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DIMENSIONAL & ALGORITHMIC REDUCTIONS FOR  
CALOGERO–RUIJSENAARS & LANDAU–GINZBURG MODELS

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# Dimensional & algorithmic reductions for Calogero–Ruijsenaars & Landau–Ginzburg models

Dimensie- en algoritmische reducties for Calogero–Ruijsenaars-  
en Landau–Ginzburgmodellen  
(met een samenvatting in het Nederlands)

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# CONTENTS

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1	INTRODUCTION	1
I	DIMENSIONAL REDUCTION FOR CALOGERO–RUIJSENAARS MODELS	
2	OVERVIEW OF PART I	5
2.1	Phase spaces and Hamiltonian reduction	5
2.2	The Calogero–Moser system	8
2.3	The Ruijsenaars–Schneider system	10
3	ON THE CALOGERO–MOSER SOLUTION BY ROOT-TYPE LAX PAIR	13
3.1	Overview	13
3.2	Example: a two-dimensional case	14
3.3	Under-determinacy of the solution	16
3.4	The non-generic case by computer search	20
3.5	Fundamental domain crossings	24
4	THE TRIGONOMETRIC RUIJSENAARS–SCHNEIDER SYSTEM	31
4.1	Overview	31
4.2	Preliminaries	32
4.2.1	Adjoint orbits in the special unitary group	32
4.2.2	Quasi-Hamiltonian reduction	36
4.2.3	The Delzant theorem	38
4.3	The special unitary fused double	39
4.4	Vertex solutions to the moment map constraint	41
4.5	Bulk solutions of the moment map constraint	44
4.6	Analytic properties of the constraint surface	52
II	ALGORITHMIC ASPECTS OF LANDAU–GINZBURG MODELS	
5	OVERVIEW OF PART II	63
5.1	Landau–Ginzburg category	63
5.2	Fusion product	64
5.3	Orbifold equivalence	65
5.4	Software package	65
6	COMPUTATIONAL ASPECTS OF ORBIFOLD EQUIVALENCE	67
6.1	Overview	67
6.2	Orbifold equivalence	68
6.3	Search algorithm	74
6.4	Computational feasibility	76

7	A SOFTWARE PACKAGE FOR LANDAU–GINZBURG EXPLO-	
	RATION	81
7.1	Overview	81
7.2	Software environment and use	81
7.3	Stable equivalence	83
7.4	The tensor product operation	86
7.5	The unit matrix factorization	89
7.6	A library of orbifold equivalences	91
8	SAMENVATTING	93
9	ACKNOWLEDGEMENTS	97
A	PACKAGE DOCUMENTATION	99
A.1	Features	99
A.2	Limitations	99
A.3	Getting Started	99
A.4	Overview	100
A.5	Operations	102
A.6	Library	108
A.7	Internal functions	108
	BIBLIOGRAPHY	111

## INTRODUCTION

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In this thesis we look at two classes of models in which we explain *complicated* behaviour of a *low-dimensional* system by relating it to *simple* behaviour of a *high-dimensional* system. In both cases, the high-dimensional system provides insight that is hard to obtain directly in the low-dimensional system. The two classes to which we apply this pattern are Calogero–Ruijsenaars models and Landau–Ginzburg systems.

The various **Calogero–Ruijsenaars models** describe  $n$  indistinguishable particles in one dimension subject to (in the simplest case) pairwise interactions. They are *integrable systems*: each has  $n$  mutually compatible conservation laws associated to its equations of motion. In the native description, however, these conservation laws are by no means obvious.

From the point of view described above, the Calogero–Ruijsenaars models each arise from a higher-dimensional model by identifying orbits of a group action. The high-dimensional model is much simpler: in the simplest case it is free motion of a single particle. The higher-dimensional model therefore has “obvious” conservation laws. Because of the specifics of the group action and identification process (namely “Hamiltonian reduction”), these conservation laws carry over to the smaller system. This yields both an explanation for the conservation laws as well as an explicit way to compute them.

In Part I of this thesis we give a detailed description of two instances of this process: the *rational Calogero–Moser system* and the *trigonometric Ruijsenaars–Schneider system*. Chapter 3 describes work previously published in [27], and largely follows the exposition there. As a new addition, we include a description of the search process that was used to find the non-generic counterexample from that article. Moreover, we describe improved optimizations and list more examples, including an example for a larger root system. This forms Section 3.4.

We also offer an improved understanding of the material related to fundamental domain crossings. In [27], we distinguished an “exact but unfeasible” approach and an “inexact but feasible”

one. Section 3.5 replaces this with an exact treatment of the feasible approach.

Chapter 4 describes work previously published in [17], a paper which builds on work in [16]. To ensure the chapter can be read independently, results from both papers are included. The exposition is new, and in particular the description in terms of what we call *quasi-Cauchy matrices* (Definition 4.5.4) has, to the author's knowledge, not been previously presented.

In Part II of this thesis, we consider the **Landau–Ginzburg model**. It describes  $n$  scalar fields on a two-dimensional space-time with a polynomial in  $n$  variables as their interaction term. For compactifying such a model, we are led to consider an object called a *matrix factorization* of the polynomial. And from there it is a small step to generalize as follows: we consider several distinct domains of space-time in which different polynomials govern the interactions. This is possible as long as we find matrix factorizations connecting the polynomials wherever the domains share a boundary.

The view we described at the start of this introduction now applies to the operation of fusing two boundaries. In computational terms, this fusion corresponds to the composition of the two associated matrix factorizations. This composition has a simple formula, but it results in an infinite-rank matrix factorization. We need to apply a reduction step to get a workable matrix factorization, and the result is very non-obvious.

In this case, as in the previous, we see an interesting interplay between the high-dimensional description and the low-dimensional one. For example, one needs the high-dimensional version to establish basic properties such as associativity of this fusion process, but it is the low-dimensional version that gives computable results.

We study algorithms related to this model, discussing their complexity as well as providing practical implementations. Chapter 6 describes work that is also part of the pre-print [28]. It appears virtually unmodified. Chapter 7 is entirely new. The appendix contains the reference documentation for the software package described.



Part I

DIMENSIONAL REDUCTION FOR  
CALOGERO–RUIJSENAARS MODELS



## OVERVIEW OF PART I

---

The objective of this chapter is to describe the main results included in Part I of this thesis. The second and third sections contains statements of these main results, while the first section provides a description of Hamiltonian reduction as a preliminary topic. A standard reference for the latter is [35].

### 2.1 PHASE SPACES AND HAMILTONIAN REDUCTION

In an isolated system, continuous symmetries correspond to conserved quantities. This familiar fact is the content of Emmy Noether's theorem. For example, applying this theorem to time invariance yields a quantity equal to total energy, and applying it to translation invariance yields a quantity equal to momentum. A natural mathematical setting for this is given by *symplectic manifolds*:

**Definition 2.1.1.** A *symplectic manifold*  $(M, \omega)$  is a real  $C^\infty$ -manifold  $M$  together with a closed, non-degenerate 2-form  $\omega$ . For every function  $f: M \rightarrow \mathbb{R}$ , we write  $h_f$  for the vector field  $h_f = \omega^{-1}(df)$ . Every vector field arising in this way is called a *Hamiltonian vector field*.

**Example 2.1.2.** Let us illustrate the usefulness of symplectic manifolds through an example. Consider a system with two particles in three dimensions. We encode their space coordinates as points

$$(q_{1,x}, q_{1,y}, q_{1,z}, q_{2,x}, q_{2,y}, q_{2,z}) \quad (2.1)$$

on a manifold  $X = \mathbb{R}^3 \oplus \mathbb{R}^3$ . If the system is translation-invariant along the  $x$ -axis, this means that it is invariant under *simultaneous* translation of both particles. Mathematically, it is invariant along the vector field  $\partial_{q_{1,x}} + \partial_{q_{2,x}}$ .

A tangent vector to  $X$  represents the velocities of the two particles, so it has units

$$[\text{space}]/[\text{time}] \quad (2.2)$$

We choose units

$$[\text{mass}][\text{space}]/[\text{time}] \quad (2.3)$$

for co-tangent vectors, because then the pairing between vectors and co-vectors is energy-valued, i.e., in units

$$[\text{mass}][\text{space}]^2/[\text{time}]^2 \quad (2.4)$$

Therefore co-tangent vectors corresponds to momentum, and the co-tangent bundle  $M = T^\vee X$  encodes position and momentum for the particles. It allows describing the full evolution of the system as a path on this manifold, and through every point passes a single path. We call it the *phase space* for this system.

Writing  $p_\alpha$  for linear coordinates on a dual basis of  $\partial_{q_\alpha}$ , this co-tangent bundle has a natural symplectic form:  $\omega = dp_\alpha \wedge dq^\alpha$  (where  $\alpha$  ranges over  $(1, x), \dots, (2, z)$ ). Therefore every function  $f : M \rightarrow \mathbb{R}$  induces a vector field  $h_f$  as defined in Definition 2.1.1. The function  $f : M \rightarrow \mathbb{R}$  given by  $f = p_{1,x} + p_{2,x}$  has  $h_f = \partial_{q_{1,x}} + \partial_{q_{2,x}}$ . In other words, we see that translation invariance is associated to total momentum. Although we have not related this to Noether's theorem (or even to the equations of motion) as of yet, this illustrates how symplectic manifolds are useful when studying mechanical systems.

**Definition 2.1.3.** Two functions  $f, g : M \rightarrow \mathbb{R}$  are said to *commute* when  $g$  is constant along  $f$ 's Hamiltonian vector field and  $f$  is constant along  $g$ 's Hamiltonian vector field: that is, when  $h_f g = h_g f = 0$  everywhere on  $M$ .

In fact, these two conditions are equivalent and one needs to check only one or the other. This can be seen from the following:

**Lemma 2.1.4.** *Let  $(M, \omega)$  a symplectic manifold and  $f, g : M \rightarrow \mathbb{R}$ . Then  $h_f g = -h_g f$ .*

*Proof.* This follows immediately from the definition of  $h_f$  in terms of  $\omega$  which is anti-symmetric.  $\square$

In particular, we see that a function is constant along *its own* Hamiltonian flow, as  $h_f f = 0$ . This invites us to define *Hamiltonian systems*. They are the topic of Part I of this thesis.

**Definition 2.1.5.** A *Hamiltonian system* is a symplectic manifold  $(M, \omega)$  together with a function  $H : M \rightarrow \mathbb{R}$ . This function is called the *Hamiltonian*.

Given a Hamiltonian system, the integral curves of  $h_H$  are interpreted as *time evolution*. Any function  $f: M \rightarrow \mathbb{R}$  that commutes with  $H$  is interpreted as a *conserved quantity*. In particular,  $H$  itself is interpreted as the system's *total energy*.

Let us now turn our attention to actions of Lie groups on symplectic manifolds.

**Definition 2.1.6.** Let  $G$  be a connected Lie group and  $(M, \omega, H)$  a Hamiltonian system with a  $G$ -action (i.e.,  $G$  acts on  $M$  with  $\omega$  and  $H$  invariant). For an element  $\xi \in \text{Lie } G$  of the Lie group of  $G$ , there is an induced vector field  $\xi_M$  on  $M$ . Suppose that there is a  $G$ -equivariant map

$$\mu: M \rightarrow (\text{Lie } G)^\vee \quad (2.5)$$

(where  $\cdot^\vee$  is the vector space dual) such that, at every  $m \in M$  and for every  $\xi \in \text{Lie } G$  we have

$$h_{\mu(m)\xi} = \xi_M \quad (2.6)$$

Then  $\mu$  is called a *moment map* for the  $G$ -action on  $M$ .

In particular, this definition requires that the  $G$ -action has every vector field  $\xi_M$  Hamiltonian: the function  $m \mapsto \mu(m)\xi$  induces  $\xi_M$ .

**Example 2.1.7.** Following up on Example 2.1.2, let us consider a rotation around the  $z$ -axis. The Lie group  $G = S^1$  has  $(\text{Lie } G)^\vee \cong \mathbb{R}$ . The map  $\mu = q_{1,x}p_{1,y} - q_{1,y}p_{1,x} + q_{2,x}p_{2,y} - q_{2,y}p_{2,x}$  is a moment map for this action. To see this, note that  $\text{Lie } G$  can be generated by a  $\xi \in \text{Lie } G$  such that

$$\xi_M = q_{1,x}\partial_{q_{1,y}} - q_{1,y}\partial_{q_{1,x}} + q_{2,x}\partial_{q_{2,y}} - q_{2,y}\partial_{q_{2,x}}. \quad (2.7)$$

We see that the value of the moment map represents angular momentum.

The existence of a moment map allows creating a new symplectic manifold from the  $G$ -orbits of  $M$  as follows.

**Definition 2.1.8.** Let  $(M, \omega)$  a symplectic manifold,  $G$  a connected Lie group that acts on  $M$ , and  $\mu: M \rightarrow (\text{Lie } G)^\vee$  a moment map for this action. Pick  $\mu_0 \in (\text{Lie } G)^\vee$ , write  $G_{\mu_0}$  for its stabilizer and  $\mathcal{O}_{\mu_0}$  for its orbit. Then the *Hamiltonian reduction*  $\widetilde{M}$  of  $M$  along  $\mu$  at  $\mu_0$  is

$$\widetilde{M} = \mu^{-1}(\{\mu_0\}) / G_{\mu_0} = \mu^{-1}(\mathcal{O}_{\mu_0}) / G \quad (2.8)$$

**Proposition 2.1.9** (Proposition 5.4.15 in [35]). Suppose  $(M, \omega, G, \mu)$  are as in Definition 2.1.8. If  $G$  acts freely and properly on  $\mathcal{O}_{\mu_0}$  then the Hamiltonian reduction  $\widetilde{M}$  is naturally a symplectic manifold, and  $G$ -invariant functions on  $M$  induce Hamiltonian vector fields on  $\widetilde{M}$ .

**Example 2.1.10.** In Example 2.1.7, this procedure fixes a value  $\mu_0$  for total angular momentum around the  $z$ -axis, while simultaneously replacing two absolute angular coordinates  $\theta_1, \theta_2$  for the two particles by a single parameter, say  $\vartheta$ , representing their angular separation.

In keeping with the theme of this thesis as described in Chapter 1, in the next two chapters, we will study cases where *complicated* behaviour on  $\widetilde{M}$  is explained by showing that it arises from *simpler* behaviour on some  $M$  through a reduction procedure. So in a sense, the motivation flows in the opposite direction from the mathematical exposition. Hamiltonian reduction will appear unmodified in Chapter 3, and in a slightly modified form (*quasi-Hamiltonian reduction*) in Chapter 4. We will describe the main results of these chapters in the following two sections.

## 2.2 THE CALOGERO–MOSER SYSTEM

The *Calogero–Moser system* describes  $n + 1 \geq 3$  indistinguishable particles in a 1-dimensional space with centre-of-mass fixed at the origin. Its phase space is given by  $(T^\vee \mathbb{R}^{n+1}) / S_{n+1}$  (where  $S_{n+1}$  acts by permuting the coordinates) and the  $S_{n+1}$ -invariant Hamiltonian is given by

$$H = \sum_{i=1}^{n+1} \frac{1}{2} p_i^2 + \sum_{i \neq j=1}^{n+1} \frac{y}{(q_i - q_j)^2} \quad (2.9)$$

for some coupling parameter  $y \in \mathbb{R}$ . It arises as a Hamiltonian reduction from free motion on the co-tangent bundle to the unitary Lie algebra  $T^\vee \mathfrak{u}(n+1)$ . The group  $SU(n+1)$  acts on this system by conjugation, and in fact there exists a Hamiltonian on  $T^\vee \mathfrak{u}(n+1)$ , a moment map  $\mu$ , and a value  $\mu_0$  which together yield the system just described. A key point is this: the unordered real eigenvalues of the matrices in  $\mathfrak{u}(n+1)$  exactly describe a configuration of  $n+1$  particles, and they also exactly describe a conjugation orbit.

The discussion in Chapter 3 focuses on a generalization of this model. This generalization exists for any irreducible simply-laced root system  $\Phi \subseteq \mathbb{R}^n$ . The phase space is  $(T^\vee \mathbb{R}^n)/W$  with  $W$  the Weyl group of  $\Phi$  the Hamiltonian is equal to

$$H_\Phi = \frac{1}{2}|p|^2 + \sum_{\alpha \in \Phi} \frac{y}{(\alpha, q)^2} \quad (2.10)$$

with  $(\cdot, \cdot)$  is the standard inner product on  $\mathbb{R}^n$ . The original Calogero–Moser model is equal to the one associated to the  $A_n$  root system, because  $A_n$  consists of the elements

$$A_n = \{e_i - e_j\}_{i \neq j=1}^{n+1} \quad (\text{where } \{e_i\}_{i=1}^{n+1} \text{ is an orthonormal basis of } \mathbb{R}^{n+1}) \quad (2.11)$$

and  $S_{n+1}$  as presented above is its Weyl group.

In [3] and [43], the authors prove that the Hamiltonian system associated to any  $\Phi$  is a Hamiltonian reduction of a free system. Or, more accurately, they first map the coordinates  $q \in \mathbb{R}^n$  into  $\mathbb{R}^\Phi$  through  $v: q \mapsto ((\alpha, q))_{\alpha \in \Phi} \in \mathbb{R}^\Phi$  (with  $\mathbb{R}^\Phi$  the set of maps  $\Phi \rightarrow \mathbb{R}$  which is an  $\mathbb{R}$ -vector space of dimension  $|\Phi|$ ). Then, they shift attention to its quotient by the symmetric group  $\mathbb{R}^\Phi/S_\Phi$ , and eventually describe *that* system as such a reduction. This provides meaningful insight into the system’s structure, but it is not complete without a full understanding of the properties of  $v$  with respect to the group actions on both sides. Chapter 3 is therefore devoted to a thorough study of this map, and proves the following:

**Theorem A** (Theorems 3.3.5 and 3.4.3). *Let  $\Phi \subseteq \mathbb{R}^n$  be a root system of rank  $n$  and let  $v$  be the map*

$$v: \mathbb{R}^n \rightarrow \mathbb{R}^\Phi \quad (2.12)$$

$$q \mapsto ((\alpha, q))_{\alpha \in \Phi} \in \mathbb{R}^\Phi \quad (2.13)$$

*Then  $v$  descends to a map*

$$\widehat{v}: \mathbb{R}^n / \text{Weyl}(\Phi) \rightarrow \mathbb{R}^\Phi / S_\Phi \quad (2.14)$$

*Writing  $k = [\text{Aut}(\Phi) : \text{Weyl}(\Phi)]$ , for a dense set  $U \subseteq \text{im}(v)$  this map is exactly  $k : 1$ . Moreover,  $\text{im}(\widehat{v}) \setminus U$  is non-empty at least for  $\Phi = \Phi(A_5)$  and for  $\Phi = \Phi(A_6)$ .  $\square$*

The last part of this statement may be mildly surprising—it certainly was to the author.

The number  $k$  in the statement is equal to the number of Dynkin diagram automorphisms. The fact that we cannot distinguish points in the same Dynkin orbit is a bit unsatisfactory: for instance, in the  $A_n$  case, we do want to distinguish a solution from its mirror image. Our approach is to partition the Weyl chamber into fundamental domains and find out when the configuration passes from one into the other.

**Theorem B** (Proposition 3.5.6, statement 1). *Let  $\Phi$  be a root system equal to  $A_n$  ( $n \geq 2$ ),  $D_n$  ( $n \geq 5$ ), or  $E_6$ ; in these cases  $[\text{Aut}(\Phi) : \text{Weyl}(\Phi)] = 2$  and the  $\text{Aut}(\Phi)/\text{Weyl}(\Phi)$ -action on a Weyl chamber has two fundamental domains separated by a hyperplane. Let initial conditions  $p_0, q_0 \in \mathbb{R}^n$  be given for the Calogero–Moser system associated to  $\Phi$  and write  $t \mapsto q(t)$  for its time evolution. Write  $\mathbb{Q}(p_0, q_0)$  for the smallest field containing all coefficients of  $p_0$  and  $q_0$ . Then the values of  $t$  where  $q(t)$  passes this hyperplane are algebraic over  $\mathbb{Q}(p_0, q_0)$ .  $\square$*

### 2.3 THE RUIJSENAARS–SCHNEIDER SYSTEM

The *Ruijsenaars–Schneider system* is, like the Calogero Moser system, a system of  $n$  indistinguishable particles. We are interested in its trigonometric version, whose particles exist on the unit circle (i.e., the Hamiltonian is periodic) and whose Hamiltonian  $H$  is given by

$$H_{\text{RS}}(p, q) = \sum_{i=1}^n \cos(p_i) \sqrt{\prod_{j \neq i} \left( 1 - \frac{\sin(\pi y)^2}{\sin(q_j - q_i)^2} \right)} \quad (2.15)$$

Note that the Hamiltonian is periodic even in the momentum variables, and in fact we will work with a *compact* phase space without a straightforward physical interpretation in the style of Example 2.1.2.

Prior to the work described in this thesis, the trigonometric Ruijsenaars–Schneider system was considered for values of  $y$  satisfying  $0 < y < 1/n$ , and it has a physical interpretation as the minimal angular separation of the particles. We here explain that the same reduction works for any generic parameter  $0 < y < 1$  and results in new compact integrable Hamiltonian systems.

The reduction procedure used for this result is called *quasi-Hamiltonian reduction* and it is different from the one used for



the Calogero–Moser model as described above; it is often considered a generalization. In this case, the unreduced space, i.e., the space corresponding to the symplectic manifold with a Hamiltonian group action, is an object called a *quasi-Hamiltonian space*. It is a manifold with a group action and a 2-form but the axioms are a bit more intricate; it is discussed in Section 4.3. For our purposes, the specific example is the *special unitary fused double* which, as a manifold, is just  $SU(n) \times SU(n)$ .

Section 4.3 also contains the details of a reduction process quite similar to Hamiltonian reduction. Together with the following definition this allows us to state a main result.

**Definition 2.3.1.** Let  $n \geq 1$  and suppose the  $n$ -dimensional torus group  $\mathbb{T}^n$  acts effectively on a  $2n$ -dimensional compact connected symplectic manifold  $M$  with moment map  $\mu$ . Then  $M$  is called a *Hamiltonian toric manifold*.

**Theorem C** (Theorems 4.6.12 and 4.5.12). *Suppose  $y \in [0, 1]$  is such that*

$$\frac{p}{n} - \frac{1}{nq} < y < \frac{p}{n} + \frac{1}{(n-q)n} \quad (2.16)$$

*for some  $p, q \in \{1, \dots, n-1\}$  such that  $pq \equiv 1 \pmod{n}$ . Define  $\gamma = e^{-2\pi i y}$  and consider special unitary fused double together with the moment map value*

$$\mu_\gamma = \text{diag}(\gamma^{-1}, \gamma^{-1}, \dots, \gamma^{-1}, \gamma^{n-1}) \in SU(n) \quad (2.17)$$

*Then the reduction at  $\mu_\gamma$  is a compact Hamiltonian toric manifold whose algebra of conserved functions contains  $H_{RS}$ .*  $\square$

We describe the compactification of this phase space in terms of what we call *quasi-Cauchy matrices* (Definition 4.5.4). This does not give new results but streamlines existing derivations through Lemma 4.5.8 and Propositions 4.5.6 and 4.5.7. To the author's knowledge, this has not been previously presented.



## ON THE CALOGERO–MOSER SOLUTION BY ROOT-TYPE LAX PAIR

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### 3.1 OVERVIEW

Throughout this chapter, we write  $(\cdot, \cdot)$  for the usual inner product on  $\mathbb{R}^n$ , and  $|\cdot|$  for the associated norm defined by  $|x|^2 = (x, x)$ . We consider the following variant of the Calogero–Moser system.

**Definition 3.1.1** ([36]). Let  $\Phi \subseteq \mathbb{R}^n$  be a simply-laced root system of rank  $n$ . Then the *Calogero–Moser system associated to  $\Phi$*  is the Hamiltonian system with phase space equal to  $T^\vee \mathbb{R}^n$  and Hamiltonian given by

$$H_\Phi = \frac{1}{2} |p|^2 + \sum_{\alpha \in \Phi} \frac{y}{(\alpha, q)^2} \quad (3.1)$$

where  $p$  and  $q$  are vector coordinates for  $T^\vee \mathbb{R}^n$  and  $y \in \mathbb{R} \setminus \{0\}$  is a coupling constant.

*Remark 3.1.2.* There exist several other versions of the Calogero–Moser system; e.g., the trigonometric and the elliptic one. When there is a need to distinguish them, the one from Definition 3.1.1 is called the *rational* Calogero–Moser system. For a review, see [20].

In [3] and [43], the authors introduce the following approach for describing the time evolution  $q(t)$  of this system. Suppose we are given initial conditions  $q(0) = q_0$  and  $p(0) = p_0$ . Let  $|\Phi| = 2N$  and let  $Q$  be the vector space with a basis indexed by  $\Phi$ , i.e.,

$$Q = \mathbb{R}^\Phi \cong \mathbb{R}^{2N} \quad (3.2)$$

Define the matrices  $W_0, L_0 \in \text{Mat}_{\mathbb{R}}(2N, 2N)$  by

$$(W_0)_{\alpha, \beta} = \delta_{\alpha, \beta} \cdot (\alpha, q_0) \quad (3.3)$$

$$(L_0)_{\alpha, \beta} = \delta_{\alpha, \beta} \cdot (\alpha, p_0) + iy \cdot \sum_{\eta \in \Phi} \frac{\delta_{\alpha-\beta, \eta}}{(\eta, q_0)} + \frac{2\delta_{\alpha-\beta, 2\eta}}{(\eta, q_0)} \quad (3.4)$$

and define a time evolution through

$$W(t) = W_0 + tL_0 \quad (3.5)$$

(“free motion of  $W$ ”). In this situation, these authors prove [3, Section 3] that the multi-set of eigenvalues  $\Lambda(W(t))$  is equal to the multi-set of real numbers

$$\Lambda(W(t)) = \{(\alpha, q(t)) \mid \alpha \in \Phi\} \quad (3.6)$$

We see that under this scheme, the study of the dynamics of the position  $q(t)$  has been replaced with a study of the dynamics of the values of these inner products; we are studying the time evolution of  $v(t) \in \mathbb{R}^\Phi$  where  $v(t)_\alpha = (\alpha, q(t))$ . This vector  $v$  is of a much higher dimension than  $q$ : the number of roots is a quadratic function of the rank  $n$  as  $n \rightarrow \infty$ . Because the simple roots are linearly independent, the map  $q \mapsto v(q)$  is injective and in fact a linear embedding.

The particular problem that interests us in this chapter is as follows. The configuration  $q$  is naturally defined up to the action of the Weyl group of  $\Phi$ , whereas  $v$ , when computed from the multi-set of eigenvalues  $\Lambda$ , is naturally defined up to the action of the permutation group  $S_\Phi$  on  $\Phi$ . The map  $q \mapsto v(q)$  intertwines these two actions, but it is not clear whether quotienting out on both sides preserves injectivity. This is an important point because if we lose injectivity, we lose the ability to compute a solution for  $q$  (our original objective) from having found the solution for  $v$ .

As it turns out, injectivity fails; in fact, this map is no less than  $k : 1$  where  $k = [\text{Aut}(\Phi) : \text{Weyl}(\Phi)]$ , and we have exactly  $k : 1$  only generically. These two statements are Theorems 3.3.5 and 3.4.3. The  $k : 1$  ambiguity can be resolved by computing a certain indicator function; in Proposition 3.5.6 we prove that it has algebraic roots.

### 3.2 EXAMPLE: A TWO-DIMENSIONAL CASE

To help the reader visualize the situation, let us discuss the case where  $\Phi$  is the root system associated to  $A_2$  in some detail and with imagery.

