\mathbb{R} \times S^1 \times \text{TN}_m$  [121]. The metric is given by

$$d\hat{s}_6^2 = 2H^{-1}du \left( dv - \frac{1}{2}H_5 du \right) + H ds_4^2 , \quad (7.21)$$

with a Taub-NUT metric of (positive) charge  $m$ ,<sup>49</sup>

$$ds_4^2 = H_2^{-1}m^2 (d\psi + \cos(\theta)d\phi)^2 + H_2 (dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2) . \quad (7.22)$$

Furthermore, we have

$$H = \left( \Omega_{\alpha\beta} H_1^\alpha H_1^\beta \right)^{1/2} , \quad (7.23)$$

and the harmonic functions on the base  $\mathbb{R}^3$  of  $\text{TN}_m$ , denoted by  $H_1^\alpha$ ,  $H_2$  and  $H_5$ , are given by

$$H_1^\alpha = \mu_\infty^\alpha + \frac{Q^\alpha}{4r} , \quad H_2 = m_\infty + \frac{m}{r} , \quad H_5 = -1 + \frac{n}{r} . \quad (7.24)$$

We also impose the restriction

$$\Omega_{\alpha\beta} \mu_\infty^\alpha \mu_\infty^\beta = 1 , \quad (7.25)$$

in order to get the correct asymptotics. The coordinate ranges are given by  $0 \leq u < \ell$  for a length parameter  $\ell$ ,  $-\infty < v < \infty$ ,  $0 \leq r < \infty$ ,  $0 \leq \psi < \frac{4\pi}{m}$ ,  $0 \leq \theta < \pi$  and  $0 \leq \phi < 2\pi$ . We will use the following dreibein for the part of the

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<sup>49</sup>Here we choose coordinates for the Taub-NUT space which differ from the ones usually found in the literature.

metric parametrized by  $\psi, \phi, \theta$  (which we henceforth will refer to as the spherical part  $\text{TN}_m^{\text{sph}}$ )

$$\begin{aligned}\hat{e}^1 &= \sqrt{HH_2} r \left( \sin(\psi) d\theta - \cos(\psi) \sin(\theta) d\phi \right), \\ \hat{e}^2 &= \sqrt{HH_2} r \left( \cos(\psi) d\theta + \sin(\psi) \sin(\theta) d\phi \right), \\ \hat{e}^3 &= \sqrt{H/H_2} m (d\psi + \cos(\theta) d\phi).\end{aligned}\tag{7.26}$$

The near horizon geometry of the metric (7.21) which is obtained in the limit  $r \rightarrow 0$  is  $\text{AdS}_3 \times S^3/\mathbb{Z}_m$  with the radius of  $S^3/\mathbb{Z}_m$  given by  $R^2 = m\sqrt{\Omega_{\alpha\beta}Q^\alpha Q^\beta}$ . In addition to a non-trivial metric background the solution also requires a radial profile for the scalars  $\hat{j}^\alpha$  given by [121]

$$\hat{j}^\alpha = \frac{H_1^\alpha}{H},\tag{7.27}$$

and non-vanishing three-form backgrounds<sup>50</sup>

$$\hat{G}^\alpha = -dv \wedge du \wedge d \left( H_1^\alpha H^{-2} \right) - \star_4 d \left( H_1^\alpha \right),\tag{7.28}$$

where  $\star_4$  denotes the Hodge dual with respect to the Taub-NUT metric  $ds_4^2$ . We also note that all hypermultiplet scalars are taken to be constant whereas all fermions vanish in the background.

Let us comment on the geometric properties of the Taub-NUT space (7.22). Firstly, we note that the Taub-NUT space has conical singularities for  $m > 1$ . In order to avoid these one can consider multi-centered solutions. One can see the metric (7.22) for general  $m$  as an  $m$ -centered Taub-NUT space in the limit in which all centers are taken to be coincident. The singularity then arises from the collapsing two-cycles between the centers of the multi-centered Taub-NUT space. Secondly, we recall that topologically Taub-NUT space is a circle fibration over  $\mathbb{R}^3$  and the radius of the circle at infinity is  $r_\infty = 1/\sqrt{m_\infty}$ . Varying this parameter  $r_\infty$  there are two interesting limits which one can consider. The first limit arises when  $m_\infty \ll \frac{m}{r}$ , i.e. the NUT circle decompactifies. In this limit the metric (7.22) approaches (after an additional coordinate transformation) the metric on  $\mathbb{R}^4/\mathbb{Z}_m$ . In particular, for the case  $m = 1$  one recovers the black string in flat space. The opposite limit is approached when  $m_\infty \gg \frac{m}{r}$ . This limit is implemented if the circle radius  $r_\infty$  is much smaller than the typical length scale of  $\mathbb{R}^3$  and leads to an effective dimensional reduction of the six-dimensional theory on this circle.

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<sup>50</sup>Note that the last term comes with a minus sign since our (anti-)self-duality conventions are opposite to those of [121].

The charges corresponding to the three-forms can be calculated by integrating over the spherical part

$$-(2\pi)^2 Q^\alpha = \int_{\text{TN}_m^{\text{sph}}} \hat{G}^\alpha \quad (7.29)$$

and are related to the microscopic charges  $q^\alpha$  via

$$Q^\alpha = q^\alpha - \frac{m}{2} c^\alpha, \quad (7.30)$$

as we will demonstrate in the following. Consider type IIB supergravity compactified on the Kähler surface  $B$ , which is the base of the elliptically fibered  $CY_3$  in F-theory. Working in conventions  $\text{vol}(B) = \frac{1}{2}$  and  $\ell_s^2 = 2\pi$  one can expand the type IIB (R,R) four-form  $C_4$  and the Kähler form  $J_B$  of the base  $B$  in harmonic  $(1,1)$ -forms on  $B$

$$C_4 = \hat{B}^\alpha \wedge \omega_\alpha, \quad J_B = \hat{j}^\alpha \omega_\alpha, \quad \text{with} \quad \omega_\alpha \in H^{1,1}(B). \quad (7.31)$$

The two-forms  $\hat{B}^\alpha$  are upon dimensional reduction on  $B$  identified with the (anti)-self-dual tensors in the six-dimensional gravity- and tensor multiplets, whereas the Kähler moduli  $\hat{j}^\alpha$  are interpreted as the scalars in the tensor multiplets. In addition to the bulk type IIB supergravity action there are also localized sources, namely D3-branes, D7-branes and O7-planes, in our setup. The presence of these ten-dimensional localized sources leads to additional six-dimensional couplings, which are crucial for the identification of the macroscopic with the microscopic charges. The D3-brane action contains the standard Chern-Simons action. Now consider  $N$  D3-branes with world-volume  $W_{D3} = \Sigma \times C$ , where  $\Sigma$  is a two-dimensional world-sheet in the six uncompactified dimensions and  $C \subset B$  is the curve in the base. Dimensionally reducing the Chern-Simons coupling we obtain

$$\begin{aligned} S_{\text{string}}^{\text{CS}} &= -\frac{N}{2\pi} \int_{W_{D3}} C_4 = -\frac{N}{2\pi} \int_{\Sigma} \hat{B}^\alpha \int_C \omega_\alpha = -\frac{N}{2\pi} \int_{\Sigma} \hat{B}^\alpha q^\beta \int_B \omega_\alpha \wedge \omega_\beta \\ &= -\frac{N}{2\pi} \int_{\Sigma} \eta_{\alpha\beta} q^\alpha \hat{B}^\beta \end{aligned} \quad (7.32)$$

for the string in six dimensions arising from wrapping the D3-brane over the curve  $C$ . We obtain further six-dimensional couplings of the two-forms  $\hat{B}^\alpha$  by taking into account higher curvature corrections on D7-branes and O7-planes. Expanding again the type IIB four-form  $C_4 = \hat{B}^\alpha \wedge \omega_\alpha$  and summing over all higher-curvature contributions from D7-branes and O7-planes, as dictated by the F-theory analogue of the D7-brane tadpole cancellation condition (see e.g. [155])

$$[\text{D7}] + 2[\text{O7}] = 12c_1(B), \quad (7.33)$$

one obtains the six-dimensional higher-curvature term relevant in the generalized Green-Schwarz mechanism (7.18). The total six-dimensional action is then the bulk part (7.18) coupled to the localized action (7.32)

$$S_{\text{tot}}^{(6)} = S^{(6)} + S_{\text{string}}^{\text{CS}}. \quad (7.34)$$

Deriving the equations of motion of the (anti-)self-dual tensors we obtain<sup>51</sup>

$$d(g_{\alpha\beta} \hat{*}\hat{G}^\beta) = (2\pi)^2 N \eta_{\alpha\beta} q^\beta \delta(\Sigma) + \frac{1}{8} \eta_{\alpha\beta} c^\beta \text{tr } \hat{\mathcal{R}} \wedge \hat{\mathcal{R}}. \quad (7.35)$$

where  $\delta(\Sigma)$  is a four-form delta current with support on the worldsheet of the string propagating in six-dimensions. Integrating the resulting equation over  $\text{TN}_m$  leads to

$$\begin{aligned} \frac{1}{(2\pi)^2} \int_{\text{TN}_m} d(g_{\alpha\beta} \hat{*}\hat{G}^\beta) &= \frac{1}{(2\pi)^2} \eta_{\alpha\beta} \int_{\text{TN}_m} d\hat{G}^\beta = -\frac{1}{(2\pi)^2} \int_{\text{TN}_m^{\text{sph}}} \eta_{\alpha\beta} \hat{G}^\beta = \eta_{\alpha\beta} Q^\beta \\ &= N \eta_{\alpha\beta} q^\beta + \frac{1}{8} \frac{1}{(2\pi)^2} \eta_{\alpha\beta} c^\beta \int_{\text{TN}_m} \text{tr } \hat{\mathcal{R}} \wedge \hat{\mathcal{R}}. \end{aligned} \quad (7.36)$$

Using furthermore that the first Pontryagin number of Taub-NUT is given by

$$p_1(\text{TN}_m) = -\frac{1}{2} \frac{1}{(2\pi)^2} \int_{\text{TN}_m} \text{tr } \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} = 2m, \quad (7.37)$$

we arrive at

$$Q^\alpha = N q^\alpha - \frac{m}{2} c^\alpha, \quad (7.38)$$

which is the desired relation between the macroscopic charge  $Q^\alpha$  and the microscopic charge  $q^\alpha$ . Most importantly, the classical two-derivative relation  $Q^\alpha = q^\alpha$ , which we found in the case of type IIB on K3 in (6.14), obtains a shift proportional to the first Chern class of the base due to the non-trivial topology of the transverse Taub-NUT space.

The relation between the macroscopic and microscopic charges (7.30) can also be derived from making contact with the five-dimensional M-theory description. This can be achieved by doing the reduction along the NUT-circle parametrized by  $\psi$  to five dimensions and using the dictionary derived in [50]. Similar shifts have been noticed in related settings [156–158].

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<sup>51</sup>We followed footnote 6 in [61] and implement the self-duality of the tensors by effectively dividing the source terms in the naive equations of motion derived from (7.34) by a factor two.

### 7.2.3 Classical contributions to central charges and levels

In the following we will compute the classical contributions to the central charges and levels. By ‘classical’ we mean those contributions which can be obtained from the six-dimensional  $\mathcal{N} = (1, 0)$  supergravity action describing our F-theory setup. This is done by extracting coefficients of Chern-Simons terms in three dimensions, which arise upon dimensionally reducing the six-dimensional action.

As already explained in section 6.2 in the computation of levels and central charges of type IIB on K3, we perform the reduction to three dimensions at asymptotic infinity. This amounts in our current F-theory setting to sending the dimensionless quantity

$$r' \equiv \frac{m_\infty r}{m} \rightarrow \infty. \quad (7.39)$$

Concretely, this means that the reduction is done on the spherical part at large  $r'$ . We will see, that the only part of the action, which leads for the reduction in the near horizon and asymptotic geometry to different results for the terms scaling with the charges, is the the higher-derivative part. This is different from previous work [94], where the higher derivative part leads to the same results in the near horizon and asymptotic reductions. In [94] the difference of the asymptotic and near horizon reduction manifested itself at the level of  $\mathcal{O}(1)$  contributions, which we do not consider in the following. This qualitative difference in our setup compared to the existing literature is due to the non-trivial geometry transverse to the string in six dimensions.

The fact that the six-dimensional near horizon geometry does not reproduce the microscopic results for four-dimensional black holes can also be understood from a different perspective. The microscopic derivation in M-theory was performed in the regime, where all volumes of the  $CY_3$  are sufficiently large. This in particular includes the volume of the elliptic fiber. The duality to F-theory then implies that we have to consider backgrounds on a small NUT circle. Therefore, we expect that the solutions (7.21) can only be used to reproduce the microscopic quantities in the limit (7.39).

Furthermore, the reduced six-dimensional effective action can only be matched to the five-dimensional effective action after adding one-loop corrections coming from the compactification circle, as shown in [50, 83]. Therefore there is no classical lift of the five-dimensional black string and four-dimensional black hole solution of M-theory to a six-dimensional F-theory solution. The microscopic central charges and levels will therefore not just follow from a reduction of the six-dimensional supergravity action. However, they do follow when one-loop corrections coming from integrating out massive Kaluza-Klein modes on the

compact space are also taken into account. Calculating these one-loop effects will be the subject of section 7.3.

### Ansatz for the reduction

We now present our ansatz for the metric and three-form field strength in order to perform the reduction in the asymptotic geometry, given as a suitable generalization of the ansatz for the near horizon geometry  $\text{AdS}_3 \times S^3/\mathbb{Z}_m$ . We will do the reduction at an arbitrary radius and compare the asymptotic and near horizon results.

**Near horizon geometry.** The near horizon geometry of the black string solution (7.21) is  $\text{AdS}_3 \times S^3/\mathbb{Z}_m$ . First consider the simplest case where  $m = 1$ . This near horizon geometry has an  $\text{SO}(4)$  isometry group which is identified with rotations on  $S^3$ . Once perturbations of this background are included, the isometries are gauged and one obtains  $\text{SO}(4)$  gauge fields. At the level of the algebra, one has  $\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ , such that we effectively have two sets of  $\mathfrak{su}(2)$  gauge fields. The ansatz for the dimensional reduction on  $\text{AdS}_3 \times S^3$  can be found in e.g. [93, 94]. We will make use of this ansatz in the following and adapt it appropriately to our setting.

For general NUT-charge  $m$  the isometry group  $\text{SO}(4)$  is broken to  $\text{U}(1)_L \times \text{SU}(2)_R$ . The unbroken  $\mathfrak{u}(1)_L \subset \mathfrak{su}(2)_L$  algebra is generated by the generator  $J_L^{(3)}$  of the original  $\mathfrak{su}(2)_L$  and the total algebra is generated by the Killing vectors

$$\begin{aligned} K_L &= \partial_\psi, \\ K_R^1 &= \sin(\phi)\partial_\theta + \cos(\phi)\cot(\theta)\partial_\phi - \frac{\cos(\phi)}{\sin(\theta)}\partial_\psi, \\ K_R^2 &= \cos(\phi)\partial_\theta - \sin(\phi)\cot(\theta)\partial_\phi + \frac{\sin(\phi)}{\sin(\theta)}\partial_\psi, \\ K_R^3 &= -\partial_\phi, \end{aligned} \tag{7.40}$$

which we collectively denote by  $K^i = (K_R^I, K_L)$  and similarly  $A^i = (A_R^I, A_L)$ ,  $F^i = (F_R^I, F_L)$ . The appropriate ansatz is [93, 94]

$$ds_6^2 = ds_{\text{AdS}_3}^2 + \delta_{ab}e^ae^b, \tag{7.41}$$

$$\hat{G}^\alpha = -Q^\alpha \left[ (2\pi)^2 m (e_3^{(m)} - \chi_3) + \text{*dvol}(S^3/\mathbb{Z}_m) \right], \tag{7.42}$$

where

$$e_3^{(m)} = \frac{1}{2\pi^2} \left[ e^1 \wedge e^2 \wedge e^3 - \frac{1}{2} K_{La} e^a \wedge F_L + \frac{1}{2} K_{Ra}^I e^a \wedge F_R^I \right]. \tag{7.43}$$

The dreibein is now given by

$$e^a = \hat{e}^a - K_L^a A_L - K_R^{Ia} A_R^I, \quad (7.44)$$

with  $\hat{e}^a$  the dreibein (7.26) in the near horizon limit  $r \rightarrow 0$ . The three-form  $e_3^{(m)}$  has the same form as  $e_3$ , which is used for a reduction on the three-sphere [93], but since  $0 \leq \psi < \frac{4\pi}{m}$ , the integral of  $e_3^{(m)}$  over the Lens space is given by

$$\int_{S^3/\mathbb{Z}_m} e_3^{(m)} = \frac{1}{m}. \quad (7.45)$$

It is also invariant under  $U(1)_L \times SU(2)_R$  transformations and its exterior derivative is given by

$$de_3^{(m)} = \frac{1}{16\pi^2} F_L \wedge F_L + \frac{1}{8\pi^2} \text{tr} F_R \wedge F_R. \quad (7.46)$$

The three-form  $\chi_3$  in the ansatz (7.42) is defined by

$$\chi_3 = \frac{1}{16\pi^2} A_L \wedge F_L + \frac{1}{8\pi^2} \text{tr} \left( A_R \wedge dA_R + \frac{2}{3} A_R^3 \right), \quad (7.47)$$

and ensures that the ansatz for the tensors satisfies the Bianchi identity.

**Spherical part of Taub-NUT.** Consider now a reduction on the spherical part of the metric (7.21)  $TN_m^{\text{sph}}$  parametrized by  $\psi, \phi, \theta$ . The Killing vectors of Taub-NUT spacetime are still given by (7.40) and form  $U(1)_L \times SU(2)_R$ . This implies that the ansatz of the previous section for the three-forms is still suitable. The metric of course needs to be adapted and can be taken as (7.41), but now with the vielbein  $\hat{e}^a$  of the spherical part of Taub-NUT space (7.26). The ansatz is thus a straightforward generalization of the one in the near horizon geometry (7.41)

$$\begin{aligned} d\hat{s}_6^2 &= ds_{\mathcal{M}_3}^2 + \delta_{ab} e^a e^b, \\ \hat{G}^\alpha &= -Q^\alpha \left[ (2\pi)^2 m (e_3^{(m)} - \chi_3) + \text{dvol}(\mathcal{M}_3) \right], \end{aligned} \quad (7.48)$$

with the difference, that we do not take the near horizon limit  $r \rightarrow 0$  now. The total metric is therefore  $TN_m^{\text{sph}}$  fibered over the non-spherical part of the metric, denoted by  $\mathcal{M}_3$ . We will in the following use this ansatz to calculate the classical parts of the levels and central charges.



### Classical contribution from two- and higher-derivative action

The classical contributions stem from the six-dimensional supergravity action. Besides the leading two-derivative action, also a four-derivative coupling in six dimensions will be of importance to us. We will perform the reduction of the two- and four-derivative action separately, and read their contributions to the levels and central charges from coefficients of three-dimensional Chern-Simons terms off.

