Lombardi Drawings of Knots and Links

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Abstract. Knot and link diagrams are projections of one or more 3-dimensional simple closed curves into $\mathbb{R}^2$, such that no more than two points project to the same point in $\mathbb{R}^2$. These diagrams are drawings of 4-regular plane multigraphs. Knots are typically smooth curves in $\mathbb{R}^3$, so their projections should be smooth curves in $\mathbb{R}^2$ with good continuity and large crossing angles: exactly the properties of Lombardi graph drawings (defined by circular-arc edges and perfect angular resolution).

We show that several knots do not allow plane Lombardi drawings. On the other hand, we identify a large class of 4-regular plane multigraphs that do have Lombardi drawings. We then study two relaxations of Lombardi drawings and show that every knot admits a plane 2-Lombardi drawing (where edges are composed of two circular arcs). Further, every knot is *near-Lombardi*, that is, it can be drawn as Lombardi drawing when relaxing the angular resolution requirement by an arbitrary small angular offset ε, while maintaining a 180° angle between opposite edges.

1 Introduction

A *knot* is an embedding of a simple closed curve in 3-dimensional Euclidean space $\mathbb{R}^3$. Similarly, a *link* is an embedding of a collection of simple closed curves in $\mathbb{R}^3$. A *drawing of a knot (link)* (also known as *knot diagram*) is a projection of the knot (link) to the Euclidean plane $\mathbb{R}^2$ such that for any point of $\mathbb{R}^2$, at most two points of the curve(s) are mapped to it [6,15,16]. From a graph drawing perspective, drawings of knots and links are drawings of 4-regular plane multigraphs that contain neither loops nor cut vertices. Likewise, every 4-regular plane multigraph without loops and cut vertices can be interpreted as a link. Unless specified otherwise, we assume that a multigraph has no self-loops or cut vertices.

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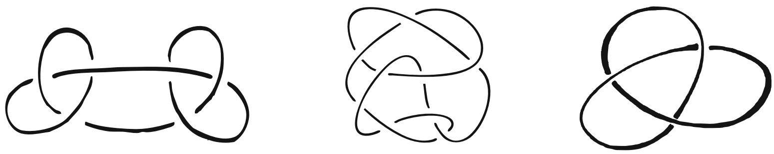

Fig. 1. Hand-made drawings of knots from the books of Rolfsen [15] (left), Livingston [14] (middle), and Kauffman [11] (right).

In this paper, we address a question that was recently posed by Benjamin Burton: "Given a drawing of a knot, how can it be redrawn *nicely* without changing the given topology of the drawing?" We do know what a drawing of a knot is, but what is meant by a *nice* drawing? Several graphical annotations of knots and links as graphs have been proposed in the knot theory literature, but most of the illustrations are hand-drawn; see Fig. 1. When studying these drawings, a few desirable features become apparent: (i) edges are typically drawn as smooth curves, (ii) the angular resolution of the underlying 4-regular graph is close to 90°, and (iii) the drawing preserves the continuity of the knot, that is, in every vertex of the underlying graph, opposite edges have a common tangent.

There already exists a graph drawing style that fulfills the requirements above: a *Lombardi drawing* of a (multi-)graph $G = (V, E)$ is a drawing of G in the Euclidean plane with the following properties:

1. The vertices are represented as distinct points in the plane.
2. The edges are represented as circular arcs connecting the representations of their end vertices (and not containing the representation of any other vertex); note that a straight-line segment is a circular arc with radius infinity.
3. Every vertex has *perfect angular resolution*, i.e., its incident edges are equiangularly spaced. For knots and links this means that the angle between any two consecutive edges is 90°.

A Lombardi drawing is plane if none of its edges intersect. Note that we are particularly interested in plane Lombardi drawings, since crossings change the topology of the drawn knot.

Knot drawing software: Software for generating drawings for knots and links exists. One powerful package is `KnotPlot` [16], which provides several methods for drawing knot diagrams. It contains a library of over 1,000 precomputed knots and can also generate knot drawings of certain families, such as torus knots. `KnotPlot` also provides methods for drawing general knots based on the embedding of the underlying plane multigraph. By replacing every vertex by a 4-cycle, the multigraph becomes a simple planar 3-connected graph, which is then drawn using Tutte's barycentric method [18]. In the end, the modifications are reversed and a drawing of the knot is obtained with edges drawn as polygonal arcs. The author noticed that this method "... does not yield 'pleasing' graphs or knot diagrams." In particular, he noticed issues with vertex and angular resolution [16, pg. 102]. Another approach was used by Emily Redelmeier [1].