The  $A_2$  root system is most naturally described in the Euclidean subspace  $\mathbb{R}^2 \subseteq \mathbb{R}^3$  satisfying  $q_1 + q_2 + q_3 = 0$ . Here, the root system

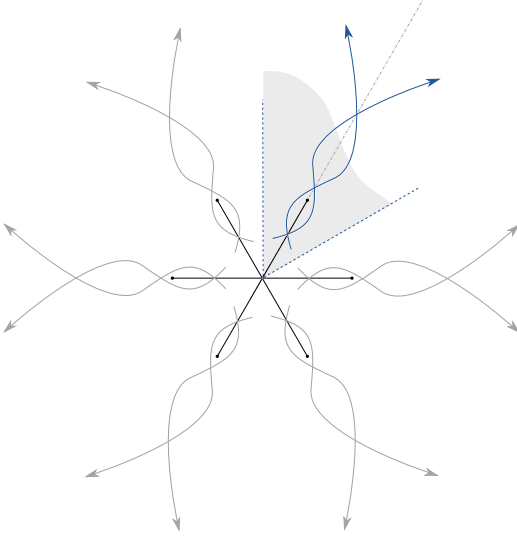


Figure 3.1: A set of paths  $q(t)$  that are in the same  $\text{Aut}(\Phi)$ -orbit, for  $\Phi = \Phi(A_2)$ . The shaded area is a Weyl chamber. The Dynkin diagram automorphism acts on the Weyl chamber by reflection through the dotted line in the middle.

is given by the set of vectors  $\pm(1, -1, 0)$ ,  $\pm(0, 1, -1)$ ,  $\pm(1, 0, -1)$  in  $\mathbb{R}^2$ , and the Hamiltonian on  $T^\vee \mathbb{R}^2$  is

$$H_{\Phi(A_2)} = \frac{1}{2} |p|^2 + \sum_{i \neq j=1}^3 \frac{y}{(q_i - q_j)^2} \quad (3.7)$$

The Weyl group is  $S_3$  and it acts by permuting the coordinates (which clearly leaves the Hamiltonian invariant). The non-trivial Dynkin diagram automorphism acts by sending  $q \mapsto -q$ . Together, they generate the root system automorphisms  $\text{Aut}(\Phi)$ . Note that a Weyl chamber is a fundamental domain for the action of the Weyl group, whereas either of the two halves of the Weyl chamber is a fundamental domain for the action of the entire  $\text{Aut}(\Phi)$ .

Figure 3.1 depicts a path  $q(t)$  in  $\mathbb{R}^2$  and the other paths in its  $\text{Aut}(\Phi)$ -orbit. The orbit consists of six paths because of the  $S_3$  group action alone, and this number is doubled by the Dynkin diagram automorphism. It is clear that at a given time  $t$ , all these values for  $q(t)$  yield the same set for  $\Lambda(t) = \{(\alpha, q(t)) \mid \alpha \in \Phi\}$ .

Conversely, in this case of  $\Phi(A_2)$ , two values giving the same set are in the same  $\text{Aut}(\Phi)$ -orbit. Namely, after choosing a Weyl chamber, the maximal value in  $\Lambda$  must be associated to the maxi-

mal root. After this, there are only two positive values left, which can be assigned in exactly two ways to the two positive roots. As noted in the previous section, in larger root systems, this converse is true generically but false in general. We explore this in Sections 3.3 and 3.4.

The shaded area is a Weyl chamber, given by  $q_1 < q_2 < q_3$ . It is easy to distinguish points on paths in different Weyl chambers, because the Hamiltonian is infinite along the borders; if the initial value of a path is in a particular Weyl chamber, the path will stay there for all time. However, this does not allow us to distinguish the two paths in the same Weyl chamber, related by  $(q_1, q_2, q_3) \mapsto (-q_3, -q_2, -q_1)$ .

Another way of describing this issue is that we obtain a value for  $q$  up to the action of  $\text{Aut}(\Phi)$ , but we are actually interested in  $q$  up to  $\text{Weyl}(\Phi)$ .

For this, we identify a hyperplane that separates one fundamental domain from the other. In the case of  $\Phi(A_2)$ , this hyperplane is just the line of fixed points, but in general, the fixed points are only contained in this hyperplane and the hyperplane is not unique. Next, we try to identify the times at which the paths cross the hyperplane. The result obtained in Section 3.5 is a real polynomial with simple zeros exactly where this happens. Therefore, this polynomial takes positive values at times when  $q(t)$  is in one fundamental domain, and negative values when it is in the other.

### 3.3 UNDER-DETERMINACY OF THE SOLUTION

For any root system  $\Phi$  we can fix a choice of *simple roots*  $\Delta \subseteq \Phi$ . This is a set of  $n = \text{rank} \Phi$  roots such that every  $\alpha \in \Phi$  is a linear combination

$$\alpha = \sum_{\delta \in \Delta} c_\delta \delta \tag{3.8}$$

such that either all  $c_\delta \geq 0$  or all  $c_\delta \leq 0$  for  $\delta \in \Delta$ . A choice for  $\Delta$  therefore induces a decomposition of  $\Phi$  as

$$\Phi = \Phi^+ \cup \Phi^- \tag{3.9}$$

containing the roots with only non-negative, resp. only non-positive coefficients. Clearly,  $\Delta \subseteq \Phi^+$ .

**Lemma 3.3.1** (Corollary in Section 10.2 of [25]). *Let  $\Delta \subseteq \Phi$  be a choice of simple roots for a root system  $\Phi$ . Then every positive*

root  $\alpha \in \Phi^+$  is the last element of a sequence of positive roots  $\alpha_0, \alpha_1, \dots, \alpha_k = \alpha$  where  $\alpha_0 \in \Delta$  and where for every  $1 \leq \ell < k$  there is a  $\delta \in \Delta$  with  $\alpha_\ell = \alpha_{\ell-1} + \delta$ .  $\square$

**Definition 3.3.2.** Let  $\Phi$  be a root system and  $G$  an abelian group. We call a map  $\sigma: \Phi \rightarrow G$  *additive* if it satisfies these conditions:

1.  $\sigma(\alpha + \beta) = \sigma(\alpha) + \sigma(\beta)$  for all  $\alpha, \beta \in \Phi$  for which  $\alpha + \beta \in \Phi$
2.  $\sigma(-\alpha) = -\sigma(\alpha)$  for all  $\alpha \in \Phi$

For a map  $\sigma: \Phi \rightarrow H$  where  $H \subseteq G$  is an arbitrary subset of a group  $G$ , we say it is *additive* if the composition with the inclusion  $H \hookrightarrow G$  is additive.

A particular case of the latter situation is  $H = \Phi$  as a subset of the group  $G = \mathbb{R}^n$ .

**Lemma 3.3.3.** Let  $\Phi \subseteq \mathbb{R}^n$  be a root system and suppose  $\sigma: \Phi \rightarrow \mathbb{R}^n$  is additive and bijective. Then  $\sigma$  is an automorphism of  $\Phi$ .

*Proof.* An automorphism of  $\Phi$  is equivalent to a linear automorphism of  $\mathbb{R}^n$  that leaves  $\Phi$  invariant [25, Section 9.2]. We apply this as follows. Let  $\Delta \subseteq \Phi$  be a choice of a set of simple roots; it forms a basis of  $\mathbb{R}^n$ . We define the linear map  $L: \mathbb{R}^n \rightarrow \mathbb{R}^n$  by  $L\delta = \sigma(\delta)$  for  $\delta \in \Delta$ .

Applying additivity inductively on the length of the constructing sequence in Lemma 3.3.1, we find that  $L\alpha = \sigma(\alpha)$  for all positive roots  $\alpha \in \Phi^+$ . Applying the second condition of additivity yields the same conclusion for the negative roots.

The map  $L$ , like  $\sigma$ , is bijective on  $\Phi$ , and therefore its image spans  $\mathbb{R}^n$ . It is therefore a linear automorphism of  $\mathbb{R}^n$  that leaves  $\Phi \subseteq \mathbb{R}^n$  invariant. Then its restriction  $L|_\Phi = \sigma$  is an automorphism of  $\Phi$ .  $\square$

With an eye towards eventually using this in the situation of equation (3.6), suppose that we are given an additive map  $\phi: \Phi \rightarrow \mathbb{R}$  and write  $\Lambda$  for the multi-set that is equal to its image counted with multiplicity. Let  $\Delta \subseteq \Phi$  be a choice of simple roots; then the equations

$$\phi(\delta) = (\delta, q) \text{ for all } \delta \in \Delta \quad (3.10)$$

determines a unique  $q \in \mathbb{R}^n$ . In fact, by  $\phi$ 's additivity, this  $q$  will even satisfy the extended set of equations

$$\phi(\alpha) = (\alpha, q) \text{ for all } \alpha \in \Phi \quad (3.11)$$

It is clear that if  $\sigma \in \text{Aut}(\Phi)$  is a root system automorphism then  $\phi \circ \sigma$  is another additive bijection. The converse is false in general but true generically:

**Proposition 3.3.4.** *Let  $\Phi \subseteq \mathbb{R}^n$  be a root system of rank  $n$ . Write  $|\Phi| = 2N$  and realize the set of multi-sets  $\Lambda$  with  $|\Lambda| = 2N$  as the quotient*

$$\mathbb{R}^{2N}/S_{2N} \quad (3.12)$$

(with  $S_{2N}$  the symmetric group on  $2N$  objects) with the quotient topology. Then there exists a dense  $U \subseteq \mathbb{R}^{2N}/S_{2N}$  such that if  $\Lambda \in U$  and  $\phi_1, \phi_2$  are two additive bijections  $\Phi \rightarrow \Lambda \subseteq \mathbb{R}$ , then there is an automorphism  $\sigma \in \text{Aut}(\Phi)$  such that  $\phi_1 \circ \sigma = \phi_2$ .

*Proof.* Choose any  $\sigma$  such that  $\phi_1 \circ \sigma = \phi_2$ . We intend to prove that  $\sigma$  is additive, in which case 3.3.3 yields the theorem. Note that even though  $\phi_1$  is assumed additive, there is no straightforward similar statement for its inverse  $\phi_1^{-1}$ : in general there may be triples  $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$  with  $\lambda_1 + \lambda_2 = \lambda_3$  without that being enforced by additivity of  $\phi_1$  and  $\phi_2$ . This is why  $\sigma$ 's additivity (and the theorem) is not trivial.

We write  $\mathbb{Z}^\Phi$  for the free abelian group with a set of generators indexed by  $\Phi$ . There is a canonical map  $\pi: \mathbb{Z}^\Phi \rightarrow \mathbb{Z} \cdot \Phi$  to the root lattice, whose kernel  $\ker \pi$  is generated by the additivity relations. The map  $\sigma$  induces a map  $\sigma_*: \mathbb{Z}^\Phi \rightarrow \mathbb{Z}^\Phi$ , and  $\sigma$  is additive if and only if  $\sigma_*$  maps  $\ker \pi$  to itself. In other words,  $\sigma$  is additive if and only if

$$\ker \pi \subseteq \ker \pi \circ \sigma_* \quad (3.13)$$

With the objective of proving this, consider  $\phi_1$  and  $\phi_2$ . Because it is assumed additive,  $\phi_i$  ( $i = 1, 2$ ) extends linearly to a map  $\phi_{i*}: \mathbb{Z} \cdot \Phi \rightarrow \mathbb{R}$ . With  $\sigma$  chosen such that  $\phi_1 \circ \sigma = \phi_2$  we have the following commutative diagram:

$$\begin{array}{ccccccc} \Phi & \hookrightarrow & \mathbb{Z}^\Phi & \xrightarrow{\pi} & \mathbb{Z} \cdot \Phi & \xrightarrow{\phi_{1*}} & \mathbb{R} \\ \downarrow \sigma & & \downarrow \sigma_* & & \downarrow \text{dotted} & & \parallel \\ \Phi & \hookrightarrow & \mathbb{Z}^\Phi & \xrightarrow{\pi} & \mathbb{Z} \cdot \Phi & \xrightarrow{\phi_{2*}} & \mathbb{R} \end{array} \quad (3.14)$$

The dotted arrow is a map that completes the diagram commutatively if and only if  $\sigma$  is additive.

We observe from the diagram that

$$\ker \phi_{1*} \circ \pi = \ker \phi_{2*} \circ \pi \circ \sigma_* \quad (3.15)$$



This means that

$$\begin{aligned} \ker \pi &\subseteq \ker \phi_{1*} \circ \pi \\ &= \ker \phi_{2*} \circ \pi \circ \sigma_* \text{ (by our observation)} \\ &= \ker \pi \circ \sigma_* + \sigma_*^{-1} \circ \pi^{-1} (\ker \phi_{2*}) \end{aligned}$$

so it is sufficient if we can prove that  $\ker \phi_{2*}$  is trivial.

Note that  $\ker \pi$  is generated by linear combinations of at most 3 generators. This means that it is actually sufficient to show that  $\sigma_*^{-1} \circ \pi^{-1} (\ker \phi_{2*})$  does not contain elements that small. In fact,  $\sigma_*$  preserves norms and  $\pi$  only makes them smaller, so it is sufficient if  $\ker \phi_{2*}$  does not contain elements of length smaller than  $\sqrt{3}$ .

Now,  $\ker \phi_{2*}$  is a hyperplane of co-dimension 1 in the ambient space  $F$  of the root lattice  $\mathbb{Z} \circ \Phi$ . For generic values of  $\phi_2$ 's coefficients  $\Lambda$ , this hyperplane has trivial intersection with

$$\mathbb{Z} \cdot \Phi \cap \{x \in F \mid |x| \leq \sqrt{3}\} \quad (3.16)$$

This means that

$$\ker \pi \subseteq \ker \pi \circ \sigma_* \quad (3.17)$$

for generic  $\Lambda$ . This implies that  $\sigma$  is additive. Together with Lemma 3.3.3 this proves the proposition.  $\square$

We need only slightly re-word this proposition to achieve the objective of this section.

**Theorem 3.3.5.** *Let  $\Phi \subseteq \mathbb{R}^n$  be a root system of rank  $n$  and let  $v$  be the map*

$$v: \mathbb{R}^n \rightarrow \mathbb{R}^\Phi \quad (3.18)$$

$$q \mapsto ((\alpha, q))_{\alpha \in \Phi} \in \mathbb{R}^\Phi \quad (3.19)$$

*Then  $v$  descends to a map*

$$v: \mathbb{R}^n / \text{Weyl}(\Phi) \rightarrow \mathbb{R}^\Phi / S_\Phi \quad (3.20)$$

*where  $S_\Phi$  is the symmetric group acting on  $\Phi$ . Writing*

$$k = [\text{Aut}(\Phi) : \text{Weyl}(\Phi)] \quad (3.21)$$

*there is a dense subset  $U \subseteq \text{im}(v)$  on which this map is exactly  $k : 1$ .*

*Proof.* As  $v$  factors through the map  $\mathbb{R}^n / \text{Weyl}(\Phi) \rightarrow \mathbb{R}^n / \text{Aut}(\Phi)$ , it is at least  $k : 1$  away from the fixed points of  $\text{Aut}(\Phi) / \text{Weyl}(\Phi)$ ; the set of fixed points is sparse for all cases where  $k > 1$ . A point  $\Lambda \in \mathbb{R}^\Phi / S_\Phi$  can be identified with a multi-set  $\Lambda \subseteq \mathbb{R}$  of size  $|\Lambda| = |\Phi|$ , and then 3.3.4 implies that generically, all pre-images are in the same  $\text{Aut}(\Phi)$ -orbit. Letting  $U \subseteq \text{im}(v)$  be the set of such points that are not fixed under  $\text{Aut}(\Phi) / \text{Weyl}(\Phi)$ , their pre-images are of size exactly  $k$ . This yields the statement.  $\square$

### 3.4 THE NON-GENERIC CASE BY COMPUTER SEARCH

The question remains whether the proof of Theorem 3.3.4 can be made to work for all, instead of just generic,  $\Lambda$ . The answer to this question is negative, at least for the cases  $\Phi = \Phi(A_5)$  and  $\Phi = \Phi(A_6)$ . In both of these cases, we present counter-examples in the proof of Theorem 3.4.3 below. It remains an open question whether counter-examples exist for any (or all) other root systems.

Let us first describe a naive way of finding these examples. Writing  $n$  for the rank of  $\Phi$  and  $N$  for the number of positive roots in  $\Phi$ , we randomly select an ordered set of values  $(v_i)_{i=1}^n \in \mathbb{N}^n$  for the simple roots  $(\alpha_i)_{i=1}^n$ . Using the additivity relations, these extend to an ordered set  $(v_j)_{j=1}^N$  of values for all the positive roots. We try all assignments  $(w_i)_{i=1}^n$  of  $n$  out of these  $N$  values to the  $n$  simple roots, and extend this assignment additively to  $(w_j)_{j=1}^N$ . Then, we count the cases where  $\{v_j\}_{j=1}^N$  and  $\{w_j\}_{j=1}^N$  are equal as *unordered* multi-sets. If this number is greater than  $[\text{Aut}(\Phi) : W]$ , we have found a counter-example.

Because trying all assignments for  $(w_i)_{i=1}^n$  results in a combinatorial explosion, we should expect that a computer search needs to be carefully implemented to make a dent in the search space—it quickly loses feasibility as the root system increases in size. Lemma 3.4.2 helps bounding the search space, and its formulation makes use of the following Definition.

**Definition 3.4.1.** For a root system  $\Phi$  and a choice of simple roots  $\alpha_1, \dots, \alpha_n$ , the sum of all positive roots in  $\Phi$ , although not a root itself, is a positive, integer linear combination of the simple

roots. We define the *auxiliary coefficients of  $\Phi$*  to be the coefficients  $(c_i)_{i=1}^n \in \mathbb{Z}_{>0}^n$  that satisfy the relation

$$\sum_{\beta \in \Phi^+} \beta = \sum_{i=1}^n c_i \alpha_i \quad (3.22)$$

where  $\Phi^+$  are the positive roots with respect to this choice of simple roots. When using these, we re-number the  $\alpha_1, \dots, \alpha_n$  such that  $c_1 \leq \dots \leq c_n$ .

**Lemma 3.4.2.** *Let  $\Phi$  be a root system with  $|\Phi^+| = N$  and  $c_1 \leq \dots \leq c_n$  be the auxiliary coefficients of  $\Phi$ . If  $f: \Phi \rightarrow \mathbb{N}$  is additive and its multi-set of values on  $\Phi^+$  is equal to*

$$\Lambda = \{\lambda_1 \leq \dots \leq \lambda_N\} \quad (3.23)$$

then

$$f(\alpha_i) \leq \frac{\sum_{\lambda \in \Lambda} \lambda - \sum_{j=1}^{n-1} c_{j+1} \lambda_{n-j}}{c_1} \quad (3.24)$$

for every simple root  $\alpha_k \in \Phi$ .

*Proof.* Let  $\Phi, \Lambda, f$  given as in the statement of the lemma, and let  $\alpha_1, \dots, \alpha_n$  be the simple roots of  $\Phi$ . By  $f$ 's additivity, we have

$$\sum_{\lambda \in \Lambda} \lambda = \sum_{i=1}^n c_i f(\alpha_i) \quad (3.25)$$

and for every  $1 \leq k \leq n$  isolating  $f(\alpha_k)$  gives

$$f(\alpha_k) = \frac{\sum_{\lambda \in \Lambda} \lambda - \sum_{i=1, i \neq k}^n c_i f(\alpha_i)}{c_k} \quad (3.26)$$

This implies that

$$f(\alpha_k) \leq \frac{\sum_{\lambda \in \Lambda} \lambda - \min_{\sigma \in S_N} \sum_{i=1, i \neq k}^n c_i \lambda_{\sigma(i)}}{c_k}$$

since  $S_N$  contains permutations  $\sigma$  for which  $\lambda_{\sigma(i)} = f(\alpha_i)$ . Moreover,

$$\min_{\sigma \in S_N} \sum_{i=1, i \neq k}^n c_i \lambda_{\sigma(i)} \geq \min_{\sigma \in S_N} \sum_{i=1}^{n-1} c_i \lambda_{\sigma(i)}$$

because on the right-hand side the set of  $c_i$  appearing in the summation are pair-wise no greater than the ones appearing on the left-hand side. Finally, we clearly have

$$\min_{\sigma \in S_N} \sum_{i=1}^{n-1} c_i \lambda_{\sigma(i)} = \sum_{i=1}^{n-1} c_i \lambda_{n-i} \quad (3.27)$$

as this pairs the smallest value  $\lambda_1 \in \Lambda$  with the largest coefficient  $c_{n-1}$ , the second smallest  $\lambda_2$  with the second largest  $c_{n-2}$ , et cetera.

We conclude that

$$f(\alpha_k) \leq \frac{\sum_{\lambda \in \Lambda} \lambda - \sum_{i=1}^{n-1} c_i \lambda_{n-i}}{c_k} \leq \frac{\sum_{\lambda \in \Lambda} \lambda - \sum_{i=1}^{n-1} c_i \lambda_{n-i}}{c_1} \quad (3.28)$$

(with the last inequality following from  $c_1 \leq c_k$ ) as claimed.  $\square$

The upshot is that we do not need to try all assignments of the values to the simple roots; only assignments involving values smaller than this upper bound need to be considered. In practice, we have observed that the upper bound is typically small enough to make this non-trivial, i.e., it is smaller than  $\max_{\lambda \in \Lambda} \lambda$ .

A further optimization is based on the realization that, if  $(w_i)_{i=1}^n$  extends in such a way that the multi-sets  $\{w_j\}_{j=1}^N$  and  $\{v_j\}_{j=1}^N$  are equal, then in particular the sums of their respective elements are equal. By additivity we have

$$\sum_{i=1}^n c_i w_i = \sum_{j=1}^N v_j \quad (3.29)$$

and the right-hand side of this equation does not depend on the assignments on the left-hand side. Because of this, we can loop over only assignments for the first  $n-1$  values  $(w_i)_{i=1}^{n-1}$ , and infer the value for  $w_n$  from (3.29). When this inferred value is not an integer or not in  $\Lambda$ , we can discard the assignment. Moreover, we

can efficiently compute the value of the left-hand side of (3.29) by traversing the permutations in such order that subsequent ones differ by a single swaps only (e.g., Heap's algorithm [24]). Namely, upon swapping  $w_i \leftrightarrow w_j$ , the left hand-side changes by  $+(c_i - c_j)(w_j - w_i)$ .

With these optimizations, our concrete implementation of this algorithm has the test for  $w_n \in \mathbb{Z}$  (i.e., a `div` instruction) as its hotspot, and a moderate computer can do 10-100 million of these operations per CPU per second. We used the Julia programming language [2] to achieve this performance. The algorithm is available on the author's github page.

The result of this is the following

**Theorem 3.4.3.** *Let  $\Phi = \Phi(A_5) \subseteq \mathbb{R}^5$  or  $\Phi = \Phi(A_6) \subseteq \mathbb{R}^6$ . For  $n = 5$  or  $n = 6$  respectively, let  $v$  be the map*

$$v: \mathbb{R}^n \rightarrow \mathbb{R}^\Phi \quad (3.30)$$

$$q \mapsto ((\alpha, q))_{\alpha \in \Phi} \in \mathbb{R}^\Phi \quad (3.31)$$

which descends to a map

$$v: \mathbb{R}^n / \text{Weyl}(\Phi) \rightarrow \mathbb{R}^\Phi / S_\Phi \quad (3.32)$$

Then there is a point  $\Lambda \in \mathbb{R}^\Phi / S_\Phi$  such that the pre-image  $v^{-1}(\Lambda)$  has more than  $[\text{Aut}(\Phi) : \text{Weyl}(\Phi)] = 2$  points.

*Proof.* Because the simple roots  $\Delta \subseteq \Phi$  form a basis, we can uniquely specify a point  $q$  in this pre-image by giving the values for  $((\alpha, q))_{\alpha \in \Delta}$ , i.e., the values on this basis. We first look at the case  $\Phi = \Phi(A_5)$ . In this case, the points  $q_1, q_2, q_3, q_4$  specified by

$$((\alpha, q_1))_{\alpha \in \Delta} = (7, 10, 9, 3, 2) \quad (3.33)$$

$$((\alpha, q_2))_{\alpha \in \Delta} = (5, 14, 3, 7, 2) \quad (3.34)$$

$$((\alpha, q_3))_{\alpha \in \Delta} = (2, 3, 9, 10, 7) \quad (3.35)$$

$$((\alpha, q_4))_{\alpha \in \Delta} = (2, 7, 3, 14, 5) \quad (3.36)$$

all map to a permutation of

$$\Lambda = \pm \{2, 3, 5, 7, 9, 10, 12, 14, 17, 19, 22, 24, 26, 29, 31\} \quad (3.37)$$

and  $4 > 2$  so this proves the statement for the  $A_5$  case. For the case  $\Phi = \Phi(A_6)$ , the points

$$((\alpha, q_1))_{\alpha \in \Delta} = (3, 3, 4, 1, 1, 1) \quad (3.38)$$

$$((\alpha, q_2))_{\alpha \in \Delta} = (2, 1, 2, 1, 6, 1) \quad (3.39)$$

$$((\alpha, q_3))_{\alpha \in \Delta} = (1, 1, 1, 4, 3, 3) \quad (3.40)$$

$$((\alpha, q_4))_{\alpha \in \Delta} = (1, 6, 1, 2, 1, 2) \quad (3.41)$$

all map to a permutation of

$$\Lambda = \pm \{1, 1, 1, 2, 2, 3, 3, 3, 4, 5, 6, 6, 7, 7, 8, 9, 10, 10, 11, 12, 13\} \quad (3.42)$$

proving the  $A_6$  case and the theorem.  $\square$

### 3.5 FUNDAMENTAL DOMAIN CROSSINGS

Having studied the combinatorial properties of  $v: \mathbb{R}^n / \text{Weyl}(\Phi) \rightarrow \mathbb{R}^\Phi / S_\Phi$  at a fixed time  $t$ , we now return to the time evolution of the Hamiltonian system. Given a point  $(p_0, q_0) \in \mathbb{R}^n / \text{Weyl}(\Phi)$  for  $t = 0$ , there is a continuous path  $(p(t), q(t)) \in \mathbb{R}^n / \text{Weyl}(\Phi)$ . Given  $t \in \mathbb{R}$ , our approach is to compute  $\Lambda(t)$  as in equation (3.6) so that by Theorem 3.3.5 there are generically  $k = [\text{Aut}(\Phi) : \text{Weyl}(\Phi)]$  options for  $q(t)$ . But only one is the right one, and we intend to find out which.

We restrict the following discussion to simply-laced, irreducible root systems. Moreover, we exclude the cases  $E_7, E_8$  (which are uninteresting as  $k = 1$ ) and the case  $D_4$  (which has  $k = 6$  and should be studied separately) so that  $k = 2$ .

**Proposition 3.5.1.** *Let  $\Phi \subseteq \mathbb{R}^n$  be a root system of rank  $n$  equal to  $A_n$  ( $n \geq 2$ ),  $D_n$  ( $n \geq 5$ ), or  $E_6$ , and let  $C \subseteq \mathbb{R}^n$  be the Weyl chamber associated to its simple roots. Choose an orbit  $\{\beta_1, \beta_2\}$  for the  $\text{Aut}(\Phi) / \text{Weyl}(\Phi) \cong \mathbb{Z}/2\mathbb{Z}$  action on the simple roots, and let  $H \subseteq C$  be the equidistant hyperplane between  $\beta_1$  and  $\beta_2$ . Write  $H_+$  and  $H_-$  for its two sides. Writing  $\text{Aut}(\Phi) / \text{Weyl}(\Phi) = \langle g \rangle$ , we then have  $H^g = H$ ,  $H_+^g = H_-$  and  $H_-^g = H_+$ .*

*Proof.* Because  $\Phi$  is of  $ADE$ -type all roots have the same length and so in particular  $H$  passes through the origin. In fact,  $H$  is the kernel of the map

$$h: a \in \mathbb{R}^n \mapsto (\beta_1 - \beta_2, a) \quad (3.43)$$

and  $H_{\pm}$  are the points where  $h$  takes positive and negative values. Because  $g$  is represented on  $\mathbb{R}^n$  by an isometry of order 2,  $h(a^g) = (\beta_1 - \beta_2, a^g) = (\beta_1^g - \beta_2^g, a) = -h(a)$ . This proves the proposition.  $\square$

We see that the  $k = 2$  pre-images are separated from each other by an equidistant hyperplane between two roots. With that in mind, let us have a more general look at equidistant hyperplanes associated to a root system. From now on, let  $A$  be a field and consider the transcendental extension  $A(q_1, \dots, q_n)$  where  $q_1, \dots, q_n$  are formal variables. We realize the root system  $\Phi$  as vectors in

$$\bigoplus_{i=1}^n \mathbb{Q} \cdot q_i \subseteq A(q_1, \dots, q_n) \quad (3.44)$$

and define the polynomial  $\chi_{\Phi} \in A(q_1, \dots, q_n)[\lambda]$  by

$$\chi_{\Phi} = \prod_{\beta \in \Phi} (\lambda - \beta) \quad (3.45)$$

We will eventually identify this polynomial with the characteristic polynomial of  $W(t)$  by substituting  $q_i \mapsto q_i(t)$ , but for now these variables remain formal. Let  $\mathcal{P}_{\Phi}$  be the set of unordered pairs of distinct roots in  $\Phi$ .

