Graphing formulas to give meaning to algebraic formulas

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Many students have difficulties giving meaning to algebra. In this study, we investigated how graphing formulas by hand can help grade 11 students to give meaning to algebraic formulas, defined as the ability to "read through" an algebraic formula: i.e., to recognize its structure and to link it to graphical features. During five 90-minute lessons, 21 students worked on graphing tasks focusing on recognition and heuristic search. To assess the effects of this intervention, a graphing task, and a card-sorting task were administered to the students. In addition, six students were asked to think aloud during the graphing task. The results of the card-sorting task showed that 14 students used categories similar to the ones experts use, but had trouble in consistently categorizing all formulas. The thinking-aloud protocols showed that the students improved their recognition of basic functions and graph features, and their qualitative reasoning, which allowed them to give meaning to algebraic formulas.

Keywords: Algebra, giving meaning to algebraic formulas, graphing formulas.

Introduction

Many secondary school students have problems with algebraic formulas, which are very abstract for them (Arcavi, Drijvers, & Stacey, 2017; Kieran, 2006; Thompson, 2013). Students, also those in upper secondary school, have difficulties giving meaning to algebraic formulas: they lack symbol sense. Symbol sense has to do with the ability to read through a formula, to recognize its structure and its characteristics (Arcavi, 1994) and would allow students to give meaning to algebraic formulas. It is insufficiently known how to teach these aspects of symbol sense (Arcavi et al., 2017; Hoch & Dreyfus, 2010). In regular education, teaching for giving meaning often starts with realistic contexts and manipulation of formulas. In the current study, we used graphs to give meaning to formulas and used basic functions as building blocks in reasoning with and about algebraic formulas.

Theory

Giving meaning to algebraic formulas

Acquiring meaning is synonymous with assimilating to a cognitive schema, which is a (hierarchal) network of concepts and procedures. Assimilation is the integration of new information into a schema, which results in an adjusted schema (Thompson, 2013). To give meaning to algebraic formulas, different sources are suggested in the literature: for instance, via the problem context, via the algebraic structure of the expression, via multiple representations, and via linguistic activity, gestures, body language, metaphors, etc. (Kieran, 2006). Ernest (1990) used a syntactical analysis to decompose algebraic expressions into meaningful sub-expressions (building blocks). Hoch and

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Dreyfus (2010) used structure sense to describe the ability to recognize familiar structures by using compound terms as single entities. Janvier (1987) and many others have shown that functions can be investigated by linking different representations. We chose to link algebraic formulas to graphs, which seem to be more accessible for students than formulas (Leinhardt, Zaslavsky, & Stein, 1990), as they appeal more to the Gestalt aspect of a function and visualize the "story" that an algebraic formula tells. Therefore, graphs can be used to give more meaning to algebraic formulas (Eisenberg & Dreyfus, 1994; Kieran, 2006; Heid, Thomas, & Zbiek, 2013). To link formulas to graphs in order to give meaning to the formulas, graphing tools such as graphic calculators can be used (Heid et al., 2013). Using these tools, it seems easy to make a graph from a formula. However, research has shown that one must know what aspects of graphs to look for (Stylianou & Silver, 2004), and that students establish the connection between formula and graph more effectively when they do graphing by hand than when they only perform computer graphing (Goldenberg, 1988).

The aim of the current study was to improve students' abilities to give meaning to algebraic formulas by teaching them to graph formulas by hand. We defined giving meaning to algebraic formulas as the ability to "read through" an algebraic formula: that is, to recognize its structure and to link it to graphical features.

Teaching graphing formulas for giving meaning

In expertise research, it has been established that for effective and efficient problem solving one needs recognition, and heuristics when recognition falls short (Chi, Feltovich, & Glaser, 1981). Based on this idea, a two-dimensional framework to describe strategies for graphing formulas was formulated (Kop, Janssen, Drijvers, Van Driel, & Veenman, 2015). In this framework, one dimension is about levels of recognition: from complete or function family recognition, to no recognition of the graph. The other dimension is about heuristic search, from strong heuristics, which provide information about large parts of the graph (e.g., qualitative reasoning about infinity behaviour and/or symmetry), to weak heuristics, which give only local information (e.g., calculating points). See Figure 1.

