Autumn School 2015
Julius-Maximilians-Universität
Würzburg

Generalized complex geometry vs. holomorphic Poisson geometry—with integrations

Michael Bailey (joint w. Marco Gualtieri)

Department of Mathematics, Utrecht University (October 2015)

Universiteit Utrecht

Summary

Generalized complex (G.C.) geometry [4], a generalization of symplectic and complex geometry, was originally defined via a complex structure on a *Courant algebroid*. As a consequence of the first author's local structure theorem, G.C. geometry may alternatively be understood as a "twisted" or *localized* version of holomorphic Poisson geometry. Therefore, we can integrate G.C. structures to certain holomorphic symplectic groupoids in analogy with holomorphic Poisson structures integrating to holomorphic symplectic groupoids. However, the "holomorphic" part of the integration is defined only up to (holomorphic) Morita equivalence, i.e., as a holomorphic stack. Nonetheless, with these data packaged as a ***weakly holomorphic symplectic groupoid***, we can recover the underlying G.C. structure, establishing a full Lie correspondence, with the correct notions of uniqueness and equivalence. This answers a longstanding question as to what is the natural object integrating G.C. structures.

Because of generalized complex geometry's connection to physics via string theory and mirror symmetry, it is of special interest to quantize generalized complex manifolds. In future work, we hope to use these holomorphic symplectic integrations to produce deformation quantizations via the construction of Kontsevich [5].

Generalized complex structures

A generalized complex structure, $\mathbb{I}$, on a smooth manifold M is a complex structure,

$$\mathbb{I} : TM \oplus T^*M \longrightarrow TM \oplus T^*M,$$

with $\mathbb{I}^2 = -\mathrm{Id}$, $\mathbb{I}$ orthogonal w.r.t. the standard pairing on $TM \oplus T^*M$, and which satisfies an integrability condition. Integrability may be expressed by considering the equivalent data of $\mathbb{I}$'s $+i$-eigenbundle

$$L \subset \mathbb{C} \otimes (TM \oplus T^*M)\,.$$

L is a complex Dirac structure, involutive w.r.t. a (possibly twisted) Courant bracket, such that $L \cap \bar{L} = 0$. The "real rank zero" condition ensures that it will be the $+i$-eigenbundle of some $\mathbb{I}$.
Examples include symplectic structures (L = graph$(i\omega)$), complex structures ($L = T^{0,1} \oplus T^*_{1,0}$) and holomorphic Poisson structures ($L = T^{0,1} \oplus$ graph$\left(\pi : T^*_{1,0} \longrightarrow T^{1,0}\right)$).

Gauge Symmetry: B-transforms

The extended symmetry group of $TM \oplus T^*M$ is taken to include, not just the diffeomorphisms acting by pushforward on TM and inverse pullback on T^*M, but also the *B-transforms*, where a closed 2-form B acts via

$$e^B \cdot (X + \xi) := X + \iota_X B + \xi \qquad (X \in TM, \xi \in T^*M).$$

This preserves the Courant bracket, the anchor map to TM, and the canonical symmetric pairing between TM and T^*M. Thus, it takes generalized complex structures to generalized complex structures.

The associated real Poisson structure and the real symplectic integration

A generalized complex structure, $\mathbb{I} : T \oplus T^* \longrightarrow T \oplus T^*$, has an associated real Poisson structure: if $a : T \oplus T^* \longrightarrow T$ is the projection to T, then the Poisson anchor map is

$$P = a \circ \mathbb{I}|_{T^*}.$$

Note that P is invariant under B-transforms. In the case of a hol. Poisson structure $\pi = IP + iP : T^*_{1,0} \longrightarrow T^{1,0}$, this is just the imaginary part.
If P satisfies known integrability conditions [2] then it integrates to a symplectic groupoid G. However, G, with its symplectic form is not enough to recover $\mathbb{I}$. (Note that if G were *holomorphic symplectic* it would differentiate to a holomorphic Poissson structure, which would be generalized complex.)

B-transform of holomorphic Poisson

If I_0 is a complex structure and $\pi_0 = I_0 P + iP$ a hol. Poisson structure, with (I_0, π_0) represented as generalized complex, and if B is a closed 2-form such that

$$BI_0 + I_0^* B + BPB = 0$$

then the B-transform of L is once again hol. Poisson, with complex structure $I_1 = I_0 + PB$ and hol. Poisson structure $\pi_1 = I_1 P + iP$. [3]

Local holomorphic structure

In earlier work [1], we showed:
Theorem (B. 2012) *In a small enough neighbourhood of any point, a generalized complex structure is equivalent, up to gauge, to a product of a symplectic manifold with a hol. Poisson manifold whose Poisson tensor vanishes at the point in question.*

