Universiteit Utrecht

Faculty of Science

Unitary Representations of Lie Groups

Representation Theory in Quantum Mechanics

Bachelor Thesis TWIN

Lucas Smits

Physics and Astronomy // Mathematics

Supervisor:

Dr. J.W. van de Leur
Mathematical Institute

January 15, 2020

Contents

Introduction

Lie groups were first introduced by the Norwegian mathematican Sophus Lie in the final years of the nineteenth century in his study on the symmetry differential equations. Heuristically, Lie groups are groups whose elements are organized in a smooth way, contrary to discrete groups. Their symmetric nature makes them a crucial ingrediant in today's study of geometry. Lie groups are often studied through representation theory, that is, the identification of group elements with linear transformations of a vector space. This enables to reduce problems in abstract algebra to linear algebra, which is a well-understood area in math.

Nowadays, Lie groups play a dominant role in theoretical physics, as symmetry groups of physical systems. This started off with Wigner's influential 1939 paper *On Unitary Representations of the Inhomogenous Lorentz Group*. In this paper, Wigner made the connection between certain representations of the Poincaré group, the symmetry group of special relativity, and elementary particles. This has for a long time been the attempt to define elementary particles, and to classify them accordingly. This is nowadays known in theoretical physics as *Wigner's classification*.

Structure of this thesis

In this thesis, we will explore the fundamental ideas of the theory of Lie groups and we will qualitatively discuss the aspects of Wigner's classification. The first chapter serves as an introduction in the theory of Lie groups, whereas the second chapter gives an introduction to representation theory of Lie groups. We will develop a way to analyse the necessary representations of the Poincaré group in chapter 3, by what is known as *Mackey Theory*. We will bundle all results of the first 3 chapters in the last chapter to discuss Wigner's classification. We will assume that the reader is known with basic differential geometry as can be found in chapter 1 up to 4 in [2] and basic representation theory as can be found in [10].

Chapter 1

Lie Groups and Lie Algebras

In this chapter, we will introduce the fundamental concepts which will play a central role throughout the rest of this thesis: *Lie groups* and *Lie algebras*. As we will see in chapter 2, these concepts will appear naturally in quantum mechanics. In this chapter, we will introduce Lie groups and Lie algebras as individual objects and we will study their basic properties. Then, we will show how every Lie group gives rise to a Lie algebra which we call the Lie algebra of the Lie group. We mainly follow Lee [2] and van den Ban [3] and try to be brief and concise whenever it is possible, but still attempt to present the material exhaustively and rigorously.

1.1 Lie Groups

In this section, we introduce Lie groups and study their basic properties. We assume the reader is familiar with basic group theory as can be found in [1]. Nevertheless, let us recall the definition of a group.

Definition 1.1 (Group [1])**.** *A group G is a set G together with a multiplication $\mu : G \times G \to G, (g, h) \mapsto \mu(g, h) = gh$ and an inversion $\iota : G \to G, g \mapsto \iota(g) = g^{-1}$ which satisfies:*

i. Associativity: $(gh)k = g(hk)$ for all $g, h, k \in G$.

ii. Identity element: There exists $e \in G$ such that $eg = ge = g$ for all $g \in G$.

iii. Inverse element: $gg^{-1} = g^{-1}g = e$ for all $g \in G$.

Example 1.2. *Consider the circle group $\mathbb{T} := \{z \in \mathbb{C} \colon |z| = 1\}$. Then $\mathbb{T}$ forms an abelian group under multiplication of complex numbers with identity element 1, since $|zw| = |z||w| = 1$ for all $z, w \in \mathbb{T}$ and every $z \in \mathbb{T}$ has an inverse $\frac{1}{z}$.*

When we restrict the Euclidean topology on $\mathbb{C}$ to $\mathbb{T}$, the multiplication and inversion map become continuous. This turns $\mathbb{T}$ in a *topological group*.

Definition 1.3 (Topological group [3])**.** *A topological group G is a group G together with a Hausdorff topology on G such that the multiplication map $(g, h) \mapsto gh$ and the inversion map $g \mapsto g^{-1}$ are continuous.*

Where discrete groups only allow us to describe discrete symmetries of an object, topological groups enable us to consider continuous symmetries as well and therefore appear naturally in many areas in phsyics. When we return to our example $\mathbb{T}$, both the group multiplication and the inversion are smooth maps, which means that $\mathbb{T}$ is a *Lie group*.

Definition 1.4 (Lie group [2])**.** *A Lie group G is a finite dimensional smooth manifold equippped with a group structure such that the multiplication map $(g, h) \mapsto gh$ and the inversion map $g \mapsto g^{-1}$ are smooth.*

Apart from $\mathbb{T}$, we will discuss two other important examples of Lie groups.

Example 1.5 ([2])**.** *The space $GL_n(\mathbb{R})$ of invertible $n \times n$-matrices with real entries is a group under matrix multiplication. The multiplication $(A, B) \mapsto AB$ is smooth, as the entries of AB are polynomials in the entries of A and B. The inversion map $A \mapsto A^{-1}$ as the entries of A^{-1} are given by ${a^{-1}}_{ij} = \frac{1}{\det(A)} C_{ji}$, with C the cofactor matrix, so the entries of A^{-1} are polynomials in the entries of A, hence smooth.*

Given Lie groups $G_1, ..., G_k$, the direct product $G_1 \times ... \times G_k$ with componentwise multiplication $(g_1, ..., g_k)(h_1, ..., h_k) = (g_1 h_1, ..., g_k h_k)$ is a Lie group.

If G is a Lie group and H is a subgroup of G, we would like to know whether H is a Lie group itself. This happens to be the case when H is a smooth submanifold of G, since the restrction $\mu_G|_H$ and the inversion $\iota_G|_H$ are smooth.

Theorem 1.6 ([3])**.** *Let G be a Lie group and $H \subset G$ a subgroup. Then the following statements are equivalent.*

i. H is topologically closed.

ii. H is a smooth submanifold of G

Proof. For the proof, we refer to Theorem 2.16 in [3]. □

We can infer from Theorem 1.6 that every closed subgroup of a Lie group is again a Lie group.

Definition 1.7 (Lie group homomorphism [2])**.** *Let G and H be Lie groups, a Lie group homomorphism is a smooth group homomorphism $f : G \to H$. If f is a diffeomorphism, we call f a Lie group isomorphism.*

Definition 1.8 (Lie subgroup, [3])**.** *Let G be a Lie group and $H \subset G$ a subgroup equipped with the structure of a Lie group. We call H a Lie subgroup if the inclusion map $\iota : H \hookrightarrow G$ is a Lie group homomorphism.*

Example 1.9. *we see that $\mathbb{T}$ is a Lie subgroup of $\mathbb{C}^*$ since $\mathbb{T}$ is a subgroup of $\mathbb{C}^*$ and the inclusion map $\iota : \mathbb{T} \hookrightarrow \mathbb{C}^*$ is a Lie group homomorphism.*

Definition 1.10 (One-parameter subgroup,[3])**.** *Let G be a Lie group. A one-parameter subgroup is a Lie group homomorphism $\alpha : \mathbb{R} \to G$.*

Example 1.11. *A more interesting example of a Lie group homomorphism is the map* $\zeta : \mathbb{R} \to \mathbb{T}$ *given by* $\zeta(x) = e^{2\pi i x}$*. This map is also a one-parameter subgroup of* $\mathbb{T}$*.*

If M and N are smooth manifolds and $f : M \to N$ is a smooth map, we denote the tangent space at $p \in M$ by T_pM and the tangent map of f at p by $(df)_p : T_pM \to T_{f(p)}M$. We denote the collection of all vector fields on M by $\mathfrak{X}(M)$. Whenever f is a diffeomorphism, we can transfer a vector field X on M to a vector field $f_*(X)$ on N where $f_*(X)_{f(p)} = (df)_p X_p$. In the case of Lie groups, it turns out that there exists an important one-to-one correspondence between one-parameter subgroups and T_eG. Before we are able to describe this, we need the notion of *left-invariant* vector fields.

Let G be a Lie group and let $g \in G$. Then the *left translation* by g, denoted by $L_g : G \to G$, $h \mapsto gh$ is a smooth map. The map L_g happens to be a diffeomorphism since the map $L_{g^{-1}}$ is a smooth inverse of L_g. Likewise, the *right translation* by g, denoted by $R_g : G \to G$, $h \mapsto hg$ is a diffeomorphism as well. So, for every pair $x, y \in G$, the maps $L_{yx^{-1}}$ and $R_{x^{-1}y}$ are Lie group isomorphisms mapping x to y. As we will see, many important theories in Lie theory arise from the fact that any point can be mapped to any other point by a global diffeomorphism.

Definition 1.12 (Left-invariant vector field [3])**.** *Let* $v \in \mathfrak{X}(G)$ *be a vector field on* G*. We say* v *is left-invariant if* $v = (L_x)_* v$ *for all* $x \in G$*.*

Note that Definition 1.12 is equivalent to saying that

$$v_{xy} = (dL_x)_y v_y. \tag{1.1.1}$$

The collection of left-invariant vector fields forms a linear subspace of $\mathfrak{X}(G)$ which we denote by $\mathfrak{X}_L(G)$. The following result shows that there exists a one-to-one correspondence between T_eG and $\mathfrak{X}_L(G)$.

Proposition 1.13 ([3])**.** *The spaces* $\mathfrak{X}_L(G)$ *and* T_eG *are isomorphic vector spaces.*

Proof. We mainly follow [3] for this proof. Define the map $\mathrm{ev}_e : \mathfrak{X}_L(G) \to T_eG$ by $v \mapsto v_e$. Linearity of this map is clear. We will first show injectivity. Let $v \in \mathfrak{X}_L(G)$. From (1.1.1) with $y = e$, it follows that $v_x = (dL_x)_e v_e$, hence a left-invariant vector field is completely determined by its value at the identity. It follows that ev_e is injective.

For surjectivity, let $X \in T_eG$ and define the vector field v^X on G by $v^X_x = (dL_x)_e X$. Since the map $(x, y) \mapsto L_x(y)$ is a smooth map $G \times G \to G$, it follows from differentiating this map with respect to y at $y = e$ that $x \mapsto (dL_x)_e X$ is a smooth map $G \to TG$. Therefore, v^X indeed defines a vector field on G. We will now show that v^X is left-invariant. Remark that $L_{xy} = L_x \circ L_y$, hence $(dL_{xy})_e = (dL_x)_y \circ (dL_y)_e$. A quick computation shows that

$$v^X_{xy} = (dL_{xy})_e X = (dL_x)_y (dL_y)_e X = (dL_x)_y v^X_y,$$

hence $v^X \in \mathfrak{X}_L(G)$. Also note that $v^X_e = X$ and it follows that ev_e is not only injective, but surjective too. $\square$

For $X \in T_eG$, we define α^X to be the *maximal integral curve* of v^X starting at e.

Lemma 1.14 ([3])**.** *Let G be a Lie group and let $X \in T_eG$. The integral curve α^X is a one-parameter subgroup of G for which the map $(X, t) \mapsto \alpha^X(t)$ is a smooth map $T_eG \times \mathbb{R} \to G$.*

Proof. We refer to Lemma 3.2 in [3]. $\square$

We are now able to define the *exponential map* of a Lie group, which allows us to pass information from the tangent space of a Lie group to the Lie group itself.

Definition 1.15 (Exponential map [3])**.** *Let G be a Lie group. We define the exponential map $\exp_G : T_eG \to G$ by $\exp_G(X) = \alpha^X(1)$.*

We will summarize the main properties of the exponential map in the following proposition.

Proposition 1.16 (Properties of the exponential map [3])**.** *Let G be a Lie group. Then for all $s, t \in \mathbb{R}$ and $X \in T_eG$, we have that:*

i. $\exp_G(sX) = \alpha^X(s)$.

ii. $\exp_G((s+t)X) = \exp_G(sX)\exp_G(tX)$.

iii. The map $\exp_G$ is smooth and $(d\exp)_0 = Id_{T_eG}$.

iv. There exists an open neighbourhood $V \subset T_eG$ of 0 and $W \subset G$ of e such that $\exp_G : V \to W$ is a diffeomorphism.

Proof. We mainly follow [3] for this proof. For *i*, we consider the curve $\gamma : t \mapsto \alpha^X(st)$. Then $\gamma(0) = e$ and

$$\frac{d}{dt}\gamma(t) = s\dot{\alpha}^X(st) = sv^X_{\gamma(t)} = v^{sX}_{\gamma(t)}.$$

Since the domain of γ is $\mathbb{R}$, we see that γ is the maximal integral curve of v^{sX} starting at e, hence $\gamma = \alpha^{sX}$. Evaluating at $t = 1$ yields $\exp_G(sX) = \alpha^{sX}(1) = \gamma(1) = \alpha^X(s)$.

Formula *ii* follows directly from *i* and Lemma 1.14: $\exp_G\big((s+t)X\big) = \alpha^X(s+t) = \alpha^X(s)\alpha^X(t) = \exp_G(sX)\exp_G(tX)$.

For *iii* and *iv*, we know from Lemma 1.14 that the map $(X, t) \mapsto \alpha^X(t)$ is a smooth map $T_eG \times \mathbb{R} \to G$, hence the map $X \mapsto \alpha^X(1) = \exp_G(X)$ is smooth. Besides, we find that

$$(d\exp_G)_0 X = \frac{d}{dt}\bigg|_{t=0} \exp_G(tX) = \dot{\alpha}^X(0) = v^X_e = X,$$

hence $(d\exp_G)_0 = Id_{T_eG}$. This last map is an isomorphism. By the inverse function theorem, there exists an open neighbourhood $V \subset T_eG$ of 0 and an open neighbourhood $W \subset G$ of e such that $\exp_G : V \to W$ is a diffeomorphism. $\square$

The exponential map turns out to be important in establishing the relation between Lie groups and Lie algebras, as we will see in Section 1.3. We finish with our discussion on one-parameter subgroups with the following proposition which establishes the one-to-one correspondence between T_eG and one-parameter subgroups of G.

Proposition 1.17 ([3]). *Let G be a Lie group and let $X \in T_eG$. Then the we have the following:*

i. The map $\alpha : t \mapsto \exp_G(tX)$ is a one-parameter subgroup of G.

ii. Let α be a one-parameter subgroup and let $X = \dot{\alpha}(0)$. Then $\alpha(t) = \exp_G(tX)$.

Proof. We mainly follow [3] for this proof. We first proof *i*. By combining Lemma 1.14 and Proposition 1.16, we immediately have that $\alpha : t \mapsto \exp(tX) = \alpha^X(t)$ is smooth and that $\alpha(t+s) = \alpha(t)\alpha(s)$, hence α is a one-parameter subgroup of G.

For *ii*, let α be a one-parameter subgroup with $\dot{\alpha}(0) = X$. Note that $\alpha(0) = e$ and that

$$\frac{d}{dt}\alpha(t) = \left.\frac{d}{ds}\right|_{s=0} \alpha(t+s) = \left.\frac{d}{ds}\right|_{s=0} \alpha(t)\alpha(s) = (dL_{\alpha(t)})_e\dot{\alpha}(0) = (dL_{\alpha(t)})X = v^X(\alpha(t)),$$

so α is an integral curve of v^X starting at e. It follows by uniqueness of integral curves that $\alpha = \alpha^X$ and we use Proposition 1.11 to conclude that $\alpha(t) = \alpha^X(t) = \exp_G(tX)$. $\square$

What the lemma above tells us is that for each $X \in T_eG$, there exists a unique one-parameter subgroup α with $\dot{\alpha}(0) = X$. Now, we come to an important result about Lie groups in which we need all the concepts we have developed up to this point.

Theorem 1.18. *Let G and H be Lie groups and let $\phi : G \to H$ be a Lie group homomorphism. Then the following diagram commutes:*

$$\begin{tikzcd} G \arrow[r, "\phi"] & H \\ T_eG \arrow[u, "\exp_G"] \arrow[r, "(d\phi)_e"] & T_eH \arrow[u, "\exp_H"'] \end{tikzcd}$$

Proof. Let $X \in T_eG$ and put $Y = (d\phi)_eX$. Then $\alpha : t \mapsto \exp_H(tY)$ is the unique one-parameter subgroup of H with $\dot{\alpha}(0) = Y$. We see that $\beta : t \mapsto \phi(\exp_G(tX))$ is also a one-parameter subgroup of H as ϕ is a Lie group homomorphism, with $\dot{\beta}(0) = (d\phi)_e(d\exp)_0X = (d\phi)_eX = Y$. It follows by Proposition 1.14 that $\alpha = \beta$ and the result follows by setting $t = 1$. $\square$

We denote by C_g the conjugation map $h \mapsto ghg^{-1}$, which is readily seen to be a diffeomophism fixing the identity element e.

Definition 1.19 (Adjoint representation [4]). *Let G be a Lie group and let $x \in G$. We define the adjoint mapping of x by $Ad(x) := (dC_x)_e : T_eG \to T_eG$. The mapping $Ad : G \to GL(T_eG)$, $x \mapsto Ad(x)$ is called the adjoint representation of G in T_eG.*

Proposition 1.20 ([3])**.** *let G be a Lie group. The adjoint representation of G in T_eG is a Lie group homomorphism.*

Proof. We refer to Lemma 4.4 in [3]. □

Since $Ad(e) = Id_{T_eG}$ and $T_{Id_{T_eG}}GL(T_eG) = end(T_eG)$, we define the linear map $ad : T_eG \to end(T_eG)$ by $ad = (dAd)_e$. In Section 1.3, we will use the linear map ad to equip T_eG with some extra structure to turn it into a Lie algebra.

1.2 Lie Algebras

In this section, we will introduce *Lie algebras* as individual objects and study their structure and properties, preparatory to relating them to Lie groups. In this section, we will mostly follow [4] and [5].

Definition 1.21 (Lie algebra [5])**.** *A Lie algebra $\mathfrak{g}$ is a finite dimensional vector space over $\mathbb{K}$ together with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:*

i. Anticommutavity: For all $X, Y \in \mathfrak{g}$, we have $[X, Y] = -[Y, X]$.

ii. Jacobi identity: For all $X, Y, Z \in \mathfrak{g}$, we have $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.

The Jacobi identity is in some sense a substitution for associativity. In general a Lie algebra need not be commutative. If a Lie algebra $\mathfrak{g}$ is commutative, then $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. We will now discuss some examples of Lie algebras.

Example 1.22. *The vector space $\mathbb{R}^3$ equipped with the cross product forms a Lie algebra. The reader can easily verify that $x\times y = -y\times x$ and that $x\times(y\times z)+y\times(z\times x)+z\times(x\times y) = 0$.*

The vector space $M_n(\mathbb{R})$ of real $n\times n$-matrices together with the commutator bracket $[A, B] = AB - BA$ becomes an n^2-dimensional Lie algebra, denoted by $\mathfrak{gl}_n(\mathbb{R})$.

Similar to Lie groups, we want to define when Lie algebras are *homomorphic Lie algebras.*

Definition 1.23 (Lie algebra homomrphism [5])**.** *Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is called a Lie algebra homomorphism if it preserves the Lie bracket, i.e. $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$. We call ϕ a Lie algebra isomorphism if ϕ is bijective.*

Important in Lie theory are the notions of a *Lie subalgebra* and an *ideal.*

Definition 1.24 (Lie subalgebra and ideal [5])**.** *A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$; it is called an ideal in $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}, Y \in \mathfrak{h}$.*

Example 1.25. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then $\phi(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{h}$ and $\ker(\phi)$ is an ideal in $\mathfrak{g}$.*

Proof. Let $X, Y \in \phi(\mathfrak{g})$. There exist $\tilde{X}, \tilde{Y} \in \mathfrak{g}$ such that $X = \phi(\tilde{X})$ and $Y = \phi(\tilde{Y})$. Then $[X, Y] = [\phi(\tilde{X}), \phi(\tilde{Y})] = \phi([\tilde{X}, \tilde{Y}]) \in \phi(\mathfrak{g})$, hence $\phi(\mathfrak{g})$ is a Lie subalgebra of $\mathfrak{h}$.

Let $X \in \mathfrak{g}$ and $Y \in \ker(\phi)$. We see that $\phi([X, Y]) = [\phi(X), \phi(Y)] = [\phi(X), 0] = 0$, hence $[X, Y] \in \ker(\phi)$, so $\ker(\phi)$ is an ideal in $\mathfrak{g}$. □

We see from Definition 1.24 that every ideal is a subalgebra, but the reverse need not be the case. If $\mathfrak{g}$ is a Lie algebra, we see that $\{0\}$ and $\mathfrak{g}$ are always ideals in $\mathfrak{g}$. This motivates us to define *simple* and *semisimple* Lie algebras.

