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Uniform interpolation and the existence of sequent calculi

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ABSTRACT

This paper presents a uniform and modular method to prove uniform interpolation for several intermediate and intuitionistic modal logics. The proof-theoretic method uses sequent calculi that are extensions of the terminating sequent calculus G4ip for intuitionistic propositional logic. It is shown that whenever the rules in a calculus satisfy certain structural properties, the corresponding logic has uniform interpolation. It follows that the intuitionistic versions of K and KD (without the diamond operator) have uniform interpolation. It also follows that no intermediate or intuitionistic modal logic without uniform interpolation has a sequent calculus satisfying those structural properties, thereby establishing that except for the seven intermediate logics that have uniform interpolation, no intermediate logic has such a sequent calculus.

1. Introduction

In [22], Andrew Pitts established that intuitionistic propositional logic IPC has uniform interpolation. His proof was the first syntactic or proof–theoretic proof of a result of that kind. This paper shows that several intermediate and intuitionistic modal logics have uniform interpolation by providing a direct connection, for a given logic, between the property of having uniform interpolation and the existence of sequent calculi for the logic. The method developed to prove these results is uniform, and, perhaps more importantly, provides a way to prove negative results concerning proof systems: logics without uniform interpolation cannot have sequent calculi of a certain form. The methods used in this paper are proof–theoretic, uniform and modular, and are inspired by Pitts' proof–theoretic proof from 1992.

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Uniform interpolation is a strengthening of interpolation, and a logic L is said to satisfy or have the property if the propositional quantifiers $\exists p$ and $\forall p$ are definable in the logic, where $\exists p\varphi$ and $\forall p\varphi$ are defined by requiring that they do not contain p and are such that for all ψ not containing p:

$$\vdash_{\mathsf{L}} \varphi \to \psi \;\Leftrightarrow\; \vdash \exists p\varphi \to \psi \qquad \vdash_{\mathsf{L}} \psi \to \varphi \;\Leftrightarrow\; \vdash \psi \to \forall p\varphi.$$

This implies that $\vdash \varphi \to \exists p\varphi$ and $\vdash \forall p\varphi \to \varphi$. Therefore, $\exists p_1 \dots \exists p_n\varphi$ is an interpolant for any derivable implication $\varphi \to \psi$ for which ψ does not contain any p_i and all other atoms in φ occur in ψ. This shows that uniform interpolation implies interpolation. It also shows that $\exists p_1 \dots \exists p_n\varphi$ is an interpolant that does not depend on the logical structure of the consequent of an implication, just on the variables it contains. Likewise, $\forall p_1 \dots \forall p_n\varphi$ is an interpolant that does not depend on the structure of the antecedent of an implication. In the literature, $\exists p$ is also called the *post* or *right* interpolant and $\forall p$ the *pre* or *left* interpolant.

Uniform interpolation as a property is stronger than interpolation, as there are modal logics, for example $\mathsf{S4}$ and $\mathsf{K4}$, that do not satisfy the stronger property, but do have interpolation [11,6].

In [17], a method has been developed to prove uniform interpolation for any modal logic with a sequent calculus consisting of so–called focused and focused modal rules. This method provides a single framework via which to prove in a uniform way existing and new results on uniform interpolation, such as the result from [6] that K has uniform interpolation and the new result that KD has uniform interpolation. But the most important use of the method lies in its contraposition: it implies that no logic without uniform interpolation has a sequent calculus consisting of focused and focused modal rules. Since there are many modal logics without uniform interpolation, it follows that none of these logics can have a calculus of this kind.

In this paper we extend the method of [17] to intermediate and intuitionistic modal logics, where the latter are modal logics that contain IPC. This is not a straightforward extension, since in contrast to CPC, already for IPC itself the proof of uniform interpolation is highly nontrivial. The intricate proof in [22] makes use of a terminating calculus for IPC developed independently by Dyckhoff [10] and Hudelmaier [14,15,16] and, much earlier in a somewhat different form, by Vorob'ev [28,29]. In [18] we have extended that calculus to terminating calculi for intuitionistic modal logics, and these are the calculi we use in this paper. Our method is uniform in the sense that it does not establish uniform interpolation based on a specific calculus, but based on certain structural properties of the calculus. In this way one can prove uniform interpolation for several logics at once, namely for all those that have calculi that satisfy these requirements.

We show how via our method Pitts' result can be obtained, that the method can be extended to other intermediate and intuitionistic modal logics, and show that the diamond–free fragments of what in the literature are called iK and iKD, have uniform interpolation.[1]

1.1. Main aim

Rather than proving uniform interpolation for intermediate and intuitionistic modal logics, the main aim of this paper is in fact the opposite: to prove that intermediate and intuitionistic modal logics without uniform interpolation do not have certain sequent calculi. The idea is simple. We provide sufficient conditions such that whenever a calculus satisfies these conditions its logic has uniform interpolation. So that for a logic not having uniform interpolation it can be concluded it does not have a calculus satisfying those constraints.

Thus this enterprise can be viewed as a possible approach to establish what, if any, sequent calculi nonclassical logics can have. The calculi we are interested in here are calculi with good properties, meaning without a cut rule and satisfying some form of the subformula property. From the definition of focused and focused modal rules below it will be clear that the calculi we consider have such properties.

[1] Most intuitionistic modal logics occur under various different names in the literature. References and alternative names will be given in Section 8.

Although in this paper we focus on sequent calculi, we conjecture that our method can be adapted to certain other proof systems as well. The general idea being that a proof system with certain *structural properties* (such as the subformula property or closure under weakening) implies that the corresponding logic has certain *regular properties* (such as uniform interpolation). The more general the requirements on the proof system, the stronger the result. This works in both ways: If many logics have a proof system with certain structural properties, then the method establishes that many logics satisfy the corresponding regular properties, and if many logics do not satisfy certain regular properties, as in the case of uniform interpolation in intermediate logics, then the method shows that none of these many logics can have a proof system with the corresponding structural properties.

In this paper the regular property is uniform interpolation, and the proof systems are extensions of G4ip and $\mathsf{G4iK}_\Box$ by focused and focused modal rules, notions that are defined below. Since there are only seven intermediate logics with uniform interpolation, our method in particular shows (Corollary 19) that except for these seven logics, no intermediate logic has a sequent calculus of that particular form.

1.2. Related work

In the literature there is quite some work on uniform interpolation for classical modal logics and intermediate logics, but for intuitionistic modal logics far less is known. In this section we discuss results from these areas that are relevant to our results.

For several modal logics, uniform interpolation has been established in various ways. The results on K and GL by Shavrukov [24] and Visser [26,27], obtained around the same time as Pitts' result, use semantical techniques. This is in contrast to Pitts' result for IPC, which is syntactic in nature. A similar syntactic method was shown to apply to K, T, GL, and S4Grz in [5,6] and to substructural logics in [1].

An algebraic or categorical approach can be found in the work of Ghilardi and Zawadowski [11,12] and in [13]. The former proved that S4, which has interpolation, does not have uniform interpolation, a fact used by Bílková [5] to show that neither has K4. In [21] it has been shown that there are only seven propositional intermediate logics with interpolation, and in [12] it was shown that there are exactly that many logics with uniform interpolation. In the algebraic setting, the quantifiers $\forall p$ and $\exists p$ can be seen to be adjoints of a certain embedding operation.

Propositional quantification in modal and intuitionistic logic has been studied in various contexts. Since there are several possible ways to define quantification, one has to be careful in comparing the different approaches. In [23] it is shown that the usual uniform interpolants do not coincide with topological quantification. The paper [19], in which it is proved that a certain version of propositional quantified intuitionistic logic is recursively isomorphic to full second order classical logic, is a good source for references to the literature on the topic.

Several intuitionistic modal logics have been introduced in the literature. Often, they consist of the modal axioms of well–known classical modal logics, but with intuitionistic logic as the underlying propositional logic [3,4,7,9,25,30]. Litak [20] provides a nice overview of the work of the Georgian School on intuitionistic modal logic, in particular on fixed point theorems for such logics.

2. Preliminaries

2.1. Language and sequents

The logics we consider are (modal) propositional logics, formulated in a language $\mathcal{L}$ that contains constants $\top$ and $\bot$, propositional variables or atoms $p, q, r, \dots$ and the connectives $\wedge, \vee, \neg, \to$, and the modal operator $\Box$ in case of modal logics. $\mathcal{F}$ denotes the set of formulas in $\mathcal{L}$ and $\mathcal{M}$ is the set of all finite multisets of formulas in $\mathcal{F}$. Given a set of atoms $\mathcal{P}$, $\mathcal{F}(\mathcal{P})$ denotes all formulas in $\mathcal{L}$ in which all atoms belong to $\mathcal{P}$.

The language $\mathcal{L}_{\mathsf{qf}}$ is defined to be the extension of $\mathcal{L}$ with propositional quantifiers $\forall p$ and $\exists p$ for every atom p, and $\mathcal{F}_{\mathsf{qf}}$ is the set of formulas in that language.

Sequents are expressions of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas in $\mathcal{F}_{\mathsf{qf}}$, which are interpreted as $I(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma \to \bigvee \Delta)$. We say that a sequent *is in* $\mathcal{L}$ when all its formulas belong to $\mathcal{L}$. In this paper we only consider single-conclusion sequents, meaning that the succedent Δ contains at most one formula. We denote finite multisets by $\Gamma, \Pi, \Delta, \Sigma$. We denote by $\Gamma \cup \Pi$ the multiset that contains only formulas φ that belong to Γ or Π and the number of occurrences of φ in $\Gamma \cup \Pi$ is the sum of the occurrences of φ in Γ and in Π. In a sequent, notation Π, Γ is short for $\Gamma \cup \Pi$. We also define (a for antecedent, s for succedent):

$$(\Gamma \Rightarrow \Delta)^a \equiv_{df} \Gamma \qquad (\Gamma \Rightarrow \Delta)^s \equiv_{df} \Delta.$$

Expression $S_0 \subseteq S_1$ denotes that $S_0^a \subseteq S_1^a$ and $S_0^s \subseteq S_1^s$, and $S_0 \subset S_1$ denotes: $S_0^a \subset S_1^a$ and $S_0^s \subseteq S_1^s$, or $S_0^s \subset S_1^s$ and $S_0^a \subseteq S_1^a$. When sequents are used in the setting of formulas, we often write S for $I(S)$, such as in $\vdash \bigvee_i S_i$, which thus means $\vdash \bigvee_i I(S_i)$. Multiplication of sequents is defined as

$$S_1 \cdot S_2 \equiv_{df} (S_1^a \cup S_2^a \Rightarrow S_1^s \cup S_2^s).$$

For a multiset Γ, let $\Gamma_\Box$, $\Box\Gamma$ and $\boxdot\Gamma$ denote the multisets $\{\varphi \mid \Box\varphi \in \Gamma\}$, $\{\Box\varphi \mid \varphi \in \Gamma\}$ and $\Gamma \cup \Box\Gamma$, respectively. $\boxdot\varphi$ is short for $\varphi \wedge \Box\varphi$, but if the expression occurs as an element of a sequent it stands for $\varphi, \Box\varphi$. For example, $(\Gamma, \boxdot\varphi \Rightarrow \Delta)$ should be read as $(\Gamma, \Box\varphi, \varphi \Rightarrow \Delta)$. For a sequent S, we define

$$\Box S \equiv_{df} (\{\Box\varphi \mid \varphi \in S^a\} \Rightarrow \{\Box\psi \mid \psi \in S^s\})$$
$$\boxdot S \equiv_{df} (\{\boxdot\varphi \mid \varphi \in S^a\} \Rightarrow \{\boxdot\psi \mid \psi \in S^s\}).$$

This implies that $\Box(\Gamma \Rightarrow) = (\Box\Gamma \Rightarrow)$ and $\Box(\Rightarrow \Delta) = (\Rightarrow \Box\Delta)$, and similarly for $\boxdot$.

The set $\mathcal{F}_{\mathsf{ex}}$ is the smallest set of expressions that contains all formulas in the language $\mathcal{L}$, is closed under the connectives (and modal operator, if present), and if S is a sequent in $\mathcal{L}$ and p an atom, then $\forall pS$ and $\exists pS$ belong to $\mathcal{F}_{\mathsf{ex}}$. For example, when S is a sequent in $\mathcal{L}$ and φ a propositional formula, then $(\varphi \to \exists pS)$ belongs to $\mathcal{F}_{\mathsf{ex}}$, as does $\Box(\varphi \wedge \forall pS)$, but $\exists p \exists qS$ does not. The interpretation of $\mathcal{F}_{\mathsf{ex}}$ into $\mathcal{F}_{\mathsf{qf}}$ is the identity on formulas in $\mathcal{F}$, commutes with the connectives and the modal operator and interprets quantified sequents as

$$\forall pS \equiv_{df} \forall p I(S) \qquad \exists pS \equiv_{df} \exists p (\bigwedge S^a).$$

We say that a sequent *is in* $\mathcal{L}_{\mathsf{ex}}$ when all its formulas belong to $\mathcal{F}_{\mathsf{ex}}$.

2.2. Rules and instances

For a proper treatment of our proof systems we need to make a distinction between the object–language and the meta–language, where the latter is the language in which the sequent calculi will be defined. $\overline{\mathcal{L}}$ consists of infinitely many formula symbols $\overline{\varphi}, \overline{\psi}, \overline{\chi}, \overline{\varphi}_1, \overline{\varphi}_2, \dots$, constants $\top$ and $\bot$, the connectives $\wedge, \vee, \neg, \to$, and the modal operator $\Box$ in the case of modal logics. The set $\overline{\mathcal{F}}$ of meta–formulas in this language is defined as usual: the constants and all formula symbols are meta–formulas, and if φ and ψ are meta–formulas, then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$ and $\neg\varphi$. $\overline{\mathcal{M}}$ is an infinite set of symbols for *meta–multisets*, the elements we denote by $\overline{\Gamma}, \overline{\Pi}, \overline{\Delta}, \overline{\Sigma}$. A *meta–sequent* $\overline{S}$ is an expression $\overline{S} = (X \Rightarrow Y)$, where X and Y are finite multisets consisting of elements in $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$.

A *substitution* σ is a map from $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$ to $\mathcal{F} \cup \mathcal{M}$ that maps constants to themselves, meta–formula to formulas, that commutes with the connectives and modal operator, and that maps meta–multisets to multisets of formulas. Thus $\sigma[\overline{\mathcal{F}}] \subseteq \mathcal{F}$ and $\sigma[\overline{\mathcal{M}}] \subseteq \mathcal{M}$. *Sub* is the set of all substitutions. Given finite multisets X and Y of elements in $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$, we write σX for $\{\sigma A \mid A \in X\}$, and $\sigma(X \Rightarrow Y)$ for $(\sigma X \Rightarrow \sigma Y)$. Since in this paper only single-conclusion sequents are considered, for a substitution σ that is applied to $X \Rightarrow Y$, it is tacitly assumed that in case Y consists of a meta–multiset symbol $\overline{\Delta}$, σ maps $\overline{\Delta}$ to a multiset that contains at most one formula.

2.3. Sequent calculi and rules

A *sequent calculus* is a set of *rules*, which are expressions of the form

$$\frac{\overline{S}_1 \quad \overline{S}_2 \quad \dots \quad \overline{S}_n}{\overline{S}_0}\ \mathcal{R} \tag{1}$$

for some meta-sequents $\overline{S}_0, \overline{S}_1, \dots, \overline{S}_n$. It is a *right rule* if $\overline{S}_0^s$ contains a meta–formula and a *left rule* if $\overline{S}_0^a$ does. Thus if $\overline{S}_0 = (\overline{\Gamma} \Rightarrow \overline{\Delta})$ for meta–multisets $\overline{\Gamma}$ and $\overline{\Delta}$, then the rule is neither left nor right. But if we assume, and we will do so in this paper, that no rule in a calculus has a conclusion that consists of meta–multisets only, then this possibility disappears and all rules are left or right (or both). A rule is called an *axiom* in case there are no premisses. Thus axioms are considered to be special cases of rules.

For any substitution σ, the inference

$$\frac{\sigma\overline{S}_1 \quad \sigma\overline{S}_2 \quad \dots \quad \sigma\overline{S}_n}{\sigma\overline{S}_0}\ \sigma\mathcal{R}$$

is an *instance* of $\mathcal{R}$. Throughout this paper we denote rules by $\mathcal{R}$ and instances of rules by R. Given a rule $\mathcal{R}$, $\mathcal{R}_{\mathsf{ins}}$ denotes the set of instances of $\mathcal{R}$.

An example of a rule could be

$$\frac{\overline{\Gamma} \Rightarrow \neg\neg\overline{\varphi}}{\overline{\Gamma} \Rightarrow \overline{\varphi}}$$

Two possible instances of the rule are

$$\frac{q, q, r \to p \Rightarrow \neg\neg p}{q, q, r \to p \Rightarrow p} \qquad \frac{\Rightarrow \neg\neg(r_1 \wedge r_2)}{\Rightarrow r_1 \wedge r_2}$$

with respective substitutions σ_1 and σ_2, where

$$\sigma_1(\overline{\varphi}) = p \quad \sigma_1(\overline{\Gamma}) = \{q, q, r \to p\} \qquad \sigma_2(\overline{\varphi}) = r_1 \wedge r_2 \quad \sigma_2(\overline{\Gamma}) = \varnothing.$$

When a rule comes with a side condition, such as the axiom

$$\overline{\Gamma}, \overline{\varphi} \Rightarrow \overline{\varphi} \quad (\overline{\varphi} \text{ is an atom}),$$

the side condition has to be interpreted as a restriction on the substitutions that correspond to the instances of the rule. In the example, this would mean restricting the instances of the axiom to those substitutions that map $\overline{\varphi}$ to an atom.

A sequent S is *derivable* in a sequent calculus G from a set of sequents $\mathcal{S}$, written $\mathcal{S} \vdash_{\mathsf{G}} S$, if there is a finite tree labeled with sequents such that the root has label S and every node either belongs to $\mathcal{S}$ and is a leaf, or is the conclusion of an instance of a rule in G and the premisses of that instance are exactly the labels of the immediate successors of the node. Such a tree is a *derivation* of S from $\mathcal{S}$ in G. Sequent S is *derivable* in G if $\varnothing \vdash_{\mathsf{G}} S$, which we write as $\vdash_{\mathsf{G}} S$. A sequent is *free* if it is not the conclusion of any instance of any rule.

A *ca (cut admissible) calculus* is a calculus in which the cut rule (Fig. 2) is admissible.

2.3.1. Principal formulas and sequents

In the definitions and proofs below we often use a case distinction based on a sequent being principal or nonprincipal for an instance of a rule, a notion that is defined as follows. Every instance of any rule in this paper comes with the notion of *principal formulas*, which are one or more formula occurrences singled out in the conclusion of the instance, and which are defined per rule.

A sequent S is *principal* for an instance R of a rule if the conclusion of R is of the form $S' \cdot S$ for some sequent S' and all principal formulas of R belong to S. For example, suppose R has conclusion $(\Gamma, \varphi \Rightarrow \Delta)$ and φ is the principal formula of R, then any sequent of the form $(\Gamma', \varphi \Rightarrow \Delta')$, where $(\Gamma' \Rightarrow \Delta') \subseteq (\Gamma \Rightarrow \Delta)$, is principal for R. A sequent S is *nonprincipal* for R if the conclusion of R is of the form $S' \cdot S$ for some sequent S' and not all principal formulas of R occur in S.

Note that for any given instance of a rule and any sequent S, S being *non*principal for R implies S being *not* principal for R, but not vice versa. For example, the sequent $S = (\varphi \wedge \psi \Rightarrow\ )$ is not principal for any instance $R = \big(\varphi', \psi' \Rightarrow\ \ /\varphi' \wedge \psi' \Rightarrow\ \big)$ of the standard left rule for conjunction in case $\varphi \neq \varphi'$ or $\psi \neq \psi'$, simply because the conclusion of R is not of the form $S \cdot S'$ for any S'. But it is not nonprincipal for such instances, for the same reason. Thus nonprincipality is not the negation of principality. If, however, the conclusion of an instance of a rule is of the form $S \cdot S'$, then S is either principal or nonprincipal for that instance, which provides the case distinction mentioned at the beginning of this section.