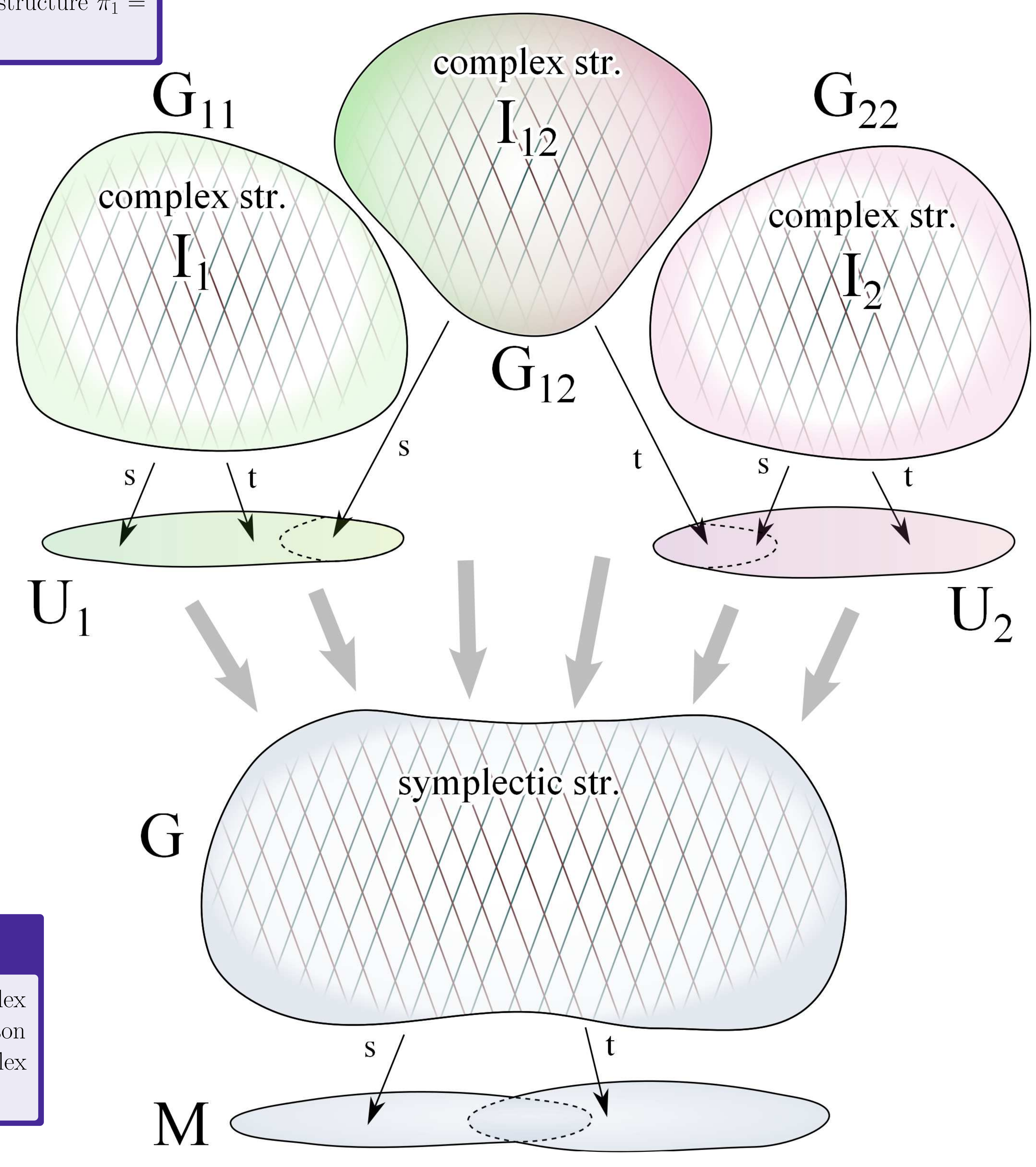
Assuming the G.C. structure has "Poisson rank" $0 \mod 4$ (which for out purposes does not lose generality), a neighbourhood of ***any*** point may be represented as hol. Poisson. However, this gauge– or B–transform can take one hol. Poisson structure to another with an entirely different complex structure! ***The local complex structure is noncanonical,*** though it is constrained by the G.C. structure. M may be covered by local complex charts, but the transition maps will only be generalized complex (i.e., with a B-transform), not holomorphic, and M may not even admit a complex structure globally.