Here, every arc, crossing, and face of the knot diagram is associated with a disk. The drawing is then generated from the implied circle packing as a circular arc drawing. As a result of the construction, every edge in the diagram is made of three circular arcs with common tangents at opposite edges. Since no further details are given, it is hard to evaluate the effectiveness of this approach, although as we show in this paper, three circular arcs per edge are never needed. A related drawing style for knots are the so-called *arc presentations* [5]. An arc presentation is an orthogonal drawing, that is, all edges are sequences of horizontal and vertical segments, with the additional properties that at each vertex the vertical segments are above the horizontal segments in the corresponding knot and that each row and column contains exactly one horizontal and vertical segment, respectively. However, these drawings might require a large number of bends per edge.

Lombardi drawings: Lombardi drawings were introduced by Duncan et al. [8]. They showed that 2-degenerate graphs have Lombardi drawings and that all d-regular graphs, with $d \not\equiv 2 \pmod 4$, have Lombardi drawings with all vertices placed along a common circle. Neither of these results, however, is guaranteed to result in plane drawings. Duncan et al. [8] also showed that there exist planar graphs that do not have plane Lombardi drawings, but restricted graph classes (e.g., Halin graphs) do. In subsequent work, Eppstein [9,10] showed that every (simple) planar graph with maximum degree three has a plane Lombardi drawing. Further, he showed that a certain class of 4-regular planar graphs (the medial graphs of polyhedral graphs) also admit plane Lombardi drawings and he presented an example of a 4-regular planar graph that does not have a plane Lombardi drawing. A generalization of Lombardi drawings are *k-Lombardi drawings.* Here, every edge is a sequence of at most k circular arcs that meet at a common tangent. Duncan et al. [7] showed that every planar graph has a plane 3-Lombardi drawing. Related to k-Lombardi-drawings are *smooth-orthogonal drawings of complexity k* [4]. These are plane drawings where every edge consists of a sequence of at most k quarter-circles and axis-aligned segments that meet smoothly, edges are axis-aligned (emanate from a vertex either horizontally or vertically), and no two edges emanate in the same direction. Note that in the special case of 4-regular graphs, smooth-orthogonal drawings of complexity k are also plane k-Lombardi drawings.

Our Contributions: The main question we study here is motivated by the application of the Lombardi drawing style to knot and link drawings: Given a 4-regular plane multigraph G without loops and cut vertices, does G admit a plane Lombardi drawing with the same combinatorial embedding? In Sect. 2 we start with some positive results on extending a plane Lombardi drawing, as well as composing two plane Lombardi drawings. In Sect. 3, by extending the results of Eppstein [9,10], we show that a large class of multigraphs, including 4-regular polyhedral graphs, does have plane Lombardi drawings. Unfortunately, there exist several small knots that do not have a plane Lombardi drawing. Section 4 discusses these cases but also lists a few positive results for small examples.

In Sect. 5, we show that every 4-regular plane multigraph has a plane 2-Lombardi drawing. In Sect. 6, we show that every 4-regular plane multigraph can be drawn with non-crossing circular arcs, so that the perfect angular resolution criterion is violated only by an arbitrarily small value ε, while maintaining that opposite edges have common tangents.

2 General Observations

If a knot or a link has an embedding with minimum number of vertices that admits a plane Lombardi drawing, we call it a *plane Lombardi knot* (*link*). We further call the property of admitting a plane Lombardi drawing *plane Lombardiness*. If two vertices in a plane Lombardi drawing of a knot are connected by a pair of multi-edges, we denote the face enclosed by these two edges as a *lens*.

There exist a number of operations that maintain the plane Lombardiness of a 4-regular multigraph. Two knots A and B can be combined by cutting one edge of each of them open and gluing pairwise the loose ends of A with the loose ends of B. This operation is known as a *knot sum* $A + B$. Knots that cannot be decomposed into a sum of two smaller knots are known as *prime knots*. By Schubert's theorem, every knot can be uniquely decomposed into prime knots [17]. The smallest prime knot is the trefoil knot with three crossings or vertices; see Fig. 1 (right). Rolfsen's knot table[1] lists all prime knots with up to ten vertices. The Alexander-Briggs-Rolfsen notation [3,15] is a well established notation that organizes these knots by their vertex number and a counting index, e.g., the trefoil knot 3_1 is listed as the first (and only) knot with three vertices.