2.3.2. Convention

As is often done implicitly in papers on proof systems, to keep the notation light, from now on the terminology for the object–language is also used for the meta–language: over scores and the word “meta” are omitted, trusting that it will always be clear from the context (or does not matter) which language we are concerned with. For example, an axiom such as $(\overline{\Gamma}, \overline{\varphi} \Rightarrow \overline{\varphi})$ will simply be written as $(\Gamma, \varphi \Rightarrow \varphi)$.

2.4. Logics

Logics are given by consequence relations closed under substitution, where $\vdash_{\mathsf{L}}$ denotes the consequence relation for logic L. Thus $\vdash_{\mathsf{L}}$ is a relation between sets of formulas and formulas, where $\Gamma \vdash_{\mathsf{L}} \varphi$ means that formula φ follows in L from the set of formulas Γ. If $\vdash_{\mathsf{L}} \varphi$, then φ is a *theorem* of the logic.

An *intermediate logic* is a logic in the language of propositional logic such that its set of theorems contains the theorems of IPC and is contained in the set of theorems of classical propositional logic CPC. An *intuitionistic modal logic* is a logic in the language of modal logic (the language of propositional logic plus the operator $\Box$) such that its set of theorems contains the theorems of IPC. Every logic in this paper is either an intermediate logic or an intuitionistic modal logic.

A logic L is said to *have* a calculus G if for any formula φ: $\vdash_{\mathsf{L}} \varphi$ if and only if $\vdash_{\mathsf{G}} (\ \Rightarrow \varphi)$. When a logic L has a sequent calculus with respect to which it is sound and complete, then we assume that the consequence relation is such that for every instance $S_1 \dots S_n/S_0$ of a rule in the calculus, $I(S_1), \dots, I(S_n) \vdash_{\mathsf{L}} I(S_0)$ holds. This requirement implies that in the case of logics with a sequent calculus that contains a rule that expresses necessitation, like $(\Rightarrow \varphi)/(\Rightarrow \Box\varphi)$, the inference $\varphi \vdash_{\mathsf{L}} \Box\varphi$ holds for all φ, a fact that we will often use.

We show that for any calculus at least one consequence relation with that property exists. Given a calculus G, where $\mathsf{G}_{\mathsf{ins}}$ is the set of instances of its rules, define the *logic corresponding to* G, denoted L_{G}, as the consequence relation given by

$$\Gamma \vdash_{\mathsf{L_G}} \varphi \equiv_{df} \{(\Rightarrow \psi) \mid \psi \in \Gamma\} \vdash_{\mathsf{G+Cut}} (\Rightarrow \varphi).$$

Lemma 1. *If* G *extends* G4ip *(Fig. 1), then* $\vdash_{\mathsf{L_G}}$ *is a consequence relation that satisfies the following:*

$$\text{for all } S_1, \dots, S_n/S_0 \in \mathsf{G_{ins}}: \; I(S_1), \dots, I(S_n) \vdash_{\mathsf{L_G}} I(S_0). \tag{2}$$

If G *is ca (Section 2.3), which means that the cut rule (Fig. 2) is admissible in* G*, then*

$$\vdash_{\mathsf{L_G}} \varphi \text{ if and only if } \vdash_{\mathsf{G}} (\Rightarrow \varphi).$$

Proof. The last part of the lemma is immediate from the definition $\vdash_{\mathsf{L_G}}$. We prove the other parts. For any set of formulas Γ, let Γ_{sq} denote the set of sequents $\{(\Rightarrow \psi) \mid \psi \in \Gamma\}$.

First, we prove that $\vdash_{\mathsf{L_G}}$ is a consequence relation. That it is reflexive, meaning $\Gamma, \varphi \vdash_{\mathsf{L_G}} \varphi$ holds, is clear from the definition of $\vdash_{\mathsf{G+Cut}}$. That it satisfies weakening, meaning that $\Gamma \vdash_{\mathsf{L_G}} \varphi$ implies $\Gamma, \Pi \vdash_{\mathsf{L_G}} \varphi$, is also straightforward. It remains to be proven that it is transitive, that is, if $\Gamma \vdash_{\mathsf{L_G}} \psi$ and $\Pi, \psi \vdash_{\mathsf{L_G}} \varphi$, then $\Gamma, \Pi \vdash_{\mathsf{L_G}} \varphi$. Therefore assume the former, which means that $\Gamma_{sq} \vdash_{\mathsf{G+Cut}} (\Rightarrow \psi)$ and $\Pi_{sq}, (\Rightarrow \psi) \vdash_{\mathsf{G+Cut}} (\Rightarrow \varphi)$. Let T_ψ and T_φ be derivations in $\mathsf{G} + \mathsf{Cut}$ of $(\Rightarrow \psi)$ from Γ_{sq} and of $(\Rightarrow \varphi)$ from $\Pi_{sq}, (\Rightarrow \psi)$, respectively. To prove that $\Gamma_{sq}, \Pi_{sq} \vdash_{\mathsf{G+Cut}} (\Rightarrow \varphi)$, replace in T_φ the leafs labeled with $(\Rightarrow \psi)$ by T_ψ. The root of the new tree has label $(\Rightarrow \varphi)$ and every node either belongs to Γ_{sq}, Π_{sq} and is a leaf, or is the conclusion of an instance of a rule in G+Cut and the premisses of that instance are exactly the labels of the immediate successors of the node. In other words, $\Gamma, \Pi \vdash_{\mathsf{L_G}} \varphi$.

It remains to prove (2), for which we consider an instance $S_1, \dots, S_n/S_0 \in \mathsf{G_{ins}}$. We have to show that $\{(\Rightarrow I(S_1)), \dots, (\Rightarrow I(S_n))\} \vdash_{\mathsf{G+Cut}} (\Rightarrow I(S_0))$. Observe that $S_1, \dots, S_n \vdash_{\mathsf{G}} S_0$ and because of the presence of the rules in G4ip, also $S_0 \vdash_{\mathsf{G}} (\Rightarrow I(S_0))$. Thus it suffices to show that $(\Rightarrow I(S_i)) \vdash_{\mathsf{G+Cut}} S_i$ for any $i = 1, \dots, n$. In fact, we show that for any sequent S: $(\Rightarrow I(S)) \vdash_{\mathsf{G+Cut}} S$. The rules in G4ip imply that $\vdash_{\mathsf{G+Cut}} S^a, I(S) \Rightarrow S^s$. Therefore the presence of Cut implies the desired. □

If a logic L with consequence relation $\vdash_{\mathsf{L}}$ has a sequent calculus G in which Cut is admissible, then $\vdash_{\mathsf{L_G}}$ does not have to be equal to $\vdash_{\mathsf{L}}$, but $\vdash_{\mathsf{L}}$ and $\vdash_{\mathsf{L_G}}$ do have the same theorems.

By $\vdash^{\mathcal{R}}_{\mathsf{IPC}}$ we denote the smallest consequence relation containing $\mathcal{R}$ and such that $\varphi_1, \dots, \varphi_n \vdash^{\mathcal{R}}_{\mathsf{IPC}} \psi$ holds whenever $(\bigwedge \varphi_i \to \psi)$ holds in IPC.

2.5. Reductive calculi

An order $\prec$ on sequents is *reductive* if

- it is well-founded;
- all proper subsequents of a sequent come before that sequent;
- whenever all formulas in S occur boxed in S', then $S \prec S'$;
- for all multisets Γ, Δ, formulas φ and atoms q: $(\Gamma, \varphi \Rightarrow \Delta) \prec (\Gamma, q \to \varphi \Rightarrow \Delta)$.

A calculus is *terminating* with respect to an order $\prec$ on sequents if

- it is finite;
- for all sequents S and all rules in the calculus there are at most finitely many instances of the rule with conclusion S;
- in every instance of a rule in the calculus the premisses come before the conclusion in the order $\prec$.

A calculus is *reductive* if it is terminating with respect to an order that is reductive.

A typical example of a rule that in general cannot belong to a reductive calculus is the cut rule, as in most common orders on sequents the premisses of that rule do not come before its conclusion. We will see that many standard cut-free calculi for modal logics are reductive.

Example 1. In all concrete examples in this paper we use the following reductive order on formulas in $\mathcal{F}$ based on a *weight* function which is a combination of the weight functions from Bílková [6] and Dyckhoff [10]: $\varphi \prec \psi \equiv_{df} w(\varphi) < w(\psi)$, where

$$\begin{array}{ll} w(p) = w(\bot) = 1 & \\ w(\varphi \circ \psi) = w(\varphi) + w(\psi) + 1 & \circ \in \{\vee, \to\} \\ w(\varphi \wedge \psi) = w(\varphi) + w(\psi) + 2 & \\ w(\Box\varphi) = w(\varphi) + 1. & \end{array}$$

We extend the weight to multisets as in [8]: $\Delta \prec \Gamma$ iff Δ is the result of replacing one or more formulas in Γ by zero or more formulas of lower weight. Sequents inherit this ordering by defining:

$$S_0 \prec S_1 \equiv_{df} S_0^a \cup S_0^s \prec S_1^a \cup S_1^s.$$

In this paper, whenever a general result about reductive calculi is applied to a concrete calculus, the reductive order that is used is the one in this example. Although most theorems hold for any reductive order, it may be helpful to keep this concrete order in mind throughout the paper.

Returning to general reductive orders, a reductive order $\prec$ is extended to an order on formulas in $\mathcal{F}_{\mathsf{ex}}$ as follows. First, we associate the following set of formulas with a formula φ in $\mathcal{F}_{\mathsf{ex}}$: $\mathsf{qf}(\varphi)$ denotes the multiset consisting of all occurrences of subformulas of the form QpS in φ, where $Q \in \{\exists, \forall\}$. The order on multisets of the form $\mathsf{qf}(\varphi)$ again is in the style of [8]: $\mathsf{qf}(\varphi) \prec_{\mathsf{qf}} \mathsf{qf}(\psi)$ iff $\mathsf{qf}(\varphi)$ is the result of replacing one or more formulas of the form QpS in $\mathsf{qf}(\psi)$ by zero or more formulas of the form $Q'qS'$ with $S' \prec S$, where $Q, Q' \in \{\exists, \forall\}$. This order is well–defined since by definition such S and S' are sequents in $\mathcal{L}$ and therefore can be compared via $\prec$. The order on $\mathcal{F}_{\mathsf{ex}}$, that is also denoted by $\prec$, can now be defined: if $\varphi, \psi \in \mathcal{F}$, then $\varphi \prec \psi$ iff $(\Rightarrow \varphi) \prec (\Rightarrow \psi)$; if $\varphi \in \mathcal{F}$ and $\psi \notin \mathcal{F}$, then $\varphi \prec \psi$ and not $\psi \prec \varphi$; if $\varphi, \psi \notin \mathcal{F}$, then $\varphi \prec \psi$ if $\mathsf{qf}(\varphi) \prec_{\mathsf{qf}} \mathsf{qf}(\psi)$. When $\varphi \prec \psi$, we say that φ is of *lower rank* than ψ. Clearly, if the order $\prec$ on sequents is well–founded, then so is the order $\prec$ on $\mathcal{F}_{\mathsf{ex}}$.

3. Uniform interpolants

A logic has *uniform interpolation* if for any atom p and any set of atoms $\mathcal{P}$ not containing p, the embedding of $\mathcal{F}(\mathcal{P})$ into $\mathcal{F}(\mathcal{P} \cup \{p\})$ has a right and a left adjoint: For any formula φ and any atom p there exist formulas χ_r and χ_l in the language of the logic, that do not contain p and such that for all ψ not containing p:

$$\vdash \psi \to \varphi \;\Leftrightarrow\; \vdash \psi \to \chi_r \qquad \vdash \varphi \to \psi \;\Leftrightarrow\; \vdash \chi_l \to \psi.$$

These formulas are usually denoted by $\forall p\varphi$ and $\exists p\varphi$, respectively, and thus we have

$$\vdash \psi \to \varphi \;\Leftrightarrow\; \vdash \psi \to \forall p\varphi \qquad \vdash \varphi \to \psi \;\Leftrightarrow\; \vdash \exists p\varphi \to \psi.$$

Given a formula φ, its *universal uniform interpolant with respect to* $p_1 \dots p_n$ is $\forall p_1 \dots p_n \varphi$, which is short for $\forall p_1(\forall p_2(\dots(\forall p_n \varphi)\dots))$, and its *existential uniform interpolant with respect to* $p_1 \dots p_n$ is $\exists p_1 \dots p_n \varphi$, short for $\exists p_1(\exists p_2(\dots(\exists p_n \varphi)\dots))$. The requirements above could be replaced by the following four requirements.

$$\vdash \forall p\varphi \to \varphi \qquad \vdash \psi \to \varphi \;\Rightarrow \vdash \psi \to \forall p\varphi. \tag{$\forall$}$$

$$\vdash \varphi \to \exists p\varphi \qquad \vdash \varphi \to \psi \;\Rightarrow \vdash \exists p\varphi \to \psi. \tag{$\exists$}$$

In classical logic one only needs one quantifier, as $\exists p$ can be defined as $\neg\forall p\neg$ and vice versa. Although in the intuitionistic setting $\exists p$ can also be defined in terms of $\forall p$, namely as $\exists p\varphi = \forall q(\forall p(\varphi \to q) \to q)$ for a q not in φ, having it as a separate quantifier is convenient in the proof-theoretic approach presented here (we follow [22], which also uses both quantifiers).

3.1. Partitions

To define uniform interpolants in the setting of sequents, we introduce the notion of a partition, which applies to sequents and to rules. The notion for sequents is treated in this section and the one for rules later on.

Intuitively, if in the statement of uniform interpolation the implication is replaced by a sequent arrow, then $(\psi \Rightarrow \forall p\varphi)$, for ψ not containing p, can be viewed as partitioning the sequent $S = (\psi \Rightarrow \varphi)$ in two sequents $S^r = (\psi \Rightarrow\ )$ and $S^i = (\ \Rightarrow \varphi)$, and applying universal quantification to the second one. Likewise for $\exists p\varphi$. The definition of a partition is a generalization of that idea to arbitrary sequents.

A *partition* of a sequent S is an ordered pair (S^r, S^i) (i for *interpolant*, r for *rest*) such that $S = S^r \cdot S^i$. It is a p*–partition* if p does not occur in S^r. For any sequent S and partition (S^i, S^r) we use the abbreviation:

$$S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing) \equiv_{df} \begin{cases} S^r \cdot (\exists pS^i \Rightarrow \forall pS^i) & \text{if } S^s \neq \varnothing \text{ and } S^{rs} = \varnothing \\ S^r \cdot (\exists pS^i \Rightarrow\ ) & \text{if } S^s = \varnothing \text{ or } S^{rs} \neq \varnothing. \end{cases}$$

A *(p-)partition* of an instance $R = (S_1 \dots S_n / S_0)$ of a rule is a (p-)partition of the sequents in the rule. Given such a partition and $\star \in \{r, i\}$, let (R^r, R^i) and $R^\star$ respectively denote the expressions

$$\frac{(S_1^r, S_1^i) \quad \dots \quad (S_n^r, S_n^i)}{(S_0^r, S_0^i)}\ (R^r, R^i) \qquad\qquad \frac{S_1^\star \quad \dots \quad S_n^\star}{S_0^\star}\ R^\star$$

3.1.1. The interpolant properties

Recall from Section 2.1 that $\forall pS$ and $\exists pS$ are defined to be $\forall pI(S)$ and $\exists p(\bigwedge S^a)$, respectively. In particular, $\forall p(\Rightarrow \varphi)$ is equivalent to $\forall p\varphi$ and $\exists p(\varphi \Rightarrow)$ to $\exists p\varphi$. As will be shown in Lemma 2, ($\forall$) and ($\exists$) can be replaced by the following three requirements, the *interpolant properties*.

($\forall$l) For all p: $\vdash S^a, \forall pS \Rightarrow S^s$;
($\exists$r) For all p: $\vdash S^a \Rightarrow \exists pS$;
($\forall\exists$) If S is derivable, for all p and all p–partitions (S^r, S^i): $\vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$.

Properties ($\forall$l) and ($\exists$r) are the *independent* (from partitions) *interpolant properties*, and ($\forall\exists$) is the *dependent interpolant property.*

A partition (S^r, S^i) of S *satisfies* the interpolant properties if, in the case of the independent property, S satisfies them (in which case we also say that S satisfies them), and in case of the dependent property, it holds for that particular partition. A sequent *satisfies* a property if every possible partition of the sequent satisfies it.

Lemma 2. *If all sequents satisfy the interpolant properties, then* L *has uniform interpolation.*

Proof. ($\exists$) Consider $S = (\varphi \Rightarrow)$. By ($\exists$r) we have $\vdash I(\varphi \Rightarrow \exists pS)$, and since $\exists p(\varphi \Rightarrow) = \exists p\varphi$, we have thereby shown $\varphi \to \exists p\varphi$ to be derivable.

Consider a ψ not containing p such that $\vdash \varphi \to \psi$. Let $S = (\varphi \Rightarrow \psi)$ and consider the p–partition (S^r, S^i), where $S^i = (\varphi \Rightarrow)$ and $S^r = (\Rightarrow \psi)$. Hence $\exists p\varphi = \exists pS^i$ by definition and $S^{rs} \neq \varnothing$. As $\vdash (\exists p\varphi \Rightarrow) \cdot (\Rightarrow \psi)$ by ($\forall\exists$), $\vdash \exists p\varphi \to \psi$ follows.

($\forall$) Consider $S = (\Rightarrow \varphi)$. By ($\forall$l) we have $\vdash I(\forall pS \Rightarrow \varphi)$, and since $\forall p(\Rightarrow \varphi) = \forall p\varphi$, we have thereby shown $\forall p\varphi \to \varphi$ to be derivable.

Consider a ψ not containing p such that $\vdash \psi \to \varphi$. Let $S = (\psi \Rightarrow \varphi)$ and consider the p–partition (S^r, S^i), where $S^i = (\Rightarrow \varphi)$ and $S^r = (\psi \Rightarrow)$. Hence $\forall p\varphi = \forall pS^i$ by definition and $S^{rs} = \varnothing$. Thus $\vdash (\psi \Rightarrow) \cdot (\exists pS^i \Rightarrow \forall pS^i)$ by ($\forall\exists$), that is, $\vdash \psi, \exists pS^i \Rightarrow \forall p\varphi$. But $\vdash (\Rightarrow \exists pS^i)$ by ($\exists$r). Therefore $\vdash \psi \to \forall p\varphi$. □

Fact 1. All free sequents satisfy the dependent interpolant properties.

3.2. Interpolant assignments

Let G be a sequent calculus. Recall that given a rule $\mathcal{R}$, $\mathcal{R}_{\mathsf{ins}}$ denotes the set of instances of $\mathcal{R}$ and $\mathsf{G}_{\mathsf{ins}}$ denotes the set of instances of rules in G. An *interpolant assignment* ι for G, assigns, for every atom p and sequent S, $\iota\exists pS = \top$ and $\iota\forall pS = \bot$ in case S is empty, and in case S is not empty:

- for every $R \in \mathsf{G}_{\mathsf{ins}}$ with conclusion S, to each of the expressions $\exists \overset{R}{p} S$ and $\forall \overset{R}{p} S$ a formula in $\mathcal{F}_{\mathsf{ex}}$ that is of lower rank than $\exists pS$ (or, equivalently, of lower rank than $\forall pS$), which are denoted by $\iota\exists \overset{R}{p} S$ and $\iota\forall \overset{R}{p} S$, respectively, and
- for every $\mathcal{R} \in \mathsf{G}$ such that S is nonprincipal for at least one instance of $\mathcal{R}$, to each of the expressions $\exists \overset{\overline{\mathcal{R}}}{p} S$ and $\forall \overset{\overline{\mathcal{R}}}{p} S$ a formula in $\mathcal{F}_{\mathsf{ex}}$ that is of lower rank than $\exists pS$, which are denoted by $\iota\exists \overset{\overline{\mathcal{R}}}{p} S$ and $\iota\forall \overset{\overline{\mathcal{R}}}{p} S$, respectively.