**Two-derivative contribution.** We calculate the contribution of the two-derivative action to the levels by determining the gauge variation of the reduced action under a  $U(1)_L \times SU(2)_R$  gauge transformation. We will do this by integrating the variation of the six-dimensional Lagrangian over the spherical part  $TN_m^{\text{sph}}$  to obtain the lower dimensional variation. Since  $e_3^{(m)}$  is gauge invariant by construction, the only source for a variation under a combined  $U(1)_L \times SU(2)_R$  gauge transformation, which is parametrized by  $\Lambda$ , is  $\chi_3$ . We therefore obtain<sup>52</sup>

$$\begin{aligned} \delta_\Lambda \mathcal{L}_{\text{CS}} \star_3 1 &= -\frac{1}{16\pi^3} \int_{TN_m^{\text{sph}}} g_{\alpha\beta} \delta_\Lambda \hat{G}^\alpha \wedge \hat{*} \hat{G}^\beta = \pi m^2 \eta_{\alpha\beta} Q^\alpha Q^\beta \int_{TN_m^{\text{sph}}} \delta_\Lambda \chi_3 \wedge e_3^{(m)} \\ &= \pi m \eta_{\alpha\beta} Q^\alpha Q^\beta \delta_\Lambda \chi_3, \end{aligned} \quad (7.49)$$

where in the second equality we used the (anti-)self-duality condition (7.20). The three dimensional variation (7.49) is nothing but the gauge variation of a three-dimensional action of the form

$$\begin{aligned} S_{\text{CS}} &= \pi m \eta_{\alpha\beta} Q^\alpha Q^\beta \int_{\mathcal{M}_3} \chi_3 \\ &= \frac{k_L^{\text{class}}}{8\pi} \int_{\mathcal{M}_3} A_L \wedge F_L + \frac{k_R^{\text{class}}}{4\pi} \int_{\mathcal{M}_3} \text{tr} \left( A_R \wedge F_R + \frac{2}{3} A_R^3 \right), \end{aligned} \quad (7.50)$$

with levels and central charges

$$\begin{aligned} k_L^{2\text{-der}} &= \frac{1}{2} m \eta_{\alpha\beta} Q^\alpha Q^\beta = \frac{1}{2} m \eta_{\alpha\beta} \left( q^\alpha - \frac{1}{2} m c^\alpha \right) \left( q^\beta - \frac{1}{2} m c^\beta \right), \\ c_L^{2\text{-der}} &= c_R^{2\text{-der}} = 6 k_R^{2\text{-der}} = 3 m \eta_{\alpha\beta} \left( q^\alpha - \frac{1}{2} m c^\alpha \right) \left( q^\beta - \frac{1}{2} m c^\beta \right). \end{aligned} \quad (7.51)$$

**Higher-derivative contribution.** In order to find the contribution to the levels and central charges stemming from higher-derivative terms we consider the

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<sup>52</sup>In our conventions  $\int_{\mathcal{M}_6} = \int_{\mathcal{M}_3} \cdot \int_{TN_m^{\text{sph}}}$ .

piece in the six-dimensional action

$$S^{(6)} \supset \frac{1}{64\pi^3} \int_{M_6} \eta_{\alpha\beta} c^\alpha \hat{G}^\beta \wedge \hat{\omega}_{\text{grav}}^{\text{CS}}, \quad (7.52)$$

where  $\hat{\omega}_{\text{grav}}^{\text{CS}}$  is the gravitational Chern-Simons three-form built of the six-dimensional spin connection. We will compute Chern-Simons terms in three dimensions by integrating (7.52) over the spherical part  $\text{TN}_m^{\text{sph}}$  for general  $r$ , in particular not taking the near horizon limit. One finds

$$\begin{aligned} \mathcal{L}_{\text{h.d.}}^{\text{CS}} \star_3 1 &= \frac{1}{64\pi^3} \int_{\text{TN}_m^{\text{sph}}} \eta_{\alpha\beta} c^\alpha \hat{G}^\beta \wedge \hat{\omega}_{\text{grav}}^{\text{CS}} \\ &= \frac{1}{16\pi} \eta_{\alpha\beta} c^\alpha \left( q^\beta - \frac{1}{2} m c^\beta \right) \left[ \omega_{\text{grav}}^{\text{CS}} - \frac{1 + 4r' + 2r'^2}{(1 + r')^4} A_L \wedge F_L \right. \\ &\quad \left. + 2 \frac{1 + 4r' + 10r'^2 + 8r'^3 + 2r'^4}{(1 + r')^4} \omega^{\text{CS}}(A_R) \right], \end{aligned} \quad (7.53)$$

where we used  $r' = \frac{m_\infty}{m} r$ . In the second equality in (7.53) we only collected terms in  $\hat{G}^\beta$  and  $\hat{\omega}_{\text{grav}}^{\text{CS}}$  that lead to Chern-Simons terms in three-dimensions. With the choice of dreibein (7.26) one obtains proper Chern-Simons terms after the reduction. Details of this calculation can be found in appendix 7.A.

Now there are two limits in (7.53) which are of interest to us: the near horizon limit  $r' \rightarrow 0$ , where we effectively go to  $\text{AdS}_3 \times S^3/\mathbb{Z}_m$  and the  $r' \rightarrow \infty$  limit corresponding to performing the ‘reduction at infinity’. In the near horizon limit  $r' \rightarrow 0$  we obtain

$$\mathcal{L}_{\text{h.d.}}^{\text{CS}} \star_3 1 = \frac{1}{16\pi} \eta_{\alpha\beta} c^\alpha \left( q^\beta - \frac{1}{2} m c^\beta \right) \left[ \omega_{\text{grav}}^{\text{CS}} - A_L \wedge F_L + 2 \omega^{\text{CS}}(A_R) \right] \quad (7.54)$$

from which we read off the following contributions to the central charges and levels

$$\begin{aligned} (c_L - c_R)^{4\text{-der}} &= 6 \eta_{\alpha\beta} c^\alpha \left( q^\beta - \frac{1}{2} m c^\beta \right), \\ k_R^{4\text{-der}} &= \frac{1}{2} \eta_{\alpha\beta} c^\alpha \left( q^\beta - \frac{1}{2} m c^\beta \right), \\ k_L^{4\text{-der}} &= -\frac{1}{2} \eta_{\alpha\beta} c^\alpha \left( q^\beta - \frac{1}{2} m c^\beta \right), \end{aligned} \quad (7.55)$$

where we used the fact that the coefficient of the three-dimensional gravitational Chern-Simons term determines the difference between the left- and right-moving central charges. The latter difference can be read off from the gravitational

Chern-Simons term by comparing it to

$$\mathcal{L}_{\text{h.d.}}^{\text{CS}} \star_3 1 \supset \frac{c_L - c_R}{96\pi} \omega_{\text{grav}}^{\text{CS}}. \quad (7.56)$$

Setting  $m = 1$  and dropping the charge shift, this is the result obtained in [94, 15]. The shift in the charges is absent in these settings, which involve black holes in asymptotically flat spacetime, as opposed to our case. We therefore recover prefactors which are in agreement with their results.

The near horizon results (7.55) turn out not to give the correct classical higher derivative correction to the central charges and levels. In contrast, taking the limit  $r' \rightarrow \infty$  in (7.53) one finds

$$\mathcal{L}_{\text{h.d.}}^{\text{CS}} \star_3 1 = \frac{1}{16\pi} \eta_{\alpha\beta} c^\alpha \left( q^\beta - \frac{1}{2} m c^\beta \right) \left[ \omega_{\text{grav}}^{\text{CS}} + 4\omega^{\text{CS}}(A_R) \right], \quad (7.57)$$

such that we obtain

$$\begin{aligned} (c_L - c_R)^{4\text{-der}} &= 6 \eta_{\alpha\beta} c^\alpha \left( q^\beta - \frac{1}{2} m c^\beta \right), \\ k_R^{4\text{-der}} &= \eta_{\alpha\beta} c^\alpha \left( q^\beta - \frac{1}{2} m c^\beta \right), \\ k_L^{4\text{-der}} &= 0. \end{aligned} \quad (7.58)$$

The total classical contributions from the reduction in the asymptotic geometry are therefore given by

$$\begin{aligned} c_L^{\text{class}} &= 3mC^2 - 3m^2 c_1(B) \cdot C + \frac{3}{4} m^3 c_1(B)^2 + 12c_1(B) \cdot C - 6mc_1(B)^2, \\ c_R^{\text{class}} &= 6k_R^{\text{class}} = 3mC^2 - 3m^2 c_1(B) \cdot C + \frac{3}{4} m^3 c_1(B)^2 + 6c_1(B) \cdot C - 3mc_1(B)^2, \\ k_L^{\text{class}} &= \frac{1}{2} mC^2 - \frac{1}{2} m^2 c_1(B) \cdot C + \frac{1}{8} m^3 c_1(B)^2. \end{aligned} \quad (7.59)$$

This is obviously not the full answer, as it does not match the microscopic results (7.3) and (7.10). The mismatch is not surprising because we know that in order to match the six- and five-dimensional effective actions one has to add one-loop corrections to the dimensionally reduced six-dimensional action [50, 83]. The results (7.59) are actually equal to the central charges and levels, which one would find from the five-dimensional action before adding these one-loop corrections. To reproduce the microscopic results one also has to include the one-loop Chern-Simons terms that arise from integrating out the massive Kaluza-Klein modes. This is what we will do in the next section. Adding the classical asymptotic contributions derived in this section to the one-loop induced contributions will lead to a matching of microscopic and macroscopic quantities up to linear order in the charges  $(q^\alpha, m)$ .

### 7.3 Macroscopics in F-theory from 6d: quantum contributions

We now wish to include one-loop Chern-Simons terms in three dimensions and interpret them as additional contributions to the central charges and levels. These loop-induced Chern-Simons terms arise from integrating out massive Kaluza-Klein modes, which run in the loops of the relevant two-point functions. Since Chern-Simons terms are intimately linked to anomalies in higher dimensions, we anticipate that the relevant three-dimensional fields to be integrated out are KK modes of chiral fields in six dimensions, which can contribute to anomalies. These fields include the six-dimensional gravitino, spin- $\frac{1}{2}$  fermions in the tensor- and hypermultiplets, and the (anti-)self-dual two-forms. Upon reduction to three dimensions these fields lead to massive spin- $\frac{3}{2}$ , spin- $\frac{1}{2}$ , and three-dimensional chiral vector fields.

We calculate the loop-induced Chern-Simons terms in the near horizon geometry, but argue that the result is still valid for a reduction at asymptotic infinity. To do the calculation, we first determine the relevant KK-spectrum by truncating the KK-spectrum found in [123, 124] and used in section 6.4.1 for the case of  $\mathcal{N} = (2, 0)$  supergravity on  $AdS_3 \times S^3$  to the corresponding  $\mathcal{N} = (1, 0)$  spectrum. Besides the local Lorentz group representations of the massive fields in three dimensions, we also extract the representations of the fields under the (gauged)  $\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$  isometry of  $S^3$ , as well as the signs of the three-dimensional masses. We then use the contribution of a single field of each type to the three-dimensional  $\mathfrak{u}(1)_L$ ,  $\mathfrak{su}(2)_R$  and gravitational Chern-Simons terms. These single field contributions were computed in section 6.4.2 using the Atiyah-Patodi-Singer (APS) index theorem [159–161]. Armed with these results we then sum the contributions over all KK-towers and determine the total contribution employing Zeta-function regularization. In particular, we implement the  $\mathbb{Z}_m$  quotient in the sum over KK states. Adding these quantum corrections to the classical ones obtained in section 7.2, we find agreement with the microscopic results up to and including terms linear in the charges.

#### 7.3.1 Kaluza-Klein spectrum

We now determine the  $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$  representations of the massive spin- $\frac{1}{2}$ , spin- $\frac{3}{2}$  and two-form Kaluza-Klein modes before taking the  $\mathbb{Z}_m$  quotient. The six-dimensional fields that give rise to relevant Kaluza-Klein modes are the gravitino and self-dual two-form in the gravity multiplet, the tensorinos and anti-self-dual

tensors in the tensor multiplets, and the hyperinos in the hypermultiplets. The gravitino, tensorinos and hyperinos are all given by two Weyl fermions subject to a symplectic-Majorana condition. The tensors obey a reality condition. The  $\mathcal{N} = (1, 0)$  theory coupled to tensor multiplets can be obtained as a truncation of the  $\mathcal{N} = (2, 0)$  theory. The spectrum of  $\mathcal{N} = (2, 0)$  supergravity on  $S^3$  was worked out in [123, 124]. The extra content we have are the hypermultiplets, but for now we assume that the modes associated to the fermions in these multiplets fall in the same representations as the fermions in the tensor multiplets. Since the spectrum analysis of the hyperinos involves, just like for the tensorinos, an analysis of the linearized Dirac equation, and the tensorinos and hyperinos also have the same chirality, this is a fairly mild assumption. We now list the massive modes that are obtained without taking into account the symplectic-Majorana and reality conditions, and denote the spectrum in terms of  $\mathfrak{so}(4) = \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$  representations  $(j_L, j_R)^{\text{sgn}(M)}$  [123, 124]<sup>53</sup>, where  $\text{sgn}(M)$  denotes the sign of the mass.

- Spin- $\frac{3}{2}$ :

$$2 \bigoplus_{j_L=\frac{1}{2}}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\mp}.$$

- Spin- $\frac{1}{2}$ :

$$\begin{aligned} & 2 \bigoplus_{j_L=\frac{3}{2}}^{\infty} (j_L, j_L \pm \frac{3}{2})^{\mp} \oplus 2 \bigoplus_{j_L=0}^1 (j_L, j_L + \frac{3}{2})^{-} \oplus 2 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \\ & \oplus 2(\frac{1}{2}, 1)^{+} \oplus 2(0, \frac{1}{2})^{+} \oplus 2(n_T + n_H) \bigoplus_{j_L=\frac{1}{2}}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \\ & \oplus 2(n_T + n_H) (0, \frac{1}{2})^{+}. \end{aligned} \quad (7.60)$$

- Chiral vectors:

$$\begin{aligned} & \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm 1)^{\mp} \oplus (\frac{1}{2}, \frac{3}{2})^{-} \oplus (0, 1)^{-} \oplus n_T \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm 1)^{\pm} \\ & \oplus n_T (\frac{1}{2}, \frac{3}{2})^{+} \oplus n_T (0, 1)^{+}. \end{aligned}$$

<sup>53</sup>In [123] the spectrum is denoted in terms of the highest weight vector  $(l_1, l_2)$  of  $\mathfrak{so}(4)$  which is related to our notation by  $l_1 = j_L + j_R$ ,  $l_2 = j_L - j_R$ .

We remind the reader of the notation introduced in (6.25) which we use to characterize the KK spectrum.

When performing the sums in the loop computations, we still need to project the spectrum on  $S^3$  in (7.60) on its  $\mathbb{Z}_m$ -invariant sub-sector to obtain the spectrum on  $S^3/\mathbb{Z}_m$ . We can again draw from the results summarized in appendix C and the discussion under (6.24) and conclude, that we only need to consider states in the spectrum (7.60), which satisfy  $j_L^{(3)} = \frac{1}{2}mk$  for some  $k \in \mathbb{Z}_{\geq 0}$ . Here  $j_L^{(3)}$  is again the eigenvalue of a state under the Cartan generator of  $\mathfrak{su}(2)_L$ .

If one performs the reduction in the asymptotic geometry one reduces on a squashed Lens space, where the radius of the two-sphere inside the squashed three-dimensional geometry is taken to be large. We expect that the representation content of the KK spectrum does not get altered by the squashing. Note that, due to the asymptotic NUT circle, the masses of the Kaluza-Klein modes remain finite. In addition we assume, that the squashing of the  $S^3/\mathbb{Z}_m$  does not change the sign of the mass of the KK states. These assumptions essentially imply, that we can do the loop computation in the near horizon geometry and use the spectrum on  $S^3/\mathbb{Z}_m$ .

### 7.3.2 One-loop Chern-Simons terms from KK spectrum

We are now almost in a position to compute the one loop corrections to the individual Chern-Simons levels by summing over the whole Kaluza-Klein spectrum. The contribution of a single massive spin- $\frac{3}{2}$ , spin- $\frac{1}{2}$  and chiral vector in three dimensions to the Chern-Simons levels  $k_L, k_R$  and the gravitational Chern-Simons term was already computed in section 6.4.2 when type IIB string theory on a K3 manifold was considered. The results obtained there remain valid, but we nevertheless quote the result again for the ease of the reader in table 7.1.

	spin- $\frac{1}{2}$	spin- $\frac{3}{2}$	chiral vectors
$\alpha_L$	$\frac{1}{2}(j_L^{(3)})^2 \dim \mathbf{R}_{j_R}$	$\frac{3}{2}(j_L^{(3)})^2 \dim \mathbf{R}_{j_R}$	$-(j_L^{(3)})^2 \dim \mathbf{R}_{j_R}$
$\alpha_R$	$-\frac{j_R}{3}(j_R + 1) \dim \mathbf{R}_{j_R}$	$-j_R(j_R + 1) \dim \mathbf{R}_{j_R}$	$\frac{2j_R}{3}(j_R + 1) \dim \mathbf{R}_{j_R}$
$\alpha_{\text{grav}}$	$\frac{1}{48} \dim \mathbf{R}_{j_R}$	$-\frac{7}{16} \dim \mathbf{R}_{j_R}$	$\frac{1}{12} \dim \mathbf{R}_{j_R}$

Table 7.1: Contributions of a single field to the left-, right- and gravitational Chern-Simons terms. The table should be read as  $\alpha_I = \frac{\text{sgn}(M)}{4\pi} \times (\text{entry of table})$ .

The relations between the coefficients  $\alpha_I$  and the levels and central charges is

given by

$$k_L = 8\pi\alpha_L, \quad k_R = 4\pi\alpha_R \quad c_L - c_R = 96\pi\alpha_{\text{grav}}. \quad (7.61)$$

We also remind the reader that  $\dim \mathbf{R}_{j_R} = 2j_R + 1$  is the dimension of the spin- $j_R$  representation of  $\text{SU}(2)_R$ , and  $j_L^{(3)}$  is the charge under  $\text{U}(1)_L \subset \text{SU}(2)_L$ . We now have all ingredients to perform the computation of the one loop corrections to the relevant Chern-Simons terms. We do this, by summing the corresponding one loop corrections given in table 7.1 over the spectrum (7.60). The implementation of the projection on  $\mathbb{Z}_m$ -invariant states, i.e. states satisfying  $j_L^{(3)} = \frac{1}{2}mk$  for some  $k \in \mathbb{Z}$ , is furthermore included in the summation. The strategy how to perform the summation explicitly is the same as described above (6.46), and we again use the values of the sums (6.44). When the one-loop corrections are computed, we also make use of the identities

$$\begin{aligned} n_H &= 273 - 29n_T, \\ n_T &= h^{1,1}(B) - 1 = 9 - c_1(B)^2, \end{aligned} \quad (7.62)$$

in order to express the final result entirely in terms of the charge  $m$  and  $c_1(B)$ . The first of these identities is the condition which ensures the cancellation of the gravitational anomaly in 6d. The detailed evaluation of the sums, which enter the computation of the one-loop Chern-Simons levels, is performed in appendix 7.B. The final results are given by

$$\begin{aligned} \Delta k_L^{\text{loop}} &= -\frac{m^3}{8}c_1(B)^2, \\ \Delta k_R^{\text{loop}} &= \frac{m^3}{24}c_1(B)^2 + \frac{m}{3}c_1(B)^2 + m, \\ \Delta(c_L - c_R)^{\text{loop}} &= 6m + 2m c_1(B)^2, \end{aligned} \quad (7.63)$$

up to terms of  $\mathcal{O}(1)$  which are independent of the charges  $(q^\alpha, m)$ . The contributions (7.63) added to the classical supergravity contribution (7.59) are the main results of this chapter.

**Summary of 4d black holes from F-theory.** In this part of the chapter, we computed the microscopic and macroscopic central charges and current algebra levels of a family of 2d  $\mathcal{N} = (0, 4)$  SCFT, which govern the entropy of a four-dimensional black hole. Microscopically, this black hole is realized as a D3-brane wrapping a curve  $C$  in the base  $B$  of an elliptically fibered Calabi–Yau threefold

$CY_3$ . In addition, the worldvolume of the string wraps an additional  $S^1$  and probes a Taub-NUT space with NUT charge  $m$  in the transverse space. This setup is dual to an M5-brane wrapping the surface  $\hat{C} + mB$  in the Calabi–Yau threefold. The dual M-theory setting allowed us to determine the microscopic values of the central charges and levels using the famous results of MSW, see (7.3) and (7.10).

We found that the microscopic results can only be recovered from a macroscopic computation, if one-loop Chern-Simons terms, whose coefficients determine the central charges and levels, are included. These one-loop Chern-Simons terms are induced by massive Kaluza-Klein modes in the Kaluza-Klein reduction of 6d  $\mathcal{N} = (1, 0)$  supergravity to three dimensions. Adding the classical supergravity result (7.59) and the one-loop contribution (7.63) up we find from the macroscopic side

$$\begin{aligned} c_L &= 3mC^2 - 3m^2c_1(B) \cdot C + m^3c_1(B)^2 + 12c_1(B) \cdot C + 12m - 2mc_1(B)^2, \\ c_R &= 6k_R = 3mC^2 - 3m^2c_1(B) \cdot C + m^3c_1(B)^2 + 6c_1(B) \cdot C + 6m - mc_1(B)^2, \\ k_L &= \frac{1}{2}mC^2 - \frac{1}{2}m^2c_1(B) \cdot C, \end{aligned} \tag{7.64}$$

for the central charges, again up to  $\mathcal{O}(1)$  contributions independent of the charges. We therefore reproduced the microscopic results (7.3) up to terms of  $\mathcal{O}(1)$ . Curiously, we reproduced the microscopic result for the left level (7.10) *including* the  $\mathcal{O}(1)$  contributions, once the one-loop Chern-Simons terms are added to the classical supergravity result.

## 7.4 ADE black holes from F-theory

In the rest of this chapter we will collect some yet to be published results. In the current chapter about black holes from wrapped D3-branes in F-theory compactifications on  $CY_3$  we focused entirely on the case, where the string propagating in 6d probes a Taub-NUT space with NUT charge  $m$ . However, as we have seen in chapter 6 when we studied D3-branes wrapped on a Riemann surface in a K3 manifold, we can consider a broader variety of transverse spaces. In chapter 6 we considered ALF spaces, which have a classification in terms of extended Dynkin diagrams of the A- and D-series, as well as ALE spaces, which



are classified in terms of extended ADE Dynkin diagrams.<sup>54</sup> In this section we will extend the discussion of the F-theory setting and cover the remaining ALE and ALF cases from a macroscopic perspective. However, as opposed to the case of D3-branes in the K3 compactification of type IIB in chapter 6, we are not able to give a corresponding microscopic computation of the levels and central charges. We will briefly comment on the conceptual difficulties one encounters in the microscopic settings at the end of this section. These challenges on the field theory side highlight that our macroscopic computations can be the method of choice to obtain information about the field theories. In the following, we will denote the four-manifold transverse to the string with  $M_\Gamma$ . In order to obtain the desired results for the central charges and levels for the remaining cases, we have to carry out the same steps as in section 7.2 and section 7.3, adapted to the case of general ALE and ALF spaces.

We first note that when we derived the shift in the dictionary between microscopic and macroscopic charges (7.38), we made use of the first Pontryagin number of  $TN_m$  (7.37). This charge shift was crucial to reproduce the microscopic result. The derivation of the charge shift for the case of  $M_\Gamma$  a general ALE or ALF space is completely analogous to the case of  $TN_m$ . But instead of using the result for the first Pontryagin number of  $TN_m$  we now obtain in general

$$Q^\alpha = q^\alpha - \frac{1}{4}p_1(M_\Gamma)c^\alpha \quad (7.65)$$

for the relation between macro- and microscopic charges. The Pontryagin numbers of ALE and ALF spaces were computed in [162] and are summarized for all relevant discrete subgroups  $\Gamma \subset \text{SU}(2)$  in table 7.2. We will now proceed and calculate the classical supergravity and quantum contributions separately.