Definition 1.26 (Simple and semisimple Lie algebra, [3]). *Let $\mathfrak{g}$ be a Lie algebra. We call $\mathfrak{g}$ simple if it is not abelian and its only ideals are $\{0\}$ and $\mathfrak{g}$. We call $\mathfrak{g}$ semisimple if $\mathfrak{g}$ has no non-zero abelian ideals.*

It can become quite tedious to prove that a Lie algebra is semisimple. We will therefore introduce a criterion known as Cartan's criterion to check whether a given Lie algebra is semisimple. This is based on the *Killing form* κ of a Lie algebra. First, we introduce the adjoint map.

Definition 1.27 (Adjoint map, [5]). *Let $\mathfrak{g}$ be a Lie algebra. We denote the linear map $ad : \mathfrak{g} \to end(\mathfrak{g})$ by $ad(X)Y = [X, Y]$.*

The observant reader will note that the notation used in Definition 1.27 is the same as in Section 1.1. We will return to this issue in Section 1.3 and give reasons to justify it. We see that the adjoint map is well-defined and linear, since the Lie bracket on $\mathfrak{g}$ is a bilinear map. The kernel of the adjoint map consists of all $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. This is called the *centre* of the Lie algebra $\mathfrak{g}$, denoted by $Z(\mathfrak{g})$. Using Example 1.25, we see that $Z(\mathfrak{g})$ is an ideal in $\mathfrak{g}$. Now, we define the Killing form of a Lie algebra.

Definition 1.28 (Killing form, [4]). *Let $\mathfrak{g}$ be a Lie algebra. Its Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ is defined by $\kappa(X, Y) = Tr(ad(X)ad(Y))$, where Tr is the trace operator.*

We see that κ is independent of the choice of basis, since the trace operator is basis independent. The proposition below summarizes the basic properties of the Killing form.

Proposition 1.29 ([4],[9]). *Let $\mathfrak{g}$ be a Lie algebra and $\kappa_{\mathfrak{g}}$ be its Killing form. Then we have the following:*

i. The Killing form is bilinear and symmetric.

ii. The Killing form is invariant under automorphisms of $\mathfrak{g}$, i.e. if $\phi \in aut(\mathfrak{g})$, then $\kappa(\phi(X), \phi(Y)) = \kappa(X, Y)$

iii. For all $X, Y, Z \in \mathfrak{g}$, we have that $\kappa([X, Y], Z) = \kappa(X, [Y, Z])$.

iv. If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ with Killing form $\kappa_{\mathfrak{g}}$, then $\kappa_{\mathfrak{g}}|_{\mathfrak{h}\times\mathfrak{h}} = \kappa_{\mathfrak{h}}$.

Proof. For i, bilinearity follows from the linearity of the trace and the adjoint map. The symmetry follows from the property that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.

For ii, note that $ad(\phi(X))Y = [\phi(X), Y] = \phi([X, \phi^{-1}(Y)])$, so $ad(\phi(X)) = \phi \circ ad(X) \circ \phi^{-1}$ for all $\phi \in aut(\mathfrak{g})$. Therefore, we see that

$$\kappa(\phi(X), \phi(Y)) = \mathrm{Tr}(ad(\phi(X))ad(\phi(Y)) = \mathrm{Tr}(\phi \circ ad(X)ad(Y) \circ \phi^{-1}) = \kappa(X, Y),$$

by cyclicity of the trace.

For *iii*, note that $ad([X,Y]) = ad(X)ad(Y) - ad(X)ad(Y)$, hence

$$\kappa([X,Y],Z) = \mathrm{Tr}(ad(X)ad(Y)ad(Z) - ad(Y)ad(X)ad(Z)) = \\ \mathrm{Tr}(ad(X)ad(Y)ad(Z) - ad(X)ad(Z)ad(Y)) = \kappa([X,[Y,Z]]),$$

by cyclicity of the trace.

For *iv*, we refer to [9]. □

In most cases, the Killing form is not hard to compute. The following theorem connects the Killing form to the semisimplicity of Lie algebras and is therefore useful to determine whether a Lie algebra is semisimple.

Theorem 1.30 (Cartan's criterion, [5])**.** *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if its Killing form κ is nondegenerate.*

Proof. We refer to Theorem 3.9.2 in [5]. □

Definition 1.31 (derivation, [3])**.** *Let $\mathfrak{g}$ be a Lie algebra. A derivation is a map $\delta \in end(\mathfrak{g})$ satisfying the Leibniz rule:*

$$\delta([X,Y]) = [X, \delta(Y)] + [\delta(X), Y], \textit{ for all } X, Y \in \mathfrak{g}.$$

We denote the set of all derivations on $\mathfrak{g}$ by Der($\mathfrak{g}$).

It follows readily that $Der(\mathfrak{g})$ is a linear subspace of $end(\mathfrak{g})$. If δ_1 and δ_2 are derivations, $\delta_1\delta_2 - \delta_2\delta_1$ is again a derivation, hence $Der(\mathfrak{g})$ is a Lie subalgebra of $end(\mathfrak{g})$. The following proposition shows how conveniant Killing forms and Cartan's criterion can be.

Proposition 1.32. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then the linear map $ad : \mathfrak{g} \to Der(\mathfrak{g})$ is a Lie algebra isomorphism.*

Proof. We will first show that $ad(X)$ is a derivation for all $X \in \mathfrak{g}$. We fix $X \in \mathfrak{g}$. By the Jacobi identity, we have that

$$ad(X)[Y,Z] = [X,[Y,Z]] = [Y,[X,Z]] + [[X,Y],Z] = [Y, ad(X)Z] + [ad(X)Y, Z] \text{ for all } Y, Z \in \mathfrak{g},$$

hence $ad(X)$ is a derivation and satisfies $ad([X,Y]) = [ad(X), ad(Y)]$, as we saw earlier hence ad is a Lie algebra homomorphism.

Since $\mathfrak{g}$ is semisimple, it must be centreless since the centre of a Lie algebra is an abelian ideal. Therefore, $\ker(ad) = Z(\mathfrak{g}) = \{0\}$, so ad is injective. For surjectivity, let $I = \mathrm{Im}(ad)$. We will show that $I = Der(\mathfrak{g})$. We remark that ad is a Lie algebra isomorphism $\mathfrak{g} \to I$, so I is semisimple. Note that I is an ideal in $Der(\mathfrak{g})$, since

$$[\delta, ad(X)](Y) = \delta(ad(X)Y) - ad(X)(\delta(Y)) = \delta([X,Y]) - [X, \delta(Y)] = [\delta(X), Y] = ad(\delta(X))Y,$$

for all $Y \in \mathfrak{g}$, so $[\delta, ad(X)] = ad(\delta(X)) \in I$ for all $\delta \in Der(\mathfrak{g})$ and $X \in \mathfrak{g}$, so $[\delta, ad(X)] \in I$ and I is an ideal of $Der(\mathfrak{g})$. We denote by $I^\perp$ the orthogonal complement of I with respect to $\kappa_{Der(\mathfrak{g})}$. Then $I^\perp$ is an ideal of $Der(\mathfrak{g})$ by *iii* of Proposition 1.29. By part *iv* of the same proposition, we see that if $X \in I \cap I^\perp$, we have that $\kappa_{Der(\mathfrak{g})}(X, Y) = \kappa_I(X, Y) = 0$ for all $Y \in I$.

Since I is semisimple, Cartan's criterion implies that κ_I is nondegenerate, hence $X = 0$, thus $I \cap I^\perp = \{0\}$. Therefore, we can write $Der(\mathfrak{g}) = I \oplus I^\perp$, and we are done if we can show that $I^\perp = \{0\}$. Note that $[I, I^\perp] \subseteq I$ and $[I, I^\perp] \subseteq I^\perp$ since I and $I^\perp$ are both ideals, hence $[I, I^\perp] \subseteq I \cap I^\perp = \{0\}$. If $\delta \in I^\perp$, we have for all $X \in \mathfrak{g}$ that

$$0 = [\delta, ad(X)] = ad(\delta(X)).$$

By injectivity of ad, it follows that $\delta(X) = 0$ for all $X \in \mathfrak{g}$, so $\delta = 0$. It follows that $I^\perp = \{0\}$ so $Der(\mathfrak{g}) = I$ and it follows that ad is surjective.

□

1.3 The Lie Algebra of a Lie Group

In the previous sections, we introduced Lie groups and Lie algebras as individual objects. In this section, we will see that there exists an inevitable connection between Lie groups and Lie algebras. We will see that for every Lie group G, we can equip T_eG with a Lie bracket which turns T_eG into a Lie algebra. We will call T_eG equipped with this Lie bracket the Lie algebra of the Lie group. Moreover, we will see that every Lie group homomorphisim can be differentiated and the resulting tangent map will be a Lie algebra homomorphism. Then, we prove a reverse theorem which provides a sufficient condition for Lie groups to be isomorphic if their Lie algebras are. We finish this section with the introduction of *semidirect products* and the centre of a Lie group. We start with the following useful lemma.

Lemma 1.33 ([3])**.** *Let G be a Lie group. For all $x \in G$ and $X \in T_eG$, we have that $x \exp_G(X) x^{-1} = \exp_G(Ad(x)X)$.*

Proof. Since C_x is a Lie group homomorphism, we use Theorem 1.18 to conclude that the following diagram commutes,

$$\begin{tikzcd} G \arrow[r, "C_x"] & G \\ T_eG \arrow[u, "\exp_G"] \arrow[r, "Ad(x)"] & T_eG \arrow[u, "\exp_G"'] \end{tikzcd}$$

hence the result follows. □

Our next aim is to equip T_eG with a Lie bracket operation.

Definition 1.34 (Lie bracket on T_eG [3])**.** *Let $ad : T_eG \to end(T_eG)$ be as in section 1.1. We define $[\cdot, \cdot] : T_eG \times T_eG \to T_eG$ by $[X, Y] = ad(X)Y$*

Note that the operation $[\cdot,\cdot]$ is well-defined. Before showing that $[\cdot,\cdot]$ is actually a Lie bracket, we will first prove that $[\cdot,\cdot]$ is preserved under the tangent map of a Lie group homomorphism.

Proposition 1.35 ([3])**.** *Let G and H be Lie groups and let $f : G \to H$ be a Lie group homomorphism. Then $(d\phi)_e([X,Y]) = [(d\phi)_e X, (d\phi)_e Y]$ for all $X, Y \in T_eG$.*

Proof. We mainly follow [3]. We denote the adjoint representations of G and H by Ad_G and Ad_H. We can easily deduce that $\phi \circ C_x = C_{\phi(x)} \circ \phi$, hence $(d\phi)_e \circ Ad_G(x) = Ad_H(\phi(x)) \circ (d\phi)_e$ as maps $T_eG \to T_eH$. If we fix some $X \in T_eG$, we can view them as maps $G \to T_eG$. When we differentiate in $x = e$, we see that $(d\phi)_e \circ ad_G(X) = ad_H((d\phi)_e X) \circ (d\phi)_e$. When we apply these maps to $Y \in T_eG$ we get exactly that $(d\phi)_e([X,Y]) = [(d\phi)_e X, (d\phi)_e Y]$ for all $X, Y \in T_eG$. □

The following proposition shows that $[\cdot,\cdot]$ is a Lie bracket on T_eG.

Proposition 1.36 ([3])**.** *Let G be a Lie group. Then T_eG together with $[\cdot,\cdot]$ is a Lie algebra.*

Proof. We follow [3]. The bilinearity of $[\cdot,\cdot]$ follows from linearity of the map $ad : T_eG \to end(T_eG)$. By Lemma 1.30, we have for all $Z \in T_eG$ and $s, t \in \mathbb{R}$ that

$$\exp(tZ) = \exp(sZ)\exp(tZ)\exp(-sZ) = \exp_G(Ad(\exp_G(sZ))(tZ)).$$

When differentiating with respect to t in $t = 0$, we obtain that $Z = Ad(\exp_G(sZ))Z$. Differentiating once more with respect to s in $s = 0$, we see that $0 = ad(Z)Z = [Z, Z]$. Substituting $Z = X + Y$, we see that

$$0 = [X+Y, X+Y] = [X,X] + [X,Y] + [Y,X] + [Y,Y] = [X,Y] + [Y,X],$$

hence $[X,Y] = -[Y,X]$ for all $X, Y \in T_eG$.

By Proposition 1.20, we know that the adjoint representation of G in T_eG is a Lie group homomorphism, hence $ad([X,Y]) = [ad(X), ad(Y)]$ by Proposition 1.35. Since the Lie bracket on $end(T_eG)$ is just the commutator of linear maps, we get that $ad([X,Y]) = ad(X)ad(Y) - ad(Y)ad(X)$. We apply this to an arbitrary $Z \in T_eG$ and the Jacobi identity follows from rearranging and using the anticommutativity relation. □

We call T_eG together with $[\cdot,\cdot]$ the Lie algebra of the Lie group G. We will call a Lie group *semisimple* if its Lie algebra is semisimple. From now on, we will denote a Lie group by a roman capital letter and use the calligraphic lower case letter to denote its Lie algebra. We can summarize this whole discussion with the following theorem.

Theorem 1.37 ([3])**.** *Let G and H be Lie groups and let $f : G \to H$ be a Lie group homomorphism. Its tangent map $(df)_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism and the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ \mathfrak{g} & \xrightarrow{(df)_e} & \mathfrak{h} \end{array}$$

Proof. The first assertion follows from Proposition 1.35. The second assertion follows from Theorem 1.18. □

We see in Theorem 1.37 that the Lie algebras of isomorphic Lie groups are isomorphic as well. It is natural to ask the reverse question, that is, whether G and H are isomorphic Lie groups if their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic. This need not always be the case, as we can easily show that S^1 and $\mathbb{R}$ have the same Lie algebra up to isomorphism. This follows simply from the fact that all one-dimensional Lie algebras carry the trivial Lie bracket structure and are therefore isomorphic. However, the converse statement is true if G and H are *simply connected*. The proof of this statement lies in the following theorem.

Theorem 1.38 (Lie's second fundamental theorem[4])**.** *Let G and H be Lie groups and assume that G is simply connected. For every Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$, there exists a unique Lie group homomorphism $F : G \to H$ such that $f = (dF)_e$.*

In the proof of Theorem 1.38, we need the following result.

Theorem 1.39 ([4])**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For each Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a unique connected Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$. The group H is generated by $\exp_G(\mathfrak{h})$.*

This result is proven in [4], Section 1.10.

The following lemma is needed in the proof of Theorem 1.38.

Lemma 1.40. *Let G, H be connected Lie groups and suppose $\phi : G \to H$ is a Lie group homomorphism. If $(d\phi)_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism, then ϕ is a covering map.*

Proof. Since $(d\phi)_e$ is an isomorphism, the inverse function gives us an open neighbourhood $\mathcal{U}$ of e_G and $\mathcal{V}$ of e_H such that $\phi : \mathcal{U} \to \mathcal{V}$ is a Lie group isomorphism. Since L_h is an automorphism of H, it suffices to check the covering property at $e_H \in H$. Denote by P the subgroup $P = \ker(\phi)$. Moreover, we see that $(\phi \circ L_a)(g) = \phi(ag) = \phi(a)\phi(g) = \phi(g)$ for all $g \in G, a \in P$. Therefore, we see that $\phi^{-1}(\mathcal{V}) = \bigcup_{a \in P} L_a(\mathcal{U})$ and $L_a(\mathcal{U})$ is homeomorphic to $\mathcal{V}$ for all $a \in P$. What remains to show is that $L_{a_1}\mathcal{U} \cap L_{a_2}\mathcal{U} = \emptyset$ for $a_1 \neq a_2 \in P$. We will reason by contradiction. Suppose there exist $a_1 \neq a_2 \in P$ such that $L_{a_1}\mathcal{U} \cap L_{a_2}\mathcal{U} \neq \emptyset$. We set $a = a_1 a_2^{-1}$, then $L_a\mathcal{U} \cap \mathcal{U} \neq \emptyset$. We pick $x_1 \in \mathcal{U}$ such that $x_2 = ax_1 \in \mathcal{U}$. Then we see that $\phi(x_2) = \phi(ax_1) = \phi(x_1)$. However, $\phi : \mathcal{U} \to \mathcal{V}$ is bijective, hence $x_1 = x_2$, so $a = e$. But this means that $a_1 = a_2$, which is a contradiction. □

Note that if G is connected and H is simply connected, ϕ is a homeomorphism and hence ϕ is a Lie group isomorphism [8]. We therefore have the following corollary of Lemma 1.40.

Corollary 1.41. *If G, H are Lie Groups with G connected and H simply connected and $\phi : G \to H$ a Lie group homomorphism with $(d\phi)_e : \mathfrak{g} \to \mathfrak{h}$ a Lie algebra isomorphism. Then ϕ is a Lie group isomorphism.*

We can now prove Theorem 1.38.

Proof. We mainly follow [4]. Define $\mathfrak{a} = \{(g, f(g)) : g \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{h}$. Then $\mathfrak{a}$ is a vector space by linearity of f. Note that we can equip $\mathfrak{a}$ with a Lie bracket in the following way: $[(g, f(g)), (h, f(h))] := ([g, h], [f(g), f(h)])$ for all $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$. Since f is a Lie algebra homomorhpism, we have that $[(g, f(g)), (h, f(h))] = ([g, h], f([g, h])) \in \mathfrak{a}$ for all $(g, f(g)), (h, f(h)) \in \mathfrak{a}$ hence $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{h}$. We remark that $\mathfrak{g} \oplus \mathfrak{h}$ is the Lie algebra of the Lie group $G \times H$. By Theorem 1.39, there exists a unique connected Lie subgroup $A \subset G \times H$ which has Lie algebra $\mathfrak{a}$. Consider the map $\pi = \mathrm{pr}_1 \circ \iota_A$ where pr_1 is the projection on the first coordinate and ι_A the inclusion of A in $G \times H$. This is a Lie group homomorphism since ι_A and pr_1 are Lie group homomorphisms. Its tangent map $(d\pi)_{e_A}$ is the projection of $\mathfrak{a}$ on $\mathfrak{g}$, hence a Lie algebra isomorphism. Then π is a Lie group isomorphism by Corallary 1.41. Then the map $F = \mathrm{pr}_2 \circ \pi^{-1} : G \to H$ is a Lie group homomorphism with $(dF)_e = f$. For uniqueness, remark that $\{(g, F(g)) : g \in G\}$ is a Lie subgroup of $G \times H$ with Lie algebra $\mathfrak{a}$, so it follows from the uniqueness in Theorem 1.39 that $A = \{(g, F(g)) : g \in G\}$, hence F is unique. □

Theorem 1.38 can be used to show the following nice result which can be seen as the converse of Theorem 1.37.

Corollary 1.42. *If G and H are simply connected Lie groups with isomorphic Lie algebras, then they are isomorphic Lie groups.*

Proof. If $f : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra isomorphism, Theorem 1.38 ensures that there exists a Lie group homomorphism $F : G \to H$. By Corollary 1.41, the map F is a Lie group isomorphism. □

Apart from the direct product, there exists another construction of Lie groups, known as the *semidirect product.*

Definition 1.43 (Semidirect product of Lie groups [14])**.** *Let G and H be Lie groups and suppose G acts smoothly on H by automorphisms, i.e. there is a Lie group homomorphism $\phi : G \times H \to H$ such that $\phi_g \in Aut(H)$ for all $g \in G$. Then the group $G \ltimes_\phi H$, which has as underlying set $G \times H$ and multiplication $(g, h)(g', h') = (gg', h\phi_g(h'))$ is called the semi-direct product of G and H.*

Note that the semidirect product of G and H depends explicitly on the action of G. If the action of G is obvious, we will write $G \ltimes H$ instead. It is readily verified that $G \ltimes_\phi H$ is again a group with identity (e_G, e_H) and $(g, h)^{-1} = (g^{-1}, \phi_{g^{-1}}(h^{-1}))$. In fact, it is a Lie group since $\phi : G \times H \to H$ is smooth.