We use the following abbreviations for certain formulas in $\mathcal{F}_{\mathsf{ex}}$. Recall that p and q range over atoms.

$$
\begin{aligned}
\forall^+ pS &\equiv_{df} \bigvee\{\iota\forall \overset{R}{p} S \mid R \in \mathsf{G}_{\mathsf{ins}},\ S \text{ is the conclusion of } R\}\\
\forall^- pS &\equiv_{df} \bigvee\{\iota\forall \overset{\overline{\mathcal{R}}}{p} S \mid \mathcal{R} \in \mathsf{G},\ S \text{ is nonprincipal for some instance of } \mathcal{R}\}\\
\exists^+ pS &\equiv_{df} \bigwedge\{\iota\exists \overset{R}{p} S \mid R \in \mathsf{G}_{\mathsf{ins}},\ S \text{ is the conclusion of } R\}\\
\exists^- pS &\equiv_{df} \bigwedge\{\iota\exists \overset{\overline{\mathcal{R}}}{p} S \mid \mathcal{R} \in \mathsf{G},\ S \text{ is nonprincipal for some instance of } \mathcal{R}\}\\
\forall^{at} pS &\equiv_{df} \bigvee\{q \in S^s \mid q \text{ an atom and } q \neq p, \text{ or } q = \top\} \vee\\
&\qquad \bigvee\{q \wedge \forall p(\varphi, S^a\backslash\{q \to \varphi\} \Rightarrow S^s) \mid (q \to \varphi) \in S^a, q \neq p\}\\
\exists^{at} pS &\equiv_{df} \bigwedge\{q \in S^a \mid q \text{ an atom and } q \neq p, \text{ or } q = \bot\} \wedge\\
&\qquad \bigwedge\{q \to \exists p(\varphi, S^a\backslash\{q \to \varphi\} \Rightarrow S^s) \mid (q \to \varphi) \in S^a, q \neq p\}.
\end{aligned}
$$

Observe that there could be more than one instance of a single rule $\mathcal{R}$ that has S as a conclusion, in which case every instance corresponds to a separate disjunct or conjunct of the interpolant assignment. The definition above is well-defined for reductive calculi, because for such calculi all sets over which the big conjunctions and disjunctions range are finite.

We define a rewrite relation $\rightarrowtail$ on $\mathcal{F}_{\mathsf{ex}}$ that is the smallest relation on $\mathcal{F}_{\mathsf{ex}}$ that preserves the logical operators and satisfies:

$$\forall pS \ \rightarrowtail\ \forall^+ pS \vee \forall^- pS \vee \forall^{at} pS \qquad \exists pS \ \rightarrowtail\ \exists^+ pS \wedge \exists^- pS \wedge \exists^{at} pS.$$

Example 2. Suppose the calculus only contains the rule $\mathcal{R}$ for conjunction on the left:

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}$$

Consider the sequent $S = (\varphi_1 \wedge \psi_1, \varphi_2 \wedge \psi_2 \Rightarrow\ )$. Let R_i stand for the instance of $\mathcal{R}$ with $\varphi_i \wedge \psi_i$ as the principal formula, and define sequents $S_1 = (\varphi_1, \psi_1, \varphi_2 \wedge \psi_2 \Rightarrow\ )$ and $S_2 = (\varphi_1 \wedge \psi_1, \varphi_2, \psi_2 \Rightarrow\ )$. By the above definition,

$$\forall p S \rightarrowtail \iota\forall \overset{R_1}{p} S \vee \iota\forall \overset{R_2}{p} S \vee \iota\forall \overset{\overline{\mathcal{R}}}{p} S \vee \iota\forall^{at} p S.$$

Using the order in Example 1, the standard interpolant assignment introduced below satisfies $\iota\forall \overset{R_i}{p} S = \forall p S_i$ and $\iota\forall \overset{\overline{\mathcal{R}}}{p} S = \bot$. This implies that

$$\forall p S \rightarrowtail \forall p S_1 \vee \forall p S_2 \vee \bot \vee \bot.$$

3.3. Reduction

The following lemma shows that every $\varphi \in \mathcal{F}_{\mathsf{ex}}$ either belongs to $\mathcal{F}$ or reduces via $\rightarrowtail$ to a unique $\psi \in \mathcal{F}$. In the latter case, the ψ will be denoted by $\delta\varphi$. Slightly abusing notation, we will mostly omit the δ, especially in the setting of derivability. For example, under this convention, $\vdash \varphi \to \bot$ abbreviates $\vdash \delta\varphi \to \bot$.

Lemma 3. *In any reductive calculus, the relation $\rightarrowtail$ on $\mathcal{F}_{\mathsf{ex}}$ is confluent and strongly normalizing.*

Proof. Let $\prec$ be the extension to $\mathcal{F}_{\mathsf{ex}}$ of the order with respect to which the calculus is reductive, as defined in Section 2.5, and recall that it is by definition well-founded. In the terminology of [2] (Definition 4.2.2), $\rightarrowtail$ determines a rewrite relation (the set V of variables, in their sense, is empty in our setting). From the definition of interpolant assignments it follows that $\varphi \rightarrowtail \psi$ implies $\psi \prec \varphi$, and thus the rewrite relation is terminating. Since no rules overlap, it has no critical pairs (Definition 6.2.1 of the same volume), and therefore (Corollary 6.2.5) the rewrite relation is confluent. Since the relation is also normalizing (as it is terminating), it follows that $\rightarrowtail$ is strongly normalizing, implying that every term has a unique normal form. □

3.4. Explanation

As is clear from the definition above, for a sequent S and atom p, the uniform interpolants $\forall p S$ and $\exists p S$ are a disjunction, respectively conjunction of formulas of lower rank than $\exists p S$, also if S is free. The role of these expressions in a proof of the interpolant properties is as follows. Clearly, if only the dependent properties have to be satisfied, then taking $\bot$ for $\exists p S$ for all sequents S suffices. If only the independent properties have to be satisfied, then assigning $\bot$ to $\forall p S$ and $\top$ to $\exists p S$ suffices. The interplay between the independent and the dependent properties is what makes the definition of the uniform interpolants difficult. It is based on the following observation.

For the dependent interpolant property, there are, for every derivable sequent S and p–partition (S^r, S^i), two cases given some derivation of S: for R being the last inference of the derivation and an instance of a rule $\mathcal{R}$, either S^i is principal for R or it is nonprincipal for R. Suppose that in the first case one can show that for some instance R^i of $\mathcal{R}$ with conclusion S^i, $\vdash S^{ra}, \exists \overset{R^i}{p} S^i \Rightarrow \forall \overset{R^i}{p} S^i$, and in the second case that $\vdash S^{ra}, \exists \overset{\overline{\mathcal{R}}}{p} S^i \Rightarrow \forall \overset{\overline{\mathcal{R}}}{p} S^i$. Then the dependent interpolant property, $\vdash (S^{ra}, \exists p S^i \Rightarrow \forall p S^i)$, holds for (S^r, S^i), as $\exists \overset{R^i}{p} S^i$ is a conjunct of $\exists p S^i$ and $\forall \overset{R^i}{p} S^i$ is a disjunct of $\forall p S^i$ in the first case, and $\exists \overset{\overline{\mathcal{R}}}{p} S^i$ is a conjunct of $\exists pS^i$ and $\forall\overset{\mathcal{R}}{p}S^i$ is a disjunct of $\forall pS^i$ in the second case. The same strategy can be used to show that $\vdash S^{ra}, \exists pS^i \Rightarrow S^{rs}$ in case $S^s = \varnothing$ or $S^{rs} \neq \varnothing$. This is how the dependent interpolant property will be proved.

The role of the disjuncts $\forall^{at}p$ and conjuncts $\exists^{at}p$ lies in certain particular cases. For example, given an instance of an axiom $(q \Rightarrow q)$ and partition $S^r = (q \Rightarrow)$, $S^i = (\ \Rightarrow q)$, the sequent $(q, \exists pS^i \Rightarrow \forall pS^i)$ has to be derivable, and $\forall^{at}pS^i$ and $\exists^{at}pS^i$ take care of that case.

3.5. The inductive properties

In order to develop a modular method for proving uniform interpolation, we introduce the following six properties of rules, where $\varnothing \vdash \varphi$ should be read as $\vdash \varphi$. Recall that $(\Gamma \Rightarrow \Delta) \subseteq (\Gamma' \Rightarrow \Delta')$ denotes that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$ (Section 2.1). Given an instance $R = (S_1 \dots S_n/S_0)$ of a rule $\mathcal{R}$, we define

$$\mathfrak{I}^p_R \equiv_{df} \{S_j \cdot (\forall pS_j \Rightarrow), (S^a_j \Rightarrow \exists pS_j) \mid 1 \leq j \leq n\} \cup$$
$$\{S^a \Rightarrow \exists p(S^a \Rightarrow) \mid S \subset S_0 \text{ or } \Box S \subseteq S_0 \text{ or } S \subseteq S_j \text{ for some } 1 \leq j \leq n\}$$
$$\mathfrak{D}^p_R \equiv_{df} \bigcup_{j=1}^{n} \{S^r_j \cdot (\exists pS^i_j \Rightarrow \forall pS^i_j \mid \varnothing) \mid (S^r_j, S^i_j) \text{ a } p\text{–partition of } S_j\}.$$

In $\mathfrak{I}^p_R$, requirement $\Box S \subseteq S_0$ is only included in the case of modal logic. For $\mathfrak{D}^p_R$, note that it contains the sequent $S^r_j \cdot (\exists pS^i_j \Rightarrow \forall pS^i_j \mid \varnothing)$ for *any* possible p-partition (S^r_j, S^i_j) of a premiss S_j of R. And that for S with empty succedent, $S^r \cdot (\exists pS^i \Rightarrow)$ derives $S^r \cdot (\exists pS^i \Rightarrow \forall pS^i)$.

The sets $\mathfrak{I}^p_R$ and $\mathfrak{D}^p_R$ contain the sequents to which, in a proof of the interpolant properties that uses induction along $\prec$, the induction hypothesis needs to be applied. In such a proof, the assumption that the interpolant properties hold for all sequents below S implies that the sequents in $\mathfrak{I}^p_R$ and $\mathfrak{D}^p_R$ are derivable. Note that in the case that R is an instance of an axiom, the latter set is empty, but the former is not, as it contains all sequents of the form $S^a \Rightarrow \exists p(S^a \Rightarrow)$ for S such that $S \subset S_0$ or $\Box S \subseteq S_0$.

- $(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ $\mathfrak{I}^p_R \vdash S \cdot (\forall\overset{R}{p}S \Rightarrow)$ for every instance R of $\mathcal{R}$ with conclusion S.
- $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$ If S is nonprincipal for some instance of $\mathcal{R}$, then the assumption that all sequents below S satisfy the interpolant properties implies $\vdash S \cdot (\forall\overset{\mathcal{R}}{p}S \Rightarrow)$.
- $(\mathrm{IPP})^{\exists}_{\mathcal{R}}$ $\mathfrak{I}^p_R \vdash (S^a \Rightarrow \exists\overset{R}{p}S)$ for every instance R of $\mathcal{R}$ with conclusion S.
- $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$ If S is nonprincipal for some instance of $\mathcal{R}$, then the assumption that all sequents below S satisfy the interpolant properties implies $\vdash (S^a \Rightarrow \exists\overset{\mathcal{R}}{p}S)$.
- $(\mathrm{DPP})_{\mathcal{R}}$ For every sequent S that has a derivation of which the last inference is an instance R of $\mathcal{R}$, and for every p–partition (S^r, S^i) such that sequent S^i is principal for R: $\mathfrak{D}^p_R \vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$.
- $(\mathrm{DPN})_{\mathcal{R}}$ For every sequent S that has a derivation of which the last inference is an instance R of $\mathcal{R}$, and for every p–partition (S^r, S^i) such that sequent S^i is nonprincipal for R: if all sequents that are below S satisfy the interpolant properties, then $\vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$.

These six properties are called the *inductive properties* in this paper. "IP" stands for *independent property*, "DP" for *dependent property* "P" and "N" for *principal* and *nonprincipal*, respectively.

An interpolant assignment is *sound* for a rule $\mathcal{R}$ in a calculus, if the six inductive properties hold for $\mathcal{R}$, where $\vdash$ equals $\vdash_{\mathsf{L_G}}$, the consequence relation corresponding to the calculus G (Section 2.4). It is *sound* for a calculus if it is sound for all the rules of the calculus. Sometimes the following strengthening of $(\mathrm{DPN})_{\mathcal{R}}$ holds:

$(\mathrm{DPN})^+_{\mathcal{R}}$ For every sequent S that has a derivation of which the last inference is an instance R of $\mathcal{R}$, for every p–partition (S^r, S^i) such that sequent S^i is nonprincipal for R: $\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$.

This is a strengthening of $(\mathrm{DPN})_{\mathcal{R}}$ because under the assumption that all sequents lower than S satisfy the interpolant properties, all sequents in the set $\mathfrak{D}^p_R$ become derivable.

Remark 1. The following observation will be used to prove $(\mathrm{DPP})_{\mathcal{R}}$ and $(\mathrm{DPN})_{\mathcal{R}}$. Consider a sequent S with partition (S^r, S^i), which has a derivation of which the last inference is an instance $R = (S_1 \dots S_n/S)$ of $\mathcal{R}$. To prove $(\mathrm{DPP})_{\mathcal{R}}$, thus in case S^i is principal for R, in order to prove

$$\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$$

it suffices to show that

$$\mathfrak{D}^p_R \vdash S^r \cdot (\exists \overset{R^i}{p} S^i \Rightarrow \forall \overset{R^i}{p} S^i \mid \varnothing)$$

for some partition (R^r, R^i) of R with conclusion (S^r, S^i) such that R^i is an instance of $\mathcal{R}$. The reason being, that for such an R^i, $\exists \overset{R^i}{p} S^i$ is a conjunct of $\exists p S^i$ and $\forall \overset{R^i}{p} S^i$ a disjunct of $\forall p S^i$. Likewise, to prove $(\mathrm{DPN})_{\mathcal{R}}$, thus in case S^i is nonprincipal for R, to prove that

$$\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$$

it suffices to prove that

$$\vdash S^r \cdot (\exists \overset{\overline{\mathcal{R}}}{p} S^i \Rightarrow \forall \overset{\overline{\mathcal{R}}}{p} S^i \mid \varnothing).$$

3.6. Soundness

Lemma 4. *If a logic* L *has a reductive ca calculus* G *for which there exists a sound interpolant assignment, then all sequents satisfy the interpolant properties.*

Proof. Let $\vdash_{\mathsf{L}_\mathsf{G}}$ be the consequence relation corresponding to G as defined in Section 2.4. We have to prove the interpolant properties for $\vdash_\mathsf{L}$. Since $\vdash_\mathsf{L} \varphi$ exactly if $\vdash_\mathsf{G} (\,\Rightarrow \varphi)$ and Cut is admissible in G because it is a ca calculus, the theorems of $\vdash_\mathsf{L}$ and $\vdash_{\mathsf{L}_\mathsf{G}}$ are equal. Thus it suffices to prove the interpolant properties for $\vdash_{\mathsf{L}_\mathsf{G}}$. In the rest of the proof, let $\vdash$ denote $\vdash_{\mathsf{L}_\mathsf{G}}$.

We use induction along the well-founded order $\prec$ on sequents with respect to which the calculus is reductive. Therefore assume that all sequents lower than S satisfy the interpolant properties. We have to show that so does S. Note that all sequents in the sets $\mathfrak{I}^p_R$ are derivable because they express interpolant properties of sequents that come before S in the order.

(∀l) We have to show that

1. $\vdash S^a, \forall^{at} p S \Rightarrow S^s$,
2. $\vdash S^a, \forall \overset{R}{p} S \Rightarrow S^s$ for all instances R with conclusion S,
3. $\vdash S^a, \forall \overset{\overline{\mathcal{R}}}{p} S \Rightarrow S^s$ for all rules $\mathcal{R}$ such that S is nonprincipal for some instance of $\mathcal{R}$.

2. follows from $(\mathrm{IPP})^\forall_{\mathcal{R}}$ and 3. from $(\mathrm{IPN})^\forall_{\mathcal{R}}$. For 1., first consider its disjuncts of the form q for some $q \neq p$ that belongs to S^s. Then $S^a, q \Rightarrow S^s$ clearly holds. Second, consider disjuncts of the form $(q \wedge \forall p(\Gamma, \varphi \Rightarrow \Delta))$, where $S = (\Gamma, q \to \varphi \Rightarrow \Delta)$ for some $q \neq p$. Let $S' = (\Gamma, \varphi \Rightarrow \Delta)$. Since $S' \prec S$, the assumption that all

sequents lower than S satisfy the interpolant properties implies that $(\Gamma, \varphi, \forall p S' \Rightarrow \Delta)$ is derivable. Thus so is $(\Gamma, q \to \varphi, q \wedge \forall p S' \Rightarrow \Delta)$.

($\exists$r) We have to show that

1. $\vdash S^a \Rightarrow \exists^{at} p S$,
2. $\vdash S^a \Rightarrow \exists^{R}_{p} S$ for all instances R with conclusion S,
3. $\vdash S^a \Rightarrow \exists^{\mathcal{R}}_{p} S$ for all rules $\mathcal{R}$ such that S is nonprincipal for some instance of $\mathcal{R}$.

2. follows from $(\text{IPP})^{\exists}_{\mathcal{R}}$ and 3. from $(\text{IPN})^{\exists}_{\mathcal{R}}$. For 1., first consider conjuncts of the form q, where $q \in S^a$ and $q \neq p$. Then $\vdash S^a \Rightarrow q$ clearly holds. The remaining conjuncts of $\exists^{at} p S$ are of the form $(q \to \exists p(\Gamma, \varphi \Rightarrow \Delta))$, where $S = (\Gamma, q \to \varphi \Rightarrow \Delta)$ and $q \neq p$. Let $S' = (\Gamma, \varphi \Rightarrow \Delta)$. Since $S' \prec S$, the assumption that all sequents lower than S satisfy the interpolant properties implies that $(\Gamma, \varphi \Rightarrow \exists p S')$ is derivable. Thus $(\Gamma, q \to \varphi \Rightarrow q \to \exists p S')$ is derivable.

($\forall\exists$) Assume that S is derivable and let $R = (S_1 \dots S_n / S)$ be the last inference of some derivation of S. Suppose R is an instance of rule $\mathcal{R}$. Consider an arbitrary p–partition (S^r, S^i) of S. Either S^i is principal for R or it is nonprincipal for R. Since all sequents in $\mathfrak{D}^p_R$ are derivable, $\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$ follows from $(\text{DPP})_{\mathcal{R}}$ or $(\text{DPN})_{\mathcal{R}}$. □

Theorem 5. *If a logic* L *has a reductive ca calculus for which there exists a sound interpolant assignment, then* L *has uniform interpolation.*

Proof. This follows from Lemmas 2 and 4. □

3.7. Modularity

Note that when a rule $\mathcal{R}$ in a calculus G satisfies $(\text{IPP})^{\exists}_{\mathcal{R}}$, $(\text{IPP})^{\forall}_{\mathcal{R}}$, $(\text{IPN})^{\exists}_{\mathcal{R}}$, $(\text{IPN})^{\forall}_{\mathcal{R}}$ for an interpolant assignment, then it does so in all extensions of G under the same interpolant assignment. In other words, the four properties are *modular*. This does not hold for the dependent properties, because a sequent not derivable in the original calculus can become derivable in the extension, and therefore has to be treated in a proof of $(\text{DPP})_{\mathcal{R}}$ or $(\text{DPN})_{\mathcal{R}}$. However, for all rules $\mathcal{R}$ treated in this paper, that is the rules of $\mathsf{G4iK}_\Box$ and all focused (modal) rules, $(\text{DPP})_{\mathcal{R}}$ and $(\text{DPN})_{\mathcal{R}}$ are modular too: they hold not only in the main calculus, $\mathsf{G4iK}_\Box$, but in any balanced extension of it.