Theorem 1. *Let A and B be two 4-regular multigraphs with plane Lombardi drawings. Let a be an edge of A and b an edge of B. Then the knot sum $A+B$, obtained by connecting A and B along edges a and b, admits a plane Lombardi drawing.*

Sketch of Proof. The idea of the composition is sketched in Fig. 2. We first apply Möbius transformations, rotations, and translations to the two drawings so that edges a and b can be aligned along a circle with infinite radius. This can be done such that the drawings of A and B do not intersect after removing a and b. We can now reconnect the two drawings into a single plane Lombardi drawing by introducing two edges c and d along the straight line.

Another operation that preserves the plane Lombardiness is *lens multiplication*. Let $G = (V, E)$ be a 4-regular plane multigraph with a lens between two vertices u and v. A lens multiplication of G is a 4-regular plane multigraph that is obtained by replacing the lens between u and v with a chain of lenses.

Lemma 1. *Let $G = (V, E)$ be a 4-regular plane multigraph with a plane Lombardi drawing Γ. Then, any lens multiplication G' of G also admits a plane Lombardi drawing.*

[1] http://katlas.org/wiki/The_Rolfsen_Knot_Table.

Sketch of Proof. Let f be a lens in Γ spanned by two vertices u and v as shown in Fig. 3. We draw a bisecting circular arc b that splits the angles at u and v into two 45° angles. Now we can draw any chain of lenses inside f by placing the additional vertices on b. The resulting drawing is a plane Lombardi drawing.

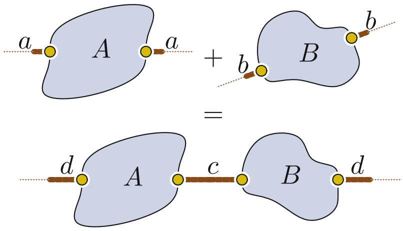

Fig. 2. Adding two plane Lombardi drawings of 4-regular multigraphs.

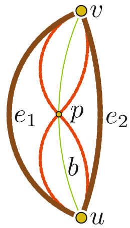

Fig. 3. Subdividing a lens between u and v by a new vertex p.

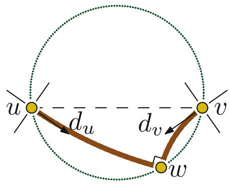

Fig. 4. Placement circle for neighbor w of u and v in a 4-regular graph.

We will use the following property several times throughout the paper.

Property 1 (Property 2 in [7,8]*).* Let u and v be two vertices with given positions that have a common, unplaced neighbor w. Let d_u and d_v be two tangent directions and let θ be a target angle. Let C be the locus of all positions for placing w so that (i) the edge (u, w) is a circular arc leaving u in direction d_u, (ii) the edge (v, w) is a circular arc leaving v in direction d_v, and (iii) the angle formed at w is θ. Then C is a circle, the so-called *placement circle* of w.

Duncan et al. [7] further specify the radius and center of the placement circle by the input coordinates and angles. For the special case that the two tangent directions d_u and d_v are symmetric with respect to the line through u and v, and that the angle θ is 90° or 270°, the corresponding placement circle is such that its tangent lines at u and v form an angle of 45° with the arc directions d_u and d_v. In particular, the placement circle bisects the right angle between d_u (resp. d_v) and its neighboring arc direction. Figure 4 illustrates this situation.

3 Plane Lombardi Drawings via Circle Packing

Recall that *polyhedral graphs* are simple planar 3-connected graphs, and that those graphs have a unique (plane) combinatorial embedding. The (plane) *dual graph* M' of a plane graph M has a vertex for every face of M and an edge between two vertices for every edge shared by the corresponding faces in M. In the "classic" drawing $D(M, M')$ of a primal-dual graph pair (M, M'), every vertex of M' lies in its corresponding face of M and vice versa, and every edge of M' intersects exactly its corresponding edge of M. Hence, every cell of $D(M, M')$

has exactly two such edge crossings and exactly one vertex of each of M and M' on its boundary. The *medial graph* of a primal-dual graph pair (M, M') has a vertex for every crossing edge pair in $D(M, M')$ and an edge between two vertices whenever they share a cell in $D(M, M')$; see Fig. 5a. Every cell of the medial graph contains either a vertex of M or a vertex of M' and every edge in the medial graph is incident to exactly one cell in $D(M, M')$.