**Lemma 3.5.2.** *Let  $\Phi$  be as in 3.5.1 and consider the discriminant  $\Delta(\chi_{\Phi}) \in A(q_1, \dots, q_n)$ . We have*

$$\Delta(\chi_{\Phi}) = \prod_{\{\beta_1, \beta_2\} \in \mathcal{P}_{\Phi}} (\beta_1 - \beta_2)^2 \quad (3.46)$$

*Proof.* This follows immediately from the definition of the discriminant and from (3.45).  $\square$

Next, let  $\mathcal{H}_{\Phi}$  be the set of equidistant hyperplanes, i.e.,  $H \in \mathcal{H}_{\Phi}$  if and only if  $H$  is an affine hyperplane in  $\mathbb{R}^n$  and there are  $\beta_1, \beta_2 \in \Phi$  such that  $H$  is equidistant between  $\beta_1$  and  $\beta_2$ . As we remarked in the proof of Proposition 3.5.1, all  $H \in \mathcal{H}_{\Phi}$  pass through the origin. The group  $\text{Aut}(\Phi)$  acts both on  $\mathcal{P}_{\Phi}$  and on  $\mathcal{H}_{\Phi}$ , and there is an obvious map  $\mathcal{P}_{\Phi} \rightarrow \mathcal{H}_{\Phi}$  that respects these actions. We call the size of the pre-image of  $H \in \mathcal{H}_{\Phi}$  its *multiplicity*  $\mu(H)$ . It is  $\text{Aut}(\Phi)$ -invariant.

**Lemma 3.5.3.** *Let  $\Phi$  be as Proposition 3.5.1. Then in the field  $A(q_1, \dots, q_n)$ , the discriminant  $\Delta(\chi_\Phi)$  can be decomposed into factors associated to the hyperplanes:*

$$\Delta(\chi_\Phi) = a \prod_{H \in \mathcal{H}_\Phi} (f_H)^{\mu(H)} \quad (3.47)$$

with  $a \in \mathbb{Q}$  and  $f_H \in A(q_1, \dots, q_n)$  for all  $H \in \mathcal{H}_\Phi$ . Moreover, with  $\mu \text{ Aut}(\Phi)$ -invariant, we have

$$\Delta(\chi_\Phi) = a \prod_{\mathcal{O} \in \mathcal{H}_\Phi / \text{Aut}(\Phi)} \left( \prod_{H \in \mathcal{O}} f_H \right)^{\mu(\mathcal{O})} \quad (3.48)$$

*Proof.* The pairs  $\{\beta_1, \beta_2\}$  and  $\{\beta_3, \beta_4\}$  have the same image  $H \in \mathcal{H}_\Phi$  if and only if  $\beta_1 - \beta_2$  is a scalar multiple of  $\beta_3 - \beta_4$ . Applying this to equation (3.46) and absorbing all such scalar multiples into  $a \in \mathbb{Q}$  yields equation (3.47). From this, equation (3.48) and the lemma follow immediately.  $\square$

**Lemma 3.5.4.** *Let  $\Phi = \Phi(A_n)$  for some  $n \geq 2$ . Let  $H \in \mathcal{H}_\Phi$  and suppose either  $n = 2$  and  $\mu(H) = 4$ , or  $n \geq 3$  and  $\mu(H) = 2$ . Then  $H$  separates the fundamental domains of a Weyl chamber.*

*Proof.* We realize  $\Phi$  as the set of vectors in  $\mathbb{R}^{n+1}$  given by

$$\{e_i - e_j\}_{1 \leq i \neq j \leq n+1} \quad (3.49)$$

For a pair of distinct roots  $\{e_i - e_j, e_k - e_l\}$  with  $i \leq k$  for uniqueness, the associated hyperplane is the orthoplement of  $e_i + e_l - e_j - e_k$ , and  $\text{Aut}(\Phi)$  acts by permutations of the coordinates and by  $x \in \mathbb{R}^{n+1} \mapsto -x$ . The orbits are parametrized by the set of non-zero coordinate values up to simultaneous scaling. We find the following orbits:



Orbit	Pairs	Multiplicity	Size
$e_1 - e_2$	$\{e_1 - e_2, e_2 - e_1\}$ $\{e_1 - e_j, e_2 - e_j\}$ for $3 \leq j \leq n+1$ $\{e_j - e_2, e_j - e_1\}$ for $3 \leq j \leq n+1$	$1 + 2(n-1)$	$\frac{(n+1)n}{2}$
$2e_1 - e_2 - e_3$	$\{e_1 - e_2, e_3 - e_1\}$ $\{e_2 - e_1, e_3 - e_1\}$	2	$\frac{(n+1)n(n-1)}{2}$
$e_1 + e_2 - e_3 - e_4$	$\{e_1 - e_3, e_4 - e_2\}$ $\{e_1 - e_4, e_3 - e_2\}$ $\{e_2 - e_3, e_4 - e_1\}$ $\{e_2 - e_4, e_3 - e_1\}$	4	$\frac{(n+1)n(n-1)(n-2)}{8}$

When  $n$  is odd, both the first and the last equivalence classes separate the fundamental domain crossing; when  $n$  is even, it is both the middle and the last one. This yields the lemma.  $\square$

**Proposition 3.5.5.** *Write  $N = |\Phi|$  and suppose*

$$\chi_\Phi = \lambda^N + c_{N-1}\lambda^{n-1} + \cdots + c_0 \quad (3.50)$$

for  $c_0, \dots, c_{N-1} \in \mathbb{Q}(q_1, \dots, q_n)$ . Then the decomposition of  $\Delta(\chi_\Phi)$  given by (3.48) has factors in  $\mathbb{Q}[c_0, \dots, c_{N-1}] \subseteq \mathbb{Q}(q_1, \dots, q_n)$ .

*Proof.* It is well-known that the discriminant of any polynomial is a polynomial in the coefficients, so clearly  $\Delta(\chi_\Phi) \in \mathbb{Q}[c_0, \dots, c_{N-1}]$ . For its decomposition, we apply Galois theory to the field extension

$$\mathbb{Q}(q_1, \dots, q_n)/\mathbb{Q}(c_0, \dots, c_{N-1}) \quad (3.51)$$

Clearly  $\mathbb{Q}(q_1, \dots, q_n)$  contains the splitting field of  $\chi_\Phi$ , because  $\chi_\Phi$  splits according to (3.45), and in fact it is equal to it because we can conversely express the  $q_i$  as linear combinations of the  $\beta \in \Phi$  as the root system spans the ambient space  $\bigoplus_{i=1}^n \mathbb{Q} \cdot q_i$ .

The group  $\text{Aut}(\Phi)$  acts as a permutation on  $\Phi$  and, because they extend to invertible maps on  $\bigoplus_{i=1}^n \mathbb{Q} \cdot q_i$  by definition of  $\text{Aut}(\Phi)$ , every such permutation even extends uniquely to a field automorphism. As such,  $\text{Aut}(\Phi)$  is a subset of the Galois group of the field extension. In fact, it is the full group: if conversely  $\sigma$  is a field automorphism that permutes  $\Phi$ , then it is additive by virtue of being a morphism of fields. So Lemma 3.3.3 implies  $\sigma \in \text{Aut}(\Phi)$ . We see that

$$\text{Gal}(\mathbb{Q}(q_1, \dots, q_n)/\mathbb{Q}(c_0, \dots, c_{N-1})) \cong \text{Aut}(\Phi) \quad (3.52)$$

This means that any element in  $\mathbb{Q}(q_1, \dots, q_n)$  that is invariant under  $\text{Aut}(\Phi)$  is an element of  $\mathbb{Q}(c_0, \dots, c_{N-1})$ . The factor

$$\prod_{H \in \mathcal{O}} f_H \quad (3.53)$$

is clearly invariant as  $\text{Aut}(\Phi)$  permutes its orbit  $\mathcal{O}$ . The proposition follows.  $\square$

Taking it all together, we are now in a position to formulate and prove the main proposition about fundamental domain crossings.

**Proposition 3.5.6.** *Let  $\Phi \subseteq \mathbb{R}^n$  be a root system of rank  $n$  equal to  $A_n$  ( $n \geq 2$ ),  $D_n$  ( $n \geq 5$ ), or  $E_6$ . Let initial conditions  $p_0, q_0 \in \mathbb{R}^n$  be given for the Calogero–Moser system associated to  $\Phi$  and write  $t \mapsto q(t)$  for its time evolution. Write  $\mathbb{Q}(p_0, q_0)$  for the smallest field containing all coefficients of  $p_0$  and  $q_0$ . Let  $H$  be the hyperplane separating the fundamental domains of the Weyl chamber containing  $q_0$ , and  $h$  a co-vector that has  $H$  as its kernel. Then:*

1. *The zeros of the function  $t \mapsto h(q(t))$  are algebraic over  $\mathbb{Q}(p_0, q_0)$ ;*
2. *For  $\Phi = A_2$  the function  $t \mapsto h(q(t))^2$  is a polynomial with coefficients in  $\mathbb{Q}(p_0, q_0)$ ;*

*Proof.* Equations (3.3) and (3.4) define Hermitian matrices  $L_0$  and  $W_0$  with coefficients in  $\mathbb{Q}(i, p_0, q_0)$  and  $W(t) = W_0 + tL_0$  with coefficients in  $\mathbb{Q}(i, p_0, q_0)[t]$ . It then follows that

$$\chi(W(t)) \in \mathbb{Q}(p_0, q_0)[t, \lambda] \quad (3.54)$$

and

$$\Delta(\chi(W(t))) \in \mathbb{Q}(p_0, q_0)[t] \quad (3.55)$$

With  $\chi_\Phi$  as in Equation (3.45), (3.6) and (3.45) together imply

$$\chi_\Phi|_{q_1=q_1(t), \dots, q_n=q_n(t)} = \chi(\hat{W}(t)) \quad (3.56)$$

and

$$\Delta(\chi_\Phi)|_{q_1=q_1(t), \dots, q_n=q_n(t)} = \Delta(\chi(\hat{W}(t))) \quad (3.57)$$

and in particular

$$\Delta(\chi_\Phi)|_{q_1=q_1(t), \dots, q_n=q_n(t)} \in \mathbb{Q}(p_0, q_0)[t] \quad (3.58)$$

Since  $h$  is of the form  $(\beta_1 - \beta_2, \cdot)$  for some roots  $\beta_1, \beta_2 \in \Phi$  we have that  $h(q(t)) \mid \Delta(\chi_\Phi)$ . Then clearly its zeros are algebraic over  $\mathbb{Q}(p_0, q_0)$  which is the first statement.

As for the second statement, for  $\Phi = A_2$  Lemma 3.5.4 shows the following hyperplane orbits: an orbit of multiplicity 3 and length 3 that coincides with the Weyl chamber boundaries, and an orbit of multiplicity 2 and length 3. Then decomposition (3.48) is

$$\Delta(\chi_{A_2}) = \left( \prod_{i < j} (q_i(t) - q_j(t))^2 \right)^3 \left( \prod_{i \neq j < k} (2q_i(t) - q_j(t) - q_k(t))^2 \right)^2 \quad (3.59)$$

and the second factor is equal to  $t \mapsto h(q(t))^2$ . Then Proposition 3.5.5 yields the second statement.  $\square$



## THE TRIGONOMETRIC RUIJSENAARS–SCHNEIDER SYSTEM

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### 4.1 OVERVIEW

The Ruijsenaars–Schneider system, introduced in [41], first appeared as a relativistic generalization of the Calogero–Moser system, in the sense that its phase space carries a Poisson representation of the Poincaré group instead of the Galilean group. These two families are now jointly referred to as Calogero–Ruijsenaars systems. In each case, the interaction is defined by some pairwise and symmetric potential energy function proportional to  $r^{-2}$ :

$$V(r) = yr^{-2} + \mathcal{O}(r^{-4}) \quad (4.1)$$

for some coupling constant  $y$ . This distinguishes the hyperbolic ( $V(r) = y \cosh(r)^{-1}$ ), the trigonometric ( $V(r) = y \cos(r)^{-1}$ ) and the elliptic ( $V = \wp$ , the Weierstrass  $\wp$ -function for an elliptic curve) cases. A review for these different models and how some are obtained as an appropriate limit of others can be found in [20].

In the present chapter, we study the trigonometric version, whose Hamiltonian is given by

$$H_{\text{RS}}(p, q) = \sum_{i=1}^n \cos(p_i) \sqrt{\prod_{j \neq i} \left( 1 - \frac{\sin(\pi y)^2}{\sin(q_j - q_i)^2} \right)} \quad (4.2)$$

With the pairwise potential being periodic, the most natural physical interpretation is that the particles are constrained to a circle. The system has two additional interesting properties: first of all, it admits a duality between the angle coordinates (“positions”) and the action coordinates (“conserved quantities”), so that these can be interchanged while preserving the qualitative properties of the system. Secondly, it is possible to extend the phase space and its action/angle coordinates so that the result is compact.

In this chapter the compactified form is derived through a reduction process. The main benefit of the reduction approach is that

the action/angle duality is obvious on the unreduced space, providing an explanation for its existence. Our derivation is slightly different than the original work in [16, 17]: we make use of a generalization of Cauchy-like matrices that describe the compactification. To our knowledge, this generalization was not previously described in the literature.

After discussing some background material in the first section, we introduce the reduction process in the second. In the third section we discuss some points of the phase space that exist only in the compactification, and we discuss the points of the open (i.e., non-compactified) phase space in the fourth. The more technical proofs are postponed to Section 5.

## 4.2 PRELIMINARIES

In this section, we review *quasi-Hamiltonian reduction* [1] and the *Delzant theorem* [13, 23], and fix notation for the other sections in this chapter. First, we review parametrizations of the adjoint orbits of the special unitary group.

### 4.2.1 Adjoint orbits in the special unitary group

The special unitary group  $\mathrm{SU}(n)$  consists of complex  $n \times n$  matrices  $g$  for which  $\det(g) = 1$  and  $g^\dagger g = 1$ , where  $^\dagger$  is the Hermitian conjugate. Writing  $\mathbb{T}^{n-1} \subseteq \mathrm{SU}(n)$  for the subgroup of diagonal matrices, it is well-known that every  $b \in \mathrm{SU}(n)$  can be diagonalized by some  $g_b \in \mathrm{SU}(n)$ ; we write  $b^{g_b} = g_b^{-1} b g_b \in \mathbb{T}^{n-1}$  for diagonalization by the adjoint right-action. The multi-set of unordered diagonal entries are the eigenvalues of  $b$  and this multi-set characterizes the adjoint orbit uniquely. Writing  $S_n$  for the symmetric group on  $n$  objects,  $\mathbb{T}^{n-1}/S_n$  is a parametrization of the adjoint orbits in  $\mathrm{SU}(n)$ .

An element  $b$  is called *regular* if all its eigenvalues are distinct, and we write  $\mathrm{SU}(n)_{\mathrm{reg}}$  for the set of all regular elements. If  $b$  has duplicated eigenvalues it is called *non-regular*.

We now describe a linearised representation of these same adjoint orbits.

**Convention 4.2.1.** As a set, we realize the Lie algebra  $\mathfrak{su}(n)$  as

$$\mathfrak{su}(n) = \{\eta \in \mathfrak{u}(n) \mid \mathrm{tr}(\eta) = 2\pi i n\} \quad (4.3)$$

i.e., as the  $n \times n$  anti-Hermitian matrices with trace equal to  $2\pi i n$ . It is more common (and in many cases more convenient) to represent it as the *trace-less* anti-Hermitian matrices. Indeed,  $\mathfrak{su}(n)$  as defined above is not even closed under vector space operations or under the Lie bracket inherited from  $\mathfrak{u}(n)$ . However, this choice is convenient for us later (see Definition 4.2.4 and the related propositions) and it is made possible by the following. The translation by the matrix  $\xi^1 \in \mathfrak{su}(n)$  defined as

$$\xi^1 = 2\pi i \cdot \text{diag}(1, 1, \dots, 1) \quad (4.4)$$

gives a bijection between  $\mathfrak{su}(n)$  and the trace-less matrices in  $\mathfrak{u}(n)$ . This bijection intertwines the exponential map:

$$\exp(\eta - \xi^1) = \exp(\eta) \text{ for all } \eta \in \mathfrak{su}(n) \quad (4.5)$$

We can therefore endow  $\mathfrak{su}(n)$  with a vector space structure and a Lie bracket using this bijection as an identification. This preserves the usual relationship between  $\exp$  and the Lie algebra structure.

The matrix  $\xi^1$  together with the following  $n - 1$  matrices spans the maximal torus  $\mathfrak{t}^{n-1} \subseteq \mathfrak{su}(n)$ :

$$\begin{aligned} \xi^2 &= \frac{2\pi i}{n} \text{diag}(1, n+1, n+1, \dots, n+1) \\ \xi^3 &= \frac{2\pi i}{n} \text{diag}(2, 2, n+2, \dots, n+2) \\ \xi^4 &= \frac{2\pi i}{n} \text{diag}(3, 3, 3, n+3, \dots, n+3) \\ &\vdots \\ \xi^n &= \frac{2\pi i}{n} \text{diag}(n-1, n-1, \dots, n-1, n-1, 2n-1) \end{aligned} \quad (4.6)$$

which is to say that every  $\tau \in \mathfrak{t}^{n-1}$  can be expressed as an affine combination of these matrices, i.e.,

$$\tau = \sum_{i=1}^n \xi(\tau)_i \xi^i \quad (4.7)$$

for for some vector  $\xi(\tau) \in \mathbb{R}^n$  with  $\sum_{i=1}^n \xi(\tau)_i = 1$ . This defines a map  $\xi: \mathfrak{t}^{n-1} \rightarrow \mathbb{R}^n$ .

**Proposition 4.2.2.** *The exponential map intertwines the  $\mathbb{Z}/n\mathbb{Z}$ -action on the coordinates of  $\xi(\tau)$  with the multiplication action of the centre  $Z(\text{SU}(n)) \cong \mathbb{Z}/n\mathbb{Z}$  on  $\mathbb{T}^{n-1}/S_n$ .*

*Proof.* Let  $\tau \in \mathfrak{t}^{n-1}$  be given. It is sufficient if we prove the statement for a generator of  $\mathbb{Z}/n\mathbb{Z}$ ; so let  $\tau'$  be such that

$$\xi(\tau')_i = \xi(\tau)_{i+1} \text{ for } 1 \leq i \leq n-1, \xi(\tau')_n = \xi(\tau)_1 \quad (4.8)$$

We will show that the  $k$ th entry of  $\exp(\tau')$  and the  $(k+1)$ th entry of  $\exp(\tau)$  are related by multiplication by  $\zeta_n = e^{2\pi i/n}$ ; then an element of  $S_n$  maps  $\exp(\tau')$  to  $\zeta \exp(\tau)$  and the statement follows. From the definitions of the  $\xi^i$  it follows that

$$\xi^i - \xi^{i-1} = \frac{2\pi i}{n} \text{diag}(1, 1, \dots, 1) - 2\pi i \cdot \text{diag}(0, \dots, 0, 1, 0, \dots, 0) \quad (4.9)$$

for  $2 \leq i \leq n$ , where the non-zero element is at the  $(i-1)$ th place, and

$$\xi^1 - \xi^n = \frac{2\pi i}{n} \text{diag}(1, 1, \dots, 1, 1-n) \quad (4.10)$$

Now we compute

$$\tau - \tau' = \sum_{i=1}^n (\xi(\tau)_i - \xi(\tau')_i) \xi^i \quad (4.11)$$

$$= (\xi(\tau)_n - \xi(\tau)_1) \xi^n + \sum_{i=1}^{n-1} (\xi(\tau)_i - \xi(\tau)_{i+1}) \xi^i \quad (4.12)$$

$$= \xi(\tau)_1 (\xi^1 - \xi^n) + \sum_{i=2}^n \xi(\tau)_i (\xi^i - \xi^{i-1}) \quad (4.13)$$

and so on the  $k$ th diagonal entry we find

$$(\tau - \tau')_k = \frac{2\pi i}{n} - 2\pi i \xi(\tau)_{k+1} \text{ for } 1 \leq k \leq n-1 \quad (4.14)$$

$$(\tau - \tau')_n = \frac{2\pi i}{n} - 2\pi i \xi(\tau)_1 \quad (4.15)$$

(where we used  $\sum_{i=1}^n \xi(\tau)_i = 1$ ) and since

$$\exp(\tau)_k e^{2\pi i \xi(\tau)_{k+1}} = \exp(\tau)_{k+1} \quad (4.16)$$

we find that

$$\exp(\tau)_{k+1} = \zeta_n \exp(\tau')_k \quad (4.17)$$

This proves the proposition.  $\square$



**Proposition 4.2.3.** *Suppose the eigenvalues of  $b \in \mathrm{SU}(n)$  can be ordered as  $\lambda_1, \dots, \lambda_n$  such that the subsequent arc length differences are equal to  $2\pi\xi_1, \dots, 2\pi\xi_n$  with  $\sum_{i=1}^n \xi_i = 1$ . Then  $b$  is conjugate to*

$$\exp \left( \sum_{i=1}^n \xi_{\sigma(i)} \xi^i \right) \quad (4.18)$$

for some cyclic permutation  $\sigma$ , and also to

$$\zeta \exp \left( \sum_{i=1}^n \xi_i \xi^i \right) \quad (4.19)$$

for some  $\zeta \in Z(\mathrm{SU}(n))$ .

*Proof.* The arc differences fix all eigenvalues as soon as one of them is known; that means they determine the eigenvalues up to simultaneous  $S^1$ -multiplication. It follows from the definitions of the  $\xi^i$  (Equations (4.4) (4.6)) that if  $\delta \in \mathrm{SU}(n)$  is such that

$$\delta = \exp \left( \sum_{i=1}^n \xi_i \xi^i \right) \quad (4.20)$$

with  $\sum_{i=1}^n \xi_i = 1$ , then  $2\pi\xi_k$  is exactly the arc length between the  $(k-1)$ th and the  $k$ th diagonal entry of  $\delta$ . Therefore  $\delta$  has the same subsequent arc lengths as  $b$  and by the preceding observation it is related to  $b$  by multiplication by  $\zeta \in S^1$ ; and since both matrices have unit determinant actually  $\zeta \in Z(\mathrm{SU}(n))$ . This proves the second statement; together with Proposition 4.2.2 the first follows as well.  $\square$

**Definition 4.2.4.** The *standard  $n-1$  simplex*  $\Delta_{n-1} \subseteq \mathfrak{t}^{n-1}$  is the convex span of  $\xi^1, \dots, \xi^n$ . We write  $\partial\Delta_{n-1}$  for its boundary and  $\Delta_{n-1}^\circ$  for its interior.

**Proposition 4.2.5.** *The image of the restriction of the exponential map  $\exp: \mathfrak{su}(n) \rightarrow \mathrm{SU}(n)$  to  $\Delta_{n-1}$  intersects every adjoint orbit in exactly one point. Therefore, the induced map  $\exp: \Delta_n \rightarrow \mathbb{T}^{n-1}/S_n$  is bijective.*

*Proof.* Let  $\tau \in \Delta_{n-1}$ ; then  $\exp(\tau)$  has a set of eigenvalues whose consecutive arc-length differences are  $2\pi(\xi_\tau)_1, \dots, 2\pi(\xi_\tau)_n$ . Since these are all non-negative, this property determines the set of eigenvalues up to a simultaneous  $S^1$ -action; and since we additionally know that the determinant is equal to 1, it determines

the set of eigenvalues up to an  $n$ th root of unity—that is, up to multiplication by  $Z(\mathrm{SU}(n))$ . Since the set of eigenvalues determines an adjoint orbit, Proposition 4.2.2 finishes the proof for injectivity. Surjectivity follows similarly.  $\square$

**Definition 4.2.6.** We define the map  $\Xi: \mathrm{SU}(n) \rightarrow \Delta_n$  as the unique map such that  $\exp(\Xi(g))$  is in the adjoint orbit of  $g$  for all  $g \in \mathrm{SU}(n)$ .

An element  $\tau \in \Delta_{n-1}$  is an element of  $\partial\Delta_{n-1}$  exactly when one or more coefficients of  $\xi(\tau)$  vanish. In this case,  $\exp(\tau)$  has duplicated eigenvalues and so  $\exp(\tau) \notin \mathrm{SU}(n)_{\mathrm{reg}}$ . Similarly, the exponential map maps  $\Delta_{n-1}^\circ$  into  $\mathbb{T}_{\mathrm{reg}}^{n-1}$ .

#### 4.2.2 Quasi-Hamiltonian reduction

Let  $G$  be a connected Lie group. Following [1], we attempt to define a notion of a moment map similar to Definition 2.1.6 that takes values in  $G$  instead of  $(\mathrm{Lie} G)^\vee$ . For concreteness, consider a conjugation orbit  $\mathcal{O} \subseteq G$ , in which case the inclusion  $\mu: \mathcal{O} \hookrightarrow G$  is a natural candidate for such a map. We want to combine  $\mu$  with a two-form  $\omega$  on  $\mathcal{O}$  such that they recover the conjugation action in a way similar to Definition 2.1.6.

For  $\xi \in \mathrm{Lie} G$ , let  $v_\xi$  represent the infinitesimal left conjugation action. If  $g$  is a matrix coordinate for  $G$ , this tangent vector can be expressed as  $v_\xi = \xi g - g\xi$ . In this same coordinate system, the Maurer–Cartan forms  $\theta_\ell$  and  $\theta_r$  take the forms  $\theta_\ell = g^{-1}dg$  and  $\theta_r = dg g^{-1}$ . Applying contraction gives

$$\iota(v_\xi)(\theta_\ell) = \mathrm{Ad}_{g^{-1}} \xi - \xi \quad (4.21)$$

$$\iota(v_\xi)(\theta_r) = \xi - \mathrm{Ad}_g \xi \quad (4.22)$$

Using these identities, we would like to recover  $v_\xi$  through some linear function of  $\xi$  by applying a pull-back through  $\mu$ . The obvious *Ansatz* is

$$\omega(v_\xi, \cdot) = (\mu^* \langle \text{some } \mathfrak{g}\text{-valued one-form on } G \rangle, \xi) \quad (4.23)$$

As a canonical  $\mathfrak{g}$ -valued one-form on  $G$ , one is tempted to take, say, the left Maurer–Cartan form as follows

$$\omega(v_\xi, \cdot) = (\mu^* \theta_\ell, \xi) \quad (4.24)$$

However, this is not (manifestly) anti-symmetric:

$$\omega_g(v_\xi, v_\eta) = \iota(v_\eta)(\mu^*\theta_\ell, \xi) \quad (4.25)$$

$$= (\text{Ad}_{g^{-1}} \eta - \eta, \xi) \quad (4.26)$$

When enforcing anti-symmetry by subtracting the result of swapping  $\xi \leftrightarrow \eta$ , the term  $(\eta, \xi)$  cancels and we end up with the definition

$$\omega_g(v_\xi, v_\eta) = \frac{1}{2} ((\xi, \text{Ad}_g \eta) - (\eta, \text{Ad}_g \xi)) \quad (4.27)$$

which can be expressed in terms of  $\theta_\ell$  and  $\theta_r$  as

$$\omega(v_\xi, \cdot) = (\mu^*\vartheta, \xi) \text{ where } \vartheta := \frac{1}{2} (\theta_\ell + \theta_r) \quad (4.28)$$

The pair  $(\mathcal{O}, \omega)$  is not a symplectic manifold: the form  $\omega$  is neither closed nor non-degenerate. The latter of these means a loss of information because it means equation (4.28) does not uniquely determine the  $G$ -action (i.e.,  $v_\xi$ ) through  $\mu$  and  $\omega$ ; it is only defined up to

$$\ker \omega_g = \{v_\xi \mid \xi + \text{Ad}_g \xi = 0\} \quad (4.29)$$

(This loss can be recuperated e.g., through Proposition 4.6 in [1].) Even though  $\omega$  is not closed, we do have some control over its exterior derivative, namely

$$d\omega = -\mu^*\chi \text{ where } \chi = \frac{1}{12}(\theta_\ell, [\theta_\ell, \theta_\ell]) = \frac{1}{12}(\theta_r, [\theta_r, \theta_r]) \quad (4.30)$$

This example of  $\mu: \mathcal{O} \hookrightarrow G$  motivates the following definition:

**Definition 4.2.7.** Let  $(X, G, \omega, \mu)$  be a tuple of a manifold  $X$ , a connected Lie group  $G$  which acts on  $X$ , a  $G$ -invariant 2-form  $\omega$  on  $X$  and a  $G$ -equivariant map  $\mu: X \rightarrow G$  (w.r.t. conjugation on  $G$ ) such that

1.  $\omega(v_\xi, \cdot) = (\mu^*\vartheta, \xi)$  where  $v_\xi$  is the infinitesimal action of  $\xi \in \text{Lie } G$ ,
2.  $d\omega = -\mu^*\chi$ ,
3.  $\ker \omega_g = \{v_\xi \mid \xi + \text{ad}_g \xi = 0\}$ ;

then  $(X, \omega, \mu)$  is called a *quasi-Hamiltonian  $G$ -space*.

