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Non reduced plane curve singularities with $b_1(F) = 0$ and Bobadilla's question

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If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to x^r.

1. Introduction

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function germ. What can be said about functions whose Milnor fibre F has the property $b_i(F) = 0$ for all $i \geq 1$? If F is connected then f is non-singular and equivalent to a linear function by A'Campo's trace formula. The remaining question: *What happens if F is non-connected?* is only relevant for non-reduced plane curve singularities.

This question is related to a recent paper [3]. That paper contains a statement about the so-called Bobadilla conjectures [2] in case of plane curves. The invariant $\beta = 0$, used by Massey [4] should imply that the singular set of f is a smooth line.

In this note we give a short topological proof of a stronger statement.

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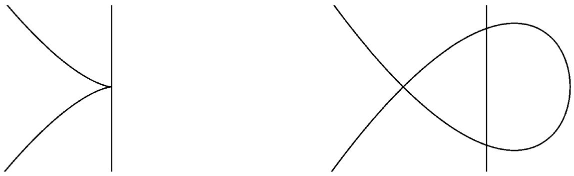

Fig. 1. Deformation to maximal number of double points.

Proposition 1.1. *If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to* x^r.

Corollary 1.2. *In the above case the singular set is a smooth line and the system of transversal singularities is trivial.*

2. Non-reduced plane curves

Non-isolated plane curve singularities have been thoroughly studied by Rob Schrauwen in his dissertation [5]. Main parts of it are published as [6] and [7]. The above Proposition 1.1 is an easy consequence of his work.

We can assume that $f = f_1^{m_1}.\cdots.f_r^{m_r}$ (partition in powers of reduced irreducible components).

Lemma 2.1. *Let* $d = \gcd(m_1, \cdots, m_r)$

(a.) *F has d components, each diffeomorphic to the Milnor fibre G of* $g = g_1^{\frac{m_1}{d}}.\cdots.g_r^{\frac{m_r}{d}}$. *The Milnor monodromy of f permutes these components,*

(b.) *if* $d = 1$ *then F is connected.*

Proof. (a.) Since $f = g^d$ the fibre F consists of d copies of G.

(b.) We recall here the reasoning from [5]. Deform the reduced factors f_i into $\hat{f}_i$ such that the product $\hat{f}_1.\cdots.\hat{f}_r = 0$ contains the maximal number of double points (cf. Fig. 1). This is called a network deformation by Schrauwen. The corresponding deformation $\hat{f}$ of f near such a point has local equation are of the form $x^p y^q = 0$ (point of type $D[p,q]$).

Near every branch $\hat{f}_i = 0$ the Milnor fibre is a m_i-sheeted covering of the zero-locus, except in the $D[p,q]$-points. We construct the Milnor fibre F of f starting with $S = \sum m_i$ copies of the affine line $\mathbb{A}$. Cover the ith branch with m_i copies of $\mathbb{A}$ and delete $(p+q)$ small discs around the $D[p,q]$-points. Glue in the holes $\gcd(p,q)$ small annuli (the Milnor fibres of $D[p,q]$). The resulting space is the Milnor fibre F of f.

A hyperplane section of a generic at a generic point of $\hat{f}_i = 0$ defines a transversal Milnor fibre $F_1^{\pitchfork}$. Start now the construction of F from $F_1^{\pitchfork}$, which consists of m_1 cyclic ordered points. As soon as $f_1 = 0$ intersects $f_k = 0$ it connects the sheets of $f_1 = 0$ modulo m_k. Since $\gcd(m_1, \cdots, m_r) = 1$ we connect all sheets. □

Proof of Proposition 1.1. If $b_1(F) = 0$, then also $b_1(G) = 0$. The Milnor monodromy has $\text{trace}(T_g) = 1$. According to A'Campo's observation [1] g is regular: $g = x$. It follows that $f = x^r$. □

3. Relation to Bobadilla's question

We consider first in any dimension $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with a 1-dimensional singular set, see especially the 1991-paper [8] for definitions, notations and statements.