Lemma 1.44. *Let G and H be Lie groups and suppose G acts smoothly on H by automorphisms. Then there exists a natural smooth action of $G \ltimes_\phi H$ on H given by $(g, h) \cdot h' = h\phi_g(h')$.*

Proof. The proof is straightforward. We see that

$$(e_G, e_H) \cdot h' = e_H \phi_{e_G}(h') = e_H h' = h'$$

and

$$(g_1,h_1)\cdot((g_2,h_2)\cdot h') = (g_1,h_1)\cdot(h_2\phi_{g_2}(h')) = h_1\phi_{g_1}(h_2\phi_{g_2}(h')) = h_1\phi_{g_1}(h_2)\phi_{g_1g_2}(h') = (g_1g_2, h_1\phi_{g_1}(h_2))\cdot h' = ((g_1,h_1)(g_2,h_2))\cdot h,$$

hence $G \ltimes_\phi H$ acts on H. □

Example 1.45. *The special Euclidean group $SE(2) = SO(2) \ltimes \mathbb{R}^2$, where $SO(2)$ acts smoothly on $\mathbb{R}^2$ by multiplcation with an invertible matrix on a vector in $\mathbb{R}^2$ is the semidirect product of $SO(2)$ and $\mathbb{R}^2$. Lemma 1.42, it acts naturally on $\mathbb{R}^2$ by $(T,v)x = v+Tx$, hence can be seen as a rotation followed by a translation and is therefore the total group of isometries of $\mathbb{R}^2$. We will we see a similar action in Minkowski space in chapter 4 by the Poincaré group.*

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and suppose G acts smoothly on H by automorphisms. We will give a description of the Lie algebra of $G \ltimes_\phi H$, which we will currently denote by $\mathrm{Lie}(G\ltimes_\phi H)$. Since $G\ltimes_\phi H$ has $G\times H$ as underlying set, it is diffeomorphic with $G\times H$ hence $\mathrm{Lie}(G\ltimes_\phi H)$ has $\mathfrak{g}\oplus\mathfrak{h}$ as an underlying vector space. However, $G\ltimes H$ and $G\times H$ need not be isomorphic Lie groups, so the bracket structure may differ. Therefore, we introduce a new construction of Lie algebras, known as the *semidirect sum*.

Definition 1.46 (Semidirect sum of Lie algebras,[14])**.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and let be $\theta : \mathfrak{g} \to Der(\mathfrak{h})$ a Lie algebra homomorphism. Then $\mathfrak{g}\oplus\mathfrak{h}$ together with the bracket*

$$[(X_1,Y_1),(X_2,Y_2)] = ([X_1,X_2],[Y_1,Y_2] + \theta(X_1)Y_2 - \theta(X_2)Y_1)$$

is called the semidirect product of $\mathfrak{g}$ and $\mathfrak{h}$, denoted by $\mathfrak{g}\oplus_\theta\mathfrak{h}$.

It is not difficult to see that $\mathfrak{g}\oplus_\theta\mathfrak{h}$ is again a Lie algebra. Bilinearity follows from the bilinearity of the Lie brackets on $\mathfrak{g}$ and $\mathfrak{h}$ and linearity of θ. Anticommutativity follows from the Lie brackets on $\mathfrak{g}$ and $\mathfrak{h}$ as well. The Jacobi identity holds since $\theta(X)$ is a derivation for all $X\in\mathfrak{g}$. In fact, we have the following nice result that the Lie algebra of the semidirect product of two groups is the semidirect sums of their Lie algebras.

Proposition 1.47. *Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and suppose G acts smoothly on H by automorphisms. Define $\psi : \mathfrak{g} \to end(\mathfrak{h})$ by $\psi(\xi) = (d\phi)_{(e_G,e_H)}(\xi,\cdot)$. Then $Lie(G\ltimes_\phi H) = \mathfrak{g}\oplus_\psi\mathfrak{h}$.*

Proof. For notation, we define the two maps $\phi_g : H \to H$ and $\phi^h : G \to H$ as the restriction of ϕ to $\{g\}\times H$ and $G\times\{h\}$ for all g,h. Then ϕ_g and ϕ^h are smooth maps. We denote the associated differentials at the identity by $\phi'_g = (d\phi_g)_{e_H}$ and $\dot{\phi}^h = (d\phi^h)_{e_G}$. Let $C_{(g,h)} : G\ltimes_\phi H \to G\ltimes_\phi H$ be the conjugation by (g,h). Then

$$C_{(g,h)}(k,l) = \big(gkg^{-1}, h\phi_g(l)\phi_{gk}(\phi_{g^{-1}}(h^{-1}))\big)$$

Taking derivative of this map yields

$$Ad(g,h)(X,Y) = \big(Ad(g)(X), Ad(h)(Y) + \sigma_h(Ad(g)(X))\big),$$

where $\sigma_h : T_eG \to T_eH$ is given by $\sigma_h(\xi) = (dL_h)_{e_H}(\dot{\phi}^{h^{-1}}(\xi))$. Then taking derivatives again, we obtain the Lie bracket on $Lie(G \ltimes_\phi H)$.

$$[(X_1,Y_1),(X_2,Y_2)] = ad(X_1,Y_1)(X_2,Y_2) = \big([X_1,X_2],[Y_1,Y_2]+$$
$$(d\phi)_{(e_G,e_H)}(X_1,Y_2) - (d\phi)_{(e_G,e_H)}(X_2,Y_1)\big) = \big([X_1,X_2],[Y_1,Y_2]+\psi(X_1)Y_2 - \psi(X_2)Y_1\big).$$

We are done if we show that $\psi(\xi)$ is a derivation for all $\xi \in \mathfrak{g}$. Since $\phi_g \in Aut(H)$, we have that $\phi_g \circ C_h = C_{\phi_g(h)} \circ \phi_g$, hence $\phi_g' \circ Ad(h) = Ad(\phi_g(h)) \circ \phi_g'$. Differentiating this relation with respect to h yields

$$\phi_g'([\xi_1,\xi_2]) = [\phi_g'(\xi_1), \phi_g'(\xi_2)].$$

Differentiating with respect to g gives the required property of ψ. □

We will finish this chapter with the following two propositions. The first proposition will give a characterization of the Lie algebra of a Lie subgroup and it will be used to prove the second proposition, which says that Lie$(Z(G)) = Z(\mathfrak{g})$ and will be used in Section 2.4 to pass from Lie groups to Lie algebras and back.

Proposition 1.48 ([2])**.** *Let G be a Lie group and H a Lie subgroup of G with Lie algebra $\mathfrak{h}$. Then $\mathfrak{h} = \{X \in \mathfrak{g} \colon \exp_G(tX) \in H,\, t \in \mathbb{R}\}$.*

Proof. We refer to theorem 20.9 in [2]. □

Proposition 1.49 ([3])**.** *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then the centre of $\mathfrak{g}$ is the Lie algebra of the centre of G.*

Proof. By Proposition 1.48, the Lie algebra of $Z(G)$ is given by $\{X \in \mathfrak{g} : \exp(tX) \in Z(G),\, t \in \mathbb{R}\}$. Note that for a connected Lie group G, we have that $\ker(Ad) = \{x \in G : Ad(x) = Id\} = \{x \in G : xgx^{-1} = g \,\text{for all}\, g \in G\} = Z(G)$. If we apply Theorem 1.34 with the Lie group homomorphism Ad, we see that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{Ad} & GL(T_eG) \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_{GL(T_eG)}} \\ \mathfrak{g} & \xrightarrow{ad} & end(T_eG) \end{array}$$

So, we derive the relation $Ad(\exp_G(X)) = \exp_{GL(T_eG)}(ad(X))$ for all $X \in \mathfrak{g}$. If $X \in Z(\mathfrak{g})$, note that $ad(X) = 0$. For all $t \in \mathbb{R}$, we see that

$$Ad(\exp_G(tX)) = \exp_{GL(T_eG)}(ad(tX))\exp_{GL(T_eG)}(t\, ad(X)) = Id,$$

hence we see that $\exp_G(tX) \in \ker(Ad) = Z(G)$ for all $t \in \mathbb{R}$. The reverse inclusion is shown similarly. □

To summarize this chapter, we have introduced Lie groups and we have discussed several examples of them. We have given several constructions of forming new Lie groups out of old ones such as taking direct products and semidirect products. We have in particular had a detailed discussion about subgroups of Lie groups. We have introduced Lie algebras and discussed the basic notions of ideals and subalgebras. Then we have given a description of how to turn the tangent space of a Lie group into a Lie algebra. We have shown that isomorphic Lie groups have isomorphic Lie algebras and that the reverse statement holds for simply connected Lie groups.

Chapter 2

Unitary Representation Theory

In this chapter, we will study the *representation theory* of topological groups and Lie algebras. Representation theory aims to describe abstract algebraic objects such as groups and Lie algebras more concrete by an action on a, not necessarily finite dimensional, vector space. We will start this chapter by introducing representation theory of finite groups, topological groups and Lie algebras. Then, we will restrict ourselves to representations over *Hilbert spaces*, which allows us to talk about *unitary representations*. Next, we will motivate why quantum mechanics demands for a so-called *projectivized* Hilbert space and we will describe projective representations of topological groups. We will finish the chapter by showing that every *projective representaion* of a simply-connected semisimple Lie group lifts to a unitary representation. In this chapter, we mainly follow [7].

2.1 Introduction to representation theory

In this section, we will give the definition of a representation and study some basic examples and properties. All vector spaces we consider will be over $\mathbb{K}$. We will assume that the reader is familiar with representation theory of finite groups as can be found in [10]. Nevertheless, we wish to recall the definition of a representation.

Definition 2.1 (Representation of finite group [10])**.** *Let G be a finite group and V a vector space. A representation of G in V is a pair (ρ, V) consisting of a vector space V and a group homomorphism $\rho : G \to GL(V)$.*

If G is a topological group, we want the representation (ρ, V) to be continuous, too. To make this exact, we need to equip the vector space V with a topology, turning V into a *topological vector space.*

Definition 2.2 (Topological vector space [11])**.** *A topological vector space V is a vector space over $\mathbb{K}$ equipped with a topology such that the addition map $V \times V \to V$ and the scalar multiplication $\mathbb{K} \times V \to V$ are continuous.*

If V is a topological vector space, a topological automorphism of V is a vector space automorphism $V \to V$ which is a homeomorphism. We denote the automorphism group of V by $Aut(V)$.

Definition 2.3 (Representation of a topological group [7])**.** *Let G be a topological group. A representation of G in V is a pair (ρ, V) consisting of a topological vector space V and a group homomorphism $\rho : G \to GL(V)$ such that the action map $(g, v) \to \rho(g)v$ is a continuous map $G \times V \to V$.*

Example 2.4. *Consider the circle group $\mathbb{T}$. For $n \in \mathbb{N}$, define $\rho_n : \mathbb{T} \to GL(\mathbb{R}^2)$ by*

$$\rho_n(e^{i\theta}) = \begin{pmatrix} \cos(n\theta) & \sin(n\theta) \\ -\sin(n\theta) & \cos(n\theta) \end{pmatrix}.$$

Then ρ_n is a group homomorphism since $\rho_n(e^{i\theta}e^{i\phi}) = \rho_n(e^{i\theta})\rho_n(e^{i\phi})$ and the map $(e^{i\theta}, v) \mapsto \rho_n(e^{i\theta})v$ is continuous as a map $\mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}^2$ for each each $n \in \mathbb{N}$, hence $(\rho_n, \mathbb{R}^2)$ is a representation of $\mathbb{T}$ in $\mathbb{R}^2$.

Lemma 2.5 ([7])**.** *Let G be a locally compact topological group, V be a Banach space and $\rho : G \to GL(V)$ a group homomorphism. Then the following statements are equivalent:*

i. The pair (ρ, V) is a representation of G.

ii. For every $x \in G$, the map $\rho(x)$ is continuous for each x and the map $(x, v) \mapsto \rho(x)v$ is continuous at the group identity for all $v \in V$.

Proof. We mainly follow [7]. By definition of a representation of G, *ii* follows directly from *i*. Now, we assume *ii*. It suffices to prove that the map $(x, v) \mapsto \rho(x)v$ is continuous. For $v \in V$, the map $x \mapsto \rho(x)v = \rho(x_0)\rho(x_0^{-1}x)v$ is continuous at x_0. Now, fix $v_0 \in V$ and let U be a compact neighbourhood of x_0. Then $\{\rho(x) \colon x \in U\}$ is a collection of continuous linear maps $V \to V$. For all $v \in V$, the map $x \mapsto \|\rho(x)v\|$ is continuous, hence bounded on the compact set U. By the uniform boundedness principle [12], the collection of $\{\|\rho(x)\| \colon x \in U\}$ is bounded by some $M > 0$. We see that for all $x \in U$, $v \in V$, we have that

$$\begin{aligned}\|\rho(x)v - \rho(x_0)v_0\| \leq & \|\rho(x)v - \rho(x)v_0\| + \|\rho(x)v_0 - \rho(x_0)v_0\| \leq \\ & M\|v - v_0\| + \|\rho(x)v_0 - \rho(x_0)v_0\|,\end{aligned}$$

from which we conclude that the action map $(x, v) \mapsto \rho(x)v$ is continuous at (x_0, v_0). □

Apart from representations of topological groups, we will also consider representations of Lie algebras.

Definition 2.6 (Lie algebra representation[3])**.** *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. A Lie algebra representation is a pair (π, V) where V is a vector space and $\pi : \mathfrak{g} \to end(V)$ is a Lie algebra homomorphism.*

Remark 2.7. *Let G be a Lie group and (ρ, V) a continuous representation of G such that ρ is a Lie group homomorphism[1], then $(d\rho)_e$ is a Lie algebra representation of $\mathfrak{g}$. The converse hold if G is simply-connected, by Theorem 1.38.*

If (ρ, V) is a representation of G, we call V a *G-module* and we will often use the notation $g \cdot v$ for $\rho(g)v$. Similar, if (π, V) is a representation of $\mathfrak{g}$, we call V a *$\mathfrak{g}$-module* and use the notation Xv for $\pi(X)v$. A representation is said to be n-dimensional if $\dim(V) = n$.

[1] In fact, this is always the case if V is finite dimensional, see Lemma 20.4 in [3]

Definition 2.8 (Invariant subspace and irreducible representations. [3])**.** *Let (ρ, V) be a representation of a finite group, topological group or Lie algebra. A linear subspace $W \subset V$ is called an invariant subspace if $gW \subset W$ for all $g \in G$ (respectively $XW \subset W$ for all $X \in \mathfrak{g}$). We call a representation irreducible if $\{0\}$ and V are the only closed invariant subspaces.*

If (ρ, V) is a representation of a topological group G and W is a closed invariant subspace of V, then $(\rho|_W, W)$ where $\rho|_W(g) = \rho(g)|_W$ is a representation of G.

Example 2.9 ([3])**.** *Let $\mathfrak{g}$ be a simple Lie algebra. The adjoint representation $ad : \mathfrak{g} \to end(\mathfrak{g})$ is irreducible.*

Proof. Since $end(\mathfrak{g})$ is finite dimensional, every invariant subspace will be closed. Let $W \subset end(\mathfrak{g})$ be an invariant subspace. Then it follows that $[X, W] \subset W$ for all $X \in \mathfrak{g}$, which means that W is an ideal in $end(\mathfrak{g})$. Since $\mathfrak{g}$ is simple, it follows that $W = \{0\}$ or $W = \mathfrak{g}$ so ad is irreducible. □

Let G be a topological group and (ρ_1, V_1) and (ρ_2, V_2) two representations of G. A continuous linear map $T : V_1 \to V_2$ satisfying $T \circ \rho_1(g) = \rho_2(g) \circ T$ for all $g \in G$ is called *G-equivariant.*

Definition 2.10 (Equivalent representations [7])**.** *Let G be a topological group and (ρ_1, V_1) and (ρ_2, V_2) two representations of G. We say that the representations are equivalent if there exists a continuous linear isomorphisim $T : V_1 \to V_2$ which is G-equivariant.*

We finish this section with one more result of irreducible representations, which is known as Schur's lemma in the finite dimensional case.

Lemma 2.11 ([10])**.** *Let G be a group or topological group and let (ρ_1, V_1) and (ρ_2, V_2) be finite dimensional representations of G and suppose $T : V_1 \to V_2$ is G-equivariant. Then $ker(T)$ is an invariant subspace of V_1 and $Im(T)$ an invariant subspace of V_2.*

Proof. The proof is an easy computation. Let $v \in \ker(T)$ and let $g \in G$. Then $T(gv) = gT(v) = 0$, so $gv \in \ker(T)$. We conclude that $\ker(T)$ is an invariant subspace of V_1. Similar, let $v \in \mathrm{Im}(T)$. Then there exists $w \in V_1$ such that $v = T(w)$. Then, $T(gw) = gT(w) = gv$, hence $gv \in \mathrm{Im}(T)$, and $\mathrm{Im}(T)$ is a G-equivariant subspace of V_2. □

If V_1 is irreducible, Lemma 2.11 tells us that $\ker(T) = \{0\}$ or $\ker(T) = V_1$. In the first case, T is injective and in the second case, $T = 0$.

Lemma 2.12 (Schur's lemma [10])**.** *Let G be a group or topological group and let (ρ_1, V_1) be an irreducible representation. If $T : V \to V$ is G-equivariant and admits an eigenvalue $\lambda \in \mathbb{K}$, then $T = \lambda Id_{V_1}$.*

Proof. We follow [10]. If T is G-equivariant, then $T - \lambda Id_{V_1}$ is G-equivariant as well. Since V_1 is irreducible, $T - \lambda Id_{V_1}$ is either injective or zero. Note that $T - \lambda Id_{V_1}$ has nontrivial kernel, so it cannot be injective. Therefore, it must be zero hence $T = \lambda Id_{V_1}$. □

2.2 Unitary Representations

In this section, we will restrict to representations over complex *Hilbert spaces*. By an *operator*, we will always mean a linear map between two Hilbert spaces. Recall the following definition of a Hilbert space.

Definition 2.13 (Complex Hilbert space[12]). *A complex Hilbert space is a vector space $\mathcal{H}$ over $\mathbb{C}$ with a complete inner product denoted by $(\cdot,\cdot)$, i.e. the inner product induces a complete metric.*

A Hilbert space is said to be *separable* if it contains a countable, dense subset. Recall that every separable Hilbert space has a countable orthonormal basis. For details, see [12]. Throughout the rest of this thesis, we will assume all Hilbert spaces are separable. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are both complex Hilbert spaces and if $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded operator, there exists a unique bounded operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ satisfying $(Tx, y)_2 = (x, T^*y)_1$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$ [12]. We call T^* the *adjoint* of T.

Definition 2.14 ((Anti)-unitary and self-adjoint operator [12]). *Let $\mathcal{H}_1, \mathcal{H}_2$ be complex Hilbert spaces and let $T : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator. We call T unitary if $TT^* = Id_{\mathcal{H}_2}$ and $T^*T = Id_{\mathcal{H}_1}$. If $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded antilinear map, that is, $T(\lambda x + y) = \bar{\lambda}T(x) + T(y)$ for all $\lambda \in \mathbb{C}$ and $x, y \in \mathcal{H}_1$, satisfying $TT^* = Id_{\mathcal{H}_2}$ and $T^*T = Id_{\mathcal{H}_1}$, we call T anti-unitary. If $T : \mathcal{H}_1 \to \mathcal{H}_1$ is a bounded operator satisfying $T = T^*$, we call T self-adjoint.*

Proposition 2.15. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces and $T : \mathcal{H}_1 \to \mathcal{H}_2$ a bounded operator. Then T is unitary if and only if T is surjective and*

$$(Tx, Ty)_2 = (x, y)_1$$

for all $x, y \in \mathcal{H}_1$.

Proof. If T is unitary, $TT^* = Id_{\mathcal{H}_2}$, so T is surjective and satisfies

$$(x, y)_1 = (x, T^*Ty)_1 = (Tx, Ty)_2.$$

We now prove the converse assertion. By hypothesis, we have for all $x, y \in \mathcal{H}_1$ that

$$(x, y)_1 = (Tx, Ty)_2 = (x, T^*Ty)_1,$$

from which it follows that $y = T^*Ty$, hence $T^*T = Id_{\mathcal{H}_1}$.