4. Focused rules

In this section we introduce the class of *one-sided unary thinnable* rules and their *standard* interpolant assignment, which is sound for these rules. Many well–known rules of sequent calculi are of this form.

4.1. Properties of rules

A rule $\mathcal{R}$ that is not an axiom is *thinnable* if it is of the form

$$\frac{S \cdot S_1 \quad S \cdot S_2 \quad \dots \quad S \cdot S_n}{S \cdot S_0} \tag{3}$$

where $S_0, S_1, \dots, S_n$ are meta–sequents such that

- if $S^s \neq \varnothing$, then $S = (\Gamma \Rightarrow \Delta)$ for two distinct meta–multisets Γ and Δ that do not occur in any of $S_0, S_1, \dots, S_n$,

- if $S^s = \varnothing$, then $S = (\Gamma \Rightarrow\ )$ for a meta–multiset Γ that does not occur in any of $S_0, S_1, \dots, S_n$.

A thinnable rule (3) is *unary* if moreover:

- S_0 consists of exactly one meta–formula, which is not an atom, and in the setting of modal logic it is not boxed either,
- any variable in any of $S_1, \dots, S_n$ occurs in S_0.

Note that a thinnable unary rule is either a left rule or a right rule, and not both. A unary thinnable rule (3) is *one-sided* if moreover:

- if $\mathcal{R}$ is a left rule, the succedents of all $S_0, \dots, S_n$ are empty,
- if $\mathcal{R}$ is a right rule, the antecedents of all $S_0, \dots, S_n$ are empty.

A rule is *focused* if either it is a one-sided unary thinnable rule that is not an axiom, or it is an axiom of the form $(\Gamma, r \Rightarrow r)$, $(\Gamma, \bot \Rightarrow \Delta)$ or $(\Gamma \Rightarrow \top)$, with Γ a meta–multiset.[2] In an instance $R = (S \cdot S_1 \dots S \cdot S_n / S \cdot S_0)$ of $\mathcal{R}$, the *principal* formula of R is the formula in S_0. All other occurrences in R of the formula in S_0 are not principal. Thus although we speak of the principal formula, it is in fact an occurrence of a formula that is principal. In axiom $(\Gamma, r \Rightarrow r)$ both occurrences of r are *principal* and in $(\Gamma, \bot \Rightarrow \Delta)$ and $(\Gamma \Rightarrow \top)$ the indicated occurrence of $\bot$ and $\top$, respectively, are *principal*.

Example 3. Typical focused rules are the left and right rules of Gentzen calculi. The right conjunction rule

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ \mathrm{R}\wedge$$

is clearly focused, as one can take $(\Gamma \Rightarrow\ )$ for S, $(\Rightarrow \varphi \wedge \psi)$ for S_0 and $(\Rightarrow \varphi)$ and $(\Rightarrow \psi)$ for S_1 and S_2. Note that what we defined to be the principal formula of an instance of R$\wedge$ coincides with what is usually called the principal formula of such an instance. An example of a less standard rule that is focused is the rule

$$\frac{\Gamma \Rightarrow \neg\chi \to \varphi \vee \psi}{\Gamma \Rightarrow (\neg\chi \to \varphi) \vee (\neg\chi \to \psi)}$$

A rule that is not focused is the right implication rule

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}$$

as it is a right rule, but the antecedent of $S_1 = (\varphi \Rightarrow \psi)$ is not empty. This does not mean that this rule blocks uniform interpolation, it just means that it is not covered by the general treatment that we develop for focused rules, and it therefore has to be treated separately. A similar phenomenon occurs for two implication rules in the calculus G4ip [10], which is treated in Section 5.

[2] The term *focused* is used in another area of proof theory as well, where it refers to certain proof search strategies and proof systems that formalize that notion. The way the term is used here is independent of that usage, which predates ours.

4.2. Partition of focused rules

Given an instance $R = \big(S \cdot S_1 \dots S \cdot S_n / S \cdot S_0\big)$ of a focused rule $\mathcal{R}$ and a p-partition of R, where each $S \cdot S_j$ is partitioned in $\big((S \cdot S_j)^r, (S \cdot S_j)^i\big)$, then this partition is *standard* if either R^i is equal to

$$\frac{S^i \cdot S_1 \quad \dots \quad S^i \cdot S_n}{S^i \cdot S_0}\ R^i$$

and $(S \cdot S_j)^r = S^r$ for all $j = 0, \dots, n$, or R^r is equal to

$$\frac{S^r \cdot S_1 \quad \dots \quad S^r \cdot S_n}{S^r \cdot S_0}\ R^r$$

and $(S \cdot S_j)^i = S^i$ for all $j = 0, \dots, n$. The following lemma implies that instances of focused rules can always be partitioned in a way that either R^r or R^i is an instance of the same rule.

Lemma 6. *For any instance $R = \big(S \cdot S_1 \dots S \cdot S_n / S \cdot S_0\big)$ of a focused rule $\mathcal{R}$ and any p-partition $((S \cdot S_0)^r, (S \cdot S_0)^i)$ of $S \cdot S_0$, there exists exactly one standard p-partition of R with conclusion $((S \cdot S_0)^r, (S \cdot S_0)^i)$ such that either the principal formula belongs to S^i and R^i is an instance of $\mathcal{R}$ or the principal formula belongs to S^r and R^r is an instance of $\mathcal{R}$.*

Proof. Since there is only one principal formula, the one in S_0, there exists a p-partition (S^r, S^i) of S such that either $(S \cdot S_0)^r = S^r \cdot S_0$ and $(S \cdot S_0)^i = S^i$, or $(S \cdot S_0)^i = S^i \cdot S_0$ and $(S \cdot S_0)^r = S^r$.

Given partition (S^r, S^i), a partition of the premisses of R is defined as follows:

$$\begin{cases} (S \cdot S_j)^i = S^i \cdot S_j & (S \cdot S_j)^r = S^r \qquad \text{if } (S \cdot S_0)^i = S^i \cdot S_0 \\ (S \cdot S_j)^i = S^i & (S \cdot S_j)^r = S^r \cdot S_j \quad \text{otherwise.} \end{cases}$$

Note that the partition is well-defined, standard, and $(S \cdot S_j)^r$ and $(S \cdot S_j)^i$ indeed form a partition of $S \cdot S_j$. That it is a p-partition of the premisses follows from the assumption that all atoms in the S_j must occur in S_0.

As $\mathcal{R}$ is focused, in the first case of the definition of the partition, R^i is an instance of $\mathcal{R}$ and in the second case R^r is, which completes the proof. □

Example 4. Consider the following instance R of the rule L$\vee$ for disjunction on the left:

$$\frac{S_1 \quad S_2}{S_0} = \frac{\Gamma, \varphi_1 \Rightarrow \Delta \quad \Gamma, \varphi_2 \Rightarrow \Delta}{\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow \Delta}$$

Then for the partition $(S_0^r, S_0^i) = \big((\Gamma \Rightarrow \Delta), (\varphi_1 \vee \varphi_2 \Rightarrow)\big)$ of the conclusion S_0, the following is the standard partition of the rule given this partition.

$$\frac{(\varphi_1 \Rightarrow) \quad (\varphi_1 \Rightarrow)}{(\varphi_1 \vee \varphi_2 \Rightarrow)}\ R^i \qquad \frac{(\Gamma \Rightarrow \Delta) \quad (\Gamma \Rightarrow \Delta)}{(\Gamma \Rightarrow \Delta)}\ R^r$$

If the partition of the conclusion is, for example, $(S_0^r, S_0^i) = \big((\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow), (\Rightarrow \Delta)\big)$, then the standard partition of R with that particular partition of the conclusion is

$$\frac{(\Gamma, \varphi_1 \Rightarrow) \quad (\Gamma, \varphi_1 \Rightarrow)}{(\Gamma, \varphi_1 \vee \varphi_2 \Rightarrow)}\ R^r \qquad \frac{(\Rightarrow \Delta) \quad (\Rightarrow \Delta)}{(\Rightarrow \Delta)}\ R^i$$

4.3. Standard interpolant assignment for focused rules

For a focused rule $\mathcal{R}$, the *standard interpolant assignment* ι is defined as follows. If $\mathcal{R}$ is not an axiom, then for an instance

$$\frac{S_1 \quad S_2 \quad \dots \quad S_n}{S} \; R$$

of $\mathcal{R}$ we define

$$\iota\exists\overset{R}{p}S \equiv_{df} \bigvee_{i=1}^n \exists p S_i \quad \iota\forall\overset{R}{p}S \equiv_{df} \bigwedge_{i=1}^n (\exists p S_i \to \forall p S_i).$$

If $\mathcal{R}$ is an axiom, and R is an instance of it which consists of sequent S, then

$$\iota\forall\overset{R}{p}S \equiv_{df} \top \quad \iota\exists\overset{R}{p}S \equiv_{df} \bigwedge\{\varphi \in S^a \mid \varphi \text{ does not contain } p\}.$$

For S that are nonprincipal for some instance of $\mathcal{R}$ we define

$$\iota\exists\overset{\overline{\mathcal{R}}}{p}S \equiv_{df} \top \quad \iota\forall\overset{\overline{\mathcal{R}}}{p}S \equiv_{df} \bot.$$

Although in this case the assignments $\exists\overset{\overline{\mathcal{R}}}{p}S$ and $\forall\overset{\overline{\mathcal{R}}}{p}S$ do not depend on $\mathcal{R}$ and are moreover trivial, this will no longer be the case for later rules. In order to provide a uniform approach we chose to define the assignments $\exists\overset{\overline{\mathcal{R}}}{p}S$ and $\forall\overset{\overline{\mathcal{R}}}{p}S$ for every rule $\mathcal{R}$ separately also in this case.

An interpolation assignment is *standard* if it is standard for all focused rules.

4.4. Soundness of the standard interpolant assignment

In this section we prove that the standard interpolant assignment for focused rules is sound, by proving the six inductive properties (Section 3.5).

Lemma 7. *For any focused rule $\mathcal{R}$ in a reductive ca calculus with a standard interpolant assignment, (IPP)$^{\exists}_{\mathcal{R}}$ and (IPN)$^{\exists}_{\mathcal{R}}$ hold.*

Proof. That (IPN)$^{\exists}_{\mathcal{R}}$ holds is clear. We treat with (IPP)$^{\exists}_{\mathcal{R}}$. Consider sequents $S_0, S_1, \dots, S_n$ such that $R = (S_1 \dots S_n / S_0)$ is an instance of $\mathcal{R}$. We have to show that $\mathfrak{I}^p_R \vdash S_0^a \Rightarrow \exists\overset{R}{p}S_0$. The case that R is an axiom is immediate from the definition of interpolant assignments for focused axioms. Therefore assume it is not an axiom. We distinguish the cases that $\mathcal{R}$ is a left rule and a right rule.

If $\mathcal{R}$ is a left rule, there are S_i' and S' such that the succedents of the S_i' are empty and

$$\frac{S_1 \quad \dots \quad S_n}{S_0} = \frac{S' \cdot S_1' \quad \dots \quad S' \cdot S_n'}{S' \cdot S_0'}$$

and for all S the following is an instance of $\mathcal{R}$:

$$\frac{S \cdot S_1' \dots S \cdot S_n'}{S \cdot S_0'}$$

This holds in particular for $S = (S'^{\,a} \Rightarrow \bigvee_{i=1}^n \exists p S_i)$. Since $\mathfrak{I}^p_R$ derives $(S_i^a \Rightarrow \exists p S_i)$ and $\iota\exists\overset{R}{p}S_0 = \bigvee_{i=1}^n \exists p S_i$, $\mathfrak{I}^p_R$ derives $(S_i^a \Rightarrow \exists\overset{R}{p}S_0) = ((S' \cdot S_i')^a \Rightarrow \exists\overset{R}{p}S_0) = S \cdot S_i'$, for all $i = 1, \dots, n$. An application of $\mathcal{R}$ shows that $\mathfrak{I}^p_R \vdash S \cdot S_0'$, which implies the desired.

If $\mathcal{R}$ is a right rule, there are S'_i and S' such that the antecedents of the S'_i are empty and

$$\frac{S_1 \quad \dots \quad S_n}{S_0} = \frac{S' \cdot S'_1 \quad \dots \quad S' \cdot S'_n}{S' \cdot S'_0}$$

This implies that all S^a_i are equal. And since $(S^a_i \Rightarrow \exists p S_i)$ belongs to $\mathfrak{I}^p_R$ for all $i = 1, \dots, n$, $\mathfrak{I}^p_R$ derives $(S^a_0 \Rightarrow \exists \overset{R}{p} S_0)$. □

Lemma 8. *For any instance $S_1 \dots S_n/S_0$ of a focused rule and any formulas $\varphi_1, \dots, \varphi_n$:*

$$\{S_j \cdot (\varphi_j \Rightarrow) \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S_0 \cdot (\bigwedge_{j=1}^{n} \varphi_j \Rightarrow).$$

Proof. Clearly, $\{S_1, \dots, S_n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S_0$. Let $S = (\bigwedge_{j=1}^n \varphi_j \Rightarrow)$. Since $\mathcal{R}$ is focused, we have $\{S \cdot S_1, \dots, S \cdot S_n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S \cdot S_0$. Since $S_j \cdot (\varphi_j \Rightarrow) \vdash^{\mathcal{R}}_{\mathsf{IPC}} S_j \cdot S$, the desired follows. □

Lemma 9. *For any focused rule $\mathcal{R}$ in a reductive ca calculus with a standard interpolant assignment, (IPP)$^{\forall}_{\mathcal{R}}$ and (IPN)$^{\forall}_{\mathcal{R}}$ hold.*

Proof. That (IPN)$^{\forall}_{\mathcal{R}}$ holds is clear. For (IPP)$^{\forall}_{\mathcal{R}}$ we reason as follows. Let $R = (S_1 \dots S_n/S_0)$ be an instance of a focused rule $\mathcal{R}$. If $\mathcal{R}$ is an axiom, S_0 is derivable and (IPP)$^{\forall}_{\mathcal{R}}$ clearly holds, as focused axioms are closed under left weakening. If not, $\iota \forall \overset{R}{p} S_0 = \bigwedge_{i=1}^n (\exists p S_j \to \forall p S_j)$. Since for each j,

$$\{S_j \cdot (\forall p S_j \Rightarrow), (S^a_j \Rightarrow \exists p S_j)\} \vdash S_j \cdot (\exists p S_j \to \forall p S_j \Rightarrow),$$

we can use Lemma 8 to obtain the desired result. □

Lemma 10. *For all formulas $\varphi_1, \dots, \varphi_n$ and any partition (S^r, S^i) of the conclusion of an instance $R = (S_1 \dots S_n/S)$ of a focused rule that is not an axiom and such that S^i is principal for R, for the standard partition of R:*

$$\{S^r_j \cdot (\Rightarrow \varphi_j) \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S^r \cdot (\Rightarrow \bigwedge_{j=1}^{n} \varphi_j).$$

Proof. As S^i contains the principal formula of R, $S^r = S^r_j$, which immediately implies the desired. □

Lemma 11. *For any focused rule $\mathcal{R}$ in a reductive ca calculus with a standard interpolant assignment, (DPP)$_{\mathcal{R}}$ holds.*

Proof. Consider a sequent S for which there exists a derivation of which the last inference is an instance $R = (S_1 \dots S_n/S)$ of $\mathcal{R}$, and let (S^r, S^i) be a partition of S such that S^i is principal for R. We have to show that

$$\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing). \tag{4}$$

If $\mathcal{R}$ is an axiom, then since S^i contains the principal formulas of R, S^i is an instance of it, which we denote by R^i. As the axiom is focused, $\iota \forall \overset{R^i}{p} S^i = \top$ and thus $\vdash \forall p S^i$. Therefore (4) clearly holds, at least in case that S^{rs} is empty and S^s is not. If this does not hold, which means if S^{rs} is not empty or S^s is empty, then we have to prove that $\mathfrak{D}^p_R \vdash S^r \cdot (\exists p S^i \Rightarrow)$. Note that in both cases, S^{is} is empty. Hence $\mathcal{R}$ has to be the axiom L⊥. For if it would be one of the other focused axioms, then the fact that S^i is an instance of it implies that S^{is} is not empty. Thus S and S^i are instances of L⊥. Therefore S^{ia} contains ⊥. Hence $\exists^{at} p S^i$, which is a conjunct of $\exists p S^i$, contains ⊥ as a conjunct. Therefore (4) holds. This completes the case that $\mathcal{R}$ is an axiom.

If $\mathcal{R}$ is not an axiom, consider the standard p-partition (R^r, R^i) of R with conclusion (S^r, S^i). Since S^i contains the principal formula of R, Lemma 6 implies that R^i is an instance of $\mathcal{R}$. Let the partition of the premisses S_j be denoted by (S_j^r, S_j^i). The definition of standard partition implies that $S_j^r = S^r$ for all $j = 1, \dots, n$. As R^i is an instance of $\mathcal{R}$, $\iota \exists_p^{R^i} S^i = \bigvee_{j=1}^n \exists p S_j^i$ and $\iota \forall_p^{R^i} S = \bigwedge_{j=1}^n (\exists p S_j^i \to \forall p S_j^i)$ by the definition of the standard interpolant assignment for focused rules.

We distinguish the case that $S^{rs} = \varnothing$ and $S^s \neq \varnothing$ from the case that this does not hold. In the first case, $S_j^{rs} = \varnothing$ holds for all premisses S_j because $S_j^r = S^r$, as observed in the previous paragraph. Hence $S^r \cdot (\exists p S_j^i \Rightarrow \forall p S_j^i \mid \varnothing)$ is in $\mathfrak{D}_R^p$. We show $\mathfrak{D}_R^p$ derives $S^r \cdot (\exists p S_j^i \Rightarrow \forall p S_j^i)$. For those j such that $S_j^s \neq \varnothing$, this holds by the definition of $S^r \cdot (\exists p S_j^i \Rightarrow \forall p S_j^i \mid \varnothing)$. And if $S_j^s = \varnothing$, then by definition $S^r \cdot (\exists p S_j^i \Rightarrow \forall p S_j^i \mid \varnothing) = (S^{ra}, \exists p S_j^i \Rightarrow)$, and thus $\mathfrak{D}_R^p$ derives $(S^{ra}, \exists p S_j^i \Rightarrow \chi)$ for any formula χ, in particular for $\chi = \forall p S_j^i$. This proves that also in the case $S_j^s = \varnothing$, $\mathfrak{D}_R^p \vdash S^r \cdot (\exists p S_j^i \Rightarrow \forall p S_j^i)$.

The above shows that $\mathfrak{D}_R^p \vdash (S^{ra} \Rightarrow \exists p S_j^i \to \forall p S_j^i)$ for all j. An application of Lemma 10 shows that $\mathfrak{D}_R^p \vdash S^r \cdot (\Rightarrow \forall_p^{R^i} S^i)$, which implies $\mathfrak{D}_R^p \vdash S^r \cdot (\exists_p^{R^i} S^i \Rightarrow \forall_p^{R^i} S^i)$. From Remark 1 we conclude that this implies (4).