**Classical supergravity contribution.** In the classical supergravity part we will make use of the reduction ansatz (7.48) and adapt it appropriately for our purposes. We will make use of the fact that the ALE and ALF spaces can be asymptotically obtained by a quotient of a covering space  $M$  with respect to  $\Gamma$ . The integrations over the ALE and ALF spaces are then defined as in (6.13), i.e. by dividing the integral performed over the covering space  $M$  by  $|\Gamma|$ . The reduction ansatz for the ALE and ALF series takes in the covering space the form

$$d\hat{s}_6^2 = ds_{\mathcal{M}_3}^2 + \delta_{ab}e^ae^b, \quad (7.66)$$

---

<sup>54</sup>Recall that the Taub-NUT space with NUT charge  $m$  is the ALF space corresponding to  $A_{m-1}$ . The ALE spaces are resolutions of the orbifold singularities  $\mathbb{C}^2/\Gamma$ , where  $\Gamma \subset \text{SU}(2)$  are the ADE-classified freely acting, discrete subgroups of  $\text{SU}(2)$ .

$\Gamma \subset \text{SU}(2)$	$ \Gamma $	$p_1(\text{ALE})$	$p_1(\text{ALF})$
cyclic group $\mathbb{Z}_m$	$m$	$2m - \frac{2}{m}$	$2m$
binary dihedral $\mathbb{D}_m^*$	$4m$	$2m + 6 - \frac{1}{2m}$	$2m + 6$
binary tetrahedral $\mathbb{T}^*$	24	$\frac{167}{12}$	
binary octahedral $\mathbb{O}^*$	48	$\frac{383}{24}$	
binary icosahedral $\mathbb{I}^*$	120	$\frac{1079}{60}$	

Table 7.2: The first Pontryagin numbers for ALE and ALF manifolds labelled by their corresponding discrete group  $\Gamma$ .

$$\hat{G}^\alpha = -Q^\alpha [(2\pi)^2 |\Gamma| (e_3 - \chi_3) + \text{dvol}(\mathcal{M}_3)] ,$$

where the dreibein  $e^a$  is defined by

$$\text{for ALF:} \quad e^a = \begin{cases} \hat{e}_{\text{ALF}}^a - K_L^a A_L - K^{Ia} A_R^I, & \text{for } \Gamma = \mathbb{Z}_m, \\ \hat{e}_{\text{ALF}}^a - K^{Ia} A_R^I, & \text{for } \Gamma \neq \mathbb{Z}_m \end{cases}, \quad (7.67)$$

$$\text{for ALE:} \quad e^a = \begin{cases} \hat{e}_{\text{ALE}}^a - K_L^a A_L - K^{Ia} A_R^I, & \text{for } \Gamma = \mathbb{Z}_m, \\ \hat{e}_{\text{ALE}}^a - K^{Ia} A_R^I, & \text{for } \Gamma \neq \mathbb{Z}_m \end{cases}. \quad (7.68)$$

Again,  $K_{L,R}$  are the  $\text{U}(1)_L \times \text{SU}(2)_R$  Killing vectors (7.40), and  $A_{L,R}$  are the Kaluza-Klein gauge fields which gauge these isometries. The dreibeine  $\hat{e}_{\text{ALF,ALE}}^a$  in (7.67) and (7.68) are defined in terms of  $\hat{e}^a$  in (7.26) by

$$e_{\text{ALF}}^a = \hat{e}^a|_{m=1}, \quad e_{\text{ALE}}^a = \hat{e}^a|_{m=1, m_\infty=0}, \quad (7.69)$$

and  $\chi_3$  is defined in (6.19). Furthermore, the three-form  $e_3$  satisfies  $\text{de}_3 = \chi_3$  and integrates to unity over the spherical part of  $M$ , which is denoted by  $M^{\text{sph}}$ , i.e.

$$\int_{M^{\text{sph}}} e_3 = 1. \quad (7.70)$$

With this reduction ansatz it is straightforward to compute the gauge variation of the 3d Lagrangian induced by a gauge transformation on the gauge fields  $A_{L,R}$ . We find

$$\begin{aligned} \delta_\Lambda \mathcal{L}_{\text{CS}} \star 1 &= \frac{1}{(2\pi)^3} \frac{1}{|\Gamma|} \int_{M^{\text{sph}}} -\frac{1}{2} g_{\alpha\beta} \delta_\Lambda \hat{G}^\alpha \wedge \hat{*} \hat{G}^\beta = \frac{1}{(2\pi)^3} \frac{1}{|\Gamma|} \int_{M^{\text{sph}}} -\frac{1}{2} \eta_{\alpha\beta} \delta_\Lambda \hat{G}^\alpha \wedge \hat{G}^\beta \\ &= (2\pi) \frac{1}{2} |\Gamma| \eta_{\alpha\beta} Q^\alpha Q^\beta \delta_\Lambda \chi_3, \end{aligned} \quad (7.71)$$

which is the gauge variation of a Chern-Simons action

$$\begin{aligned} S_{\text{CS}} &= (2\pi) \frac{1}{2} |\Gamma| \eta_{\alpha\beta} Q^\alpha Q^\beta \int_{\mathcal{M}_3} \chi_3 \\ &= \frac{k_L}{8\pi} \int_{\mathcal{M}_3} A_L \wedge F_L + \frac{k_R}{4\pi} \int_{\mathcal{M}_3} \text{tr} \left( A_R \wedge F_R + \frac{2}{3} A_R^3 \right). \end{aligned} \quad (7.72)$$

We therefore find from the two-derivative action for both ALE and ALF

$$\begin{aligned} k_L^{2\text{-der}} &= \frac{1}{2} |\Gamma| \eta_{\alpha\beta} Q^\alpha Q^\beta = \frac{1}{2} |\Gamma| \left( C - \frac{1}{4} p_1(M_\Gamma) c_1(B) \right)^2 \quad \text{for } \Gamma = \mathbb{Z}_m, \\ k_R^{2\text{-der}} &= \frac{1}{2} |\Gamma| \eta_{\alpha\beta} Q^\alpha Q^\beta = \frac{1}{2} |\Gamma| \left( C - \frac{1}{4} p_1(M_\Gamma) c_1(B) \right)^2, \\ (c_L - c_R)^{2\text{-der}} &= 0. \end{aligned} \quad (7.73)$$

The relation  $(c_L - c_R)^{2\text{-der}} = 0$  follows from the absence of a gravitational Chern-Simons term in (7.72).

In the computation of the contributions from the higher-derivative coupling in (7.18) we can draw from results collected in appendix 7.A due to the covering space construction which we utilize. We use that the reduction of the higher-derivative term for a discrete group  $\Gamma$  is given by

$$\mathcal{L}_{\text{h.d.}}^{\text{CS}} \star_3 1 = \frac{1}{8(2\pi)^3} \frac{1}{|\Gamma|} \int_{M^{\text{sph}}} \eta_{\alpha\beta} c^\alpha \hat{G}^\beta \wedge \hat{\omega}_{\text{grav}}^{\text{CS}}, \quad (7.74)$$

where now  $\hat{\omega}_{\text{grav}}^{\text{CS}}$  is for the ALF case computed from (7.67), and for the ALE case from (7.68). Using the general result of the integral provided in (7.127) we find

$$\text{for ALF:} \quad \mathcal{L}_{\text{h.d.}}^{\text{CS}} \star_3 1 = \frac{1}{16\pi} \eta_{\alpha\beta} Q^\alpha c^\beta \left[ \omega_{\text{grav}}^{\text{CS}} + 4\omega^{\text{CS}}(A_R) \right], \quad (7.75)$$

which leads us to the corrections to the central charges and levels for ALF spaces from higher-derivatives

$$\begin{aligned} k_L^{4\text{-der}} &= 0 \\ k_R^{4\text{-der}} &= \eta_{\alpha\beta} Q^\alpha c^\beta, \\ (c_L - c_R)^{4\text{-der}} &= 6\eta_{\alpha\beta} Q^\alpha c^\beta. \end{aligned} \quad \text{for } \Gamma = \mathbb{Z}_m, \quad (7.76)$$

Likewise, we can use the expression (7.127) to obtain the contributions of the higher-derivative term to the Chern-Simons terms for the case of a transverse ALE space.<sup>55</sup> We obtain

$$\text{for ALE:} \quad \mathcal{L}_{\text{h.d.}}^{\text{CS}} \star_3 1 = \frac{1}{16\pi} \eta_{\alpha\beta} Q^\alpha c^\beta \left[ \omega_{\text{grav}}^{\text{CS}} - A_L \wedge F_L + 2\omega^{\text{CS}}(A_R) \right], \quad (7.77)$$

<sup>55</sup>Recall, that for  $M_\Gamma$  an ALE space we now also have to set  $m_\infty = 0$  in (7.127).

where it is understood that  $A_L = 0$  for  $\Gamma \neq \mathbb{Z}_m$ . We therefore find for the ALE case the subleading contributions

$$\begin{aligned} k_L^{4\text{-der}} &= -\frac{1}{2}\eta_{\alpha\beta}Q^\alpha c^\alpha && \text{for } \Gamma = \mathbb{Z}_m, \\ k_R^{4\text{-der}} &= +\frac{1}{2}\eta_{\alpha\beta}Q^\alpha c^\beta, \\ (c_L - c_R)^{4\text{-der}} &= 6\eta_{\alpha\beta}Q^\alpha c^\beta. \end{aligned} \quad (7.78)$$

Summing up the two-derivative and four-derivative contributions, we obtain the results for the levels and  $c_L - c_R$  from the reduction of the supergravity action. In full analogy to type IIB on K3 in section 6.4.2, we still have to compute the one-loop corrections for the case of transverse ALF spaces.

**Quantum contribution.** Since the one-loop contributions to the levels and  $c_L - c_R$  in the case of a transverse Taub-NUT space with NUT charge  $m$  were already computed in detail in section 7.3 and appendix 7.B, it only remains to compute the one loop corrections for the case of the ALF spaces corresponding to the D-series. We noted above (6.48) that there is an efficient way how to obtain the loop corrections for the D-series from the expressions for the A-series. Since the group  $\mathbb{D}_m^*$  has a  $\mathbb{Z}_{2m}$  subgroup, we first take the result for the A-series (i.e. the group  $\mathbb{Z}_m$ ) and replace  $m \rightarrow 2m$ . Since for the D-series there is an additional generator in (C.5), under which the spectrum must be invariant, we also have to divide the result obtained after replacing  $m \rightarrow 2m$  by a factor two. This simplified computation gives the correct result, up to terms which are of  $\mathcal{O}(1)$ . Applying this procedure to the expressions (7.63), which we obtained for the case of a transverse space  $TN_m$ , we find

$$k_R^{\text{loop}} = \frac{m^3}{6}c_1(B)^2 + \frac{m}{3}c_1(B)^2 + m, \quad (c_L - c_R)^{\text{loop}} = 6m + 2mc_1(B)^2 \quad (7.79)$$

for the one-loop contributions in the case of the D-series. All results are again given up to  $\mathcal{O}(1)$  contributions.

We are now in a position to put all the results computed in this section together. For transverse ALE spaces labelled by a discrete group  $\Gamma$  we have to add the corresponding two-derivative results (7.73) to the four-derivative contributions

(7.78) from which we find

$$\begin{aligned}
 k_R &= \frac{1}{2}|\Gamma|C^2 - \frac{1}{2}\left(\frac{1}{2}|\Gamma|p_1(M_\Gamma) - 1\right)c_1(B) \cdot C + \frac{1}{8}\left(\frac{1}{4}|\Gamma|p_1(M_\Gamma) - 1\right)p_1(M_\Gamma)c_1(B)^2, \\
 k_L &= \frac{1}{2}|\Gamma|C^2 - \frac{1}{2}\left(1 + \frac{1}{2}|\Gamma|p_1(M_\Gamma)\right)c_1(B) \cdot C + \frac{1}{8}\left(1 + \frac{1}{4}|\Gamma|p_1(M_\Gamma)\right)p_1(M_\Gamma)c_1(B)^2, \\
 c_L &= c_R + 6c_1(B) \cdot C - \frac{3}{2}p_1(M_\Gamma)c_1(B)^2,
 \end{aligned}
 \tag{7.80}$$

where it is again understood, that the left-moving current algebra with level  $k_L$  is only present for  $\Gamma = \mathbb{Z}_m$ . For the A-series ALF spaces (i.e.  $M_\Gamma = \text{TN}_m$ ) the results were already presented in (7.64). For the ALF space corresponding to the group  $\Gamma = \mathbb{D}_m^*$  we have to add the two-derivative result (7.73), the four-derivative result (7.76) and the loop correction (7.79) up. We furthermore use the information in table 7.2, such that we find the final result

$$\begin{aligned}
 c_R &= 12mC^2 - 6(2m^2 + 6m - 1)c_1(B) \cdot C + (4m^3 + 18m^2 + 26m)c_1(B)^2 + 6m, \\
 c_L &= 12mC^2 - 6(2m^2 + 6m - 2)c_1(B) \cdot C + (4m^3 + 18m^2 + 25m)c_1(B)^2 + 12m.
 \end{aligned}
 \tag{7.81}$$

Using these results, it is possible to compute the entropy of the corresponding four- and five-dimensional black holes with the Cardy formula. It is a very interesting yet challenging task to reproduce our results from a corresponding microscopic computation. In the following we will explain why techniques, which are currently available in the literature, seem to not be sufficient to perform the aforementioned microscopic computations.

In the case of transverse ALF spaces, there are the A-series which corresponds to  $\text{TN}_m$  space, and the D-series ALF spaces. The  $\text{TN}_m$  case is dual to the MSW setting specialized to a particular four-cycle in the elliptically fibered  $\text{CY}_3$ . The MSW results were used in section 7.1 to compute the microscopic central charges and levels. In the case of an ALF space in the D-series a microscopic computation of the central charges seems currently difficult. The 4d woldvolume theory of a

D3-brane probing the aforementioned ALF space is not known. Since the dual M-theory 5-brane picture turned out to be very useful in the Taub-NUT case, we can attempt to dualize in the analogous way to M-theory for the case of the D-series ALF space. We first T-dualize along the fiber of the ALF space to type IIA, and we obtain a D4-brane wrapping  $C$  and the  $S^1$  of the original ALF fiber, and  $m$  NS5-branes wrapping the dimensions transverse to the original ALF space. The key difference between the A- and the D-series is that for the D-series we now also have an ON5-plane<sup>56</sup> on top of the  $m$  NS5-branes [140]. Uplifting to M-theory we now find an M5-brane wrapping the elliptic surface  $\hat{C}$  intersecting with a stack of  $m$  M5-branes on top of an OM5-plane wrapping the base  $B$  of the elliptically fibered Calabi–Yau threefold. This setup is in particular not in the range of MSW and therefore does not allow for a straightforward computation of the the corresponding central charges and levels.

For transverse ALE space, such a 5-brane picture in M-theory is not available. This is due to the absence of a  $S^1$  fiber for the ALE spaces, compared to the ALF spaces. A potentially more promising strategy could be to attempt a computation of the microscopic quantities in F-theory directly. The worldvolume theory of a D3-brane probing a transverse ALE space may be obtained by a suitable projection of non-abelian  $\mathcal{N} = 4$  SYM [102, 103], see section 6.5.1. However, once the 4d theory is wrapped on the curve  $C$  inside the base  $B$  of the elliptic fibration, it is not clear how some of the supercharges can be preserved, since a regular partial topological twist turns out to not be sufficient. The reason for this is that the complex structure of the elliptic fibration – or in field theory language, the complexified gauge coupling  $\tau$  – undergoes monodromies around certain loci in the base  $B$ .<sup>57</sup> The supercharges of the 4d theory also transform under these monodromy transformations, which causes them to be ill defined once the theory is wrapped on a curve  $C \subset B$ . In the case of a single D3-brane probing flat space (i.e.  $\mathcal{N} = 4$  SYM with gauge group  $U(1)$ ) the appearance of an *additional* global symmetry  $U(1)_D$ , which stems from these monodromy transformations, enables us to not only perform a twist with respect to the R-symmetry of the 4d theory, but also with respect to this  $U(1)_D$  symmetry. Using both the R-symmetry and the  $U(1)_D$  twist, some of the supercharges can be preserved [15, 163, 22]. A generalization of this *topological duality twist* to the case of quiver gauge theories, which live on the worldvolume of a D3-brane probing an ALE space, is however not known. It is the lack of a generalization of the topological duality twist, which

<sup>56</sup>This is the analogue of an orientifold plane for the NS5-branes.

<sup>57</sup>These loci are the divisors on which the  $(p, q)$ -7-branes are localized.

prevents us from performing an analysis similar to the one performed in section 6.5.1 for D3-branes on a curve inside a K3 manifold.

In the light of these comments, the results we have obtained in this section should be understood as a prediction for the computed quantities of the corresponding 2d theories. These predictions come from a macroscopic computation, and should eventually be reproduced from a field theory point of view.

## 7.5 Non-abelian flavour symmetries

We will now extend the previous sections by including vector multiplets and charged hypermultiplets in the 6d  $\mathcal{N} = (1, 0)$  supergravity theory. These supergravity theories descend from compactifications of F-theory on singular elliptically fibered Calabi–Yau threefolds. In the 6d bulk theory, these vector multiplets furnish a gauge symmetry. From the point of view of the string propagating in 6d, which is microscopically realized by a D3-brane wrapping the curve  $C$ , this gauge symmetry in the bulk acts as a flavour symmetry on the worldvolume theory living on the string.

### Classical supergravity contribution.

In the following, we will for simplicity assume that the gauge group  $G$  is a product of simple factors, i.e.

$$G = \prod_i G_i, \quad (7.82)$$

where each simple factor  $G_i$  descends from a 7-brane stack in the underlying F-theory model. Including vectors, a new sector in the 6d effective action is

$$(2\pi)^3 S^{(6)} \supset \int_{M_6} \left[ -h_{UV} \mathcal{D}q^U \wedge \hat{*} \mathcal{D}q^V - \sum_i 2\Omega_{\alpha\beta} j^\alpha s_i^\beta \text{tr} \hat{F}^i \wedge \hat{*} \hat{F}^i \right. \\ \left. - \frac{1}{4} \Omega_{\alpha\beta} \hat{B}^\alpha \wedge \hat{X}_4^\beta \right], \quad (7.83)$$

$$\hat{X}_4^\alpha = d\hat{X}_3^\alpha = \frac{1}{2} c^\alpha \text{tr} \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} - 2 \sum_i s_i^\alpha \text{tr} \hat{F}^i \wedge \hat{F}^i,$$

$$\hat{X}_3^\alpha = \frac{1}{2} c^\alpha \hat{\omega}_{\text{grav}}^{\text{CS}} - 2 \sum_i s_i^\alpha \hat{\omega}_i^{\text{CS}},$$

where  $S_i = s_i^\alpha \omega_\alpha$  are the Poincaré duals of the irreducible, effective divisors in  $B$ , on which the 7-brane stacks with gauge group  $G_i$  are localized and  $\hat{\omega}_i^{\text{CS}} \equiv \hat{\omega}^{\text{CS}}(\hat{A}^i)$  is the Chern-Simons form of the gauge field  $\hat{A}^i$ . The divisors Poincaré dual to  $S_i$  are in fact the codimension one loci in  $B$ , over which the elliptic fibration develops

a singularity. Furthermore, the hypermultiplet scalars  $q^U$  are now gauged with gauge covariant derivative  $\mathcal{D}q^U$ . We now perform a dimensional reduction of this action on  $S^3/\Gamma$  including the field strengths  $\hat{F}^i$ , which correspond to the non-abelian symmetry factors  $G_i$ . We perform the dimensional reduction and search for Chern-Simons terms of the dimensionally reduced connection. We take the trivial reduction ansatz for the gauge fields, i.e.  $\hat{F}^i = F^i = dA^i + A^i \wedge A^i$ , where now  $F^i$  is a connection purely on the non-spherical part of the spacetime  $\mathcal{M}_3$ . In the most general reduction ansatz, one would have to include all scalar and vector spherical harmonics in the ansatz. The higher harmonics however, lead to massive vectors and scalar fields in 3d. Since we are interested in the 3d vectors which generate the non-abelian flavour symmetries  $G_i$ , we only need to include the massless vector bosons in the reduction ansatz. Inspecting the action (7.83), we notice that the only potential source for a Chern-Simons term for  $F_i$ , is the Green-Schwarz-Sagnotti coupling  $\hat{X}_4^\alpha$ . We compute<sup>58</sup>

$$\begin{aligned}
(2\pi)^3 S_{CS} &= \int_{M_6} -\frac{1}{4} \Omega_\alpha \hat{B}_\Gamma^\alpha \wedge \hat{X}_4^\beta = \int_{M_6} \frac{1}{4} \Omega_{\alpha\beta} \hat{G}_\Gamma^\alpha \wedge \hat{X}_3^\beta \\
&\supset \int_{M_6} \sum_i -\frac{1}{2} \Omega_{\alpha\beta} s_i^\beta \hat{G}_\Gamma^\alpha \wedge \omega_i^{\text{CS}} = \sum_i \frac{1}{2|\Gamma|} \Omega_{\alpha\beta} s_i^\beta \int_{\mathcal{M}_3} \omega_i^{\text{CS}} \int_{S^3} \hat{G}^\alpha \\
&= (2\pi)^3 \frac{-1}{4\pi} \sum_i \Omega_{\alpha\beta} Q^\alpha s_i^\beta \int_{\mathcal{M}_3} \omega_i^{\text{CS}} = (2\pi)^3 \sum_i -\frac{k_{G_i}^{\text{sugra}}}{4\pi} \int_{\mathcal{M}_3} \omega_i^{\text{CS}}, \quad (7.84)
\end{aligned}$$

such that we find for the levels for the current algebras

$$k_{G_i}^{\text{sugra}} = \Omega_{\alpha\beta} Q^\alpha s_i^\beta. \quad (7.85)$$