Its interest for us is rooted in the following

**Theorem 4.2.8** (Theorem 5.1 in [1]). *Let  $(X, \omega, \mu)$  be a quasi-Hamiltonian  $G$ -space and let  $\mu_0 \in G$  be a regular value of  $\mu$ . Write  $\mathcal{O}_{\mu_0}$  for its conjugacy class. Define*

$$M_{\mu_0} := \mu^{-1}(\mathcal{O}_{\mu_0})/G \quad (4.31)$$

*If  $M_{\mu_0}$  is smooth then together with the restriction of  $\omega$  to  $\mu^{-1}(\mathcal{O}_{\mu_0})$  it is a symplectic manifold.*  $\square$

**Proposition 4.2.9.** *Let  $(X, G, \omega, \mu)$  be a quasi-Hamiltonian  $G$ -space, let  $U \subseteq X$  and let  $f$  be a smooth,  $G$ -invariant function on  $U$ . Then there is a unique vector field  $v_f$  on  $U$  satisfying*

1.  $\iota(v_f)\omega = df$
2.  $\iota(v_f)\mu^*(\theta_\ell) = 0$

*Proof.* This is Proposition 4.6 in [1] modified to allow for  $f$  smooth and  $G$ -invariant only locally. The cited proof works unmodified.  $\square$

#### 4.2.3 The Delzant theorem

The last preliminary subject is the Delzant theorem, which classifies certain compact symplectic manifolds with a torus action.

If  $S^1$  acts by a Hamiltonian vector field  $h_f$  on a symplectic manifold, then  $f$  itself is a moment map: after all, we can identify  $(\text{Lie } S^1)^\vee \cong \mathbb{R}$  in a suitably scaled way. More generally, if the  $n$ -dimensional torus  $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$  acts by Hamiltonian vector fields then there is a moment map  $\mu: M \rightarrow \mathbb{R}^n$ . If additionally,  $M$  is compact, its image  $\mu(M) \subseteq \mathbb{R}^n$  is compact. In fact, the image is even more restricted:

**Theorem 4.2.10** (Atiyah-Guillemin-Sternberg [23] [35, Theorem 5.5.1]). *Suppose  $\mathbb{T}^n$  acts on a  $2n$ -dimensional compact connected symplectic manifold  $M$  with moment map  $\mu$ . Then  $\mu(M) \subseteq \mathbb{R}^n$  is a convex polytope with vertices given by the images of the fixed points of the  $\mathbb{T}^n$ -action.*  $\square$

**Theorem 4.2.11** (Théorème 2.1 in [13]). *Suppose  $M$  is a Hamiltonian toric manifold with moment map  $\mu$ . Then  $M$  together with this action is determined up to symplectomorphism by the image  $\mu(M)$ .*  $\square$

These two facts together will let us classify the Hamiltonian systems we obtain from reduction processes in the rest of this chapter.

## 4.3 THE SPECIAL UNITARY FUSED DOUBLE

We now introduce the quasi-Hamiltonian system that is the topic of this chapter. For a Lie group  $G$ , the space  $G \times G$  is a quasi-Hamiltonian system over the group  $G \times G$ , and any quasi-Hamiltonian system over  $G \times G$  yields a system over  $G$  through a process called “internal fusion”. For brevity we define the resulting *internally fused double* directly, but the reader may consult [1] for the full story.

**Definition 4.3.1.** Let  $G$  be a compact Lie group and write  $\theta_\ell, \theta_r$  for the left- and right-invariant Maurer–Cartan forms. Write  $\langle \cdot, \cdot \rangle$  for an invariant inner product on  $\text{Lie } G$ . Let  $X = G \times G$  and define a  $G$ -action by component-wise conjugation; i.e.,  $g \cdot (a, b) = (gag^{-1}, gbg^{-1})$ . We write  $\pi_a, \pi_b: X \rightarrow G$  for the projections onto the factors. Let  $m_\ell, m_r: X \rightarrow G$  be given by  $m_\ell(a, b) = ab$  and  $m_r(a, b) = ba$  and  $\mu: X \rightarrow G$  be given by  $\mu(a, b) = aba^{-1}b^{-1}$ . With  $\theta_\ell, \theta_r$  the left- and right-invariant Maurer–Cartan forms on  $G$ , we define a 2-form  $\omega$  on  $X$  by

$$\omega = \langle \pi_a^* \theta_\ell, \pi_b^* \theta_r \rangle + \langle \pi_a^* \theta_r, \pi_b^* \theta_\ell \rangle - \langle m_\ell^* \theta_\ell, m_r^* \theta_\ell \rangle \quad (4.32)$$

Then  $\tilde{D}(G) = (X, \omega, \mu)$  is called the *internally fused double* of  $G$ .

**Theorem 4.3.2** (Proposition 3.2 and Theorem 6.1 in [1]). *The internally fused double  $\tilde{D}(G)$  is a quasi-Hamiltonian  $G$ -space.*  $\square$

In particular, Proposition 4.2.9 associates a vector field to every  $G$ -invariant function  $f$  on  $X$ . Consider the case  $G = \text{SU}(n)$ ; we define the following invariant functions.

**Definition 4.3.3.** Let  $\Xi$  be as in Definition 4.2.6. We define  $\alpha, \beta: \tilde{D}(\text{SU}(n)) \rightarrow \mathfrak{t}^{n-1}$  through

$$\begin{aligned} \alpha &= \Xi \circ \pi_a \\ \beta &= \Xi \circ \pi_b \end{aligned}$$

Proposition 4.2.9 associates a flow to every invariant function. For  $\alpha$  and  $\beta$  these associated flows are as follows.

**Lemma 4.3.4** (Equations (2.25) and (2.26) in [16]). *Let  $(X, \omega, \mu) = \tilde{D}(\text{SU}(n))$  the internally fused double of the special unitary group. Let  $(a, b) \in X$ , and suppose  $g_a$  diagonalizes  $a$ . Then the flow associated to  $\alpha$  is*

$$\tau \in \mathbb{T}^{n-1} \mapsto \left( a, b \exp(\tau) g_a^{-1} \right) \quad (4.33)$$

With  $g_b$  defined similarly, the flow associated to  $\beta$  is

$$\tau \in \mathbb{T}^{n-1} \mapsto \left( a \exp(\tau) g_b^{-1}, b \right) \quad (4.34)$$

□

With an eye towards Theorem 4.2.10 we compute for which points this flow is fully contained in its  $G$ -orbit, as those are fixed points after reduction.

**Proposition 4.3.5.** *Let  $(X, \omega, \mu)$  as before and let  $(a, b) \in X$  such that  $b \in \mathrm{SU}(n)_{\mathrm{reg}}$  and let  $g_b$  diagonalize  $b$ . Then the flow of  $(a, b)$  under  $v_\beta$  is fully contained in its own adjoint orbit if and only if  $a^{g_b}$  is a generalized permutation matrix (i.e.,  $a^{g_b}$  has exactly one non-zero entry in each row and column) associated to a permutation with a single cycle of length  $n$ .*

*Proof.* Since the two statements that are claimed to be equivalent in the proposition are both invariant under the adjoint action, we may as well assume that  $b$  is diagonal. Then according to Lemma 4.3.4 the torus action associated to  $\beta$  is

$$(a\phi, b) \quad (4.35)$$

for some  $\phi = \mathrm{diag}(\phi_1, \dots, \phi_n) \in \mathbb{T}^{n-1}$ .

For “if”, suppose  $a$  is a generalized permutation matrix with a single cycle of length  $n$ ; we may re-number such that  $ae_i = e_{i+1}$  (where subscripts are interpreted modulo  $n$ ). We need to find  $\psi$  that commutes with  $b$  and for which  $(a\phi)^\psi = a$ . If  $\psi$  is to commute with  $b$  it needs to be diagonal, say  $\psi = \mathrm{diag}(\psi_1, \dots, \psi_n)$ . Then  $(a\phi)^\psi = a$  holds if and only if

$$\psi_i^{-1} a_i^j \phi_j \psi_j = a_i^j \quad (4.36)$$

for all  $i, j$  which, since  $a_{j+1}^j$  (indices modulo  $n$ ) are the only non-zero entries, is equivalent to

$$\phi_j = \psi_{j+1}^{-1} \psi_j \quad (4.37)$$

for all  $j$  (indices modulo  $n$ ). These equations are consistent as  $\phi_1 \cdots \phi_n = \det(\phi) = 1$ , and they fix the subsequent ratios, that is, the subsequent arc differences, between the eigenvalues of  $\psi$ . So like in the proof of Proposition 4.2.3, if we additionally require  $\det(\psi) = 1$  this gives exactly  $n$  solutions for  $\psi$ ; in particular there

is at least one such solution. So the flow of  $(a, b)$  is contained in its own adjoint orbit.

For “only if” suppose that the flow of  $(a, b)$  is contained in its own orbit. Then for every  $\phi$  there exists a diagonal  $\psi$  such that  $(a\phi)^\psi = a$ . Suppose  $(p_1, \dots, p_n)$  is such that  $a_{ip_i} \neq 0$  for  $1 \leq i \leq n$ ; there is at least two such distinct choices if  $a$  is not a generalized permutation matrix. Then

$$\psi_i^{-1} \psi_{p_i} = \phi_{p_i}^{-1} \quad (4.38)$$

for  $1 \leq i \leq n$ . But if for two different choices  $(p_1, \dots, p_n)$  and  $(p'_1, \dots, p'_n)$  we have  $p_k \neq p'_k$  then we must still have  $\phi_{p_k} = \phi_{p'_k}$ . But this cannot be as  $\phi$  is arbitrary. This contradiction shows  $a$  is a generalized permutation matrix.  $\square$

#### 4.4 VERTEX SOLUTIONS TO THE MOMENT MAP CONSTRAINT

Pick  $\gamma \in S^1 \subseteq \mathbb{C}$  in such a way that

$$\gamma^m \neq 1 \text{ for all } 1 \leq m \leq n \quad (4.39)$$

It will play the role of a coupling parameter; we will also write  $y = y_\gamma$  such that  $0 \leq y \leq 1$  and

$$\gamma = \exp(-2\pi i y) \quad (4.40)$$

Define the matrix

$$\mu_\gamma = \text{diag}(\gamma^{-1}, \gamma^{-1}, \dots, \gamma^{-1}, \gamma^{n-1}) \in \text{SU}(n) \quad (4.41)$$

and write  $\mathcal{O}_{\mu_\gamma} \subseteq \text{SU}(n)$  for its adjoint orbit. The main technical part is the parametrization of  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})/\text{SU}(n)$ ; in other words, finding solutions  $(a, b) \in \tilde{D}(\text{SU}(n))$  with

$$aba^{-1}b^{-1} \in \mathcal{O}_{\mu_\gamma} \quad (4.42)$$

Let us first parametrize the orbit  $\mathcal{O}_{\mu_\gamma}$ .

**Lemma 4.4.1.** *For a given  $\gamma \in S^1 \subseteq \mathbb{C}$  write  $\rho_\gamma = \gamma^n - 1$ . Then any  $g \in \mathcal{O}_{\mu_\gamma}$  can be written as*

$$g = \gamma^{-1} \left( 1 + \rho_\gamma v v^\dagger \right) \quad (4.43)$$

for some  $v \in S^{2n-1} \subseteq \mathbb{C}^n$  (i.e.,  $v \in \mathbb{C}^n$  and  $v^\dagger v = 1$ ), and conversely every such element is in  $\mathcal{O}_{\mu_\gamma}$ .

*Proof.* We have

$$\mu_\gamma = \gamma^{-1} (1 + \rho_\gamma \operatorname{diag}(0, 0, \dots, 0, 1)) \quad (4.44)$$

and so under conjugation by  $g \in \operatorname{SU}(n)$  we have

$$\mu_\gamma^g = \gamma^{-1} (1 + \rho_\gamma \operatorname{diag}(0, 0, \dots, 0, 1)^g) \quad (4.45)$$

The expression  $\operatorname{diag}(0, 0, \dots, 0, 1)^g$  is equal to  $vv^\dagger$  where  $v$  is the last column of  $g^\dagger$ , and we have  $v^\dagger v = 1$  because  $g$  is unitary. Conversely, for any  $v$  that is normalized in this way there is a  $g \in \operatorname{SU}(n)$  for which  $g^\dagger$  has  $v$  as its last column.  $\square$

Then equation (4.42) is therefore equivalent to  $(a, b) \in \tilde{D}(\operatorname{SU}(n))$  satisfying

$$a (\gamma b) a^{-1} = \left(1 + \rho_\gamma vv^\dagger\right) b \quad (4.46)$$

for some  $v \in S^{2n-1}$ .

The multi-set of eigenvalues of  $\gamma b$  is just equal to

$$\{\gamma\lambda \mid \lambda \text{ is an eigenvalue of } b\} \quad (4.47)$$

In light of Equation (4.46), we compare this set to the sets of eigenvalues of  $(1 + \rho_\gamma vv^\dagger) b$  as  $v$  ranges over  $S^{2n-1}$ . An interesting special case is the following.

**Lemma 4.4.2.** *Let  $b \in \operatorname{SU}(n)$  have eigenvalues  $\lambda_1, \dots, \lambda_n$  and suppose  $v \in S^{2n-1}$  is an eigenvector of  $b$  with  $bv = \lambda_n v$ . Then  $(1 + \rho_\gamma vv^\dagger) b$  is simultaneously diagonalizable with  $b$  and its eigenvalues are  $\lambda_1, \dots, \lambda_{n-1}, \gamma^n \lambda_n$ .*

*Proof.* Let  $w_1, \dots, w_{n-1}, w_n = v$  be a unitary basis for  $b$  such that  $bw_i = \lambda_i w_i$ . Then by unitarity,  $v^\dagger w_i = 0$  for all  $1 \leq i \leq n-1$ , and so

$$\left(1 + \rho_\gamma vv^\dagger\right) bw_i = \left(1 + \rho_\gamma vv^\dagger\right) \lambda_i w_i = \lambda_i w_i \quad (4.48)$$

for  $1 \leq i \leq n-1$ . For  $w_n = v$  we have

$$\left(1 + \rho_\gamma vv^\dagger\right) bv = \left(1 + \rho_\gamma vv^\dagger\right) \lambda_n v = (1 + \rho_\gamma) \lambda_n v \quad (4.49)$$

and since by definition  $1 + \rho_\gamma = \gamma^n$  the statement about eigenvalues follows. As the eigenvectors are identical,  $b$  and  $(1 + \rho_\gamma vv^\dagger)b$  are simultaneously diagonalizable.  $\square$



We conclude from this lemma that an eigenvector  $v$  associated to  $\lambda_n$  yields a solution  $a$  to Equation (4.46) if and only if the following equality of multi-sets holds:

$$\{\gamma\lambda_1, \dots, \gamma\lambda_n\} = \{\lambda_1, \dots, \lambda_{n-1}, \gamma^n\lambda_n\} \quad (4.50)$$

This is sufficiently restrictive that we can enumerate the finite number of possibilities that result.

**Lemma 4.4.3.** *Let  $b \in \mathrm{SU}(n)$  and  $v \in S^{2n-1}$  be an eigenvector of  $b$  such there exists an  $a \in \mathrm{SU}(n)$  satisfying Equation (4.46). Let  $\gamma$  satisfy Equation (4.39) and  $y = y_\gamma$  as in Equation (4.40). Let  $\tau \in \mathfrak{su}(n)$  be defined through*

$$\tau = (1 - (n-1)y)\xi^1 + y \sum_{i=2}^n \xi^i \in \mathfrak{t}^{n-1} \quad (4.51)$$

*Then  $b$  is conjugate to  $\exp(\tau^\sigma)$  for some cyclic  $\sigma \in \mathbb{Z}/n\mathbb{Z}$ . Moreover, if  $b$  is diagonal, then  $a$  is a generalized permutation matrix associated to a cyclic permutation of order  $n$ .*

*Proof.* Suppose  $b$  is as in the statement of the lemma and that has eigenvalues  $\lambda_1, \dots, \lambda_n$ . As observed above, these eigenvalues must satisfy

$$\{\gamma\lambda_1, \dots, \gamma\lambda_n\} = \{\lambda_1, \dots, \lambda_{n-1}, \gamma^n\lambda_n\} \quad (4.52)$$

and that means that there is an index  $i_1$  such that  $\gamma^n\lambda_n = \gamma\lambda_{i_1}$ . The arc difference between  $\lambda_n$  and  $\lambda_{i_1}$  is then  $2\pi(n-1)y$ . There is also an  $i_2$  such that  $\lambda_{i_1} = \gamma\lambda_{i_2}$ , and because of Equation (4.39),  $i_2 \neq i_1$ . The arc difference between  $\lambda_{i_1}$  and  $\lambda_{i_2}$  is  $2\pi y$ . Continuing this way, we find an ordering  $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_{n-1}}$ , with all indices distinct, and we may complete this with  $i_n = n$ . For this ordering, the subsequent arc-length differences are

$$2\pi(n-1)y, 2\pi y, 2\pi y, \dots, 2\pi y \quad (4.53)$$

which coincides with the arc differences of  $\exp(\tau)$ . Then by Proposition 4.2.3  $b$  is conjugate to  $\exp(\tau^\sigma)$  for some  $\sigma \in \mathbb{Z}/n\mathbb{Z}$ .

If  $b$  is diagonal and  $v$  an eigenvector, then in Equation (4.46) the adjoint action of  $a$  sends the diagonal matrix  $(1 + \rho_\gamma vv^\dagger)b$  to the diagonal matrix  $\gamma b$ . This implies  $a$  is a generalized permutation

matrix. Given the mapping of the eigenvalues described above, it must realize the permutation

$$i_k \mapsto i_{k+1} \quad (4.54)$$

$$i_n \mapsto i_1 \quad (4.55)$$

This is a cyclic permutation of order  $n$  as claimed.  $\square$

The result is that, *if* the reduced space is a Hamiltonian toric manifold, we know its fixed points and therefore its entire structure. We summarize this in the following theorem.

**Theorem 4.4.4.** *Consider internally fused double  $\tilde{D}(\mathrm{SU}(n))$  with the invariant function  $\beta$  as in Definition 4.3.3. Let  $\gamma \in S^1$  distinguish a moment map value  $\mu_\gamma$  according to Equation (4.41) such that  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  is smooth and connected, and such that the restriction of  $\beta$  to  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  is smooth. Then  $\beta$  descends to a smooth function  $\tilde{\beta}: \mu^{-1}(\mathcal{O}_{\mu_\gamma}) / \mathrm{SU}(n) \rightarrow \mathfrak{t}^{n-1}$ , and  $\mu^{-1}(\mathcal{O}_{\mu_\gamma}) / \mathrm{SU}(n)$  together with  $\tilde{\beta}$  is a Hamiltonian toric manifold. The image of the toric moment map  $\tilde{\beta}$  is given by the convex span of the images of  $\exp(\tau^\sigma)$ , i.e., the exponential image of the cyclic permutations of  $\tau \in \mathfrak{t}^{n-1}$  defined by*

$$\tau = (1 - (n-1)y)\xi^1 + y \sum_{i=2}^n \xi^i \in \mathfrak{t}^{n-1} \quad (4.56)$$

*Proof.* Since we assume that  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  is contained in the subset where  $\beta$  is smooth, the fact that  $\mu^{-1}(\mathcal{O}_{\mu_\gamma}) / \mathrm{SU}(n)$  together with  $\tilde{\beta}$  is a Hamiltonian toric manifold is immediate. The fixed points from Proposition 4.3.5 coincide with the case of Lemma 4.4.3 which is exactly the cyclic permutations of  $\tau$ . Then the image of  $\tilde{\beta}$  is their convex span by Theorem 4.2.10.  $\square$

#### 4.5 BULK SOLUTIONS OF THE MOMENT MAP CONSTRAINT

Knowing the shape of the toric manifold, we would now like to compute an explicit form of the angle coordinates by giving an explicit expression for  $a$  in Equation (4.42). With this objective we take a small detour through  $\mathrm{GL}(n, \mathbb{C})$ . For any diagonal  $b = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$  with  $\lambda_i \neq \lambda_j$  and  $\lambda_i \neq \gamma\lambda_j$  for all  $i \neq j$ , we consider the matrix  $L_\gamma(b) \in \mathrm{GL}(n, \mathbb{C})$  defined by

$$L_\gamma(b)_{ij} = \frac{1}{\lambda_i - \gamma\lambda_j} \quad (4.57)$$

**Lemma 4.5.1.** *Let  $b \in \mathbb{T}_{\text{reg}}^{n-1}$  (i.e.,  $b \in \mathbb{T}^{n-1} \cap \text{SU}(n)_{\text{reg}}$ ) and write  $a = L_\gamma(b)$ . Let  $\mu_\gamma$  be as in Equation (4.41). Then in  $\text{GL}(n, \mathbb{C})$ ,  $aba^{-1}b^{-1}$  is conjugate to  $\mu_\gamma$ .*

*Proof.* We compute the eigenspaces of  $aba^{-1}b^{-1}$ . For any  $\lambda \in \mathbb{C}$ , a vector  $v \in \mathbb{C}^n$  is in the kernel of  $(aba^{-1}b^{-1} - \lambda)$  if and only if  $a^{-1}b^{-1}v$  is in the kernel of  $(ab - \lambda ba)$ . The coefficients of this matrix are given by

$$(ab - \lambda ba)_{ij} = \frac{\lambda_j - \lambda \lambda_i}{\lambda_i - \gamma \lambda_j} \quad (4.58)$$

and substituting  $\lambda \mapsto \gamma^{-1}$  yields

$$(ab - \gamma^{-1}ba)_{ij} = -\gamma^{-1} \quad (4.59)$$

This matrix with all coefficients equal to  $-\gamma^{-1}$  has an  $(n-1)$ -dimensional kernel, and it follows that  $aba^{-1}b^{-1}$  has a  $(n-1)$ -dimensional eigenspace associated to the eigenvalue  $\gamma^{-1}$ . Since  $\det(aba^{-1}b^{-1}) = 1$  its last eigenvalue is  $\gamma^{n-1}$  with a one-dimensional eigenspace. It follows that  $aba^{-1}b^{-1}$  has a basis of eigenvectors and its eigenvalues are equal to those of the diagonal matrix  $\mu_\gamma$ ; they are therefore conjugate.  $\square$

We conclude that in  $\text{GL}(n, \mathbb{C})$ , the equation

$$aba^{-1}b^{-1} = \mu_\gamma^g \quad (4.60)$$

for some  $g \in \text{GL}(n, \mathbb{C})$  does not put a restriction on the eigenvalues of  $b$  as it does in the  $\text{SU}(n)$  case of Theorem 4.4.4. The matrix  $L_\gamma(b)$  is an example of a *Cauchy matrix* and such matrices lend themselves well to computation because explicit formulae are known for inverse and determinant.

**Definition 4.5.2.** An  $n \times n$  matrix  $a$  with coefficients in a field  $k$  is called *Cauchy matrix* if it is of the form

$$a_{ij} = \frac{1}{x_i - y_j} \quad (4.61)$$

for some vectors  $x, y \in k^n$  with all  $x_i \neq x_j$  and  $y_i \neq y_j$  for  $i \neq j$ , and  $x_i \neq y_j$  for any  $1 \leq i, j \leq n$ . For any such  $a$  we call its orbit under left- and right-multiplication by invertible diagonal

matrices its *Cauchy-like orbit*. Explicitly, any such matrix  $a$  is of the form

$$a_{ij} = \frac{s_i t_j}{x_i - y_j} \quad (4.62)$$

for some vectors  $s, t \in (k \setminus \{0\})^n$ . Any matrix in a Cauchy-like orbit is called *Cauchy-like*.

**Proposition 4.5.3** ([9, 44, Theorem 1]). *The determinant of a Cauchy matrix  $a$  as in Equation (4.61) is given by*

$$\det(a) = \frac{\prod_{i < j} (x_i - x_j)(y_j - y_i)}{\prod_{i,j=1}^n (x_i - y_j)} \quad (4.63)$$

and its inverse  $a^{-1} = \left( (a^{-1})_{ji} \right)_{j,i=1}^n$  is given by

$$(a^{-1})_{ji} = (x_i - y_j) \prod_{m \neq i} \frac{y_j - x_m}{x_i - x_m} \prod_{m \neq j} \frac{x_i - y_m}{y_j - y_m} \quad (4.64)$$

In particular, every Cauchy matrix is invertible.

We need the following generalization of this concept.

**Definition 4.5.4.** Let  $k$  be a field and vectors  $x, y \in k^n$  be given such that all coefficients  $x_i$  are distinct and all coefficients  $y_j$  are distinct. We call  $a \in \text{GL}(n, k)$  *quasi-Cauchy with respect to  $(x, y)$*  if there are non-zero vectors  $s, t \in k^n \setminus \{0\}$  such that

$$a_{ij} (x_i - y_j) = s_i t_j \quad (4.65)$$

for all  $1 \leq i, j \leq n$ . The *quasi-Cauchy orbit of  $(x, y)$*  is the set of all matrices that are quasi-Cauchy with respect to  $(x, y)$ .

*Remark 4.5.5.* Another way of describing such matrices is through the requirement that, for

$$\begin{cases} x = \text{diag}(x_1, \dots, x_n) \\ y = \text{diag}(y_1, \dots, y_n) \end{cases} \quad (4.66)$$

the matrix  $xa - ay$  has rank 1. This latter condition is called a *displacement structure for  $a$*  [26] and means that certain computations can be done through efficient algorithms even for large  $n$ . This point of view is related to the explicit formulae of Proposition 4.5.3 but we will not use it otherwise.

**Proposition 4.5.6.** *Every quasi-Cauchy matrix is a limit point of the set of Cauchy-like matrices.*

*Proof.* Suppose  $a$  is quasi-Cauchy and satisfies Equation (4.65). Choose two vectors  $u, v \in (k \setminus \{0\})^n$  such that

$$u_i = 1 \text{ if } s_i \neq 0 \quad (4.67)$$

$$v_j = 1 \text{ if } t_j \neq 0 \quad (4.68)$$

$$u_i v_j = a_{ij} \text{ if } x_i = y_j \quad (4.69)$$

This is possible because  $x_i = y_j$  implies  $s_i t_j = 0$ , and because all entries of  $x$  and  $y$  are distinct.

With  $u$  and  $v$  defined as such and with  $x, y, s, t$  as in Equation (4.65), define a matrix  $a(\varepsilon)$  through

$$a(\varepsilon)_{ij} = \frac{(s_i + u_i \varepsilon)(t_j + v_j \varepsilon)}{(x_i + \varepsilon s_i + \frac{1}{2} \varepsilon^2) - (y_j - \varepsilon t_j - \frac{1}{2} \varepsilon^2)} \quad (4.70)$$

Then  $a(\varepsilon)$  is a Cauchy-like matrix and one can check that

$$\lim_{\varepsilon \rightarrow 0} a(\varepsilon) = a \quad (4.71)$$

This proves that  $a$  is a limit point of the set of Cauchy-like matrices.  $\square$

A quasi-Cauchy matrix generalizes a Cauchy-like matrix by allowing  $x_i = y_j$  for some  $i, j$ : in this case  $s_i = 0$  or  $t_j = 0$  and so the  $i$ th row or the  $j$ th column of  $a$  has  $a_{ij}$  as its only non-zero entry. Moreover, in this case  $a_{ij}$  is an additional parameter of  $a$  unconstrained by the values of  $x, y, s, t$ .