We turn to the case that $S^{rs} \neq \varnothing$ or $S^s = \varnothing$, where the former implies $S_j^{rs} \neq \varnothing$ for all $j = 1, \dots, n$, as $S^r = S_j^r$, and the latter implies $S_j^s = \varnothing$, for all $j = 1, \dots, n$, by the definition of focused rules. Using that $S^r \cdot (\exists p S_j^i \Rightarrow)$ belongs to $\mathfrak{D}_R^p$, we conclude that $\mathfrak{D}_R^p$ derives $S^r \cdot (\exists_p^{R^i} S^i \Rightarrow)$. Again, Remark 1 implies that (4) holds. □

Lemma 12. *For the standard partition of any instance $R = (S_1 \dots S_n / S_0)$ of any focused rule $\mathcal{R}$ that is not an axiom and such that S_0^i is nonprincipal for R: $S_j^i = S_0^i$ for all $j = 1, \dots, n$ and for all sequents S,*

$$\{S_j^r \cdot S \mid j = 1, \dots, n\} \vdash_{\mathsf{IPC}}^{\mathcal{R}} S_0^r \cdot S.$$

Proof. As S_0^i does not contain the principal formula of R, R^r is an instance of $\mathcal{R}$ and $S_j^i = S_0^i$ for all $j = 1, \dots, n$ by Lemma 6. As $\mathcal{R}$ is focused, $(S \cdot S_1^r, \dots, S \cdot S_n^r / S \cdot S_0^r)$ is an instance of $\mathcal{R}$, which implies that what we had to show. □

Lemma 13. *For any focused rule $\mathcal{R}$ in a reductive ca calculus with a standard interpolant assignment, $(DPN)_{\mathcal{R}}^+$ holds.*[3]

Proof. Consider a sequent S_0 for which there exists a derivation of which the last inference is an instance $R = (S_1 \dots S_n / S_0)$ of $\mathcal{R}$ and let (S_0^r, S_0^i) be a partition of S_0 such that S_0^i is nonprincipal for R. We have to show that

$$\mathfrak{D}_R^p \vdash S_0^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing). \tag{5}$$

First consider the case that $\mathcal{R}$ is an axiom. If the axiom is of the form $(\Gamma, \bot \Rightarrow \Delta)$ or $(\Gamma \Rightarrow \top)$, then the fact that S_0^i is not instance of it implies that S^{ra} contains ⊥ or S^{rs} consists of ⊤. In both cases (5) holds. Therefore consider the remaining case that the axiom is of the form $(\Gamma, q \Rightarrow q)$. Since S_0^i is not instance of R, $(q \Rightarrow q)$ cannot be a subsequent of S_0^i. If $(q \Rightarrow q)$ is a subsequent of S_0^r, then $S_0^r \cdot (\exists p S_0^i \Rightarrow)$ is derivable, and we are done. If $(q \Rightarrow q)$ is neither a subsequent of S_0^i nor of S_0^r, either $q \in S_0^{ra}$, $S_0^{is} = \{q\}$, and $S_0^{rs} = \varnothing$, or

[3] $(\mathrm{DPN})_{\mathcal{R}}^+$ is the strengthening of $(\mathrm{DPN})_{\mathcal{R}}$ defined in Section 3.5.

$q \in S_0^{ia}$ and $S_0^{rs} = \{q\}$. Since S_0^r does not contain p we have that $q \neq p$. Hence $\vdash^{\mathcal{R}}_{\mathsf{IPC}} S_0^r \cdot (\exists^{at} p S_0^i \Rightarrow \forall^{at} p S_0^i)$ in the first case and $\vdash^{\mathcal{R}}_{\mathsf{IPC}} S_0^r \cdot (\exists^{at} p S_0^i \Rightarrow)$ in the second. As $\exists^{at} p S_0^i$ is a conjunct of $\exists p S_0^i$ and $\forall^{at} p S_0^i$ is a disjunct of $\forall p S_0^i$, this implies (5).

The case that $\mathcal{R}$ is not an axiom remains. We have to show that $S_0^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing)$ is derivable from $\mathfrak{D}_R^p$. By Lemma 12, the fact that S_0^i does not contain the principal formula of R implies that for the standard partition of R: $\{S \cdot S_j^r \mid j = 1, \dots, n\} \vdash^{\mathcal{R}}_{\mathsf{IPC}} S \cdot S_0^r$ for any S and $S_0^i = S_j^i$. Thus $\exists p S_j^i = \exists p S_0^i$ and $\forall p S_j^i = \forall p S_0^i$. Therefore, $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing)$ belongs to $\mathfrak{D}_R^p$.

If $S_0^{rs} = \varnothing$ and $S_0^s \neq \varnothing$, we have to show that $\mathfrak{D}_R^p \vdash S_0^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i)$. By the observation above for $S = (\exists p S_0^i \Rightarrow \forall p S_0^i)$, it suffices to show that $\mathfrak{D}_R^p \vdash S_j^r \cdot S$ for all j. The assumption on S_0 implies that $S_0^{is} = S_j^{is} \neq \varnothing$ for all j. Thus $S_j^{rs} = \varnothing$. Hence $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing) = S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i)$, which proves that $\mathfrak{D}_R^p \vdash S_j^r \cdot S$.

If $S_0^{rs} \neq \varnothing$ or $S_0^s = \varnothing$, we have to show that $\mathfrak{D}_R^p \vdash S_0^r \cdot (\exists p S_0^i \Rightarrow)$. By the observation above, for $S = (\exists p S_0^i \Rightarrow)$, it suffices to show that $\mathfrak{D}_R^p \vdash S_j^r \cdot S$ for all j. Since $\mathcal{R}$ is focused, R is of the form

$$\frac{S_1 \quad \dots \quad S_n}{S_0} = \frac{S \cdot S_1' \quad \dots \quad S \cdot S_n'}{S \cdot S_0'}$$

where S_0' consists of one formula and either $S_j'^{\,a} = \varnothing$ for all $j = 0, \dots, n$ or $S_j'^{\,s} = \varnothing$ for all $j = 0, \dots, n$. Therefore, if $S_0^s = \varnothing$, then $S_j^s = \varnothing$ for all j. Hence $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing) = S_j^r \cdot (\exists p S_0^i \Rightarrow)$ belongs to $\mathfrak{D}_R^p$, and we are done. If $S_0^{rs} \neq \varnothing$, then $S_0^{is} = S_j^{is} = \varnothing$. We distinguish the cases that $\mathcal{R}$ is a right and a left rule. If $\mathcal{R}$ is a right rule, none of the $S_j'^{\,s}$ is empty. Thus the S_j^{rs} are all not empty, and $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing) = S_j^r \cdot (\exists p S_0^i \Rightarrow)$ for all j, which is what we had to show. If, on the other hand, $\mathcal{R}$ is a left rule, then $S_0'^{\,s} = \varnothing$, which implies that $S_0^{rs} \subseteq S^s$. Hence $S_j^{rs} \neq \varnothing$ by the definition of the standard partition, and again $S_j^r \cdot (\exists p S_0^i \Rightarrow \forall p S_0^i \mid \varnothing) = S_j^r \cdot (\exists p S_0^i \Rightarrow)$ for all j follows. □

Theorem 14. *A logic* L *with a reductive ca calculus with a standard interpolant assignment that is sound with respect to all rules that are not focused, has uniform interpolation.*

Proof. By Theorem 5 it suffices to prove that the interpolant assignment is sound. This follows from Lemmas 7, 9, 11, 13. □

Corollary 15. *A logic* L *with a reductive ca calculus that consists of focused rules only, has uniform interpolation.*

Corollary 16. *Any intermediate logic that does not have uniform interpolation cannot have a reductive ca calculus consisting of focused rules.*

We do not know whether there are examples of interesting logics that have calculi that consist solely of focused rules. In the classical setting [17] we use a notion of focused[4] that extends the one in this paper and is such that all rules of G3p are focused. The same does not hold for G3ip, but in the next section we show that IPC has a calculus consisting of focused and nonfocused rules, for which there exists a sound interpolant assignment. This brings us to the class of logics that we set out to study: the intermediate and intuitionistic modal logics.

[4] In that paper under the name *focused*.

$$\Gamma, q \Rightarrow q \quad \text{At} \quad (q \text{ an atom}) \qquad\qquad \Gamma, \bot \Rightarrow \Delta \quad \text{L}\bot$$

$$\frac{\Gamma \Rightarrow \varphi \quad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}\ \text{R}\wedge \qquad\qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \text{L}\wedge$$

$$\frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_0 \vee \varphi_1}\ \text{R}\vee\ (i=0,1) \qquad\qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \text{L}\vee$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}\ \text{R}\!\to$$

$$\frac{\Gamma, q, \psi \Rightarrow \Delta}{\Gamma, q, q \to \psi \Rightarrow \Delta}\ \text{L}_1\!\to (q \text{ an atom}) \qquad \frac{\Gamma, \varphi \to (\psi \to \gamma) \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \to \gamma \Rightarrow \Delta}\ \text{L}_2\!\to$$

$$\frac{\Gamma, \varphi \to \gamma, \psi \to \gamma \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \to \gamma \Rightarrow \Delta}\ \text{L}_3\!\to \qquad \frac{\Gamma, (\psi \to \gamma) \Rightarrow \varphi \to \psi \quad \gamma, \Gamma \Rightarrow \Delta}{\Gamma, (\varphi \to \psi) \to \gamma \Rightarrow \Delta}\ \text{L}_4\!\to$$

Fig. 1. The Gentzen calculus G4ip. In all L-rules $|\Delta| \leq 1$. In all rules except $\text{L}_1\!\to$ and the axioms, the formula displayed in the conclusion is the *principal* formula. In $\text{L}_1\!\to$ both formulas q and $q \to \psi$ in the conclusion are *principal*. In axiom At both q's are *principal*, and $\bot$ is *principal* in $\text{L}\bot$.

$$\frac{\Gamma \Rightarrow \psi \quad \Pi, \psi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Delta}\ \text{Cut} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \text{LW}$$

Fig. 2. The rules Cut and Left Weakening (LW), which do not belong to G4ip but are admissible in it.

5. Intuitionistic logic

As a first application of the method developed thus far we establish that intuitionistic propositional logic has uniform interpolation, a fact first proved by Pitts [22]. We use the same calculus as Pitts does in his article, the calculus G4ip developed independently by Dyckhoff [10] and Hudelmaier [15,16] and given in Fig. 1. The calculus has no structural rules, but they are admissible in it, as is the cut rule. Recall that sequents are assumed to have at most one formula in the succedent. Thus $|\Delta| \leq 1$ for the Δ in Fig. 1.

The interpolant assignments for the nonfocused rules $\text{R}\!\to, \text{L}_1\!\to, \text{L}_4\!\to$ of G4ip as defined in the proof of Theorems 17 are called the *standard* interpolant assignments for these rules.

Theorem 17. *For any extension of the calculus* G4ip *there exists for any of the rules of* G4ip *a sound interpolant assignment that is standard for focused rules.*

Proof. As explained in Section 3.7, we have to extend the standard interpolant assignment to the rules in the calculus that are not focused and show that the assignment is sound with respect to the new rules, that is, that any new rule $\mathcal{R}$ satisfies the six inductive properties (Section 3.5).

The three rules in question are $\text{R}\!\to$, $\text{L}_1\!\to$ and $\text{L}_4\!\to$: for $\text{R}\!\to$ ($\text{L}_4\!\to$) the requirement for focused rules that in right (left) rules the antecedents (succedents) of the sequents should be empty is violated, and in $\text{L}_1\!\to$ the requirement that S_0 consists of one formula is violated. For all three rules the assignment $\iota\forall\overset{\mathcal{R}}{p}S \equiv_{df} \bot$ is as for focused rules, Section 4.3, and $(\text{IPN})^{\forall}_{\mathcal{R}}$ is easily seen to hold. Assignments of the form $\iota\exists\overset{\mathcal{R}}{p}S$, $\iota\forall\overset{R}{p}S$, $\iota\exists\overset{R}{p}S$ are defined as follows, where we first treat $\text{R}\!\to$, then $\text{L}_1\!\to$, and then $\text{L}_4\!\to$.

Suppose $\mathcal{R} = \text{R}\!\to$. For an instance $R = (S_1/S) = (\Gamma, \varphi \Rightarrow \psi)/(\Gamma \Rightarrow \varphi \to \psi)$ of $\mathcal{R}$ define $\iota\exists\overset{\mathcal{R}}{p}S \equiv_{df} \top$ as for focused rules and furthermore

$$\begin{array}{lll} \iota\exists\overset{R}{p}S \equiv_{df} \varphi \to \exists p S_1 & \iota\forall\overset{R}{p}S \equiv_{df} \varphi \to \forall p S_1 & \text{if } p \text{ does not occur in } \varphi \\ \iota\exists\overset{R}{p}S \equiv_{df} \top & \iota\forall\overset{R}{p}S \equiv_{df} \exists p S_1 \to \forall p S_1 & \text{if } p \text{ occurs in } \varphi. \end{array}$$

Clearly, $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$ holds. We have to prove the remaining four properties.

$(\mathrm{IPP})^{\exists}_{\mathcal{R}}$ We have to show that $\mathfrak{I}^p_R$ derives $\Gamma \Rightarrow \exists\overset{R}{p}S$. The case that p occurs in φ is trivial. If p does not occur in φ, then we use that $\mathfrak{I}^p_R$ contains $(\Gamma, \varphi \Rightarrow \exists p S_1)$, and thus derives $(\Gamma \Rightarrow \exists\overset{R}{p}S)$.

$(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ We have to show that $\mathfrak{I}^p_R$ derives $(\Gamma, \forall p S \Rightarrow \varphi \to \psi)$, for which we use that $(\Gamma, \varphi, \forall p S_1 \Rightarrow \psi)$ belongs to $\mathfrak{I}^p_R$. If p occurs in φ, we use that $\mathfrak{I}^p_R$ contains $(\Gamma, \varphi \Rightarrow \exists p S_1)$, and therefore derives sequents $(\Gamma, \varphi, \exists p S_1 \to \forall p S_1 \Rightarrow \psi)$ and $(\Gamma, \exists p S_1 \to \forall p S_1 \Rightarrow \varphi \to \psi)$. If p does not occur in φ, then we use that $\mathfrak{I}^p_R$ derives $(\Gamma, \varphi, \varphi \to \forall p S_1 \Rightarrow \psi)$, and thus also $(\Gamma, \varphi \to \forall p S_1 \Rightarrow \varphi \to \psi)$.

For $(\mathrm{DPP})_{\mathcal{R}}$ and $(\mathrm{DPN})_{\mathcal{R}}$, consider a derivation of S of which the last inference is an instance $R = (S_1/S) = (\Gamma, \varphi \Rightarrow \psi)/(\Gamma \Rightarrow \varphi \to \psi)$ of $\mathcal{R}$, and let (S^r, S^i) be a partition of S.

$(\mathrm{DPP})_{\mathcal{R}}$ Suppose that S^i is principal for R. Choose the p–partition (S^r_1, S^i_1) of S_1 for which $S^r_1 = S^r$. It suffices to show that

$$S^r_1 \cdot (\exists p S^i_1 \Rightarrow \forall p S^i_1 \mid \varnothing) \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing).$$

Since S^i contains the principal formula of R, S^{is} consists of $\varphi \to \psi$, and thus S^{rs} is empty. Let R^i be instance (S^i_1/S^i) of $\mathcal{R}$. Hence $\exists\overset{R^i}{p}S^i$ and $\forall\overset{R^i}{p}S^i$ are a conjunct and a disjunct of $\exists p S^i$ and $\forall p S^i$, respectively. Thus it suffices to show that

$$S^{ra}_1, \exists p S^i_1 \Rightarrow \forall p S^i_1 \vdash S^{ra}, \exists\overset{R^i}{p}S^i \Rightarrow \forall\overset{R^i}{p}S^i.$$

In case p does not occur in φ, the above clearly holds. In the other case, note that the left side derives $(S^{ra}_1 \Rightarrow \exists p S^i_1 \to \forall p S^i_1)$. Since $S^r_1 = S^r$, it also derives $(S^{ra}, \exists\overset{R^i}{p}S^i \Rightarrow \exists p S^i_1 \to \forall p S^i_1)$, which implies $(S^{ra}, \exists\overset{R^i}{p}S^i \Rightarrow \forall\overset{R^i}{p}S^i)$, which is what had to be shown.

$(\mathrm{DPN})_{\mathcal{R}}$ Suppose that S^i is nonprincipal for R. Assume that all sequents lower than S satisfy the interpolant properties. We have to show that

$$\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing).$$

Since S^i does not contain the principal formula of R, S^{is} is empty and S^{rs} consists of $\varphi \to \psi$. Hence p cannot occur in φ. Let (S^r_1, S^i_1) be the corresponding partition of S_1 such that $S^i_1 = S^i$. It suffices to show $\vdash S^{ra}, \exists p S^i \Rightarrow \varphi \to \psi$, which follows from $\vdash S^{ra}_1, \exists p S^i_1, \varphi \Rightarrow \psi$, which again follows from the assumption about sequents lower than S and $S^i_1 = S^i$.

We turn to the case $\mathcal{R} = \mathrm{L}_1\!\to$. For $R = (S_1/S) = (\Gamma, q, \psi \Rightarrow \Delta)/(\Gamma, q, q \to \psi \Rightarrow \Delta)$ an instance of $\mathcal{R}$, define

$$\begin{array}{lll} \iota\forall\overset{R}{p}S \equiv_{df} q \to \forall p S_1 & \iota\exists\overset{R}{p}S \equiv_{df} q \wedge \exists p S_1 & \text{if } q \neq p \\ \iota\forall\overset{R}{p}S \equiv_{df} \forall p S_1 & \iota\exists\overset{R}{p}S \equiv_{df} \exists p S_1 & \text{if } q = p \\ \iota\exists\overset{\overline{\mathcal{R}}}{p}S \equiv_{df} \bigwedge\{q \in S^a \mid q \neq p\}. & & \end{array}$$

It is clear that $(\mathrm{IPN})^{\exists}_{\mathcal{R}}$ holds. We prove the remaining four inductive properties.

We treat $(\mathrm{IPP})^{\exists}_{\mathcal{R}}$ and leave $(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ to the reader. To show $\mathfrak{I}^p_R \vdash \Gamma, q, q \to \psi \Rightarrow \exists\overset{R}{p}S$, note that $\mathfrak{I}^p_R$ contains $(\Gamma, q, \psi \Rightarrow \exists p S_1)$. Hence $\mathfrak{I}^p_R \vdash \Gamma, q, q \to \psi \Rightarrow q \wedge \exists p S_1$, which is what we had to show.

For $(\mathrm{DPP})_{\mathcal{R}}$ and $(\mathrm{DPN})_{\mathcal{R}}$, consider a derivation of S of which the last inference is an instance $R = (S_1/S) = (\Gamma, q, \psi \Rightarrow \Delta)/(\Gamma, q, q \to \psi \Rightarrow \Delta)$ of $\mathcal{R}$, and let (S^r, S^i) be a partition of S.

$(\mathrm{DPP})_{\mathcal{R}}$ Suppose that S^i is principal for R. Thus $S^i = (\Pi, q, q \to \psi \Rightarrow \Sigma)$ for some multisets Π, Σ. Choose the p–partition (S^r_1, S^i_1) of S_1 for which $S^r_1 = S^r$. Thus $S^i_1 = (\Pi, q, \psi \Rightarrow \Sigma)$. It suffices to show that

$$S^r_1 \cdot (\exists p S^i_1 \Rightarrow \forall p S^i_1 \mid \varnothing) \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing).$$

Let R^i denote the instance S_1^i/S^i of $\mathcal{R}$ and note that $\exists \overset{R^i}{p} S^i$ and $\forall \overset{R^i}{p} S^i$ are a conjunct and a disjunct of $\exists p S^i$ and $\forall p S^i$, respectively. Thus it suffices to show that

$$S^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing) \vdash S^r \cdot (\exists \overset{R^i}{p} S^i \Rightarrow \forall \overset{R^i}{p} S^i \mid \varnothing).$$

As R^i is an instance of $\mathrm{L}_1{\rightarrow}$, the definition of $\exists \overset{R^i}{p} S^i$ and $\forall \overset{R^i}{p} S^i$ implies the above, both in the case that $q = p$ and that $q \neq p$.