Groupoid localizations

If $G \rightrightarrows X$ is a [hol./symplectic/etc.] Lie groupoid and $\mathcal{U} = \{U_i, \ldots\}$ is an open cover of X, then we have the *localization*,

$$G_{\mathcal{U}} = \bigsqcup_{i,j} G_{ij} = \bigsqcup_{i,j} s^{-1}(U_i) \cap t^{-1}(U_j),$$

a new [hol./symplectic/etc.] groupoid over base $\bigsqcup \mathcal{U}$ consisting of elements of G labeled with indices ij to designate source in U_i and target in U_j.
In the picture, we see the groupoid G (below) localizes over $\{U_1, U_2\}$ to a groupoid with 4 components (3 shown), G_{11} and G_{22}—full subgroupoids over U_1 and U_2 respectively—G_{12}, consisting of elements going from U_1 to U_2, and similarly G_{21} (not shown).



The Holomorphic Integration of L

If $G \rightrightarrows M$ is the real symplectic groupoid integrating the real Poisson structure of a generalized complex manifold M, then G may not admit a compatible complex structure, but (if the Poisson rank of M is $0 \mod 4$) it admits a localization that does.

Theorem. *If $(M, \mathbb{I})$ is a generalized complex manifold, G is a source-simply-connected symplectic groupoid integrating the real Poisson structure associated to $\mathbb{I}$, and $\mathcal{U} = \{U_i, \ldots\}$ is a cover of M by open sets, on each of which $\mathbb{I}$ is gauge-equivalent to a hol. Poisson structure for some $\mathbb{C}$-structure I_i, then the localization $G_{\mathcal{U}}$ is hol. symplectic in a natural way.*

The "diagonal" components $G_{ii} \rightrightarrows U_i$ are (formal completions of) the usual hol. symplectic integrations of the hol. Poisson structures [6], whereas $G_{ij} = s^{-1}(U_i) \cap t^{-1}(U_j)$ has a complex structure mixing I_i with I_j.

Lie Correspondence

Given the data of a symplectic groupoid and a holomorphic localization of it, one can recover a generalized complex manifold by a kind of "differentiation." One takes the usual differentiation of a hol. symplectic groupoid to a hol. Poisson manifold on each $G_{ii} \rightrightarrows U_i$, and on the intersections $U_i \cap U_j$ the two hol. Poisson structures glue as generalized complex structures via a B-transform—data which can be recovered from the hol. symplectic structure on G_{ij}.

The Holomorphic Stack

Localization is a kind of *Morita equivalence* of groupoids, and Lie groupoids up to Morita equivalence are *differential stacks*. Then the integrating object of a generalized complex manifold (given the parity condition) is a symplectic Lie groupoid together with a "stacky" complex strcture, i.e., a Morita-equivalent holomorphic stack. We call these data a ***weakly holomorphic symplectic groupoid***.

References

[1] Michael Bailey.
Local classification of generalized complex structures.
J. Differential Geom., 95(1):1–37, 08 2013.

[2] Marius Crainic and Rui Loja Fernandes.
Integrability of poisson brackets.
J. Differential Geom., 66(1):71–137, 01 2004.

[3] Marco Gualtieri.
Branes on poisson varieties.
In *The Many Facets of Geometry*, pages 368–394. Oxford University Press (OUP), jul 2010.

