

Quantum Brownian motion in a Landau level

E. Cobanera,* P. Kristel, and C. Morais Smith

*Institute of Theoretical Physics, Center for Extreme Matter and Emergent Phenomena, Utrecht University,
Leuvenlaan 4, 3584 CE Utrecht, The Netherlands*

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Motivated by questions about the open-system dynamics of topological quantum matter, we investigated the quantum Brownian motion of an electron in a homogeneous magnetic field. When the Fermi length $l_F = \hbar/(v_F m_{\text{eff}})$ becomes much longer than the magnetic length $l_B = (\hbar c/eB)^{1/2}$, then the spatial coordinates X, Y of the electron cease to commute, $[X, Y] = i l_B^2$. As a consequence, localization of the electron becomes limited by Heisenberg uncertainty, and the linear bath-electron coupling becomes unconventional. Moreover, because the kinetic energy of the electron is quenched by the strong magnetic field, the electron has no energy to give to or take from the bath, and so the usual connection between frictional forces and dissipation no longer holds. These two features make quantum Brownian motion topological, in the regime $l_F \gg l_B$, which is at the verge of current experimental capabilities. We model topological quantum Brownian motion in terms of an unconventional operator Langevin equation derived from first principles, and solve this equation with the aim of characterizing diffusion. While diffusion in the noncommutative plane turns out to be conventional, with the mean displacement squared being proportional to t^α and $\alpha = 1$, there is an exotic regime for the proportionality constant in which it is directly proportional to the friction coefficient and inversely proportional to the square of the magnetic field: in this regime, friction helps diffusion and the magnetic field suppresses all fluctuations. We also show that quantum tunneling can be completely suppressed in the noncommutative plane for suitably designed metastable potential wells, a feature that might be worth exploiting for storage and protection of quantum information.

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I. INTRODUCTION

While topologically ordered quantum systems [1] remain a main theme of current research in condensed matter physics and quantum engineering, the focus has been shifting away from ground-state and thermal properties towards the more challenging problem of nonequilibrium properties and open-system dynamics. In addition to the well-developed Floquet theory of classically driven topological insulators and superconductors, there have been investigations of the quench dynamics of prototypical systems like the toric code [2] and honeycomb [3] models, and ideas for generalizing the Kibble-Zurek mechanism to systems without local order parameters [4]. The least pursued approach, and the one closest in spirit to this paper, has been that of coupling, implicitly by way of the Lindblad equation, quantum baths to topologically ordered systems [5] or designing the non-Hamiltonian part of the Lindblad superoperator so that it will stabilize known topologically nontrivial states [6].

Investigations of this latter type are, however, particularly intriguing because they point to a far reaching question: What is the connection between the structures associated to topological ordering in quantum systems, irreversible energy dissipation, and the theory of quantum open-system dynamics and control? In (generalized) gauge theories, for example, topological ordering is induced by local symmetries [7], whereas for topological insulators and superconductors, topological ordering is induced by global symmetries (chiral, time reversal, and charge conjugation) of the single-particle

Hamiltonian [8]. From this point of view, Refs. [5,6] may be understood as concrete explorations of the interplay between these symmetry structures and Lindblad (Markovian) open-system dynamics.

In this paper, we investigate the relation between open-system dynamics, described within the formalism of the operator Langevin equation [9], and a non-symmetry-based source of topological ordering: the dimensional reduction of phase space experienced by an electron subject to a strong, homogeneous magnetic field [10]. In the limit in which the Fermi length is much larger than the magnetic length of the electron, the position operators of the electron become noncommuting, canonically conjugate variables (with \hbar replaced by the magnetic length squared l_B^2), and its kinetic energy becomes quenched (vanishing up to a constant shift). Taken together, these two features (a noncommutative configuration space and vanishing Hamiltonian) are trademarks of topological models [12], and they impact standard theories of quantum Brownian motion [13,14] in a magnetic field [15,16] in two ways. First, the coupling of a topological model to a bath occurs through noncommuting coordinates, and second, because the Hamiltonian vanishes identically in topological models, the energetics of the diffusion process are completely controlled by the bath. We call this instance of the Brownian motion problem topological.

Topological quantum Brownian motion displays a surprising mixture of conventional and exotic behavior. On one hand, the character of the diffusion process is not changed by the passage to the noncommutative plane. An electron undergoing normal diffusion (mean-squared displacement proportional to the time elapsed) before the magnetic field is applied, continues to do so after the singular limit of very strong magnetic-field strength is taken. On the other hand, the friction coefficient can play an extremely counterintuitive role since there is a regime

*Current address: Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, New Hampshire 03755, USA; emilio.cobanera@dartmouth.edu

in which the frictional force helps rather than hinders diffusion! Moreover, in this regime the magnetic field suppresses thermal and quantum fluctuations.

The noncommutative plane is rich in surprises even before considering Brownian motion. Because the position operators do not commute, the ability to localize the electron is limited by Heisenberg uncertainty. In other words, the location of an electron in the noncommutative plane is always fuzzy. Hence, it seems somewhat paradoxical that, as we will show, quantum tunneling can be completely suppressed in the noncommutative plane. This result immediately suggests some different ideas for quantum memories. We will come back to this point in the Outlook section.

The organization of this paper is as follows. In Sec. II, we recall the topological quantum mechanics [12] of an electron subject to a strong magnetic field, discuss the physical conditions for the emergence of the noncommutative plane, and counterintuitive physical aspects like the complete suppression of quantum tunneling. We conclude by rederiving the noncommutative plane in a different way, namely, by projection onto a Landau level. In Sec. III, we further couple the electron to a bath of electrically neutral, independent oscillators and derive an operator Langevin equation appropriate for modeling topological quantum Brownian motion. Finally, in Sec. IV we use our operator Langevin equation to investigate a particular aspect of topological quantum Brownian motion: diffusion in the noncommutative plane. We conclude in Sec. V with a summary and outlook. We include three appendixes that make the paper nearly self-contained. The first one, Appendix A, is devoted to calculating statistical properties of quantum Langevin forces or velocity fields. In Appendix B, a connection is made between our results and the previously worked out case [16] of quantum Brownian motion in a weak ($l_B \gg l_F$) magnetic field (see Ref. [17] for a comprehensive discussion of classical Brownian motion in a magnetic field). Finally, in Appendix C it is explained which constraints on the bath lead to memoryless frictional forces.

II. EMERGENT PHYSICS IN STRONG MAGNETIC FIELDS: THE NONCOMMUTATIVE PLANE

The nonrelativistic motion of an electron of mass m and charge $-e$ is described by the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - \frac{e}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) + e\phi(\mathbf{r}) - V(\mathbf{r}), \quad (1)$$

where \mathbf{r} ($\dot{\mathbf{r}}$) is the position (velocity) vector of the electron, \mathbf{A} (ϕ) is the vector (scalar) potential describing an external electromagnetic field, c is the speed of light, and $V(\mathbf{r})$ is an external potential. Let us focus for concreteness on motion in a homogeneous magnetic field applied perpendicularly to the plane of motion. We write X, Y for the Cartesian coordinates in this plane, and adopt the Landau gauge $\mathbf{A} = -(BY, 0, 0)$. Then, the Lagrangian function simplifies to

$$L = \frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) + \frac{e}{c}BY\dot{X} - V(X, Y). \quad (2)$$

In the limit of large magnetic field, we may neglect the kinetic term $m(\dot{X}^2 + \dot{Y}^2)/2$. This approximation will be valid if

$$\frac{1}{2}m(\dot{X}^2 + \dot{Y}^2) \ll \frac{e}{c}BY\dot{X}. \quad (3)$$

In solids, $m = m_{\text{eff}}$ is the effective mass of the electron, and the Fermi velocity v_F provides an upper bound for its characteristic velocity. Hence, the left-hand side of Eq. (3) is maximal if $\dot{X}^2 + \dot{Y}^2 = v_F^2$. Thus, we set $\dot{X} = \dot{Y} = v_F/\sqrt{2}$, in which case we obtain

$$\frac{1}{\sqrt{2}}m_{\text{eff}}v_F \ll \frac{e}{c}BY. \quad (4)$$

If we now set $Y = l_B = \sqrt{\hbar c/(eB)}$, we obtain the criterion

$$\frac{1}{\sqrt{2}}m_{\text{eff}}v_F \ll \frac{\hbar}{l_B} \quad (5)$$

or, equivalently,

$$l_F \gg \frac{l_B}{\sqrt{2}}, \quad (6)$$

where $l_F = \hbar/(v_F m_{\text{eff}})$ is the Fermi length. Hence, if the Fermi length l_F is much larger than the magnetic length l_B , then we may use the approximate Lagrangian

$$\tilde{L} = \frac{e}{c}BY\dot{X} - V(X, Y) \quad (7)$$

for describing the motion of the electron. For the favorable case of electrons in the lowest conduction band of GaAs at room temperature [18],

$$m_{\text{eff}} = 0.063m_{\text{el}}, \quad v_F = 4.4 \times 10^5 \text{ m/s} \approx 10^{-3}c. \quad (8)$$

Hence, $l_F = l_B/\sqrt{2}$ for $B \approx 20$ T.