Every 4-regular plane multigraph G can be interpreted as the medial graph of some plane graph M and its dual M', where both graphs possibly contain multi-edges. If G contains no loops and cut vertices, then neither M nor M' contains loops. Eppstein [9] showed that if M (and hence also M') is polyhedral, then G admits a plane Lombardi drawing. We show next how to extend this result to a larger graph class. We only give a short sketch of the proof here. The full proof, as well as an example of the algorithm can be found in the arXiv version [12].

Theorem 2. *Let $G = (V, E)$ be a biconnected 4-regular plane multigraph and let M and M' be the primal-dual multigraph pair for which G is the medial graph. If one of M and M' is simple, then G admits a plane Lombardi drawing preserving its embedding.*

Sketch of Proof. Assume w.l.o.g. that M is simple. If M (and hence also M') is polyhedral, then G admits a plane Lombardi drawing Γ by Eppstein [9]. Moreover, Γ is embedding-preserving (up to Möbius transformation), as the combinatorial embedding of M is unique (up to homeomorphism on the sphere).

If M is not 3-connected, we proceed in three steps. First, we augment M to a polyhedral graph M_p by iteratively adding p edges (any added edge splits a face of size at least four into two faces of size at least three). During this process, we also iteratively modify the dual graph and the medial graph as shown in Fig. 5a–b.

Second, we apply Eppstein's result to obtain a primal-dual circle packing of M_p and M'_p, together with a Lombardi drawing Γ_p of the medial graph G_p; see Fig. 5c. Finally, we revert the augmentation process from the first step by iteratively changing the plane Lombardi drawing Γ_p of G_p to a plane Lombardi drawing Γ of G. A main ingredient for this last step is the following: In the primal-dual circle packing of M_p and M'_p, every edge g of Γ_p lies in a region $\ell(g)$ that is bounded by a primal and a dual circle. This region $\ell(g)$ is interior-disjoint with the region $\ell(g')$ of any other edge g' of Γ_p. When removing an edge from the primal graph, a vertex is removed from the Lombardi drawing and the four incident edges are replaced by two edges connecting the non-common endpoints of the four edges. We show how to draw each new edge g that replaces g_1 and g_2 in a way that again has a uniquely assigned region $\ell(g)$ that is interior-disjoint with the regions of all other edges; see Fig. 6 for a sketch.

We remark that this result is not tight: there exist 4-regular plane multigraphs whose primal-dual pair M and M' contain parallel edges that still admit plane Lombardi drawings, e.g., knots 8_{12}, 8_{14}, 8_{15}, 8_{16}; see the arXiv version [12] for illustrations.

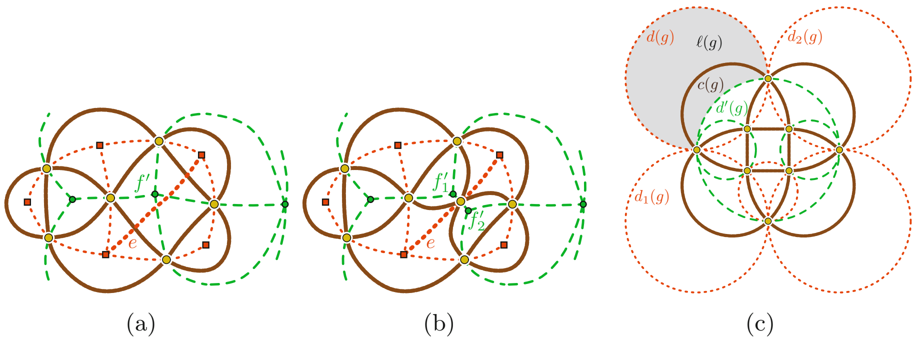

Fig. 5. (a)–(b) Modifications due to an edge addition and (c) a primal-dual circle packing. The medial graph G is drawn solid, the primal M is drawn dotted, and the dual M' is drawn dashed. The shaded area is the lens region $l(g)$.