We will now show that $TT^* = Id_{\mathcal{H}_2}$. For all $y \in \mathcal{H}_2$ there exists $x \in \mathcal{H}_1$ such that $Tx = y$, since T is surjective. Therefore,

$$TT^*y = TT^*Tx = Tx = y,$$

hence $TT^* = Id_{\mathcal{H}_2}$ and we conclude that T is unitary. □

If $\mathcal{H}$ is a complex Hilbert space, we denote by $\mathcal{U}(\mathcal{H})$ the set of all unitary automorphisms of $\mathcal{H}$. We equip this set with the *strong operator topology*. That is, the topology generated by the sets

$$U(T_0, x, \epsilon) = \{T \in \mathcal{U}(\mathcal{H})\colon \|Tx - T_0x\| < \epsilon\}, \tag{2.2.1}$$

where $T_0 : \mathcal{H} \to \mathcal{H}$ is any bounded operator, x is any element in $\mathcal{H}$ and $\epsilon > 0$. Note that $T_j \in \mathcal{H}$ converges to T in the strong operator topology if and only if $T_j(x)$ converges to $T(x)$ in $\mathcal{H}$ for all $x \in \mathcal{H}$.

Proposition 2.16. *Let $\mathcal{H}$ be a complex Hilbert space. Then the group $\mathcal{U}(\mathcal{H})$ equipped with the strong operator topology forms a topological group.*

Proof. For Hausdorffness, let $T, S \in \mathcal{U}(\mathcal{H})$. Take $x \in \mathcal{H}$ such that $Tx \neq Sx$ and set $\epsilon = \|(T-S)x\|$. Then $U(T, x, \epsilon/2)$ and $U(S, x, \epsilon/2)$ are disjoint open neighbourhoods of T and S. It follows from the separability of $\mathcal{H}$ that the strong operator topology on $\mathcal{U}(\mathcal{H})$ satisfies the axiom of first countability. Therefore, it suffices to proof that the multiplication map is sequential continuous. Let $T, S \in \mathcal{U}(\mathcal{H})$ and let $(T_k)_{k\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}}$ be sequences in $\mathcal{U}(\mathcal{H})$ that converge to T and S in the strong operator topology. For all $x \in \mathcal{H}$, we have that

$$\begin{aligned}\|T_kS_k(x) - TS(x)\| \leq &\|T_kS_k(x) - T_kS(x)\| + \|T_kS(x) - TS(x)\| = \\ &\|(S_k - S)(x)\| + \|(T_k - T)(S(x))\|.\end{aligned}$$

It follows from the estimate above that T_kS_k converges to TS in the strong operator topology, hence the multiplication map is continuous. For continuity of the inversion, let $(T_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{U}(\mathcal{H})$ converging to T. We have to show that $(T_k^{-1})_{k\in\mathbb{N}}$ converges to T^{-1}. For all $x \in \mathcal{H}$, we have that

$$\|T_k^{-1}(x) - T^{-1}(x)\| = \|T_k^{-1}(T - T_k)T^{-1}(x)\| = \|(T - T_k)T^{-1}(x)\|,$$

hence $(T_k^{-1})_{k\in\mathbb{N}}$ converges T^{-1} in the strong operator topology, so the inversion map is continuous. □

Let $\mathcal{H}$ be a complex Hilbert space with countable orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$. Then, we can define a map $J : \mathcal{H} \to \mathcal{H}$ by $J(\sum_{i\in\mathbb{N}} c_ie_i) = \sum_{i\in\mathbb{N}} \overline{c_i}e_i$. The map J is a anti-unitary automorphism of $\mathcal{H}$. We denote the set of all anti-unitary automorphisms of $\mathcal{H}$ by $\mathcal{U}^{-1}(\mathcal{H})$. In fact, it is readily verified that $\mathcal{U}^{-1} = J(\mathcal{U}(\mathcal{H}))$. The union $\mathcal{U}^{+}(\mathcal{H}) = \mathcal{U}^{-1}(\mathcal{H}) \cup \mathcal{U}(\mathcal{H})$ is again a topological group when equipped with the strong operator topology, in which $\mathcal{U}(\mathcal{H})$ is both open and closed. If we consider the map $I : \mathbb{T} \to \mathcal{U}(\mathcal{H})$, $z \mapsto zId_{\mathcal{H}}$, we can view the image of $\mathbb{T}$ under I, which we will denote by $\tilde{\mathbb{T}}$, as a topological subgroup of $\mathcal{U}(\mathcal{H})$ [7].

We go back to representations over Hilbert spaces. Let G be a group and $\mathcal{H}$ a complex Hilbert space. When we consider a representation $(\rho, \mathcal{H})$ of a topological group G, we want the structure of the inner product on $\mathcal{H}$ to be respected. In virtue of proposition 2.16, it is therefore natural to demand that $\rho(g)$ is unitary for all $g \in G$. This motivates the following definition.

Definition 2.17 (Unitary representation, [7])**.** *Let G be a topological group and $\mathcal{H}$ a complex Hilbert space. A unitary representation of G in $\mathcal{H}$ is a continuous representation $(\rho, \mathcal{H})$ such that $\rho(g)$ is unitary for all $g \in G$.*

Example 2.18. *If we endow $\mathbb{R}^2$ with the standard inner product $\langle \cdot, \cdot \rangle$, the representation of $\mathbb{T}$ in $\mathbb{R}^2$ from Example 2.4 is unitary, since $\langle \rho_n(e^{i\theta})v, \rho_n(e^{i\theta})v \rangle = \langle v, v \rangle$ as rotations preserve distances.*

The following lemma is useful in determining whether a representation is unitary.

Lemma 2.19 ([7])**.** *Let G be a topological group, let $\mathcal{H}$ be a complex Hilbert space and $\rho : G \to GL(\mathcal{H})$ a group homomorphism satisfying:*

i. the map $\rho(g)$ is unitary for all $g \in G$.

ii. There exists a dense subset $V \subseteq \mathcal{H}$ such that $\lim_{g \to e} \rho(g)v = v$ for all $v \in V$.

Then $(\rho, \mathcal{H})$ is a unitary representation of G.

Proof. We follow [7]. We have to prove that the action map $G \times \mathcal{H} \to \mathcal{H}$ is continuous. Let $h \in G$ and $w \in V$. By the unitarity of $\rho(g)$, we infer using Proposition 2.15 that

$$\|\rho(g)v - \rho(h)w\| \leq \|\rho(g)(v - w)\| + \|\rho(g)w - \rho(h)w\| = \|v - w\| + \|\rho(h^{-1}g)w - w\|.$$

We are done if we can show that $\lim_{g \to e} \rho(g)w = w$.

Let $\epsilon > 0$. By denseness of V, there exists some $v \in V$ such that $\|v - w\| < \epsilon/3$. By hypothesis, there exists an open neighbourhood U of e such that $\|\rho(g)v - v\| < \epsilon/3$ for all $g \in U$. Using *i*, we see for all $g \in U$ that

$$\|\rho(g)w - w\| \leq \|\rho(g)w - \rho(g)v\| + \|\rho(g)v - v\| + \|w - v\| = 2\|w - v\| + \|\rho(g)v - v\| < \epsilon,$$

hence $\lim_{g \to e} \rho(g)w = w$, and we are done. □

Similar to continuous representations of G, we also have a notion of equivalent unitary representations.

Definition 2.20. *Let G be a topological group and let $\mathcal{H}_1, \mathcal{H}_2$ be complex Hilbert spaces. Two representations $(\rho_1, \mathcal{H}_1)$ and $(\rho_2, \mathcal{H}_2)$ are unitarily equivalent if there exists a G-equivariant unitary isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$.*

If G is a topological group and (ρ, V) a representation of G, it is an interesting question to ask whether there exists an inner product on V which turns (ρ, V) into a unitary representation. If this happens to be the case, the representation is said to be *unitarizable*.

Example 2.21 ([10])**.** *Let G be a finite group and let (ρ, V) be a finite dimensional representation of G. Then this representation is unitarizable.*

Proof. As V is finite dimensional, we can equip V with a standard inner product $\langle \cdot, \cdot \rangle$ inherited from $\mathbb{R}^n$ and we use it to define a new inner product $(\cdot, \cdot)$ on V by

$$(v, w) = \sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle.$$

Then $(\cdot, \cdot)$ satisfies all the properties of an inner product and

$$(\rho(h)v, \rho(h)w) = \sum_{g \in G} \langle \rho(h)\rho(g)v, \rho(h)\rho(g)w \rangle = \sum_{g \in G} \langle \rho(hg)v, \rho(hg)w \rangle =$$

$$\sum_{g \in G} \langle \rho(g)v, \rho(g)w \rangle = (v, w),$$

so $\rho(g)$ is unitary for all $g \in G$, hence the representation is unitary. □

In the next section, we will see why unitary representations are of particular interest to us.

2.3 Representation Theory in Quantum Mechanics

In this section, we will discuss how we can apply representation theory to quantum mechanics. In quantum mechanics, the physical properties of a system are determined by a nonzero vector $|\Psi\rangle$ in a separable Hilbert space $\mathcal{H}$ often referred to as the state of the system. Physical observables such as energy or momentum correspond to a self-adjoint operator T acting on $\mathcal{H}$. If $|\Phi\rangle$ is an eigenstate of T, the probability that a system in a state collapses in the eigenstate $|\Phi\rangle$ during an experiment is given by the *transition probability*.

$$P(|\Psi\rangle \to |\Phi\rangle) = \frac{|\langle \Psi, \Phi \rangle|^2}{\langle \Phi, \Phi \rangle \langle \Psi, \Psi \rangle}. \tag{2.3.1}$$

Example 2.22. *Consider the Pauli matrices*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will compute the transition probabilities for σ_x. The eigenstates of σ_x are $|\Phi_1\rangle = (1,1)^T$ and $|\Phi_2\rangle = (-1,1)^T$. Consider two systems, one in state $|\Psi_1\rangle = (1,-1)^T$ and one in state $|\Psi_2\rangle = (0,2)^T$. Then

$$P(|\Psi_1\rangle \to |\Phi_1\rangle) = \frac{|(1,-1)^T \cdot (1,1)^T|^2}{4} = 0 \qquad P(|\Psi_1\rangle \to |\Phi_2\rangle) = \frac{|(1,-1)^T \cdot (-1,1)^T|^2}{4} = 1$$

and

$$P(|\Psi_2\rangle \to |\Phi_1\rangle) = \frac{|(0,2)^T \cdot (1,1)^T|^2}{8} = \frac{1}{2} \qquad P(|\Psi_2\rangle \to |\Phi_2\rangle) = \frac{|(0,2) \cdot (1,-1)|^2}{8} = \frac{1}{2}.$$

Physically, this means that the first system will always collapse in $|\Phi_2\rangle$ and the second system collapses either in $|\Phi_1\rangle$ and $|\Phi_2\rangle$ with equal probabilities.

Notice that two states $|\Psi\rangle$ and $|\chi\rangle$ in $\mathcal{H}$ that differ by a nonzero scalar $\lambda \in \mathbb{C}$ will always have the same transition probability. Physically, this means that they are the same states since all transition probabilities will be the same. This motivates us to define an equivalence relation $\sim$ on $\mathcal{H} \setminus \{0\}$ by $x \sim y$ if and only if $x = \lambda y$ for some nonzero $\lambda \in \mathbb{C}$. The resulting quotient space is denoted by $\mathbb{P}(\mathcal{H}) := (\mathcal{H} \setminus \{0\})/\sim$ is known as the *projectivized* Hilbert space. It carries the natural quotient topology inherited from $\mathcal{H}$. It is readily verified that $\mathbb{P}(\mathcal{H})$ is a Hausdorff space and the quotient map $\pi : \mathcal{H} \setminus \{0\} \to \mathbb{P}(\mathcal{H})$ is open.

Definition 2.23 (Projective homomorphism [7]). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces. A continuous map $T : \mathbb{P}(\mathcal{H}_1) \to \mathbb{P}(\mathcal{H}_2)$ is called a projective homomorphism if T preserves the transition probabilities, i.e. $P(T(|\Psi\rangle) \to T(|\Phi\rangle)) = P(|\Psi\rangle \to |\Phi\rangle)$ for all $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_1$.*

We denote the group of all projective automorphisms of $\mathbb{P}(\mathcal{H})$ by $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ which forms a group under composition. The following lemma helps us to describe the natural topology on $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$.

Lemma 2.24 ([7]). *The map $q : \mathcal{U}^+(\mathcal{H}) \to Aut(\mathbb{P}(\mathcal{H}))$ defined by $q(T)([v]) = [Tv]$ is a surjective group homomorphism with kernel $\tilde{\mathbb{T}}$.*

Proof. This is Lemma 1.12 in [7]. $\square$

Note that q is an open map, since q is a projection under a group action of $\mathcal{U}^+(\mathcal{H})$ on $Aut(\mathbb{P}(\mathcal{H}))$ We infer from Lemma 2.24 that q induces an isomorphism $\mathcal{U}^+/\tilde{\mathbb{T}} \cong \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ and we can equip $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ naturally with the quotient topology induced by the strong operator topology on $\mathcal{U}^+$ and it is readily verified that $\mathrm{Aut}(\mathbb{P})$ is again a topological group.

Lemma 2.25 ([7]). *The topology on $Aut(\mathbb{P}(\mathcal{H}))$ is generated by the sets $S(x, V) = \{T \in Aut(\mathbb{P}(\mathcal{H}))\colon Tx \in V\}$ with $x \in \mathbb{P}(\mathcal{H})$ and V an open subset of $\mathbb{P}(\mathcal{H})$.*

Proof. Let $x \in \mathbb{P}(\mathcal{H})$. Then $x = \pi(v)$ for some $v \in \mathcal{H} \setminus \{0\}$. Remark that the image under q of the set $\{A \in \mathcal{U}^+(\mathcal{H}) : A(v) \in \pi^{-1}(V)\}$ for some open $V \subset \mathbb{P}(\mathcal{H})$ is exactly the set $S(x, V)$. Since $\mathcal{U}^+(\mathcal{H})$ is a topological group and every operator is bounded, it follows that the action map $\mathcal{U}^+(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ is continuous, so $N(x, V)$ is open in $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ and we see that the topology on $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ is finer than the topology generated by the sets $S(x, V)$. To show that the topology on $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ is contained in the topology generated by the sets $S(x, V)$, we refer to Lemma 1.14 in [7]. $\square$

Suppose a quantum system with projectivized Hilbert space $\mathbb{P}(\mathcal{H})$ has a natural symmetry which can be described by a topological group G. Then it is natural to assume that G acts by a projective automorphism on $\mathbb{P}(\mathcal{H})$. This motivates the definition of a *projective representation*.

Definition 2.26 (Projective representation,[7]). *Let G be a topological group and $\mathcal{H}$ a complex Hilbert space. A projective representation of G in $\mathcal{H}$ is a pair $(\rho, \mathbb{P}(\mathcal{H}))$ where $\rho : G \to Aut(\mathbb{P}(\mathcal{H}))$ is a group homomorphism such that the map $(g, x) \mapsto \rho(g)x$ is a continuous map $G \times \mathbb{P}(\mathcal{H}) \to \mathbb{P}(\mathcal{H})$.*

If $(\rho_1, \mathcal{H}_1)$ and $(\rho_2, \mathcal{H}_2)$ are two projective representations of the same topological group G, we say that ρ_1 and ρ_2 are equivalent if there exists a projective isomorphism $T : \mathbb{P}(\mathcal{H}_1) \to \mathbb{P}(\mathcal{H}_2)$ such thath $T \circ \rho_1(g) = \rho_2(g) \circ T$. A projective representation $(\rho, \mathbb{P}(\mathcal{H}))$ is said to be irreducible if the only closed invariant subspaces are $\{0\}$ and $\mathbb{P}(\mathcal{H})$. If $(\rho, \mathcal{H})$ is a unitary representation of G, the composition $\tilde{\rho} = q \circ \rho$ is a projective representation of G. Hence, every unitary representation induces a projective representation. More intereseting is the converse question, whether a projective represenation is induced by a unitary representation. Such projective representations are said to *lift* to a unitary representation. In the next section, we will show that each projective representation of a Lie group lifts to a unitary representation if the Lie group is semisimple and simply-connected.

2.4 Lifting projective representations

In this section, we will prove a condition for when projective reperesentations lift to unitary representations. This is formulated in the following theorem.

Theorem 2.27 (Lifting projective representations [7])**.** *Let G be a semisimple simply-connected Lie group. Then every projective representation $\rho : G \to Aut(\mathbb{P}(\mathcal{H}))$ lifts to a unitary representation of $\pi : G \to \mathcal{U}(\mathcal{H})$.*

The proof of Theorem 2.27 is based on the notion of *central extensions*, which are a special kind of *exact sequences*.

Definition 2.28 (Exact sequence of groups [2])**.** *Suppose $G_0, ..., G_n$ are groups and $f_i : G_i \to G_{i+1}$ are homomorphisms. The sequence*

$$G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_3} \dots \xrightarrow{f_{n-1}} G_n$$

is called exact if $im(f_i) = ker(f_{i+1})$.

In particular, the sequence

$$\{e\} \xrightarrow{\iota} G \xrightarrow{\varphi} H$$

is exact if $\mathrm{im}(\iota) = \{e_G\} = \ker(\varphi)$, hence if φ is injective. Likewise, the sequence

$$G \xrightarrow{\varphi} H \xrightarrow{\iota} \{e\}$$

is exact if $\mathrm{im}(\varphi) = \ker(\iota) = H$, so if φ is surjective. For defining continuous central extensions, we need specific exact sequences, namely the *short exact sequences.*

Definition 2.29 (Short exact sequence [2])**.** *A short exact sequence is an exact sequence of the form*

$$\{e\} \xrightarrow{\iota_1} G \xrightarrow{f_1} H \xrightarrow{f_2} K \xrightarrow{\iota_2} \{e\}. \tag{2.4.1}$$

By our discussion above, the sequence (2.4.1) is exact only if f_1 is injective and f_2 is surjective. We can now define central extensions of topological groups.

Definition 2.30 (Central extensions of topological groups [7])**.** *Assume G and K are topological groups. A continuous extension of K by G is a triple (H, ϕ, η) where H is a topological group and $\phi : G \to H, \eta : H \to K$ are continuous homomorphisms such that*

$$\{e\} \to G \xrightarrow{\phi} H \xrightarrow{\eta} K \to \{e\} \tag{2.4.2}$$

is an exact sequence. The extension is called central if $\phi(G)$ is contained in the centre of H.

The central extension in (2.4.2) is called trivial if there exists a section $\mu : K \to H$ of η, i.e. a continuous homomorphism $\mu : K \to H$ such that $\eta \circ \mu = Id_K$.

Lemma 2.31. *Suppose the central extension in* (2.4.2) *is trivial. Then there is a continuous isomorphism $\Phi : G \times K \to H$.*

Proof. The proof is straightforward. Define $\Phi : G \times K \to H$ by $\Phi(g, k) = \phi(g)\mu(k)$. Continuity of Φ is clear. It is a homomorphism since

$$\Phi((g, k)(g', k')) = \phi(g)\phi(g')\mu(k)\mu(k') = \phi(g)\mu(k)\phi(g')\mu(k') = \Phi(g, k)\Phi(g', k').$$

We will construct a continuous inverse. For $h \in H$, note that $x = h\mu(\eta(h^{-1})) \in \ker(\eta)$, since

$$\eta(h\mu(\eta(h^{-1}))) = \eta(h)(\eta \circ \mu \circ \eta)(h^{-1}) = \eta(h)\eta(h^{-1}) = e_K.$$

Since $\ker(\eta) = \mathrm{im}(\phi)$, there exists a unique $g_h \in G$ such that $\phi(g_h) = x$. Then define $\Psi : H \to G \times K$ by $\Psi(h) = (g_h, \eta(h))$, which is evidently continuous. Then

$$\Phi(\Psi(h)) = \Phi(g_h, \eta(h)) = h\mu(\eta(h^{-1}))\mu(\eta(h)) = h$$

and

$$\Psi(\Phi(g, k)) = \Psi(\underbrace{\phi(g)\mu(k)}_{=h'}) = (g_{h'}, \eta(\phi(g)\mu(k))) = (g, k),$$

so Ψ is a continuous inverse for Φ. We conclude that $G \times K \cong H$. $\square$

Definition 2.32 (Fibered product)**.** *Let X, Y and Z be topological spaces and let $f : X \to Z$ and $g : Y \to Z$ be continuous maps, we define the fibered product by $f^*(Y) = \{(x, y) \in X \times Y : f(x) = g(y)\}$, which we endow with the subspace topology.*

Now we return to our question whether we can lift projective representations. Let $\rho : G \to Aut(\mathbb{P}(\mathcal{H}))$ be a projective representation of G in a complex Hilbert space $\mathcal{H}$ and recall that the natural map $q : \mathcal{U}^+(\mathcal{H}) \to Aut(\mathbb{P}(\mathcal{H}))$ is a continuous group homomorphism. Then the fibered product $\rho^*(\mathcal{U}^+(\mathcal{H})) = \{(g, A) \in G \times \mathcal{U}^+(\mathcal{H}) : \rho(g) = q(A)\}$ is a closed topological subgroup of $G \times \mathcal{U}^+(\mathcal{H})$. The projection $pr_G : \rho^*(\mathcal{U}^+(\mathcal{H})) \to G$ is a surjective continuous group homomorphisms with $\ker(pr_G) = \{(e_G, A) : q(A) = Id_{Aut(\mathbb{P}(\mathcal{H}))}\} = \tilde{\mathbb{T}}$. Note that $\tilde{\mathbb{T}}$ is naturally embedded in $\rho^*(\mathcal{U}^+(\mathcal{H}))$ and that this embedding is a closed map by compactness of $\tilde{\mathbb{T}}$. Therefore, $\tilde{\mathbb{T}}$ is homeomorphic to a closed normal subgroup of $\rho^*(\mathcal{U}^+(\mathcal{H}))$. It follows that

$$\{e\} \to \tilde{\mathbb{T}} \xrightarrow{\phi} \rho^*(\mathcal{U}^+(\mathcal{H})) \xrightarrow{pr_G} G \to \{e\} \tag{2.4.3}$$

is a central extension of G by $\tilde{\mathbb{T}}$.