Let us point out in closing that gauge invariance has not been broken for the sake of the approximation. The Lagrangian

$$\tilde{L} = -\frac{e}{c}\dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) + e\phi(\mathbf{r}) - V(\mathbf{r}), \quad (9)$$

of which Eq. (7) is a special case, changes by a total time derivative if $\mathbf{A} \rightarrow \mathbf{A} + \nabla\chi$.

A. Quantization in ultrahigh magnetic fields

The general problem of quantizing Lagrangians linear in the velocity has been discussed in Ref. [19]. For \tilde{L} , in particular, there is only one momentum variable

$$P_x = \frac{\partial \tilde{L}}{\partial \dot{X}} = \frac{e}{c}BY \quad (10)$$

conjugate to X because \tilde{L} does not depend on \dot{Y} . The Hamiltonian is then

$$H = P_x \dot{X} - \tilde{L} = V(X, Y). \quad (11)$$

Upon quantization, the operators X and P_x should obey the canonical commutation relations $[X, P_x] = i\hbar$. Hence,

$$[X, Y] = i\frac{c\hbar}{eB} = il_B^2, \quad (12)$$

and this is how the noncommutative planar coordinate emerges in an ultrahigh magnetic field.

The noncommutative plane is rich in counterintuitive features. Let us explore some of them.

Impossibility of perfect localization. How well, how sharply can we locate an electron moving in the noncommutative plane? Since its position operators do not commute [recall Eq. (12)], localization is limited by Heisenberg's uncertainty principle. The best one can do is to prepare the electron in a state that is peaked as sharply as possible around the mean positions $\langle X \rangle$, $\langle Y \rangle$. These states are the coherent states associated with the creation and annihilation operators

$$W = \frac{X + iY}{\sqrt{2}l_B}, \quad W^\dagger = \frac{X - iY}{\sqrt{2}l_B} \quad (13)$$

($[W, W^\dagger] = 1$). Let

$$|w\rangle = e^{-|w|^2/2} e^{wW^\dagger} |0\rangle, \quad (14)$$

where $|0\rangle$ is the unique normalized state satisfying $W|0\rangle = 0$. Then

$$\langle w|X|w\rangle = \sqrt{2}l_B \operatorname{Re}(w), \quad \langle w|Y|w\rangle = \sqrt{2}l_B \operatorname{Im}(w), \quad (15)$$

where $\operatorname{Re}(w)$ and $\operatorname{Im}(w)$ denote the real and the imaginary parts of w . Moreover, the mean-squared dispersion of the position of the particle around this average is the minimum allowed by Heisenberg's uncertainty principle.

Complete quenching of quantum tunneling. Tunneling out of a metastable equilibrium position through states that are classically forbidden is a hallmark of quantum mechanics, and seems impossible to avoid. Semiclassical reasoning like the WKB approximation shows it is possible to suppress the rate of quantum tunneling by applying a magnetic field. But, what is the precise behavior of the tunneling rate in the limit of ultrahigh fields? The answer is transparent in the noncommutative plane: the rate can converge to zero, so that quantum tunneling becomes completely suppressed.

Consider, for example, the Hamiltonian

$$H = V(X, Y) = \alpha X^2 (1 - \beta X) \quad (\alpha, \beta > 0) \quad (16)$$

in the noncommutative plane, which is independent of the Y coordinate. Normally, an electron positioned anywhere on the metastable minimum of this potential well would be able to escape by quantum tunneling. Escape is witnessed by the time evolution of $\langle X \rangle$: the electron escapes the well if $\langle X \rangle$ or $\langle X^2 \rangle$ grows beyond a certain value. However, in the noncommutative plane, the Hamiltonian of the electron is just $H = V$ and, since $[H, X] = 0$, $\langle X \rangle$ and $\langle X^2 \rangle$ are constant in time. It follows that the rate of quantum tunneling out of this metastable well vanishes in the limit in which the Fermi length greatly exceeds the magnetic length.

Another interesting example is provided by the Hamiltonian

$$H = V(X, Y) = -\alpha(R^2 - \beta)^2 \quad (\alpha, \beta > 0) \quad (17)$$

in the noncommutative plane (inverted Mexican hat). The observable

$$R^2 = X^2 + Y^2 \quad (18)$$

measures the radial distance from the origin, squared. In order to escape from this potential well, the electron must be able to change its radial distance to the origin. But, since $[H, R^2] = 0$,

this is not possible, and thus we find again that quantum tunneling has been completely quenched by the ultrahigh magnetic field.

Rotations and translations. In the noncommutative plane, R^2 is proportional to the infinitesimal generator of rotations. Let

$$\mathcal{L} = \frac{1}{2l_B^2} R^2. \quad (19)$$

It follows immediately that

$$[\mathcal{L}, X] = iY, \quad [\mathcal{L}, Y] = -iX. \quad (20)$$

Meanwhile, a translation is represented by the unitary transformation

$$U(x_0, y_0) = e^{i(y_0 X - x_0 Y)/l_B^2} \quad (21)$$

since

$$U(x_0, y_0) X U^\dagger(x_0, y_0) = X + x_0, \quad (22)$$

$$U(x_0, y_0) Y U^\dagger(x_0, y_0) = Y + y_0. \quad (23)$$

The relation

$$U(x_0, y_0) U(x'_0, y'_0) = U(x_0 + x'_0, y_0 + y'_0) e^{i(x_0 y'_0 - y_0 x'_0)/2l_B^2} \quad (24)$$

shows that this representation of planar translations is projective, as one would expect in the presence of a magnetic field.

B. An alternative point of view: Projection onto a Landau level

In this section, we establish the equivalence of projecting onto any Landau level and taking the ultrahigh-magnetic-field limit as above.

The canonical momenta corresponding to Eq. (1) are

$$p_x = m\dot{X} - \frac{e}{c} A_x, \quad p_y = m\dot{Y} - \frac{e}{c} A_y. \quad (25)$$

In the absence of an electric field and external potential, the Hamiltonian corresponding to Eq. (1) can be written as

$$H = \frac{1}{2m} (\pi_x^2 + \pi_y^2), \quad (26)$$

where π_x and π_y are the gauge-covariant momenta, defined in terms of the usual momenta $p_x = -i\hbar\partial/\partial x$ and $p_y = -i\hbar\partial/\partial y$ as

$$\pi_x := p_x + \frac{e}{c} A_x, \quad \pi_y := p_y + \frac{e}{c} A_y. \quad (27)$$

The gauge-covariant momenta obey the commutation relation

$$[\pi_x, \pi_y] = -i \frac{\hbar^2}{l_B^2}. \quad (28)$$

The guiding center operators, defined by

$$C_x := X - c \frac{\pi_y}{eB}, \quad C_y := Y + c \frac{\pi_x}{eB}, \quad (29)$$

commute with the Hamiltonian (26) and obey the commutation relation

$$[C_x, C_y] = i l_B^2. \quad (30)$$

Let us now define the Landau-level annihilation and creation operators a and a^\dagger , and the radial annihilation and creation operators b and b^\dagger , respectively, by

$$a = \frac{l_B}{\hbar\sqrt{2}}(\pi_x - i\pi_y), \quad a^\dagger = \frac{l_B}{\hbar\sqrt{2}}(\pi_x + i\pi_y), \quad (31)$$

$$b = \frac{1}{l_B\sqrt{2}}(C_x + iC_y), \quad b^\dagger = \frac{1}{l_B\sqrt{2}}(C_x - iC_y). \quad (32)$$

The commutation relations

$$[a, a^\dagger] = 1, \quad [b, b^\dagger] = 1 \quad (33)$$

are a straightforward consequence of Eqs. (28) and (30). Moreover,

$$[a, b] = 0 = [a, b^\dagger]. \quad (34)$$

The Hamiltonian (26) can now be written as

$$H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2} \right), \quad (35)$$

where $\omega_c = eB/(mc)$ is the cyclotron frequency. The identity

$$2l_B^2 \left(b^\dagger b - \frac{1}{2} \right) = C_x^2 + C_y^2 \quad (36)$$

explains why the operators b and b^\dagger are called radial. Since the a and a^\dagger commute with b and b^\dagger , a complete basis of normalized eigenvectors of the Hamiltonian of Eq. (26) is constructed as

$$|n, m\rangle = \frac{(a^\dagger)^n (b^\dagger)^m}{\sqrt{n!} \sqrt{m!}} |0, 0\rangle, \quad (37)$$

where $|0, 0\rangle$ is the unique normalized vector obeying

$$a|0, 0\rangle = b|0, 0\rangle = 0. \quad (38)$$

Hence,

$$H|n, m\rangle = \hbar\omega_c \left(n + \frac{1}{2} \right) |n, m\rangle. \quad (39)$$

The fact that the energy does not depend on m indicates that n labels the infinitely degenerate Landau levels and m labels this degeneracy. From Eq. (39), we see that the energy spacing between adjacent Landau levels is $\hbar\omega_c = \hbar eB/(mc)$, i.e., it is linear in the magnetic field. In the large magnetic field limit, we thus see that tunneling between Landau levels is suppressed. This explains, intuitively, why the large magnetic-field limit is equivalent to projecting onto a Landau level. Let us make this intuition rigorous.