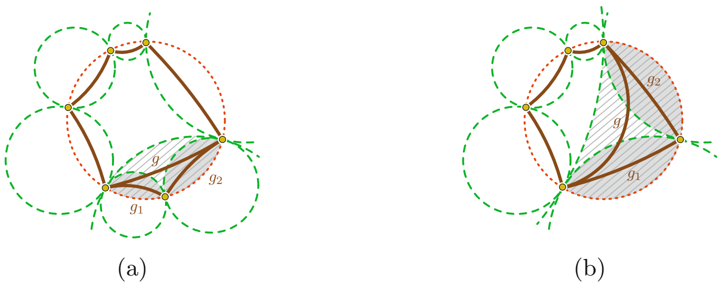

Fig. 6. Two examples of a lens region $\ell(g)$ resulting from $\ell(g_1)$ and $\ell(g_2)$: (a) convex and (b) reflex. The lens regions of g_1 and g_2 are drawn as shaded areas, while the one of g is the cross-hatched region.

We now prove that 4-regular polyhedral graphs are medial graphs of a simple primal-dual pair.

Lemma 2. *Let $G = (V, E)$ be a 4-regular polyhedral graph and let M and M' be the primal-dual pair for which G is the medial graph. If there is a multi-edge in M or in M', then the corresponding vertices of G either have a multi-edge between them or they form a separation pair of G.*

Proof. W.l.o.g., assume that there are two edges between vertices f and g in M. Let u and v be the vertices of G that these two edges pass through; see Fig. 7. The vertices f and g of M correspond to faces in the embedding of G that both contain u and v. Hence, the removal of u and v from G disconnects G into two parts: the part inside the area spanned by the two edges between f and g and the part outside this area. Both u and v have two edges in both areas, so either there is a multi-edge between u and v, or there are vertices in both parts, which makes u, v a separation pair of G.

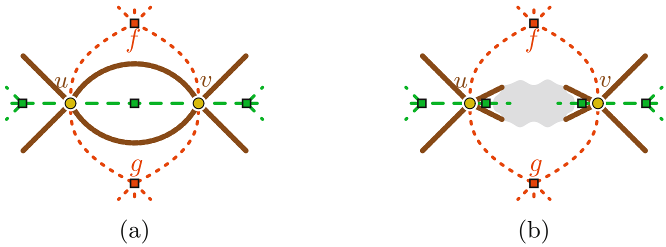

Fig. 7. If there is a multi-edge between vertices f and g in the primal, then there is a multi-edge (u, v) or a separation pair u, v in the medial.

Lemma 2 and Theorem 2 immediately give the following theorem.

Theorem 3. *Let $G = (V, E)$ be a 4-regular polyhedral graph. Then G admits a plane Lombardi drawing.*

4 Positive and Negative Results for Small Graphs

We next consider all prime knots with 8 vertices or less. We compute plane Lombardi drawings for those that have it and argue that such drawings do not exists for the others. We start by showing that no knot with a K_4 subgraph is plane Lombardi.

Lemma 3. *Every 4-regular plane multigraph G that contains K_4 as a subgraph does not admit a plane Lombardi drawing.*

Proof. Let a, b, c, d be the vertices of the K_4. Every plane embedding of K_4 has a vertex that lies inside the cycle through the other 3 vertices; let d be this vertex. Since d has degree 4, it has another edge to either one of a, b, c, or to a different vertex. In the former case, assume that there is a multi-edge between c and d. In the latter case, by 4-regularity, there has to be another vertex of a, b, c that is connected to a vertex inside the cycle through a, b, c; let c be this vertex. In both cases, c has two edges that lie inside the cycle through a, b, c.

Assume that G has a Lombardi drawing. Since Möbius transformations do not change the properties of a Lombardi drawing, we may assume that the edge (a, b) is drawn as a straight-line segment as in Fig. 8b. Since both c and d are neighbors of a and b, there are two corresponding placement circles by Property 1. In fact, since any two edges of a Lombardi drawing of a 4-regular graph must enclose an angle of 90° and since a and b have "aligned tangents" due to being neighbors themselves, the two placement circles coincide and a situation as shown in Fig. 8 arises. In particular, this means that in any Lombardi drawing of G the four vertices must be co-circular. It is easy to see that we cannot draw the missing circular arcs connecting c and d: any such arc must either lie completely inside or completely outside of the placement circle. Yet, the stubs for the two edges between c and d point inside at c and outside at d.