Lemma 2.33 ([7]). *The map $pr_G : \rho^*(\mathcal{U}^+(\mathcal{H})) \to G$ is open and hence induces an isomorphism $\rho^*(\mathcal{U}^+(\mathcal{H}))/\tilde{\mathbb{T}} \cong G$.*

Proof. We refer to Lemma 3.4 in [7]. $\square$

The following proposition forms the first connection between lifting projective representations and trivial central extensions.

Proposition 2.34 ([7]). *Let ρ be a projective representation of G in some complex Hilbert space $\mathcal{H}$. Then the following statements are equivalent.*

i. The projective representation ρ lifts to a map $\pi : G \to \mathcal{U}^+(\mathcal{H})$.

ii. The central extension in (2.4.3) *is trivial*

Proof. We mainly follow [7]. First, suppose that ρ lifts to a map $\pi : G \to \mathcal{U}^+(\mathcal{H})$, i.e. $\rho = q \circ \pi$. Define the map $\Phi : G \to \rho^*(\mathcal{U}^+(\mathcal{H}))$ by $\Phi(g) = (g, \pi(g))$. This map is well-defined since $\rho(g) = (q \circ \pi)(g)$. Moreover, it is a section of the map pr_G in (2.4.3), hence the central extension (2.4.3) is trivial.

Now suppose the central extension is trivial. Let $\Phi : G \to \rho^*(\mathcal{U}^+(\mathcal{H}))$ be a section of pr_G. Define by $pr_U : \rho^*(\mathcal{U}^+(\mathcal{H})) \to \mathcal{U}^+(\mathcal{H})$ the projection on the second coordinate. Then $\pi := pr_U \circ \Phi$ is a continuous lift of ρ. $\square$

The proposition above forms the body of the proof of Theorem 2.27. In the rest of this section, we will show that the central extension in (2.4.3) is trivial for semisimple simply-connected Lie groups.

If G is a Lie group, we want to know whether $\rho^*(\mathcal{U}^+(\mathcal{H}))$ admits the structure of a Lie group as well. This would make everything much easier since we could pass to its Lie algebra. In fact, this is question can be answerd affirmatively when we use the following result of Gleason, Montgomery and Zippin, which is a part of the solution of Hilbert's fifth problem.

Theorem 2.35 (Hilbert's fifth problem [16]). *Let H be a locally compact separable topological group. Then H has a compatible Lie group structure if and only if there exists an open neighbourhood around the identity which contains no nontrivial subgroup of H.*

Proof. In fact, necessity of the condition is fairly easy to proof. We follow [7]. The idea will be to construct an open neighbourhood which does not contain a nontrivial subgroup of H.

Suppose H has the structure of a Lie group and let $\mathfrak{h}$ be its Lie algebra. Let Ω be a bounded neighbourhood of $0 \in \mathfrak{h}$ such that $\exp_H : \Omega \to \exp(\Omega)$ is a diffeomorphism onto an open neighbourhood containing e. Define $\Omega' = \frac{1}{2}\Omega$. Suppose $\exp_H(\Omega')$ contains a nontrivial subgroup K and let $e_H \neq k \in K$, so $k = \exp_H(X)$ for some unique nonzero $X \in \Omega'$. Since Ω' is bounded, there exists a maximal $n \in \mathbb{N}$ such that $2^n X \in \Omega'$. Note that $2^{n+1}X \in \Omega \setminus \Omega'$. Since $\exp_H$ is injective on Ω, we get that $k^{2n+1} = \exp_H(2^{n+1}X) \notin \exp_H(\Omega')$. Since $k^{2n+1} \in K \subseteq \exp_H(\Omega')$, this is a contradiction. The sufficiency result is far more involved and we refer to [16]. $\square$

We can apply this result to show that $\rho^*(\mathcal{U}^+(\mathcal{H}))$ carries the structure of a Lie group.

Theorem 2.36 ([7])**.** *If G is a Lie group, then $\rho^*(\mathcal{U}^+(\mathcal{H}))$ is a Lie group as well.*

Proof. We mainly follow [7]. Every Lie group is locally Euclidean, hence locally compact. Recall from topology that G is locally compact if $H \subseteq G$ is a closed normal subgroup such that H and G/H are locally compact. Since $\tilde{\mathbb{T}}$ is locally compact, it follows by Lemma 2.33 that $\rho^*(\mathcal{U}^+(\mathcal{H}))$ is locally compact. We will now construct an open neighbourhood of the identity in $\rho^*(\mathcal{U}^+(\mathcal{H}))$ which contains no nontrivial subgroups.

By Theorem 2.35, there exists an open neighbourhood O_1 of the identity in $\tilde{\mathbb{T}}$ containing no nontrivial subgroups. The embedding $\phi : \tilde{\mathbb{T}} \to \rho^*(\mathcal{U}^+(\mathcal{H}))$ is a homeomorphism onto a compact subgroup, hence there exists an open neighbourhood O_2 of $e \in \rho^*(\mathcal{U}^+(\mathcal{H}))$ such that $\phi^{-1}(O_2) = O_1$. Since G is a Lie group, there exists an open neighbourhood O_3 of the identity which does not contain a nontrivial subgroup. Define $O = O_2 \cap (pr_G)^{-1}(O_3)$, which is an open neighbourhood of the identity of $\rho^*(\mathcal{U}^+(\mathcal{H}))$. Let H be a subgroup of $\rho^*(\mathcal{U}^+(\mathcal{H}))$ contained in O. Then $Pr_G(H)$ is a subgroup of G contained in O_3, and must be trivial. Since the sequence (2.4.3) is exact, $H \subseteq \ker(pr_G) = \operatorname{im}(\phi)$. Then $\phi^{-1}(H)$ is a subgroup of $\tilde{\mathbb{T}}$ contained in O_1, hence $\phi^{-1}(H)$ is trivial, so H is trivial. By Theorem 2.35, $\rho^*(\mathcal{U}^+(\mathcal{H}))$ has the structure of a Lie group. □

From now on, we assume that G and K are Lie groups and that $(\tilde{G}, \phi, \eta)$ is a *Lie group extension* of K by G. That is, $\tilde{G}$ is a Lie group and ϕ and η are Lie group homomorphisms such that

$$\{e\} \to G \xrightarrow{\phi} \tilde{G} \xrightarrow{\eta} K \to \{e\} \tag{2.4.4}$$

is a short exact sequence. Differentiating each map in (2.4.4) yields an *extension of Lie algebras.*

Definition 2.37 (Extension of Lie algebras [7])**.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. An extension of $\mathfrak{k}$ by $\mathfrak{g}$ is a triple $(\mathfrak{h}, \Phi, \Psi)$ where $\mathfrak{h}$ is a Lie algebra and Φ, Ψ are Lie algebra homomorphisms such that*

$$0 \to \mathfrak{g} \xrightarrow{\Phi} \mathfrak{h} \xrightarrow{\Psi} \mathfrak{k} \to 0 \tag{2.4.5}$$

is a short exact sequence of Lie algebras. The extension is called central if $\Phi(\mathfrak{g})$ is contained in the centre of $\mathfrak{h}$.

We call the Lie algebra extension (2.4.5) trivial if there exists a section $\chi : \mathfrak{k} \to \mathfrak{h}$ of Ψ. Similar to Lemma 2.31, $\mathfrak{h}$ and $\mathfrak{g} \oplus \mathfrak{k}$ are isomorphic Lie algebras if the extension (2.4.5) is trivial.

Differentiating the Lie group extension $(\tilde{G}, \phi, \eta)$ in (2.4.4) yields a Lie algebra extension $(\tilde{\mathfrak{g}}, (d\phi)_e, (d\eta)_e)$ of $\mathfrak{k}$ by $\mathfrak{g}$. If the original extension $(\tilde{G}, \eta, \phi)$ was central, the Lie algebra extension $(\tilde{\mathfrak{g}}, (d\phi)_e, (d\eta)_e)$ is central as well by Proposition 1.49.

Proposition 2.38 ([7])**.** *The extension of Lie groups (2.4.4) with K simply-connected is trivial if and only if the extension of Lie algebras $(\tilde{\mathfrak{g}}, (d\phi)_e, (d\eta)_e)$ is trivial.*

Proof. We mainly follow [7]. Suppose the Lie group extension (2.4.4) is trivial and let μ be a section of η. Then $\eta \circ \mu = Id_K$. Differentiating at the identity yields that $(d\eta)_e \circ (d\mu)_e = Id_{\mathfrak{k}}$, so $(d\mu)_e$ is a section of $(d\eta)_e$ and the extension of Lie algebras $(\tilde{\mathfrak{g}}, (d\phi)_e, (d\eta)_e)$ is trivial.

Conversely, assume that the extension of Lie algebras $(\tilde{\mathfrak{g}}, (d\phi)_e, (d\eta)_e)$ is trivial and let μ be a section of $(d\eta)_e$. Since K is simply-connected, Theorem 1.38 ensures that there exists a unqiue Lie group homomorphism $\psi : K \to \tilde{G}$ such that $(d\psi)_e = \mu$. Since $(d\eta)_e \circ (d\psi)_e = Id_{\mathfrak{k}}$ and K is simply-connected, we have that $\eta \circ \psi = Id_K$ so the Lie group extension (2.4.4) is trivial. □

By Proposition 2.38, we have the following corollary to Proposition 2.34.

Corollary 2.39. *Let G be a simply-connected Lie group and let ρ be a projective representation of G in a complex Hilbert space $\mathcal{H}$. Then the following statements are equivalent.*

i. The projective representation ρ lifts to a map $\pi : G \to \mathcal{U}^+(\mathcal{H})$.

ii. The central extension

$$0 \to \tilde{\mathfrak{k}} \xrightarrow{(d\phi)_e} Lie(\rho^*(\mathcal{U}^+(\mathcal{H}))) \xrightarrow{(dpr_G)_e} \mathfrak{g} \to 0$$

is trivial.

In view of Corollary 2.39, we will further investigate central extensions of Lie algebras. Suppose $\mathfrak{g}$ and $\mathfrak{k}$ are Lie algebras and suppose that $(\tilde{\mathfrak{g}}, \phi, \eta)$ is a Lie algebra extension of $\mathfrak{k}$ by $\mathfrak{g}$. Let $\{k_1, ..., k_n\}$ be a basis of $\mathfrak{g}$. Since η is surjective, there exist $\{\widetilde{g_1}, ..., \widetilde{g_n}\} \subset \tilde{\mathfrak{g}}$ such that $\eta(\widetilde{g_i}) = k_i$. We define the linear map $\beta : \mathfrak{k} \to \tilde{\mathfrak{g}}$ by $\beta(k_i) = \widetilde{g_i}$. Note that we do not require β to be a Lie algebra homomorphism. We remark that $\eta \circ \beta = Id_{\mathfrak{k}}$. We use the map β to define the bilinear map $\Theta : \mathfrak{k} \times \mathfrak{k} \to \tilde{\mathfrak{g}}$ by

$$\Theta(X, Y) = [\beta(X), \beta(Y)] - \beta([X, Y]).$$

Note that β is a Lie algebra homomorphism if and only if $\Theta = 0$, hence the central extension $(\tilde{\mathfrak{g}}, \phi, \eta)$ is trivial if and only if $\Theta = 0$.

Remark 2.40. *The map Θ satisfies the following two properties.*

i. The map Θ is anticommutative, i.e. $\Theta(X, Y) = -\Theta(Y, X)$ for all $X, Y \in \mathfrak{k}$.

ii. The map Θ satisfies

$$\Theta(X, [Y, Z]) + \Theta(Y, [Z, X]) + \Theta(Z, [X, Y]) = 0 \qquad \textit{for all} X, Y, Z \in \mathfrak{k}.$$

The first property follows from the anticommutativity of the Lie bracket and the linearity of β. For the second property, remark that

$$\eta(\Theta(X, Y)) = \eta([\beta(X), \beta(Y)]) - \eta(\beta(X, Y)) = [\eta(\beta(X)), \eta(\beta(Y))] - [X, Y] = [X, Y] - [X, Y] = 0,$$

for all $X, Y \in \mathfrak{k}$, since η is a Lie algebra homomorphism and $\eta \circ \beta = Id_{\mathfrak{k}}$. So, $\Theta(X,Y) \in \ker(\eta) = \mathrm{im}(\phi)$ for all $X, Y \in \mathfrak{g}$. Since $\mathrm{im}(\phi)$ is central in $\tilde{\mathfrak{g}}$, we have that $[f, \Theta(X,Y)] = 0$ for all $f \in \tilde{\mathfrak{g}}$. Therefore,

$$\begin{aligned}\Theta(X,[Y,Z]) =& [\beta(X), \beta([Y,Z])] - \beta([X,[Y,Z]]) = [\beta(X), -\Theta(Y,Z)] + \\ & [\beta(X), [\beta(Y), \beta(Z)]] - \beta([X,[Y,Z]]) = [\beta(X), [\beta(Y), \beta(Z)]] - \beta(X,[Y,Z]).\end{aligned}$$

Now, the second property in Remark 2.40 follows from the bilinearity of Θ and the Jacobi identity on $\tilde{\mathfrak{g}}$ and $\mathfrak{k}$.

Definition 2.41 (Cocycle)**.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A bilinear map $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ satisfying property i and ii in Remark 2.40 is called a cocycle. We call a cocycle Θ exact if there exists a linear map $\mu : \mathfrak{g} \to \mathfrak{h}$ satisfying $\Theta(X,Y) = \mu([X,Y])$.*

If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, we denote by $Z^2(\mathfrak{g}, \mathfrak{h})$ the vector space of all cocycles $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ and we denote by $B^2(\mathfrak{g}, \mathfrak{h})$ the vector space of all bilinear maps $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ such that $\Theta(X,Y) = \mu([X,Y])$ for some linear map $\mu : \mathfrak{g} \to \mathfrak{h}$.. It follows by the Jacobi identity on $\mathfrak{g}$ that $B^2(\mathfrak{g}, \mathfrak{h}) \subset Z^2(\mathfrak{g}, \mathfrak{h})$.

Definition 2.42 (Second cohomology group)**.** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. We define by*

$$H^2(\mathfrak{g}, \mathfrak{h}) = Z^2(\mathfrak{g}, \mathfrak{h}) / B^2(\mathfrak{g}, \mathfrak{h})$$

the second cohomology group of $\mathfrak{g}$ in $\mathfrak{h}$.

If Θ and Ψ are two cocycles, we have that $[\Theta] = [\Psi]$ if and only if $\Theta - \Psi \in B^2(\mathfrak{g}, \mathfrak{h})$. In particular if $H^2(\mathfrak{g}, \mathfrak{h}) = 0$, every cocycle is exact.

We return to the question whether a given central extension is trivial. As the following theorem shows, this is the case if the second cohomology group is 0.

Theorem 2.43 ([7])**.** *Let $\mathfrak{g}$ and $\mathfrak{k}$ be Lie algebras such that $H^2(\mathfrak{k}, \mathbb{R}) = 0$. Then every central extension $(\tilde{\mathfrak{g}}, \phi, \eta)$ of $\mathfrak{k}$ by $\mathfrak{g}$ is trivial.*

Proof. We choose a linear map $\beta : \mathfrak{k} \to \tilde{\mathfrak{g}}$ such that $\eta \circ \beta = Id_{\mathfrak{k}}$. We define the cocycle Θ as above. Since $\Theta(X,Y) \in \ker(\eta) = \mathrm{im}(\phi)$ for all $X, Y \in \mathfrak{g}$ by Remark 2.40, there exists a bilinear map $\bar{\Theta} : \mathfrak{k} \times \mathfrak{k} \to \mathfrak{g}$ such that $\phi(\bar{\Theta}(X,Y)) = \Theta(X,Y)$. By bilinearity and injectivity of ϕ, the map $\bar{\Theta}$ is a cocycle $\mathfrak{k} \times \mathfrak{k} \to \mathfrak{h}$. We canonically identify $H^2(\mathfrak{k}, \mathfrak{g}) \cong H^2(\mathfrak{k}, \mathbb{R}) \otimes \mathfrak{g}$, so $H^2(\mathfrak{k}, \mathfrak{g}) = 0$. It follows that the cocycle $\bar{\Theta}$ is exact, so there exists a linear map $\xi : \mathfrak{k} \to \mathfrak{g}$ such that $\bar{\Theta}(X,Y) = \xi([X,Y])$. Now, we claim that the map $\Psi : \mathfrak{k} \to \tilde{\mathfrak{g}}$ defined by $\Psi(X) = \beta(X) + (\phi \circ \xi)(X)$ is a Lie algebra homomorphism satisfying $\eta \circ \Psi = Id_{\mathfrak{k}}$. We compute that

$$\Psi([X,Y]) = \beta([X,Y]) + \phi(\xi([X,Y])) = \beta([X,Y]) + \Theta(X,Y) = [\beta(X), \beta(Y)].$$

On the other hand, since $\mathrm{im}(\phi)$ is in the centre of $\tilde{\mathfrak{g}}$, we get

$$[\Psi(X), \Psi(Y)] = [\beta(X) + \phi(\xi(X)), \beta(Y) + \phi(\xi(Y))] = [\beta(X), \beta(Y)],$$

so Ψ is a Lie algebra homomorphism. For the second part of the claim, remark that $\mathrm{im}(\phi) = \ker(\eta)$, so

$$\eta(\Psi(X)) = (\eta \circ \beta)(X) + \eta(\phi(\xi(X))) = X + 0 = X,$$

so $\eta \circ \Psi = Id_{\mathfrak{k}}$. We conclude that $(\tilde{\mathfrak{g}}, \phi, \eta)$ is trivial. $\square$

We can now prove Theorem 2.27

Proof. The proof will consist of two parts. In the first part, we will show that $H^2(\mathfrak{g}, \mathbb{R}) = 0$. Then, we will invoke Theorem 2.43 to conclude that

$$0 \to \tilde{\mathfrak{t}} \xrightarrow{(d\phi)_e} Lie(\rho^*(\mathcal{U}^+(\mathcal{H}))) \xrightarrow{(dpr_G)_e} \mathfrak{g} \to 0$$

is trivial and show that ρ lifts to a unitary representation π.

If G is a semisimple Lie group, its Lie algebra $\mathfrak{g}$ is semisimple. Let $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be a cocycle. Since the Killing form is non degenerate, there exists a linear map $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $\omega(X, Y) = \kappa_{\mathfrak{g}}(X, \phi(Y))$ for all $X, Y \in \mathfrak{g}$. Since ω is a cocycle, it follows that ϕ is a derivation. By Proposition 1.32, there exists $Z \in \mathfrak{g}$ such that $\phi = ad(Z)$, so

$$\omega(X, Y) = \kappa_g(X, ad(Z)(Y)) = \kappa_{\mathfrak{g}}(X, [Z, Y]) = -\kappa_{\mathfrak{g}}(X, [Y, Z]) = -\kappa_{\mathfrak{g}}([X, Y], Z) = \lambda([X, Y]),$$

where $\lambda(\cdot) = -\kappa_{\mathfrak{g}}(\cdot, Z)$. It follows that ω is exact and since ω was arbitrary, $H^2(\mathfrak{g}, \mathbb{R}) = 0$.