There are no quasi-Cauchy matrices associated to  $(x, y)$  if every  $x_i$  coincides with some  $y_j$ , as it would follow that  $s = t = 0$ . When  $n - 1$  values coincide, this enforces a sufficient number of zeros that  $a$  is a generalized permutation matrix. This shows that quasi-Cauchy matrices are a generalization both of Cauchy-like matrices and of permutation matrices. It allows us to unify the cases from Lemma 4.5.1 (Cauchy-like) and Lemma 4.4.3 (permutation), namely:

**Proposition 4.5.7.** *Let  $b = \text{diag}(\lambda_1, \dots, \lambda_n) \in \text{SU}(n)_{\text{reg}}$  be given. Then  $a \in \text{GL}(n)$  solves*

$$aba^{-1}b^{-1} = \mu_{\gamma}^g \quad (4.72)$$

*for some  $g \in \text{GL}(n, \mathbb{C})$  if and only if  $a$  is quasi-Cauchy with respect to  $(x, y)$  given by  $x_i = \lambda_i$  and  $y_j = \gamma \lambda_j$ .*

*Proof.* By the same reasoning as in the proof of Lemma 4.5.1 we find that  $a$  solves this equation if and only if

$$ab - \gamma^{-1}ba \quad (4.73)$$

is a matrix of rank 1; say

$$ab - \gamma^{-1}ba = -\gamma^{-1}st^\top \quad (4.74)$$

for some non-zero vectors  $s, t \in \mathbb{C}^n$ . Then for the coefficient at  $(i, j)$  for  $1 \leq i, j \leq n$  we find

$$a_{ij}(\lambda_i - \gamma\lambda_j) = s_it_j \quad (4.75)$$

and so  $a$  is in the quasi-Cauchy orbit as claimed. This argument can be read in reverse to prove the converse.  $\square$

In view of this proposition, we would like to see when a given  $(x, y)$  has any associated quasi-Cauchy matrices that are also unitary. Let us first compute the analogue to Proposition 4.5.3.

**Lemma 4.5.8.** *The inverse of a quasi-Cauchy matrix  $a$  satisfies*

$$s_it_j(a^{-1})_{ji} = (x_i - y_j) \prod_{m \neq i} \frac{y_j - x_m}{x_i - x_m} \prod_{m \neq j} \frac{x_i - y_m}{y_j - y_m} \quad (4.76)$$

for all  $1 \leq i, j \leq n$ . Moreover, for any non-zero vectors  $s, t$  such that  $s_it_j = 0$  for every  $1 \leq i, j \leq n$  with  $x_i = y_j$  there exists an associated quasi-Cauchy matrix.

*Proof.* The first statement is the  $\varepsilon \rightarrow 0$  limit of Equation (4.70) and Proposition 4.5.3, and so it follows from continuity of inversion.

Equation (4.65) fixes  $a_{ij}$  whenever  $x_i \neq y_j$ ; the second statement asserts that where  $x_i = y_j$ , we can complete this with choices

for  $a_{ij}$  to make  $a$  invertible and therefore quasi-Cauchy. In the case  $s_i t_j = 0$  the computation of the same limit yields

$$u_i v_j (a^{-1})_{ji} = \begin{cases} 1 & \text{if } x_i = y_j \\ \frac{-t_q}{t_j} \prod_{m \neq i} \frac{y_j - x_m}{x_i - x_m} \prod_{m \neq j, q} \frac{x_i - y_m}{y_j - y_m} & \text{if } \begin{cases} x_i = y_q \ (q \neq j) \\ t_j \neq 0 \end{cases} \\ \frac{-s_p}{s_i} \prod_{m \neq i, p} \frac{y_j - x_m}{x_i - x_m} \prod_{m \neq j} \frac{x_i - y_m}{y_j - y_m} & \text{if } \begin{cases} y_j = x_p \ (p \neq i) \\ s_i \neq 0 \end{cases} \\ \frac{s_p t_q}{(x_i - y_j)} \prod_{m \neq i, p} \frac{y_j - x_m}{x_i - x_m} \prod_{m \neq j, q} \frac{x_i - y_m}{y_j - y_m} & \text{if } \begin{cases} (x_i, y_j) = (y_q, x_p) \\ (p \neq i, q \neq j) \\ (s_i, t_j) = (0, 0) \end{cases} \end{cases} \quad (4.77)$$

This yields finite values for the coefficients of  $a^{-1}$ , proving invertibility.  $\square$

**Lemma 4.5.9.** *Suppose that  $x, y$  are the respective diagonal entries of two elements of  $\mathbb{T}_{\text{reg}}^{n-1}$ . Then the quasi-Cauchy orbit of  $(x, y)$  intersects  $\text{SU}(n)$  if and only if*

$$(1 - \bar{x}_i y_i) (1 - \bar{y}_j x_j) \left( \prod_{m \neq i} \frac{x_i - y_m}{x_i - x_m} \right) \left( \prod_{m \neq j} \frac{y_j - x_m}{y_j - y_m} \right) \in \mathbb{R}_{\geq 0} \quad (4.78)$$

for all  $1 \leq i \leq n$ . This intersection is connected.

*Proof.* Pick  $a$  in the quasi-Cauchy orbit as specified by Equation (4.65). Then  $a$  is unitary by definition if  $\bar{a}_{ij} = (a^{-1})_{ji}$  for all  $1 \leq i, j \leq n$ ; and by Definition 4.5.4 we have

$$\bar{a}_{ij} (\bar{x}_i - \bar{y}_j) = \bar{s}_i \bar{t}_j \quad (4.79)$$

Together with Lemma 4.5.8 this implies

$$|s_i|^2 |t_j|^2 = |x_i - y_j|^2 \prod_{m \neq i} \frac{y_j - x_m}{x_i - x_m} \prod_{m \neq j} \frac{x_i - y_m}{y_j - y_m} \quad (4.80)$$

for all  $1 \leq i, j \leq n$ . For all  $x_i, y_j \in S^1$  one readily checks that that

$$|x_i - y_j|^2 = -\bar{x}_i \bar{y}_j (x_i - y_j)^2 \quad (4.81)$$

and using this we find equivalently

$$|s_i|^2 |t_j|^2 = \left( \bar{x}_i (x_i - y_i) \prod_{m \neq i} \frac{x_i - y_m}{x_i - x_m} \right) \left( \bar{y}_j (y_j - x_j) \prod_{m \neq j} \frac{y_j - x_m}{y_j - y_m} \right) \quad (4.82)$$

after grouping the factors depending on  $i$  and those depending on  $j$ . The left-hand side is clearly real and non-negative, so the case  $j = i$  implies that (4.78) holds.

Conversely, suppose (4.78) holds. We choose complex vectors  $s$  and  $t$  satisfying

$$|s_i|^2 = \left| \bar{x}_i (x_i - y_i) \prod_{m \neq i} \frac{x_i - y_m}{x_i - x_m} \right|$$

$$|t_i|^2 = \left| \bar{y}_i (y_i - x_i) \prod_{m \neq i} \frac{y_i - x_m}{y_i - y_m} \right|$$

It is clear that  $x_i = y_j$  implies  $s_i t_j = 0$  so according to Lemma 4.5.8 there exists a quasi-Cauchy matrix  $a$  associated to  $x, y, s, t$ . Moreover, for  $1 \leq i, j \leq n$  with  $x_i = y_j$  it holds that  $s_i = t_j = 0$ . Then according to Equation (4.65),  $a_{ij}$  is the only non-zero entry in its row and column and it is necessary that  $|a_{ij}|^2 = 1$  for unitarity. Moreover, the preceding derivation implies that this is a sufficient condition.

This proves the intersection with  $\mathrm{SU}(n)$  is non-empty. It is also clear that  $s_i$  and  $t_j$  are each fixed by this requirement up to multiplication by  $\mathbb{T}^n$ ; the intersection is therefore a continuous image of  $\mathbb{T}^n \times \mathbb{T}^n$  which proves connectedness.  $\square$

Applying this result to the case of  $L_\gamma(b)$  yields the following.

**Proposition 4.5.10.** *Let  $b = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathrm{SU}(n)_{\mathrm{reg}}$ . For  $1 \leq i \leq n$  let the functions  $z_\gamma(b)_i$  be defined by*

$$z_\gamma(b)_i = (1 - \gamma) \prod_{m \neq i} \frac{\lambda_i - \gamma \lambda_m}{\lambda_i - \lambda_m} \quad (4.83)$$

$$z_{\gamma^{-1}}(b)_i = (1 - \gamma^{-1}) \prod_{m \neq i} \frac{\lambda_i - \gamma^{-1} \lambda_m}{\lambda_i - \lambda_m} \quad (4.84)$$

Then we have

$$z_\gamma(b)_i z_{\gamma^{-1}}(b)_i \in \mathbb{R} \quad (4.85)$$



for all  $1 \leq i \leq n$ . Moreover, there is an  $a \in \mathrm{SU}(n)$  solving (4.42) if and only if

$$z_\gamma(b)_i z_{\gamma^{-1}}(b)_i \in \mathbb{R}_{\geq 0} \quad (4.86)$$

for all  $1 \leq i \leq n$ .

*Proof.* The second statement follows immediately from Lemma 4.5.9 by applying it to  $x_i = \lambda_i$  and  $y_i = \gamma \lambda_i$ .

As for reality, we compute

$$z_\gamma(b)_i z_{\gamma^{-1}}(b)_i = (1 - \gamma) (1 - \gamma^{-1}) \prod_{m \neq i} \frac{(\lambda_i - \gamma \lambda_m) (\lambda_i - \gamma^{-1} \lambda_m)}{(\lambda_i - \lambda_m)^2}$$

The factor  $(1 - \gamma) (1 - \gamma^{-1})$  is real since  $\bar{\gamma} = \gamma^{-1}$ . For reality of the other factors, pick some  $\varpi, \kappa_{im} \in S^1$  with  $\varpi^2 = \gamma$  and  $\kappa_{im}^2 = \lambda_i \lambda_m^{-1}$ . Then for every  $1 \leq i, m \leq n$  we have

$$\begin{aligned} \frac{(\lambda_i - \gamma \lambda_m) (\lambda_i - \gamma^{-1} \lambda_m)}{(\lambda_i - \lambda_m)^2} &= \left( \frac{\varpi^{-1} \lambda_i - \varpi \lambda_m}{\lambda_i - \lambda_m} \right) \left( \frac{\varpi \lambda_i - \varpi^{-1} \lambda_m}{\lambda_i - \lambda_m} \right) \\ &= \left( \frac{\varpi^{-1} \kappa_{im} - \varpi \kappa_{im}^{-1}}{\kappa_{im} - \kappa_{im}^{-1}} \right) \left( \frac{\varpi \kappa_{im} - \varpi^{-1} \kappa_{im}^{-1}}{\kappa_{im} - \kappa_{im}^{-1}} \right) \end{aligned}$$

For both factors, both numerator and denominator pick up a minus sign under complex conjugation. Then both factors are real.  $\square$

Putting this all together, we now have a way to parametrize  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})/\mathrm{SU}(n)$  and to recognize the Ruijsenaars–Schneider system.

**Definition 4.5.11.** For  $b = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathbb{T}_{\mathrm{reg}}^{n-1}$ , the *Ruijsenaars–Schneider Lax matrix* (or *Lax matrix* for short)  $\mathcal{L}_\gamma(b)$  is the quasi-Cauchy matrix associated to  $x, y, s, t, u, v$  given by

$$x_i = \lambda_i \quad (4.87)$$

$$y_i = \gamma \lambda_i \quad (4.88)$$

$$s_i = \lambda_i \sqrt{|z_\gamma(b)_i|} \quad (4.89)$$

$$t_i = \varpi \sqrt{|z_{\gamma^{-1}}(b)_i|} \quad (4.90)$$

$$u_i = v_i = 1 \quad (4.91)$$

for  $1 \leq i \leq n$ , where  $\varpi \in S^1$  is such that  $\varpi^2 = \gamma$ .

**Theorem 4.5.12.** *Let  $p = (p_i)_{i=1}^n$  and  $q = (q_i)_{i=1}^n$  both be vector coordinates of  $\mathfrak{t}^{n-1}$ . Let*

$$\begin{aligned}\theta(p) &= \exp(p) \\ b(q) &= \exp(2q)\end{aligned}$$

*and consider the Lax matrix  $\mathcal{L}_\gamma(b(q))\theta(p)$ . Writing  $\Re$  for the real part of a complex number, on the domain where it is defined the function*

$$H(p, q) = \Re \operatorname{tr} (\mathcal{L}_\gamma(b(q))\theta(p)) \quad (4.92)$$

*is equal to the trigonometric Ruijsenaars–Schneider Hamiltonian (4.2).*

*Proof.* On the specified domain the Lax matrix is a Cauchy-like matrix so, writing  $\lambda_i = \exp(2q_i)$  we find

$$\begin{aligned}\operatorname{tr}(\mathcal{L}_\gamma(b(q))\theta(p)) &= \sum_{i=1}^n \frac{\theta(p)_i}{\varpi^{-1} - \varpi} \sqrt{(1 - \gamma)(1 - \gamma^{-1}) \prod_{m \neq i} \frac{(\lambda_i - \gamma \lambda_m)(\lambda_i - \gamma^{-1} \lambda_m)}{(\lambda_i - \lambda_m)^2}} \\ & \quad (4.93)\end{aligned}$$

$$\begin{aligned}&= \sum_{i=1}^n \frac{\theta(p)_i}{\varpi^{-1} - \varpi} \sqrt{(\varpi^{-1} - \varpi)^2 \prod_{m \neq i} \frac{(\varpi^{-1} \lambda_i - \varpi \lambda_m)(\varpi \lambda_i - \varpi^{-1} \lambda_m)}{(\lambda_i - \lambda_m)^2}} \\ & \quad (4.94)\end{aligned}$$

Applying elementary trigonometry yields

$$\operatorname{tr}(\mathcal{L}_\gamma(b(q))\theta(p)) = \sum_{i=1}^n \theta(p)_i \sqrt{\prod_{m \neq i} \left(1 - \frac{\sin(\pi y)^2}{\sin(q_i - q_m)^2}\right)} \quad (4.95)$$

and clearly the real part is equal to  $H_{\text{RS}}$  as in Equation (4.2).  $\square$

*Remark 4.5.13.* This is suggestive of the fact that this reduced model coincides with the Ruijsenaars–Schneider model, but it is not a proof unless one also proves that the symplectic structures are equal. This is [16, Theorem 4].

#### 4.6 ANALYTIC PROPERTIES OF THE CONSTRAINT SURFACE

In Theorem 4.4.4 we derived, with relative ease, the complete shape of the quasi-Hamiltonian reduction at  $\mu_\gamma$ , under the assumption that this reduction is smooth and connected and that

the toric moment map is smooth. This section studies the validity of these assumptions for the possible values of the parameter  $\gamma \in S^1$ .

For points  $(a, b) \in \tilde{D}(\mathrm{SU}(n))$ , the non-regular values for  $b$  are problematic in two ways. First of all, this is where  $\beta$  from Definition 4.3.3 is not a smooth function. Secondly, Lemma 4.5.9, our main tool, does not hold for non-regular  $b$ , and this is not easily fixable. It is therefore convenient to limit ourselves to regular solutions.

**Definition 4.6.1.** The *regular part* of a subset  $U \subseteq \mathrm{SU}(n)$  is  $U \cap \mathrm{SU}(n)_{\mathrm{reg}}$ . The *regular part* of a subset  $U \subseteq \Delta_{n-1}$  is  $U \cap \Delta_{n-1}^\circ$ . The  *$\beta$ -regular part* of a subset  $U \subseteq \tilde{D}(\mathrm{SU}(n))$  is  $U \cap \beta^{-1}(\Delta_{n-1}^\circ)$ .

Fortunately, limiting our analysis to the  $\beta$ -regular part is not only convenient, but also reasonable. We will prove in Corollary 4.6.10 that for many cases, the  $\beta$ -regular part of our solution surface is a connected component of  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  and we may therefore study it independently; in particular Delzant's theorem applies to it.

**Theorem 4.6.2** ([16, Theorem 1]). *Let  $\gamma$  be as in Equation (4.39). Then the  $\beta$ -regular part of the inverse image  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  is an embedded smooth sub-manifold of  $\tilde{D}(\mathrm{SU}(n))$ , and the quotient  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})/\mathrm{SU}(n)$  is a smooth manifold.*  $\square$

The cited Theorem 1 is stated for  $0 < y < \frac{1}{n}$  but its proof works for any value of  $\gamma = e^{-2\pi i y}$  as long as Equation (4.39) holds.

**Lemma 4.6.3.** *The regular part of the image of  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  under  $\beta$ , i.e., the set*

$$\beta(\mu^{-1}(\mathcal{O}_{\mu_\gamma})) \cap \Delta_{n-1}^\circ \quad (4.96)$$

*is convex. In particular, it is connected.*

*Proof.* By Proposition 4.5.10, the regular part of the image of  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  under  $\beta$  consists of those points  $\xi \in \Delta_{n-1}^\circ$  for which  $z_\gamma(\exp(\xi))_i$  and  $z_{\gamma^{-1}}(\exp(\xi))_i$  from Equation (4.86) are non-negative for all  $1 \leq i \leq n$ . Define  $y \in [0, 1]$  such that  $\gamma = e^{-2\pi i y}$ ; the zero loci of  $(z_{\gamma^{-1}} \circ \exp)_i$  are given by the affine hyperplanes

$$H_{i,k} = \{\xi \in \Delta_{n-1} \mid \xi_i + \cdots + \xi_{i+k-1} = y\} \quad (4.97)$$

for  $1 \leq k \leq n$ , where subscripts are interpreted modulo  $n$ . For every point  $\xi$  and every  $i$  there is a number  $\kappa(\xi)_i$  such that

$$\xi_i + \cdots + \xi_{i+\kappa(\xi)_i-1} < y \quad (4.98)$$

$$\xi_i + \cdots + \xi_{i+\kappa(\xi)_i-1} + \xi_{i+\kappa(\xi)_i} > y \quad (4.99)$$

If  $\xi \in \Delta_{n-1}$ , then  $\kappa(\xi)_i \equiv \kappa(\xi)_j \pmod{2}$  for all  $i, j$  because

$$(z_{\gamma^{-1}} \circ \exp)_i > 0 \text{ for all } 1 \leq i \leq n \quad (4.100)$$

Moreover,  $\kappa(\xi)_i \leq \kappa(\xi)_{i+1}$ : suppose on the contrary that  $\kappa(\xi)_{i+1} < \kappa(\xi)_i$ ; because they are equal modulo 2 we have  $\kappa(\xi)_{i+1} + 2 \leq \kappa(\xi)_i$ . Then

$$y < \xi_{i+1} + \cdots + \xi_{i+1+\kappa(\xi)_{i+1}} \leq \xi_i + \cdots + \xi_{i+\kappa(\xi)_i-1} < y \quad (4.101)$$

and this is a contradiction. The same argument also proves  $\kappa(\xi)_n \leq \kappa(\xi)_1$  and we find that  $\kappa(\xi)_i = \kappa(\xi)$  for some  $\kappa(\xi)$  independent of  $i$ .

The sum of the  $n$  cyclic permutations of these inequalities together with  $\sum_{i=1}^n \xi_i$  shows

$$\kappa(\xi) < ny < \kappa(\xi) + 1 \quad (4.102)$$

which shows that the integer  $\kappa(\xi)$  only depends on  $y$  through

$$\frac{\kappa}{n} < y < \frac{\kappa + 1}{n} \quad (4.103)$$

Then the image of  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  under  $\beta$  is equal to the subset of  $\Delta_{n-1}$  that is the intersection of the half-spaces

$$\xi_i + \cdots + \xi_{i+\kappa-1} < y \quad (4.104)$$

$$\xi_i + \cdots + \xi_{i+\kappa-1} + \xi_{i+\kappa} > y \quad (4.105)$$

for all  $1 \leq i \leq n$ . In particular, it is convex and connected.  $\square$

**Corollary 4.6.4.** *The  $\beta$ -regular part of the inverse image  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  is connected.*

*Proof.* The map  $\beta$  is a map from the compact Hausdorff space  $\tilde{D}(\mathrm{SU}(n))$  to the Hausdorff space  $\Delta_{n-1}$ , and so  $\beta(U)$  is closed for every closed  $U$ . Write  $A \subseteq \mu^{-1}(\mathcal{O}_{\mu_\gamma})$  for the  $\beta$ -regular part, and suppose  $A$  is not connected, say  $A \subseteq U \cup V$  with  $U$  and  $V$  disjoint, closed, and  $A \cap U$  and  $A \cap V$  non-empty. By Lemma 4.6.3 the closed

sets  $\beta(U)$  and  $\beta(V)$  are not disjoint, say  $\xi \in \beta(U) \cap \beta(V)$ . Then the closed set  $\beta^{-1}(\{\xi\})$  is connected by Lemma 4.5.9 but it is also the union of its intersections with the closed sets  $U$  and  $V$ . This contradiction proves the corollary.  $\square$

**Proposition 4.6.5.** *For every admissible value of  $\gamma$ , the point*

$$\bar{\xi} = \frac{1}{n} \sum_{i=1}^n \xi^i \quad (4.106)$$

*is in image of (the regular component of)  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  under  $\beta$ .*

*Proof.* The set  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$  is non-empty by Lemma 4.4.3. It is also invariant under the  $\mathbb{Z}/n\mathbb{Z}$ -action, and convex by Lemma 4.6.3. For any point in the  $\beta$ -image,  $\bar{\xi}$  is equal to the average of its orbit, and this average is also within the convex span of the orbit. Then it is in the  $\beta$ -image.  $\square$

*Remark 4.6.6.* The eigenvalues of  $\exp(\bar{\xi})$  are exactly the  $n$  distinct  $n$ th roots of unity. This coincides with the eigenvalues of a permutation matrix associated to an  $n$ -cycle. Swapping the roles of  $\alpha$  and  $\beta$  we conclude that the value  $\beta = \bar{\xi}$  is associated to the vertices of the image of  $\alpha$  in the sense that any  $(a, b) \in \tilde{D}(\mathrm{SU}(n))$  for which  $\alpha(a, b)$  is a vertex of  $\mathrm{im}(\alpha)$  has  $b$  conjugate to  $\exp(\bar{\xi})$ .

Having settled the smoothness of the reduced manifold, we now turn our attention to the smoothness of the toric moment map  $\hat{\beta}$ . For this we take a closer look at  $\bar{\xi}$  and the hyperplanes  $H_{i,k}$  from the proof of Lemma 4.6.3.

**Definition 4.6.7.** The *cyclic  $(n-1)$ -polytope of parity  $p$  and scale  $y$*  is the convex, possibly unbounded subset  $\mathcal{B}_y^{n-1}(p) \subseteq \mathfrak{t}^{n-1}$  that contains  $\bar{\xi}$  and that is bounded by the hyperplanes

$$H_{i,p} = \{\xi \in \mathfrak{su}(n) \mid \xi_i + \xi_{i+1} + \cdots + \xi_{i+p-1} = y\} \quad (4.107)$$

where  $1 \leq i \leq n$  and where the indices of  $\xi$  are understood modulo  $n$ . For  $p = 0$  or  $p = n$  we set  $\mathcal{B}_y^{n-1}(0) = \mathcal{B}_y^{n-1}(n) = \mathfrak{t}^{n-1}$ .

We will use the following matrix  $M(n, p)$  to put  $\mathcal{B}_y^{n-1}(p)$  in a better position.

**Lemma 4.6.8.** *Suppose  $1 \leq p \leq n$  with  $\gcd(p, n) = 1$ , and consider the matrix  $M(n, p)$  whose  $i$ th row consists of  $p$  cyclic-consecutive non-zero entries between columns  $j$  and  $j + p - 1$ :*

$$M(n, p)_{ij} = \begin{cases} \frac{1}{p} & \text{if } \text{mod}(j - i, n) < p \\ 0 & \text{otherwise} \end{cases} \quad (4.108)$$

where  $\text{mod}(k, n)$  is the unique number  $0 \leq m < n$  such that  $m \equiv k \pmod{n}$ . Then  $M(n, p)$  is invertible.

*Proof.* The matrix  $M(n, p)$  is a circulant matrix [22] and by [22, Theorem 3.1] its eigenvalues are equal to

$$\frac{1}{p} \sum_{k=0}^{p-1} \zeta^k \quad (4.109)$$

5, where  $\zeta$  ranges over the  $n$ th roots of unity. Suppose  $M(n, p)$  is not invertible, then one of these eigenvalues vanishes, so there is an  $n$ th root of unity  $\zeta \neq 1$  such that  $\sum_{k=0}^{p-1} \zeta^k = 0$ . Then also

$$0 = (\zeta - 1) \sum_{k=0}^{p-1} \zeta^k = \zeta^p - 1 \quad (4.110)$$

so  $\zeta \neq 1$  is also a  $p$ th root of unity. This contradicts the assumption that  $\gcd(p, n) = 1$ . We conclude that  $M(n, p)$  is invertible.  $\square$

**Lemma 4.6.9.** *Suppose  $1 \leq p \leq n$  with  $\gcd(p, n) = 1$ . Write  $1 \leq q \leq n$  such that  $pq \equiv 1 \pmod{n}$ . Then  $\mathcal{B}_y^{n-1}(p)$  is a simplex, and its vertices are given by the cyclic permutations of*

$$v = \sum_{i=1}^n v_i \xi^i \in \mathfrak{t}^{n-1}$$

$$\text{where } v_i = \begin{cases} a & \text{if } 1 \leq \text{mod}(qi, n) \leq n - q \\ b & \text{otherwise} \end{cases} \quad (4.111)$$

where  $a, b \in \mathbb{R}$  are defined through

$$\begin{cases} (n - q)a + qb & = 1 \\ b - a & = p - ny \end{cases} \quad (4.112)$$

*Proof.* It follows immediately from the definitions that the image of  $\mathcal{B}_y^{n-1}(p)$  under  $M(n, p)$  is bounded by the hyperplanes  $x_i = y$  for  $1 \leq i \leq n$ . This cuts out a simplex  $\Delta_y$  from the hyperplane  $\sum_{i=1}^n x_i = 1$ ; Since  $M$  is invertible by Lemma 4.6.8,  $\mathcal{B}_y^{n-1}(p)$  is isomorphic to  $\Delta_y$  as a convex polytope and therefore a simplex.

The vertices of  $\mathcal{B}_y^{n-1}(p)$  are given by the points forming the intersection of  $n - 1$  out of the  $n$  bounding hyperplanes, and it is clear that they form the cyclic orbit of any given one. We compute the one that lies on  $H_{i,p}$  for all  $1 \leq i \leq n - 1$ . Write  $v$  as in the statement of the lemma; it satisfies

$$v_1 + \cdots + v_p = y = v_2 + \cdots + v_{p+1} \quad (4.113)$$

from which we conclude  $v_1 = v_{1+p}$ . Similarly,  $v_2 = v_{2+p}$ , and we find  $n - 2$  such relations ending with  $v_{n-2} = v_{n-2+p}$  (where all indices are understood modulo  $n$ ). Since  $n$  and  $p$  are mutually prime, this means  $v_i = v_{i+p}$  for almost all  $1 \leq i \leq n$ ; we only (possibly) have  $v_{n-1} \neq v_{p-1}$  and  $v_n \neq v_p$ . We may therefore summarize these relations as

$$v_p = v_{2p} = \cdots = v_{(n-q)p} = a \quad (4.114)$$

$$v_{(n-q+1)p} = v_{(n-q+2)p} = \cdots = v_{np} = b \quad (4.115)$$

for some  $a, b \in \mathbb{R}$ . Since  $qkp \equiv k \pmod{n}$ , this proves the shape of  $v$  from the statement of the lemma.