We focus on the group $H_n(F, F^\pitchfork)$ which occurs in two exact sequences on p. 468 of [8]:

$$0 \to H^{\mathrm{f}}_{n-1}(F) \to H_{n-1}(F^\pitchfork) \to H_n(F) \oplus H^{\mathrm{t}}_{n-1}(F) \to 0$$
$$0 \to H_n(F) \to H_n(F, F^\pitchfork) \to H_{n-1}(F^\pitchfork) \to H_{n-1}(F) \to 0$$

Here $F^\pitchfork$ is the disjoint union of the transversal Milnor fibres $F_i^\pitchfork$, one for each irreducible branch of the 1-dimensional singular set.[1]

Note that $H_n(F)$, $H_n(F, F^\pitchfork)$ and $H_{n-1}(F^\pitchfork)$ are free groups. $H_{n-1}(F)$ can have torsion, we denote its free part by $H_{n-1}(F)^{\mathrm{f}}$ and its torsion part by $H_{n-1}(F)^{\mathrm{t}}$. All homologies here are taken over $\mathbb{Z}$, but also other coefficients are allowed.

From both sequences it follows that the β-invariant, introduced in [4] has a 25 years history, since is nothing else than:

$$\dim H_n(F, F^\pitchfork) = b_n - b_{n-1} + \sum \mu_i^\pitchfork := \beta$$

From this definition is immediately clear that $\beta \geq 0$ and that β is topological. The topological definition has as direct consequence:

Proposition 3.1. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with a 1-dimensional singular set, then:*

$$\beta = 0 \quad \Leftrightarrow \quad \chi(F) = 1 + (-1)^n \sum \mu_i^\pitchfork \quad \Leftrightarrow \quad H_n(F, \mathbb{Z}) = 0 \text{ and } H_{n-1}(F, \mathbb{Z}) = \mathbb{Z}^{\sum \mu_i}$$

The original Bobadilla conjecture C [2] was in [4] generalized to the reducible case as follows: *Does $\beta = 0$ imply that the singular set is smooth?* As consequence of our main Proposition 1.1 we have:

Corollary 3.2. *In the curve case $\beta = 0$ implies that the singular set is smooth; and that the function is equivalent to x^r.*

Remark 3.3. In [3] the first part of this corollary was obtained with the help of Lê numbers.

Remark 3.4. From the definition $\beta = H_n(F, F^\pitchfork)$ follow direct and short proofs of several statements from [4].

An other consequence from [8] is the composition of surjections:

$$H_{n-1}(F^\pitchfork) = \oplus \mathbb{Z}^{\mu_i} \twoheadrightarrow H_{n-1}(\partial_2 F) = \oplus \frac{\mathbb{Z}^{\mu_i}}{A_i - I} \twoheadrightarrow H_{n-1}(F)$$

From this follows:

Proposition 3.5. *If* $\dim H_{n-1}(F) = \sum \mu_i$ *(upper bound) then*

a. $H_{n-1}(\partial_2 F)$ and $H_{n-1}(F)$ are free and isomorphic to $\mathbb{Z}^{\sum \mu_i}$.
b. All transversal monodromies A_i are the identity.

The second part of [4] contains an elegant statement about $\beta = 1$ via the A'Campo trace formula. Also the reduction of the generalized Bobadilla conjecture to the (irreducible) Bobadilla conjecture. As final remark: The great work (the irreducible case) has still has to be done! Together with the Lê-conjecture this seems to be an important question in the theory of hypersurfaces 1-dimensional singular sets.

[1] $F^\pitchfork$ was originally denoted by F'. In the second sequence a misprint n in the third term has been changed to $n - 1$.

References

[1] N. A'Campo, Le nombre de Lefschetz d'une monodromie, Indag. Math. 35 (1973) 113–118.
[2] J. Fernandez de Bobadilla, Topological equisingularity of hypersurfaces with 1-dimensional critical set, Adv. Math. 248 (2013) 1199–1253.
[3] B. Hepler, D. Massey, Some special cases of Bobadilla's conjecture, Topol. Appl. 217 (2017) 59–69.
[4] D.B. Massey, A new conjecture, a new invariant, and a new non-splitting result, in: J.L. Cisneros-Molina, et al. (Eds.), Singularities in Geometry, Topology, Foliations and Dynamics: A Celebration of the 60th Birthday of José Seade, Merida, Mexico, December 2014, Springer International Publishing, 2017, pp. 171–181.
[5] R. Schrauwen, Series of Singularities and Their Topology, Dissertation, Universiteit Utrecht, 1991.
[6] R. Schrauwen, Deformations and the Milnor number of nonisolated plane curve singularities, in: Singularity Theory and Its Applications, Part I, Coventry, 1988/1989, in: Lecture Notes in Math., vol. 1462, Springer, Berlin, 1991, pp. 276–291.
[7] R. Schrauwen, Topological series of isolated plane curve singularities, Enseign. Math. (2) 36 (1–2) (1990) 115–141.
[8] D. Siersma, Variation mappings on singularities with a 1-dimensional critical locus, Topology 30 (3) (1991) 445–469.