		Heuristic search (strong ® weak)				
Levels of Recognition (low ß high)	A	A1. Graph is instantly recognized as a whole				
	B	B1. Recognition of family (with characteristics)	B2. Search for 'parameters' of the graph	B3. Investigate the family characteristics, for instance via zeroes		
	C	C1. Split formula in sub-formulas	C2. Compose the graphs by qualitative reasoning	C3. Compose the graphs by making a table		
	D	D1. Graph unknown; Characteristic aspect of graph is recognized.				
	E	E1. Graph unknown; strategic exploration of algebraic formula	E2. Qualitative reasoning about domain, or vertical asymptote, or symmetry, or infinity behaviour, …	E3. Algebraic manipulation	E4. Strategic search, for zeroes or extreme values	E5. Calculate strategically chosen point(s)
	F	F1. No recognition at all	F2. Standard repertoire of research	F3. Make random table		

Figure 1: Two-dimensional framework for strategies to graph formulas

For recognition, a repertoire of basic function families and knowledge of attributes to describe graphs are needed (Eisenberg & Dreyfus, 1994; Slavit, 1997). Kop, Janssen, Drijvers, and Van

Driel (2017) found, using a card-sorting task, that experts' repertoires of basic function families resembled the basic function families taught in secondary school, like exponential, logarithmic, and polynomial functions. Experts seem to have linked prototypes of these function families to a set of critical graph attributes. For instance, a prototypical logarithmic graph has a vertical asymptote, only positive x-values as a domain, and is concave down. They use their repertoire of basic function families as building blocks in working with formulas, for instance, when decomposing complex functions into simpler basic functions and when reasoning about characteristic graph features from formulas (Kop et al., 2015, 2017).

Experts often start with a global perspective that enables high levels of recognition (Even, 1998). With the global perspective the whole function and/or graph is in view, whereas with a local perspective only a part is in view. The global perspective is more powerful and gives a better understanding of the relation between formulas and graphs, but the pointwise approach is needed to monitor naïve and/or immature interpretations (Even, 1998; Slavit, 1997).

The question in the current study was whether we could teach these experts' strategies to students. This led to the following research question: How can graphing formulas based on recognition and heuristic search help pre-university students to develop a knowledge base to give meaning to algebraic formulas?

Method

The teaching experiment was performed in a grade 11 mathematics B class, a regular class of 21 students who were 16- and 17-years old. Mathematics B is a course that prepares students in the Netherlands for university studies in mathematics, science, and engineering. The intervention took place during five lessons of 90 minutes. The students were obliged to work in pairs or groups of three which required them to exchange ideas. Each day, the lesson started with a short plenary discussion (max. 10 minutes) with general feedback on the students' work, reflection on the tasks, and modelling of expert thinking processes aloud. During the rest of the lesson, the students worked on tasks, related to the levels of recognition of the two-dimensional framework (Figure 1). These tasks were formulated as exploratory whole tasks with help, concluded with a reflection question. After the lesson, each pair and group handed in their work for feedback.

To assess the effects of the intervention, we collected students' responses to a written graphing task in a pre-test, post-test, and retention-test (four months after the intervention). Each graph in these graphing tasks was reviewed as correct (score=1) or incorrect (score=0), and a total score was calculated for each student.

Written tests give only limited information about the students' thinking processes. Therefore, six students were asked to think aloud during the graphing task in the pre-test and post-test. These protocols were transcribed and analyzed based on the two-dimensional framework (Figure 1): that is, on the use of recognition (function families, formula decomposed into sub-formulas, and/or graph features), qualitative reasoning, and calculation (points and/or derivatives).