The operator that projects a state onto the n th Landau level is given by

$$P_n = \sum_{m=0}^{\infty} |n, m\rangle \langle n, m|. \quad (40)$$

It follows that

$$P_n a P_n = 0, \quad [P_n, b] = 0. \quad (41)$$

Combining Eqs. (29) and (31), we see that

$$X = C_x + i \frac{l_B}{\sqrt{2}}(a - a^\dagger), \quad (42)$$

$$Y = C_y - \frac{l_B}{\sqrt{2}}(a + a^\dagger), \quad (43)$$

which may be combined with Eqs. (32) and (41) to yield

$$P_n X P_n = C_x P_n = P_n C_x, \quad (44)$$

$$P_n Y P_n = C_y P_n = P_n C_y. \quad (45)$$

Now, one may verify that

$$[P_n X P_n, P_n Y P_n] = i l_B^2 P_n, \quad (46)$$

which is isomorphic to Eq. (12) on the range of P_n .

Let us now include an external potential in the Hamiltonian (26), and project the Hamiltonian

$$H = \frac{1}{2m}(\pi_x^2 + \pi_y^2) + V(X, Y) \quad (47)$$

onto the n th Landau level

$$P_n H P_n = \hbar\omega_c \left(n + \frac{1}{2} \right) + P_n V(X, Y) P_n. \quad (48)$$

If we make the approximation

$$P_n V(X, Y) P_n \approx V(P_n X P_n, P_n Y P_n) \quad (49)$$

of further neglecting any virtual transitions to other levels, then the resulting projected approximate Hamiltonian describes precisely the dynamical problem associated to the approximate Lagrangian of Eq. (7). It is in this sense that this paper investigates Brownian motion restricted to a Landau Level.

III. LANGEVIN EQUATION IN THE NONCOMMUTATIVE PLANE

In classical mechanics, Brownian motion of a charged particle is described by the Langevin equation

$$m\ddot{\mathbf{r}} = -\eta\dot{\mathbf{r}} - e\mathbf{E} - \frac{e}{c}\dot{\mathbf{r}} \times \mathbf{B} - \nabla V + \mathbf{f}, \quad (50)$$

where η is the friction constant, and where \mathbf{f} is the random Langevin force satisfying

$$\langle\langle \mathbf{f} \rangle\rangle = 0, \quad \langle\langle f^\alpha(t) f^\beta(t') \rangle\rangle = 2\eta k_B T \delta^{\alpha\beta} \delta(t - t'). \quad (51)$$

The operation $\langle\langle \dots \rangle\rangle$ denotes averaging with respect to the probability distribution of the random force.

The classical Langevin equation can be derived from a model where the charged particle is coupled to a bath of electrically neutral, independent harmonic oscillators. A Lagrangian representation of this model is

$$L_{\text{Brownian}} = \frac{1}{2} m \dot{\mathbf{r}}^2 - \frac{e}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) + e\phi(\mathbf{r}) - V(\mathbf{r}) + \sum_j \frac{m_j}{2} [\dot{\mathbf{x}}_j^2 - \omega_j^2 (\mathbf{x}_j - \mathbf{r})^2], \quad (52)$$

where the harmonic oscillators have coordinates \mathbf{x}_j , masses m_j , and frequencies ω_j . Now, suppose that the oscillators are in a thermal Gibbs state. Then, the Lagrangian (52) can indeed

be used as the starting point for deriving Eq. (50) (see for example Appendix C in Ref. [20]).

This result is remarkable because the Lagrangian L_{Brownian} can be quantized, and the procedure for deriving the classical Langevin equation can be adapted in order to derive an operator Langevin equation [15]. This standard operator Langevin equation is by now textbook material [9]. Nonetheless, we will briefly recall its derivation (in one space dimension for simplicity) in Sec. III A in order to clarify some delicate mathematical and physical points and make certain ideas and notations readily available for the rest of the paper.

Then, in Sec. III B we will follow the same procedure, but starting from the approximation

$$\tilde{L}_{\text{Brownian}} = -\frac{e}{c} \dot{\mathbf{r}} \cdot \mathbf{A}(\mathbf{r}) + e\phi(\mathbf{r}) - V(\mathbf{r}) + \sum_j \frac{m_j}{2} [\dot{\mathbf{x}}_j^2 - \omega_j^2 (\mathbf{x}_j - \mathbf{r})^2], \quad (53)$$

appropriate for describing motion in an ultrahigh magnetic field. Within this approximation, the bath is coupled to the electron by way of two noncommuting observables. This is how, building on a sound foundation, we arrive to a Langevin equation for the noncommutative plane.

A. Standard operator Langevin equation

Let us focus on one-dimensional motion, for clarity of presentation. The starting point is the Hamiltonian

$$H = \frac{P^2}{2M} + V(X) + \frac{1}{2} \sum_j \left[\frac{p_j^2}{m_j} + m_j \omega_j^2 (x_j - X)^2 \right] \quad (54)$$

for a particle with momentum and position operators P and X , respectively, singled out for observation and coupled to a bath of independent oscillators labeled by j . In the following, we will write \hat{O} for observables in the Schrödinger picture, $\hat{O}(t) = e^{iHt} \hat{O} e^{-iHt}$ for the Heisenberg picture, and $\tilde{O}(t) = e^{iH_B t} \hat{O} e^{-iH_B t}$ for the interaction picture, with

$$H_B = \frac{1}{2} \sum_j \left[\frac{p_j^2}{m_j} + m_j \omega_j^2 x_j^2 \right]. \quad (55)$$

All three pictures coincide at $t = 0$.

In order to derive the operator Langevin equation, we will begin by investigating the dynamics of this closed quantum system in the Heisenberg picture. Hence, the state ρ of the system, a density matrix acting on the total Hilbert space

$$\mathcal{H} = \bigotimes_j \mathcal{H}_{x_j} \otimes \mathcal{H}_X \quad (56)$$

is independent of time. As a consequence, the presence or absence of entanglement between parts of the system at any time other than $t = 0$ is not directly encoded in ρ . The Heisenberg equations of motion are

$$\ddot{x}_j = -\omega_j^2 (x_j - X), \quad (57)$$

$$M\ddot{X} = -V'(X) + \sum_j m_j \omega_j^2 (x_j - X). \quad (58)$$

Let us solve the set of equations in the first line. Since we would like to use standard results available for differential equations involving functions, it is safer to start by solving the associated differential equations for transition amplitudes. If $|\Phi\rangle, |\Psi\rangle$ are normalizable, time-independent states, then

$$\frac{d^2}{dt^2} \langle \Phi | x_j(t) | \Psi \rangle = -\omega_j^2 [\langle \Phi | x_j(t) | \Psi \rangle - \langle \Phi | X(t) | \Psi \rangle]. \quad (59)$$

The delicate point is whether these matrix elements define well-behaved functions of time for which standard manipulations hold. The quick answer is yes, *as long as the number of oscillators making up the bath is finite*.