The sign in the Chern-Simons term (7.84) implies that the corresponding current algebra in the 2d theory is left-moving. We note, that in the dictionary between the macroscopic charge  $Q^\alpha$  with the microscopic charge  $q^\alpha$ , which we used in the previous section, a shift proportional to the first Pontryagin number of the transverse space of the string was present

$$Q^\alpha = q^\alpha - \frac{1}{4} p_1(M_\Gamma) c^\alpha. \quad (7.86)$$

Recall, that this shift was due to the coupling  $\hat{B}^\alpha \wedge \text{tr } \hat{\mathcal{R}} \wedge \hat{\mathcal{R}}$  in the 6d action. When the equation of motion of the two form  $\hat{B}^\alpha$  in the presence of a string-like

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<sup>58</sup>Recall that  $\hat{G}_\Gamma^\alpha = d\hat{B}_\Gamma^\alpha$  is the field strength on the quotient space.



source with (microscopic) charges  $q^\alpha$  is integrated over the space transverse to the string  $M_\Gamma$ , a non-trivial charge shift is found if

$$p_1(M_\Gamma) = -\frac{1}{2} \frac{1}{(2\pi)^2} \int_{M_\Gamma} \text{tr } \hat{\mathcal{R}} \wedge \hat{\mathcal{R}} \neq 0. \quad (7.87)$$

In the presence of vector multiplets in 6d, we have a similar coupling, now involving  $\hat{B}^\alpha$  and the gauge fields  $\hat{F}^i$  in (7.83). This coupling is

$$\int_{M_6} \sum_i \Omega_{\alpha\beta} s_i^\alpha B^\beta \wedge \text{tr } \hat{F}^i \wedge \hat{F}^i, \quad (7.88)$$

and contributes to the equation of motion of  $B^\alpha$  similar to the  $\text{tr } \hat{\mathcal{R}} \wedge \hat{\mathcal{R}}$  coupling. These couplings would also contribute with independent shifts in the dictionary between  $Q^\alpha$  and  $q^\alpha$ , provided that

$$\int_{M_\Gamma} \text{tr } \hat{F}^i \wedge \hat{F}^i \neq 0 \quad (7.89)$$

in the background. From an F-theory perspective, such an instanton can be viewed as a gauge instanton on the 7-branes. This instanton configuration can induce D3-brane charge localized on the 7-brane dimensions transverse to the instanton configuration through a Chern-Simons coupling of the (R,R) four-form to the gauge fields.<sup>59</sup> However, we consider 6d solutions without gauge instantons, such that (7.88) does not contribute to the dictionary.<sup>60</sup> We therefore conclude that the original identification of micro- and macroscopic charges (7.86) remains valid. We find from supergravity the levels

$$k_{G_i}^{\text{sugra}} = \Omega_{\alpha\beta} \left( q^\alpha - \frac{1}{4} p_1(M_\Gamma) c^\alpha \right) s_i^\beta = \left( C - \frac{1}{4} p_1(M_\Gamma) c_1(B) \right) \cdot S_i. \quad (7.90)$$

### Quantum contribution.

We next wish to include (up to  $\mathcal{O}(1)$ ) the quantum contributions to the Chern-Simons terms in 3d. These are induced by chiral fields which couple to the gauge fields  $\hat{A}^i$ . These are given by the gauginos in the adjoint representations of  $G_i$ , and the hyperinos in hypermultiplets transforming in representations

$$\mathbf{R} = \bigotimes_i \mathbf{R}_i \quad (7.91)$$

<sup>59</sup>This is similar to D(-1)-brane charge induced on D3-branes via gauge instantons in  $\mathcal{N} = 4$  SYM. In the case of the 7-branes, the coupling in question has the form  $C_4 \wedge \text{tr } F \wedge F$ .

<sup>60</sup>Such instanton configurations on Taub-NUT spaces were discussed in [164–166].

of  $G$ . Here,  $\mathbf{R}_i$  is the representation of the fermion under the gauge group factor  $G_i$ . We label the spin- $\frac{1}{2}$  field content in 3d from the harmonic expansion on  $S^3$  with the notation

$$(j_L, j_R, \mathbf{R})^{\text{sgn}(M)}. \quad (7.92)$$

We find the following spin- $\frac{1}{2}$  spectrum, which transforms under  $G$ :

$$2 \bigoplus_{\mathbf{R}} x_{\mathbf{R}} \bigoplus_{j_L=\frac{1}{2}}^{\infty} (j_L, j_L \pm \frac{1}{2}, \mathbf{R})^{\pm} \oplus 2 \bigoplus_i \bigoplus_{j_L=\frac{1}{2}}^{\infty} (j_L, j_L \pm \frac{1}{2}, \mathbf{ad}_{G_i})^{\mp}. \quad (7.93)$$

Here,  $x_{\mathbf{R}}$  denotes the multiplicity of the representation  $\mathbf{R}$  in the spectrum of the 6d supergravity theory and  $\mathbf{ad}_{G_i}$  denotes the representation, which is the adjoint of the gauge group factor  $G_i$ , but in the singlet representation of all other gauge group factors  $G_{j \neq i}$ .

Along the lines of section 6.4.2, we compute the contribution of a single massive spin- $\frac{1}{2}$  field, which transforms in the representation  $\mathbf{R} = \bigotimes_i \mathbf{R}_i$  under  $G$ , to the Chern-Simons terms in 3d. The parity anomaly from such a massive field can be cancelled by a counter-term [128]

$$\pi \text{sgn}(M) \int_{M_3} Q_{\frac{1}{2}}(\{A^i\}, \omega), \quad (7.94)$$

where

$$dQ_{\frac{1}{2}}(A, \omega) = \hat{A}(M_3) \wedge \text{ch}(F)|_{4\text{-form}} = \hat{A}(M_3) \wedge \bigwedge_i \text{ch}(F^i)|_{4\text{-form}} \quad (7.95)$$

$$\hat{A}(M_1) = 1 + \frac{1}{(4\pi)^2} \frac{1}{12} \text{tr} \mathcal{R} \wedge \mathcal{R} + \dots \quad (7.96)$$

$$\text{ch}(F^i) = \dim \mathbf{R}_i + \frac{i}{2\pi} \text{tr}_{\mathbf{R}_i} F^i - \frac{1}{2} \frac{1}{(2\pi)^2} \text{tr}_{\mathbf{R}_i} F^i \wedge F^i + \dots \quad (7.97)$$

Note that  $\text{tr}_{\mathbf{R}_i} F^i = 0$ , since we assumed that the factors  $G_i$  are simple. We then find

$$\begin{aligned} dQ_{\frac{1}{2}}(A, \omega)|_{4\text{-form}} &= \frac{\dim \mathbf{R}}{(4\pi)^2} \frac{1}{12} \text{tr} \mathcal{R} \wedge \mathcal{R} - \frac{1}{2} \frac{1}{(2\pi)^2} \sum_i d_i(\mathbf{R}) \text{tr}_{\mathbf{R}_i} F^i \wedge F^i, \\ &= \frac{\dim \mathbf{R}}{(4\pi)^2} \frac{1}{12} \text{tr} \mathcal{R} \wedge \mathcal{R} - \frac{1}{2} \frac{1}{(2\pi)^2} \sum_i d_i(\mathbf{R}) A_{\mathbf{R}_i} \lambda_i \text{tr} F^i \wedge F^i, \end{aligned} \quad (7.98)$$

where now

$$\dim \mathbf{R} = \prod_i \dim \mathbf{R}_i, \quad d_i(\mathbf{R}) = \prod_{j \neq i} \dim \mathbf{R}_j. \quad (7.99)$$

Furthermore,  $\lambda_i$  are normalization constants of  $\text{tr}$  with respect to the trace over the fundamental representation of  $G_i$ , i.e.  $\text{tr}_f = \lambda_i \text{tr}$ . This normalization constant depends on  $G_i$ , and  $\text{tr}$  is used in the Chern-Simons terms in (7.84). Furthermore, the constants  $A_{\mathbf{R}_i}$  are defined via the relation

$$\text{tr}_{\mathbf{R}_i} F^i \wedge F^i = A_{\mathbf{R}_i} \text{tr}_f F^i \wedge F^i. \quad (7.100)$$

We therefore conclude that a single spin- $\frac{1}{2}$  fermion in the representation  $\mathbf{R}$  of  $G$  contributes with

$$\alpha_{G_i}^{(1/2)} = -\frac{1}{8\pi} \text{sgn}(M) d_i(\mathbf{R}) A_{\mathbf{R}_i} \lambda_i \quad (7.101)$$

to the Chern-Simons term of the gauge field  $A^i$ .

**One-loop corrections for  $\Gamma = \mathbb{Z}_m$ .**

The contribution from the hyperinos is given by

$$\begin{aligned} (\text{hyper}) &= -2 \times \frac{\lambda_i}{8\pi} \sum_{\mathbf{R}} x_{\mathbf{R}} d_i(\mathbf{R}) A_{\mathbf{R}_i} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} [(2(j_L + \frac{1}{2}) + 1) - (2(j_L - \frac{1}{2}) + 1)] \\ &= -\frac{\lambda_i}{2\pi} \sum_{\mathbf{R}} x_{\mathbf{R}} d_i(\mathbf{R}) A_{\mathbf{R}_i} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} 1 = -\frac{\lambda_i}{48\pi} m \sum_{\mathbf{R}} x_{\mathbf{R}} d_i(\mathbf{R}) A_{\mathbf{R}_i} + \mathcal{O}(1). \end{aligned} \quad (7.102)$$

We can now use

$$\sum_{\mathbf{R}} x_{\mathbf{R}} d_i(\mathbf{R}) A_{\mathbf{R}_i} = \sum_{\mathbf{R}_i} x_{\mathbf{R}_i} A_{\mathbf{R}_i} \quad (7.103)$$

to trade the sum over all product representation  $\mathbf{R}$  of  $G$  in favour of the sum over all representations  $\mathbf{R}_i$  of the gauge group factor  $G_i$ . Here we defined  $x_{\mathbf{R}_i}$  as the multiplicity of all hypermultiplet fermions, which transform in the representation  $\mathbf{R}_i$  of  $G_i$ . We therefore find for the contribution from the hypermultiplet fermions

$$(\text{hyper}) = -\frac{\lambda_i}{48\pi} m \sum_{\mathbf{R}_i} x_{\mathbf{R}_i} A_{\mathbf{R}_i} + \mathcal{O}(1). \quad (7.104)$$

Likewise, we find for the contribution from the gauginos

$$\begin{aligned} (\text{gauge}) &= 2 \times \frac{\lambda_i}{8\pi} A_{\mathbf{ad}_{G_i}} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} [(2(j_L + \frac{1}{2}) + 1) - (2(j_L - \frac{1}{2}) + 1)] \\ &= \frac{\lambda_i}{48\pi} m A_{\mathbf{ad}_{G_i}} + \mathcal{O}(1). \end{aligned} \quad (7.105)$$

The correction to the level  $k_{G_i}$  is then

$$k_{G_i}^{\text{loop}} = -4\pi[(\text{hyper}) + (\text{gauge})] = \frac{m}{12}\lambda_i \left( \sum_{\mathbf{R}_i} x_{\mathbf{R}_i} A_{\mathbf{R}_i} - A_{\mathbf{ad}_{G_i}} \right). \quad (7.106)$$

We now make use of the anomaly cancellation condition

$$\lambda_i \left( \sum_{\mathbf{R}_i} x_{\mathbf{R}_i} A_{\mathbf{R}_i} - A_{\mathbf{ad}_{G_i}} \right) = 6 \Omega_{\alpha\beta} c^\alpha s_i^\beta = 6 c_1(B) \cdot S_i, \quad (7.107)$$

and we find

$$k_{G_i}^{\text{loop}} = \frac{m}{2} c_1(B) \cdot S_i. \quad (7.108)$$

The final result is

$$k_{G_i} = \begin{cases} k_{G_i}^{\text{sugra}} + k_{G_i}^{\text{loop}} = C \cdot S_i & \text{for } M_\Gamma = \text{TN}_m \\ k_{G_i}^{\text{sugra}} = (C - \frac{m}{2} c_1(B)) \cdot S_i & \text{for } M_\Gamma = \mathbb{C}^2/\mathbb{Z}_m \end{cases}. \quad (7.109)$$

This result is in agreement with results stated in the literature. Such a result is  $k_{G_i} = C \cdot S_i$  for  $M_\Gamma = \mathbb{R}^4$  [167–169]. Our result for  $\text{TN}_{m=1}$  matches with the flat space result trivially, as expected. This matching is due to a non-trivial cancellation between the charge shift (7.86) and the one-loop correction (7.108).

### One-loop corrections for $\Gamma = \mathbb{D}_m^*$ .

In the computation of one-loop corrections we noticed that it is sufficient (up to  $\mathcal{O}(1)$ ) to take the result obtained for  $\mathbb{Z}_m$ , replace  $m \rightarrow 2m$  in the expression for the loop correction, and then divide by a factor of 2. As the correction (7.108) is linear in  $m$ , we obtain with this procedure the identical result for  $\mathbb{D}_m^*$ . We therefore find for  $\mathbb{D}_m^*$

$$k_{G_i} = \begin{cases} k_{G_i}^{\text{sugra}} + k_{G_i}^{\text{loop}} = C \cdot S_i & \text{for } M_\Gamma = \text{ALF}_{\mathbb{D}_m^*} \\ k_{G_i}^{\text{sugra}} = (C - \frac{m}{2} c_1(B)) \cdot S_i & \text{for } M_\Gamma = \mathbb{C}^2/\mathbb{D}_m^* \end{cases},$$

$$(7.110)$$

again, up to  $\mathcal{O}(1)$ .

### One-loop corrections to central charges and universal levels.

It is easy to see (e.g. using the index theorem again, or by thinking about the structure of Feynman diagrams) that a spin- $\frac{1}{2}$  state with  $U(1)_L$  charge  $j_L^{(3)}$ , in the spin  $j_R$  representation of  $SU(2)_R$  and  $\mathbf{R}$  of  $G$  contributes with

$$\begin{aligned}\alpha_L^{(1/2)} &= \frac{1}{8\pi} \text{sgn}(M) \dim \mathbf{R} (2j_R + 1) (j_L^{(3)})^2, \\ \alpha_R^{(1/2)} &= -\frac{1}{12\pi} \text{sgn}(M) \dim \mathbf{R} j_R (j_R + 1) (2j_R + 1), \\ \alpha_{\text{grav}}^{(1/2)} &= \frac{1}{192\pi} \text{sgn}(M) \dim \mathbf{R} (2j_R + 1),\end{aligned}\tag{7.111}$$

to the  $U(1)_L$ ,  $SU(2)_R$  and gravitational Chern-Simons terms, respectively. These are just the values from table 7.1 multiplied by  $\dim \mathbf{R}$ . When summing over the spectrum (7.93) we therefore get the same sums as in appendix 7.B, except with all terms proportional to  $n_H$  replaced by

$$n_H \rightarrow \sum_{\mathbf{R}} x_{\mathbf{R}} \dim \mathbf{R} - \sum_i \dim \mathbf{ad}_{G_i} = n_H - n_V. \tag{7.112}$$

In the derivation of the final expressions for the central charges and levels in appendix 7.B, we used

$$n_H = 273 - 29n_T, \tag{7.113}$$

$$n_T = 9 - c_1(B)^2, \tag{7.114}$$

i.e. the condition which ensures the cancellation of the gravitational anomaly in 6d for the case  $n_V = 0$  (7.113), and the number of tensor multiplets in terms of topological data on  $B$  (7.114), to eliminate  $n_H$  and  $n_T$  from the final expressions. Now, if  $n_V \neq 0$  the conditions we need to use are

$$n_H - n_V = 273 - 29n_T, \tag{7.115}$$

$$n_T = 9 - c_1(B)^2. \tag{7.116}$$

Note that the condition imposed by cancellation of the gravitational anomaly in the case  $n_V \neq 0$  is simply (7.113) with  $n_H$  replaced by  $n_H \rightarrow n_H - n_V$ . Since the only effect of including charged matter and vectors in the loop sums is (7.112), we find that the expressions for  $c_L, c_R$  and  $k_L$  (given in terms of  $C, c_1(B), m$ ) are still valid for  $n_V \neq 0$  and with charged matter in 6d.

## 7.A 6d to 3d reduction higher derivative term

In this appendix we give some more details of the reduction of the six-dimensional higher derivative term to three dimensions. In particular, we calculate the part of the integral

$$\int_{\text{TN}_m^{\text{sph}}} \hat{G}^\alpha \wedge \hat{\omega}_{\text{grav}}^{\text{CS}} \quad (7.117)$$

that leads to three-dimensional Chern-Simons terms. In order to do the reduction we first decompose the spin connection corresponding to the ansatz (7.48) to determine the parts that lead to Chern-Simons terms in three dimensions. Denoting indices of the non-spherical part  $\mathcal{M}_3$  of the black string solution by  $\tilde{a} = 1, 2, 3$  and a vielbein of  $\mathcal{M}_3$  by  $\hat{e}^{\tilde{a}}$ , the spin connection  $\omega$  with respect to the vielbein of the ansatz,  $e^{\tilde{a}} \equiv \hat{e}^{\tilde{a}}, e^a$ , can be expressed as [170]

$$\begin{aligned} \omega_{\tilde{a}\tilde{b}} &= \hat{\omega}_{\tilde{a}\tilde{b}} + \frac{1}{2} F_{\tilde{a}\tilde{b}}^i K_c^i e^c, \\ \omega_{\tilde{a}b} &= \frac{1}{2} F_{\tilde{a}c}^i K_b^i \hat{e}^{\tilde{c}}, \\ \omega_{ab} &= \hat{\omega}_{ab} + \left( \hat{\nabla}_a K_b^i \right) A^i. \end{aligned} \quad (7.118)$$

Here  $\hat{\omega}_{\tilde{a}\tilde{b}}$  are the components of the spin connection  $\hat{\omega}_{\mathcal{M}_3}$  with respect to the vielbein  $\hat{e}^{\tilde{a}}$  of  $\mathcal{M}_3$  and  $\hat{\omega}_{ab}$  are the components of the spin connection  $\hat{\omega}_{\text{sph}}$  with respect to the vielbein  $\hat{e}^a$  of the spherical part of the black string solution. From the expression of the gravitational Chern-Simons term,

$$\hat{\omega}_{\text{grav}}^{\text{CS}} = \text{tr} \left( \omega \wedge d\omega + \frac{2}{3} \omega^3 \right), \quad (7.119)$$

it is immediately clear that if we are interested in three-dimensional Chern-Simons terms, we can restrict to

$$\begin{aligned} \omega_{\tilde{a}\tilde{b}} &= \hat{\omega}_{\tilde{a}\tilde{b}}, \\ \omega_{\tilde{a}b} &= 0, \\ \omega_{ab} &= \hat{\omega}_{ab} + \left( \hat{\nabla}_a K_b^i \right) A^i. \end{aligned} \quad (7.120)$$

This is a direct sum connection, hence

$$\hat{\omega}_{\text{grav}}^{\text{CS}} = \omega^{\text{CS}}(\hat{\omega}_{\mathcal{M}_3}) + \omega^{\text{CS}}(X), \quad (7.121)$$

where  $X$  is a connection with components  $\hat{\omega}_{ab} + \left( \hat{\nabla}_a K_b^i \right) A^i$  and

$$\omega^{\text{CS}}(\hat{\omega}_{\mathcal{M}_3}) \equiv \text{tr} \left( \hat{\omega}_{\mathcal{M}_3} \wedge d\hat{\omega}_{\mathcal{M}_3} + \frac{2}{3} \hat{\omega}_{\mathcal{M}_3}^3 \right),$$

$$\omega^{\text{CS}}(X) \equiv \text{tr} \left( X \wedge dX + \frac{2}{3} X^3 \right). \quad (7.122)$$

Notice that  $\omega^{\text{CS}}(\hat{\omega}_{\mathcal{M}_3}) = \omega_{\text{grav}}^{\text{CS}}$  is the gravitational Chern-Simons term of  $\mathcal{M}_3$ .