It follows by Theorem 2.43 that the central extension

$$0 \to \tilde{\mathfrak{t}} \xrightarrow{(d\phi)_\eta} Lie(\rho^*(\mathcal{U}^+(\mathcal{H}))) \xrightarrow{(dpr_G)_e} \mathfrak{g} \to 0$$

is trivial, hence ρ lifts to a map $\pi : G \to \mathcal{U}^+(\mathcal{H})$. Since $\pi^{-1}(\mathcal{U}(\mathcal{H}))$ is both open and closed in G, it equals G and it follows that $\pi : G \to \mathcal{U}(\mathcal{H})$ so every projective representation lifts to a unitary representation. $\square$

The final goal of this thesis will be to find the physical relevant projective representations of the connected Poincaré group $SO(3,1)^\circ \ltimes \mathbb{R}^4$. We will prove that it is semisimple and that its second cohomology group vanishes. Unfortunately, this group is not simply connected. Fortunately, the following theorem solves this problem.

Theorem 2.44 (Universal cover group, [2])**.** *Let G be a connected Lie group. The pointed universal cover $(\tilde{G}, \tilde{e})$ of (G, e) is a Lie group and the covering map $p : \tilde{G} \to G$ is a Lie group homomorphism.*

Proof. Since G is locally Euclidean, there exists a univeral covering manifold [8]. The rest of this proof is standard in literature, see for example Theorem 7.7 in [2]. $\square$

We will show that the univeral cover of the connected Poincaré group equals $SL(2, \mathbb{C}) \ltimes \mathbb{R}^4$. If ρ is a projective representation of $SO(3,1)^\circ \ltimes \mathbb{R}^4$, $\tilde{\rho} = \rho \circ p$ is a projective representation of $SL(2, \mathbb{C}) \ltimes \mathbb{R}^4$ and lifts to a unitary representation.

To summarize this chapter, we have introduced continuous representations of topological groups. First over any vector space, and later restricted ourselves to Hilbert spaces which allow us to introduce unitary representations. Then, these unitary representations are closely related to the projective representations which appear naturally in quantum mechanics. In the case of Lie groups, we have shown that each projective representation has a lift to a unitary representation if the Lie group is semisimple and simply-connected. We will apply this theorem to the connected Poincaré group $SO(3,1)^\circ \ltimes \mathbb{R}^4$, the symmetry group of flat spacetime. This will allow us to qualitatively discuss the classification of elementary particles, as we see in Chapter 4. In the next chapter, we will develop the necessary techniques to find all irreducible unitary representations of semidirect products.

Chapter 3

Representations of semidirect products

In this chapter, we will start with an investigate unitary representations of semidirect products. We will show that every irreducible unitary representation of a semidirect product $G = H \ltimes N$ with N abelian, is induced by a representation of a certain subgroup. The route we take is a little unorthodox: all groups in this chapter will be finite to stay away from the more advanced techniques in measure theory and functional analysis which will not contribute directly to our understanding what is happening. We will first describe intuitively what is happening for finite groups, omitting the proofs. Then, we will in fact develop the techniques neccessary to prove a classification theorem on the irreducible unitary representations of G. Although we do not treat the case in full generality, this chapter may serve as an upshot for a generalization to locally compact groups. For more details on this, see [7].

In this chapter, we will first recall some basics from representation theory of finite groups. We will introduce *induced* and *restricted* representations and prove *Frobenius reciprocity* which relates them. We will provide a description how to construct all irreducible representations of G, but we omit the standard proof, as it cannot be generalized to Lie groups.

We will actually give a proof in the second part of this chapter, which can be extended to Lie groups. We will introduce *projection valued measures* and *systems of imprimitivity* and prove the Imprimitivity Theorem, which is a further tool for analyzing induced representations. Then we will focus on systems of imprimitivity of a semidirect products and invoke the Imprimitivity Theorem to classify the irreducible unitary representations of G.

3.1 Character theory

This section is focused on the convenient property of a finite group that it admits only finitely many non-equivalent irreducible representations. More precisely, we have the following result.

Theorem 3.1 ([10]). *Let G be a finite group. There are only finitely many non-equivalent irreducible representations $(\rho_i, V_i)_{1\leq i\leq n}$ where n is the number of conjugacy classes of G. If (ρ, V) is any finite dimensional representation of G, we have a unique decomposition in* *irreducible representations*

$$V \cong \bigoplus_{i=1}^{k} V_i^{n_i} \qquad V_i^{n_i} = \underbrace{V_i \oplus V_i \oplus ... \oplus V_i}_{n_i times}.$$

Moreover, we have

$$|G| = \sum_{i=1}^{n} \dim(V_i)^2.$$

Proof. This is a combination of Corollary 10.7, Theorem 11.12 and Theorem 15.3 in [10]. □

It is insightful to study a representation of a finite group in terms of its *character*. We will see that a representation of a finite group is completely determined by its character values. Moreover, we can use its character as a useful tool to see whether the representation is irreducible.

Definition 3.2 (Character of a representation, [17]). *Let G be a finite group and let (ρ, V) be a representation. The character χ of (ρ, V) is the function $\chi : G \to \mathbb{C}$ defined by $\chi(g) = Tr(\rho(g))$, where Tr is the trace operator.*

The most elementary properties of a character are summarized in the following proposition.

Proposition 3.3 ([17]). *Let G be a finite group and (ρ, V) an n-dimensional representation of G with character χ. We have*

i. $\chi(e) = n$.

ii. $\chi(g^{-1}) = \overline{\chi(g)}$ *for all* $g \in G$.

iii. The character χ is a class function, i.e. $\chi(ghg^{-1}) = \chi(h)$ for all $g, h \in G$.

Proof. For *i*, we remark that $\rho(1) = Id_V$, hence $\chi(1) = Tr(Id_V) = n$ since V is n-dimensional. For *ii*, note that $\rho(g)$ has finite order for all $g \in G$. Therefore, the same must be true for the eigenvalues $\lambda_{g,1}, ..., \lambda_{g,n}$ of $\rho(g)$ hence they have absolute value one. Then

$$\overline{\chi(g)} = \overline{\mathrm{Tr}(\rho(g))} = \sum_{i=1}^{n} \overline{\lambda}_{g,i} = \sum_{i=1}^{n} \lambda_{g,i}^{-1} = \mathrm{Tr}(\rho(g^{-1})) = \chi(g^{-1}).$$

Property *iii* follows from the cyclicity of the trace, so

$$\chi(ghg^{-1}) = \mathrm{Tr}(\rho(ghg^{-1})) = \mathrm{Tr}(\rho(g)\rho(h)\rho(g)^{-1}) = \mathrm{Tr}(\rho(g)) = \chi(g).$$

□

We denote by $\mathbb{C}(G)$ the vector space of functions $\phi : G \to \mathbb{C}$.

Definition 3.4 (Inner product on $\mathbb{C}(G)$ [17]). *Let G be a finite group and let. We define the map $\langle \cdot, \cdot \rangle_G : \mathbb{C}(G) \times \mathbb{C}(G) \to \mathbb{C}$ by*

$$\langle \phi, \psi \rangle_G = \frac{1}{|G|} \sum_{g \in G} \phi(g)\overline{\psi(g)}. \tag{3.1.1}$$

It is not difficult to check that (3.1.1) is an inner product on $\mathbb{C}(G)$. Moreover, it is a real number if ϕ, ψ are characters of G. This inner product can be used for further analysis of characters.

Theorem 3.5 ([10]). *Let G be a finite group and let (ρ_V, V) and (ρ_W, W) be representations of G. We have*

i. If the representation V is irreducible, then $\langle \chi_V, \chi_V \rangle_G = 1$.

ii. If V and W are not equivalent, $\langle \chi_V, \chi_W \rangle_G = 0$.

iii. If V is irreducible the number of times that V occurs in the decomposition of W is the number $\langle \chi_V, \chi_W \rangle_G \in \mathbb{N}$.

iv. If $\langle \chi_V, \chi_V \rangle_G = 1$, the representation V is irreducible.

Proof. The first two assertions follow from Theorem 14.12 in [10]. For *iii*, we decompose W in irreducible representations as in theorem 3.1, so $W \cong (V_1 \oplus ... \oplus V_1) \oplus (V_2 \oplus ... \oplus V_2) \oplus ... \oplus (V_k \oplus ... \oplus V_k)$ where for each i, there are d_i factors of V_i. Then $\chi_W = \sum_{i=1}^{k} d_i \chi_{V_i}$ and using i and ii, we see that $\langle \chi_{V_i}, \chi_W \rangle_G = d_i \in \mathbb{N}$, so $\langle \chi_V, \chi_W \rangle_G$ is the number of times that V occurs in the decomposition of W.

For *iv*, We again decompose $V \cong \bigoplus_{i=1}^{k} V_i^{d_i}$ with each V_i irreducible. Then $1 = \langle \chi_V, \chi_V \rangle_G = \sum_{i=1}^{k} d_i^2$, hence there is a unique i such that $d_i = 1$ and all other d_i are zero, so $V \cong V_i$ and V is irreducible. □

We call characters corresponding to 1-dimensional representations *linear characters*.

Lemma 3.6. *Let G be a group and let χ be any character of G and suppose that λ is a linear character of G. Then $\chi\lambda$ is a character of G which is irreducible if and only if χ is irreducible.*

Proof. Let $\rho : G \to GL(V)$ be the representation corresponding to χ. Then $\lambda\rho : G \to GL(V)$ defined by $\lambda\rho(g) = \lambda(g)\rho(g)$. Since ρ and λ are both homomorphisms, it follows directly that $\lambda\rho$ is a representation of G with character $\chi\lambda$. Since λ is a root of unity, we get

$$\langle \chi\lambda, \chi\lambda \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi(g)\lambda(g)\overline{\chi(g)\lambda(g)} = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\chi(g)} = \langle \chi, \chi \rangle_G,$$

so $\chi\lambda$ is irreducible if and only if χ is irreducible, by the previous theorem. □

By Schur's lemma, all irreducible representations of an abelian group G are linear. Thus, the irreducible characters of G are group homomorphisms $G \to \mathbb{C}^*$. By the previous lemma, they form a group $\widehat{G} = \hom(G, \mathbb{C}^*)$, which we call the *character group*. Since an abelian group G has $|G|$ different conjugacy classes, G has $|G|$ different characters hence $|G| = |\widehat{G}|$.

We end this section with the following example, which shows how to apply the basic aspects of representation theory we have just discussed to the group D_8.

Example 3.7. *Consider the dihedral group $D_8 = \langle a, b | a^4 = b^2 = e, bab = a^{-1}\rangle$. Define the matrices A and B by*

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define the function $\zeta : D_8 \to GL(2, \mathbb{R})$ by $\zeta(b^i a^j) = B^i A^j$. Since $A^4 = B^2 = Id_{\mathbb{R}^2}$ and $BAB = A^3$, ζ defines a representation of D_8. We will now construct its character, χ_ζ. By part iii of Proposition 3.3, we only have to determine χ_ζ on the conjugacy classes of D_8, which are $\{e\}, \{a, a^3\}, \{a^2\}, \{b, ba^2\}, \{ba, ba^3\}$. We can now construct the character table of χ_ζ.

x^G	e	a	a^2	b	ba
$\lvert x^G \rvert$	*1*	*2*	*1*	*2*	*2*
χ_ζ	*2*	*0*	*-2*	*0*	*0*

We compute that $\langle \chi_\zeta, \chi_\zeta \rangle_{D_8} = \frac{1}{8}(4+0+4+0+0) = 1$, so the representation ζ is irreducible.

3.2 Representations of semidirect products

In this section, we will show how to obtain all irreducible representations of a semidirect product $G = H \ltimes N$, with N abelian, from certain subgroups. To describe this in more detail, we first have to introduce the notions of restricted and induced representations.

Suppose that G is a finite group with a subgroup H. If (ρ, V) is a representation of a group G, the restriction $(\rho|_H, V)$ is a representation of H. We will denote this representation by $\mathrm{Res}_H^G(\rho)$ and we call it the restricted representation. More fascinating is the fact that every representation of H induces a representation of G, known as the induced representation. We will now give a characterization of this representation.

Definition 3.8 (Induced representation [17])**.** *Let G be a finite group with subgroup H and suppose (ρ, V) is a representation of H. Define the vector space*

$$Ind_H^G(V) = \{\phi : G \to V : \phi(hg) = \rho(h)\phi(g) \text{ for all } g \in G, h \in H\}.$$

We define the induced representation $Ind_H^G(\rho)$ of G by H to be the vector space $Ind_H^G(V)$ together with the group action $(g \cdot \phi)(g') = \phi(g'g)$.

Proposition 3.9. *The action of G on $Ind_H^G(V)$ is well-defined action.*

Proof. We check that $g \cdot \phi \in \mathrm{Ind}_H^G(V)$ for all $g \in G$. We see that

$$(g \cdot \phi)(hg') = \phi(hg'g) = \rho(h)\phi(g'g) = \rho(h)(g \cdot \phi)(g')$$

for all $g' \in G$ and $h \in H$, so $g \cdot \phi \in \mathrm{Ind}_H^G(V)$. A similar computation shows that $e \cdot \phi = \phi$ and $g_2 \cdot g_1 \cdot \phi = g_1 g_2 \cdot \phi$, so the action is well-defined. □

If G is a finite group and if H is a subgroup of G, we can consider a partition of G in right cosets of H in G. That is,

$$G = \coprod_{i=1}^{d} H g_i$$

for some representatives $\{g_1, ..., g_d\}$ where $d = |G|/|H|$. We set $C_i = Hg_i$, then we define the right coset space $H\backslash G = \{C_1, ..., C_d\}$.

Definition 3.10 (Support)**.** *Let G be a finite group and let V be a vector space. Let $f : G \to V$ be a map. Then we we define its support, denoted by supp(f), to be $\{g \in G \colon f(g) \neq 0\}$.*

If $f \in \mathrm{Ind}_H^G(V)$, then we have that $g \in \mathrm{supp}(f)$ if and only if $hg \in \mathrm{supp}(f)$ for all $h \in H$. In particular $\mathrm{supp}(f)$ is a union of right cosets of H in G. For $C \in H\backslash G$, we set V_C to be the subspace of functions $f \in \mathrm{Ind}_H^G(V)$ supported in C.

Proposition 3.11 ([19])**.** *Let G be a finite group and let H be a subgroup of H and let (V, ρ) be a representation of H. Then we have the following.*

i. The map

$$\bigoplus_{C \in H\backslash G} V_C \to Ind_H^G(V), \quad (f_C)_{C \in H\backslash G} \mapsto \sum_{C \in H\backslash G} f_C$$

is an isomorphism of vector spaces.

ii. Let $C \in H\backslash G$ and choose $g_C \in G$ such that $C = Hg_C$. Then the map $ev_{g_C} : V_C \to V$, $f \mapsto f(g_C)$ is an isomorphism of vector spaces.

Proof. The map in *i* is clearly linear. We will give an inverse map for the map in *i*. Let $C \in H\backslash G$ be a right coset of H in G. For $f \in \mathrm{Ind}_H^G(V)$, we set f_C be the map which equals f on C and is zero everywhere else. Then $f_C \in V_C$ for all $C \in H\backslash G$. Define the linear map $\mathrm{Ind}_H^G(V) \to \bigoplus_{C \in H\backslash G} V_C$ by $f \mapsto (f_C)_{C \in H\backslash G}$. This map is clearly linear and it is a two-sided inverse for the map in *i* since $\sum_{C \in H\backslash G} f_C = f$.

The map in *ii* is linear. We will again give an inverse map. Let $v \in V$ and define $f_v \in V_C$ by $f_v(hg_C) = \rho(h)v$ and f_v is zero elsewhere. Then the map $v \mapsto f_v$ is a two-sided inverse for the map in *ii*. □

We deduce from Proposition 3.11 that

$$\dim(\mathrm{Ind}_H^G(V)) = \sum_{C \in H\backslash G} \dim(V_C) = \sum_{i=1}^{d} \dim(V) = d \dim(V) = \frac{|G|}{|H|} \dim(V). \tag{3.2.1}$$

Definition 3.12 ([10])**.** *Let G be a finite group and let (ρ, V) and (ρ', V') be two representations of G. Then we set $\hom_G(V, V')$ to be the vector space space of G-equivariant maps $V \to V'$.*

Proposition 3.13 ([10]). *Let G be a finite group and suppose that (ρ, V) and (ρ', V') are two representations of G with characters χ_V and $\chi_{V'}$. Then*

$$\langle \chi_V, \chi_{V'} \rangle_G = \dim \hom_G(V, V').$$

Proof. This is Theorem 14.24 in [10]. □

In the setting of the lemma above, we agree to write $\langle V, V' \rangle_G = \langle \chi_V, \chi_{V'} \rangle_G$. We call two representations disjoint if $\langle V, V' \rangle_G = 0$.

The following theorem shows that the induced and restricted represenations are closely related.

Theorem 3.14 (Frobenius reciprocity,[17]). *Let G be a finite group and let H be a finite subgroup of G. If (ρ, V) is a representation of G and (π, W) a representation of H, we have that*

$$\hom_G(V, Ind_H^G(W))) \cong \hom_H(Res_H^G(V), W) \tag{3.2.2}$$

Proof. We present a proof which is based on [17] using character theory. Suppose $f : H \to \mathbb{C}$ is a class function. By Proposition 20 in [17], the function

$$\operatorname{Ind}_H^G(f)(g) = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}gt \in H}} f(t^{-1}gt). \tag{3.2.3}$$

is a class function on G. If χ is the character of π, then $\operatorname{Ind}_G^H(\chi)$ is the character of $\operatorname{Ind}_G^H(\pi)$ [17].

Now, we can extend the character χ to a function $\chi^\circ : G \to \mathbb{C}$ by setting $\chi^\circ(g) = 0$ on for $g \in G \setminus H$. Suppose ψ is the character of ρ. Then, we compute that

$$\begin{aligned}
\langle \psi, \operatorname{Ind}_H^G(\chi) \rangle_G = \langle \operatorname{Ind}_H^G(\chi), \psi \rangle_G &= \frac{1}{|G|} \sum_{g \in G} \operatorname{Ind}_H^G(\chi)(g) \psi(g^{-1}) \\
&= \frac{1}{|G||H|} \sum_{g \in G} \Bigg(\sum_{\substack{t \in G \\ t^{-1}gt \in H}} \chi(t^{-1}gt) \Bigg) \psi(g^{-1}) \\
&= \frac{1}{|G||H|} \sum_{g \in G} \sum_{t \in G} \chi^\circ(t^{-1}gt) \psi(g^{-1}) \\
&= \frac{1}{|G||H|} \sum_{t \in G} \sum_{h \in H} \chi(h) \psi(th^{-1}t^{-1}) \\
&= \frac{1}{|H|} \sum_{h \in H} \chi(h) \psi(h^{-1}) = \langle \chi, \operatorname{Res}_H^G(\psi) \rangle_H = \langle \operatorname{Res}_H^G(\psi), \chi \rangle_H,
\end{aligned}$$

where we have used that $t^{-1}gt \in H$ if and only if there exists $h \in H$ such that $g = tht^{-1}$. By proposition 3.13, it follows that

$$\dim \hom_G(V, \operatorname{Ind}_H^G(W)) = \dim \hom_H(\operatorname{Res}_H^G(V), W) < \infty,$$

from which we deduce that $\hom_G(V, \operatorname{Ind}_H^G(W)) \cong \hom_H(\operatorname{Res}_H^G(V), W)$. □

Suppose G is a finite group with subgroup H. If (ρ, W) is an irreducible representation of H, it is an interesting question to ask under which conditions the induced representation of G by H is irreducible as well. If χ is a character of ρ, we have by Frobenius reciprocity that

$$\langle \mathrm{Ind}_H^G(\chi), \mathrm{Ind}_H^G(\chi)\rangle_G = \langle \mathrm{Res}_H^G(\mathrm{Ind}_H^G(\chi)), \chi\rangle_H. \tag{3.2.4}$$

This means that our question simply reduces to the question how $\mathrm{Res}_H^G(\mathrm{Ind}_H^G(W))$ splits into irreducible representations. For our purposes (classifying the irreducible representations of semidirect products), we can restrict ourselves to the case where H is a normal subgroup of G.