Hence, let us proceed for now under the assumption that this is the case. Then, the solution

$$\langle \Phi | x_j(t) | \Psi \rangle = \langle \Phi | \bar{x}_j^h | \Psi \rangle + \int_0^t ds \omega_j \sin[\omega_j(t-s)] \langle \Phi | X(s) | \Psi \rangle, \quad (60)$$

with

$$\bar{x}_j^h(t) = \hat{x}_j \cos(\omega_j t) + \frac{\hat{p}_j}{m_j \omega_j} \sin(\omega_j t), \quad (61)$$

properly incorporates the boundary condition that the Heisenberg and Schrödinger pictures should coincide at $t = 0$. Integrating by parts, we obtain the alternative representation

$$\begin{aligned} \langle \Phi | x_j(t) | \Psi \rangle - \langle \Phi | X(t) | \Psi \rangle &= \langle \Phi | \bar{x}_j^h | \Psi \rangle - \langle \Phi | \hat{X} | \Psi \rangle \cos(\omega_j t) \\ &\quad - \int_0^t ds \cos[\omega_j(t-s)] \langle \Phi | \dot{X}(s) | \Psi \rangle. \end{aligned} \quad (62)$$

The operator $\bar{x}_j^h(t)$, a solution of the homogeneous version of Eq. (57), evolves in time according to the interaction picture, as defined at the beginning of this section.

For a finite bath, these time-dependent matrix elements are reasonably well behaved in general. Hence, we can promote the family of c -number solutions of Eq. (62) to operator status:

$$\begin{aligned} x_j(t) - X(t) &= \bar{x}_j^h(t) - \cos(\omega_j t) \hat{X} \\ &\quad - \int_0^t ds \cos[\omega_j(t-s)] \dot{X}(s). \end{aligned} \quad (63)$$

For an infinite bath, this step is ungranted: The many-body amplitude $\langle \Phi | \dot{X}(s) | \Psi \rangle$ will require renormalization in general. We will come back to this point near the end of this section.

The next step is to substitute Eq. (63) in (58). In terms of the definitions

$$\mu(t) = \sum_j m_j \omega_j^2 \cos(\omega_j t), \quad (64)$$

$$\bar{F}(t) = \sum_j m_j \omega_j^2 \bar{x}_j^h(t), \quad (65)$$

one obtains

$$M\ddot{X}(t) + V'(X(t)) = - \int_0^t ds \mu(t-s) \dot{X}(s) - \mu(t) \hat{X} + \bar{F}(t). \quad (66)$$

Hence, μ is the memory kernel. It also appears in the commutator

$$[\bar{F}(t), \bar{F}(s)] = i\hbar \frac{d}{dt} \mu(t-s). \quad (67)$$

Equation (66) is the standard operator Langevin equation and the foundation of the Ford-Kac-Mazur approach [9,14,21] to modeling dissipation in quantum mechanics. It is a peculiar equation of motion because it mixes together the Heisenberg, Schrödinger, and interaction pictures. In particular, the quantum Langevin force \bar{F} evolves in time according to the interaction picture in which the electron acts as a perturbation on the bath.

Let us take a closer look at Eq. (66), as it is used for modeling diffusion [9]. Hence, we set $V = 0$, and

$$\mu(t) = 2\eta\delta(t), \quad (68)$$

so that

$$M\ddot{X}(t) = -\eta\dot{X}(t) - 2\eta\delta(t)\hat{X} + \bar{F}(t), \quad (69)$$

according to the rule $\int^t ds \delta(t-s)f(s) = f(t)/2$. The memory kernel of Eq. (68) can only be obtained by letting the number of oscillators in the bath become infinite, and the term $-2\eta\delta(t)\hat{X}$ is typical of the problems associated to this limiting procedure. Taken at face value, it indicates that $P(t)$ is discontinuous at $t = 0$ [assuming that $X(t)$ is continuous at $t = 0$], and so we need to be more careful in specifying boundary conditions. If we impose

$$\lim_{t \rightarrow 0^+} P(t) = \hat{P}, \quad (70)$$

then the solution of the operator Langevin equation for $t \geq 0$ is

$$P(t) = e^{-\eta t/M} \hat{P} + \int_0^t ds e^{-\eta(t-s)/M} \bar{F}(s), \quad (71)$$

$$X(t) = \hat{X} + \frac{1}{\eta}(1 - e^{-\eta t/M})\hat{P} + \frac{1}{M} \int_0^t ds'' \int_0^{s''} ds' e^{-\eta(s''-s')/M} \bar{F}(s'). \quad (72)$$

Using these explicit solutions and Eq. (67), one can show that

$$[X(t), P(t)] = i\hbar + i\hbar e^{-\eta t/M} (e^{-\eta t/M} - 1) \quad (t \geq 0). \quad (73)$$

The commutator is canonical precisely at $t = 0$, and for long times up to exponentially small errors. The transient period during which the deviation from $i\hbar$ is appreciable is very short. This result indicates quite correctly that making the bath infinite spoils, to some extent, the simple microscopic derivation of the operator Langevin equation. As we mentioned already, the passage from Eq. (62) to (63) is not well justified if the bath is infinite. In spite of this complication, Eq. (73) reassures us that the operator Langevin equation remains a phenomenologically sound starting point for describing quantum Brownian motion at times $t \gg 0$.

The discontinuity at $t = 0$ and the partial failure of $[X(t), P(t)] = i\hbar$ are two *separate* subtleties that one has to deal with in this approach. Let us make a final remark on the first of these issues. Within the influence functional formalism

advanced by Caldeira and Leggett [20], the same kind of problem arises when the bath and the system of interest are considered to be decoupled at $t = 0$ (or, equivalently, when the initial state is factorizable). In this case, one also has to perform certain integrals from $t = 0^+$. A less artificial way to resolve the problem is to consider a state where the system and the bath are initially in thermal equilibrium (see Refs. [22,23]).

B. Operator Langevin equations in the noncommutative regime $l_F \gg l_B$

In this section, we will go through the steps of deriving an operator Langevin equation starting from the approximate Lagrangian $\tilde{L}_{\text{Brownian}}$ of Eq. (53). The associated Hamiltonian is

$$H = V(X, Y) + \sum_j \left[\frac{1}{2m_j} p_j^x{}^2 + \frac{1}{2} m_j \omega_j^x{}^2 (x_j - X)^2 \right] + \sum_j \left[\frac{1}{2m_j} p_j^y{}^2 + \frac{1}{2} m_j \omega_j^y{}^2 (y_j - Y)^2 \right]. \quad (74)$$

The Heisenberg equations of motion for the oscillators are

$$\ddot{x}_j + \omega_j^x{}^2 x_j = \omega_j^x{}^2 X, \quad (75)$$

$$\ddot{y}_j + \omega_j^y{}^2 y_j = \omega_j^y{}^2 Y; \quad (76)$$

and the equations of motion for the electron are

$$\dot{X} = \frac{i}{\hbar} [V, X] + \frac{l_B^2}{\hbar} \sum_j m_j \omega_j^y{}^2 (y_j - Y), \quad (77)$$

$$\dot{Y} = \frac{i}{\hbar} [V, Y] - \frac{l_B^2}{\hbar} \sum_j m_j \omega_j^x{}^2 (x_j - X). \quad (78)$$

The idea is to reduce this system of equations by solving the equations of motion for the oscillators. Let us introduce some notation before we proceed. First,

$$\bar{U}_\alpha(t) = \frac{l_B^2}{\hbar} \sum_j m_j \omega_j^{\alpha 2} \bar{r}_j^\alpha(t), \quad (79)$$

with

$$\bar{r}_j^\alpha(t) = \hat{r}_j^\alpha \cos(\omega_j^\alpha t) + \hat{p}_j^\alpha \frac{\sin(\omega_j^\alpha t)}{m_j^\alpha \omega_j^\alpha} \quad (80)$$

and

$$\hat{r}_j^x = \hat{x}_j, \quad \hat{r}_j^y = \hat{y}_j \quad (81)$$

defines a random velocity field. The operators $\hat{x}_j, \hat{p}_j^x, \hat{y}_j, \hat{p}_j^y$ are standard, time-independent Schrödinger position and momentum operators for the oscillators in the bath. As a consequence, this random velocity field evolves in time according to the interaction picture. Second,

$$\Omega_\alpha = \frac{l_B^2}{\hbar} \sum_j m_j \omega_j^{\alpha 2} \quad (\alpha = x, y) \quad (82)$$

is a quantity with the dimension of angular frequency, and finally

$$v_\alpha(t) = \frac{l_B^2}{\hbar} \sum_j m_j \omega_j^{\alpha 3} \sin(\omega_j^\alpha t) \quad (\alpha = x, y) \quad (83)$$

is a function of time with the dimension of angular frequency squared.