As for the equations for  $a$  and  $b$ : since  $v \in \mathfrak{t}^n$  we have  $1 = \sum_{i=1}^n v_i = (n - q)a + qb$  which is the first equation. For the second, we sum all cyclic permutations of  $v_1 + \cdots + v_p$ . On the one hand, the result is  $p$  since  $v \in \mathfrak{t}^{n-1}$ . On the other hand, we have  $n - 1$  cyclic permutations that each sum to  $y$  and the  $n$ th (omitted) cyclic permutation satisfies

$$v_n + v_1 + \cdots + v_{p-1} = v_n - v_p + (v_1 + \cdots + v_p) \quad (4.116)$$

$$= b - a + y \quad (4.117)$$

and so in total we obtain the second equation.  $\square$

One sees from Lemma 4.6.9 that as  $y$  approaches  $\frac{p}{n}$  the simplex  $\mathcal{B}_y^{n-1}(p)$  contracts onto the point  $\bar{\xi}$ . Then as  $y$  moves away from  $\frac{p}{n}$  the simplex grows and at some value of  $y$  its vertices reach  $\partial\Delta_{n-1}$ . The range of  $y$  for which it stays inside the interior  $\Delta_{n-1}^\circ$  of  $\Delta_{n-1}$  is described as follows.

**Corollary 4.6.10.** For  $\gcd(n, p) = 1$  and  $pq \equiv 1 \pmod{n}$ , the simplex  $\mathcal{B}_y^{n-1}(p)$  is contained in  $\Delta_{n-1}^\circ$  if and only if  $y \neq \frac{p}{n}$  belongs to the open interval  $\left(\frac{p}{n} - \frac{1}{nq}, \frac{p}{n} + \frac{1}{(n-q)n}\right)$ .

*Proof.* The simplex  $\mathcal{B}_y^{n-1}(p)$  is contained in  $\Delta_{n-1}^\circ$  if and only if its vertices are contained in  $\Delta_{n-1}^\circ$ , which means that both  $a$  and  $b$  in Equation (4.111) are positive. This happens exactly on the stated interval.  $\square$

**Lemma 4.6.11.** Suppose  $\gcd(n, p) = 1$  and take  $y$  from the interval given in Corollary 4.6.10 such that it is not an integer multiple of  $1/n$ . Then:

- If  $\frac{p}{n} < y < \frac{p}{n} + \frac{1}{(n-q)n}$ , then  $\mathcal{B}_y^{n-1}(p) \subseteq \mathcal{B}_y^{n-1}(p+1)^\circ$ .
- If  $\frac{p}{n} - \frac{1}{nq} < y < \frac{p}{n}$ , then  $\mathcal{B}_y^{n-1}(p) \subseteq \mathcal{B}_y^{n-1}(p-1)^\circ$ .

*Proof.* Let us pick a vertex  $v$  of the simplex  $\mathcal{B}_y^{n-1}(p)$  and recall that it is on all but one of the hyperplanes  $H_{i,p}$ . In particular, it satisfies at least one of the following two equations:

$$v_\ell + \cdots + v_{\ell+p-1} = y \quad (4.118)$$

or

$$v_{\ell+1} + \cdots + v_{\ell+p} = y \quad (4.119)$$

for each  $\ell = 1, \dots, n$ . Suppose now that  $\frac{p}{n} < y < \frac{p}{n} + \frac{1}{(n-q)n}$ , which entails that the polyhedron  $\mathcal{B}_y^{n-1}(p+1)^\circ$  is given by the inequalities

$$\xi_\ell + \cdots + \xi_{\ell+p} > y \quad (4.120)$$

The fact that all components of  $v$  are positive implies by that the vertex  $v$  of  $\mathcal{B}_y^{n-1}(p)$  lies in  $\mathcal{B}_y^{n-1}(p+1)^\circ$ . The case  $\frac{p}{n} - \frac{1}{nq} < y < \frac{p}{n}$  is settled similarly.  $\square$

**Theorem 4.6.12.** Suppose  $\gamma = e^{-2\pi i y}$  is such that

$$\frac{p}{n} - \frac{1}{nq} < y < \frac{p}{n} + \frac{1}{(n-q)n} \quad (4.121)$$

for some  $p, q \in \{1, \dots, n-1\}$  such that  $pq \equiv 1 \pmod{n}$ . Write  $V$  for the  $\beta$ -regular part of  $\mu^{-1}(\mathcal{O}_{\mu_\gamma})$ . Then the class functions  $\beta$  descend to a smooth function  $\tilde{\beta}: V/\mathrm{SU}(n)$ , and  $V/\mathrm{SU}(n)$  together with  $\tilde{\beta}$  is a Hamiltonian toric manifold.



*Proof.* It follows from the proof of Lemma 4.6.3 that  $V$  is given by

$$V = \mathcal{B}_y^{n-1}(\kappa) \cap \mathcal{B}_y^{n-1}(\kappa + 1) \quad (4.122)$$

where  $\kappa$  is defined through

$$\frac{\kappa}{n} < y < \frac{\kappa + 1}{n} \quad (4.123)$$

We see that on the interval under consideration,  $\kappa = p - 1$  or  $\kappa = p$ ; we will focus our proof on the second case but the first can be treated similarly. In the second case,

$$\frac{p}{n} < y < \frac{p}{n} + \frac{1}{(n - q)n} \quad (4.124)$$

and by Lemma 4.6.11 and Corollary 4.6.10 we have

$$V = \mathcal{B}_y^{n-1}(p) \cap \mathcal{B}_y^{n-1}(p + 1) = \mathcal{B}_y^{n-1}(p) \subseteq \Delta_{n-1}^\circ \quad (4.125)$$

Then  $\beta$  is smooth on  $V$  and the theorem follows.  $\square$



Part II

ALGORITHMIC ASPECTS OF  
LANDAU–GINZBURG MODELS



## OVERVIEW OF PART II

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The objective of this chapter is to describe the main results included in Part II of this thesis. This second part describes a different class of models, namely *Landau–Ginzburg models*. This second part is more algebraic and less analytic in nature than the first.

### 5.1 LANDAU–GINZBURG CATEGORY

We now describe a pedestrian version of the *Landau–Ginzburg category* over an algebraically closed field  $k$ . The description is only intended to provide background about the fusion product and about orbifold equivalence, which are the topics of the subsequent chapters. As such, the description is incomplete and omits technical details that would otherwise be essential. In particular, we omit its bi-category structure. The reader is referred to [6] for the full picture.

Consider  $(R, f)$  with  $R$  a polynomial ring over  $k$  and  $f \in R$  satisfying a certain finiteness criterion (Definition 6.2.3). Such  $f$  is called a *potential*. A *matrix factorization of  $f$*  is a finitely generated, free  $R$ -module  $X$  together with an endomorphism  $d_X: X \rightarrow X$  that has a prescribed block structure (it is *odd* as in Definition 6.2.13) such that  $d_X^2 = f \cdot \text{id}$ .

The category of interest has objects equal to pairs  $(R, f)$ , i.e., potentials. For two objects  $(R, f)$  and  $(S, g)$ , a morphism between them is a matrix factorization of  $f - g$  with coefficients in  $R \otimes S$ . In other words, it is a free, f.g.,  $R \otimes_k S$ -module  $Q$  together with an odd endomorphism  $d_Q$  that squares to the identity matrix multiplied by  $f \otimes 1 - 1 \otimes g \in R \otimes_k S$ . We will usually write  $(f - g) \text{id}$  or even just  $f - g$ .

Composition in this category is provided by the following. Let  $Q_1$  be a morphism between  $(R, f)$  and  $(S, g)$  and  $Q_2$  between  $(S, g)$  and  $(T, h)$ . As a module, define  $Q_3 = Q_1 \otimes_S Q_2$  and let

$$d_{Q_1 \otimes Q_2} = d_{Q_1} \otimes_S 1 + 1 \otimes_S d_{Q_2} \quad (5.1)$$

(with a sign convention 6.2.14). It has the property that  $g$  cancels in such a way that

$$d_{Q_1 \otimes Q_2}^2 = (f - h) \text{id} \quad (5.2)$$

which is what a morphism between  $(R, f)$  and  $(T, h)$  should satisfy. The module  $Q_3$  is a f.g., free  $R \otimes_k S \otimes_k T$ -module, and so in particular it is a free  $R \otimes_k T$ -module—but it is not finitely generated over the latter. However, it is *homotopy equivalent* (Definition 7.3.6) to a finitely generated matrix factorization  $Q'_3$  over  $R \otimes_k T$ , and it is this latter matrix factorization that we use as a concrete realization of the composition of  $Q_1$  and  $Q_2$ . We write  $Q_1 \otimes Q_2$  (without a subscript) for this composition.

## 5.2 FUSION PRODUCT

The concrete realization  $Q'_3$  of  $Q_1 \otimes Q_2$  was constructed in [14] in two steps. In the first, a matrix factorization  $Q$  of finite rank over  $S \otimes_k T$  is constructed together with a map of modules  $e: Q \rightarrow Q$ . This map has the property that  $ed_Q = d_Q e$  (it is an *endomorphism of the matrix factorization*) and moreover it is *idempotent up to homotopy*, i.e.,  $e^2 - e$  is homotopy equivalent to the zero map. Moreover, they show that  $Q_1 \otimes Q_2$  is a splitting of  $e$ .

As a splitting is defined up to isomorphism, we may compute any splitting of  $e$  to compute  $Q_1 \otimes Q_2$ . This is the second step. This thesis's contribution is the following.

**Theorem D** (Proposition 7.4.6). *Let  $Q$  be a quasi-homogeneous matrix factorization and let  $e: Q \rightarrow Q$  be a morphism idempotent up to homotopy. Then there is an algorithm that computes a splitting of  $e$ . Moreover, this algorithm applies only field operations on the coefficients of  $d_Q$ , and in particular the resulting matrix factorization has coefficients in the the smallest field over which  $d_Q$  is defined.*

The importance of this is the following theorem.

**Theorem E** (Corollary 7.4.7). *Let  $Q_1$  and  $Q_2$  be quasi-homogeneous, composable matrix factorizations defined over fields  $k_1, k_2 \subseteq k$  respectively. Then there is an algorithm that computes a finite-rank representative of  $Q_1 \otimes Q_2$ . Moreover, the resulting matrix  $d_{Q_1 \otimes Q_2}$  has coefficients in the smallest field containing both  $k_1$  and  $k_2$ .*

### 5.3 ORBIFOLD EQUIVALENCE

Orbifold equivalence is an equivalence relation on the objects of the Landau–Ginzburg category. Two potentials  $f$  and  $g$  are equivalent if and only if there exists a morphism between them (i.e., a matrix factorization of  $f - g$ ) with a certain non-degeneracy property (Definition 6.2.20).

In Chapter 6 we present a search algorithm for determining whether a given pair  $f$  and  $g$  are orbifold equivalent. This search algorithm follows parallel work from [38].

**Theorem F** (Theorem 6.3.2; [38]). *Given two potentials  $f$  and  $g$ , there exists an algorithm that terminates if and only if  $f$  and  $g$  are orbifold equivalent.*

Moreover, we present heuristic arguments indicating that the required Gröbner basis computations are out of reach of current computation hardware by comparing to cryptology research.

### 5.4 SOFTWARE PACKAGE

In addition to the text of this thesis, we make a concrete implementation of the algorithms described available as a software package. This package also helpfully contains a library of known orbifold equivalences to aid the user with experimentation. Some comments on the concrete implementation are contained in Chapter 7.





## COMPUTATIONAL ASPECTS OF ORBIFOLD EQUIVALENCE

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### 6.1 OVERVIEW

Initially a model to describe superconductivity, Landau–Ginzburg models were promoted in the late 80s to 2-dimensional  $(2, 2)$ -supersymmetric sigma models completely characterized by a polynomial  $W$  called *potential* [47]. Landau–Ginzburg models gained importance in string theory and algebraic geometry as they form a family of quantum field theories which are related under homological mirror symmetry [15, 48]. Furthermore, they are connected to cohomological field theories via [37]. This makes it natural to ask whether we can define some notion of “equivalence” between different potentials. The notion of orbifold equivalence was inspired by the study of (defects in) topological quantum field theories (see [8, 12, 19]) and it was first defined in the context of the study of equivariant and orbifold completions of the bicategory of Landau–Ginzburg models. Several examples have been explored in detail in the recent years [7, 38–40].

An extra reason to study orbifold equivalences is that it may be used to generate examples of the so-called Landau–Ginzburg/conformal field theory (LG/CFT) correspondence. This physics result states that the infrared fixed point of a Landau–Ginzburg model with potential  $f$  is a 2-dimensional rational conformal field theory (CFT) with central charge  $c_f$ . At the defects level, this predicts some relation between two seemingly different mathematical entities: matrix factorizations (which describe defects for Landau–Ginzburg models [4]) and representations of the vertex operator algebra of the CFT (describing defects for the rational CFT). We lack a precise mathematical statement for this result, yet there are several promising examples available of this correspondence. In the particular case of simple singularities, it was proven in [7] that via orbifold equivalence one finds exactly the predicted equivalences for the  $N = 2$  supersymmetric minimal models. Furthermore, there are physics results suggesting that this might not be the only case, involving Landau–Ginzburg models with poten-

tials describing singularities of modality greater than 0 [10, 32, 33]. Hence, finding further orbifold equivalences is potentially a source of further examples of equivalences within the LG/CFT correspondence. This would strongly enhance our mathematical understanding of this intriguing physics result.

The present chapter is concerned with finding orbifold equivalences using computer search. The current state of the art is the algorithm presented in [38]. As recorded in Theorem 6.3.2, this algorithm terminates if and only if two potentials are orbifold equivalent. In pertinent examples, we quantify the size of these computations, and compare these sizes to current bests in solving these systems: the Fukuoka MQ challenge [49]. As such, we show that experimental infeasibility was not an accident that can be solved by choosing a different implementation (as was speculated in [38]) but that these computations lie well beyond what current technology enables.

## 6.2 ORBIFOLD EQUIVALENCE

### *Potentials*

Let  $k$  be an algebraically closed field of characteristic zero.

**Definition 6.2.1.** A *graded polynomial ring* is a polynomial ring in a finite number of variables over  $k$ , each variable endowed with a fixed grading in  $\mathbb{Q}_{>0}$ . We write  $\mathcal{R}$  for the category of graded polynomial rings, and

$$R = \bigoplus_{q \in \mathbb{Q}_{\geq 0}} R_q \tag{6.1}$$

for the equal-grading direct summands of  $R \in \mathcal{R}$ . We call the elements of  $R_q$  *quasi-homogeneous (of grade  $q$ )*. Note that  $R_0 = k$ .

*Remark 6.2.2.* We use *quasi-homogeneous* to emphasize the fact that variables do not necessarily have grading equal to 1 and that these gradings can take values in  $\mathbb{Q}$ . In some literature the elements of uniform degree of a graded ring are called *homogeneous* regardless of the specific grading. The same holds true for (quasi-)homogeneous ideals, morphisms, et cetera, which will play a role later.

**Definition 6.2.3.** For  $R = k[x_1, \dots, x_n] \in \mathcal{R}$  and  $f \in R$ , the *Jacobian ideal*  $I_f$  of  $f$  is the ideal generated by the partial derivatives of  $f$ :

$$I_f = (\partial_{x_1} f, \dots, \partial_{x_n} f) \quad (6.2)$$

The *Jacobian* of  $f$  is  $\text{Jac } f = R/I_f$ . We call  $(R, f)$  a *potential* if  $f$  is quasi-homogeneous and if  $\text{Jac } f$  is a non-trivial finite-dimensional  $k$ -vector space. We often write  $f$  to represent the pair  $(R, f)$ , and we may similarly write ‘let  $f \in R$  be a potential’. We write  $\mathcal{P}_k$  for the set of potentials.

*Remark 6.2.4.* The polynomial  $f$  is quasi-homogeneous of degree  $d \in \mathbb{Q}$  if and only if it satisfies:

$$\frac{|x_1|}{d} x_1 \partial_{x_1} f + \dots + \frac{|x_n|}{d} x_n \partial_{x_n} f = f \quad (6.3)$$

In particular, this implies that  $f \in I_f$ . We have an interesting converse in the case of power series[42]: there is a coordinate transformation making  $f$  quasi-homogeneous if and only if  $f \in I_f$ .

For future use, we record the following result.

**Lemma 6.2.5.** *If  $f$  is a potential, then there exists an  $N$  such that  $(x_1, \dots, x_n)^N \subseteq I_f$ .*

*Proof.* This only uses the facts that  $I_f$  is quasi-homogeneous (i.e., for every  $g \in I_f$  with quasi-homogeneous decomposition  $g = \sum_{\ell} g_{\ell}$ , we have  $g_{\ell} \in I_f$  for all  $\ell$ ) and that  $R/I_f$  is finite dimensional over  $k$ .

Pick a variable  $x_i$ . We will first prove that  $x_i^{N_i} \in I_f$  for some  $N_i$ . For this, pick a lexicographical monomial order such that  $x_i$  is smaller than all other variables. Under this order, a power of  $x_i$  can only be a leading monomial of a polynomial  $g$  if  $g$  is a function of only  $x_i$  and no other variables. Let  $G$  be a Gröbner basis of  $I_f$  with respect to this monomial order. Because  $I_f$  is quasi-homogeneous, we may choose  $G$  such that every  $g \in G$  is quasi-homogeneous as well.

Because  $R/I_f$  is finite-dimensional, for large  $N_i$ ,  $x_i^{N_i}$  must be reducible by  $G$ . That means there is a  $g \in G$  that has a power of  $x_i$  as a leading monomial, and we may as well fix  $N_i$  so that this leading monomial is  $x_i^{N_i}$ . But with the chosen monomial order  $g$  is a function of only  $x_i$ , and with  $g$  being quasi-homogeneous, we find  $g = c x_i^{N_i}$  for some  $c \in k^*$ . Then  $x_i^{N_i} \in I_f$ .

To see that  $(x_1, \dots, x_n)^N \subseteq I_f$ , we need to show that monomials of total degree  $N$  are in  $I_f$  for large enough  $N$ . But for

$$N > n \sum_i N_i \quad (6.4)$$

at least one variable  $x_i$  has, in such a monomial, an exponent greater than  $N_i$ , and so the monomial is a multiple of  $x_i^{N_i} \in I_f$ . It is therefore an element of  $I_f$ .  $\square$

### *Graded modules*

**Convention 6.2.6.** While  $R$  has a grading with values in  $\mathbb{Q}_{\geq 0}$ , *graded  $R$ -modules* have a  $\mathbb{Q}$ -grading.

**Definition 6.2.7.** For  $q \in \mathbb{Q}$  we define the graded  $R$ -module  $R(n)$  ( $n \in \mathbb{Q}$ ) as follows. As a non-graded  $R$ -module, it is isomorphic to  $R$ , and its grading is given by

$$R(n)_m = R_{n+m}. \quad (6.5)$$

A choice of grading on two  $R$ -modules induces a unique grading on the space of maps between such modules. Let us make this explicit for maps from  $R(n)$  to  $R(m)$ . As non-graded modules we have

$$\mathrm{Hom}_R(R(n), R(m)) \cong \mathrm{Hom}_R(R, R) \cong R. \quad (6.6)$$

Comparing the quasi-homogeneous components of the left hand side and the right hand side, one readily obtains the following explicit form:

$$\mathrm{Hom}_R(R(n), R(m))_\ell \cong R_{m-n+\ell}. \quad (6.7)$$

**Convention 6.2.8.** We use the term *quasi-homogeneous map* for maps of any degree, whereas *morphism* is reserved for quasi-homogeneous maps of degree zero.

In particular, this convention implies that even though there is an invertible map between  $R(n)$  and  $R(m)$  for any  $n, m$ , they are *isomorphic* if and only if  $n = m$ .

**Definition 6.2.9.** A *finitely generated, free, graded  $R$ -module*  $X$  is a graded  $R$ -module  $X$  that has a decomposition

$$X \cong R(n_1) \oplus \dots \oplus R(n_\ell) \quad (6.8)$$

for some  $n_1 \geq \dots \geq n_\ell \in \mathbb{Q}$ .

The choice of such a decomposition is equivalent to the choice of an  $R$ -basis consisting of quasi-homogeneous elements.

### *Multi-variate residues*

We will make use of the multi-variate residue signal as described by Lipman [30]. It is completely characterized by three simple facts that we describe in this section. With a view towards our computational objective, we will prove that this characterization is effective, i.e., it gives an algorithm for computing it.

These three facts are as follows:

(F1)

$$\text{Res} \left( \frac{g dx_1 \wedge \cdots \wedge dx_q}{f_1, \dots, f_q} \right) = 0 \text{ if } g \in (f_1, \dots, f_q) \quad (6.9)$$

(F2)

$$\text{Res} \left( \frac{g dx_1 \wedge \cdots \wedge dx_q}{x_1^{d_1}, \dots, x_q^{d_q}} \right) = \left( \text{the } x_1^{d_1-1} \cdots x_q^{d_q-1}\text{-coefficient of } g \right) \quad (6.10)$$

for all  $d_1, \dots, d_q \in \mathbb{N}$ .

(F3) The transformation rule:

$$\text{Res} \left( \frac{g \det(M) dx_1 \wedge \cdots \wedge dx_q}{M(f_1, \dots, f_q)} \right) = \text{res} \left( \frac{g dx_1 \wedge \cdots \wedge dx_q}{f_1, \dots, f_q} \right) \quad (6.11)$$

for any  $R$ -linear transformation  $M: R^q \rightarrow R^q$ .

*Remark 6.2.10.* Note that (F3) preserves the applicability of (F1): if  $g \in (f_1, \dots, f_q)$ , then also  $g \det(M) \in (Mf_1, \dots, Mf_q)$ . Namely, write  $f = (f_1, \dots, f_q)$  and suppose  $g = \beta f$  for some  $R$ -linear  $\beta: R^q \rightarrow R$ . Writing  $M^\dagger$  for the adjoint of  $M$ , we have  $M^\dagger M = \det(M) \text{id}$ , and so we can write  $g \det(M) = (\beta M^\dagger)(Mf)$ , which expresses  $g \det(M)$  in the generators of  $Mf = (Mf_1, \dots, Mf_q)$ .

These facts suffice to compute any residue symbol:

**Lemma 6.2.11.** *Let  $R \in \mathcal{R}$  and let  $f_1, \dots, f_q \in I$  be generators for an ideal  $I \subseteq R$  such that  $(x_1, \dots, x_n)^N \subseteq I$  for some  $N \in \mathbb{N}$ . Then there exists a  $q \times q$  matrix  $M$  with coefficients in  $R$  such that for every  $i$ ,  $\sum_j M_{ij} f_j = x_i^{d_i}$  for some  $d_i \in \mathbb{N}$ . Moreover, this matrix can be computed explicitly.*

*Proof.* The assumption guarantees that for every  $i$ , some power  $x_i^{d_i}$  is an element of  $I$ , and this power  $d_i$  can be found algorithmically by a Gröbner basis computation as outlined in the proof of Lemma 6.2.5. This computation yields the coefficients  $M_{ij}$  for all  $j$ . Repeating the computation for all  $i$  yields the matrix  $M$ .  $\square$

**Proposition 6.2.12.** *For given  $g \in R$  and  $I = (f_1, \dots, f_q)$  such that  $R/I$  is finite-dimensional, the residue symbol*

$$\text{Res} \left( \frac{g dx_1 \wedge \dots \wedge dx_q}{f_1, \dots, f_q} \right) \quad (6.12)$$

*can be computed algorithmically.*

*Proof.* Write  $I = (f_1, \dots, f_q)$ . We first compute a Gröbner basis  $G$  of  $I$ . Then, we check whether  $g \in I$ . If it is, the residue is 0 and we have finished the computation.

If  $g \notin I$ , then we compute the matrix  $M$  such that  $M \cdot (f_1, \dots, f_q)$  consists of a vector of monomials (Lemma 6.2.11). We can then use (F3) to replace  $g$  by  $g \det(M)$ , and (F2) to compute the residue as the appropriate coefficient of  $g \det(M)$ .  $\square$

### *Matrix factorizations*

**Definition 6.2.13.** A finitely generated, free, graded  $R$ -module  $X$  is *super-graded* if it has a decomposition

$$X = X_+ \oplus X_- \quad (6.13)$$

into an *even* and *odd* part, respectively, both of which are f.g., free, graded  $R$ -modules themselves.

**Convention 6.2.14.** There is some risk of confusion from using two gradings: the  $\mathbb{Q}$ -grading on  $R$ -modules and maps between them is not to be confused with the super-grading on  $X_+ \oplus X_-$ . These are our conventions:

- We use “grade”, “grading”, and “quasi-homogeneous” exclusively to refer to the  $\mathbb{Q}$ -grading. We use “even” and “odd” exclusively to refer to the super-grading. We use “even/odd” for super-homogeneity.
- Just like in the case of the  $\mathbb{Q}$ -grading (see Convention 6.2.8), maps may be even or odd, but morphisms are assumed even.

- We use the Koszul sign rule for tensor products of super-graded modules. In order to highlight its effect on the trace operator, we write  $\text{str}$  or *super-trace* to emphasize this. Explicitly, it is given by

$$\text{str } e_i \otimes e^j = (-1)^{\text{sign}(e_i) \text{sign}(e^j)} \delta_i^j \quad (6.14)$$

for a basis  $\{e_i\}_i$  with dual basis  $\{e^i\}_i$ .

**Definition 6.2.15.** Let  $f \in R$  be a potential. A *matrix factorization of  $f$*  is a finitely generated, graded, super-graded  $R$ -module  $X$  together with an odd, homogeneous map  $d_X$  such that

$$d_X^2 = f \cdot \text{id}_X \quad (6.15)$$

*Notation 6.2.16.* We will write  $X$  to represent the pair  $(X, d_X)$  from this definition.

*Orbifold equivalence*

**Definition 6.2.17.** Let two potentials  $f \in R$  and  $g \in S$  be given. Write  $T = R \otimes_k S$ . Then a *matrix factorization of  $f - g$*  is a matrix factorization  $Q$  over  $T$  of the potential

$$f \otimes 1 - 1 \otimes g \in T \quad (6.16)$$

Note that the existence of  $Q$  implies that  $f$  and  $g$  have the same grading, since  $d_Q$  and therefore  $d_Q^2$  are quasi-homogeneous endomorphisms by assumption, and therefore so is  $(f - g) \cdot \text{id}_Q$ .

**Definition 6.2.18.** Let  $f \in k[x_1, \dots, x_m]$ ,  $g \in k[y_1, \dots, y_n]$ , and  $Q$  a matrix factorization of  $f - g$ . Its *quantum dimension with respect to  $f$*  is

$$\text{qdim}_f Q = \text{Res} \left( \frac{\text{str } \partial_{x_1} Q \cdots \partial_{x_m} Q \partial_{y_1} Q \cdots \partial_{y_n} Q dx_1 \wedge \cdots \wedge dx_m}{\partial_{x_1} f, \dots, \partial_{x_m} f} \right) \quad (6.17)$$

The *left* and *right quantum dimensions* are, respectively, the quantum dimensions w.r.t.  $f$  and w.r.t.  $g$ .

*Remark 6.2.19.* Since at present we are only interested in the (non)zero-ness of quantum dimensions, we omit the signs [6, 7].

**Definition 6.2.20.** The potentials  $f$  and  $g$  are *orbifold equivalent* if there is a matrix factorization of  $f - g$  with non-zero left and right quantum dimensions.

It is not quite trivial to see that this is an equivalence relation; in fact, even reflexivity already requires a rather complicated matrix factorization  $Q$ . Similarly, transitivity is “almost” easy to obtain, namely through a suitably defined tensor product of bimodules, but this results in a non-finitely generated module and quite some machinery is needed to reduce it to one.

Here, we will content ourselves with citing the proof, contained at section 2.1 of [7]:

**Theorem 6.2.21.** *Orbifold equivalence is an equivalence relation on the set of potentials  $\mathcal{P}_k$ .*  $\square$

### 6.3 SEARCH ALGORITHM

Our task is as follows: given potentials  $f \in R$  and  $g \in S$ , find out whether they are orbifold equivalent. We will present an algorithm that finishes in finite time if they are. It is not a decision procedure, however: the algorithm does not terminate if they are not. This section offers an exposition of parts of [38], tailored towards our use in Section 6.4.

Let’s first describe an easy instance of the algorithm.