The only part of  $\hat{G}^\alpha$  relevant for Chern-Simons terms in three dimensions is

$$-Q^\alpha (2\pi)^2 m \left( e_3^{(m)} - \chi_3 \right). \quad (7.123)$$

Here  $\chi_3$  has all its legs on  $\mathcal{M}_3$  which means that its wedge product with  $\hat{\omega}_{\text{grav}}^{\text{CS}}$  only gets a contribution of  $\omega^{\text{CS}}(\hat{\omega}_{\text{sph}})$ . We can then expand

$$\begin{aligned} \int_{\text{TN}_m^{\text{sph}}} \hat{G}^\alpha \wedge \hat{\omega}_{\text{grav}}^{\text{CS}} = & -Q^\alpha (2\pi)^2 m \int_{\text{TN}_m^{\text{sph}}} \left[ e_3^{(m)} \wedge \omega_{\text{grav}}^{\text{CS}} + e_3^{(m)} \wedge \omega^{\text{CS}}(X) \right. \\ & \left. - \chi_3 \wedge \omega^{\text{CS}}(\hat{\omega}_{\text{sph}}) \right] \end{aligned} \quad (7.124)$$

The separate integrals are given by<sup>61</sup>

$$\begin{aligned} \int_{\text{TN}_m^{\text{sph}}} e_3^{(m)} \wedge \omega_{\text{grav}}^{\text{CS}} &= -\frac{1}{m} \omega_{\text{grav}}^{\text{CS}}, \\ \int_{\text{TN}_m^{\text{sph}}} e_3^{(m)} \wedge \omega^{\text{CS}}(X) &= \frac{2}{m} A_L \wedge F_L, \\ \int_{\text{TN}_m^{\text{sph}}} \chi_3 \wedge \omega^{\text{CS}}(\hat{\omega}_{\text{sph}}) &= \frac{1 + 4r' + 10r'^2 + 8r'^3 + 2r'^4}{m(1+r')^4} \times 16\pi^2 \chi_3, \end{aligned} \quad (7.125)$$

where we introduced  $r' \equiv m_\infty r/m$ . Using that

$$\chi_3 = \frac{1}{16\pi^2} A_L \wedge F_L + \frac{1}{8\pi^2} \omega^{\text{CS}}(A_R), \quad (7.126)$$

we find that (7.124) becomes

$$\begin{aligned} \int_{\text{TN}_m^{\text{sph}}} \hat{G}^\alpha \wedge \hat{\omega}_{\text{grav}}^{\text{CS}} = & Q^\alpha (2\pi)^2 \left[ \omega_{\text{grav}}^{\text{CS}} - \frac{1 + 4r' + 2r'^2}{(1+r')^4} A_L \wedge F_L \right. \\ & \left. + 2 \frac{1 + 4r' + 10r'^2 + 8r'^3 + 2r'^4}{(1+r')^4} \omega^{\text{CS}}(A_R) \right]. \end{aligned} \quad (7.127)$$

This leads to the expression (7.53).

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<sup>61</sup>We used Mathematica to calculate the second and third integral.

## 7.B Summation of 6d to 3d one-loop corrections

To perform the sum of the one-loop corrections over the Kaluza-Klein spectrum, we use the regularization procedure described in the main text. We need the sums

$$\begin{aligned} \sum_{j_L=\frac{1}{2}mk}^{\infty} 1 &= \frac{1}{2} - \frac{1}{2}km, & \sum_{j_L=\frac{1}{2}mk}^{\infty} j_L &= \frac{1}{24}(-2 + 6km - 3k^2m^2), \\ \sum_{j_L=\frac{1}{2}mk}^{\infty} j_L^2 &= \frac{1}{24}(-2km + 3k^2m^2 - k^3m^3), \end{aligned} \quad (7.128)$$

where the sum is over integers (half integers) when  $\frac{1}{2}mk$  is integer (half integer). The sums used in this section can then be calculated using (6.44):

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} 1 &= -\frac{1}{4} + \frac{1}{24}m, & \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} j_L &= \frac{1}{24} - \frac{1}{48}m, \\ \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} j_L^2 &= -\frac{m^3}{24 \cdot 120} + \frac{m}{144}, & \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} k^2 &= -\frac{1}{2} \frac{m}{120}. \end{aligned} \quad (7.129)$$

We calculate the corrections  $\Delta k_L^{\text{loop}}, \Delta k_R^{\text{loop}}$  to the levels and the correction  $\Delta(c_L - c_R)^{\text{loop}}$  separately. Using the projection condition  $j_L^{(3)} = \frac{1}{2}mk$ , we calculate every time the contribution of the  $k = 0$  representations first, and after that the contribution of  $k \neq 0$ . Since the structure of representations for small values of  $j_L, j_R$  becomes more complicated, we first calculate the corrections for  $m \geq 3$  and do the cases  $m = 1, 2$  separately.

### 7.B.1 Corrections for $m \geq 3$

We list the spectrum, which includes a state with  $k = 0$ . All the sums are over integers

- Spin- $\frac{3}{2}$ :

$$2 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\mp}.$$

- Spin- $\frac{1}{2}$ :

$$2 \bigoplus_{j_L=2}^{\infty} (j_L, j_L \pm \frac{3}{2})^{\mp} \oplus 2 \bigoplus_{j_L=0}^1 (j_L, j_L + \frac{3}{2})^{-} \oplus 2 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \quad (7.130)$$



$$\oplus 2(0, \frac{1}{2})^+ \oplus 2(n_T + n_H) \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \oplus 2(n_T + n_H) (0, \frac{1}{2})^+.$$

- vectors:

$$\bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm 1)^{\mp} \oplus (0, 1)^- \oplus n_T \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm 1)^{\pm} \oplus n_T (0, 1)^+.$$

Note that the  $(0, 1)^- \oplus n_T (0, 1)^+$  vector representations are mapped to itself when applying the reality condition. Hence their contribution comes with an extra factor  $\frac{1}{2}$ .

When  $k > 0$  the projection condition gives that  $j_L^{(3)} = \frac{1}{2}mk \geq \frac{m}{2}$  which means that when  $m \geq 3$  we only need the representations (again the sums go with integer steps)

- Spin- $\frac{3}{2}$ :

$$2 \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\mp}.$$

- Spin- $\frac{1}{2}$ :

$$2 \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm \frac{3}{2})^{\mp} \oplus 2 \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm} \oplus 2(n_T + n_H) \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm}.$$

- vectors:

$$\bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm 1)^{\mp} \oplus n_T \bigoplus_{j_L=\frac{1}{2}mk}^{\infty} (j_L, j_L \pm 1)^{\pm}.$$

**Correction to left level.** In this case we do not have a contribution of the  $k = 0$  modes. We thus only have to calculate the contribution from the  $k > 0$  modes and we will do this separately for the various types of fields contributing to the left level. For the spin- $\frac{3}{2}$  fermions we get

$$\begin{aligned} \alpha_L^{(3/2)} &= 2 \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \frac{3}{8\pi} \left(\frac{1}{2}mk\right)^2 \left[2\left(j_L - \frac{1}{2}\right) + 1 - 2\left(j_L + \frac{1}{2}\right) - 1\right] \\ &= -\frac{3m^2}{8\pi} \sum_{k=1}^{\infty} \sum_{j=\frac{1}{2}mk}^{\infty} k^2 = \frac{1}{8\pi} \frac{m^3}{80}. \end{aligned} \tag{7.131}$$

In the same way the spin- $\frac{1}{2}$  fermions give

$$\begin{aligned}\alpha_L^{(1/2)} &= \frac{1}{8\pi} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \left[ -12\left(\frac{1}{2}mk\right)^2 + 4\left(\frac{1}{2}mk\right)^2 + 4(n_T + n_H) \left(\frac{1}{2}mk\right)^2 \right] \\ &= -\frac{m^2}{8\pi} (2 - n_T - n_H) \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} k^2 = \frac{1}{8\pi} \frac{m^3}{240} (2 - n_T - n_H). \quad (7.132)\end{aligned}$$

Finally, the vectors contribute with

$$\alpha_L^{(\text{vect})} = 4(1 - n_T) \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \frac{1}{4\pi} \left(\frac{1}{2}mk\right)^2 = -\frac{1}{8\pi} \frac{m^3}{120} (1 - n_T). \quad (7.133)$$

Adding the contributions (7.131), (7.132) and (7.133) up, we find

$$\Delta k_L^{\text{loop}} = 8\pi \cdot \left( \alpha_L^{(3/2)} + \alpha_L^{(1/2)} + \alpha_L^{(\text{vect})} \right) = -\frac{m^3}{8} c_1(B)^2,$$

where we used the identities (7.62).

**Correction to right level.** We get the contribution of the  $k = 0$  modes by summing over the representations listed around (7.130). We first do this for the  $2(j_L, j_L \pm \frac{1}{2})^{\mp}$  representations for the spin- $\frac{3}{2}$  fermions, the  $2(j_L, j_L \pm \frac{3}{2})^{\mp}$  representations for the spin- $\frac{1}{2}$  fermions and the  $(1 - n_T)(j_L, j_L \pm 1)^{\mp}$  representations for the vectors, which are in this order given by

$$\begin{aligned}& -\frac{1}{2\pi} \sum_{j_L=1}^{\infty} \left[ -\left(j_L + \frac{1}{2}\right)\left(j_L + \frac{3}{2}\right)(2j_L + 2) + \left(j_L - \frac{1}{2}\right)\left(j_L + \frac{1}{2}\right)(2j_L) \right] = -\frac{5}{8\pi}, \\ & -\frac{1}{6\pi} \sum_{j_L=2}^{\infty} \left[ -\left(j_L + \frac{3}{2}\right)\left(j_L + \frac{5}{2}\right)(2j_L + 4) + \left(j_L - \frac{3}{2}\right)\left(j_L - \frac{1}{2}\right)(2j_L - 2) \right] = -\frac{83}{8\pi}, \\ & \frac{1}{6\pi} (1 - n_T) \sum_{j_L=1}^{\infty} \left[ -(j_L + 1)(j_L + 2)(2j_L + 3) \right. \\ & \quad \left. + (j_L - 1)j_L(2j_L - 1) \right] = \frac{2}{3\pi} (1 - n_T). \quad (7.134)\end{aligned}$$

For the spin- $\frac{1}{2}$  fields we then also need to sum over the other infinite towers of states, namely the  $2(j_L, j_L \pm \frac{1}{2})^{\pm}$  and  $2(n_T + n_H)(j_L, j_L \pm \frac{3}{2})^{\mp}$  representations. These can be determined by inserting the right relative factors in the first of the sums above. We also add the contributions from the isolated representations,

which are not part of an infinite tower in the spectrum. These are in the case of spin- $\frac{1}{2}$  fields the  $2 \left(0, \frac{3}{2}\right)^- \oplus 2 \left(1, \frac{5}{2}\right)^- \oplus 2(n_T + n_H + 1) \left(0, \frac{1}{2}\right)^+$  representations. Their contribution is given by

$$\frac{5}{12\pi} - (n_T + n_H) \frac{11}{24\pi}. \quad (7.135)$$

Lastly, we need to sum over the isolated  $(0, 1)^- \oplus n_T (0, 1)^+$  representations for the vectors. Since they are mapped to itself when applying the reality condition, we have to add an extra factor  $\frac{1}{2}$ . This results in

$$- \frac{1 - n_T}{2\pi}. \quad (7.136)$$

Summing all the different contributions up gives

$$\alpha_R^{k=0} = \frac{1}{8\pi} \left( 3 - \frac{1}{3}n_H - \frac{5}{3}n_T \right). \quad (7.137)$$

We now calculate the  $k \neq 0$  contributions in the same way as for the left level. The spin- $\frac{3}{2}$  fermions contribute

$$\begin{aligned} \alpha_R^{(3/2)} &= -\frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \left[ -\left(j_L + \frac{1}{2}\right)\left(j_L + \frac{3}{2}\right)(2j_L + 2) + \left(j_L - \frac{1}{2}\right)\left(j_L + \frac{1}{2}\right)2j_L \right] \\ &= \frac{1}{2\pi} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \left( \frac{3}{2} + 6j_L + 6j_L^2 \right) = \frac{1}{4\pi} \left( -\frac{1}{4} - \frac{m}{24} - \frac{m^3}{240} \right). \end{aligned} \quad (7.138)$$

The contribution of the spin- $\frac{1}{2}$  fermions is obtained by first summing over the  $(j_L, j_L \pm \frac{3}{2})^\mp$  representations

$$\begin{aligned} \alpha_R^{(1/2)} &= -\frac{1}{6\pi} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \left[ -\left(j_L + \frac{3}{2}\right)\left(j_L + \frac{5}{2}\right)(2j_L + 4) \right. \\ &\quad \left. + \left(j_L - \frac{3}{2}\right)\left(j_L - \frac{1}{2}\right)(2j_L - 2) \right] \\ &\quad - \frac{1}{6\pi} (1 + n_T + n_H) \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \left[ \left(j_L + \frac{1}{2}\right)\left(j_L + \frac{3}{2}\right)(2j_L + 2) \right. \\ &\quad \left. - \left(j_L - \frac{1}{2}\right)\left(j_L + \frac{1}{2}\right)2j_L \right] \\ &= \frac{1}{4\pi} \left( -\frac{9}{4} + \frac{7m}{24} - \frac{m^3}{240} \right) - \frac{1}{4\pi} (1 + n_T + n_H) \left( -\frac{1}{12} - \frac{m}{72} - \frac{m^3}{720} \right). \end{aligned} \quad (7.139)$$

Lastly, the vectors contribute with

$$\begin{aligned}
\alpha_R^{(\text{vect})} &= \frac{1}{6\pi} (1 - n_T) \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \left[ - (j_L + 1) (j_L + 2) (2j_L + 3) \right. \\
&\quad \left. + (j_L - 1) j_L (2j_L - 1) \right] \\
&= -\frac{1}{6\pi} (1 - n_T) \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} (6 + 12j_L + 12j_L^2) \\
&= \frac{1}{4\pi} (n_T - 1) \left( -\frac{2}{3} + \frac{m}{18} - \frac{m^3}{360} \right). \tag{7.140}
\end{aligned}$$

Adding the contributions (7.137), (7.138), (7.139) and (7.140), we get the following correction to the right level

$$\begin{aligned}
\Delta k_R^{\text{loop}} &= 4\pi \left[ \alpha_R^{k=0} + \alpha_R^{(3/2)} + \alpha_R^{(1/2)} + \alpha_R^{(\text{vect})} \right] \\
&= \frac{m^3}{24} c_1(B)^2 + \frac{m}{3} c_1(B)^2 + m - c_1(B)^2 - 14. \tag{7.141}
\end{aligned}$$

**Correction to  $c_L - c_R$ .** We do these calculations in the same way as for the left and right levels. We first calculate the contribution of the  $k = 0$  states. Summing over these states in the spin- $\frac{1}{2}$ , spin- $\frac{3}{2}$  and vector spectrum results in

$$\begin{aligned}
\alpha_{\text{grav}}^{k=0} &= \frac{1}{192\pi} \left[ -12 \sum_{j_L=2}^{\infty} 1 + 4(1 + n_T + n_H) \sum_{j_L=1}^{\infty} 1 \right. \\
&\quad \left. - 2 \cdot 4 - 2 \cdot 6 + 2 \cdot 2 + 2(n_H + n_T) \cdot 2 \right] \\
&\quad + 4 \sum_{j_L=1}^{\infty} \frac{7}{64\pi} - 4(1 - n_T) \sum_{j_L=1}^{\infty} \frac{1}{48\pi} - \frac{1}{2} \cdot \frac{1}{48\pi} (1 - n_T) \cdot 3 \\
&= \frac{1}{96\pi} (n_H + n_T) - \frac{7}{32\pi} + \frac{1}{96\pi} (1 - n_T) = -\frac{1}{96\pi} (20 - n_H). \tag{7.142}
\end{aligned}$$

We now calculate the contribution of the  $k \neq 0$  representations. Spin- $\frac{3}{2}$  fermions give the following correction

$$\alpha_{\text{grav}}^{(3/2)} = \frac{7}{16\pi} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} 1 = \frac{7}{16\pi} \left( -\frac{1}{4} + \frac{1}{24}m \right) = \frac{1}{96\pi} \left( -\frac{21}{2} + \frac{7}{4}m \right). \tag{7.143}$$

From spin- $\frac{1}{2}$  fermions we obtain

$$\alpha_{\text{grav}}^{(1/2)} = -\frac{1}{4 \cdot 48\pi} \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} \left[ 2 \cdot 6 - 2 \cdot 2 - 2(n_T + n_H) \cdot 2 \right]$$

$$\begin{aligned}
&= -\frac{1}{96\pi} (4 - 2(n_T + n_H)) \sum_{k=1}^{\infty} \sum_{j=\frac{1}{2}mk}^{\infty} 1 \\
&= -\frac{1}{96\pi} (4 - 2(n_T + n_H)) \left( -\frac{1}{4} + \frac{1}{24}m \right). \tag{7.144}
\end{aligned}$$

Finally, for the vectors we find

$$\alpha_{\text{grav}}^{(\text{vect})} = -\frac{1}{48\pi} (1 - n_T) \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}mk}^{\infty} 4 = -\frac{1}{96\pi} (1 - n_T) \left( -2 + \frac{m}{3} \right). \tag{7.145}$$

Adding the results (7.142), (7.143), (7.144) and (7.145) yields the following correction to  $c_L - c_R$

$$\begin{aligned}
\Delta(c_L - c_R) &= 96\pi \cdot \left( \alpha_{\text{grav}}^{k=0} + \alpha_{\text{grav}}^{(3/2)} + \alpha_{\text{grav}}^{(1/2)} + \alpha_{\text{grav}}^{(\text{vect})} \right) \\
&= 6m + (2m + 17) c_1(B)^2 - 44. \tag{7.146}
\end{aligned}$$

### 7.B.2 Corrections for $m = 2$

For  $m = 2$  the representations we need to take into account for  $k = 0$  stay the same. For  $k > 0$  only the spin- $\frac{1}{2}$  contribution changes. Summing over the correct representations in the spin- $\frac{1}{2}$  sector, we again find the one-loop corrections.

**Correction to left level.** For the spin- $\frac{1}{2}$  fermions we now find

$$\begin{aligned}
\alpha_L^{(1/2)} &= -12 \sum_{k=2}^{\infty} \sum_{j_L=k}^{\infty} \frac{1}{8\pi} k^2 - 12 \sum_{j_L=2}^{\infty} \frac{1}{8\pi} - 12 \frac{1}{8\pi} + 4 \sum_{k=1}^{\infty} \sum_{j_L=k}^{\infty} \frac{1}{8\pi} k^2 \\
&\quad + 4(n_T + n_H) \sum_{k=1}^{\infty} \sum_{j_L=k}^{\infty} \frac{1}{8\pi} k^2 = \frac{1}{120\pi} - \frac{n_T + n_H}{240\pi}, \tag{7.147}
\end{aligned}$$

which is exactly the same as (7.132) for  $m = 2$ .

**Correction to right level.** The change in contribution of the spin- $\frac{1}{2}$  fermions is caused by the  $(j_L, j_L \pm \frac{3}{2})^{\mp} \oplus 2(1, \frac{5}{2})^{-}$  representations. Their contributions are given by

$$\begin{aligned}
&-\frac{1}{6\pi} \sum_{k=2}^{\infty} \sum_{j_L=k}^{\infty} \left[ -\left(j_L + \frac{3}{2}\right)\left(j_L + \frac{5}{2}\right)(2j_L + 4) + \left(j_L - \frac{3}{2}\right)\left(j_L - \frac{1}{2}\right)(2j_L - 2) \right] \\
&-\frac{1}{6\pi} \sum_{j_L=2}^{\infty} \left[ -\left(j_L + \frac{3}{2}\right)\left(j_L + \frac{5}{2}\right)(2j_L + 4) + \left(j_L - \frac{3}{2}\right)\left(j_L - \frac{1}{2}\right)(2j_L - 2) \right] + \frac{35}{4\pi}, \tag{7.148}
\end{aligned}$$

which, when combined with the other spin- $\frac{1}{2}$  representations (for which the summation goes the same as in the  $m \geq 3$  case), gives

$$\alpha_R^{(1/2)} = -\frac{71}{180\pi} + \frac{11(n_H + n_T)}{360\pi}. \quad (7.149)$$

This is again the same as the contribution (7.139) for  $m = 2$ .

**Correction to  $c_L - c_R$ .** This time, we find the following contribution for the spin- $\frac{1}{2}$  fermions:

$$\begin{aligned} \alpha_{\text{grav}}^{(1/2)} &= -\frac{1}{4 \cdot 48\pi} \left[ 2 \sum_{k=2}^{\infty} \sum_{j_L=k}^{\infty} 6 + 2 \sum_{j_L=2}^{\infty} 6 + 12 - 2 \sum_{k=1}^{\infty} \sum_{j_L=k}^{\infty} 2 \right. \\ &\quad \left. - 2(n_T + n_H) \sum_{k=1}^{\infty} \sum_{j_L=k}^{\infty} 2 \right] \\ &= \frac{1}{144\pi} - \frac{n_T + n_H}{288\pi}, \end{aligned} \quad (7.150)$$

which is the same as (7.144) for  $m = 2$ .

### 7.B.3 Corrections for $m = 1$

The change in summations is again only for  $k > 0$ , but in this case it is both in the spin- $\frac{1}{2}$  sector and in the vector sector.