In terms of the explicit closed-form expression for $y_j(t)$, i.e., the operator y analog of Eq. (60), we can rewrite Eq. (77) as

$$\begin{aligned} \dot{X}(t) - \frac{i}{\hbar} [V, X](t) \\ = \bar{U}_y(t) - \Omega_y Y(t) + \int_0^t ds v_y(t-s) Y(s). \end{aligned} \quad (84)$$

Similarly,

$$\begin{aligned} \dot{Y}(t) - \frac{i}{\hbar} [V, Y] \\ = -\bar{U}_x(t) + \Omega_x X(t) - \int_0^t ds v_x(t-s) X(s). \end{aligned} \quad (85)$$

Moreover, in terms of standard memory kernels

$$\mu_\alpha(t) = \frac{l_B^2}{\hbar} \sum_j m_j \omega_j^{\alpha 2} \cos(\omega_j^\alpha t), \quad (86)$$

our kernels v_α are

$$v_\alpha(t) = -\frac{d\mu_\alpha}{dt}, \quad \mu_\alpha(0) = \Omega_\alpha. \quad (87)$$

Thus, after an integration by parts, we obtain

$$\begin{aligned} \dot{X}(t) - \frac{i}{\hbar} [V, X](t) \\ = \bar{U}_y(t) - \mu_y(t) \hat{Y} - \int_0^t \mu_y(t-s) \dot{Y}(s) ds, \end{aligned} \quad (88)$$

$$\begin{aligned} \dot{Y}(t) - \frac{i}{\hbar} [V, Y](t) \\ = -\bar{U}_x(t) + \mu_x(t) \hat{X} + \int_0^t \mu_x(t-s) \dot{X}(s) ds, \end{aligned} \quad (89)$$

where

$$\hat{X} = X(0), \quad \hat{Y} = Y(0) \quad (90)$$

are time-independent Schrödinger operators. Equations (88) and (89) are the operator Langevin equations for the noncommutative plane. Unlike the standard operator Langevin equations, they are of first order in time derivatives, which is why the random driving term is a velocity field rather than a Langevin force.

Correlators of the random velocity field for a thermal bath. The statistical properties of the operator-valued random velocity fields \bar{U}_x and \bar{U}_y are described by their symmetrized expectation value. To compute this quantity, it is necessary to specify the state of the bath. A common choice is to assume that the harmonic oscillators are canonically distributed, at

temperature T , with respect to the Hamiltonian

$$\begin{aligned} H_B = \sum_j \left[\frac{1}{2m_j} p_j^{x2} + \frac{1}{2} m_j \omega_j^{x2} (x_j - X)^2 \right] \\ + \sum_j \left[\frac{1}{2m_j} p_j^{y2} + \frac{1}{2} m_j \omega_j^{y2} (y_j - Y)^2 \right], \end{aligned} \quad (91)$$

so that the expectation values of a bath observable O is

$$\langle O \rangle = \frac{\text{Tr}\{O e^{-H_B/k_B T}\}}{\mathcal{Z}_B}, \quad (92)$$

where \mathcal{Z}_B is the partition function of the bath. In particular,

$$\begin{aligned} \frac{1}{2} \langle U_\alpha(t) U_\beta(t') + U_\beta(t') U_\alpha(t) \rangle \\ = \delta_{\alpha\beta} \frac{l_B^4}{2\hbar} \sum_j m_j \omega_j^{\alpha 3} \cos[\omega_j^\alpha(t-t')] \coth\left(\frac{\hbar\omega_j^\alpha}{2k_B T}\right) \end{aligned} \quad (93)$$

(see Appendix A for a derivation of this result).

The continuum frequency limit. If one would like to allow for true dissipation of energy by way of the bath, then it becomes necessary to let the number of oscillators become infinite in terms of a continuum range of frequencies. Let us introduce an interpolating mass function $m_\alpha(\omega)$, such that

$$m_j = m_\alpha(\omega_j^\alpha). \quad (94)$$

The spectral density of the bath is

$$\rho_\alpha(\omega) = \sum_j \delta(\omega - \omega_j^\alpha). \quad (95)$$

It is understood that $\rho(\omega) = 0$ if $\omega \leq 0$. With the help of these definitions, all of our previous expressions may be rewritten in terms of integrals involving ρ and κ , so that the properties of the bath are encoded in the spectral density ρ . For example, the memory kernel of Eq. (86) becomes

$$\mu_\alpha(t) = \frac{l_B^2}{\hbar} \int_0^\infty d\omega \rho_\alpha(\omega) m_\alpha(\omega) \omega^2 \cos(\omega t). \quad (96)$$

At this point, it is interesting to recall Ref. [16] for comparison. In this paper, the conventional Brownian motion of an electron in a magnetic field \mathbf{B} is investigated, away from the singular limit $l_F \gg l_B$ characteristic of the quantum Hall regime. For more details, see Appendix B and the original sources.

IV. DIFFUSION IN THE NONCOMMUTATIVE PLANE

In this section, we will use our Langevin equation [Eqs. (88) and (89)] for investigating an important aspect of topological Brownian motion: diffusion in the noncommutative plane. According to our results in Sec. II B, we can think of this phenomenon as describing the emergent properties of normal electronic diffusion in the situation in which an applied magnetic field is strong enough to project the Brownian motion of the electron to a fixed Landau level.

Let us assume the oscillator bath to be isotropic and $V = 0$, so that the total system as described by the Hamiltonian of Eq. (74) is rotationally and translationally invariant. Moreover,

we assume that the frequency distribution of the bath is such that the memory kernels are

$$\mu_x(t) = \mu_y(t) = 2\gamma\delta(t). \quad (97)$$

Recalling the definition of these memory kernels [Eq. (86)], we see that a particular distribution of frequencies and masses with this property is

$$\omega_j^x = \omega_j^y = j, \quad m_j = \frac{2}{j} \frac{\gamma\hbar}{\pi l_B^2}. \quad (98)$$

This statement is rather qualitative, but suffices for our purposes (see Appendix C for the justifications of these claims).

Under these conditions, our Langevin equation reduces to

$$\dot{X}(t) + \gamma\dot{Y}(t) = \bar{U}_y(t) - 2\gamma\delta(t)\hat{Y}, \quad (99)$$

$$\dot{Y}(t) - \gamma\dot{X}(t) = -\bar{U}_x(t) + 2\gamma\delta(t)\hat{X}. \quad (100)$$

The meaning of terms proportional to $\delta(t)$ was explained in Sec. III A. In the following, we will discard these terms with the understanding that $t > 0$ always. Then, an elementary calculation yields

$$X(t) = \frac{1}{\gamma^2 + 1} \int_0^t [\bar{U}_y(s) + \gamma\bar{U}_x(s)]ds + \hat{X}, \quad (101)$$

$$Y(t) = \frac{1}{\gamma^2 + 1} \int_0^t [-\bar{U}_x(s) + \gamma\bar{U}_y(s)]ds + \hat{Y}. \quad (102)$$

While the parameter γ has the interpretation of a friction coefficient, the Hamiltonian of a free electron vanishes identically in the noncommutative plane. Hence, the electron has no energy to give to the bath, and cannot take energy from the bath either. It is a bizarre setup for diffusion, and further insight would be very desirable.

One of the hallmarks of classical Brownian motion is that its mean-squared displacement grows linearly as a function of time. Hence, let us compute

$$\begin{aligned} \langle R^2(t) \rangle &= \lim_{t' \rightarrow t} \frac{1}{2} \langle X(t)X(t') + X(t')X(t) \rangle \\ &\quad + \frac{1}{2} \langle Y(t)Y(t') + Y(t')Y(t) \rangle, \end{aligned} \quad (103)$$

the mean displacement (from the origin) squared. Since the system is rotationally invariant, this expression simplifies to

$$\begin{aligned} \langle R^2(t) \rangle &= \lim_{t' \rightarrow t} \langle X(t)X(t') + X(t')X(t) \rangle \\ &= \lim_{t' \rightarrow t} \langle Y(t)Y(t') + Y(t')Y(t) \rangle. \end{aligned} \quad (104)$$

At this point, it becomes necessary to make an explicit choice of state for the system. We take the product state

$$\rho = \frac{e^{-H_B/k_B T}}{\mathcal{Z}_B} \otimes |0\rangle\langle 0|, \quad (105)$$

where $|0\rangle$ denotes the coherent state centered at the origin of the noncommutative plane (see Sec. II A). Since the operator Langevin equation is derived in the Heisenberg picture, this state has the following interpretation at $t = 0$: The electron is maximally localized at the origin, the bath is in a thermal Gibbs state at temperature T , and there is no entanglement between the two. Recall that, by convention, $t = 0+$ is the time when the Heisenberg, Schrödinger, and interaction pictures entering the operator Langevin equation coincide (the reasons for writing $t = 0+$ are explained in Sec. III A).