**Example 6.3.1.** Out of reflexivity of equivalence relations, it is clear that  $x^3$  is orbifold equivalent to  $y^3$ , but let us analyse this case as an illustration. One way of finding an orbifold equivalence is splitting the total grading 2 into  $\frac{4}{3} + \frac{2}{3}$  and then writing the most general rank 2 odd matrix with entries of those gradings respectively:

$$d_Q = \begin{pmatrix} 0 & c_1x + c_2y \\ c_3x^2 + c_4xy + c_5y^2 & 0 \end{pmatrix} \quad (6.18)$$

with indeterminates  $c_1, \dots, c_5 \in k$ . Then the equation

$$d_Q^2 = (x^3 - y^3) \cdot \text{id}_Q \quad (6.19)$$

(i.e., Equation (6.15)) is equivalent to a set of equations in the variables  $c_1, \dots, c_5$ . In detail, we find 4 distinct quadratic equations – one for each degree-3 monomial – in 5 variables.

We add to these equations the requirement that the quantum dimensions do not vanish. Thanks to Proposition 6.2.12, we can



compute e.g., the left quantum dimension. It is a polynomial  $q_\ell$  in  $c_1, \dots, c_5$ , namely

$$q_\ell = -\frac{2}{3}c_2c_3 + \frac{1}{3}c_1c_4 \quad (6.20)$$

Following [38], we encode the non-vanishing by adding a helper variable  $c_\ell$  and adding

$$c_\ell q_\ell - 1 = 0 \quad (6.21)$$

to our equations. This has at least one solution for  $c_\ell, c_1, \dots, c_5$  if and only if the original system has at least one solution for which  $q$  does not vanish.

Adding two such equations, for left and right quantum dimension respectively, we find 6 equations in 7 variables, and if they admit a solution in  $k^7$ , we found a matrix factorization proving orbifold equivalence of  $x^3$  and  $y^3$ .

The existence of such a solution can be established or refuted, thanks to the weak *Nullstellensatz*, by checking whether the ideal generated by these equations is not equal to the trivial ideal (1). Algorithmically, this can be decided by computing a Gröbner basis.

It is straightforward to generalize this example to a search procedure. For this, we note the following:

- There are only countably many ranks  $2m \in 2\mathbb{N}$  for  $Q$ ;
- For every  $m$ , we can enumerate the possible tuples of gradings  $(n_1, \dots, n_{2m})$  of the free summands in

$$Q = R(n_1) \oplus \dots \oplus R(n_{2m}) \quad (6.22)$$

Through a standard diagonal procedure, we can enumerate the union of all modules appearing in this way. The tuple  $n_1, \dots, n_{2m}$  fixes the grading of the entries in  $d_Q$  through  $|(d_Q)_{ij}| = n_j - n_i + |d_Q|$ . Then a most general version of  $(d_Q)_{ij}$  for these gradings is given by a polynomial with

$$\dim_k T_{n_j - n_i + |d_Q|} \quad (6.23)$$

free variables at the  $(i, j)$  entry – one free variable for every quasi-homogeneous monomial of grading  $n_j - n_i + |d_Q|$  in  $T$ .

Having found this most general form, we compute the coefficient equations from Equation (6.15). Suppose they are given by

$$\{s_i(c_1, \dots, c_N) = 0\}_{i \in I} \quad (6.24)$$

for some polynomials  $s_i$  in  $c_1, \dots, c_N$  and for some finite index set  $I$ . We augment this set with the two equations

$$\begin{cases} c_\ell q_\ell(c_1, \dots, c_N) - 1 = 0 \\ c_r q_r(c_1, \dots, c_N) - 1 = 0 \end{cases} \quad (6.25)$$

Just like in the example, the weak *Nullstellensatz* implies that determining whether these allow a simultaneous solution in  $k^{N+2}$  is a finite computation.

We can summarize the discussion above in the following result:

**Theorem 6.3.2** ([38]). *There is an algorithm that, given two potentials  $f \in k[x_1, \dots, x_q]$  and  $g \in k[y_1, \dots, y_n]$ , terminates if and only if  $f$  and  $g$  are orbifold equivalent.*  $\square$

#### 6.4 COMPUTATIONAL FEASIBILITY

The algorithm described above consists of a discrete part and a continuous part: The discrete part is concerned with enumerating possible ranks and gradings, and the continuous part is concerned with solving geometric equations.

Compared to the way it is described above, it is possible to significantly optimize the enumeration of possible gradings by taking into account the possible factorizations of the monomials appearing in  $f$  and  $g$ . In fact, it is necessary to do so to avoid a combinatorial explosion. Details for such a significant optimization are provided in [38].

In this section we look at the feasibility of the continuous part. It is well known that Gröbner basis computations have a tendency to blow up; in fact, doubly-exponential runtime has been proved for pathological cases [34]. For this reason, algebraic problems such as the present one have attracted the interest of the cryptology community as a potentially quantum-computer resistant replacement for digital signatures now commonly implemented through a discrete logarithm problem [31, 45].

To quantify computational difficulty and feasibility, this community maintains lists of open problems for the public at large to

submit solutions. One of these challenges is the *Fukuoka MQ Challenge* [49]. One of their published lists consists of  $2N$  quadratic equations in  $N$  variables – much like the ones we encountered in the previous section – for ever increasing  $N$ .

In the remainder of this section, we will compare the difficulty of the Gröbner basis computation corresponding to known matrix factorizations to the top contenders in the MQ Challenge as of December 2018. This should give an indication of the workability of this algorithm in practice.

*Remark 6.4.1.* In contrast to our present work, cryptology focuses on finite fields and the MQ Challenge is no exception. We believe that a comparison for feasibility still makes sense, as finite fields often have very efficient computer implementations. If anything, a problem stated over a field of characteristic zero will be *less* feasible. If this belief holds true, the MQ Challenge offers a *lower bound* for the difficulty of the problem we are trying to tackle.

Another difference is that the MQ Challenge concerns itself with dense polynomials; i.e., with polynomials where almost all monomials of degree at most two have a non-zero coefficient. The polynomials that appear for us are less dense than that. In particular, no linear terms appear. We still believe that denseness is a reasonable comparison.

For explaining Table 6.1 let us go over one of its entries in detail. The three-variable potentials describing the singularities  $Q_{10}$  and  $E_{14}$  are known to be orbifold equivalent [40]. Explicitly, they are given by  $f_{E_{14}} = x^4 + y^3 + xz^2$  and  $f_{Q_{10}} = u^4w + v^3 + w^2$  respectively.

The matrix factorization testifying that is given by

$$\begin{aligned} Q = & T \oplus T\left(\frac{1}{4}\right) \oplus T\left(\frac{1}{3}\right) \oplus T\left(\frac{7}{12}\right) \\ & \oplus T \oplus T\left(\frac{1}{4}\right) \oplus T\left(\frac{1}{3}\right) \oplus T\left(\frac{7}{12}\right) \end{aligned} \quad (6.26)$$

as a  $\mathbb{Q}$ -graded module over  $T = k[x, y, z, u, v, w]$ . That implies that  $d_Q$ 's entries have gradings given by the following matrix:

$$\frac{1}{12} \begin{pmatrix} & & & & 12 & 15 & 16 & 19 \\ & & & & 9 & 12 & 13 & 16 \\ & & & & 8 & 11 & 12 & 15 \\ & & & & 5 & 8 & 9 & 12 \\ 12 & 15 & 16 & 19 & & & & \\ 9 & 12 & 13 & 16 & & & & \\ 8 & 11 & 12 & 15 & & & & \\ 5 & 8 & 9 & 12 & & & & \end{pmatrix} \quad (6.27)$$

Following the procedure from the last section, this results in the variables  $c_1, \dots, c_{106}$  to describe the most general version of  $d_Q$  with these gradings.

When taken coefficient-by-coefficient (both of the matrix and of the polynomial entries), the equation

$$d_Q^2 = (f_{E_{14}} - f_{Q_{10}}) \cdot \text{id}_Q \quad (6.28)$$

gives 470 equations in  $c_1, \dots, c_{106}$ . Adding the quantum dimension helper variables and constraints, we are faced with a system of 472 equations in 108 variables.

A significant optimization can be made. Since  $d_Q$  is odd, it is of the form

$$d_Q = \begin{pmatrix} 0 & d_Q^\sharp \\ d_Q^\flat & 0 \end{pmatrix} \quad (6.29)$$

and  $d_Q^2 = (f - g)\text{id}_Q$  reduces to the two equations

$$d_Q^\sharp d_Q^\flat = (f_{E_{14}} - f_{Q_{10}}) \cdot \text{id}_{Q_+} \quad (6.30)$$

$$d_Q^\flat d_Q^\sharp = (f_{E_{14}} - f_{Q_{10}}) \cdot \text{id}_{Q_-} \quad (6.31)$$

However, these two equations are equivalent to one another. We may therefore consider only the constraints arising from either one of them, and this cuts the number of independent constraints on  $c_1, \dots, c_{106}$  roughly in half. In the specific case above, we are left with 237 equations in 108 variables.

For comparison, the current top contender in the MQ Challenge solved a system of 148 equations in 74 variables over the field of

Equivalence	indeterminates	equations
$f_{Q_{10}} \sim f_{E_{14}}$	108	237
$f_{Q_{18}} \sim f_{E_{30}}$	140	341
$f_{Q_{12}} \sim f_{E_{18}}$	116	263

Table 6.1: Gröbner basis challenge size for known orbifold equivalences.

2 elements. This strongly suggests that the described algorithm would not have been able to find this orbifold equivalence within reasonable time.

Table 6.1 lists similar outcomes for different equivalences.



## A SOFTWARE PACKAGE FOR LANDAU–GINZBURG EXPLORATION

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### 7.1 OVERVIEW

Any mathematical endeavour benefits from exploration of concrete examples, and Landau–Ginzburg models are no exception. As part of the work described in this thesis, we make available a software package to explore known matrix factorizations and their compositions. The present chapter provides a description of it.

The package consists of two independent parts: one part provides operations on matrix factorizations, and the other part provides a library of known orbifold equivalences. In addition to implementing the search algorithm from Chapter 6, the former part extends the capabilities of the software made available alongside [5] by being able to compute finite-rank representatives of tensor products for *all* quasi-homogeneous finite-rank matrix factorizations. The second part records known results from the literature for convenience. As an improvement on published work, in certain cases it represents coefficients in explicit number fields. This explicitness is needed for the former operations to work.

### 7.2 SOFTWARE ENVIRONMENT AND USE

The package `LandauGinzburgCategories.jl` is made available for use with the Julia language [2]. In Chapter 3 we described significant speed benefits obtained from moving a specific algorithm from Sage/Python [11, 46] (as was used for [27]) to Julia; we expect that these benefits carry over to more domains including the present one. We built a package `LandauGinzburgCategories.jl` containing this work and a package for polynomial rings and number fields called `PolynomialRings.jl`.

An example interactive session might look as follows. The user loads the known orbifold equivalences between  $A_2 \times A_2$  and  $A_5$  [38] and between  $A_5$  and  $D_4$  [7, 8]. Then, she computes their composition as described in Section 5.1 through the `fuse` function. Under the hood, this function applies the procedure from [14]

to obtain an abstract representation, and applies the procedure described in Section 7.4 below for obtaining an explicit one.

```
$ julia
julia> using LandauGinzburgCategories,
        LandauGinzburgCategories.Library
julia> @ring! QQ[x,y,s,t,u,v]QQ
[x,y,s,t,u,v]
julia> f = TwoVariables.22AA; f(x, y)
x^3 + y^3
julia> g = TwoVariables.5A; g(s, t)
s^6 + -t^2
julia> h = TwoVariables.4D; h(u, v)
u^3 + -u*v^2
julia> A = orbifold_equivalence(f, g, (x, y), (s, t));
julia> quantum_dimensions(A, (x, y), (s, t))α
(-134192032//8347670061*^11 + α21597568//2782556687*^10 +
  α-2024128640//25043010183*^9 + α534444476//2782556687*^8
  + α-4954210880//25043010183*^7 +
  α-382643680//2782556687*^6 + α-1679203988//8347670061*^5
  + α-27815697416//25043010183*^4 +
  α-7056195320//25043010183*^3 + α1518961426//2782556687*^2
  + α15110718884//25043010183* + 291607726//229752387,
  α-20421120//2782556687*^11 + α-278881144//8347670061*^10
  + α-78928070//2782556687*^9 + α-756438488//8347670061*^8
  + α805212776//2782556687*^7 + α-3849952723//8347670061*^6
  + α-5144306252//8347670061*^5 +
  α-9866973193//8347670061*^4 +
  α-19482111790//8347670061*^3 +
  α-15826622593//8347670061*^2 + α-9022452568//8347670061*
  + 63368099//153168258)
julia> B = orbifold_equivalence(g, h, (s, t), (u, v));
julia> quantum_dimensions(B, (s, t), (u, v))
(1//1, 2//1)
julia> AB = fuse(A, B, s, t);
julia> quantum_dimensions(AB, (x, y), (u, v))α
(-1060955753//75129030549*^11 + α170755772//25043010183*^10
  + α-16003267060//225387091647*^9 +
  α33803613107//200344081464*^8 +
  α-39169229770//225387091647*^7 +
  α-3025276595//25043010183*^6 +
  α-106209652241//601032244392*^5 +
  α-879671430781//901548366588*^4 +
  α-223152176995//901548366588*^3 +
  α192148620389//400688162928*^2 +
  α955752969413//1803096733176* + 36888377339//33084343728,
  α-2978080//8347670061*^11 +
```



```

α-244021001//150258061098*^10 +
α-276248245//200344081464*^9 +
α-661883677//150258061098*^8 + α704561179//50086020366*^7
+ α-26949669061//1202064488784*^6 +
α-9002535941//300516122196*^5 +
α-69068812351//1202064488784*^4 +
α-68187391265//601032244392*^3 +
α-110786358151//1202064488784*^2 +
α-7894645997//150258061098* + 443576693//22056229152)

```

Installation and usage instructions are available to the public on the Internet<sup>1</sup>.

As an aside, we would like to highlight the OSCAR project, which is a major effort in the direction of applying Julia to abstract algebra. At the time of development, OSCAR's packages were not yet suitable for our use; as a particular example, certain choices in its type representation make it unsuitable for interacting with Julia's support for sparse matrices. We expect this to change in the future and interested readers should consult [18] for an overview.

### 7.3 STABLE EQUIVALENCE

In the description of the algorithms that follows in subsequent sections, the concepts of stable equivalence and homotopy equivalence play an important role. In this section, we record their relation in slightly more generality than we need.

**Definition 7.3.1.** Given two matrix factorizations  $X$  and  $Y$ , a *morphism (of matrix factorizations)* is a morphism  $\phi: X \rightarrow Y$  of graded, super-graded  $R$ -modules such that  $d_Y \circ \phi = \phi \circ d_X$ .

**Definition 7.3.2.** Two matrix factorizations  $X$  and  $Y$  are *stably equivalent* if

$$X \oplus A \cong Y \oplus B \quad (7.1)$$

where both  $A$  and  $B$  are equal to finite direct sums of any combination of copies of the *trivial matrix factorizations*

$$T_1 = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} \text{ or } T_2 = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} \quad (7.2)$$

<sup>1</sup> <https://github.com/tkluck/LandauGinzburgCategories.jl/tree/release/>

**Definition 7.3.3.** A matrix factorization  $X$  is called *reduced* if there is no matrix factorization  $Y$  such that

$$X \cong Y \oplus T_i \quad (7.3)$$

for any  $i \in \{1, 2\}$ .

Recall Definition 6.2.1 of a graded polynomial ring. Since  $R_0 = k$ , we can decompose  $R$  as

$$R = k \oplus \mathfrak{m} \quad (7.4)$$

where  $\mathfrak{m}$  is the ideal

$$\mathfrak{m} = \bigoplus_{q \in \mathbb{Q}_{>0}} R_q \quad (7.5)$$

This decomposition induces two projections, and for  $r \in R$  we write

$$r = \bar{r} + r^* \quad (7.6)$$

with  $\bar{r} \in k$  and  $r^* \in \mathfrak{m}$ . Both projections are homomorphisms of graded  $R$ -modules. Moreover, the projection to  $k$  is a homomorphism of rings, and we call it the “residue map”.

*Remark 7.3.4.* Such a ring  $R$  is generally not a local ring (as  $\mathfrak{m}$  is not unique), but every proper *homogeneous* ideal *is* contained in  $\mathfrak{m}$ . For this reason, we borrow notation (“ $\mathfrak{m}$ ”) and terminology (“residue map”) from local rings. This should not lead to confusion with the multi-variate residue as defined in Section 6.2.

If  $X$  is a finitely generated, free, graded  $R$ -module (Definition 6.2.9), then we can form the tensor product  $X \otimes_R k$  where  $k$  is an  $R$ -module through the residue map. A morphism of f.g., free, graded  $R$ -modules  $\phi: X \rightarrow Y$  descends to a morphism

$$\bar{\phi}: X \otimes_R k \rightarrow Y \otimes_R k \quad (7.7)$$

and when expressed as a matrix on a free homogeneous basis,  $\bar{\phi}$  is obtained by applying the residue map to each coefficient.

**Proposition 7.3.5.** *A matrix factorization  $X$  is reduced if and only if*

$$\overline{d_X} = 0 \quad (7.8)$$

*Proof.* This is [29, exercise 8.35] in the case of  $R$  a local ring, but the same proof works for  $R$  a graded ring.  $\square$

**Definition 7.3.6.** Two endomorphisms  $g$  and  $h$  of the  $R$ -module  $X$  are called *homotopy equivalent* if there exists an odd map  $h: X \rightarrow X$  such that

$$f - g = d_X h + h d_X. \quad (7.9)$$

Two matrix factorizations  $X$  and  $Y$  are *homotopy equivalent* if there exist morphisms of matrix factorizations  $\phi: X \rightarrow Y$  and  $\psi: Y \rightarrow X$  such that  $\phi \circ \psi$  is homotopy equivalent to  $\text{Id}_Y$ , and  $\psi \circ \phi$  is homotopy equivalent to  $\text{Id}_X$ .

**Proposition 7.3.7.** *The trivial matrix factorizations  $T_1$  and  $T_2$  are null-homotopic, i.e., they are equivalent to the null module.*

*Proof.* We will only prove the statement for  $T_1$ . For this case, write  $Y$  for the null module and let  $\phi: T_1 \rightarrow Y$  be the projection morphism and  $\psi: Y \rightarrow T_1$  be the inclusion morphism.

In the direction  $Y \rightarrow T_1 \rightarrow Y$ , it is obvious that  $\text{Id}_Y = \phi \circ \psi$  by lack of endomorphisms on  $Y$ .

For the other direction  $T_1 \rightarrow Y \rightarrow T_1$ , we define

$$h := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (7.10)$$

to obtain

$$\text{Id}_{T_1} - \psi \circ \phi = h \circ d_{T_1} + d_{T_1} \circ h. \quad (7.11)$$

This equation proves that  $\text{Id}_{T_1}$  is homotopy equivalent to  $\psi \circ \phi$ . These two directions together prove that  $T_1$  and  $Y$  are homotopy equivalent.  $\square$

**Proposition 7.3.8.** *Let  $X$  and  $Y$  be matrix factorizations of  $f$ . Then  $X$  and  $Y$  are homotopy equivalent if and only if they are stably equivalent.*

*Proof.* The “if” part follows at once from Proposition 7.3.7 so we only prove the “only if” part.

So assume that  $X$  and  $Y$  are homotopy equivalent. By definition, we have

$$\phi \circ \psi - \text{Id}_Y = d_Y j + j d_Y \quad (7.12)$$

$$\psi \circ \phi - \text{Id}_X = d_X h + h d_X \quad (7.13)$$

for some  $j$  and  $h$ . By Proposition 7.3.5, we may, without changing the stable equivalence class of either  $X$  or  $Y$ , assume that  $X$  and  $Y$  are reduced.

With this assumption, applying the residue map yields

$$\overline{\phi} \circ \overline{\psi} - \text{Id}_Y = 0 \quad (7.14)$$

$$\overline{\psi} \circ \overline{\phi} - \text{Id}_X = 0 \quad (7.15)$$

and it follows that  $\overline{\phi}$  and  $\overline{\psi}$  are inverse to each other.

The inclusion  $k \hookrightarrow R$  splits the residue map, and we can use this to regard  $\overline{\phi}$  and  $\overline{\psi}$ , which *a priori* are matrices with coefficients in  $k$ , as maps between  $X$  and  $Y$ . In fact, since they are inverse to each other, we may use these maps to completely identify  $X$  with  $Y$ . After this identification,  $\phi$  and  $\psi$  satisfy

$$\overline{\phi} = 1 \quad (7.16)$$

$$\overline{\psi} = 1 \quad (7.17)$$

and therefore

$$\phi = 1 + \phi^* \quad (7.18)$$

$$\psi = 1 + \psi^* \quad (7.19)$$

Since  $\phi$  is a morphism of graded rings,  $\phi^*$  is strictly upper-triangular. Then  $\det(\phi) = 1$  and so  $\phi$  is invertible. It follows that  $d_X$  and  $d_Y$  are conjugate. In particular, they are stably equivalent.  $\square$

#### 7.4 THE TENSOR PRODUCT OPERATION

The tensor product operation in the category of Landau–Ginzburg models is the crucial ingredient in the transitivity of orbifold equivalence 6.2.21. It has a straightforward expression when we allow infinite-rank matrix factorizations; however, every such infinite-rank matrix factorization is *homotopy equivalent* to a finite-rank matrix factorization [14], and this is the expression that interests us. (Section 7.3 below defines the relevant notion of homotopy equivalence.) The approach taken by the software package is described below.

**Definition 7.4.1.** Let  $Q$  be an object in an abelian category and let  $e: Q \rightarrow Q$  be a morphism satisfying  $e^2 = e$ . Then an object  $X$  together with an epimorphism  $\hat{e}: Q \rightarrow X$  and a monomorphism  $\iota: X \rightarrow Q$  such that  $\iota \circ \hat{e} = e$  and  $\hat{e} \circ \iota = \text{id}_X$  is called a *splitting of  $e$* .

If a splitting exists, it is unique up to unique isomorphism, so we may also just refer to  $X$  as a splitting of  $e$ .

**Theorem 7.4.2** ([14]). *Let  $A$  and  $B$  be composable matrix factorizations. Then there is an algorithm that computes a finitely generated, free matrix factorization  $Q$  and a morphism  $e: Q \rightarrow Q$  such that  $e^2 = e$  (up to homotopy) and such that  $A \otimes B$  is a splitting of  $e$ . Moreover, if  $A$  and  $B$  are quasi-homogeneous, then so are  $Q$  and  $e$ .*  $\square$

In `LandauGinzburgCategories.jl`, this algorithm is available through the `fuse_abstract` function, which takes matrices  $d_A$  and  $d_B$  as arguments and returns a matrix  $d_Q$  and a matrix representing  $e$ .

**Lemma 7.4.3.** *Write  $x' := 2e - \text{id}$  and suppose  $y: Q \rightarrow Q$  satisfies*

$$x - y(x')^2 + y^2(x')^2 = 0 \quad (7.20)$$

*and commutes with  $e$ . Let  $E = e - yx'$ . Then it holds that  $E^2 = E$  strictly, i.e., not only up to homotopy.*

*Proof.* We compute

$$E^2 - E = (e - yx')^2 - e + yx' \quad (7.21)$$

$$= e^2 - 2eyx' + y^2(x')^2 \quad (7.22)$$

$$- e + yx' \quad (7.23)$$

$$= x - y(x')^2 + y^2(x')^2 \quad (7.24)$$

and so  $E^2 - E = 0$  exactly when Equation (7.20) holds.  $\square$

**Lemma 7.4.4.** *Suppose  $Q$  is a quasi-homogeneous matrix factorization, and suppose  $x: Q \rightarrow Q$  is a null-homotopic morphism. Then  $Q$  is homotopy equivalent to a matrix factorization  $Q'$  such that under this equivalence,  $x$  maps to a strictly upper-triangular matrix  $x'$ .*

*Proof.* Let  $Q'$  be a reduced matrix factorization (Definition 7.3.3) in the same homotopy equivalence class as  $Q$ . Concretely, one computes it by taking every entry of  $d_Q$  that is a scalar as a pivot and sweeping its row and column: the scalar entry will then be part of a block matrix of the form  $T_1$  or  $T_2$  as in Definition 7.3.2 and we may remove this null-homotopic block. Each row/column operation is applied to  $e$  as well.

Since  $x$  is null-homotopic, by Definition 7.3.6 there is  $h$  such that

$$x = d_Q h + h d_Q \quad (7.25)$$

and since  $Q$  is now reduced we have  $\overline{d_Q} = 0$  by Proposition 7.3.5. It follows that  $\bar{x} = 0$ . As  $x$  is a graded morphism, it is then strictly upper-triangular.  $\square$

**Lemma 7.4.5.** *Suppose  $Q$  is a matrix factorization and  $e: Q \rightarrow Q$  is idempotent up to homotopy. Moreover, suppose that  $x = e^2 - e$  is strictly upper-triangular. Then there exists a solution  $y$  to Equation (7.20). Moreover, this solution can be computed algorithmically using only additions and multiplications of the coefficients of  $x$ .*

*Proof.* The expansion

$$p(z) = \sum_{i=0}^{\infty} \binom{1/2}{i} z^i \quad (7.26)$$

satisfies  $p(z)^2 = 1 + z$ , and the expansion

$$q(z) = \sum_{i=0}^{\infty} (-z)^i \quad (7.27)$$

satisfies  $(1+z)q(z) = 1$ . As  $x$  is strictly upper-triangular, it is nilpotent in particular, and we can compute

$$y = q(p(4x) - 1) \quad (7.28)$$

as a finite computation. The fact that it satisfies Equation (7.20) is a straightforward verification.  $\square$

In `LandauGinzburgCategories.jl`, these lemmas have been implemented in a function called `fuse`. The result is the following.

**Proposition 7.4.6.** *Let  $Q$  be a quasi-homogeneous matrix factorization and let  $e: Q \rightarrow Q$  be a morphism idempotent up to homotopy. Then there is an algorithm that computes a splitting of  $e$ . Moreover, this algorithm applies only field operations on the coefficients of  $d_Q$ , and in particular the resulting matrix factorization has coefficients in the smallest field over which  $d_Q$  is defined.*  $\square$

**Corollary 7.4.7.** *Let  $Q_1$  and  $Q_2$  be quasi-homogeneous, composable matrix factorizations defined over fields  $k_1, k_2 \subseteq k$  respectively. Then there is an algorithm that computes a finite-rank representative of  $Q_1 \otimes Q_2$ . Moreover, the resulting matrix  $d_{Q_1 \otimes Q_2}$  has coefficients in the smallest field containing both  $k_1$  and  $k_2$ .  $\square$*

*Remark 7.4.8.* The approach described above follows quite naturally from the *Ansatz*

$$E = e - f(x)x' \quad (7.29)$$

which in turn may be obtained from Newton iterations and/or Hensel’s lemma. From software made available alongside [5] we can infer that (part of) this approach was known to these authors at that time. In their software, they restrict to the case where  $x$  is nilpotent and apply the power series expansion

$$f(x) = \frac{1}{2} \sum_{i=1}^{\infty} (-1)^{i-1} \binom{2i}{i} x^i \quad (7.30)$$

which is a finite computation in the nilpotent case. We observed many cases, including the simple case of the composition of two unit matrix factorizations of  $x^3$ , where  $x$  is not nilpotent, necessitating the extended approach described in the preceding text.

In our practical implementation, the reduction step of “sweeping” rows and columns as in the proof of Lemma 7.4.4 is currently the slowest, taking several seconds for small examples and several minutes for larger ones. However, it is not an obstacle for practical computations.

## 7.5 THE UNIT MATRIX FACTORIZATION

The `LandauGinzburgCategories.jl` package includes a way to obtain the unit matrix factorization for any given potential in the form of the function `unit_matrix_factorization`. It implements the definition from [6] in a straightforward manner, but there is one detail of the implementation that is interesting to add to this chapter.