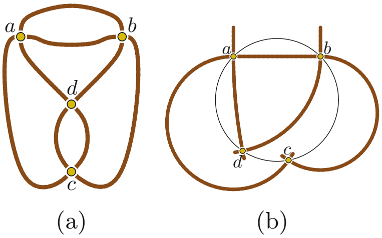

Fig. 8. Knot 4_1 (left) has no Lombardi drawing.

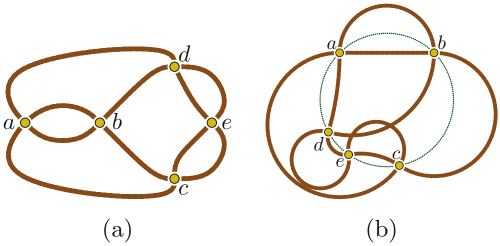

Fig. 9. Knot 5_2 and a non-plane Lombardi drawing.

The full proof for the following lemma is given in in the arXiv version [12].

Lemma 4. *Knots* 4_1 *and* 5_2 *are not plane Lombardi knots.*

Sketch of Proof. For knot 4_1, the claim immediately follows from Lemma 3. For knot 5_2 (see Fig. 9) we can argue that all five vertices must be co-circular. Unlike knot 4_1 we can geometrically draw all edges of knot 5_2 as Lombardi arcs, see the non-plane drawing in Fig. 9b. However, by carefully considering the smooth chain of arcs $a - c - e - d - b$ and their radii, we can prove that this path must self-intersect in any Lombardi drawing, so the claim follows.

As the above lemma shows, even very small knots may not have a plane Lombardi drawings. However, most knots with a small number of crossings are indeed plane Lombardi. In the arXiv version [12], we provide plane Lombardi drawings of all knots with up to eight vertices except 4_1 and 5_2. Most of these drawings can actually be obtained using the techniques from Sect. 2 and 3.

Theorem 4. *All prime knots with up to eight vertices other than* 4_1 *and* 5_2 *are plane Lombardi knots.*

Note that Theorem 4 implies that each of these knots has a combinatorial embedding that supports a plane Lombardi drawing. It is not true, however, that any embedding admits a plane Lombardi drawing. In fact, the knot 7_5 has an embedding that cannot be drawn plane Lombardi; details can be found in the arXiv version [12].

Theorem 5. *There exists an infinite family of prime knots and links that have embeddings that do not support plane Lombardi drawings.*

5 Plane 2-Lombardi Drawings of Knots and Links

Since not every knot admits a plane Lombardi drawing, we now consider plane 2-Lombardi drawings; see Fig. 10a for an example. Bekos et al [4] recently introduced *smooth orthogonal drawings of complexity k*. These are drawings where every edge consists of a sequence of at most k circular arcs and axis-aligned segments that meet smoothly with horizontal or vertical tangents, and where at every vertex, each edge emanates either horizontally or vertically and no two edges emanate in the same direction. For the special case of 4-regular graphs, every smooth orthogonal drawing of complexity k is also a plane k-Lombardi drawing. Alam et al. [2] showed that every plane graph with maximum degree 4 can be redrawn as a plane smooth-orthogonal drawing of complexity 2. Their algorithm takes as input an orthogonal drawing produced by the algorithm of Liu et al. [13] and transforms it into a smooth orthogonal drawing of complexity 2. We show how to modify the algorithm by Liu et al., to compute an orthogonal drawing for a 4-regular plane multigraph and then use the algorithm by Alam et al. to transform it into a smooth orthogonal drawing of complexity 2. Details are given in the arXiv version [12].

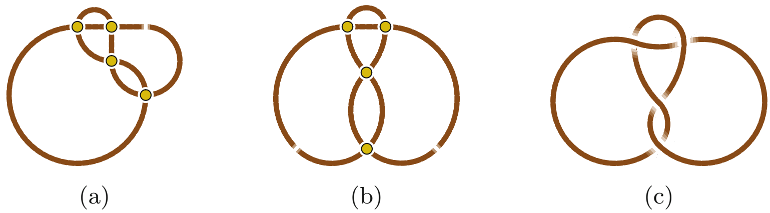

Fig. 10. Drawings of knot 4_1 which by Lemma 4 does not admit a plane Lombardi drawing. (a) A smooth orthogonal drawing of complexity 2, (b) a different plane 2-Lombardi drawing, and (c) a plane ε-angle Lombardi drawing.