Our choice of state for the system implies

$$\langle \hat{X} \rangle = 0 = \langle \hat{Y} \rangle, \quad \langle \hat{X}^2 \rangle = \frac{l_B^2}{2} = \langle \hat{Y}^2 \rangle, \quad \langle \bar{U}_x(t)\bar{U}_y(t') \rangle = 0. \quad (106)$$

Combining this information with Eq. (101), we obtain

$$\begin{aligned} \frac{1}{2} \langle X(t)X(t') + X(t')X(t) \rangle &= \frac{1}{2(\gamma^2 + 1)^2} \int_0^t ds \int_0^{t'} ds' \langle \bar{U}_y(s)\bar{U}_y(s') + \bar{U}_y(s')\bar{U}_y(s) \\ &\quad + \gamma^2 \bar{U}_x(s)\bar{U}_x(s') + \gamma^2 \bar{U}_x(s')\bar{U}_x(s) \rangle + \frac{l_B^2}{2}. \end{aligned} \quad (107)$$

With our current choice of distribution for the oscillator masses and frequencies, Eq. (93) reduces to

$$\begin{aligned} \frac{1}{2} \langle \bar{U}_\alpha(t)\bar{U}_\beta(t') + \bar{U}_\beta(t')\bar{U}_\alpha(t) \rangle &= \delta_{\alpha\beta} \frac{\gamma l_B^2}{\pi} \sum_j j^2 \cos[j(t - t')] \coth\left(\frac{\hbar j}{2k_B T}\right) \\ &= \delta_{\alpha\beta} \frac{\gamma l_B^2}{\pi} \int_0^\infty \omega \cos[\omega(t - t')] \coth\left(\frac{\hbar\omega}{2k_B T}\right) d\omega. \end{aligned} \quad (108)$$

Strictly speaking, this integral diverges. However, one can make sense of it by declaring it to be the Fourier cosine transform of $\omega \coth(\hbar\omega/(2k_B T))$. This yields (see Ref. [14])

$$\begin{aligned} \frac{1}{2} \langle \bar{U}_\alpha(t)\bar{U}_\beta(t') + \bar{U}_\beta(t')\bar{U}_\alpha(t) \rangle \\ = \delta_{\alpha\beta} k_B T \frac{\gamma l_B^2}{\hbar} \frac{d}{dt} \coth\left(\frac{\pi k_B T(t - t')}{\hbar}\right). \end{aligned} \quad (109)$$

The symmetrized velocity correlator does not distinguish between the x and y directions, as was to be expected, since

the distributions for the masses and frequencies were chosen to be identical as well [see Eq. (98)]. Hence, Eq. (107) becomes

$$\begin{aligned} \frac{1}{2} \langle X(t)X(t') + X(t')X(t) \rangle \\ = \frac{\gamma k_B T l_B^2}{\hbar(1 + \gamma^2)} \int_0^{t'} ds' \coth\left(\frac{\pi k_B T(t - s')}{\hbar}\right), \end{aligned} \quad (110)$$

where we have dropped the unimportant constant $\langle \hat{X}^2 \rangle$. Finally, by performing the integral in Eq. (110) for $0 < t' < t$, we

obtain

$$\begin{aligned} & \frac{1}{2} \langle X(t)X(t') + X(t')X(t) \rangle \\ &= \frac{\gamma l_B^2}{\pi(1+\gamma^2)} \ln \left[\operatorname{csch} \left(\frac{\pi(t-t')Tk_B}{\hbar} \right) \sinh \left(\frac{\pi t Tk_B}{\hbar} \right) \right]. \end{aligned} \quad (111)$$

The next step in order to compute the mean-squared displacement is taking the limit $t' \rightarrow t$. However, our current expression is not valid for $t = t'$. Hence, we set $t' = t - \varepsilon$, with ε some small positive time, so that

$$\begin{aligned} & \frac{1}{2} \langle X(t)X(t-\varepsilon) + X(t-\varepsilon)X(t) \rangle \\ &= C(\varepsilon) + \frac{\gamma l_B^2}{\pi(1+\gamma^2)} \ln \left[\sinh \left(\frac{\pi t Tk_B}{\hbar} \right) \right]. \end{aligned} \quad (112)$$

The point to notice is that

$$C(\varepsilon) = \frac{\gamma l_B^2}{\pi(1+\gamma^2)} \ln \left[\operatorname{csch} \left(\frac{\pi \varepsilon Tk_B}{\hbar} \right) \right] \quad (113)$$

is independent of t . For large times $t \rightarrow \infty$,

$$\begin{aligned} \ln \left[\sinh \left(\frac{\pi t Tk_B}{\hbar} \right) \right] &\approx \ln \left[\frac{1}{2} \exp \left(\frac{\pi t Tk_B}{\hbar} \right) \right] \\ &= \frac{\pi Tk_B}{\hbar} t - \ln[2], \end{aligned} \quad (114)$$

which leads to

$$\frac{1}{2} \langle X(t)X(t-\varepsilon) + X(t-\varepsilon)X(t) \rangle = \frac{\gamma l_B^2 Tk_B}{\hbar(1+\gamma^2)} t + C'(\varepsilon), \quad (115)$$

with

$$C'(\varepsilon) = \frac{\gamma l_B^2}{\pi(1+\gamma^2)} \ln \left[\frac{1}{2} \operatorname{csch} \left(\frac{\pi \varepsilon Tk_B}{\hbar} \right) \right] \quad (116)$$

again independent of time. Hence, to leading order in t ,

$$\langle R^2(t) \rangle = 2 \langle X^2(t) \rangle = 2 \frac{\gamma l_B^2 k_B T}{\hbar(1+\gamma^2)} t. \quad (117)$$

This is precisely the behavior characteristic of normal diffusion!

In closing, we would like to compare the result (117) with the analogous result for the standard quantum Langevin equation (69). For this equation of motion, in the classical limit one obtains [9]

$$\langle X^2(t) \rangle = 2 \frac{k_B T}{\eta} t. \quad (118)$$

Now, by dimensional analysis,

$$\gamma = \frac{l_B^2}{\hbar} \eta. \quad (119)$$

Hence, we can reexpress our result (117) as

$$\langle R^2(t) \rangle = \frac{2k_B T}{\eta + \frac{\hbar^2}{l_B^2} \frac{1}{\eta}} t. \quad (120)$$

This shows, on one hand, that

$$\langle R^2(t) \rangle = 2 \frac{k_B T}{\eta} t \quad (121)$$

in the limit $\eta \gg \hbar/l_B^2$, in which the friction coefficient becomes dominant. This is precisely the classical result just mentioned above. On the other hand, if the friction constant is small, $0 < \eta \ll \hbar/l_B^2$, then

$$\langle R^2(t) \rangle \approx 2 \frac{l_B^4 k_B T \eta}{\hbar^2} t = 2 \frac{c^2 k_B T \eta}{(eB)^2} t. \quad (122)$$

This is in stark contrast with the classical result: if η is small, the magnetic field suppresses fluctuations. This is a very important result of this work.

V. SUMMARY AND OUTLOOK

In this paper, we have investigated an instance of topological quantum Brownian motion: an electron subjected to an ultrahigh magnetic field and coupled to a thermal bath of independent oscillators. The operator Langevin equations that we derive for modeling this system are unconventional. They are first- rather than second-order differential equations, and can be interpreted as describing quantum Brownian motion projected onto a Landau level. In spite of the differences between the standard and our Langevin equation, diffusion in the noncommutative plane, or equivalently, in a Landau level, is conventional, i.e., the mean-squared displacement is proportional to t . However, the proportionality constant displays a bizarre regime as a function of the friction coefficient: For strong dissipation, friction reduces diffusion, as expected, and the magnetic field plays no role, but for weak dissipation, friction enhances diffusion and the magnetic field suppresses it. It would be remarkable to observe this regime experimentally, but it is far from clear how to do so. Possibly, this is yet another problem for the fast growing field of quantum simulations.