(DPN)$_{\mathcal{R}}$ Suppose that S^i is nonprincipal for R. We have to show that under the assumption that all sequents lower than S satisfy the interpolant properties we have:

$$\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing).$$

As S^i does not contain all principal formulas of R, at least one of q and $q \to \psi$ belongs to S^{ra}. As we consider a p–partition, $q \neq p$. We distinguish three cases.

If both q and $q \to \psi$ belong to S^{ra}, then $S^i = S_1^i$ and $S^r = (\Gamma_1, q, q \to \psi \Rightarrow \Delta_1)$ and $S_1^r = (\Gamma_1, q, \psi \Rightarrow \Delta_1)$ for certain multisets Γ_1, Δ_1. Clearly,

$$\frac{S_1^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing)}{S^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing)}$$

is an application of $\mathcal{R}$. The premiss is derivable because the interpolant properties hold for S_1 by assumption. Hence the conclusion is derivable too. Since $S^i = S_1^i$, this proves the desired.

If $q \to \psi$ belongs to S^{ra} but q does not, then $S^r = (\Gamma_1, q \to \psi \Rightarrow \Delta_1)$ and $S^i = (\Gamma_2, q \Rightarrow \Delta_2)$ for certain multisets $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$. Consider the partition of S_1 given by $S_1^r = (\Gamma_1, \psi \Rightarrow \Delta_1)$ and $S_1^i = S^i$. Let Σ be such that $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing) = S^r \cdot (\exists p S^i \Rightarrow \Sigma)$. We have to show that $\vdash S^r \cdot (\exists p S^i \Rightarrow \Sigma)$. We have $S_1^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing) = S_1^r \cdot (\exists p S_1^i \Rightarrow \Sigma)$ because $S_1^{rs} = S^{rs}$ and $S_1^s = S^s$. By assumption, $S_1^r \cdot (\exists p S_1^i \Rightarrow \Sigma)$ is derivable. Therefore sequent $(\Gamma_1, q, \psi \Rightarrow \Delta_1) \cdot (\exists p S_1^i \Rightarrow \Sigma)$ is derivable too. An application of $\mathcal{R}$ proves that $S^r \cdot (\exists p S_1^i, q \Rightarrow \Sigma)$ is derivable. Since $q \in S_1^{ia}$ and $q \neq p$, q is a conjunct of $\exists \overset{\overline{\mathcal{R}}}{p} S_1^i$, which is a conjunct of $\exists p S_1^i$. Thus $S^r \cdot (\exists p S_1^i \Rightarrow \Sigma)$ is derivable. Together with $S_1^i = S^i$, this establishes what we had to show.

If q belongs to S^{ra} but $q \to \psi$ does not, then we have $S^r = (\Gamma_1, q \Rightarrow \Delta_1)$ and $S^i = (\Gamma_2, q \to \psi \Rightarrow \Delta_2)$ for certain multisets $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$. Consider the partition of S_1 given by $S_1^r = S^r$ and $S_1^i = (\Gamma_2, \psi \Rightarrow \Delta_2)$. Let Σ, Σ_1 be such that $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing) = S^r \cdot (\exists p S^i \Rightarrow \Sigma)$ and $S_1^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i \mid \varnothing) = S_1^r \cdot (\exists p S_1^i \Rightarrow \Sigma_1)$, respectively. As $S_1^{rs} = S^{rs}$ and $S_1^s = S^s$, we have $\Sigma = \{\forall p S^i\}$ and $\Sigma_1 = \{\forall p S_1^i\}$, or $\Sigma = \Sigma_1 = \Delta_1$. We have to show that $\vdash S^r \cdot (\exists p S^i \Rightarrow \Sigma)$. By assumption, $S_1^r \cdot (\exists p S_1^i \Rightarrow \Sigma_1)$ is derivable. The definition of $\exists^{at} p S^i$ shows that $\exists p S^i$ implies the formula $q \to \exists p S_1^i$. Together with the fact that $q \in S^{ra}$ and $S^r = S_1^r$, we obtain the derivability of $S^r \cdot (\exists p S^i \Rightarrow \Sigma_1)$. In the case that $\Sigma_1 = \Sigma$, we are done. In the case that $\Sigma = \{\forall p S^i\}$ and $\Sigma_1 = \{\forall p S_1^i\}$, the definition of $\forall^{at} p S^i$ shows that $q \wedge \forall p S_1^i$ implies $\forall p S^i$. As $q \in S^{ra}$, it follows that $S^r \cdot (\exists p S^i \Rightarrow \Sigma)$ is derivable in this case as well.

We turn to the case that $\mathcal{R} = \mathrm{L}_4{\rightarrow}$. For an instance

$$\frac{S_1 \quad S_2}{S} = \frac{\Gamma, \psi \to \gamma \Rightarrow \varphi \to \psi \quad \Gamma, \gamma \Rightarrow \Delta}{\Gamma, (\varphi \to \psi) \to \gamma \Rightarrow \Delta} \; R$$

of $\mathcal{R}$ define

$$\iota \forall \overset{R}{p} S \equiv_{df} \bigwedge_{i=1}^{2} (\exists p S_i \to \forall p S_i)$$

$$\iota \exists \overset{R}{p} S \equiv_{df} \exists p S_1 \wedge (\forall p S_1 \to \exists p S_2)$$

$$\iota\exists\overset{\overline{\mathcal{R}}}{p}S \equiv_{df} \begin{cases} \top & \text{if } S^s = \varnothing \\ \bigwedge\{\exists p(\Pi \Rightarrow\,) \mid \Pi \subseteq S^a\} & \text{if } S^s \neq \varnothing. \end{cases}$$

Note that $\iota\exists\overset{\overline{\mathcal{R}}}{p}S$ and $\iota\exists\overset{R}{p}S$ are well–defined since $(\Pi \Rightarrow\,) \prec S$ for all $\Pi \subseteq S^a$ in case $S^s \neq \varnothing$. Since $(\mathrm{IPN})^{\forall}_{\mathcal{R}}$ has been treated at the beginning of the proof, we have to prove the remaining five inductive properties.

$(\mathrm{IPP})^{\exists}_{\mathcal{R}}$ We have to prove that $\mathfrak{I}^p_R$ derives $(S^a, \forall pS_1 \Rightarrow \exists pS_2)$ and $(S^a \Rightarrow \exists pS_1)$. We use the obvious fact that $\vdash \bigwedge S^a \to \bigwedge S^a_1$. That $\mathfrak{I}^p_R$ derives $(S^a \Rightarrow \exists pS_1)$ thus follows from the fact that $(S^a_1 \Rightarrow \exists pS_1)$ belongs to $\mathfrak{I}^p_R$. For the other case, the fact that $\mathfrak{I}^p_R$ contains $(S^a_1, \forall pS_1 \Rightarrow S^s_1)$ implies that it derives $S^a, \forall pS_1 \Rightarrow S^s_1$. Since $S^a, S^s_1 \vdash \bigwedge S^a_2$ and $\mathfrak{I}^p_R$ contains $(S^a_2 \Rightarrow \exists pS_2)$, it follows that $\mathfrak{I}^p_R \vdash (S^a, \forall pS_1 \Rightarrow \exists pS_2)$.

$(\mathrm{IPP})^{\forall}_{\mathcal{R}}$ We have to prove that $\mathfrak{I}^p_R$ derives $(S^a, \{\exists pS_i \to \forall pS_i \mid i = 1, 2\} \Rightarrow S^s)$. As the previous case showed that $\mathfrak{I}^p_R$ derives $(S^a \Rightarrow \exists pS_1)$ and $(S^a, \forall pS_1 \Rightarrow \exists pS_2)$, it suffices to prove that $\mathfrak{I}^p_R \vdash S^a, \forall pS_1, \forall pS_2 \Rightarrow S^s$. Since $\mathfrak{I}^p_R$ contains $(S^a_j, \forall pS_j \Rightarrow S^s_j)$ for $j = 1, 2$ and $\vdash \bigwedge S^a \to \bigwedge S^a_1$, $\mathfrak{I}^p_R$ derives $S^a, \forall pS_1 \Rightarrow S^s_1$. As $S^a, S^s_1 \vdash \bigwedge S^a_2$, $\mathfrak{I}^p_R$ also derives $S^a, \forall pS_1, \forall pS_2 \Rightarrow S^s_2$, that is, $S^a, \forall pS_1, \forall pS_2 \Rightarrow S^s$.

$(\mathrm{IPN})^{\exists}_{\mathcal{R}}$ We have to show that under the assumption that all sequents lower than S satisfy the interpolant properties we have $\vdash S^a \Rightarrow \exists\overset{\overline{\mathcal{R}}}{p}S$. If $S^s \neq \varnothing$, then $(\Pi \Rightarrow\,) \prec S$ for all $\Pi \subseteq S^a$. Therefore $\vdash \Pi \Rightarrow \exists p(\Pi \Rightarrow\,)$ and thus $\vdash S^a \Rightarrow \exists p(\Pi \Rightarrow\,)$.

For $(\mathrm{DPP})_{\mathcal{R}}$ and $(\mathrm{DPN})_{\mathcal{R}}$, consider a derivation of S of which the last inference is an instance $R = (S_1\ S_2/S)$ of $\mathcal{R}$, and let (S^r, S^i) be a partition of S.

$(\mathrm{DPP})_{\mathcal{R}}$ Suppose S^i is principal for R. We have to show that

$$\mathfrak{D}^p_R \vdash S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing). \tag{6}$$

Consider the following partition of R:

$$S^i = (\Gamma^i, (\varphi \to \psi) \to \gamma \Rightarrow \Delta^i) \quad S^i_1 = (\Gamma^i, \psi \to \gamma \Rightarrow \varphi \to \psi) \quad S^i_2 = (\Gamma^i, \gamma \Rightarrow \Delta^i)$$
$$S^r = S^r_2 = (\Gamma^r \Rightarrow \Delta^r) \quad S^r_1 = (\Gamma^r \Rightarrow\,).$$

As $S^s_1 \neq \varnothing = S^{rs}_1$, $\mathfrak{D}^p_R$ contains $(\Gamma^r, \exists pS^i_1 \Rightarrow \forall pS^i_1)$. Sequent $S^r_2 \cdot (\exists pS^i_2 \Rightarrow \forall pS^i_2 \mid \varnothing)$ belongs to $\mathfrak{D}^p_R$ too. Let R^i be the instance $(S^i_1\ S^i_2/S^i)$ of $\mathcal{R}$.

First we treat the case that $S^s \neq \varnothing$ and $S^{rs} = \varnothing$. We have to show that $\mathfrak{D}^p_R$ derives $S^r \cdot (\exists pS^i \Rightarrow \forall pS^i) = (\Gamma^r, \exists pS^i \Rightarrow \forall pS^i)$. As $S^r = S^r_2$ and $S^{rs} = S^{rs}_2$, $S^r \cdot (\exists pS^i_2 \Rightarrow \forall pS^i_2 \mid \varnothing)$ is equal to $(\Gamma^r, \exists pS^i_2 \Rightarrow \forall pS^i_2)$. Thus $(\Gamma^r, \exists pS^i_1 \Rightarrow \forall pS^i_1)$ and $(\Gamma^r, \exists pS^i_2 \Rightarrow \forall pS^i_2)$ belong to $\mathfrak{D}^p_R$. Since $\forall\overset{R^i}{p}S^i = (\exists pS^i_1 \to \forall pS^i_1) \wedge (\exists pS^i_2 \to \forall pS^i_2)$ is a disjunct of $\forall pS^i$, we have that $\mathfrak{D}^p_R$ derives $(\Gamma^r \Rightarrow \forall pS^i)$, and therefore also $(\Gamma^r, \exists pS^i \Rightarrow \forall pS^i)$.

Second, we treat the case that $S^s = \varnothing$ or $S^{rs} \neq \varnothing$. We have to show that $\mathfrak{D}^p_R$ derives $S^r \cdot (\exists pS^i \Rightarrow \Delta^r) = (\Gamma^r, \exists pS^i \Rightarrow \Delta^r)$. Because $S^r = S^r_2$ and $S^{rs} = S^{rs}_2$, we have $S^r \cdot (\exists pS^i_2 \Rightarrow \forall pS^i_2 \mid \varnothing) = (\Gamma^r, \exists pS^i_2 \Rightarrow \Delta^r)$. Thus both $(\Gamma^r, \exists pS^i_1 \Rightarrow \forall pS^i_1)$ and $(\Gamma^r, \exists pS^i_2 \Rightarrow \Delta^r)$ belong to $\mathfrak{D}^p_R$. Because $\exists pS^i$ has conjunct $\exists\overset{R^i}{p}S^i$, which has conjuncts $\exists pS^i_1$ and $\forall pS^i_1 \to \exists pS^i_2$, we have that $\mathfrak{D}^p_R$ derives $(\Gamma^r, \exists pS^i \Rightarrow \Delta^r)$, which is what we had to show.

$(\mathrm{DPN})_{\mathcal{R}}$ Suppose that S^i is nonprincipal for R. We show that under the assumption that all sequents lower than S satisfy the interpolant properties, $S^r \cdot (\exists pS^i \Rightarrow \forall pS^i \mid \varnothing)$ is derivable. Consider the following partition of R, where $S^r = (\Gamma^r, (\varphi \to \psi) \to \gamma \Rightarrow \Delta^r)$ and $S^i = (\Gamma^i \Rightarrow \Delta^i)$:

$$S^r_1 = (\Gamma^r, \psi \to \gamma \Rightarrow \varphi \to \psi) \quad S^r_2 = (\Gamma^r, \gamma \Rightarrow \Delta^r) \quad S^i_1 = (\Gamma^i \Rightarrow\,) \quad S^i_2 = S^i = (\Gamma^i \Rightarrow \Delta^i).$$

First, we treat the case that $S^s \neq \varnothing$ and $S^{rs} = \varnothing$. Thus we have to show that sequent $S^r \cdot (\exists pS^i \Rightarrow \forall pS^i)$, which is equal to $(\Gamma^r, \exists pS^i \Rightarrow \forall pS^i)$, is derivable. As $S^{rs} = S^{rs}_2$ and $S^s = S^s_2$, sequent $S^r \cdot (\exists pS^i_2 \Rightarrow \forall pS^i_2 \mid \varnothing)$ is equal to $(\Gamma^r, \exists pS^i_2 \Rightarrow \forall pS^i_2)$. Since the sequents $(\Gamma^r, \psi \to \gamma, \exists pS^i_1 \Rightarrow \varphi \to \psi)$ and $(\Gamma^r, \gamma, \exists pS^i_2 \Rightarrow \forall pS^i_2)$ are derivable by assumption, $\mathcal{R}$ can be applied to them, showing the derivability of

$$\Gamma^r, (\varphi \to \psi) \to \gamma, \exists p S_1^i, \exists p S_2^i \Rightarrow \forall p S_2^i.$$

As $S_2^i = S^i$, it follows that $\Gamma^r, (\varphi \to \psi) \to \gamma, \exists p S_1^i, \exists p S^i \Rightarrow \forall p S^i$ is derivable. Thus it suffices to show that $\exists p S^i$ derives $\exists p S_1^i$. In case Δ^i is empty, $S_1^i = S^i$, and we are done. In case Δ^i is not empty, S^s is not empty, and thus $\exists p S_1^i = \exists p(\Gamma^i \Rightarrow)$ is a conjunct of $\exists \overset{\mathcal{R}}{p} S^i$, which is a conjunct of $\exists p S^i$. Thus also in this case $\exists p S^i$ derives $\exists p S_1^i$. □

Note that for the above result one cannot use the propositional part of Gentzen's LK or other calculi that contain the Cut Rule, as it is not clear that such calculi are reductive.

Theorems 14 and 17 and the fact that G4ip is a reductive calculus in which Cut is admissible [18] imply the following.

Corollary 18. *[22] Intuitionistic propositional logic has uniform interpolation.*

Since IPC, Sm, LC, GSc, KC, Bd_2, CPC are the only intermediate logics with uniform interpolation, the contraposition of Theorem 17 and Theorem 14 imply the following.

Corollary 19. *If an intermediate logic is not equal to one of the seven logics* IPC, Sm, LC, GSc, KC, Bd_2, CPC*, then it does not have a reductive ca calculus that is an extension of* G4ip *by focused rules.*

6. Intuitionistic modal logic

In this and the next section we extend the method developed thus far to intuitionistic modal logics by extending the class of rules to which Theorem 14 applies. To this end we first develop a reductive calculus based on G4ip for the intuitionistic normal modal logic iK without the diamond operator. Let $\mathsf{G4iK}_\Box$ be the calculus G4ip but then for the language of propositional modal logic, extended by the following two rules:

$$\frac{\Gamma \Rightarrow \varphi}{\Pi, \Box\Gamma \Rightarrow \Box\varphi}\ \mathcal{R}_K \qquad \frac{\Gamma \Rightarrow \varphi \quad \Pi, \Box\Gamma, \psi \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Delta}\ \mathrm{L}_\Box\!\to$$

The *principal formula* in $\mathcal{R}_K$ is $\Box\varphi$ and in $\mathrm{L}\Box\!\to$ it is $\Box\varphi \to \psi$. Note that $\mathsf{G4iK}_\Box$ again is a reductive calculus in the order on sequents defined in Section 2.5. The following are two well–known modal rules.

$$\frac{\Gamma, \varphi \Rightarrow}{\Pi, \Box\Gamma, \Box\varphi \Rightarrow \Delta}\ \mathcal{R}_D \qquad \frac{\boxdot\Gamma \Rightarrow \varphi}{\Pi, \Box\Gamma \Rightarrow \Box\varphi}\ \mathcal{R}_{K4}$$

The calculus $\mathsf{G3iK}_\Box$ consists of G3ip (for the language of modal logic) plus the rule $\mathcal{R}_K$, and $\mathsf{G3iKD}_\Box$ is $\mathsf{G3iK}_\Box$ plus the rules $\mathcal{R}_D$. The calculus $\mathsf{G4iKD}_\Box$ is the extension of $\mathsf{G4iK}_\Box$ by $\mathcal{R}_D$.

In [18] it is shown that for $\mathsf{X} \in \{\mathsf{K}, \mathsf{KD}\}$, the calculi $\mathsf{G3iX}_\Box$ and $\mathsf{G4iX}_\Box$ are equivalent. Section 8 discusses the other names under which these logics occur in the literature.

Theorem 20. *[18] For $\mathsf{X} \in \{\mathsf{K}, \mathsf{KD}\}$: $\mathsf{G4iX}_\Box$ is a reductive sequent calculus (with respect to the order in Example 1) in which Cut, Left Weakening, and Left and Right Contraction are admissible.*

6.1. Interpolant assignment for $\mathrm{L}_\Box\!\to$

Before considering other modal rules, we extend the interpolant assignment to the new implication rule $\mathrm{L}_\Box\!\to$. For this, we first define the *standard partition* of the rule. Given an instance R of $\mathrm{L}_\Box\!\to$

$$\frac{S_1 \qquad S_2}{S} = \frac{\Gamma \Rightarrow \varphi \qquad \Pi, \Box\Gamma, \psi \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Delta} \tag{7}$$

and a partition $S = (S^r, S^i)$ of the conclusion, the standard partition is defined as follows.

$$\begin{array}{lll} S_1^i = (\Gamma^i \Rightarrow \varphi) & S_2^i = (\Pi^i, \Box\Gamma^i, \psi \Rightarrow \Delta^i) & \text{if } S^{ia} = (\Pi^i, \Box\Gamma^i, \Box\varphi \to \psi \Rightarrow \Delta^i) \\ S_1^i = (\Gamma^i \Rightarrow \ ) & S_2^i = S^i & \text{if } \Box\varphi \to \psi \notin S^{ia} = (\Pi^i, \Box\Gamma^i \Rightarrow \Delta^i). \end{array}$$

Given such a partition, R^i and R^r denote $S_1^i \ S_2^i/S^i$ and $S_1^r \ S_2^r/S^r$, respectively. The following lemma is easy to prove.