Suppose that H is a normal subgroup of G and that (ρ, W) is a representation of H. For each $g \in G$, we define a representation (ρ^g, W^g) of H over the same vector space W by $\rho^g(h) = \rho(ghg^{-1})$. If g and g' belong to the same coset C of H in G, there exists some $h \in H$ such that $g' = hg$. It follows that the invertible map $\rho(h) : W^g \to W^{g'}$ is an H-equivariant isomorphism, so $W^g \cong W^{g'}$. The following proposition shows how $\mathrm{Res}_H^G(\mathrm{Ind}_H^G(W))$

Proposition 3.15 (Mackey's restriction formula, [17])**.** *Let G be a finite group and let H be a normal subgroup of G. Let $\{g_1, ..., g_d\}$ be a set of representatives for the set of right cosets $H\backslash G$. Suppose that (ρ, W) is a representation of H. We have the following decomposition of $Res_H^G(Ind_H^G(W))$.*

$$Res_H^G(Ind_H^G(W)) \cong \bigoplus_{i=1}^{d} W^{g_i}.$$

Proof. By Theorem 14.21 in [10], it suffices to proof that the characters of both sides are the same. Let χ be the character of (ρ, W). Since H is a normal subgroup, the character on the left hand side is the function $H \to \mathbb{C}$ which maps $h \mapsto \frac{1}{|H|}\sum_{g\in G}\chi(ghg^{-1}) = \sum_{i=1}^{d}\chi(g_i h g_i^{-1})$, which is exactly the character of the right hand side, so we are done. □

Now that we know that the decomposition of $\mathrm{Res}_H^G(\mathrm{Ind}_H^G(W)))$ for normal subgroups, we can infer necessary and sufficient conditions for the induced representation to be irreducible.

Theorem 3.16 (Mackey's irreducibility criterion, [17])**.** *Let G be a finite group and let H be a normal subgroup of G. Let $\{g_1, ..., g_d\}$ be representatives for the coset space $H\backslash G$. If (ρ, W) is a representation of H, the induced representation $Ind_H^G(W)$ of G is irreducible if and only if the following two condintions hold.*

i. The representation (ρ, W) is irreducible.

ii. The two representations W and W^{g_i} of H are disjoint for all $g_i \neq e$.

Proof. We follow [17] for this proof. The representation $\mathrm{Ind}_H^G(W)$ is irreducible if and only if

$$\langle \mathrm{Ind}_H^G(W), \mathrm{Ind}_H^G(W)\rangle_G = 1.$$

By (3.2.4) and Proposition 3.15, we have that

$$\langle \mathrm{Ind}_H^G(W), \mathrm{Ind}_H^G(W)\rangle_G = \langle \mathrm{Res}_H^G(\mathrm{Ind}_H^G(W)), W\rangle_H = \sum_{i=1}^{d}\langle W^{g_i}, W\rangle_H.$$

We may assume that $g_1 = e$, so we have that

$$\langle \mathrm{Ind}_H^G(W), \mathrm{Ind}_H^G(W)\rangle_G = \langle W, W\rangle_H + \sum_{i=2}^{d} \langle W^{g_i}, W\rangle_H \geq 1 + \sum_{i=2}^{d} \langle W^{g_i}, W\rangle,$$

so $\langle \mathrm{Ind}_H^G(W), \mathrm{Ind}_H^G(W)\rangle_G = 1$ if and only if $\langle W, W\rangle_H = 1$ and $\langle W^{g_i}, W\rangle_H = 0$ for all $g_i \neq e$, which are precisely the conditions *i* and *ii*. □

Armed with Mackey's irreducibility criterion, we can give a classification of all irreducible representation of a semidirect product $G = H \ltimes N$, with N abelian. The following description is based on [17].

Recall from Definition 1.43 that H acts on N by automorphisms. This action induces a natural action of H on the character group $\widehat{N}$ given by

$$(h \cdot \chi)(n) = \chi(\phi_{h^{-1}}(n)).$$

The inverse is necessary to obtain a well-defined left action. For $\chi \in \widehat{N}$, we denote the stabilizer subgroup by $H_\chi = \{h \in H \colon h \cdot \chi = \chi\}$. Now, let $(\chi_i)_{1\leq i\leq k}$ be the set of representatives of the orbits of H in $\widehat{N}$. For each i, we define $G_i = H_{\chi_i} \ltimes N$ as a subgroup of G. and extend each character χ_i to a function $\chi_i^\circ : G_i \to \mathbb{C}^*$ by $\chi_i^\circ(h, n) = \chi_i(n)$. Since H_{χ_i} stabilizes χ_i, we see that

$$\begin{aligned}\chi_i^\circ\big((h_1, n_1)(h_2, n_2)\big) &= \chi_i^\circ(h_1h_2, n_1\phi_{h_2}(n_2)) = \chi_i(n_1\phi_{h_2}(n_2)) \underbrace{=}_{\chi_i \text{ is linear}} \chi_i(n_1)\chi_i(\phi_{h_2}(n_2)) \\ &= \chi_i(n_1)(h_2^{-1} \cdot \chi_i)(n_2) = \chi_i(n_1)\chi_i(n_2) = \chi_i^\circ(h_1, n_1)\chi_i^\circ(h_2, n_2),\end{aligned}$$

so χ_i° defines a one-dimensional representation of G_i. Now, let ρ_i be an irreducible representation of H_{χ_i} We compose ρ_i with the natural projection $G_i \to H_{\chi_i}$ to obtain an irreducible representation $\tilde{\rho}_i$ of G_i. Note that the irreducibility is preserved since the characters of $\tilde{\rho}_i$ and ρ_i agree. Then, consider the tensor product representation $\chi_i^\circ \otimes \tilde{\rho}_i$ of G_i. By irreducibility of $\tilde{\rho}_i$, this representation is again irreducible. Now, let $\theta_{i,\rho} = \mathrm{Ind}_{G_i}^G(\chi_i^\circ \otimes \tilde{\rho}_i)$ be the corresponding induced representation of G. The following theorem shows that we have indeed classified all irreducible representations of G.

Theorem 3.17 ([17]). *For $\theta_{i,\rho}$ as above, we have the following.*

i. The representation $\theta_{i,\rho}$ is irreducible for each i and each ρ.

ii. If $\theta_{i,\rho}$ and $\theta_{i',\rho'}$ are isomorphic, then $i = i'$ and ρ is isomorphic to ρ'.

iii. Every irreducible representation of G is isomorphic to one of the $\theta_{i,\rho}$.

Thus, we have classified all irreducible representations of G.

Proof. A proof of this theorem can be found in [17]. We do not prove it here, since the proof uses techniques which are not generalizable to Lie groups. The proof in [17] primarily relies on Mackey's irreducibility criterion (Theorem 3.16) and Theorem 3.1. □

3.3 Systems of Imprimitivity

The classification of all irreducible representations of $G = H \ltimes N$ (Theorem 3.17), uses specific results which only apply for finite groups. Unfortunately, this method cannot be extended to Lie groups, which is what we aim for after all. In this section, we will introduce *systems of imprimitivity*, which are a method of analyzing induced representations. These systems will allow us to derive the result of Theorem 3.17 in such a way that this can be generalized to Lie groups. This generalization is performed in [7]. In this section, we will reduce the arguments in [7] to just finite groups to abstain ourselves from the functional analysis and topological measure theory which is required for a full description, but it will still give a good overview of what is happening in the general case.

Essential for a system of imprimitivity is the notion of a *projection valued measure.*

Definition 3.18 (Projection valued measure,[7])**.** *Let X be a set and let $\mathcal{H}$ be a finite dimensional Hilbert space. A projection valued measure on X is a map $P : \mathcal{P}(X) \to S(\mathcal{H}) = \{A \in end(\mathcal{H})\colon A^* = A = A^2\}$, where $\mathcal{P}(X)$ is the power set of X, satisfying*

i. $P(V_1 \cup V_2) = P(V_1) + P(V_2)$ *for all disjoint* $V_1, V_2 \in \mathcal{P}(X)$.

ii. $P(V_1 \cap V_2) = P(V_1)P(V_2)$ *for all* $V_1, V_2 \in \mathcal{P}(X)$.

iii. $P(X) = Id_{\mathcal{H}}$

It follows directly from property i that $P(\emptyset) = 0$. With the definition of a projection valued measure, we can define systems of imprimitivity.

Definition 3.19 (System of imprimitivity, [7])**.** *Let G be a group which acts on some set X. A system of imprimitivity of G based on X is a pair (ρ, P) where ρ is a unitary representation of G in a Hilbert space $\mathcal{H}$ and P is a projection valued measure $\mathcal{P}(X) \to S(\mathcal{H})$ on X satisfying*

$$P(g \cdot V) = \rho(g)P(V)\rho(g)^{-1} \tag{3.3.1}$$

for all $g \in G$ and all $V \in \mathcal{P}(X)$.

If the group G acts on some set X, we call X a G-set. As in representation theory, we have a similar notion of equivalent and irreducible systems of imprimitivity.

Definition 3.20 (Equivalent and irreducible systems of imprimitivity, [7])**.** *Suppose that G is a finite group and that $(\rho_1, \mathcal{H}_1)$ and $(\rho_2, \mathcal{H}_2)$ are two unitary representations of G. If (ρ_1, P_1) and (ρ_2, P_2) are two systems of imprimitivity based on a G-set X, we call them equivalent if there exists an isometric isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that:*

i. T *is G-equivariant, so* $T \circ \rho_1(g) = \rho_2(g) \circ T$ *for all* $g \in G$.

ii. T *intertwines* P_1 *and* P_2*. That is,* $T \circ P_1(E) = P_2(E) \circ T$ *for all* $E \subset X$.

If (ρ, P) is a system of imprimitivity of G based on a G-set X, an invariant subspace of the system (ρ, P) is a linear subspace $W \subset \mathcal{H}$ such that *i. W is invariant for ρ, i.e. $\rho(g)W \subset W$ for all $g \in G$.*

ii. W is invariant for P, i.e. $P(E)W \subset W$ for all $E \subset X$.

We call the system (ρ, P) irreducible if $\{0\}$ and $\mathcal{H}$ are the only invariant subspaces.

If G is a finite group with a subgroup H, we will show that each unitary representation of H induces a system of imprimitivity of G based on the coset space $H\backslash G$. Then, we will prove the Imprimitivity Theorem (Theorem 3.22) which states that each irreducible system of imprimitivity of G based on $H\backslash G$ is induced by an irreducible unitary representation of H, and that this representation is unique up to equivalence. Then, we will use the Imprimitivity Theorem to rederive the classification of Theorem 3.17.

We will first show how to induce a system of imprimitivity by a unitary representation. Suppose G is a group with subgroup H and suppose (ρ, W) is a unitary representation of H. We endow the vector space $\mathrm{Ind}_H^G(W)$ with the following inner product to turn it into a Hilbert space:[1]

$$\langle \phi, \psi \rangle_{\mathrm{Ind}_H^G(W)} = \frac{1}{|H|} \sum_{g \in G} \langle \phi(g), \psi(g) \rangle_W.$$

Then the induced representation of G by H is unitary, since

$$\langle g \cdot \phi, g \cdot \psi \rangle_{\mathrm{Ind}_H^G(W)} = \frac{1}{|H|} \sum_{g' \in G} \langle \phi(g'g), \psi(g'g) \rangle_W = \frac{1}{|H|} \sum_{g'' \in G} \langle \phi(g''), \psi(g'') \rangle_W = \langle \phi, \psi \rangle_{\mathrm{Ind}_H^G(W)}.$$

We have a natural left action of G on the coset space $H\backslash G$ given by $g \cdot Hg' = Hg'g^{-1}$. To finish this construction, define the map $P^\rho : \mathcal{P}(H\backslash G) \to S(\mathrm{Ind}_H^G(W))$ by $P^\rho(E)\phi = \mathbb{I}_E \phi$ where $\mathbb{I}_E$ is the indicator function of E, where E is seen as a subset of G.

Proposition 3.21. *In the setting above, the pair $(Ind_H^G(\rho), P^\rho)$ is a system of imprimitivity of G based on the coset space $H\backslash G$.*

Proof. We should prove that P^ρ is a projection valued measure satisfying (3.3.1). we remark that $P^\rho(E)$ is linear for all $E \subset H\backslash G$ and that $\mathbb{I}_E\phi \in \mathrm{Ind}_H^G(W)$ for all $E \subset H\backslash G$ and all $\phi \in \mathrm{Ind}_H^G(W)$, so P^ρ is well-defined. Next, we note that $\mathbb{I}_{H\backslash G}\phi = \phi$ for all $\phi \in \mathrm{Ind}_H^G(W)$, so $P^\rho(H\backslash G) = Id_{\mathrm{Ind}_H^G(W)}$. Note that $\mathbb{I}_{V_1 \cap V_2} = \mathbb{I}_{V_1}\mathbb{I}_{V_2}$ for all $V_1, V_2 \subseteq H\backslash G$, so $P^\rho(V_1 \cap V_2) = P^\rho(V_1)P^\rho(V_2)$. If V_1 and V_2 are disjoint, we have that $\mathbb{I}_{V_1 \cup V_2} = \mathbb{I}_{V_1} + \mathbb{I}_{V_2}$, so $P^\rho(V_1 \cup V_2) = P^\rho(V_1) + P^\rho(V_2)$. It follows similarly that $(P^\rho(V))^2 = P^\rho(V)$ for all $V \subset H\backslash G$.

Now, we show that $P^\rho(V)$ is self-adjoint for all V. Recall that $(P^\rho(V))^*$ is the unique map such that

$$\langle P^\rho(V)\phi, \psi \rangle_{\mathrm{Ind}_H^G(W)} = \langle \phi, P^\rho(V)^*\psi \rangle_{\mathrm{Ind}_H^G(W)}.$$

Writing out the left hand side yields

$$\langle P^\rho(V)\phi, \psi \rangle_{\mathrm{Ind}_H^G(W)} = \frac{1}{|H|} \sum_{g \in G} \langle \mathbb{I}_V \phi(g), \psi(g) \rangle_W = \frac{1}{|H|} \sum_{g \in G} \langle \phi(g), \mathbb{I}_V \psi(g) \rangle_W = \langle \phi, P^\rho(V)\psi \rangle_{\mathrm{Ind}_H^G(W)},$$

[1] This can be done for Lie groups by changing the summation into an integral. The requirement that ρ is unitary is only essential when one is dealing with Lie groups, which is Proposition 9.4 in [7]. We choose not to omit it as it will not turn out to be a burden (by Example 2.21) and it resembles the general theory more.

so $P^\rho(V)^* = P^\rho(V)$ for all $V \subset H\backslash G$. We conclude that P^ρ is a projection valued measure.

To show (3.3.1), we have to show that $(\mathbb{I}_{g\cdot E}\phi)(x) = (g \cdot \mathbb{I}_E(g^{-1}\cdot\phi))(x)$ for all $x \in G$. The left hand side equals $\phi(x)$ for $x \in g\cdot E$ and zero otherwise. On the other hand, the right hand side equals $(g\cdot\mathbb{I}_E(g^{-1}\cdot\phi))(x) = \mathbb{I}_E(g^{-1}\cdot\phi)(xg)$, so is equal to $\phi(x)$ for $xg \in E$ and to zero for xg not in E. Note that $xg \in E$ if and only if $x \in Eg^{-1} = g\cdot E$, so indeed we have that $\mathbb{I}_{g\cdot E}\phi = g\cdot\mathbb{I}_E(g^{-1}\cdot\phi)$. We conclude that $(\mathrm{Ind}_H^G(\rho), P^\rho)$ is a system of imprimitivity of G based on the coset space $H\backslash G$. □

We call the system of imprimitivity $(\mathrm{Ind}_H^G(\rho), P^\rho)$ the system of imprimitivity *induced by* ρ. Now, we arrive at the promised Imprimitivity Theorem.

Theorem 3.22 (Imprimitivity theorem, [7])**.** *Let G be a finite group with subgroup H. Suppose that (ρ, P) is a system of imprimitivity of G based on the coset space $H\backslash G$. There exists a unitary representation ξ of H such that $(Ind_H^G(\xi), P^\xi)$ is equivalent to (ρ, P). This equivalence determines ξ uniquely (up to equivalence). Moreover, the system (ρ, P) is irreducible if and only if the representation ξ is.*

Before we can present a proof of Theorem 3.22, we need an equivalent characterization of a system of imprimitivity. Let G be a finite group with subgroup H and let $(\rho, \mathcal{H})$ be a unitary representation of G. Suppose that $P : \mathcal{P}(H\backslash G) \to S(\mathcal{H})$ is a projection valued measure on $H\backslash G$. Then P induces a linear map $\bar{P} : \mathbb{C}(H\backslash G) \to end(\mathcal{H})$ by

$$\bar{P}(f) = \sum_{x\in H\backslash G} f(x)P(x)$$

The action of G on the coset space $H\backslash G$ induces an action on $\mathbb{C}(H\backslash G)$ by

$$(g'\cdot f)(Hg) = f(Hgg'^{-1}).$$

Then (ρ, P) is a system of imprimitivity of G based on $H\backslash G$ if and only if

$$\bar{P}(g\cdot f) = \rho(g)\bar{P}(f)\rho(g)^{-1} \tag{3.3.2}$$

for all $f \in \mathbb{C}(H\backslash G)$ and all $g \in G$, so (3.3.1) is equivalent with (3.3.2).

We will now present a proof of the Imprimitivity Theorem.

Proof. The proof will go in several steps. First, we will construct a unitary representation ξ of H. Then, we will prove that $(\rho, P) \cong (\mathrm{Ind}_H^G(\xi), P^\xi)$ and that this equivalence determines ξ uniquely. We finish this proof by showing that (ρ, P) is irreducible if and only if ξ is irreducible.

Suppose $\mathcal{H}$ is the representation space of ρ. We define the sesquilinear form $\beta : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ by

$$\beta(v, w) = \langle \bar{P}(\mathbb{I}_{He})v, w\rangle_{\mathcal{H}}.$$

Then its kernel $\ker(\beta) = \{v \in \mathcal{H} \colon \beta(v,w) = 0 \text{ for all } w \in \mathcal{H}\}$ is an H-invariant linear subspace of $\mathcal{H}$, since

$$\beta(h \cdot v, h \cdot w) = \langle \bar{P}(\mathbb{I}_{He})(h \cdot v), h \cdot w\rangle_{\mathcal{H}} = \langle h \cdot \bar{P}(\mathbb{I}_{He})v, h \cdot w\rangle_{\mathcal{H}} = \langle \bar{P}(\mathbb{I}_{He})v, w\rangle_{\mathcal{H}} = \beta(v,w),$$

where we have used that $(\rho, \mathcal{H})$ is a unitary representation and that $\bar{P}(\mathbb{I}_{He}) = \bar{P}(h \cdot \mathbb{I}_{He}) = h \cdot \bar{P}(\mathbb{I}_{He}) \cdot h^{-1}$. The form β factors to a well-defined sesquilinear form $\bar{\beta} : \mathcal{H}/\ker(\beta) \times \mathcal{H}/\ker(\beta) \to \mathbb{C}$. Indeed, this map is well defined since if $v_1 - v_2 \in \ker(\beta)$, then $\beta(v_1, w) = \beta(v_2, w)$ for all $w \in \mathcal{H}$. In fact, $\bar{\beta}$ is positive definite, hence an inner product on $\mathcal{H}/\ker(\beta)$, since

$$\bar{\beta}([v],[v]) = \beta(v,v) = \langle \bar{P}(\mathbb{I}_{He})v, v\rangle_{\mathcal{H}} = \langle \bar{P}(\mathbb{I}_{He})v, \bar{P}(\mathbb{I}_{He})v\rangle_{\mathcal{H}} \geq 0,$$

and we have equality if and only if $[v] = 0$. Indeed, if $\beta([v],[v]) = 0$, we have that $\beta(v,v) = 0$, so $\bar{P}(\mathbb{I}_{He})v = 0$. It follows that $\beta(v,w) = 0$ for all $w \in \mathcal{H}$, hence $v \in \ker(\beta)$, which is equivalent with saying that $[v] = 0$.

We denote $\bar{\mathcal{H}} = \mathcal{H}/\ker(\beta)$ and we endow this vector space with the inner product $\bar{\beta}$ such that it becomes a Hilbert space. We define the representation $\xi : H \to GL(\bar{\mathcal{H}})$ by $\xi(h)\bar{v} = [\rho(h)v]$, where $[v] = \bar{v}$. Since $\ker(\beta)$ is H-invariant, the representation ξ is well-defined. By the computation above, the representation ξ is unitary.