In addition, a card-sorting task was assigned in the post-test to gather information about the students' knowledge base of function families. The students were asked to categorize 40 formulas according to their rough graph, using as many categories as they wished. In addition, the students were asked to give a description and a prototypical formula for each of their categories. Card-sorting tasks are often used in eliciting structured knowledge (Chi et al., 1981). To analyze the card-sorting task, an expert categorization with 12 main categories, similar to the one found by Kop et al. (2017), was constructed and used as the criterion categorization. For each categorization, the student's categories were compared with those of the criterion categorization and it was established which formulas were not consistently categorized. Each corresponding category was graded with a score of two points and an extra point if a distinction between increasing and decreasing had been made. Then the number of correct formulas (consistent with the category) was graded. This method of analysis has been used by others (Ruiz-Primo & Shavelson, 1996); the total score should be an indication of the correctness of a student's categorization.

Results

For the card-sorting task, we found that 14 out of the 21 students used categories almost similar to those in the expert categorization, although different names were used for the categories, like "having a horizontal and vertical asymptote" (linear broken functions), or "only a vertical asymptote" (logarithmic function). Many students made mistakes in categorizing distractors like $y = 8\sqrt{x^3}$, $y = \sqrt{8 - x^2}$, $y = 100 - e^x$, $y = x + 4/x$, and often no distinction was made between increasing and decreasing log-functions.

To illustrate the results, we report about two students: K (a high-achieving student) and M (an above-average-achieving student). Student K's categorization is shown in the Appendix. S/he distinguished increasing and decreasing exponential, root, and polynomial functions. S/he categorized only a few formulas inconsistently, but s/he did not make a distinction between logarithmic and root functions, which resulted in a total score of only 45 points. Student M's categories also resembled those of the criterion categorization. When categorizing the formulas, student M used predominantly zeroes and max/min with polynomial functions, like in $y = (x + 3)^4 - 9$, asymptotes (horizontal and vertical for broken functions, horizontal for exponential functions and vertical for logarithmic functions), and edge points for root-functions. Although, s/he had more inconsistently categorized formulas, her/his categorization had a total score of 44 points.

The thinking-aloud protocols on the graphing tasks showed that, in the pre-test, the students had trouble recognizing (basic) functions, and only high-achieving students were successful now and then because of their reasoning abilities. In the post-test, these students had improved their recognition of basic function families with their characteristics and their qualitative reasoning.

In the pre-test, K scored 9 correct graphs out of 14. As a high-achieving student, s/he was able to compensate for a lack of recognition through qualitative reasoning. However, s/he often had to calculate many points of the graph. We give two citations to illustrate K's strategies in the pre-test. K did not recognize the global shape of a logarithmic function and used calculations and reasoning about the inverse function to sketch the graph of $y = \ln(x - 3)$ correctly:

"I do not know the ln-graph anymore. When $x-3=0$, then …… When $x-3=1$, then $y=0$, so $x=4$. At x-as the x-axis is intersected. When x is increasing then y increases, so the graph increases. When x is negative … (thinking). Because something to the power of e ($e^{\cdots}$) does not give negative y-values. So, $x-3$ cannot be negative; the graph only exists from $x=3$, larger than 3. So, at $x=3$ a tangent (asymptote?) and outcomes smaller when x is in the neighbourhood of 3".

At first K showed a correct global shape by gesturing, however her calculating of individual points resulted in an incorrect sketch of $y=\sqrt{6-2x}$:

"…root-functions go like this (makes a correct gesture); the fact that it is 6-2x means that the function only exists for negative values of x, to the point where x is 3; for x larger than 3 the argument is negative; when x=3 than y=0; when x=0, y=$\sqrt{6}$, which is about 2.5; when x is more negative, the argument becomes larger; …, $\sqrt{10} \approx 3.3$, $\sqrt{12} \approx 3.5$, differences become larger (and sketches a graph concave up)"

In the post-test, K's recognition of basic functions had improved and s/he still used her/his abilities of qualitative reasoning, resulting in a score of 13 out of 14. Again, two citations give an illustration.

K sketching $y=ln(x-4)$ correctly: "ln(x) graph goes like this; $x-4$, so 4 to the right."

K used qualitative reasoning to compose two graphs while sketching $y=2x\sqrt{x+6}$ correctly:

"2x goes like this; $\sqrt{x+6}$ goes like this (sketch); here it is 0; here negative, here 0, and after this it is steeper".