Lemma 21. *For any instance (7) of $\mathcal{R} = \mathrm{L}_\Box\!\to$ and any partition (S^r, S^i) of S, for the standard partition of R, R^i is an instance of $\mathcal{R}$ if the principal formula of R belongs to S^i, and R^r is an instance of $\mathcal{R}$ otherwise.*

For an instance R of (7) and for $\mathcal{R}$ denoting $\mathrm{L}_\Box\!\to$, the *standard* interpolant assignment is defined as follows.

$$\begin{aligned} \iota\exists\overset{R}{p}S &\equiv_{df} \Box\exists p S_1 \wedge (\Box\forall p S_1 \to \exists p S_2) \wedge \bigwedge\{\Box\exists p(\Sigma \Rightarrow\ ) \mid \Sigma \subseteq S_1^a\} \\ \iota\forall\overset{R}{p}S &\equiv_{df} \Box\forall p S_1 \wedge \forall p S_2 \\ \iota\forall\overset{\overline{\mathcal{R}}}{p}S &\equiv_{df} \bot \\ \iota\exists\overset{\overline{\mathcal{R}}}{p}S &\equiv_{df} \Box\exists p(S_\Box^a \Rightarrow\ ). \end{aligned}$$

6.2. Soundness of the interpolant assignment for $\mathrm{L}_\Box\!\to$

Lemma 22. *(IPP)$_{\mathcal{R}}^{\exists}$ and (IPN)$_{\mathcal{R}}^{\exists}$ hold for $\mathcal{R} = \mathrm{L}_\Box\!\to$ in any extension of* $\mathsf{G4iK}_\Box$*.*

Proof. For (IPP)$_{\mathcal{R}}^{\exists}$, consider an instance R as in (7). For conjuncts of $\exists\overset{R}{p}S$ of the form $\Box\exists p(\Sigma \Rightarrow\ )$ for some $\Sigma \subseteq S_1^a$, note that $(\Sigma \Rightarrow \exists p(\Sigma \Rightarrow))$ belongs to $\mathfrak{I}_R^p$ by definition, as $(\Box\Sigma \Rightarrow\ ) \subseteq S$. An application of $\mathcal{R}_K$ shows that $\mathfrak{I}_R^p$ derives $(\Box\Sigma \Rightarrow \Box\exists p(\Sigma \Rightarrow))$, and thus also $(S^a \Rightarrow \Box\exists p(\Sigma \Rightarrow))$.

For the conjunct $\Box\exists p S_1$ of $\exists\overset{R}{p}S$, note that $\mathfrak{I}_R^p$ contains $(\Gamma \Rightarrow \exists p S_1)$. An application of $\mathcal{R}_K$ gives $\mathfrak{I}_R^p \vdash (\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Box\exists p S_1)$. For the conjunct $(\Box\forall p S_1 \to \exists p S_2)$ of $\exists\overset{R}{p}S$, note that $\mathfrak{I}_R^p$ contains $(\Gamma, \forall p S_1 \Rightarrow \varphi)$ and $(\Pi, \Box\Gamma, \psi \Rightarrow \exists p S_2)$. An application of $\mathrm{L}_\Box\!\to$ shows that $\mathfrak{I}_R^p$ derives $(\Pi, \Box\Gamma, \Box\varphi \to \psi, \Box\forall p S_1 \Rightarrow \exists p S_2)$. This implies that $\mathfrak{I}_R^p$ derives $(\Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Box\forall p S_1 \to \exists p S_2)$.

For (IPN)$_{\mathcal{R}}^{\exists}$, assume all sequents lower than S satisfy the interpolant properties. Thus $\vdash S_\Box^a \Rightarrow \exists p(S_\Box^a \Rightarrow\ )$. The presence of $\mathcal{R}_K$ implies that $\vdash \Pi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \Box\exists p(S_\Box^a \Rightarrow\ )$, and since $\Box\exists p(S_\Box^a \Rightarrow\ ) = \exists\overset{\overline{\mathcal{R}}}{p}S$, we are done. □

Lemma 23. *In any extension of* $\mathsf{G4iK}_\Box$*, (IPP)$_{\mathcal{R}}^{\forall}$ and (IPN)$_{\mathcal{R}}^{\forall}$ hold for $\mathcal{R} = \mathrm{L}_\Box\!\to$.*

Proof. It is easy to see that (IPN)$_{\mathcal{R}}^{\forall}$ holds. For (IPP)$_{\mathcal{R}}^{\forall}$, consider an instance R as in (7) and note that $(\Gamma, \forall p S_1 \Rightarrow \varphi)$ and $(\Pi, \Box\Gamma, \psi, \forall p S_2 \Rightarrow \Delta)$ belong to $\mathfrak{I}_R^p$. This implies that $(\Pi, \Box\Gamma, \Box\forall p S_1, \psi, \forall p S_2 \Rightarrow \Delta)$ is derivable from $\mathfrak{I}_R^p$. The presence of $\mathrm{L}_\Box\!\to$ and the fact that $\forall p S_2$ is not a boxed formula, shows that $(\Pi, \Box\Gamma, \Box\varphi \to \psi, \Box\forall p S_1, \forall p S_2 \Rightarrow \Delta)$ is derivable from $\mathfrak{I}_R^p$. This implies that (IPP)$_{\mathcal{R}}^{\forall}$ holds. □

Lemma 24. *In any extension of* $\mathsf{G4iK}_\Box$*, (DPP)$_{\mathcal{R}}$ holds for $\mathcal{R} = \mathrm{L}_\Box\!\to$.*

Proof. Consider a sequent S with a derivation of which the last inference is an instance R of $\mathcal{R}$ as in (7). Let (S^r, S^i) be a p–partition of S such that S^i is principal for R. Consider the standard partition of R such that $R^i = S_1^i\ S_2^i/S^i$ is an instance of $\mathcal{R}$, which exists by Lemma 21. We have to prove that $\mathfrak{D}_R^p \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$. We distinguish the following two cases.

First, assume S^{rs} is empty and S^s is not. We have to show that $\mathfrak{D}_R^p \vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i)$. Note that S_2^{rs} is empty and S_2^s is not. Hence $\mathfrak{D}_R^p$ contains sequent $S^r \cdot (\exists p S_2^i \Rightarrow \forall p S_2^i)$. Since S_1^{rs} is empty while S_1^s is not, $\mathfrak{D}_R^p$ contains $S_1^r \cdot (\exists p S_1^i \Rightarrow \forall p S_1^i)$ as well. This implies that $\mathfrak{D}_R^p$ derives sequent $S^r \cdot (\Box\exists p S_1^i, \exists p S_2^i \Rightarrow \Box\forall p S_1^i \wedge \forall p S_2^i)$. As sequent $\exists \overset{R^i}{p} S^i$ derives $\Box\exists p S_1^i \wedge (\Box\forall p S_1^i \rightarrow \exists p S_2^i)$ and $\forall \overset{R^i}{p} S^i = \Box\forall p S_1^i \wedge \forall p S_2^i$, it follows that $\mathfrak{D}_R^p$ derives sequent $S^r \cdot (\exists \overset{R^i}{p} S^i \Rightarrow \forall \overset{R^i}{p} S^i)$. Remark 1 then gives the desired conclusion.

Second, assume that S^{rs} is not empty or S^s is empty. Therefore we have to show that $\mathfrak{D}_R^p \vdash S^r \cdot (\exists p S^i \Rightarrow)$. As in the previous case, $\mathfrak{D}_R^p$ derives $S^r \cdot (\Box\exists p S_1^i \Rightarrow \Box\forall p S_1^i)$. As it contains $S^r \cdot (\exists p S_2^i \Rightarrow)$, it derives $S^r \cdot (\Box\exists p S_1^i, \exists p S_2^i \Rightarrow)$. As $\exists \overset{R^i}{p} S^i$ derives $\Box\exists p S_1^i \wedge (\Box\forall p S_1^i \rightarrow \exists p S_2^i)$, sequent $S^r \cdot (\exists \overset{R^i}{p} S^i \Rightarrow)$ is derivable from $\mathfrak{D}_R^p$. □

Lemma 25. *In any extension of* $\mathsf{G4iK}_\Box$, *(DPN)*$_\mathcal{R}$ *holds for* $\mathcal{R} = \mathrm{L}_\Box\!\rightarrow$.

Proof. Consider a sequent S with a derivation of which the last inference is an instance R of $\mathcal{R}$ as in (7). Let (S^r, S^i) be a p–partition of S such that S^i is nonprincipal for R. Thus

$$S^r = (\Pi^r, \Box\Gamma^r, \Box\varphi \rightarrow \psi \Rightarrow \Delta^r) \qquad S^i = (\Pi^i, \Box\Gamma^i \Rightarrow \Delta^i).$$

Assuming that all sequents below S in the ordering $\prec$ satisfy the interpolant properties, we have to show that $\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$. Consider the standard partition of the rule, which in this case means that

$$S_1^i = (\Gamma^i \Rightarrow\ ) \quad S_2^i = S^i \qquad S_1^r = (\Gamma^r \Rightarrow \varphi) \quad S_2^r = (\Pi^r, \Box\Gamma^r, \psi \Rightarrow \Delta^r).$$

First, observe that we can assume that Π^i does not contain boxed formulas. For if it does, say $\Pi^i = \Box\Sigma, \Pi'$ for some Π' not containing boxed formulas, then $S_1' = (\Gamma^r, \Sigma, \Gamma^i \Rightarrow \varphi)$ is derivable as well, because of the admissibility of left weakening, and $S_1'\ S_2/S$ is still an instance of $\mathcal{R}$. But now we can partition S_1' as $S_1^r \cdot (\Sigma, \Gamma^i \Rightarrow\ )$. This shows that we can assume that Π^i does not contain boxed formulas. In this case we have $\exists \overset{\mathcal{R}}{p} S^i = \Box\exists p S_1^i$.

We distinguish the case that $S^{rs} = \varnothing$ and $S^s \neq \varnothing$ both hold and the opposite. In the first case, $\Delta^r = \varnothing$ and $\Delta \neq \varnothing$, we have by assumption that

$$\vdash \Gamma^r, \exists p S_1^i \Rightarrow \varphi \qquad \vdash \Pi^r, \Box\Gamma^r, \psi, \exists p S_2^i \Rightarrow \forall p S_2^i.$$

$\mathcal{R}$ can be applied, and the fact that $S^i = S_2^i$ shows that

$$\vdash \Pi^r, \Box\Gamma^r, \Box\varphi \rightarrow \psi, \Box\exists p S_1^i, \exists p S^i \Rightarrow \forall p S^i.$$

Since $\Box\exists p S_1^i = \exists \overset{\mathcal{R}}{p} S^i$ is a conjunct of $\exists p S^i$, we have reached the desired conclusion. Case $\Delta^r \neq \varnothing$ and case $\Delta = \varnothing$ are analogous. □

7. Focused modal rules

In this section focused modal rules and their standard interpolant assignment are introduced, and it is shown that this interpolant assignment is sound for these rules.

7.1. Definition of focused modal rules

A rule $\mathcal{R}$ is a *focused modal rule* if it is of the following form:

$$\frac{S_u}{S_l} = \frac{\bigcirc S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0} \tag{8}$$

where S_0, S_1, S_2 are meta-sequents and

- S_0 consists of exactly one meta–formula,
- $\bigcirc S_1$ denotes either S_1 or $\boxdot S_1$,
- S_1 is of the form $(\Gamma \Rightarrow)$ or empty, for a distinct meta–multiset Γ that does not occur in S_0 and S_2,
- S_2 is of the form $(\Pi \Rightarrow \Delta)$ in case S_0^s is empty and of the form $(\Pi \Rightarrow)$ otherwise, where Π and Δ range over distinct meta–multisets that do not occur in S_0 or S_1.

A focused modal rule is a *focused $\Box$–rule* in case $\bigcirc S_1 = S_1$, and a *focused $\boxdot$–rule* in case $\bigcirc S_1 = \boxdot S_1$. Thus $\mathcal{R}_K$ and $\mathcal{R}_D$ above are focused $\Box$–rules, and $\mathcal{R}_{K4}$ is a focused $\boxdot$–rule. Note that if S_1 is the empty sequent, the rule is both a $\Box$–rule and a $\boxdot$–rule. The requirement that sequents have at most one formula on the right implies that the multiset $\Box S_0^s \cup \Box S_1^s \cup S_2^s$ consists of at most one formula.

Given an instance of a focused modal rule as in (8), the *lower sequent* is $S_2 \cdot \Box S_1 \cdot \Box S_0$ and denoted by S_l, the *upper sequent* S_u is the premiss $\bigcirc S_1 \cdot S_0$ of the rule. The formula in $\Box S_0$ is the *principal formula* of the instance. It is an r–rule if $S_0^s \neq \varnothing$, and an l–rule otherwise. It is an L-rule if S_1^a is not empty (S_1^s is required to be empty). An Lr–rule is a rule that is both an L–rule and an r–rule, and an l$\Box$–rule is a $\Box$–rule that is an l–rule, and likewise for all other combinations.

The rules $\mathcal{R}_K, \mathcal{R}_D, \mathcal{R}_{K4}$ that were defined at the beginning of Section 6 are a focused modal Lr$\Box$-rule, Ll$\Box$-rule, and Lr$\boxdot$-rule, respectively. The following is an example of an l$\Box$-rule.

$$\frac{\neg\neg\varphi \Rightarrow}{\Pi, \Box\neg\neg\varphi \Rightarrow \Delta}$$

In the following we mainly consider extensions of $\mathsf{G4iK}_\Box$ that are balanced, where a calculus is *balanced* if

- it is reductive;
- it does not contain l–rules that are not L–rules;
- contains $\mathcal{R}_{K4}$ whenever it contains some $\boxdot$–rule;
- Cut and Left Weakening (Fig. 2) are admissible in it.

7.2. Standard interpolant assignment for focused modal rules

Lemma 26. *For any instance $R = (S_u/S_l) = \big(\bigcirc S_1 \cdot S_0 / S_2 \cdot \Box S_1 \cdot \Box S_0\big)$ of a focused modal rule $\mathcal{R}$ and any p-partition (S_l^r, S_l^i) of S_l, there is a* standard partition *of R and p-partition (S_u^i, S_u^r) of S_u such that either S^i contains the principal formula, R^i is equal to*

$$\frac{\bigcirc S_1^i \cdot S_0}{S_2^i \cdot \Box S_1^i \cdot \Box S_0}\ R^i$$

and $S_l^r = S_2^r \cdot \Box S_1^r$ and $S_u^r = \bigcirc S_1^r$, or S^r contains the principal formula, R^r is equal to

$$\frac{\bigcirc S_1^r \cdot S_0}{S_2^r \cdot \Box S_1^r \cdot \Box S_0}\ R^r$$

and $S_l^i = S_2^i \cdot \Box S_1^i$ *and* $S_u^i = \bigcirc S_1^i$. *In the first case* R^i *is an instance of* $\mathcal{R}$ *and in the second case* R^r *is.*

Proof. Since S_0 contains exactly one formula, there are, given a p-partition (S_l^r, S_l^i) of S_l, partitions (S_j^r, S_j^i) of S_j for $j = 0, 1, 2$, such that either $S_l^r = S_2^r \cdot \Box S_1^r \cdot \Box S_0$ and $S_l^i = S_2^i \cdot \Box S_1^i$, or vice versa ($i$ and r interchanged). We leave it to the reader to check that these partitions indeed satisfy the lemma. □

The *standard interpolant assignment* for an instance R of a focused modal rule $\mathcal{R}$ as in (8) is defined as follows.

$$\iota\forall_p^R S_l \quad \equiv_{df} \quad \Box\forall p S_u$$

$$\iota\exists_p^R S_l \quad \equiv_{df} \quad \begin{cases} \Box\exists p S_u & \text{if } S_u^s \neq \varnothing \\ \Box\exists p S_u \wedge \Box\neg\forall p S_u & \text{if } S_u^s = \varnothing. \end{cases}$$

$$\iota\forall_p^{\overline{\mathcal{R}}} S \quad \equiv_{df} \quad \bot$$

$$\iota\exists_p^{\overline{\mathcal{R}}} S \quad \equiv_{df} \quad \Box\exists p(S_\Box^a \Rightarrow\ ).$$

7.3. Soundness of the standard interpolant assignment

Lemma 27. *For any focused modal rule* $\mathcal{R}$ *in any balanced extension of* $\mathsf{G4iK}_\Box$, *(IPP)*$_{\mathcal{R}}^{\exists}$ *and (IPN)*$_{\mathcal{R}}^{\exists}$ *hold.*

Proof. For (IPP)$_{\mathcal{R}}^{\exists}$ we have to show that $\mathfrak{I}_R^p \vdash S^a \Rightarrow \Box\exists p S_u$, and $\mathfrak{I}_R^p(S) \vdash S^a \Rightarrow \Box\neg\forall p S_u$ in case $S_u^s = \varnothing$. For the first part, it suffices to show that for any instance S_u/S of $\mathcal{R}$ and any formula φ:

$$S_u^a \Rightarrow \varphi \vdash S^a \Rightarrow \Box\varphi,$$

where we will be interested in the case that $\varphi = \exists p S_u$. In case $\mathcal{R}$ is a $\Box$–rule, we apply $\mathcal{R}_K$ to the sequent $(S_u^a \Rightarrow \varphi)$ and obtain $(S^a \Rightarrow \Box\varphi)$. In case $\mathcal{R}$ is a $\boxdot$-rule we use $\mathcal{R}_{\mathsf{K4}}$ instead. This proves $\mathfrak{I}_R^p \vdash S^a \Rightarrow \Box\exists p S_u$. To prove that also $\mathfrak{I}_R^p \vdash S^a \Rightarrow \Box\neg\forall p S_u$ in case $S_u^s = \varnothing$, note that $S_u^a, \forall p S_u \Rightarrow S_u^s$ belongs to $\mathfrak{I}_R^p$, and since $S_u^s = \varnothing$, $\mathfrak{I}_R^p$ derives $S_u^a \Rightarrow \neg\forall p S_u$. An application of $\mathcal{R}_K$ (or $\mathcal{R}_{\mathsf{K4}}$ if $\mathcal{R}$ is an $\boxdot$-rule) proves that $\mathfrak{I}_R^p \vdash S^a \Rightarrow \Box\neg\forall p S_u$.