We also investigated other physical aspects of the noncommutative plane. We explored its unconventional symmetries, analyzed the fact that the electron cannot be perfectly localized due to Heisenberg uncertainty, and showed that quantum tunneling out of a metastable minimum can be completely suppressed. Since quantum tunneling is often thought of as an unavoidable fact of (quantum) life, it is interesting to see a regime where it can, in fact, be avoided. This result further suggests that the noncommutative plane might provide fertile ground for novel designs of quantum information processing devices. Since electrons become effectively spinless (they are completely spin polarized), these devices would necessarily be based on manipulating the charge degree of freedom of the electron, which is traditionally regarded as very noisy. However, this appreciation should be reconsidered once the magnetic length becomes much shorter than the Fermi length. Just like quantum tunneling and Brownian fluctuations are suppressed in this regime, it is possible that other sources of noise are equally suppressed. If this is the case, a device along these basic lines may be worth considering: Imagine a configuration of electrons with their X coordinates pinned to the minima of a grid $V(X, Y) = A \cos(2\pi X/L)$, and perfectly delocalized in the Y direction. Then, one could quantum

manipulate this system by diminishing, on specified regions of the grid and for specified time intervals, the local magnetic field enough to restore quantum tunneling. The final configuration would be recorded simply by restoring the ultrahigh magnetic field everywhere, freezing again the X coordinate of the electrons. What is not clear about this protocol is whether it allows for entanglement manipulation.

Let us conclude with an open problem. Since the operator Langevin equation is either simple or impossible to solve, according to whether the potential $V(X, Y)$ is at most quadratic in the coordinates, it is natural to attempt a path-integral description of the problem. It is not difficult to write a suitable propagator

$$\begin{aligned} G(Q'', T; Q', 0) &= \int \mathcal{D}Q \exp \left\{ \frac{i}{\hbar} \int_0^T dt \left[\frac{\hbar}{l_B^2} Y \dot{X} - V(X, Y) \right] \right\} \\ &\times \exp \left\{ \frac{i}{\hbar} \int_0^T \sum_j \left[\frac{1}{2} m_j \dot{x}_j^2 - \frac{1}{2} m_j \omega_j^2 (x_j - X)^2 \right] \right\} \\ &\times \exp \left\{ \frac{i}{\hbar} \int_0^T \sum_j \left[\frac{1}{2} m_j \dot{y}_j^2 - \frac{1}{2} m_j \omega_j^2 (y_j - Y)^2 \right] \right\}, \end{aligned} \quad (123)$$

where $Q' = \{X', Y', x'_j, y'_j\}$ and Q'' denote collectively the initial and final values of the coordinates, respectively. Moreover, it is possible to integrate out the bath in the usual way in order to obtain a description of the electron-bath coupling in terms of influence functionals [24]. The problem is, on one hand, that the resulting effective path integral for the electron has the structure of a phase-space path integral, equivalent to a coherent-state path integral, rendering semiclassical approximations untrustworthy [25]. On the other hand, the standard perturbation theory associated to coherent-state path integrals has not been explored in the presence of influence functionals. What is then a good way of modeling nonlinear forces and friction in the noncommutative plane?

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APPENDIX A: STATISTICAL PROPERTIES OF THE VELOCITY FIELD

In this section, we recall the arguments of Ref. [21] leading to Eq. (93). Consider a system of decoupled harmonic oscillators, canonically distributed at temperature T and described by the Hamiltonian

$$H_B = \sum_j \left[\frac{p_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 x_j^2 \right]. \quad (A1)$$

If O is any observable, then its expectation value may be computed as

$$\langle O \rangle = \frac{\text{Tr}\{O \exp[-H_B/(kT)]\}}{\text{Tr}\{\exp[-H_B/(kT)]\}}. \quad (A2)$$

The usual annihilation operator

$$a_j = \sqrt{\frac{m_j \omega_j}{2\hbar}} \left(\hat{x}_j + \frac{i}{m_j \omega_j} \hat{p}_j \right) \quad (A3)$$

and its adjoint allow us to rewrite Eq. (A1) as

$$H_B = \sum_j \hbar \omega_j \left(a_j^\dagger a_j + \frac{1}{2} \right). \quad (A4)$$

The eigenvalues of the j th single-particle Hamiltonian $H_{Bj} = \hbar \omega_j (a_j^\dagger a_j + 1/2)$ are $E_{Bjn} = \hbar \omega_j (n + 1/2)$. Therefore,

$$\begin{aligned} \langle a_i^\dagger a_j \rangle &= \delta_{ij} \frac{\text{Tr}\{a_i^\dagger a_j \exp[-H_B/(kT)]\}}{\text{Tr}\{\exp[-H_B/(kT)]\}} \\ &= \delta_{ij} \frac{\sum_{n=0}^{\infty} n \exp\left[-\frac{\hbar \omega_j}{kT} \left(n + \frac{1}{2}\right)\right]}{\sum_{n=0}^{\infty} \exp\left[-\frac{\hbar \omega_j}{kT} \left(n + \frac{1}{2}\right)\right]}. \end{aligned} \quad (A5)$$

In order to evaluate this sum, we define the function

$$f(\lambda) = \sum_{n=0}^{\infty} \exp\left[-\frac{\hbar \omega_j}{kT} \left(\lambda n + \frac{1}{2}\right)\right], \quad (A6)$$

which can be evaluated using the geometric series. In terms of f , Eq. (A5) can be written as

$$\langle a_i^\dagger a_j \rangle = \delta_{ij} \frac{\partial_\lambda f(\lambda)}{f(\lambda)} \Big|_{\lambda=1}. \quad (A7)$$

The result is

$$\langle a_i^\dagger a_j \rangle = \delta_{ij} \left[\exp\left(\frac{\hbar \omega_j}{kT}\right) - 1 \right]^{-1}. \quad (A8)$$

In other words,

$$\langle a_i^\dagger a_j \rangle = \frac{1}{2} \delta_{ij} \left[\coth\left(\frac{\hbar \omega_j}{2kT}\right) - 1 \right], \quad (A9)$$

$$\langle a_i a_j^\dagger \rangle = \frac{1}{2} \delta_{ij} \left[\coth\left(\frac{\hbar \omega_j}{2kT}\right) + 1 \right], \quad (A10)$$

and so

$$\langle \hat{x}_i \hat{x}_j \rangle = \delta_{ij} \frac{\hbar}{2\omega_j m_j} \coth\left(\frac{\hbar \omega_j}{2kT}\right), \quad (A11)$$

$$\langle \hat{p}_i \hat{p}_j \rangle = \delta_{ij} \frac{1}{2} \hbar \omega_j m_j \coth\left(\frac{\hbar \omega_j}{2kT}\right), \quad (A12)$$

$$\langle \hat{x}_i \hat{p}_j \rangle = \frac{1}{2} i \hbar \delta_{ij}. \quad (A13)$$

Now, we would like to compute

$$\frac{1}{2} \langle \bar{U}_\alpha(t) \bar{U}_\beta(t') + \bar{U}_\beta(t') \bar{U}_\alpha(t) \rangle, \quad (A14)$$

where

$$\bar{U}_\alpha(t) = \frac{l_B^2}{\hbar} \sum_j m_j \omega_j^2 \left(\cos(\omega_j^\alpha t) \hat{r}_j^\alpha + \frac{\sin(\omega_j^\alpha t)}{m_j \omega_j^\alpha} \hat{p}_j^\alpha \right). \quad (A15)$$

Since harmonic oscillators in multiple dimensions can be seen as decoupled one-dimensional harmonic oscillators, generalizing Eqs. (A11), (A12), and (A13) to multiple dimensions is straightforward. This reasoning yields the expectation values

$$\begin{aligned}\langle \hat{r}_i^\alpha \hat{r}_j^\beta + \hat{r}_j^\beta \hat{r}_i^\alpha \rangle &= \delta_{\alpha\beta} \delta_{ij} \frac{\hbar}{\omega_j^\alpha m_j} \coth\left(\frac{\hbar\omega_j}{2kT}\right), \\ \langle \hat{p}_i^\alpha \hat{p}_j^\beta + \hat{p}_j^\beta \hat{p}_i^\alpha \rangle &= \delta_{\alpha\beta} \delta_{ij} \hbar \omega_j^\alpha m_j \coth\left(\frac{\hbar\omega_j}{2kT}\right), \\ \langle \hat{r}_i^\alpha \hat{p}_j^\beta + \hat{p}_j^\beta \hat{r}_i^\alpha \rangle &= 0,\end{aligned}$$

and so we obtain

$$\begin{aligned}\frac{1}{2} \langle \bar{U}_\alpha(t) \bar{U}_\beta(t') + \bar{U}_\beta(t') \bar{U}_\alpha(t) \rangle \\ = \frac{l_B^4}{2\hbar^2} \delta_{\alpha\beta} \sum_j m_j^2 \omega_j^{\alpha 4} \left[\cos(\omega_j^\alpha t) \cos(\omega_j^\alpha t') 2 \langle \hat{r}_j^\alpha \hat{r}_j^\alpha \rangle \right. \\ \left. + \sin(\omega_j^\alpha t) \sin(\omega_j^\alpha t') 2 \frac{\langle \hat{p}_j^\alpha \hat{p}_j^\alpha \rangle}{m_j^2 \omega_j^{\alpha 2}} \right] \\ = \delta_{\alpha\beta} \frac{l_B^4}{2\hbar} \sum_j m_j \omega_j^{\alpha 3} \cos[\omega_j^\alpha(t-t')] \coth\left(\frac{\hbar\omega_j^\alpha}{2kT}\right).\end{aligned}\quad (\text{A16})$$