**Correction to left level.** For the spin- $\frac{1}{2}$  fermions we find

$$\begin{aligned} \alpha_L^{(1/2)} &= -12 \sum_{k=3}^{\infty} \sum_{j_L=\frac{1}{2}k}^{\infty} \frac{1}{8\pi} \left(\frac{1}{2}k\right)^2 - 12 \sum_{j_L=2}^{\infty} \frac{1}{8\pi} - 12 \sum_{j_L=\frac{3}{2}}^{\infty} \frac{1}{8\pi} \left(\frac{1}{2}\right)^2 - \frac{3}{2\pi} \\ &\quad - 10 \frac{1}{8\pi} \left(\frac{1}{2}\right)^2 + 4 \sum_{k=2}^{\infty} \sum_{j_L=\frac{1}{2}k}^{\infty} \frac{1}{8\pi} \left(\frac{1}{2}k\right)^2 + 4 \sum_{j_L=3/2}^{\infty} \frac{1}{8\pi} \left(\frac{1}{2}\right)^2 + 6 \frac{1}{8\pi} \left(\frac{1}{2}\right)^2 \\ &\quad + 4(n_T + n_H) \sum_{k=1}^{\infty} \sum_{j_L=\frac{1}{2}k}^{\infty} \frac{1}{8\pi} \left(\frac{1}{2}k\right)^2 = \frac{121}{960\pi} - \frac{n_H + n_T}{1920\pi}, \end{aligned} \quad (7.151)$$

which is not the same as (7.132) for  $m = 1$ . The vector contribution changes to

$$\begin{aligned} \alpha_L^{(\text{vect})} &= 4(1 - n_T) \sum_{k=2}^{\infty} \sum_{j_L=\frac{1}{2}k}^{\infty} \frac{1}{4\pi} \left(\frac{1}{2}k\right)^2 + 4(1 - n_T) \sum_{j_L=\frac{3}{2}}^{\infty} \frac{1}{4\pi} \cdot \frac{1}{4} \\ &\quad + 4(1 - n_T) \frac{1}{4\pi} \cdot \frac{1}{4} = -\frac{1 - n_T}{960\pi}, \end{aligned} \quad (7.152)$$

$$(7.153)$$

which is exactly the same as (7.133) for  $m = 1$ .

**Correction to right level.** We first calculate the contribution of the spin- $\frac{1}{2}$  fermions. For the  $2 \bigoplus_{j_L=\frac{3}{2}}^{\infty} (j_L, j_L \pm \frac{3}{2})^{\mp}$  representations we find

$$\begin{aligned} & -\frac{1}{6\pi} \left[ \sum_{k=3}^{\infty} \sum_{j_L=k/2}^{\infty} + \sum_{j_L=2}^{\infty} + \sum_{j_L=3/2}^{\infty} \right] \left[ -\left(j_L + \frac{3}{2}\right)\left(j_L + \frac{5}{2}\right)(2j_L + 4) \right. \\ & \quad \left. + \left(j_L - \frac{3}{2}\right)\left(j_L - \frac{1}{2}\right)(2j_L - 2) \right] \\ & = -\frac{4557}{320\pi}. \end{aligned} \quad (7.154)$$

The  $2 \bigoplus_{j_L=1}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm}$  representations give

$$\begin{aligned} & -\frac{1}{6\pi} \left[ \sum_{k=2}^{\infty} \sum_{j_L=k/2}^{\infty} + \sum_{j_L=3/2}^{\infty} \right] \left[ -\left(j_L + \frac{1}{2}\right)\left(j_L + \frac{3}{2}\right)(2j_L + 2) \right. \\ & \quad \left. + \left(j_L - \frac{1}{2}\right)\left(j_L + \frac{1}{2}\right)(2j_L) \right] \\ & = \frac{2951}{2880\pi}. \end{aligned} \quad (7.155)$$

For the  $2 \bigoplus_{j_L=0}^1 (j_L, j_L + \frac{3}{2})^{-} \oplus 2 (\frac{1}{2}, 1)^{+}$  representations we find a contribution

$$-\frac{1}{6\pi} \left( -\frac{5}{2} \cdot \frac{7}{2} \cdot 6 - 2 \cdot 3 \cdot 5 + 1 \cdot 2 \cdot 3 \right) = \frac{51}{4\pi}. \quad (7.156)$$

The contribution of the  $2(n_T + n_H) \bigoplus_{j_L=\frac{1}{2}}^{\infty} (j_L, j_L \pm \frac{1}{2})^{\pm}$  representations stays the same. Adding all the contributions we find

$$\alpha_R^{(1/2)} = -\frac{671}{1440\pi} + (n_H + n_T) \frac{71}{2880\pi}, \quad (7.157)$$

which is the same as (7.139) for  $m = 1$ . The vector contribution changes to

$$\begin{aligned} \alpha_R^{(\text{vect})} &= \frac{1}{6\pi} (1 - n_T) \sum_{k=2}^{\infty} \sum_{j_L=\frac{1}{2}k}^{\infty} \left[ -\left(j_L + 1\right)\left(j_L + 2\right)(2j_L + 3) \right. \\ & \quad \left. + \left(j_L - 1\right)j_L(2j_L - 1) \right] \\ & \quad + \frac{1}{6\pi} (1 - n_T) \sum_{j_L=\frac{3}{2}}^{\infty} \left[ -\left(j_L + 1\right)\left(j_L + 2\right)(2j_L + 3) + \left(j_L - 1\right)j_L(2j_L - 1) \right] \\ &= (1 - n_T) \frac{221}{1440\pi}, \end{aligned} \quad (7.158)$$

which is the same as (7.140) for  $m = 1$ .

**Correction to  $c_L - c_R$ .** Now we find for the spin- $\frac{1}{2}$  fields

$$\begin{aligned}
 \alpha_{\text{grav}}^{(1/2)} &= \frac{1}{4 \cdot 48\pi} \left[ -2 \sum_{k=3}^{\infty} \sum_{j_L=k/2}^{\infty} 6 - 2 \sum_{j_L=2}^{\infty} 6 - 2 \sum_{j_L=3/2}^{\infty} 6 - 12 - 10 \right. \\
 &\quad \left. + 2 \sum_{k=2}^{\infty} \sum_{j_L=k/2}^{\infty} 2 + 2 \sum_{j_L=3/2}^{\infty} 2 + 6 - 2(n_T + n_H) \sum_{k=1}^{\infty} \sum_{j_L=k/2}^{\infty} 2 \right] \\
 &= \frac{17}{576\pi} - \frac{5}{1152\pi} (n_T + n_H) .
 \end{aligned} \tag{7.159}$$

For the vectors we get

$$\alpha_{\text{grav}}^{(\text{vect})} = -\frac{1}{12\pi} (1 - n_T) \left[ \sum_{k=2}^{\infty} \sum_{j_L=\frac{1}{2}k}^{\infty} 1 + \sum_{j_L=3/2}^{\infty} 1 + 1 \right] = \frac{5}{288\pi} - \frac{5}{288\pi} n_T ,$$

(7.160)

which is the same as (7.145) for  $m = 1$ .



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# Summary

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We conclude this thesis by giving a summary of the content discussed. Since the thesis is split into two parts, we will proceed with a summary of both parts individually.

The focus of part I of this thesis was the study of dimensional reductions on Calabi–Yau manifolds with higher derivatives. Since this is very challenging from a technical point of view, we developed a computer code which allows us to simplify the results of the dimensional reduction as much as possible. In order to have a correct starting point, we reviewed and found new solutions of supergravity theories with eight-derivative corrections in chapter 2. These solutions are the analogues of the classical Calabi–Yau backgrounds of the form  $\mathbb{R}^{1,d-1-2n} \times \text{CY}_n$  once higher derivatives are included. Obtaining these background solutions is crucial to have the correct starting point to compactify a theory with higher derivatives. If the dimensional reduction was performed on the uncorrected Calabi–Yau background, important contributions to the low-energy effective action would be missed.

In chapter 3 the focus was on type IIA supergravity and eleven-dimensional supergravity, the low-energy limit of M-theory. We performed the dimensional reduction of the latter two supergravity theories on the backgrounds discussed in chapter 2. We computed two- and four-derivative couplings of the Kähler deformations in the lower dimensional effective action. Computer generated total derivative and Schouten identities allowed us to write all couplings in terms of a single building block. While the M-theory reduction to five dimensions on  $\text{CY}_3$  turned out to be compatible with  $\mathcal{N} = 2$  supersymmetry, we found that the results from the reduction of type IIA to four dimensions are in tension with

supersymmetry. This led us to propose a new set of eight-derivative terms at tree-level in the low-energy effective action of the type IIA superstring. This proposal was confirmed in [42] by a string amplitude computation.

Subsequently we investigated in chapter 4 four-derivative couplings for the case of a single Kähler modulus, which is interpreted as the overall volume of the Calabi–Yau manifold on which the theory is compactified. The restriction to the one modulus case allowed us to overcome some technical and conceptual issues which appear in the case of generic  $h^{1,1}$ . In this particular case, we were able to determine all four derivative terms of the volume modulus  $\mathcal{V}$  for type IIA, type IIB and eleven-dimensional supergravity with eight-derivative corrections. This allowed us to make contact with a recently proposed scenario of moduli stabilization. The latter scenario uses an action, which obtains a correction to the classical scalar potential. This correction to the potential is by supersymmetry related to a certain four-derivative coupling in the action. We are, under certain assumptions, able to extract this previously unknown four-derivative coupling, which fixes the correction to the scalar potential.

Chapter 5 is devoted to the study of M-theory compactifications on Calabi–Yau fourfolds to three dimensions. These are of particular relevance, given that they can in principle be lifted to four-dimensional  $\mathcal{N} = 1$  compactifications of F-theory. We again focused on the volume modulus and aimed to find a correction to the classical Kähler potential in three dimensions. The corrections we find descend from the dimensional reduction of eight-derivative terms in eleven dimensions. We demonstrate that the correction we find is compatible with  $\mathcal{N} = 2$  supersymmetry in three dimensions and show that it breaks the no-scale property of the classical Kähler potential. The shape of our correction is surprising at first and has been overlooked in the past. We provided arguments why this correction may admit an interpretation as a one-loop correction to a Wilsonian effective action in three dimensions.

In part II of the thesis we studied black holes in type IIB and F-theory which can be constructed with D3-branes. Chapter 6 dealt with compactifications of type IIB string theory on a K3 manifold. The four- and five-dimensional black holes are formed by wrapping a D3-brane over a Riemann surface inside the K3 surface and an additional circle. As a novel feature we considered non-trivial transverse spaces which the D3-brane probes. Some of these spaces induce an additional topological charge on which the black hole entropy depends. We computed the quantities, which determine the black hole entropy from a microscopic and a macroscopic point of view. The inclusion of quantum corrections in the macroscopic description of the black holes turned out to be crucial in order to obtain the desired results.

Black holes in F-theory compactifications on smooth elliptic fibrations were the main topic of chapter 7. The black holes are again constructed by wrapping a D3-brane on a Riemann surface, which is this time embedded in a Kähler base of an elliptic fibration. The main focus of this chapter was on particular types of transverse spaces, the Taub-NUT spaces with NUT charge  $m$ . Upon dualizing to M-theory, the microscopic description of these black holes is conveniently given in terms of an M5-brane wrapped on a particular four-cycle in the underlying Calabi-Yau threefold. On the macroscopic side of the black holes we again had to include quantum corrections to obtain a matching with the microscopic results. Lastly, we studied additional types of transverse spaces and singular elliptic fibrations. This extends and generalizes the results obtained in the previous sections. In particular, we showed that the expressions for the central charges and levels which we obtained for smooth elliptic fibrations remain valid for singular elliptic fibrations. Furthermore, we computed anomaly coefficients of additional global symmetries acting on the worldvolume theory of the D3-brane when the elliptic fibration is singular.



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# Samenvatting

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We sluiten dit proefschrift af met een samenvatting van de besproken inhoud. Aangezien het proefschrift in twee delen is opgesplitst, hebben we voor beide delen een afzonderlijke samenvatting.

De focus van deel I van dit proefschrift was de studie van dimensionale reducties op Calabi-Yau variëteiten. Omdat dit technisch zeer uitdagend is, hebben we een computercode ontwikkeld waarmee we de resultaten van de dimensionale reductie zoveel mogelijk kunnen vereenvoudigen. Om een juist uitgangspunt te hebben, besproken en vonden we nieuwe oplossingen van supergravitatie theorieën met acht-afgeleide correcties in hoofdstuk 2. Deze oplossingen zijn de analogen van de klassieke Calabi-Yau achtergronden van de vorm  $\mathbb{R}^{1,d-1-2n} \times \text{CY}_n$  zodra hogere afgeleiden zijn meegenomen. Het verkrijgen van deze achtergrondoplossingen is cruciaal om het juiste startpunt te hebben om een theorie met hogere afgeleiden te compactificeren. Als de dimensionale reductie zou worden uitgevoerd op de ongecorrigeerde Calabi-Yau achtergrond, zouden belangrijke bijdragen aan lage-energie effectieve actie ontbreken.

In hoofdstuk 3 lag de focus op type IIA supergravitatie en elf-dimensionale supergravitatie, de lage-energielimit van de M-theorie. We hebben de dimensionale reductie van deze twee supergravitatie theorieën uitgevoerd op de achtergronden die worden besproken in hoofdstuk 2. We berekenden twee- en vier-afgeleide koppelingen van de Kähler vervormingen in de lager dimensionale effectieve actie. Door de computer gegenereerde totale afgeleide en Schouten identiteiten lieten ons toe om alle koppelingen te schrijven in termen van een enkele bouwsteen. Terwijl de M-theorie reductie tot vijf dimensies op  $\text{CY}_3$  compatibel bleek te zijn met  $\mathcal{N} = 2$  supersymmetrie, vonden we dat de resultaten van de reductie van

type IIA tot vier dimensies in spanning waren met supersymmetrie. Dit bracht ons ertoe een nieuwe set van laagste orde acht-afgeleide termen voor te stellen in de lage-energie effectieve actie van type IIA supersnaar. Dit voorstel werd bevestigd in [42] door een string amplitude berekening.

In het volgende hoofdstuk 4 onderzochten we vier-afgeleide koppelingen voor het geval van een enkele Kähler modulus, die geïnterpreteerd wordt als het totale volume van de Calabi-Yau variëteit waarop de theorie is gecomactificeerd. De beperking tot het één-modulus geval stelde ons in staat om enkele technische en conceptuele problemen op te lossen die normaliter voorkomen in het geval van arbitraire  $h^{1,1}$ . In dit specifieke geval konden we alle vier-afgeleide termen van het volumemodulus  $\mathcal{V}$  bepalen voor type IIA, type IIB en elf-dimensionale superzwaartekracht met acht-afgeleide correcties. Hierdoor konden we een connectie maken met een recent voorgesteld scenario van moduli stabilisatie. Dit scenario gebruikt een actie, die een correctie op de klassieke scalaire potentiaal krijgt. Deze correctie op de potentiaal is door supersymmetrie gerelateerd aan een bepaalde vier-afgeleide koppeling in de actie. We zijn, onder bepaalde veronderstellingen, in staat om deze voorheen onbekende vier-afgeleide koppeling af te leiden, wat de correctie op het scalaire potentiaal vastlegt.

Hoofdstuk 5 is gewijd aan het bestuderen van M-theorie compactificaties op Calabi-Yau viervariëteiten naar drie dimensies. Deze zijn van bijzonder belang, aangezien ze in principe kunnen worden opgetild tot vierdimensionale  $\mathcal{N} = 1$  compactificaties van F-theorie. We richten ons opnieuw op de volumemodulus en streven naar een correctie voor het klassieke Kähler potentiaal in drie dimensies. De correcties die we vinden, stammen af van de dimensionale reductie van acht-afgeleide termen in elf dimensies. We demonstreren dat de correctie die we vinden compatibel is met  $\mathcal{N} = 2$  supersymmetrie in drie dimensies en we laten zien dat het de niet-schaal eigenschap van het klassieke Kähler potentiaal breekt. De vorm van onze correctie is in eerste instantie verrassend en werd in het verleden over het hoofd gezien. We geven argumenten waarom deze correctie wellicht geïnterpreteerd kan worden als een lus-correctie tot een Wilsoniaanse effectieve actie in drie dimensies.

In deel II van het proefschrift hebben we zwarte gaten bestudeerd in type IIB en F-theorie die geconstrueerd kunnen worden met D3-branen. Hoofdstuk 6 behandelde compactificaties van type IIB snaartheorie op een K3 variëteit. De vier- en vijf-dimensionale zwarte gaten worden gevormd door een D3-braan over een Riemann-oppervlak binnen het K3-oppervlak en een extra cirkel te wikkelen. Als nieuwe eigenschap hebben we niet-triviale transversale ruimtes overwogen, welke het D3-braan doordringt. Sommige van deze ruimtes veroorzaken een extra

topologische lading die de entropie van het zwarte gat beïnvloedt. We berekenden de grootheden, die de entropie van het zwarte gat microscopisch en macroscopisch bepalen. Het meenemen van kwantumcorrecties in de macroscopische beschrijving van de zwarte gaten bleek van cruciaal belang om de gewenste resultaten te verkrijgen.

Zwarte gaten in compactificaties van F-theorie op gladde elliptische fibraties waren het onderwerp van hoofdstuk 7. De zwarte gaten worden opnieuw geconstrueerd door een D3-braan op een Riemann-oppervlak te wikkelen, dat dit keer is ingebed in een Kähler-basis van een elliptische fibratie. De belangrijkste focus van dit hoofdstuk was op specifieke soorten transversale ruimtes, de Taub-NUT-ruimtes met NUT lading  $m$ . Bij dualisatie naar de M-theorie wordt de microscopische beschrijving van deze zwarte gaten handig beschreven in termen van een M5-braan gewikkeld op een bepaalde vier-cyclus in de onderliggende Calabi-Yau drievariëteit. Aan de macroscopische kant van de beschrijving van de zwarte gaten moesten we opnieuw kwantumcorrecties opnemen om een overeenkomst met de microscopische resultaten te verkrijgen. Tenslotte bestudeerden we extra soorten transversale ruimtes en singuliere elliptische fibraties. Dit generaliseert de resultaten verkregen in de vorige secties. In het bijzonder lieten we zien dat de uitdrukkingen voor de centrale ladingen en niveaus die we verkregen voor gladde elliptische fibraties geldig blijven voor enkelvoudige elliptische fibraties. Verder berekenden we anomalie-coëfficiënten van aanvullende globale symmetrieën die werken op de wereldvolume theorie van het D3-braan als de elliptische fibratie singulier is.





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## Appendix A

# Differential geometry and Hodge theory

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### Definitions and conventions

We denote the eleven-dimensional spacetime indices by capital Latin letters  $M, N = 0, \dots, 10$ , the external ones by  $\mu, \nu = 0, 1, 2, (3), (4)$ , and the internal complex ones by  $i, j, k, l = 1, \dots, 3, (4)$  and  $\bar{i}, \bar{j}, \bar{k}, \bar{l} = 1, \dots, 3, (4)$ , if we consider threefold or fourfold compactifications, respectively. The metric signature of the eleven-dimensional space is  $(-, +, \dots, +)$ . Our conventions for the totally antisymmetric tensor in Lorentzian signature in an orthonormal frame are  $\epsilon_{012\dots 9} = \epsilon_{01234} = +1$ . The epsilon tensor in  $d$  dimensions then satisfies

$$\epsilon^{R_1 \dots R_p N_1 \dots N_{d-p}} \epsilon_{R_1 \dots R_p M_1 \dots M_{d-p}} = (-1)^s (d-p)! p! \delta^{N_1}_{[M_1} \dots \delta^{N_{d-p}}_{M_{d-p}]}, \quad (\text{A.1})$$

where  $s = 0$  if the metric has Euclidean signature and  $s = 1$  for a Lorentzian metric.

We adopt the following conventions for the Christoffel symbols and Riemann tensor

$$\begin{aligned} \Gamma^R_{MN} &= \frac{1}{2} g^{RS} (\partial_M g_{NS} + \partial_N g_{MS} - \partial_S g_{MN}), \\ R^M_{NRS} &= \partial_R \Gamma^M_{SN} - \partial_S \Gamma^M_{RN} + \Gamma^M_{RT} \Gamma^T_{SN} - \Gamma^M_{ST} \Gamma^T_{RN}, \end{aligned}$$

$$\begin{aligned} R_{MN} &= R^R{}_{MRN}, \\ R &= R_{MNg}{}^{MN}, \end{aligned} \quad (\text{A.2})$$

with equivalent definitions on the internal and external spaces. Written in components, the first and second Bianchi identity are

$$\begin{aligned} R^O{}_{PMN} + R^O{}_{MNP} + R^O{}_{NPM} &= 0 \\ \nabla_L R^O{}_{PMN} + \nabla_M R^O{}_{PNL} + \nabla_N R^O{}_{PLM} &= 0 \quad . \end{aligned} \quad (\text{A.3})$$

Differential  $p$ -forms are expanded in a basis of differential one-forms as

$$\Lambda = \frac{1}{p!} \Lambda_{M_1 \dots M_p} dx^{M_1} \wedge \dots \wedge dx^{M_p} \quad . \quad (\text{A.4})$$

In the following we will use the shorthand notation

$$d(\Lambda^{(1)}, \dots, \Lambda^{(I)}) = d\Lambda^{(1)} \wedge \dots \wedge d\Lambda^{(I)} \quad (\text{A.5})$$

in some places, where expressions become particularly lengthy. The wedge product between a  $p$ -form  $\Lambda^{(p)}$  and a  $q$ -form  $\Lambda^{(q)}$  is given by

$$(\Lambda^{(p)} \wedge \Lambda^{(q)})_{M_1 \dots M_{p+q}} = \frac{(p+q)!}{p!q!} \Lambda_{[M_1 \dots M_p}^{(p)} \Lambda_{M_1 \dots M_q]}^{(q)} \quad . \quad (\text{A.6})$$

Furthermore, the exterior derivative on a  $p$ -form  $\Lambda$  reads

$$(d\Lambda)_{NM_1 \dots M_p} = (p+1) \partial_{[N} \Lambda_{M_1 \dots M_p]} \quad , \quad (\text{A.7})$$

while the Hodge star of  $p$ -form  $\Lambda$  in  $d$  real coordinates is given by

$$(*_d \Lambda)_{N_1 \dots N_{d-p}} = \frac{1}{p!} \Lambda^{M_1 \dots M_p} \epsilon_{M_1 \dots M_p N_1 \dots N_{d-p}} \quad . \quad (\text{A.8})$$

Moreover, the identity

$$\Lambda^{(1)} \wedge * \Lambda^{(2)} = \frac{1}{p!} \Lambda_{M_1 \dots M_p}^{(1)} \Lambda^{(2) M_1 \dots M_p} * 1 \quad (\text{A.9})$$

holds for two arbitrary  $p$ -forms  $\Lambda^{(1)}$  and  $\Lambda^{(2)}$ .