For (IPN)$_{\mathcal{R}}^{\exists}$, assume that all sequents lower than S satisfy the interpolant properties. We have to show that $\vdash S^a \Rightarrow \Box\exists p(S_\Box^a \Rightarrow\ )$. Since $(S_\Box^a \Rightarrow \exists p(S_\Box^a \Rightarrow))$ is derivable by the assumption on all sequents lower than S, an application of $\mathcal{R}_K$ (or $\mathcal{R}_{K4}$ if $\mathcal{R}$ is an $\boxdot$-rule) proves that $\vdash S^a \Rightarrow \Box\exists p(S_\Box^a \Rightarrow\ )$. □

Lemma 28. *For any focused modal rule* $\mathcal{R}$ *in any balanced extension of* $\mathsf{G4iK}_\Box$, *(IPP)*$_{\mathcal{R}}^{\forall}$ *and (IPN)*$_{\mathcal{R}}^{\forall}$ *hold.*

Proof. Property (IPN)$_{\mathcal{R}}^{\forall}$ follows immediately from the fact that $\iota\forall_p^{\overline{\mathcal{R}}} S = \bot$. For (IPP)$_{\mathcal{R}}^{\forall}$, consider a sequent S that is the conclusion of an instance $R = (S_u/S)$ of $\mathcal{R}$. This implies that $S_u \cdot (\forall p S_u \Rightarrow\ )$ belongs to $\mathfrak{I}_R^p$. It suffices to show that $S_u \cdot (\forall p S_u \Rightarrow\ ) \vdash S \cdot (\Box\forall p S_u \Rightarrow\ )$ as $\forall_p^R S = \Box\forall p S_u$. In case $\mathcal{R}$ is a right $\Box$-rule, $\mathcal{R}_K$ can be applied to $S_u \cdot (\forall p S_u \Rightarrow\ )$ to obtain $S \cdot (\Box\forall p S_u \Rightarrow\ )$. In case $\mathcal{R}$ is a right $\boxdot$-rule, $\mathcal{R}_{K4}$ can be used. If $\mathcal{R}$ is a left rule, it has to be an LI-rule because the calculus is balanced. Thus it can be applied to $S_u \cdot (\forall p S_u \Rightarrow\ )$ to obtain $S \cdot (\Box\forall p S_u \Rightarrow\ )$. □

Lemma 29. *For any focused modal rule* $\mathcal{R}$ *in any balanced extension of* $\mathsf{G4iK}_\Box$, *(DPP)*$_{\mathcal{R}}$ *holds.*

Proof. Consider a sequent S with a derivation of which the last inference is an instance $R = (S_u/S)$ of $\mathcal{R}$ as in (7). Let (S^r, S^i) be a p–partition of S such that S^i is principal for R.

Consider the standard partition of S_u/S given the partition of S, and let (S_u^r, S_u^i) denote the partition of S_u. We have to show that $\mathfrak{D}_R^p$ derives $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$. We treat the case that $\mathcal{R}$ is a $\Box$-rule, the proof for a $\boxdot$-rule is analogous.

If $\mathcal{R}$ is a left rule, it is a Ll-rule and there are $\Pi^r, \Pi^i, \Gamma^r, \Gamma^i$ such that

$$S^r = (\Pi^r, \Box\Gamma^r \Rightarrow \Delta^r) \;\; S^i = (\Pi^i, \Box\Gamma^i, \Box\varphi \Rightarrow \Delta^i) \;\; S_u^r = (\Gamma^r \Rightarrow) \;\; S_u^i = (\Gamma^i, \varphi \Rightarrow).$$

We distinguish the case that $\Delta^r = \varnothing$ and $\Delta \neq \varnothing$ from the case that this does not hold. In the first case, we have to show that $\mathfrak{D}_R^p$ derives $(\Pi^r, \Box\Gamma^r, \exists p S^i \Rightarrow \forall p S^i)$. Since $S_u^s = \varnothing$, $\mathfrak{D}_R^p$ contains $(\Gamma^r, \exists p S_u^i \Rightarrow)$, and thus derives $(\Gamma^r, \exists p S_u^i \Rightarrow \forall p S_u^i)$. An application of $\mathcal{R}_K$ proves that $\mathfrak{D}_R^p$ derives $(\Pi^r, \Box\Gamma^r, \Box\exists p S_u^i \Rightarrow \Box\forall p S_u^i)$. By Lemma 26, R^i is an instance of $\mathcal{R}$, and thus $\Box\exists p S_u^i$ is a conjunct of $\exists^{R^i}_p S^i$, which is a conjunct of $\exists p S^i$, and $\Box\forall p S_u^i = \forall^{R^i}_p S^i$ is a disjunct of $\forall p S^i$. This proves the desired.

In case $\Delta^r \neq \varnothing$ or $\Delta = \varnothing$, we have to show that $\mathfrak{D}_R^p$ derives $(\Pi^r, \Box\Gamma^r, \exists p S^i \Rightarrow \Delta^r)$. Again we use that $\mathfrak{D}_R^p$ contains $(\Gamma^r, \exists p S_u^i \Rightarrow)$, and conclude from this that $\mathfrak{D}_R^p$ derives $(\Pi^r, \Box\Gamma^r, \Box\exists p S_u^i \Rightarrow \Delta^r)$ by an application of $\mathcal{R}_K$. Then the same reasoning as in the case that Δ^r is empty can be applied to obtain $\mathfrak{D}_R^p \vdash (\Pi^r, \Box\Gamma^r, \exists p S^i \Rightarrow \Delta^r)$.

If $\mathcal{R}$ is a right rule, there are $\Pi^r, \Pi^i, \Gamma^r, \Gamma^i$ such that

$$S^r = (\Pi^r, \Box\Gamma^r \Rightarrow) \;\; S^i = (\Pi^i, \Box\Gamma^i \Rightarrow \Box\varphi) \;\; S_u^r = (\Gamma^r \Rightarrow) \;\; S_u^i = (\Gamma^i \Rightarrow \varphi),$$

and we have to show that $\vdash \Pi^r, \Box\Gamma^r, \exists p S^i \Rightarrow \forall p S^i$. Since $(\Gamma^r, \exists p S_u^i \Rightarrow \forall p S_u^i)$ belongs to $\mathfrak{D}_R^p$, an application of $\mathcal{R}_K$ shows that $\mathfrak{D}_R^p$ derives $(\Pi^r, \Box\Gamma^r, \Box\exists p S_u^i \Rightarrow \Box\forall p S_u^i)$. By Lemma 26, R^i is an instance of $\mathcal{R}$, and thus $\Box\exists p S_u^i$ is a conjunct of $\exists^{R^i}_p S^i$, which is a conjunct of $\exists p S^i$, and $\Box\forall p S_u^i = \forall^{R^i}_p S^i$ is a disjunct of $\forall p S^i$, which implies what we had to show. □

Lemma 30. *For any focused modal rule $\mathcal{R}$ in any balanced extension of $\mathsf{G4iK}_\Box$, (DPN)$_\mathcal{R}$ holds.*

Proof. Assume that S has a derivation of which the last inference is an instance

$$\frac{S_u}{S} = \frac{\bigcirc S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}\; R$$

of a focused modal rule $\mathcal{R}$. Assume that all sequents lower than S satisfy the interpolant properties. Let (S^r, S^i) be a p–partition of S such that S^i is nonprincipal for R and consider the standard partition of S_u/S given the partition of S, where (S_u^r, S_u^i) denotes the partition of the upper sequent S_u. We have to prove the derivability of $S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$. If S^i is empty, then $S^r = S$, and thus S^r is derivable, and so is any weakening $S^r \cdot S'$. Therefore assume S^i is not empty.

We distinguish the cases that $\mathcal{R}$ is and is not an r-rule. First suppose it is an r-rule. We treat the case that it is a $\Box$-rule, the case that it is a $\boxdot$-rule is similar. Since S^i does not contain the principal formula of R, there are $\Pi^r, \Pi^i, \Gamma^r, \Gamma^i$ such that

$$S^r = (\Pi^r, \Box\Gamma^r \Rightarrow \Box\varphi) \;\; S^i = (\Pi^i, \Box\Gamma^i \Rightarrow) \;\; S_u^r = (\Gamma^r \Rightarrow \varphi) \;\; S_u^i = (\Gamma^i \Rightarrow).$$

Thus we have to show that $\vdash \Pi^r, \Box\Gamma^r, \exists p S^i \Rightarrow \Box\varphi$. First, observe that we can assume that Π^i does not contain boxed formulas. For if it does, say $\Pi^i = \Box\Sigma, \Pi'$ for some Π' not containing boxed formulas, then $S_u' = (\Gamma^r, \Sigma, \Gamma^i \Rightarrow \varphi)$ is derivable as well, because of the admissibility of left weakening, and S_u'/S is still an instance of $\mathcal{R}$. But now we can partition S_u' as $S_u^r \cdot (\Sigma, \Gamma^i \Rightarrow)$. This shows that we can assume that Π^i does not contain boxed formulas. In this case we have $\exists^{\mathcal{R}}_p S^i = \Box\exists p S_u^i$.

By assumption we have $\vdash \Gamma^r, \exists p S_u^i \Rightarrow \varphi$, and thus $\vdash \Pi^r, \Box\Gamma^r, \Box\exists p S_u^i \Rightarrow \Box\varphi$ by an application of $\mathcal{R}_K$. Since $\Box\exists p S_u^i$ is equal to $\exists \overline{p}^{\mathcal{R}} S^i$, which is a conjunct of $\exists p S^i$, this implies what we had to show.

Suppose that $\mathcal{R}$ is an l-rule, and thus an Ll-rule. Since S^i does not contain the principal formula of R, there are $\Pi^r, \Pi^i, \Gamma^r, \Gamma^i$ such that

$$S^r = (\Pi^r, \Box\Gamma^r, \Box\varphi \Rightarrow \Delta^r) \quad S^i = (\Pi^i, \Box\Gamma^i \Rightarrow \Delta^i) \quad S_u^r = (\Gamma^r, \varphi \Rightarrow) \quad S_u^i = (\Gamma^i \Rightarrow).$$

Like in the case of an r-rule, we can assume that Π^i does not contain boxed formulas, and thus conclude that $\exists \overline{p}^{\mathcal{R}} S^i = \Box\exists p S_u^i$. Since $S_u^s = \varnothing$, $\vdash (\Gamma^r, \varphi, \exists p S_u^i \Rightarrow)$ holds by assumption. Thus an application of $\mathcal{R}$ proves that $(\Pi^r, \Box\Gamma^r, \Box\varphi, \Box\exists p S_u^i \Rightarrow \Sigma)$ is derivable for any multiset Σ with $|\Sigma| \leq 1$. In particular, $S^r \cdot (\Box\exists p S_u^i \Rightarrow)$ is derivable in case $\Delta^r \neq \varnothing$ or $\Delta = \varnothing$, and $S^r \cdot (\Box\exists p S_u^i \Rightarrow \forall p S^i)$ is derivable otherwise. As $\Box\exists p S_u^i$ is equal to $\exists \overline{p}^{\mathcal{R}} S^i$, which is a conjunct of $\exists p S^i$, this proves $\vdash S^r \cdot (\exists p S^i \Rightarrow \forall p S^i \mid \varnothing)$. □

8. Main theorems

Theorem 31. *Any intuitionistic modal logic* L *with a balanced calculus that contains* $\mathsf{G4iK}_\Box$ *and has an interpolant assignment that is sound with respect to all rules that neither are focused, nor modal focused, nor belong to* $\mathsf{G4iK}_\Box$*, has uniform interpolation.*

Proof. Consider a calculus as in the theorem. By Theorem 17, Lemmas 22–25 and Lemmas 27–30, there exists an interpolant assignment that is sound for all rules of the calculus. Theorems 5 and 20 imply that the logic has uniform interpolation. □

Corollary 32. *Any intuitionistic modal logic* L *with a balanced calculus that consists of* $\mathsf{G4iK}_\Box$*, focused rules, and modal focused rules, has uniform interpolation.*

Recall that for intermediate logics, besides reproving Pitts' theorem, the following has been obtained.

Corollary 19 If an intermediate logic is not equal to one of the seven logics IPC, Sm, LC, GSc, KC, Bd_2, CPC, then it does not have a reductive ca calculus that is an extension of G4ip by focused rules.

For intuitionistic modal logics, Theorem 31 has the following similar corollary.

Corollary 33. *No intuitionistic modal logic* L *that does not have uniform interpolation has a balanced calculus that is an extension of* $\mathsf{G4iK}_\Box$ *by focused and focused modal rules.*

As proved in [18], $\mathsf{G4iK}_\Box$ and $\mathsf{G4iKD}_\Box$ are reductive and equivalent to $\mathsf{G3iK}_\Box$ and $\mathsf{G3iKD}_\Box$, respectively, and Cut and Left Weakening are admissible in them. It is not hard to see that they also satisfy the other requirements for a balanced calculus. Therefore Theorem 31 implies the following.

Theorem 34. *The logics* $\mathsf{L}_{\mathsf{G3iK}_\Box}$ *and* $\mathsf{L}_{\mathsf{G3iKD}_\Box}$ *have uniform interpolation.*

Since uniform interpolation is a statement about the theorems of a logic, it follows that every logic with the same theorems as $\mathsf{L}_{\mathsf{G3iK}_\Box}$ or as $\mathsf{L}_{\mathsf{G3iKD}_\Box}$ has uniform interpolation. This in particular applies to the following logics: the logic "*IK* without ◇" from [25], which is defined by a Hilbert system for IPC extended by the axiom K and the rule Necessitation; the logics $\mathbf{HK}_\Box$ and $\mathbf{HD}_\Box$ from [7,9]; the logics K^i and NV^i in [20]; and $\mathbf{IntK}_\Box$ from [30].

For a proper treatment of iK4, and possibly other transitive logics, the first question that needs to be answered is whether there exists a balanced calculus for iK4. The rule $\mathcal{R}_{K4}$ is problematic because a calculus with that rule is not reductive, at least not under the order on sequents defined in Example 1. Whether there is a balanced calculus for iK4 without such a rule is as yet unknown, and it is not known either whether $\mathsf{iK4}_\Box$ has uniform interpolation. Since Bílková [6] and Ghilardi and Zawadowski [11] have shown that K4 and S4 do not have uniform interpolation, one wonders whether the same holds for $\mathsf{iK4}_\Box$. If so, this would imply that the calculus $\mathsf{G4iK4}_\Box$ cannot be reductive.

9. Conclusion

We have presented a uniform modular method to prove that certain intermediate and intuitionistic modal logics (without the diamond operator $\Diamond$) have uniform interpolation. Using this method, we have proved that the intuitionistic versions of K and KD have uniform interpolation. The modularity of the method guarantees that when $\mathsf{G4iK}_\Box$ is extended by new rules, then in order to establish that uniform interpolation is preserved in the extension (in the case that the extension indeed has that property), only the new rules have to be proven sound for some interpolant assignment that is standard for the rules of $\mathsf{G4iK}_\Box$.

The contraposition of these results lead to the main theorem of the paper, namely that for any intermediate or intuitionistic modal logic that does not have uniform interpolation, it follows that it cannot have a reductive calculus that is an extension of G4ip by focused rules or a balanced extension of $\mathsf{G4iK}_\Box$ by focused (modal) rules, respectively. Applying this to the infinitely many intermediate logics without uniform interpolation, shows that extensions of G4ip by focused rules cannot be proof systems for these logics.

9.1. Future work

There are many directions for future research. Something that should be explored is the extension of the framework to other intuitionistic modal logics, such as iGL, and to modal logics that contain the diamond operator. Another natural continuation of the work presented here would be the extension of the method to other classes of logics, such as the substructural logics, where one could try to develop a method similar to the one in this paper to prove and generalize the results in [1]. It would also be useful to extend our method to hypersequent calculi, as there are logics with nice hypersequent calculi that have uniform interpolation, for example KC. Moreover, not having uniform interpolation would, for a given logic, exclude not only certain sequent calculi but even certain hypersequent calculi as sound and complete proof systems for the logic.

Declaration of Competing Interest

None.

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References

[1] M. Alizadeh, F. Derakhshan, H. Ono, Uniform interpolation in substructural logics, Rev. Symb. Log. 7 (3) (2014) 455–483.
[2] F. Baader, T. Nipkow, Term Rewriting and All That, Cambridge University Press, 1998.
[3] G. Bellin, V. de Paiva, E. Ritter, Extended Curry–Howard correspondence for a basic constructive modal logic, in: C. Araces, M. de Rijke (Eds.), Proceedings of Methods for Modalities 2, 2001.
[4] G. Bierman, V. de Paiva, On an intuitionistic modal logic, Studia Logica 65 (3) (2000) 383–416.
[5] M. Bílková, Interpolation in Modal Logics, PhD–thesis, Charles University, Prague, 2006.
[6] M. Bílková, Uniform interpolation and propositional quantifiers in modal logics, Studia Logica 85 (1) (2007) 259–271.
[7] M. Božić, K. Došen, Models for normal intuitionistic modal logics, Studia Logica 43 (3) (1984) 217–245.

[8] N. Dershowitz, Z. Manna, Proving termination with multiset orderings, Commun. ACM 22 (1979) 465–476.
[9] K. Došen, Models for Stronger normal intuitionistic modal logics, Studia Logica 44 (1) (1985) 39–70.
[10] R. Dyckhoff, Contraction-free sequent calculi for intuitionistic logic, J. Symbolic Logic 57 (3) (1992) 795–807.
[11] S. Ghilardi, M. Zawadowski, Undefinability of propositional quantifiers in the modal system $S4$, Studia Logica 55 (1995) 259–271.
[12] S. Ghilardi, M. Zawadowski, Sheaves Games, and Model Completions: A Categorical Approach to Nonclassical Propositional Logics, Trends in Logic (Book 14), Springer, 2002.
[13] S. van Gool, G. Metcalfe, C. Tsinakis, Uniform interpolation and compact congruences (2016), submitted, https://doi.org/10.1016/j.apal.2017.05.001.
[14] J. Hudelmaier, A Prolog program for intuitionistic propositional logic, Tübingen, 1988, SNS-Bericht 88–28.
[15] J. Hudelmaier, Bounds for cut elimination in intuitionistic propositional logic, Arch. Math. Logic 31 (1992) 331–353.
[16] J. Hudelmaier, An $O(n\, log\, n)$–space decision procedure for intuitionistic propositional logic, J. Logic Comput. 3 (1993) 63–76.
[17] R. Iemhoff, Uniform interpolation and sequent calculi in modal logic, Arch. Math. Logic 58 (1-2) (2019) 155–181.
[18] R. Iemhoff, Terminating sequent calculi for two intuitionistic modal logics, J. Logic Comput. 28 (7) (2018) 1701–1712.
[19] Kremer, On the complexity of propositional quantification in intuitionistic logic, J. Symbolic Logic 62 (2) (1997) 529–544.
[20] T. Litak, Constructive modalities with provability smack, in: G. Bezhanishvili (Ed.), Leo Esakia on Duality in Modal and Intuitionistic Logics. Outstanding Contributions to Logic 4, Springer, 2014, pp. 179–208.
[21] L.L. Maksimova, Craig's theorem in superintuitionistic logics and amalgamated varieties of pseudo-boolean algebras, Algebra Logika 16 (6) (1977) 643–681.
[22] A. Pitts, On an interpretation of second order quantification in first order intuitionistic propositional logic, J. Symbolic Logic 57 (1) (1992) 33–52.
[23] T. Połacik, Pitts' quantifiers are not topological quantification, Notre Dame J. Form. Log. 39 (4) (1998) 531–544.
[24] V.Y. Shavrukov, Subalgebras of Diagonalizable Algebras of Theories Containing Arithmetic, Dissertationes Mathematicae, vol. 323, Instytut Matematyczny Polskiej Akademi Nauk, Warszawa, 1993.
[25] A.K. Simpson, The Proof Theory and Semantics of Intuitionistic Modal Logic, PhD-thesis, University of Edinburgh, 1994.
[26] A. Visser, Bisimulations, Model Descriptions and Propositional Quantifiers, Logic Group Preprint Series, vol. 161, 1996, Utrecht University.
[27] A. Visser, Uniform interpolation and layered bisimulation, in: P. Hájek (Ed.), Gödel '96, Logical Foundations of Mathematics, Computer Science and Physics–Kurt Gödel's Legacy, in: Lecture Notes in Logic, vol. 6, Springer, Berlin, 1996, pp. 139–164.
[28] N.N. Vorob'ev, The derivability problem in the constructive propositional calculus with strong negation, Dokl. Akad. Nauk SSSR 85 (1952) 689–692.
[29] N.N. Vorob'ev, A new algorithm for derivability in the constructive propositional calculus, in: AMS Translations Ser. 2, vol. 94, 1970, pp. 37–71.
[30] F. Wolter, M. Zakharyaschev, Intuitionistic modal logic, in: A. Cantini, E. Casari, P. Minari (Eds.), Logic and Foundations of Mathematics. Synthese Library, vol. 280, 1999, pp. 227–238.