The unit matrix factorization is defined as an endomorphism on an exterior algebra. In order to compute its matrix representation, we choose the usual basis

$$\mathcal{B}_n = \{dx_{i_1} \wedge \cdots \wedge dx_{i_k} \mid 0 \leq k \leq n, 1 \leq i_1 < \cdots < i_k \leq n\} \quad (7.31)$$

There is a straightforward way of ordering this basis for obtaining a  $2^n \times 2^n$  matrix with integer-labelled rows/columns. Namely, define the map

$$a: \mathcal{B}_n \rightarrow \{0, \dots, 2^n - 1\} \quad (7.32)$$

by using an  $n$ -bit representation: the element  $dx_{i_1} \wedge \dots \wedge dx_{i_k}$  maps to the number that has the  $\ell$ th bit set if and only if  $i_\ell \in \{i_1, \dots, i_k\}$ . Unfortunately, this does not preserve the sign (evenness/oddness) of the basis element. We therefore apply the following tool.

**Definition 7.5.1** ([21]). Let  $n \in \mathbb{N}$ . The *Gray code* is the bijection

$$\text{gray}: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0} \quad (7.33)$$

given in bit representation by

$$\text{gray}(x) = x \vee (x \gg 1) \quad (7.34)$$

where  $\vee$  is the exclusive OR operation, and  $\gg$  is right bit shift.

**Fact 7.5.2.** *The Gray code satisfies the following properties:*

- *It is a bijection on the initial part  $\{0, \dots, 2^n - 1\}$  for every  $n \geq 0$ ;*
- *$\text{gray}(k)$  and  $\text{gray}(k + 1)$  differ by exactly one bit for every  $k \geq 0$ .*

Consider the map  $b: \{0, \dots, 2^n - 1\} \rightarrow \mathcal{B}$  given by  $b = a^{-1} \circ \text{gray}$ . Because of the second property of the Gray code,  $b(k)$  and  $b(k + 1)$  differ exactly by the addition or removal of  $dx_{i_\ell}$  for some  $\ell$ . In particular,  $b(k)$  and  $b(k + 1)$  have opposite sign. This means that  $b^{-1}$  preserves the even/odd structure of the exterior product into the integer-labeled rows/columns.

For aesthetic reasons, we apply one more transformation. The labelling  $b^{-1}$  gives alternating even/odd rows and columns, but we prefer even and odd blocks. In other words, it should not be the *least significant* bit that determines the sign, but the *most significant*. Composing  $b^{-1}$  with bit reversal achieves this.

In summary, this discussion proves the following.

**Proposition 7.5.3.** *Write  $c: \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  for the bit-reversal map, and let  $a$  and  $\text{gray}$  as above. Then the map*

$$c \circ \text{gray}^{-1} \circ a: \mathcal{B} \rightarrow \{0, \dots, 2^n - 1\} \quad (7.35)$$

*maps the even and odd basis elements in  $\mathcal{B}$  to consecutive blocks  $\{0, \dots, 2^{n-1} - 1\}$  and  $\{2^{n-1}, \dots, 2^n - 1\}$  respectively.  $\square$*



## 7.6 A LIBRARY OF ORBIFOLD EQUIVALENCES

As part of the same installation process, a library module<sup>2</sup> also becomes available. It exports a function `orbifold_equivalence` which, given two names of potentials, attempts to retrieve an orbifold equivalence between them. The logic is very simple: the list of known orbifold equivalences is interpreted as a graph with nodes equal to the potentials and edges equal to equivalences. Given potentials  $f$  and  $g$ , we apply a simple graph traversal to see if  $f$  and  $g$  are in the same connected component (i.e., the same orbifold equivalence class). In that case, `fuse` is applied to compute the correct result.

If  $f$  and  $g$  are not in the same connected component, the function returns `false` if the central charges of  $f$  and  $g$  are not the same, as central charge is invariant under orbifold equivalence. Otherwise, it returns `missing`, indicating that an orbifold equivalence may or may not exist.

For now, the list of known orbifold equivalences is sufficiently small that the equivalence classes can be trivially seen at first sight. We hope that future developments in the theory of orbifold equivalences will make the algorithm more useful than it is at the present.

A useful future addition might be the following. The current version of the `orbifold_equivalence` function expects to receive a potential by its name, e.g., `TwoVariables.D{5}`, as opposed to receiving it as a polynomial  $x^4 - xy^2$ . The latter case would be more interesting, but the user might also expect that other representatives obtained through substitutions (e.g,  $D_5$  can be represented by  $x^4 + xy^2$  by substituting  $y \mapsto iy$ ) give an appropriate result. This would require implementing a classification algorithm for these potentials. We have not pursued this and do not know whether this task would prove very simple, very difficult, or somewhere in between.

---

<sup>2</sup> `LandauGinzburgCategories.Library`



## SAMENVATTING

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Dit hoofdstuk bevat een samenvatting van dit proefschrift bedoeld voor niet-wiskundigen. De twee delen van dit proefschrift gaan over verschillende onderdelen van de mathematische fysica. Het eerste deel beschrijft de beweging van een aantal lichamen die elkaar op een bepaalde manier afstoten: “Calogero–Ruijsenaars-systemen”. Het tweede deel bestudeert een gereedschap om grensgebieden van snaartheorieën te bestuderen: “matrixontbindingen”.

### CALOGERO–RUIJSENAARSSYSTEMEN

In sommige gevallen kunnen we een natuurkundig systeem exact oplossen. Een bekend voorbeeld is wanneer een klein object (bijvoorbeeld een satelliet of het International Space Station) zich in de zwaartekracht van de aarde bevindt: het object beschrijft dan een rechte lijn, een cirkel, een ellips of een hyperbool, en zal dat blijven doen totdat/tenzij het de atmosfeer raakt. We noemen dit het “twee-lichamenprobleem” (de twee lichamen zijn de aarde en het object) en we kunnen het exact oplossen.

Het “drie-lichamenprobleem”, onder andere bekend van het boek van Liu Cixin met de Engelse titel *The Three Body Problem*, voegt een derde lichaam toe. Met name wanneer de drie lichamen ongeveer dezelfde massa hebben is het erg moeilijk een exacte oplossing uit te rekenen. En wat erger is: een kleine meetfout aan het begin zal zich na enige tijd in grote afwijkingen vertalen. Dit is een flink probleem voor de personages in *The Three Body Problem*: op hun planeet met drie zonnen is er geen enkele manier om seizoenen te voorspellen en hebben ze veel moeite zich tegen de grillen van het klimaat te beschermen.

Calogero–Ruijsenaarssystemen hebben hier geen last van. Het simpelste Calogero–Ruijsenaarssysteem beschrijft een systeem met zoveel lichamen als men maar wil, zolang ze maar (1) op één lijn liggen en (2) ze elkaar niet door zwaartekracht (een zogenaamde omgekeerd-kwadratische kracht) beïnvloeden, maar door een omgekeerd-kubische kracht. Niet alleen hebben dit soort systemen geen last van chaos; de posities van de lichamen kunnen op elk mo-

ment in de toekomst worden uitgerekend *zonder de tussenliggende posities te hoeven berekenen*.

Dit verschijnsel is zo interessant dat natuurkundigen en wiskundigen allerlei variaties hebben bedacht. Zo kunnen de lichamen zich ook op een cirkel of zelfs op een donut bevinden (“trigonometrische” respectievelijk “elliptische” versies); zo kunnen de lichamen zich aan Einsteins relativiteitstheorie proberen te houden (“Ruijsenaars–Schneiderversie”) en zo kunnen de lichamen elkaars invloed anders beleven dan door hun paarsgewijze afstand (“een ander wortelsysteem dan van type  $A$ ”).

Deze laatste variatie behoeft uitleg. De verzameling van alle paarsgewijze afstanden is een voorbeeld van wat wiskundigen een “wortelsysteem” noemen. Wanneer er bijvoorbeeld drie lichamen zijn op posities  $x$ ,  $y$ , en  $z$ , dan zijn alle paarsgewijze afstanden:

$$\begin{array}{ll} x - y, & x - z, & y - z \\ y - x, & z - x, & z - y \end{array} \quad (8.1)$$

Deze verzameling is erg symmetrisch: niet alleen bevat hij voor elk element een volledig spiegelbeeld (bijvoorbeeld  $x - y$  en  $y - x$ ) maar ook kan men spiegelen *langs elk van de elementen*: als men bijvoorbeeld  $x - z$  spiegelt langs  $x - y$  dan is het resultaat  $y - z$ . Elke verzameling die aan een aantal van dit soort symmetriekenmerken voldoet heet een wortelsysteem; degene die we net hebben beschreven heet  $A_2$ . Als we met vier deeltjes zouden beginnen dan is het  $A_3$ , met vijf deeltjes  $A_4$ , enzovoorts. Een ander voorbeeld van een wortelsysteem bevat niet alleen de paarsgewijze afstanden, maar ook de optelsommen. Bijvoorbeeld in het geval van vier lichamen op posities  $x$ ,  $y$ ,  $z$ , en  $w$ :

$$\begin{array}{llllll} x - y, & x - z, & x - w, & y - z, & y - w, & z - w \\ y - x, & z - x, & w - x, & z - y, & w - y, & w - z \\ x + y, & x + z, & x + w, & y + z, & y + w, & z + w \\ -x - y, & -x - z, & -x - w, & -y - z, & -y - w, & -z - w \end{array} \quad (8.2)$$

Dit is het  $D_4$  wortelsysteem. Net zoals bij  $A_2$  is er een volledig spiegelbeeld voor elk element (bijvoorbeeld  $x + w$  en  $-x - w$ ) en als men bijvoorbeeld  $y - x$  langs  $x + w$  spiegelt dan is het resultaat  $y + w$ . Er bestaan nog veel meer wortelsystemen.

In de  $D_4$ -variant van het bovenstaande systeem is er sprake van vier lichamen die een kracht ervaren gebaseerd op de onderlinge

afstand, maar ook gebaseerd op de optelsom van de onderlinge coördinaten. Op het eerste gezicht is dit wat vreemd, maar we kunnen dezelfde methode gebruiken om de toekomstige posities uit te rekenen.

Het eerste deel van dit proefschrift onderzoekt een aantal technische details van deze oplossingsmethode.

## MATRIXONTBINDINGEN

Een polynoom is een formule waarin getallen en variabelen met elkaar vermenigvuldigd en/of bij elkaar opgeteld worden. Bijvoorbeeld:

$$\begin{aligned} 3x^2 + 4y + 2 \\ xy + 1 \\ x^2y + 2z + 3 \end{aligned}$$

Soms kan een polynoom “ontbonden” worden: het kan geschreven worden als een product. Bijvoorbeeld:

$$\begin{aligned} x^2 - 1 &= (x - 1)(x + 1) \\ x^2 + 2xy + y^2 &= (x + y)(x + y) \end{aligned}$$

In de zogenaamde “Landau–Ginzburgsystemen” (een vorm van snaartheorie) is er in elk afgebakend stuk van de ruimte een polynoom dat natuurkundige eigenschappen bepaalt. Op de grensgebieden van deze afgebakende stukken is het nodig om een ontbinding *van het verschil* te vinden. Jammer genoeg kan dat niet altijd; zo is er bijvoorbeeld geen ontbinding van

$$(u^2 - uv^2) - (x^4 - y^2) \tag{8.3}$$

Een oplossing blijkt te zijn om niet alleen naar gewone ontbindingen, maar ook naar “matrixontbindingen” te kijken. Een voorbeeld ziet er zo uit:

$$\begin{aligned} &\begin{pmatrix} u^2 - uv^2 & \\ & u^2 - uv^2 \end{pmatrix} - \begin{pmatrix} x^4 - y^2 & \\ & x^4 - y^2 \end{pmatrix} \\ &= \begin{pmatrix} vx + y & u - x^2 \\ x^2 + u - v^2 & vx - y \end{pmatrix} \begin{pmatrix} y - vx & u - x^2 \\ x^2 + u - v^2 & -vx - y \end{pmatrix} \end{aligned}$$

Zelfs in dit relatief eenvoudige geval ziet het er al wat bewerkelijk uit. Voor echte voorbeelden zijn de polynomen en matrices al snel groter en wordt dat problematisch. Daarom is het belangrijk om dit soort berekeningen door een computer te kunnen laten uitvoeren.

In het tweede deel van dit proefschrift bestuderen we hoe een computer voor een gegeven polynoom een matrixontbinding kan zoeken. Het blijkt dat dit niet alleen moeilijk is omdat de matrix zo groot mag zijn als men maar wil; zelfs voor een gegeven grootte van de matrix is de berekening van een “Gröbnerbasis” nodig. Dit is een dermate moeilijke berekening dat, zelfs voor bescheiden beginwaarden, er veel rekenkracht nodig is. Het proefschrift vergelijkt de grootte van deze berekeningen met een aantal van de grootste berekeningen van Gröbnerbases in de literatuur: de *MQ Challenge*.

Naast de zoektocht naar matrixontbindingen zijn er nog een aantal andere bewerkingen waarbij een computer kan helpen: bijvoorbeeld het samenstellen van twee matrixontbindingen tot een *push-forward*. Een beschrijving van de implementatie van deze bewerkingen is ook verwerkt in het tweede deel.

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---

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## PACKAGE DOCUMENTATION

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### A.1 FEATURES

The `LandauGinzburgCategories.jl` package aims to help you experiment with matrix factorizations in the bicategory of Landau–Ginzburg models. It contains implementations of operations such as *unit*, *composition*, *fusion of variables*, *duality*, and *left/right quantum dimension*. Future additions may include *(co)evaluation maps*.

In addition, a library of well-known named potentials, as well as known orbifold equivalences, is provided.

For getting started, have a look at [the Getting Started section](#).

### A.2 LIMITATIONS

**Expensive computations.** Many known orbifold equivalences are presented by way of an *Ansatz* together with the verifiable statement that the resulting equations have a solution. We represent this by placing these coefficients in a quotient ring. Every operation in this quotient ring necessarily involves a reduction step, and this can be noticeably slow.

### A.3 GETTING STARTED

#### *Installation*

Refer to [the Julia website](#) for details on installing Julia. As soon as you have, start it and type `]` to get in package mode. Then, run

```
(v1.3) pkg> add https://github.com/tkluck/  
LandauGinzburgCategories.jl#release
```

to install `LandauGinzburgCategories` and its dependencies. To test whether it worked, type

```
julia> using LandauGinzburgCategories.Library
```

```
julia> orbifold_equivalence(TwoVariables.A{9}, TwoVariables.
    D{6})
4×4 Array{@ring(Int64[u,v,x,y]),2}:
 0          0          -v*
   x + y      -x^2 + u
 0          0          x^8
   + u*x^6 + u^2*x^4 + u^3*x^2 + u^4 + -v^2 -v*x + -y
 v*x + y      -x^2 + u  0
          0
 x^8 + u*x^6 + u^2*x^4 + u^3*x^2 + u^4 + -v^2 v*x + -y  0
          0
```

If you see a matrix factorization, the installation worked!

#### A.4 OVERVIEW

##### *Composing two matrix factorizations*

Let us compose two simple matrix factorizations. For the first, let's take a unit matrix factorization:

```
julia> using LandauGinzburgCategories,
    LandauGinzburgCategories.Library;

julia> @ring! Int[x,y,s,t,u,v];

julia> A = unit_matrix_factorization(x^6 - y^2, x => s, y =>
    t)
4×4 Array{@ring(Int64[x,y,s,t,u,v]),2}:
 0          0          x^5
   + x^4*s + x^3*s^2 + x^2*s^3 + x*s^4 + s^5 -y + -t
 0          0          y +
   -t          -x + s  0
 -x + s  y + t          0
 -y + t  x^5 + x^4*s + x^3*s^2 + x^2*s^3 + x*s^4 + s^5  0
          0
```

As you see, we first load the package by writing `using .....`. Then, we declare the variables `x,y,s,t,u,v` that we will use. The last step calls the function `unit_matrix_factorization`. The result is a matrix factorization:

```
julia> A^2
4×4 Array{@ring(Int64[x,y,s,t,u,v]),2}:
```

$$\begin{array}{cccc}
-x^6 + s^6 + y^2 + -t^2 & 0 & & 0 \\
& 0 & & \\
0 & & -x^6 + s^6 + y^2 + -t^2 & 0 \\
& 0 & & \\
0 & 0 & & -x^6 + s \\
& ^6 + y^2 + -t^2 & 0 & \\
0 & 0 & & 0 \\
& & -x^6 + s^6 + y^2 + -t^2 & 
\end{array}$$

Using the function `quantum_dimensions` you can compute the left- and right quantum dimensions:

```
julia> quantum_dimensions(A, (x,y), (s,t))
(1//1, 1//1)
```

As you see, you need to pass the sets of left  $(x,y)$  and right  $(s,t)$  variables.

The polynomial  $x^6 - y^2$  is a representative of the  $A_5$  equivalence class. We now load an orbifold equivalence from the library that we can compose with it.

```
julia> B = orbifold_equivalence(TwoVariables.A{5},
    TwoVariables.D{4}, (s, t), (u, v))
4x4 Array{@ring(Int64[x,y,s,t,u,v]),2}:
 0          0      -s*v + t
          -s^2 + u
 0          0      s^4 + s^2*u + u^2 + -v
    ^2  -s*v + -t
s*v + t          -s^2 + u  0
          0
s^4 + s^2*u + u^2 + -v^2  s*v + -t  0
          0
```

The function `fuse` can compute the composition for us. The following command fuses along the  $s$  and  $t$  variables.

```
julia> fuse(A, B, s, t)
4x4 SparseMatrixCSC{@ring(Rational{Int64}[x,y,u,v]),Int64}
with 8 stored entries:
[3, 1] = -x^4 + -x^2*u + -u^2 + v^2
[4, 1] = x*v + y
[3, 2] = x*v + -y
[4, 2] = x^2 + -u
[1, 3] = x^2 + -u
[2, 3] = -x*v + -y
[1, 4] = -x*v + y
[2, 4] = -x^4 + -x^2*u + -u^2 + v^2
```

The result is a sparse matrix; for easier viewing we can call `collect` on it.

```
julia> collect(ans)
4×4 Array{@ring(Rational{Int64}[x,y,u,v]),2}:
 0          0      x^2 + -u  -x*v + y
 0          0      -x*v + -y  -x^4 + -x
      ^2*u + -u^2 + v^2
 -x^4 + -x^2*u + -u^2 + v^2  x*v + -y  0      0
 x*v + y                  x^2 + -u  0      0
```

### *Computing the central charge of a potential*

The function `centralcharge` computes the central charge. You can use it as follows:

```
julia> centralcharge(x^6 - x*y^2, x, y)
5//6
```

As you can see, it is necessary to also pass the variables with respect to which you want to compute the central charge – in this case `x,y`. For example, this is useful when you want to compute the central charge of a parametrized family:

```
julia> centralcharge(x^6 + t*x*y^2, x, y)
5//6
```

## A.5 OPERATIONS

*.Operations.:⊗ – Function.*

```
A ⊗ B
```

Graded tensor product of  $\mathbb{Z}/2$  graded block matrices. The result consists of even/odd blocks just like the input and the operation is associative.

*Graded* means that we introduce Koszul signs. Writing  $A = a_j^i e_i e^j$  and  $B = b_\ell^k f_k f^\ell$  for some (co)basis vectors, we represent the tensor product as

$$A \otimes B = (-1)^{|j|(|k|+|\ell|)} p_j^i q_\ell^k e_i f_k f^\ell e^j \quad (\text{A.1})$$

where the sign comes from commuting  $e^j$  with  $f_k f^\ell$ . This ensures that, upon matrix multiplication between two such tensor products, the contractions between  $e^j$  and  $e_j$ , and between  $f^\ell$  and  $f_\ell$ , are already adjacent and do not introduce more signs.

Mapping the multi-indices  $(i, k)$  to rows and  $(\ell, j)$  to columns requires some care to ensure that we end up with even/odd blocks. A comment in the code explains how this is done. We first interlace the rows/columns to get alternating grades in rows/columns (`to_alternating_grades` and `from_alternating_grades`). Then we need only to reverse some row/column orders. Finally, we deinterlace to get even/odd blocks. (I wonder if there is a way to avoid making the detour through the alternating signs representation, but it seems hard to maintain associativity.)

*.Operations.: $\hat{\otimes}$  – Function.*

$A \hat{\otimes} B$

Tensor product of matrix factorizations.

This is the composition operation in the bicategory or Landau Ginzburg models.

*.QuasiHomogeneous.centralcharge – Function.*

`c = centralcharge(f, vars...)`

Return the central charge of `f` with respect to the variables `vars`.

This is the value of the expression

$$\sum_{i=1}^n 1 - q_i \tag{A.2}$$

where  $q_i$  is the grading of the  $i$ th variable under a ( $\mathbb{Q}$ -valued) grading for which `f` is homogeneous of degree 2.

*Example*

```
julia> using PolynomialRings, LandauGinzburgCategories
julia> @ring! Int[x, y];
```

```
julia> centralcharge(x^5 + y^2, x, y)
3//5
```

*.Operations.dual* – Function.

```
Q' = dual(Q)
```

Return the dual (i.e., the adjoint) matrix factorization of  $Q$ .

If  $Q$  factors the potential  $f$ , then this is a matrix factorization that factors  $-f$ .

*Example*

```
julia> using LandauGinzburgCategories, PolynomialRings;
        @ring! Int[x,y]
@ring(Int64[x,y])

julia> Q = [0 x - y; x^2 + x*y + y^2 0];

julia> Q^2
2×2 Array{@ring(Int64[x,y]),2}:
 x^3 + -y^3  0
 0          x^3 + -y^3

julia> dual(Q)
2×2 Array{@ring(Int64[x,y]),2}:
 0          x^2 + x*y + y^2
 -x + y  0

julia> dual(Q)^2
2×2 Array{@ring(Int64[x,y]),2}:
 -x^3 + y^3  0
 0          -x^3 + y^3
```

*.QuasiHomogeneous.find\_quasihomogeneous\_degrees* – Function.

```
gradings = find_quasihomogeneous_degrees(f, vars...)
```

Find the assignment of gradings to the variables **vars** such that  $f$  is a quasi-homogeneous polynomial with respect to these grad-

ings, if such an assignment exists, and return it. Otherwise, raise an exception.

### Example

```
julia> using PolynomialRings, LandauGinzburgCategories

julia> @ring! Int[x, y];

julia> find_quasihomogeneous_degrees(x^4*y + x*y^9, :x, :y)
Gradings(x = 8, y = 3)
```

### *.Operations.fuse – Function.*

```
A_B = fuse(A, B, vars_to_fuse...)
```

Compute the finite-rank matrix factorization homotopy-equivalent to  $A \hat{\otimes} B$  according to the procedure made popular by [14].

In contrast to `fuse_abstract`, the result is a concrete matrix that represents  $A \hat{\otimes} B$ .

### Examples

```
julia> using LandauGinzburgCategories, PolynomialRings;

julia> @ring! Int[x,y,z];

julia> A = unit_matrix_factorization(x^3, x => y);

julia> B = unit_matrix_factorization(y^3, y => z);

julia> fuse(A, B, y) |> collect
2×2 Array{@ring(Rational{Int64}[x,z]),2}:
 0      x^2 + x*z + z^2
 -x + z  0
```

### *.Operations.fuse\_abstract – Function.*

```
Q, e = fuse_abstract(A, B, vars_to_fuse...)
```

Compute the finite-rank matrix factorization homotopy-equivalent to  $A \hat{\otimes} B$  according to the procedure made popular by [14].

$Q$  is a finite-rank matrix factorization of `getpotential(A) + getpotential(B)`.  $e$  is an endomorphism (i.e.  $Q * e == e * Q$ ) that is idempotent up to homotopy. A theorem by Dyckerhoff and Murfet [14] ensures that any splitting of  $e$  is homotopy equivalent to  $A \hat{\otimes} B$ .

In order to also compute the splitting, use the `fuse` function.

*.Operations.getpotential – Function.*

```
W = getpotential(A::AbstractMatrix)
```

Assuming  $A$  is a matrix factorization, return the associated potential. It asserts that  $A^2$  is a scalar multiple of the identity and returns that scalar.

*.QuasiHomogeneous.quasidegree – Function.*

```
d = quasidegree(f::Polynomial, g::Gradings)
```

The quasidegree of  $f$ , with variable gradings specified by  $g$ .

*Example*

```
julia> using PolynomialRings, LandauGinzburgCategories
julia> @ring! Int[x, y];
julia> quasidegree(x^2 * y^3, Gradings(x=2, y=1))
7
```

*.OrbifoldEquivalence.quantum\_dimensions – Function.*

```
lqdim, rqdim = quantum_dimensions(Q, left_vars, right_vars,
    W=getpotential(Q))
```

The left and right quantum dimensions of  $Q$ .



*Examples*

```
julia> using PolynomialRings, LandauGinzburgCategories;

julia> @ring! Int[x,y,u,v];

julia> Q = unit_matrix_factorization(x^5 - y^2, x => u, y =>
    v);

julia> quantum_dimensions(Q, (x,y), (u,v))
(1, -1)
```

*.OrbifoldEquivalence.search\_orbifold\_equivalence* – *Function*.

```
search_orbifold_equivalence(f, g, left_vars, right_vars; max
    _rank=10)
```

Perform a brute-force search for an orbifold equivalence between **f** and **g**.

This uses the algorithm made popular by [38].

*Example*

```
julia> @ring! Int[x,y];

julia> search_orbifold_equivalence(x^3, 2y^3, (x,), (y,))
```

*.Operations.unit\_matrix\_factorization* – *Function*.

```
unit_matrix_factorization(f, source_to_target...)
```

A  $\mathbb{Z}/2$ -graded matrix that squares to **substitute(f, source\_to\_target...)** - **f** times the identity matrix.

The source for this formulation is [6].

*Examples*

```
julia> using LandauGinzburgCategories, PolynomialRings;
```

```
julia> @ring! Int[x,y];

julia> unit_matrix_factorization(x^3, x => y)
2×2 Array{@ring(Int64[x,y]),2}:
 0      x^2 + x*y + y^2
 -x + y  0
```

## A.6 LIBRARY

*.Library.orbifold\_equivalence – Function.*

```
orbifold_equivalence(f, g)
```

Return a matrix representing an orbifold equivalence between **f** and **g**, if one is available in the library. Return **false** if it is known that **f** and **g** are not orbifold equivalent. Return **missing** if this is not known by this library.

*.Library.Potential – Type.*

```
Potential{NumVars}
```

A type representing a potential through a classification scheme. For example, the potential  $x^4 - y^2$  is called  $A_3$  through the ADE classification of unimodular potentials, and it is represented in Julia by the type **TwoVariables.A<sub>3</sub>**. This is a subtype of **Potential{2}**.

## A.7 INTERNAL FUNCTIONS

*.MatrixUtil.sweepscalars! – Function.*

```
sweepscalars!(M::AbstractMatrix{<:Polynomial}, Ms...)
```

For every entry of **M** for that is a scalar, sweep its row and column by conjugation operations, and apply the same operation to every matrix in **Ms**.

When **M** is a matrix factorization, the sweep by conjugation operations means that a column operation on the top-right block is applied simultaneously with a row operation on the bottom-left block. The resulting matrix factorization is isomorphic to the original one.

In a typical use case,  $\mathbf{M}$  is a matrix factorization and  $\mathbf{M}\mathbf{s}$  are morphisms. The result is that the matrices in  $\mathbf{M}\mathbf{s}$  are pushed forward under the same isomorphism.

*.Operations.flatten\_blocks – Function.*

```
M = flatten_blocks(X)
```

Construct a matrix from a matrix of matrices by concatenating them horizontally and vertically.

*.Operations.supertrace – Function.*

```
t = supertrace(Q::AbstractMatrix)
```

Trace of a of  $\mathbb{Z}/2$ -graded block matrix with the Koszul sign convention.

*.OrbifoldEquivalence.multivariate\_residue – Function.*

```
r = multivariate_residue(g, f, vars...)
```

Compute the multivariate residue

$$r = \text{res} \left( \frac{g dx_1 \wedge \cdots \wedge dx_n}{f_1, \dots, f_n} \right) \quad (\text{A.3})$$

where  $n = \text{length}(f)$  and  $\text{vars} = [x_1, \dots, x_n]$ . See [30].

*Example*

```
julia> using PolynomialRings, LandauGinzburgCategories;
julia> @ring! Int[x,y];
julia> multivariate_residue(3x^2*y^2, [x^3, y^3], x, y)
3
```

*.OrbifoldEquivalence.weight\_split\_criterion\_gradings* – *Function*.

```
for gradings in weight_split_criterion_gradings(W, N, vars
    ...)
```

Iterate over all grading matrices satisfying the weight split criterion from [38].

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