Let us specify in more detail our conventions regarding complex coordinates in the internal space. For a complex Hermitian manifold  $\mathcal{M}$  with complex dimension  $n$  the complex coordinates  $z^1, \dots, z^n$  and the underlying real coordinates  $\xi^1, \dots, \xi^{2n}$  are related by

$$(z^1, \dots, z^n) = \left( \frac{1}{\sqrt{2}} (\xi^1 + i\xi^2), \dots, \frac{1}{\sqrt{2}} (\xi^{2n-1} + i\xi^{2n}) \right) \quad . \quad (\text{A.10})$$

Using these conventions one finds

$$\sqrt{g} d\xi^1 \wedge \dots \wedge d\xi^{2n} = \sqrt{g} (-1)^{\frac{(n-1)n}{2}} i^n d(z^1, \dots, z^n, \bar{z}^1, \dots, \bar{z}^n) = \frac{1}{n!} J^n, \quad (\text{A.11})$$

with  $g$  the determinant of the metric in real coordinates and  $\sqrt{\det g_{mn}} = \det g_{i\bar{j}}$ . Note that we used in (A.11) the notation (A.5). The Kähler form is given by

$$J = i g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}. \quad (\text{A.12})$$

Let  $\omega_{p,q}$  be a  $(p, q)$ -form, then its Hodge dual is the  $(n - q, n - p)$  form

$$\begin{aligned} *\omega_{p,q} &= \frac{(-1)^{\frac{n(n-1)}{2}} i^n (-1)^{pn}}{p!q!(n-p)!(n-q)!} \omega_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} \varepsilon^{i_1 \dots i_p}_{\bar{k}_1 \dots \bar{k}_{n-p}} \\ &\quad \times \varepsilon^{\bar{j}_1 \dots \bar{j}_q}_{l_1 \dots l_{n-q}} d(z^{l_1}, \dots, z^{l_{n-q}}, \bar{z}^{\bar{k}_1}, \dots, \bar{z}^{\bar{k}_{n-p}}). \end{aligned} \quad (\text{A.13})$$

Finally, let us record our conventions regarding Chern forms. To begin with, we define the curvature two-form for Hermitian manifolds to be

$$\mathcal{R}^i_j = R^i_{j\bar{k}\bar{l}} d(z^k, \bar{z}^{\bar{l}}) \quad (\text{A.14})$$

and we set

$$\begin{aligned} \text{tr } \mathcal{R} &= R^k_{k\bar{i}\bar{j}} d(z^i, \bar{z}^{\bar{j}}), \\ \text{tr } \mathcal{R}^2 &= R^k_{l\bar{i}\bar{j}} R^l_{k\bar{i}_1\bar{j}_1} d(z^i, \bar{z}^{\bar{j}}, z^{i_1}, \bar{z}^{\bar{j}_1}), \\ \text{tr } \mathcal{R}^3 &= R^k_{l\bar{i}\bar{j}} R^l_{l_1\bar{i}_1\bar{j}_1} R^{l_1}_{k\bar{i}_2\bar{j}_2} d(z^i, \bar{z}^{\bar{j}}, z^{i_1}, \bar{z}^{\bar{j}_1}, z^{i_2}, \bar{z}^{\bar{j}_2}). \end{aligned} \quad (\text{A.15})$$

The Chern forms can then be expressed in terms of the curvature two-form as

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \frac{i}{2\pi} \text{tr } \mathcal{R}, \\ c_2 &= \frac{1}{(2\pi)^2} \frac{1}{2} (\text{tr } \mathcal{R}^2 - (\text{tr } \mathcal{R})^2), \\ c_3 &= \frac{1}{3} c_1 \wedge c_2 + \frac{1}{(2\pi)^2} \frac{1}{3} c_1 \wedge \text{tr } \mathcal{R}^2 - \frac{1}{(2\pi)^3} \frac{i}{3} \text{tr } \mathcal{R}^3, \\ c_4 &= \frac{1}{24} \left( c_1^4 - \frac{6}{(2\pi)^2} c_1^2 \wedge \text{tr } \mathcal{R}^2 - \frac{8i}{(2\pi)^3} c_1 \wedge \text{tr } \mathcal{R}^3 \right) + \frac{1}{(2\pi)^4} \frac{1}{8} ((\text{tr } \mathcal{R}^2)^2 - 2 \text{tr } \mathcal{R}^4). \end{aligned} \quad (\text{A.16})$$

The Chern forms of a complex  $n$ -dimensional Calabi-Yau manifold  $\text{CY}_n$  reduce to

$$c_3(\text{CY}_{n \geq 3}) = -\frac{1}{(2\pi)^3} \frac{i}{3} \text{tr } \mathcal{R}^3, \quad (\text{A.17})$$

$$c_4(\text{CY}_{n \geq 4}) = \frac{1}{(2\pi)^4} \frac{1}{8} ((\text{tr } \mathcal{R}^2)^2 - 2 \text{tr } \mathcal{R}^4). \quad (\text{A.18})$$

**Explicit expressions for building blocks.** In this part of the appendix we will give some explicit index structures of the building blocks, which are relevant in the dimensional reductions with higher-derivative corrections.

**second Chern form:**

$$\begin{aligned}
 (2\pi)^2 c_2 &= \frac{1}{2} \text{tr } \mathcal{R} \wedge \mathcal{R} = \frac{1}{2} R^k_{l\bar{i}\bar{j}} R^l_{k i_1 \bar{j}_1} d(z^r, \bar{z}^{\bar{s}}, z^{r_1}, \bar{z}^{\bar{s}_1}) \\
 &= \frac{1}{2! 2!} \underbrace{\left( -2 R^{\bar{l}}_{\bar{k} i \bar{j}} R^{\bar{k}}_{\bar{l} i_1 \bar{j}_1} \right)}_{=(2\pi)^2 (c_2)_{i i_1 \bar{j} \bar{j}_1}} d(z^i, z^{i_1}, \bar{z}^{\bar{j}}, \bar{z}^{\bar{j}_1}) \\
 &\Rightarrow \boxed{(2\pi)^2 (c_2)_{i i_1 \bar{j} \bar{j}_1} = -2 R^{\bar{l}}_{\bar{k} i \bar{j}} R^{\bar{k}}_{\bar{l} i_1 \bar{j}_1}}
 \end{aligned}$$

**$c_2 \wedge J$  :**

$$\begin{aligned}
 (2\pi)^2 c_2 \wedge J &= \frac{1}{2} \text{tr } (\mathcal{R} \wedge \mathcal{R}) \wedge J \\
 &= \frac{1}{2} R^k_{l i_1 \bar{j}_1} R^l_{k i_2 \bar{j}_2} i g_{i_3 \bar{j}_3} d(z^{i_1}, \bar{z}^{\bar{j}_1}, z^{i_2}, \bar{z}^{\bar{j}_2}, z^{i_3}, \bar{z}^{\bar{j}_3}) \\
 &= \frac{1}{3! 3!} \underbrace{\left( -18 i R^{\bar{l}}_{\bar{k} i_1 \bar{j}_1} R^{\bar{k}}_{\bar{l} i_2 \bar{j}_2} g_{i_3 \bar{j}_3} \right)}_{=(2\pi)^2 (c_2 \wedge J)_{i_1 i_2 i_3 \bar{j}_1 \bar{j}_2 \bar{j}_3}} d(z^{i_1}, z^{i_2}, z^{i_3}, \bar{z}^{\bar{j}_1}, \bar{z}^{\bar{j}_2}, \bar{z}^{\bar{j}_3}) \\
 &\Rightarrow \boxed{(2\pi)^2 (c_2 \wedge J)_{i_1 i_2 i_3 \bar{j}_1 \bar{j}_2 \bar{j}_3} = -18 i R^{\bar{l}}_{\bar{k} i_1 \bar{j}_1} R^{\bar{k}}_{\bar{l} i_2 \bar{j}_2} g_{i_3 \bar{j}_3}} \\
 &\Rightarrow \boxed{(2\pi)^2 *_6 (c_2 \wedge J) = \frac{1}{2} R^{\bar{j}_1 \bar{j}_2 i_3} R_{i_1 \bar{j} i_3 \bar{j}_2}}
 \end{aligned}$$

**$c_2 \wedge \omega_a$  :**

$$\begin{aligned}
 (2\pi)^2 c_2 \wedge \omega_a &= \frac{1}{2} \text{tr } (\mathcal{R} \wedge \mathcal{R}) \wedge \omega_a = \frac{1}{2} R^k_{l i_1 \bar{j}_1} R^l_{k i_2 \bar{j}_2} \omega_{a i_3 \bar{j}_3} d(z^{i_1}, \bar{z}^{\bar{j}_1}, z^{i_2}, \bar{z}^{\bar{j}_2}, z^{i_3}, \bar{z}^{\bar{j}_3}) \\
 &= \frac{1}{3! 3!} \underbrace{\left( -\frac{3! 3!}{2} R^{\bar{l}}_{\bar{k} i_1 \bar{j}_1} R^{\bar{k}}_{\bar{l} i_2 \bar{j}_2} \omega_{a i_3 \bar{j}_3} \right)}_{=(2\pi)^2 (c_2 \wedge \omega_a)_{i_1 i_2 i_3 \bar{j}_1 \bar{j}_2 \bar{j}_3}} d(z^{i_1}, z^{i_2}, z^{i_3}, \bar{z}^{\bar{j}_1}, \bar{z}^{\bar{j}_2}, \bar{z}^{\bar{j}_3}) \\
 &\Rightarrow \boxed{(2\pi)^2 (c_2 \wedge \omega_a)_{i_1 i_2 i_3 \bar{j}_1 \bar{j}_2 \bar{j}_3} = -18 R^{\bar{k}}_{\bar{l} i_1 \bar{j}_1} R^{\bar{l}}_{\bar{k} i_2 \bar{j}_2} \omega_{a i_3 \bar{j}_3}} \\
 &\Rightarrow \boxed{(2\pi)^2 *_6 (c_2 \wedge \omega_a) = i \omega_a^{\bar{j}_1 i_1} R_{i_1}^{i_2 \bar{j}_3 i_4} R_{i_2 \bar{j}_1 i_4 \bar{j}_3} - \frac{i}{2} \omega_a^{\bar{j}_3} R^{\bar{j}_1 i_1 \bar{j}_2 i_3} R_{i_1 \bar{j}_1 i_3 \bar{j}_2}}
 \end{aligned}$$

**third Chern form:**

$$(2\pi)^3 c_3 = -\frac{i}{3} \text{tr } \mathcal{R} \wedge \mathcal{R} \wedge \mathcal{R} = -\frac{i}{3} R^k_{l i_1 \bar{j}_1} R^l_{l_1 i_2 \bar{j}_2} R^{l_1}_{k i_3 \bar{j}_3} d(z^{i_1}, \bar{z}^{\bar{j}_1}, z^{i_2}, \bar{z}^{\bar{j}_2}, z^{i_3}, \bar{z}^{\bar{j}_3})$$

$$\begin{aligned}
&= \frac{i}{3} R^{\bar{l}}_{\bar{k}i_1\bar{j}_1} R^{\bar{k}}_{\bar{k}_1i_2\bar{j}_2} R^{\bar{k}_1}_{\bar{l}i_3\bar{j}_3} d(z^{i_1}, \bar{z}^{\bar{j}_1}, z^{i_2}, \bar{z}^{\bar{j}_2}, z^{i_3}, \bar{z}^{\bar{j}_3}) \\
&= \frac{1}{3!3!} \underbrace{\left( -\frac{i}{3} 3! 3! R^{\bar{l}}_{\bar{k}i_1\bar{j}_1} R^{\bar{k}}_{\bar{k}_1i_2\bar{j}_2} R^{\bar{k}_1}_{\bar{l}i_3\bar{j}_3} \right)}_{=(2\pi)^3 (c_3)_{i_1 i_2 i_3 \bar{j}_1 \bar{j}_2 \bar{j}_3}} d(z^{i_1}, z^{i_2}, z^{i_3}, \bar{z}^{\bar{j}_1}, \bar{z}^{\bar{j}_2}, \bar{z}^{\bar{j}_3}) \\
&\Rightarrow \boxed{(2\pi)^3 (c_3)_{i_1 i_2 i_3 \bar{j}_1 \bar{j}_2 \bar{j}_3} = -12i R^{\bar{l}}_{\bar{k}i_1\bar{j}_1} R^{\bar{k}}_{\bar{k}_1i_2\bar{j}_2} R^{\bar{k}_1}_{\bar{l}i_3\bar{j}_3}}
\end{aligned}$$

$$\begin{aligned}
(2\pi)^3 *_6 c_3 &= -\frac{i}{3!3!} \epsilon^{i_1 i_2 i_3} \epsilon^{\bar{j}_1 \bar{j}_2 \bar{j}_3} (c_3)_{i_1 i_2 i_3 \bar{j}_1 \bar{j}_2 \bar{j}_3} \\
&= -\frac{1}{3} \epsilon^{i_1 i_2 i_3} \epsilon^{\bar{j}_1 \bar{j}_2 \bar{j}_3} R^{\bar{l}}_{\bar{k}i_1\bar{j}_1} R^{\bar{k}}_{\bar{k}_1i_2\bar{j}_2} R^{\bar{k}_1}_{\bar{l}i_3\bar{j}_3} \\
&\Rightarrow \boxed{(2\pi)^3 *_6 c_3 = -\frac{1}{3} R^{\bar{j}i_1\bar{j}_2i_3} R_{i_1\bar{j}}^{\bar{j}_4i_5} R_{i_3\bar{j}_2i_5\bar{j}_4} - \frac{1}{3} R^{\bar{j}i_1\bar{j}_2i_3} R_{i_1i_3}^{\bar{i}_4i_5} R_{i_4\bar{j}i_5\bar{j}_2}}
\end{aligned}$$

$\mathcal{K}_{abc}, \mathcal{K}_{ab}, \mathcal{K}_a :$

$$\mathcal{K}_{abc}^{(0)} = \int_{\text{CY}_3} \omega_a \wedge \omega_b \wedge \omega_c \quad (\text{A.19})$$

$$\mathcal{K}_{ab}^{(0)} = \mathcal{K}_{abc}^{(0)} v_0^c = \int_{\text{CY}_3} \left( \omega_{a i \bar{j}} \omega_b^{\bar{j} i} - \omega_{a i}^i \omega_{b j}^j \right) *_6^{(0)} 1 \quad (\text{A.20})$$

$$\mathcal{K}_a^{(0)} = \frac{1}{2} \mathcal{K}_{abc}^{(0)} v_0^b v_0^c = -i \int_{\text{CY}_3} \omega_{a i}^i *_6^{(0)} 1 \quad (\text{A.21})$$



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## Appendix B

# Eight-derivative terms in ten and eleven dimensions

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**Higher derivatives in M-theory.** The classical eleven-dimensional supergravity action gets corrected by different contributions which should be discussed in the following. The most prominent higher curvature term in M-theory is the sector containing four Riemann tensors. These involve two different structures namely

$$S_{\hat{R}^4} = \frac{1}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \left( \hat{t}_8 \hat{t}_8 - \frac{1}{24} \epsilon_{11} \epsilon_{11} \right) \hat{R}^4 \hat{*} 1. \quad (\text{B.1})$$

In (B.1) the two quantities  $\hat{t}_8 \hat{t}_8 \hat{R}^4$  and  $\epsilon_{11} \epsilon_{11} \hat{R}^4$  have the index representation

$$\hat{t}_8 \hat{t}_8 \hat{R}^4 = t_8^{M_1 \dots M_8} t_8^{N_1 \dots N_8} \hat{R}^{N_1 N_2}_{M_1 M_2} \dots \hat{R}^{N_7 N_8}_{M_7 M_8} \quad (\text{B.2})$$

$$\epsilon_{11} \epsilon_{11} \hat{R}^4 = \epsilon_{11}^{R_1 R_2 R_3 M_1 \dots M_8} \epsilon_{11}^{R_4 R_5 R_6 N_1 \dots N_8} \hat{R}^{N_1 N_2}_{M_1 M_2} \dots \hat{R}^{N_7 N_8}_{M_7 M_8}. \quad (\text{B.3})$$

The tensor  $\hat{t}_8$  is defined in terms of the eleven-dimensional metric as

$$\begin{aligned}
\hat{t}_8^{M_1 \cdots M_8} = \frac{1}{16} \Big[ & -2(\hat{g}^{M_1 M_3} \hat{g}^{M_2 M_4} \hat{g}^{M_5 M_7} \hat{g}^{M_6 M_8} + \hat{g}^{M_1 M_5} \hat{g}^{M_2 M_6} \hat{g}^{M_3 M_7} \hat{g}^{M_4 M_8} \\
& + \hat{g}^{M_1 M_7} \hat{g}^{M_2 M_8} \hat{g}^{M_3 M_5} \hat{g}^{M_4 M_6}) \\
& + 8(\hat{g}^{M_2 M_3} \hat{g}^{M_4 M_5} \hat{g}^{M_6 M_7} \hat{g}^{M_8 M_1} + \hat{g}^{M_2 M_5} \hat{g}^{M_6 M_3} \hat{g}^{M_4 M_7} \hat{g}^{M_8 M_1} \\
& + \hat{g}^{M_2 M_5} \hat{g}^{M_6 M_7} \hat{g}^{M_8 M_3} \hat{g}^{M_4 M_1}) \\
& - (M_1 \leftrightarrow M_2) - (M_3 \leftrightarrow M_4) - (M_5 \leftrightarrow M_6) - (M_7 \leftrightarrow M_8) \Big].
\end{aligned} \tag{B.4}$$

These  $\mathcal{R}^4$ -terms are furthermore supplemented by another term quartic in the Riemann tensor. This term however also comprises a three form  $\hat{C}_3$ . This piece of the higher-derivative action then has the form

$$S_{C_3 X_8} = -\frac{3^2 2^{13}}{2\kappa_{11}^2} \int_{\mathcal{M}_{11}} \hat{C}_3 \wedge \hat{X}_8 \tag{B.5}$$

where eight form  $\hat{X}_8$  is defined as

$$\hat{X}_8 = \frac{1}{192} \left[ \text{tr } \hat{\mathcal{R}}^4 - \frac{1}{4} \left( \text{tr } \hat{\mathcal{R}}^2 \right)^2 \right] \tag{B.6}$$

which is in terms of the (real) curvature two form

$$\hat{\mathcal{R}}^M{}_N = \frac{1}{2} \hat{R}^M{}_{NN_1 N_2} dx^{N_1} \wedge dx^{N_2}. \tag{B.7}$$

In addition to these quartic Riemann tensor terms it was conjectured in [29] that the complete  $\hat{G}_4$  dependence at  $\mathcal{O}(\hat{G}_4^2)$  is captured by introducing

$$\begin{aligned}
\hat{t}_8 \hat{t}_8 \hat{G}_4^2 \hat{R}^3 = \hat{t}_8^{M_1 \cdots M_8} \hat{t}_8^{N_1 \cdots N_8} \hat{G}_4^{N_1}{}_{M_1 R_1 R_2} \hat{G}_4^{N_2}{}_{M_2}{}^{R_1 R_2} \\
\times \hat{R}^{N_3 N_4}{}_{M_3 M_4} \cdots \hat{R}^{N_7 N_8}{}_{M_7 M_8},
\end{aligned} \tag{B.8}$$

$$\begin{aligned}
\epsilon_{11} \epsilon_{11} \hat{G}_4^2 \hat{R}^3 = \epsilon_{11}^{RM_1 \cdots M_{10}} \epsilon_{11}^{RN_1 \cdots N_{10}} \hat{G}_4^{N_1 N_2}{}_{M_1 M_2} \hat{G}_4^{N_3 N_4}{}_{M_3 M_4} \\
\times \hat{R}^{N_5 N_6}{}_{M_5 M_6} \cdots \hat{R}^{N_9 N_{10}}{}_{M_9 M_{10}}.
\end{aligned} \tag{B.9}$$

Furthermore, the scalar  $\hat{Z}$  is defined as

$$\begin{aligned}
\hat{Z} = \frac{1}{12} \big( \hat{R}_{M_1 M_2}{}^{M_3 M_4} \hat{R}_{M_3 M_4}{}^{M_5 M_6} \hat{R}_{M_5 M_6}{}^{M_1 M_2} \\
- 2 \hat{R}_{M_1}{}^{M_2}{}_{M_3}{}^{M_4} \hat{R}_{M_2}{}^{M_5}{}_{M_4}{}^{M_6} \hat{R}_{M_5}{}^{M_1}{}_{M_6}{}^{M_3} \big).
\end{aligned} \tag{B.10}$$