In the pre-test student, M had many problems with graphing formulas. S/he did not know basic function family characteristics, and did not recognize graph features like zeroes and turning points. This resulted in a score of only 2 out of 14. Two citations illustrate her/his strategies.

M recognized the parabola-shape but did not know how to proceed while sketching $y=(x-3)^4-9$ (not finished):

"…at $x=3, y=-9$. (After some time) The larger x is, the larger y, so it increases. It is a parabola. (S/he stopped talking for a while; after a couple of minutes) I do not know how to proceed."

M recognized a translation but not the shape of the ln(x)-graph; s/he tried to construct the graph of $y=\ln(x-3)$ via the inverse function (not finished):

"…graph of ln(x) that is translated 3 to the right (s/he did not use this but writes $log_e(x-3)=log(e)/log(x-3); e^y=x-3$). This is an asymptote; x cannot be 3; ….; when $y=0, x-3=1$ (drew point (4,0) and stopped)".

In the post-test, M's repertoire of basic function families and her/his qualitative reasoning had improved, resulting in a score of 10 out of 14. Two citations illustrate this. In the post-test M recognized more graph features like zeroes and turning points; for instance, while sketching $y=-2x(x-2)(x-5)$ correctly: "…goes downwards; zeroes at 0, 2, 5. At 1 it is negative".

M in the post-test, sketching $y=(x^2+6)/(x^2-4)$ correctly:

"…asymptotes at 2 and -2; zero at $\sqrt{6}$; no, no zeroes, because x^2 cannot be negative; when x is smaller than 2, then it is positive here, and negative here, so it is negative; when x is a bit larger than 2, positive here, positive here, so positive; the same for -2".

Not only had these two students improved their scores in the post-test, but the results of all students showed a significant improvement in scores, with a large effect-size, from a mean score of 3.1 (SD = 2.6) in the pre-test to a mean score of 8.6 (SD = 2.8) in the post-test ($t(20) = 10.40$, $p < .001$, $d = 2.27$). As expected, the scores in the retention-test dropped: a mean score of 6.3 (SD = 3.4). This result is significant when compared with the pre-test scores: $t(14) = 3.15$, $p = .007$, $d = 0.81$.

Conclusion and discussion

The aim of the current study was to enable students to give meaning to algebraic formulas in terms of a graph. A knowledge base of basic function families with their characteristics is needed for this. In the intervention, students learned to graph formulas through recognition and qualitative reasoning. The results of the thinking-aloud protocols showed that in the pre-test students were either unable to link formulas to graphs or needed to use a lot of time-consuming reasoning and error-prone calculations. Therefore, without explicit teaching, students might not link formulas to graphs. The post-test results of the graphing task showed that the students had improved their recognition of basic function families and, in general, their reasoning about formulas. In terms of the two-dimensional framework: in the pre-test the students mostly used recognition level E, whereas in the post-test they often used higher levels of recognition (recognition of function families, decomposition of the formula, and recognition of graph features).

The categorizations gave an impression of the knowledge base and thinking processes of the students. The students' categorizations showed that they used more or less the same categories as experts, but still had problems with categorizing less familiar functions, like $y = \sqrt{8 - x^2}$, $y = 100 - e^x$, $y = 8\sqrt{x^3}$, $y = x + 4/x$. Student M used zeroes, asymptotes (horizontal and/or vertical), and edge points as main strategies when categorizing. In the post-test graphing task, s/he often used these same strategies. Student K's categorization was very detailed, although s/he combined root- and log-functions. However, in the post-test graphing task s/he showed that s/he knew the differences well (see sketching $y = \ln(x - 3)$ and $y = \sqrt{x + 6}$ in the results section). In the post-test graphing task, K used her/his repertoire of function families often. Both students' categorizations seemed to allow them to recognize function families and use these function families as building blocks in reasoning with and about formulas: these function families are objects that they can use to give meaning to algebraic formulas.

These findings suggest that teaching graphing of formulas based on recognition and heuristic search might enable students to develop a repertoire of basic function families, to identify the structure of formulas, to (qualitatively) reason with and about algebraic formulas, and to link formulas to graphical features. This could allow students to "read through" an algebraic formula, that is, to give meaning to algebraic formulas.