APPENDIX B: QUANTUM BROWNIAN MOTION IN A WEAK MAGNETIC FIELD

Of all the comprehensive work done on the canonical quantization of Brownian motion, there is one paper, Ref. [16], that is especially relevant. In this reference, the conventional quantum Brownian motion of an electron moving in a harmonic potential well $[V(\mathbf{r}) = K\mathbf{r}^2/2]$ and a magnetic field \mathbf{B} is investigated. Their starting point is the operator Langevin equation

$$M\ddot{\mathbf{r}} + \int_{-\infty}^t ds \mu(t-s) \dot{\mathbf{r}}(s) - \frac{e}{c} \dot{\mathbf{r}} \times \mathbf{B} + K\mathbf{r} = \bar{\mathbf{F}}(t), \quad (\text{B1})$$

where M is the mass of the Brownian particle, $\bar{\mathbf{F}}(t)$ is the force due to the bath, and $\mu(t-s)$ is the memory kernel. From Eq. (B1), they find an expression for the expectation value of the inequal time-symmetrized position correlator

$$\frac{1}{2} \langle \mathbf{r}^\alpha(t) \mathbf{r}^\beta(t') + \mathbf{r}^\beta(t') \mathbf{r}^\alpha(t) \rangle \quad (\alpha, \beta = x, y, z). \quad (\text{B2})$$

We will not repeat this expression here, but will consider the expression for $\mu(t) = 2M\gamma\delta(t)$, $\alpha = \beta = x$ and $\mathbf{B} = (0, 0, B)$ because this is precisely the case that we will consider in Sec. IV. Let us write $\mathbf{r}^x = X$. In this case, the result from [16] reads as

$$\begin{aligned}\frac{1}{2} \langle X(t) X(t') + X(t') X(t) \rangle \\ = \frac{\hbar}{\pi} \int_0^\infty d\omega \text{Im}[\alpha(\omega + i0^+)] \coth\left(\frac{\hbar\omega}{2kT}\right) \cos[\omega(t-t')],\end{aligned}\quad (\text{B3})$$

where $\alpha(\omega)$ is given by

$$\alpha(\omega) = \frac{1}{2} \left[\frac{1}{K - M\omega^2 - \omega \frac{eB}{c} - iM\gamma\omega} + \frac{1}{K - M\omega^2 + \omega \frac{eB}{c} - iM\gamma\omega} \right]. \quad (\text{B4})$$

For $K = 0$, Eq. (B3), which is appropriate for the conventional situation $[X, Y] = 0$, is comparable to our Eq. (110) in the noncommutative plane. We refer the reader to Ref. [16] for a detailed analysis of Eq. (B3).

APPENDIX C: MARKOVIAN LANGEVIN EQUATION

In this appendix, we show that the distributions for the masses and the frequencies of the bath oscillators given by

$$\omega_j^x = \omega_j^y = j, \quad m_j = \frac{2}{j} \frac{\gamma \hbar}{\pi l_B^2} \quad (\text{C1})$$

lead to instantaneous memory kernels. In particular, this choice gives

$$\int_0^t \mu_x(t-s) \dot{X}(s) ds = \gamma \dot{X}(t). \quad (\text{C2})$$

In the following, we will omit the labels x and y since the argument is valid for both. Substitution of Eq. (C1) into Eq. (86) yields

$$\mu(t) = \frac{2\gamma}{\pi} \sum_j \cos(jt). \quad (\text{C3})$$

Now, we take the continuum limit of this expression and perform the integral

$$\mu(t) \longrightarrow \frac{2\gamma}{\pi} \int_0^\infty \cos(jt) dj = 2\gamma \delta(t). \quad (\text{C4})$$

If we now define the Heaviside θ function by

$$\theta(t) := \int_{-1}^t \delta(s) ds, \quad (\text{C5})$$

we obtain

$$\int_0^t \mu(t-s) \dot{X}(s) ds = 2\gamma \theta(0) \dot{X}(t).$$

The value $\theta(0)$ depends on the limiting procedure used in defining the Dirac delta distribution. In this case, we may write

$$\begin{aligned}\theta(0) &= \int_{-1}^0 \delta(s) ds = \frac{1}{\pi} \int_{-1}^0 ds \int_0^\infty dk \cos[ks] \\ &= \frac{1}{\pi} \int_0^\infty dk \int_{-1}^0 \cos[ks] = \frac{1}{\pi} \int_0^\infty dk \frac{\sin[k]}{k} \\ &= \frac{1}{2}.\end{aligned}\quad (\text{C6})$$

Hence,

$$\int_0^t \mu(t-s) \dot{X}(s) ds = \gamma \dot{X}(t). \quad (\text{C7})$$

- [1] X.-G. Wen, *Quantum Field Theory of Many-body Systems: From the Origin of Sound to an Origin of Light and Electrons* (Oxford University Press, Oxford, 2007).
- [2] D. I. Tsomokos, A. Hamma, W. Zhang, S. Haas, and R. Fazio, *Phys. Rev. A* **80**, 060302(R) (2009).
- [3] G. Kells, D. Sen, J. K. Slingerland, and S. Vishveshwara, *Phys. Rev. B* **89**, 235130 (2014).
- [4] A. Chandran, F. J. Burnell, V. Khemani, and S. L. Sondhi, *J. Phys.: Condens. Matter* **25**, 404214 (2013).
- [5] O. Viyuela, A. Rivas, and M. A. Martin-Delgado, *New J. Phys.* **14**, 033044 (2012).
- [6] S. Diehl, E. Rico, M. A. Baranov, and P. Zoller, *Nat. Phys.* **7**, 971 (2011).
- [7] Z. Nussinov and G. Ortiz, *Ann. Phys. (NY)* **324**, 977 (2009).
- [8] B. Andrei Bernevig (with T. Hughes), *Topological Insulators and Topological Superconductors* (Princeton University Press, Princeton, NJ, 2013).
- [9] C. Gardiner and P. Zoller, *Quantum Noise: A Handbook of Markovian and Non-Markovian Quantum Stochastic Methods with Applications to Quantum Optics*, 3rd ed. (Springer, Berlin, 2004).
- [10] It is, however, possible also to have dimensional reduction and associated topological quantum ordering by way of symmetries [11].
- [11] Z. Nussinov, G. Ortiz, and E. Cobanera, *Ann. Phys. (NY)* **327**, 2491 (2012).
- [12] G. V. Dunne, R. Jackiw, and C. A. Trugenberger, *Phys. Rev. D* **41**, 661 (1990).
- [13] A. O. Caldeira and A. J. Leggett, *Physica A (Amsterdam)* **121**, 587 (1983).
- [14] G. W. Ford, J. T. Lewis, and R. F. O'Connell, *Phys. Rev. A* **37**, 4419 (1988).
- [15] X. L. Li, G. W. Ford, and R. F. O'Connell, *Phys. Rev. A* **41**, 5287 (1990).
- [16] X. L. Li, G. W. Ford, and R. F. O'Connell, *Phys. Rev. A* **42**, 4519 (1990).
- [17] R. Czopnik and P. Garbaczewski, *Phys. Rev. E* **63**, 021105 (2001).
- [18] J. S. Blakemore, *J. Appl. Phys.* **53**, R123 (1982).
- [19] L. Faddeev and R. Jackiw, *Phys. Rev. Lett.* **60**, 1692 (1988).
- [20] A. O. Caldeira and A. J. Leggett, *Ann. Phys. (NY)* **149**, 374 (1983).
- [21] G. W. Ford, M. Kac, and P. Mazur, *J. Math. Phys.* **6**, 504 (1965).
- [22] C. M. Smith and A. O. Caldeira, *Phys. Rev. A* **36**, 3509(R) (1987).
- [23] C. M. Smith and A. O. Caldeira, *Phys. Rev. A* **41**, 3103 (1990).
- [24] R. P. Feynman and F. L. Vernon Jr., *Ann. Phys. (NY)* **281**, 547 (2000).
- [25] L. S. Schulman, *Techniques and Applications of Path Integration* (Dover, New York, 2005).