The last eleven-dimensional eight-derivative contribution involves the tensor  $\hat{s}_{18}$  parametrized by six unknown coefficients  $a_n \in \mathbb{R}$ . We then have the additional coupling of the form

$$\hat{s}_{18} (\hat{\nabla} \hat{G}_4)^2 \hat{R}^2 = \hat{s}_{18}{}^{N_1 \cdots N_{18}} \hat{R}_{N_1 \cdots N_4} \hat{R}_{N_5 \cdots N_8} \hat{\nabla}_{N_9} \hat{G}_4{}_{N_{10} \cdots N_{13}} \hat{\nabla}_{N_{14}} \hat{G}_4{}_{N_{15} \cdots N_{18}}$$



$$= \mathcal{A} + \sum_{n=1}^6 a_n \mathcal{Z}_n, \quad (\text{B.11})$$

where the quantities  $\mathcal{A}, \mathcal{Z}_n$  are defined by

$$\begin{aligned} \mathcal{A} &= -24B_5 - 48B_8 - 24B_{10} - 6B_{12} - 12B_{13} + 12B_{14} + 8B_{16} - 4B_{20} + B_{22} \\ &\quad + 4B_{23} + B_{24} \\ \mathcal{Z}_1 &= 48B_1 + 48B_2 - 48B_3 + 36B_4 + 96B_6 + 48B_7 - 48B_8 + 96B_{10} + 12B_{12} \\ &\quad + 24B_{13} \\ \mathcal{Z}_2 &= -48B_1 - 48B_2 - 24B_4 - 24B_5 + 48B_6 - 48B_8 - 24B_9 - 72B_{10} - 24B_{13} \\ &\quad + 24B_{14} - B_{22} + 4B_{23} \\ \mathcal{Z}_3 &= +12B_1 + 12B_2 - 24B_3 + 9B_4 + 48B_6 + 24B_7 - 24B_8 + 24B_{10} + 6B_{12} \\ &\quad + 6B_{13} + 4B_{15} - 4B_{17} + 3B_{19} + 2B_{21} \\ \mathcal{Z}_4 &= +12B_1 + 12B_2 - 12B_3 + 9B_4 + 24B_6 + 12B_7 - 12B_8 + 24B_{10} + 3B_{12} \\ &\quad + 6B_{13} + 4B_{15} - 4B_{17} + 2B_{20} \\ \mathcal{Z}_5 &= +4B_3 - 8B_6 - 4B_7 + 4B_8 - B_{12} - 2B_{14} + 4B_{18} \\ \mathcal{Z}_6 &= +B_4 + 2B_{11}. \end{aligned} \quad (\text{B.12})$$

The elements  $B_i$  form a basis of the terms with the structure  $(\hat{\nabla}\hat{G}_4)^2\hat{R}^2$  given in (B.13).

$$\begin{aligned} B_1 &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7N_8} \hat{\nabla}^{N_5} \hat{G}^{N_1N_7N_8}_{N_9} \hat{\nabla}^{N_3} \hat{G}^{N_2N_4N_6N_9} \\ B_2 &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7N_8} \hat{\nabla}^{N_5} \hat{G}^{N_1N_3N_7}_{N_9} \hat{\nabla}^{N_8} \hat{G}^{N_2N_4N_6N_9} \\ B_3 &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7N_8} \hat{\nabla}^{N_5} \hat{G}^{N_1N_3N_7}_{N_9} \hat{\nabla}^{N_6} \hat{G}^{N_2N_4N_8N_9} \\ B_4 &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7N_8} \hat{\nabla}_{N_9} \hat{G}^{N_3N_4N_7N_8} \hat{\nabla}^{N_6} \hat{G}^{N_9N_1N_2N_5} \\ B_5 &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7}^{N_4} \hat{\nabla}^{N_1} \hat{G}^{N_2N_3}_{N_8N_9} \hat{\nabla}^{N_5} \hat{G}^{N_6N_7N_8N_9} \\ B_6 &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7}^{N_4} \hat{\nabla}^{N_1} \hat{G}^{N_2N_5}_{N_8N_9} \hat{\nabla}^{N_3} \hat{G}^{N_6N_7N_8N_9} \\ B_7 &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7}^{N_4} \hat{\nabla}^{N_1} \hat{G}^{N_2N_5}_{N_8N_9} \hat{\nabla}^{N_7} \hat{G}^{N_3N_6N_8N_9} \\ B_8 &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7}^{N_4} \hat{\nabla}^{N_1} \hat{G}^{N_3N_5}_{N_8N_9} \hat{\nabla}^{N_2} \hat{G}^{N_6N_7N_8N_9} \\ B_9 &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7}^{N_4} \hat{\nabla}^{N_1} \hat{G}^{N_3N_5}_{N_8N_9} \hat{\nabla}^{N_6} \hat{G}^{N_2N_7N_8N_9} \\ B_{10} &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7}^{N_4} \hat{\nabla}_{N_9} \hat{G}^{N_3N_5N_7}_{N_8} \hat{\nabla}^{N_9} \hat{G}^{N_1N_2N_6N_8} \\ B_{11} &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7}^{N_4} \hat{\nabla}_{N_8} \hat{G}^{N_1N_2N_6}_{N_9} \hat{\nabla}^{N_9} \hat{G}^{N_3N_5N_7N_8} \\ B_{12} &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5N_6N_7}^{N_4} \hat{\nabla}^{N_3} \hat{G}^{N_5N_6}_{N_8N_9} \hat{\nabla}^{N_7} \hat{G}^{N_2N_1N_8N_9} \\ B_{13} &= \hat{R}_{N_1N_2N_3N_4} \hat{R}_{N_5}^{N_1} \hat{\nabla}_{N_6}^{N_3} \hat{\nabla}_{N_9} \hat{G}^{N_2N_6}_{N_7N_8} \hat{\nabla}^{N_9} \hat{G}^{N_4N_5N_7N_8} \end{aligned}$$

$$\begin{aligned}
B_{14} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5}^{N_1} \hat{N}_6^{N_3} \hat{\nabla}_{N_9} \hat{G}^{N_2 N_4}_{N_7 N_8} \hat{\nabla}^{N_5} \hat{G}^{N_4 N_7 N_8 N_9} \\
B_{15} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5}^{N_1} \hat{N}_6^{N_3} \hat{\nabla}^{N_2} \hat{G}^{N_6}_{N_7 N_8 N_9} \hat{\nabla}^{N_5} \hat{G}^{N_4 N_7 N_8 N_9} \\
B_{16} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5}^{N_1} \hat{N}_6^{N_3} \hat{\nabla}^{N_2} \hat{G}^{N_4}_{N_7 N_8 N_9} \hat{\nabla}^{N_5} \hat{G}^{N_6 N_7 N_8 N_9} \\
B_{17} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5}^{N_1} \hat{N}_6^{N_3} \hat{\nabla}^{N_2} \hat{G}^{N_5}_{N_7 N_8 N_9} \hat{\nabla}^{N_4} \hat{G}^{N_6 N_7 N_8 N_9} \\
B_{18} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5}^{N_1} \hat{N}_6^{N_3} \hat{\nabla}_{N_9} \hat{G}^{N_5 N_6}_{N_7 N_8} \hat{\nabla}^{N_4} \hat{G}^{N_2 N_7 N_8 N_9} \\
B_{19} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5 N_6}^{N_3 N_4} \hat{\nabla}_{N_9} \hat{G}^{N_1 N_5}_{N_7 N_8} \hat{\nabla}^{N_9} \hat{G}^{N_2 N_6 N_7 N_8} \\
B_{20} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5 N_6}^{N_3 N_4} \hat{\nabla}^{N_1} \hat{G}^{N_5}_{N_7 N_8 N_9} \hat{\nabla}^{N_2} \hat{G}^{N_6 N_7 N_8 N_9} \\
B_{21} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5 N_6}^{N_3 N_4} \hat{\nabla}^{N_1} \hat{G}^{N_5}_{N_7 N_8 N_9} \hat{\nabla}^{N_6} \hat{G}^{N_2 N_7 N_8 N_9} \\
B_{22} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5}^{N_1 N_3 N_4} \hat{\nabla}^{N_2} \hat{G}^{N_6}_{N_7 N_8 N_9} \hat{\nabla}^{N_5} \hat{G}^{N_6 N_7 N_8 N_9} \\
B_{23} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}_{N_5}^{N_1 N_3 N_4} \hat{\nabla}_{N_9} \hat{G}^{N_2}_{N_6 N_7 N_8} \hat{\nabla}^{N_9} \hat{G}^{N_5 N_6 N_7 N_8} \\
B_{24} &= \hat{R}_{N_1 N_2 N_3 N_4} \hat{R}^{N_1 N_2 N_3 N_4}_{N_5} \hat{\nabla}_{N_5} \hat{G}^{N_6}_{N_7 N_8 N_9} \hat{\nabla}^{N_6} \hat{G}^{N_5 N_7 N_8 N_9} \quad (B.13)
\end{aligned}$$

**Higher derivatives in Type IIA.** The relevant structures in the gravitational sector of Type IIA supergravity are  $\hat{t}_8 \hat{t}_8 \hat{R}^4$  and  $\epsilon_{10} \epsilon_{10} \hat{R}^4$ . The  $\hat{t}_8 \hat{t}_8 \hat{R}^4$  structure has precisely the same index structure as its eleven-dimensional counterpart (B.2). The  $\epsilon_{10} \epsilon_{10} \hat{R}^4$  combination is explicitly given by

$$\epsilon_{10} \epsilon_{10} \hat{R}^4 = \epsilon_{10}^{R_1 R_2 M_1 \dots M_8} \epsilon_{10}^{R_1 R_2 N_1 \dots N_8} \hat{R}^{N_1 N_2}_{M_1 M_2} \dots \hat{R}^{N_7 N_8}_{M_7 M_8}, \quad (B.14)$$

where  $\epsilon_{10}$  denotes the Levi-Civita tensor in ten dimensions. We now proceed in introducing higher derivative terms for the NS-NS two-form. One step in the  $\hat{B}_2$  completion of higher derivative terms in Type IIA is taken by introducing the connection with torsion, as outlined in section 2.1.2,

$$\Omega_{\pm M}^{AB} = \Omega_M^{AB} \pm \frac{1}{2} \hat{H}_{3M}^{AB}, \quad (B.15)$$

where  $\Omega_M^{AB}$  are the components of the  $\mathfrak{so}(1,9)$ -valued connection one-form corresponding to the Levi-Civita connection. In this notation  $A, B$  are flat tangent space indices of  $\mathcal{M}_{10}$ . The additional structure  $\epsilon_{10} \epsilon_{10} \hat{H}_3^2 \hat{R}^3$ , which enters the replacement in (2.19), has the component form

$$\begin{aligned}
\epsilon_{10} \epsilon_{10} \hat{H}_3^2 \hat{R}^3(\Omega_{\pm}) &= \epsilon_{10}^{R_1 R_2 M_0 \dots M_8} \epsilon_{10}^{R_1 N_0 \dots N_8} \hat{H}_3^{N_1 N_2}_{M_0} \hat{H}_3^{N_3 N_4}_{M_1} \dots \hat{H}_3^{N_7 N_8}_{M_7} \\
&\quad \times \hat{R}(\Omega_+)^{M_3 M_4}_{N_3 N_4} \dots \hat{R}(\Omega_+)^{M_7 M_8}_{N_7 N_8}, \quad (B.16)
\end{aligned}$$

where the Riemann tensor  $\hat{R}(\Omega_+)$  computed with respect to the connection with torsion is given in (2.21).

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## Appendix C

# Action of $\mathbb{Z}_m$ and $\mathbb{D}_m^*$ on Kaluza-Klein spectrum

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In order to compute the one-loop quantum corrections for the ALF transverse spaces we must determine the spectrum on  $\text{AdS}_3 \times S^3/\Gamma$ . This is achieved by taking the spectrum of  $\text{AdS}_3 \times S^3$ , as given in [123, 124], and truncating out modes that are *not* invariant under  $\Gamma \subset \text{SU}(2)_L$ . We shall consider  $\Gamma = \mathbb{Z}_m$  and  $\Gamma = \mathbb{D}_m^*$  in turn. The representations are determined by  $\text{SU}(2)_L$  representations labeled by  $j_L$  with  $j_L \in \frac{1}{2}\mathbb{N}$  and with dimension  $2j_L + 1$ . The  $j_L$  representation is given by the  $2j_L$ -fold symmetrized tensor product of the fundamental representation ( $j_L = \frac{1}{2}$ ). The fundamental representation is given by the standard action of  $\text{SU}(2)_L$  on  $\mathbb{C}^2$ . Let us denote the basis of  $\mathbb{C}^2$  by

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (\text{C.1})$$

The vector space  $V_{j_L}$  corresponding to the representation labeled by  $j_L$  is given by  $V_{j_L} = \text{Sym}^{2j_L}(\mathbb{C}^2)$ . A basis for this vector space is given by

$$|j_L, j_L^{(3)}\rangle = \text{Sym}^{2j_L}(\uparrow^{j_L+j_L^{(3)}} \downarrow^{j_L-j_L^{(3)}}) \equiv \underbrace{|\uparrow\rangle \otimes \cdots \otimes |\uparrow\rangle}_{(j_L+j_L^{(3)}) \text{ factors}} \otimes \underbrace{|\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle}_{(j_L-j_L^{(3)}) \text{ factors}} + \text{sym}, \quad (\text{C.2})$$

for  $j_L^{(3)} = -j_L, -j_L + 1, \dots, j_L$ . Consider now the actions of  $\mathbb{Z}_m$  and  $\mathbb{D}_m^*$  on the above representations.

**Action  $\mathbb{Z}_m$ .** The action of the group  $\mathbb{Z}_m \subset \text{SU}(2)_L$  on the fundamental representation is generated by

$$\mathcal{A} = \begin{pmatrix} e^{\frac{2i\pi}{m}} & 0 \\ 0 & e^{-\frac{2i\pi}{m}} \end{pmatrix}. \quad (\text{C.3})$$

Using this action and the construction of the other representations in terms of the fundamental one, we can explicitly deduce how the basis states  $|j_L, j_L^{(3)}\rangle$  transform under the action of  $\mathcal{A}$ . We find

$$|j_L, j_L^{(3)}\rangle \xrightarrow{\mathcal{A}} e^{4\pi i \frac{j_L^{(3)}}{m}} |j_L, j_L^{(3)}\rangle. \quad (\text{C.4})$$

The modes invariant under  $\mathbb{Z}_m$  are those with  $j_L^{(3)} = \frac{1}{2}mk$  for  $k \in \mathbb{Z}$ .

**Action  $\mathbb{D}_m^*$ .** Consider now the action of  $\mathbb{D}_m^* \subset \text{SU}(2)_L$  on the fundamental representation. It is generated by the two generators

$$\mathcal{A} = \begin{pmatrix} e^{\frac{i\pi}{m}} & 0 \\ 0 & e^{-\frac{i\pi}{m}} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (\text{C.5})$$

Similarly we find that under the action of the generator  $\mathcal{A}$

$$|j_L, j_L^{(3)}\rangle \xrightarrow{\mathcal{A}} e^{2\pi i \frac{j_L^{(3)}}{m}} |j_L, j_L^{(3)}\rangle, \quad (\text{C.6})$$

implying the projection condition  $j_L^{(3)} = mk$  for  $k \in \mathbb{Z}$ . Note that this also implies  $j_L \in \mathbb{N}$ . For the generator  $\mathcal{B}$  we find

$$|j_L, j_L^{(3)}\rangle \xrightarrow{\mathcal{B}} (-1)^{j_L} |j_L, -j_L^{(3)}\rangle. \quad (\text{C.7})$$

To obtain invariant states under the generator  $\mathcal{B}$  we construct the linear combinations

$$|j_L, j_L^{(3)}\rangle + (-1)^{j_L} |j_L, -j_L^{(3)}\rangle. \quad (\text{C.8})$$

These states are then invariant under both generators provided  $j_L^{(3)} = mk$  for  $k \in \mathbb{Z}_{\geq 0}$ .

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## Appendix D

### 6d to 5d one-loop corrections

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In this appendix we use the index theorems to reproduce the results of [83] in which they calculated the one-loop corrections that one gets when integrating out massive chiral particles after the reduction from six to five dimensions on a circle. The gauge field that is relevant in this case is the  $u(1)$  Kaluza-Klein vector  $A^0$ . We will calculate the contributions from spin- $\frac{1}{2}$  fermions, spin- $\frac{3}{2}$  fermions and for anti-symmetric tensors separately.

**Spin- $\frac{1}{2}$  fermions.** We consider a massive spin- $\frac{1}{2}$  fermion coupled to the gauge field  $A^0$  and to an external gravitational field denoted by the vielbein  $e$ . The parity anomaly resulting from this particle can be canceled by a term

$$-i\pi \operatorname{sign}(M) \int_{\mathcal{M}_5} Q(A^0, \omega), \quad (\text{D.1})$$

where

$$dQ(A^0, \omega) = \hat{A}(\mathcal{M}_5) \wedge \operatorname{ch}(F^0)|.$$

Using that

$$\hat{A}(\mathcal{M}_5) \wedge \operatorname{ch}(F^0)| = \frac{1}{(2\pi)^3} \left( \frac{i^3}{6} F^0 \wedge F^0 \wedge F^0 + \frac{i}{48} F^0 \wedge \operatorname{tr} \mathcal{R} \wedge \mathcal{R} \right), \quad (\text{D.2})$$

we find that the counterterms should be given by

$$\frac{\text{sign}(M)}{8\pi^2} \left( -\frac{1}{6} A^0 \wedge F^0 \wedge F^0 + \frac{1}{48} A^0 \wedge \text{tr } \mathcal{R} \wedge \mathcal{R} \right). \quad (\text{D.3})$$

Comparing conventions in [83] and [128] we find that we need  $A^0 \rightarrow qiA^0$  in the counterterm above. We also need to do a Wick rotation to obtain a Lorentzian action which gives another factor  $i$ . We thus find the counterterms

$$\frac{\text{sign}(M)}{8\pi^2} \left( -\frac{q^3}{6} A^0 \wedge F^0 \wedge F^0 - \frac{q}{48} A^0 \wedge \text{tr } \mathcal{R} \wedge \mathcal{R} \right). \quad (\text{D.4})$$

**Spin- $\frac{3}{2}$  fermions.** For spin- $\frac{3}{2}$  fermions the anomaly is canceled by a term of the form (D.1) where

$$\begin{aligned} dQ(A, \omega) &= \hat{A}(\mathcal{M}_5) \wedge \left( \text{tr } e^{i\mathcal{R}/(2\pi)} - 1 \right) \wedge \text{ch}(F^0)| \\ &= \frac{1}{(2\pi)^3} \left( \frac{5i^3}{6} F^0 \wedge F^0 \wedge F^0 - \frac{19i}{48} F^0 \wedge \text{tr } \mathcal{R} \wedge \mathcal{R} \right). \end{aligned} \quad (\text{D.5})$$

The counterterms to the Lorentzian action are thus given by

$$\frac{\text{sign}(M)}{8\pi^2} \left( -\frac{5q^3}{6} A^0 \wedge F^0 \wedge F^0 + \frac{19q}{48} A^0 \wedge \text{tr } \mathcal{R} \wedge \mathcal{R} \right). \quad (\text{D.6})$$

**Anti-symmetric tensors.** In this case we were unable to find an index theorem in the literature, but following the arguments in the main text we postulate that the relevant index is given by

$$\text{ind } iD_A = \frac{1}{2} \int L(M) \wedge \text{ch}(2F^0)|. \quad (\text{D.7})$$

Since the tensors are bosons, the counterterm is now given by [129]

$$i\pi \text{sign}(M) \int_{\mathcal{M}_5} Q(A^0, \omega) \quad (\text{D.8})$$

where

$$\begin{aligned} dQ(A^0, \omega) &= \frac{1}{2} L(M) \wedge \text{ch}(2F^0)| \\ &= \frac{1}{8\pi^3} \left( \frac{2i^3}{3} F^0 \wedge F^0 \wedge F^0 - \frac{i}{6} F^0 \wedge \text{tr } \mathcal{R} \wedge \mathcal{R} \right). \end{aligned} \quad (\text{D.9})$$

This implies that the counterterms to the Lorentzian action are

$$\frac{\text{sign}(M)}{8\pi^2} \left( \frac{2q^3}{3} A^0 \wedge F^0 \wedge F^0 - \frac{q}{6} A^0 \wedge \text{tr } \mathcal{R} \wedge \mathcal{R} \right). \quad (\text{D.10})$$

The terms (D.4), (D.6) and (D.10) are precisely the one-loop contributions of table 2.2 in [83].

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## About the author

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Kilian Mayer was born on the 31<sup>st</sup> of July 1991 in Landshut, Germany. After graduating from Gymnasium Dorfen, he studied physics at the Technical University of Munich with focus on high energy physics. His Bachelor thesis was supervised by Prof. Dr. Alejandro Ibarra. He graduated in 2014 with a Bachelor degree with honors.

After completion of the Bachelor's degree, he continued his studies in the Master's program in theoretical and mathematical physics, a joint degree organized by the Ludwig-Maximilians University Munich and the Technical University of Munich in cooperation with the Elite Network of Bavaria. The main focus subjects were String Theory and Quantum Field Theory. The Master thesis was carried out at the Max-Planck-Institute for physics under supervision of Dr. Thomas Grimm. He graduated in September 2016 with honors.

From October 2016 onwards, he joined the recently moved research group of Dr. Thomas Grimm at the institute for theoretical physics at Utrecht University. The results produced during this time are presented in this book.





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