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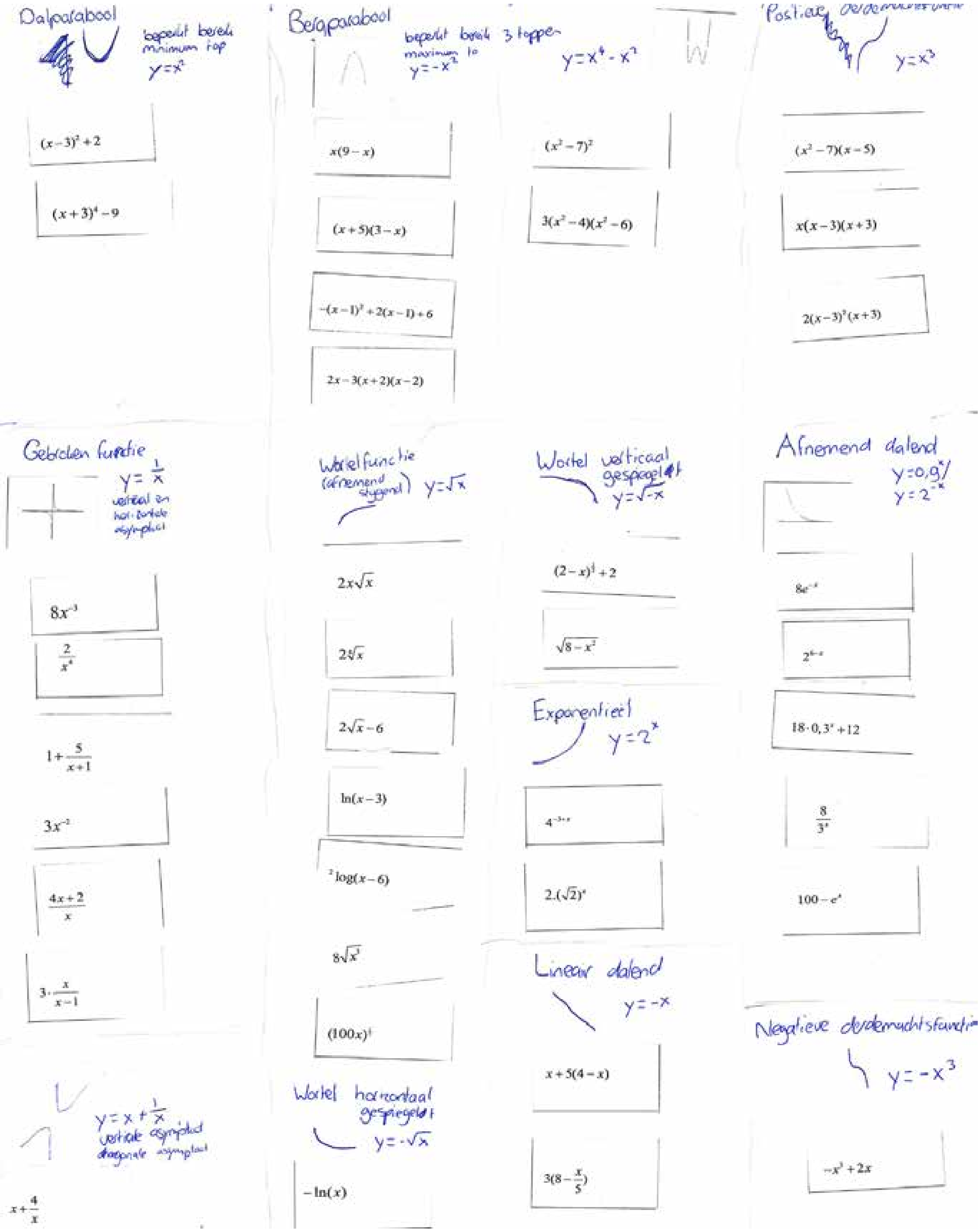
Appendix: Student K's categorization



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