Theorem 6. *Every biconnected 4-regular plane multigraph G admits a plane 2-Lombardi drawing with the same embedding.*

6 Plane Near-Lombardi Drawings

Since not all knots admit a plane Lombardi drawing, in this section we relax the perfect angular resolution constraint. We say that a knot (or a link) is *near-Lombardi* if it admits a drawing for every $\varepsilon > 0$ such that

1. All edges are circular arcs,
2. Opposite edges at a vertex are tangent;
3. The angle between crossing pairs at each vertex is at least $90° - \varepsilon$.

We call such a drawing a *ε-angle Lombardi drawing*. Note that a Lombardi drawing is essentially a 0-angle Lombardi drawing. For example, the knot 4_1 does not admit a plane Lombardi drawing, but it admits a plane ε-angle Lombardi drawing, as depicted in Fig. 10c.

Let Γ be an ε-angle Lombardi drawing of a 4-regular graph. If each angle described by the tangents of adjacent circular arcs at a vertex in Γ is exactly $90° + \varepsilon$ or $90° - \varepsilon$, then we call Γ an *ε-regular Lombardi drawing*. Note that any Lombardi drawing is a 0-regular Lombardi drawing.

We first extend some of our results for plane Lombardi drawings to plane ε-angle Lombardi drawings. The following Lemma is a stronger version of Theorem 2. The full proof is given in the arXiv version [12].

Lemma 5. *Let $G = (V, E)$ be a biconnected 4-regular plane multigraph and let M and M' be the primal-dual multigraph pair for which G is the medial graph. If one of M and M' is simple, then G admits a plane ε-regular Lombardi drawing preserving its embedding for every $0° \leq \varepsilon < 90°$.*

Sketch of Proof. We first direct the edges such that every vertex has two incoming opposite edges and two outgoing opposite edges by orienting the edges around each face that belongs to M in counter-clockwise order. We use the same primal-dual circle packing approach as in Theorem 2 to obtain a drawing of G', but instead of using the bisection of the lens region, we draw every edge with angles $45° + \varepsilon/2$ and $45° - \varepsilon/2$ around the source vertex in counter-clockwise order and around the target vertex in clockwise order. Whenever a vertex is removed, an incoming and an outgoing edge of it is substituted by a new edge between its neighbors, so the angles at the neighbors are compatible and the new edge can inherit the direction of the old edges.

The following Lemmas is a stronger version of Lemma 1 and Theorem 1. Since the proofs do not rely on 90° angles, they can be immediately applied to the stronger version. A formal proof of Lemma 6 can be found in the arXiv version [12].

Lemma 6. *Let $G = (V, E)$ be a 4-regular plane multigraph with a plane ε-angle Lombardi drawing Γ. Then, any lens multiplication G' of G also admits a plane ε-angle Lombardi drawing.*

Lemma 7. *Let A and B be two 4-regular plane multigraphs with plane ε-angle Lombardi drawings. Let a be an edge of A and b an edge of B. Then the composition $A + B$ obtained by connecting A and B along edges a and b admits a plane ε-angle Lombardi drawing.*

Let $G = (V, E)$ be a 4-regular plane multigraph and let $x \in V$ with edges (x, a), (x, b), (x, c), and (x, d) in counter-clockwise order. A *lens extension* of G is a 4-regular plane multigraph that is obtained by removing x and its incident edges from G, and adding two vertices u and v to G with two edges between u and v and the edges $(u, a), (u, b), (v, c), (v, d)$. Informally, that means that a vertex is substituted by a lens.

Lemma 8. *Let $G = (V, E)$ be a 4-regular plane multigraph with a plane ε-angle Lombardi drawing Γ. Then, any lens extension of G admits a plane $(\varepsilon+\varepsilon')$-angle Lombardi drawing for every $\varepsilon' > 0$.*

Sketch of Proof. Let $x \in V$ be the vertex that we want to perform the lens extension on, such that we get the edges $(u, a), (u, b), (v, c), (v, d)$ in the obtained graph G'. Let α be the angle between the tangents of (x, a) and (x, b) at x in Γ. Since Γ is a plane ε-angle Lombardi drawing, we have that $\alpha \le 90° + \varepsilon$. Further, the angle between the tangents of (x, c) and (x, d) at x in Γ is also α, while the angles between the tangents of (x, b) and (x, c) at x and between the tangents of (x, d) and (x, a) at x are both $180° - \alpha$. We apply the Möbius-transformation on Γ that maps the edges (x, a) and (x, d) to straight-line segments and a lies on the same y-coordinate and to the right of x; hence, d lies strictly below x.