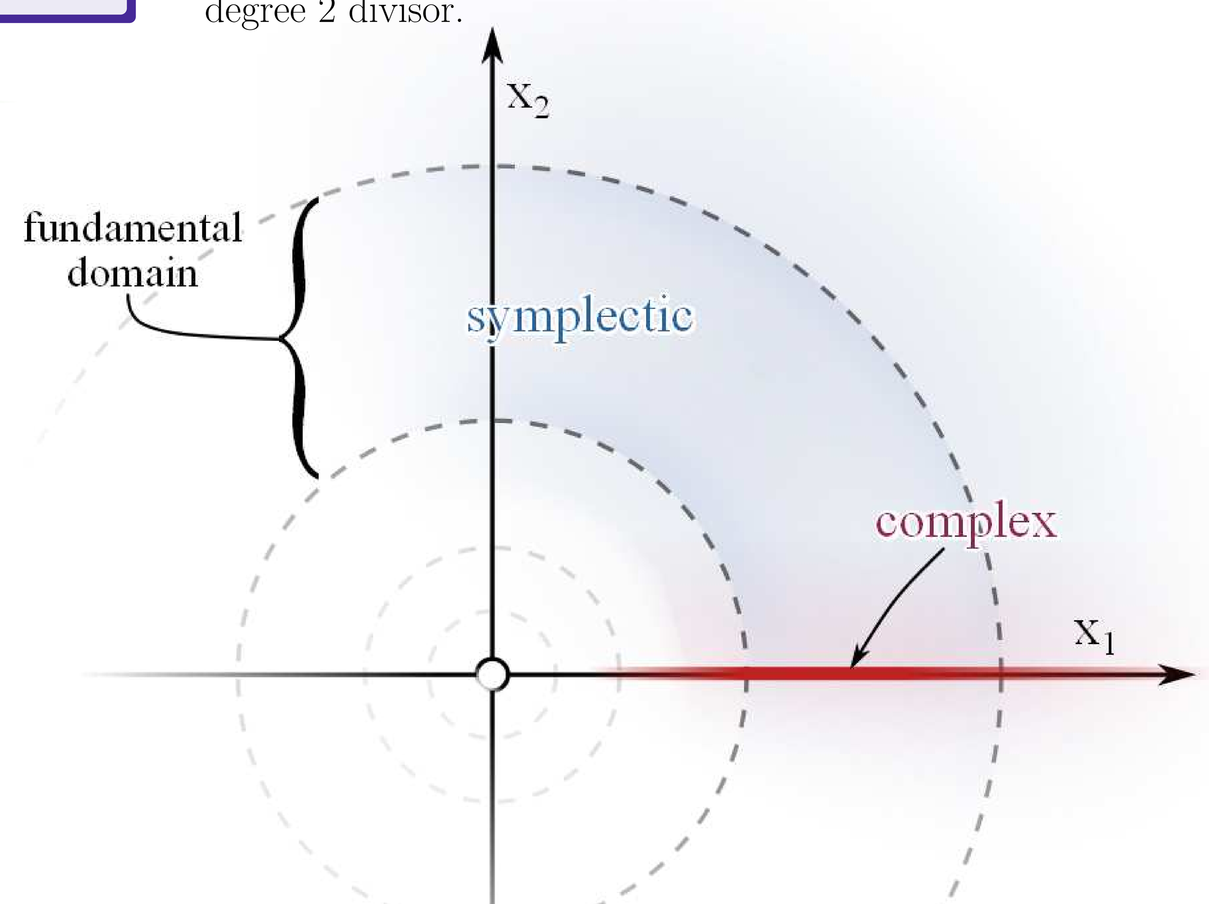
[4] Marco Gualtieri.
Generalized complex geometry.
Ann. Math., 174(1):75–123, 2011.

[5] Maxim Kontsevich.
Deformation quantization of algebraic varieties.
Letters in Mathematical Physics, 56(3):271–294, 2001.

[6] Camille Laurent-Gengoux, Mathieu StiÃĺnon, and Ping Xu.
Integration of holomorphic lie algebroids.
Mathematische Annalen, 345(4):895–923, 2009.

Example: the Hopf surface

The Hopf surface is $X = \left(\mathbb{C}^2 \setminus \{0\}\right)/\mathbb{Z}$, where $1 \in Z$ acts by multiplication by 2. X admits various holomorphic Poisson structures, in this case just hol. sections of its anticanonical bundle $\bigwedge^2 T^{1,0}$. Any such nontrivial sections will vanish on a degree 2 divisor.



X admits a G.C. structure $\mathbb{I}$ whose Poisson structure vanishes on a degree 1 divisor. If x_1, x_2 are coordinates for $\mathbb{C}^2 \setminus \{0\}$ and $R^2 = |x_1|^2 + |x_2|^2$, then $L = \mathrm{graph}(C) \subset \mathbb{C} \otimes (T \oplus T^*)$, where

$$C = \frac{1}{R^2}\left(\frac{2x_1}{\bar{x}_2} d\bar{x}_1 \wedge d\bar{x}_2 + dx_1 \wedge d\bar{x}_1 + dx_2 \wedge d\bar{x}_2\right)$$

is a complex 2-form (homogeneous of degree 0, so it passes to the quotient X).

Away from $x_2 = 0$, the imaginary part $\mathrm{Im}(C)$ gives a symplectic structure, and at $x_2 = 0$, C blows up, but L extends to $\left(T^{1,0} \oplus T^*_{0,1}\right)|_{\{x_2=0\}}$, a $\mathbb{C}$-structure.
If we remove the curve $x_1 = 0$, then on $U_1 = X \setminus \{x_2 = 0\}$ we can find a new $\mathbb{C}$-structure and a B-transform such that $e^B \cdot \mathbb{I}$ is hol. Poisson vanishing to first order at $x_2 = 0$. On the other hand, on $U_2 = X \setminus \{x_1 = 0\}$, $\mathbb{I}$ is B-equivalent to hol. symplectic, for another new $\mathbb{C}$-structure. This gives us two holomorphic local charts for X. On $U_1 \cap U_2$, the $\mathbb{C}$-structures will not agree—they are related by a B-transform. However, the localized groupoid $G_{\mathcal{U}} \rightrightarrows U_1 \sqcup U_2$ is holomorphic symplectic.