We aim to place v such that the angle between the arcs (v, c) and (v, d) is $\alpha+\lambda$ for some $0 < \lambda \le \varepsilon'$, which we will show how to choose later. We have fixed ports at c and d and a fixed angle $\alpha+\lambda$ at v. According to Property 1, all possible positions of v lie on a circle through c and d. Note that the circle through c, d, and x describes all possible positions of neighbors of c and d with angle α. Since the desired angle gets larger, the position circle for v contains a point v_d on the straight-line edge (x, d) and a point v_c on the half-line starting from x with the angle of the port used by the arc (x, c); see Fig. 11. We denote by C_v^λ the circular arc between v_c and v_d on the placement circle of v that gives the angle $\alpha+\lambda$ at v. We use the same construction for u to obtain the circular arc C_u^λ between u_a and u_b. Since the drawing of G is plane, there exists some non-empty region in which we can move x, such that the arcs $(x, a), (x, b), (x, c), (x, d)$ are drawn with the same ports at a, b, c, d and do not cross any other edge of the drawing. We choose λ as the largest value with $0 < \lambda \le \varepsilon'$ such that the two circular arcs C_u and C_v lie completely inside this region.

We show how to find a pair of points on C_v^λ and C_u^λ such that we can connect them via a lens in the arXiv version [12].

Lemma 9. *Every 4-regular plane multigraph with at most 3 vertices admits a plane ε-regular Lombardi drawing for every $0 \le \varepsilon < 90°$.*

Proof. There are only two 4-regular multigraphs with at most 3 vertices and each of them has a plane Lombardi drawing as depicted in Fig. 12a. For some $0° < \varepsilon < 90°$, we can obtain a plane ε-regular Lombardi drawing by simply making the circular arcs larger or smaller, as depicted in Fig. 12b.

We are now ready to present the main result of this section. The proof boils down to a large case distinction using the tools developed in the previous discussion. We split the original graph into biconnected components and then use Lemma 9 and 5 as base cases. With the help of lens extensions, lens multiplications, and knot sums we can combine the "near-Lombardi" drawings of the biconnected graphs to generate an "near-Lombardi" drawing of the original graph. As a consequence, every knot is near-Lombardi. The full proof is given in the arXiv version [12].

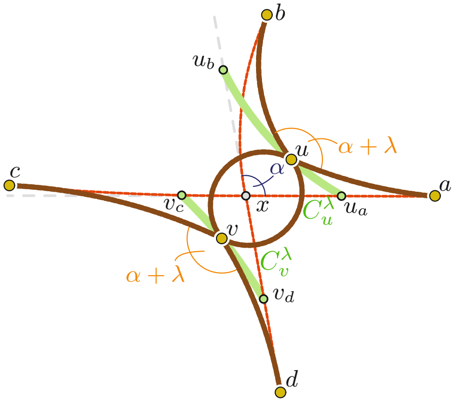

Fig. 11. The circular arc C_u^λ between u_a and u_b on the placement circles of u and the circular arc C_v^λ between v_c and v_d on the placement circles of v.

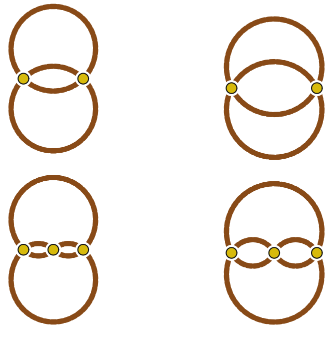

Fig. 12. The only biconnected 4-regular multigraphs with at most 3 vertices. (a) plane Lombardi and (b) plane ε-angle Lombardi drawings.

Theorem 7. *Let $G = (V, E)$ be a biconnected 4-regular plane multigraph and let $\varepsilon > 0$. Then G admits a plane ε-angle Lombardi drawing.*

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