Now, we prove that the system of imprimitivity (ρ, P) is equivalent to $(\mathrm{Ind}_H^G(\xi), P^\xi)$. Define $T : \mathcal{H} \to \mathrm{Ind}_H^G(\bar{\mathcal{H}})$ by $T(v)g = [\rho(g)v]$. Indeed, $T(v) \in \mathrm{Ind}_H^G(\bar{\mathcal{H}})$ since

$$T(v)(hg) = [\rho(hg)v] = [\rho(h)\rho(g)v] = \xi(h)[\rho(g)v] = \xi(h)\,T(v)g.$$

We will show that T is the required isomorphism. First, we show that T is a G-equivariant isometric isomorphism. The map T is G-equivariant, since

$$(g \cdot T(v))(g') = T(v)(g'g) = [\rho(g'g)v] = [\rho(g')\rho(g)v] = T(\rho(g)v)(g').$$

The map T is an isometry, since

$$\begin{aligned}\langle Tv, Tw\rangle_{\mathrm{Ind}_H^G(\bar{\mathcal{H}})} =& \frac{1}{|H|}\sum_{g\in G} \bar{\beta}([\rho(g)v],[\rho(g)w]) = \frac{1}{|H|}\sum_{g\in G}\beta(\rho(g)v, \rho(g)v) = \\ & \frac{1}{|H|}\sum_{g\in G}\langle \bar{P}(\mathbb{I}_{He})\rho(g)v, \rho(g)w\rangle_{\mathcal{H}} = \frac{1}{|H|}\sum_{g\in G}\langle \rho(g)\bar{P}(\mathbb{I}_{Hg})v, \rho(g)w\rangle_{\mathcal{H}} = \\ & \frac{1}{|H|}\sum_{g\in G}\langle \bar{P}(\mathbb{I}_{Hg})v, w\rangle_{\mathcal{H}} = \frac{1}{|H|}\langle |H| Id_{\mathcal{H}} v, w\rangle = \langle v, w\rangle_{\mathcal{H}},\end{aligned}$$

where we have used that ρ is a unitary representation and that the cosets of H in G are disjoint. Since T is an isometry, it is necessarily inective. Surjectivity follows by a similar computation, and we conclude that T is a G-equivariant isometric isomorphispm.

We still have to show that T intertwines P and P^ξ. It suffices to show that T intertwines

$\bar{P}(f)$ and $\bar{P}^\xi(f)$ for all $f \in \mathbb{C}(H\backslash G)$. Let $v' \in \mathrm{Ind}_H^G(\bar{\mathcal{H}})$ arbitrary and choose $w \in \mathcal{H}$ (T is surjective) such that $Tw = v'$. Note that for all $f \in \mathbb{C}(H\backslash G)$, we have that

$$\begin{aligned}\langle \bar{P}^\xi(f)Tv, v'\rangle_{\mathrm{Ind}_H^G(\bar{\mathcal{H}})} =& \frac{1}{|H|}\sum_{g\in G}\bar{\beta}(f(Hg)[\rho(g)v],[\rho(g)w]) = \frac{1}{|H|}\sum_{g\in G}\beta(f(Hg)\rho(g)v,\rho(g)w) = \\ & \frac{1}{|H|}\langle f(Hg)P(He)\rho(g)v,\rho(g)w\rangle_{\mathcal{H}} = \frac{1}{|H|}\sum_{g\in G}\langle \rho(g)\, f(Hg)P(Hg)v,\rho(g)w\rangle_{\mathcal{H}} = \\ & \frac{1}{|H|}\sum_{g\in G}\langle f(Hg)P(Hg)v,w\rangle_{\mathcal{H}} = \langle \bar{P}(f)v,w\rangle_{\mathcal{H}} = \langle T(\bar{P}(f)v),v'\rangle_{\mathrm{Ind}_H^G(\bar{\mathcal{H}})},\end{aligned}$$

By nondegeneracy of the inner product, we conclude that $T\circ \bar{P}(f) = \bar{P}^\xi(f)\circ T$, for all $f \in \mathbb{C}(H\backslash G)$, hence T intertwines P and P^ξ and we conclude that the systems of imprimitivity (ρ, P) and $(\mathrm{Ind}_H^G(\xi), P^\xi)$ are equivalent.

Now, we show the uniqueness of ξ. Suppose that $(\zeta, \mathcal{H}')$ is another unitary representation of H such that $(\mathrm{Ind}_H^G(\zeta), P^\zeta) \cong (\rho, P)$, there exists an isometric isomorphism $S : \mathcal{H} \to \mathrm{Ind}_H^G(\mathcal{H}')$ intertwining $\mathrm{Ind}_H^G(\zeta)$ and ρ and intertwining P and P^ζ. For arbitrary $v, w \in \mathcal{H}$, we get

$$\begin{aligned}\beta(v,v) =& \langle P(He)v,v\rangle_{\mathcal{H}} = \langle SP(He)v,Sv\rangle_{\mathrm{Ind}_H^G(\mathcal{H}')} = \frac{1}{|H|}\sum_{g\in G}\langle (SP(He)v)(g),Sv(g)\rangle_{\mathcal{H}'} = \\ & \frac{1}{|H|}\sum_{g\in G}\langle (P^\zeta(He)Sv)(g),Sv(g)\rangle_{\mathcal{H}'} = \frac{1}{|H|}\sum_{g\in G}\langle \mathbb{I}_{He}(Sv)(g),Sv(g)\rangle_{\mathcal{H}'} = \\ & \frac{1}{|H|}\sum_{h\in H}\langle Sv(h),Sv(h)\rangle_{\mathcal{H}'} = \langle Sv(e),Sv(e)\rangle_{\mathcal{H}'},\end{aligned}$$

where we used in the first line that S is an isometry, that S intertwines P and P^ζ in the second line and that ζ is unitary in the last line. It follows that the map $v \mapsto (Sv)(e)$ factors through $\ker(\beta)$ to a well-defined H-equivariant isomorphism $\mathcal{H}/\ker(\beta) \to \mathcal{H}', [v] \mapsto Sv(e)$. Indeed, if $[v] = [w]$, we have that $v - w \in \ker(\beta)$. Hence, we have that

$$\langle S(v-w)(e), S(v-w)(e)\rangle_{\mathcal{H}'} = \beta(v-w, v-w) = 0,$$

so $S(v-w)(e) = 0$. By linearity of S, it follows that $Sv(e) = Sw(e)$ hence the canonical map $\mathcal{H}/\ker(\beta) \to \mathcal{H}'$ is well-defined. It is H-invariant since S intertwines both representations and it is an isomorphism, by a similar computation as above.

Lastly, we show that (ρ, P) is irreducible if and only if $(\xi, \bar{\mathcal{H}})$ is. If $(\xi, \bar{\mathcal{H}})$ is reducible, we can decompose $\bar{\mathcal{H}}$ as $\bar{\mathcal{H}} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Let ξ_1, ξ_2 denote the restriction of ξ to $\mathcal{H}_1$ and $\mathcal{H}_2$. It follows readily that $(\mathrm{Ind}_H^G(\xi), P^\xi) \cong (\mathrm{Ind}_H^G(\xi_1), P^{\xi_1}) \oplus (\mathrm{Ind}_H^G(\xi_2), P^{\xi_2})$, hence (ρ, P) is reducible.

Conversely, assume that $(\mathrm{Ind}_H^G(\xi), P^\xi)$ is reducible, hence it splits into two nontrivial systems of imprimitivity of G based on $H\backslash G$, so $(\mathrm{Ind}_H^G(\xi), P^\xi) \cong (\xi^1, P^1) \oplus (\xi^2, P^2)$. Each of the summands is induced by a unitary representation η_i with $i = 1, 2$ of H. Let $\eta = \eta_1 \oplus \eta_2$, then $(\mathrm{Ind}_H^G(\eta), P^\eta) \cong (\xi^1, P^1) \oplus (\xi^2, P^2)$. By uniqueness, it follows that $\xi \cong \eta = \eta_1 \oplus \eta_2$, hence ξ is reducible. This concludes the proof of Theorem 3.22. □

3.4 Systems of imprimitivity and semidirect products

With the Imprimitivity Theorem, we will rederive the result of Theorem 3.17. We will *assume throughout this section* that $G = H \ltimes N$ where N is abelian. We follow Chapter 11 from [7]. First, we need the following preliminary definitions.

Definition 3.23 ([7])**.** *Let N be an abelian group and let $(\pi, \mathcal{H})$ be a unitary representation of N. For $f \in \mathbb{C}(N)$, we define*

$$\pi(f) = \sum_{n \in N} f(n)\pi(n).$$

Recall that we denoted the character group of an abelian group N by $\widehat{N}$. The character group allows us to define Fourier transforms of class functions on N.

Definition 3.24 (Discrete Fourier transform, [7])**.** *Let N be an abelian group with $f \in \mathbb{C}(N)$. We define the discrete Fourier transform of f by $\hat{f} : \widehat{N} \to \mathbb{C}$*

$$\hat{f}(\chi) = \sum_{n \in N} f(n)\chi(n).$$

Note that the discrete Fourier transform in Defintion 3.24 is bijective. By orthogonality of the irreducible characters, we can decompose f as $f = \sum_{\chi \in \widehat{N}} \langle f, \chi \rangle_N \chi$. Then the Fourier transform is given by $\hat{f}(\chi) = |N| \langle f, \chi \rangle_N$. It follows that the inverse Fourier transform is then given by $f = \frac{1}{|N|} \sum_{\chi \in \widehat{N}} \hat{f}(\overline{\chi})\chi$.

Our method will be centered around the Imprimitivity Theorem. First, we will show that for each unitary representation π of N, there exists a unique associated projection valued measure P_π based on $\widehat{N}$ such that $\pi(f) = \bar{P}_\pi(\hat{f})$. We will use this associated projection valued measure to establish a one-to-one correspondence between irreducible unitary representations of G and irreducible systems of imprimitivity of H based on $\widehat{N}$. In order to invoke the Imprimitivity Theorem, we transfer this system of imprimitivity to a system of imprimitivity of G based on a coset space of some subgroup, and we use this system of imprimitivity to classify all irreducible representations of G.

First, we describe the associated projection valued measure of a unitary representation.

Proposition 3.25 ([7])**.** *Let N be an abelian group and let $(\pi, \mathcal{H})$ be a unitary representation of N. There exists a unique associated projection valued measure P_π on $\widehat{N}$ satisfying $\pi(f) = \bar{P}_\pi(\hat{f})$ for all $f \in \mathbb{C}(N)$.*

Proof. For the moment, suppose such a projection valued measure exists. Writing out the definitions, it must satisfy for all $f \in \mathbb{C}(N)$

$$\sum_{n \in N} f(n)\pi(n) = \sum_{\chi \in \widehat{N}} \hat{f}(\chi)P_\pi(\chi) = \sum_{n \in N} \sum_{\chi \in \widehat{N}} f(n)\chi(n)P_\pi(\{\chi\}).$$

In particular, if we take $f = \delta_n^{n'}$, we obtain that it should satisfy

$$\pi(n') = \sum_{\chi \in \widehat{N}} \chi(n') P_\pi(\{\chi\}),$$

so for $\psi \in \widehat{N}$, we get that

$$\sum_{n \in N} \pi(n)\overline{\psi(n)} = \sum_{n \in N} \chi(n)\overline{\psi(n)} P_\pi(\{\chi\}) = |N| \sum_{\chi \in \widehat{N}} \langle \chi, \psi \rangle_N P_\pi(\{\chi\}) = |N| P_\pi(\{\psi\}), \tag{3.4.1}$$

since irreducible characters are orthogonal. Note that a projection valued measure is completely determined by its values on the singletons, hence P_π is uniqe if it exists. Also, we have obtained a possible definition for P_π. Thus, if we prove that P_π defined in (3.4.1) is indeed a projection valued measure, we have proven Proposition 3.25.

Note that $P_\pi(\{\chi\})^*$ is the unique linear map satisfying

$$\langle P_\pi(\{\chi\})v, w \rangle_{\mathcal{H}} = \langle v, P_\pi(\{\chi\})^* w \rangle_{\mathcal{H}}$$

for all $v, w \in \mathcal{H}$. Writing out the left hand side, we see that

$$\langle P_\pi(\{\chi\})v, w \rangle_{\mathcal{H}} = \frac{1}{|N|} \sum_{n \in N} \overline{\langle \chi(n)} \pi(n) v, w \rangle = \frac{1}{|N|} \sum_{n \in N} \langle v, \chi(n) \pi(n)^* w \rangle_{\mathcal{H}},$$

from which we deduce that

$$P_\pi(\{\chi\})^* = \frac{1}{|N|} \sum_{n \in N} \chi(n) \pi(n)^* = \frac{1}{|N|} \sum_{n \in N} \overline{\chi(n^{-1})} \pi(n^{-1}) = \frac{1}{|N|} \sum_{n \in N} \overline{\chi(n)} \pi(n) = P_\pi(\{\chi\}).$$

Likewise, we see that

$$P_\pi(\{\chi\})^2 = \frac{1}{|N|^2} \sum_{n \in N} \sum_{n' \in N} \overline{\chi(n)\chi(n')} \pi(n)\pi(n').$$

Since χ is always a linear character, we get that

$$P_\pi(\{\chi\})^2 = \frac{1}{|N|^2} \sum_{n \in N} \sum_{n' \in N} \overline{\chi(nn')} \pi(nn') = \frac{1}{|N|} \sum_{n \in N} P_\pi(\{\chi\}) = P_\pi(\{\chi\}).$$

Lastly, we see that

$$P_\pi(\widehat{N}) = P_\pi(\cup_{\chi \in \widehat{N}} \{\chi\}) = \sum_{\chi \in \widehat{N}} P_\pi(\{\chi\}) = \frac{1}{|N|} \sum_{\chi \in \widehat{N}} \sum_{n \in N} \overline{\chi(n)} \pi(n) =$$
$$\frac{1}{|N|} \sum_{n \in N} \left(\sum_{\chi \in \widehat{N}} \overline{\chi(n)} \right) \pi(n) = \frac{1}{|N|} \sum_{n \in N} \overline{\chi_{\text{reg}}(n)} \pi(n) = \frac{1}{|N|} |N| \pi(e) = Id_{\mathcal{H}}.$$

We conclude that P_π is a projection valued measure, and we have proven Proposition 3.25. $\square$

In particular, it follows from Proposition 3.25 that two unitary representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of N are equivalent if and only if their associated projection valued measures are.

As we have seen in the previous section, projection valued measures can give rise to systems of imprimitivity. This is also happens to be the case for the associated projection valued measure.

Proposition 3.26 ([7])**.** *Let $(\pi, \mathcal{H})$ be a unitary representation of G. Then $(\pi|_H, P_{\pi|_N})$ is a system of H based on the character group $\widehat{N}$.*

Proof. We have already verified that $P_{\pi|N}$ is already a projection valued measure. Thus, we only have to check that it satisfies (3.3.2). By the bijectivity of the Fourier transform, each $g \in \mathbb{C}(\widehat{N})$ is the Fourier transform of a unique $f \in \mathbb{C}(N)$, so we have to check (3.3.2) for all $\hat{f}$ with $f \in \mathbb{C}(N)$. Since H acts on $\widehat{N}$, it also acts on $\mathbb{C}(\widehat{N})$ by

$$(h \cdot \hat{f})(\chi) = \hat{f}(h^{-1} \cdot \chi) = \sum_{n \in N} f(n)\chi(\phi_h(n)) = \sum_{n \in N} f(\phi_{h^{-1}}(n))\chi(n) = \sum_{n \in N} (h \cdot f)(n)\chi(n) = \widehat{h \cdot f}(\chi),$$

hence $h \cdot \hat{f} = \widehat{h \cdot f}$. Then we see that

$$\overline{P_{\pi|_N}}(h \cdot \hat{f}) = \overline{P_{\pi|_N}}(\widehat{h \cdot f}) = \pi|_N(h \cdot f) = \sum_{n \in N} \pi|_N(n)(h \cdot f)(n) = \sum_{n \in N} \pi|_N(n) f(\phi_{h^{-1}}(n)) = \sum_{n \in N} \pi|_N(\phi_h(n))f(n) = \sum_{n \in N} h \cdot \pi|_N(n)f(n) \cdot h^{-1} = h \cdot \overline{P_{\pi|_N}}(\hat{f}) \cdot h^{-1}.$$

We conclude that $(\pi|_H, P_{\pi|_N})$ is a system of imprimitivity of H based on $\widehat{N}$. $\square$

We can analyze this system of imprimitivity even further. We use the notation from the proposition above. Suppose that the system of imprimitivity $(\pi|_H, P_{\pi|_N})$ is reducible and let $W \subset \mathcal{H}$ be an invariant subspace. By definition, we have that $\pi|_H(h)W \subseteq W$ for all $h \in H$ and that $\overline{P_{\pi|_N}}(\hat{f}) = \pi|_N(f)W \subseteq W$ for all $f \in \mathbb{C}(N)$. Now, take $f = \delta_n^{n'}$ so that we have that $\pi|_N(f) = \pi(n')$. Then we get that $n \cdot W \subset W$ for all $n \in N$. In particular, it follows that $(h,n)W = (e,n)(h,e)W \subset (h,e)W \subset W$, so the representation $(\pi, \mathcal{H})$ is a reducible representation of G. Likewise, suppose that π is a reducible representation of G, with invariant subspace $W \subset \mathcal{H}$. Then $\pi(h,n)W \subset W$ for all $(h,n) \in G$. In particular, it follows that $\pi|_H(h)W = \pi(h,e)W \subset W$ and $\overline{P_{\pi|_N}}(\hat{f})W = \sum_{n \in N} f(n)\pi(e,n)W \subset \sum_{n \in N} f(n)W = W$, so the system of imprimitivity $(\pi|_H, P_{\pi|_N})$ is reducible.

Moreover, suppose we are given a system of imprimitivity (ρ, P) based on $\widehat{N}$, with $(\rho, \mathcal{H})$ a unitary representation of H. Now, we define $\pi : N \to GL(\mathcal{H})$ by $\pi(n) = \sum_{\chi \in \widehat{N}} \chi(n)P(\chi)$. Then, we obtain a group homomorphism $\bar{\pi} : G \to GL(\mathcal{H})$ by setting $\bar{\pi}(h,n) = \pi(n)\rho(h)$. In particular, the maps $(\rho, P) \mapsto (\pi, \mathcal{H})$ and $(\pi, \mathcal{H}) \mapsto (\pi|_H, P_{\pi|_N})$ are mutual inverses, up to equivalences.

We can summarize the two important observations from above in the following corollary.

Corollary 3.27. *Let $(\pi, \mathcal{H})$ be a unitary representation of G. The system of imprimitivity $(\pi|_H, P_{\pi|_N})$ is irreducible if and only if $(\pi, \mathcal{H})$ is irreducible. Moreover, there exists a one-to-one correspondence between equivalence classes of unitary representations of G and equivalence classes of irreducible systems of imprimitivity of H based on $\widehat{N}$.*

Our next aim is to transfer the system of imprimitivity $(\pi|_H, P_{\pi|_N})$ to a system of imprimitivity of G based on the coset space of some subgroup of G. The following lemma will be useful.

Lemma 3.28 ([7])**.** *Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G. There exists a unique orbit O of the action of H on $\widehat{N}$ such that $supp(P_{\pi|_N}) = O$.*

Proof. Let O be an orbit of the action of H on $\widehat{N}$. Then the linear subspace $P_\pi(O)\mathcal{H}$ is H-invariant, since $h \cdot P_{\pi|_N}(O)\mathcal{H} = P_{\pi|_N}(h^{-1} \cdot O)(h \cdot \mathcal{H}) = P_{\pi|_N}(O)\mathcal{H}$, since $h^{-1} \cdot O = O$ and $h \cdot \mathcal{H} = \mathcal{H}$. Inspecting the proof of Proposition 3.25, we see that $P_{\pi|_N}(O)$ is also N-invariant. In particular, it follows that $(h, n)P_{\pi|_N}(O)\mathcal{H} = (e, n)(h, e)P_{\pi(O)} = P_{\pi|_N}(O)\mathcal{H}$, so all $P_{\pi|_N}(O)\mathcal{H}$ are G-invariant. Since the orbits of the action of H on $\widehat{N}$ are disjoint, we obtain a decomposition

$$\mathcal{H} = \bigoplus_O P_\pi(O)\mathcal{H}.$$

By irreducibility of the representation, there are no invariant subspaces. Therefore, there must be exactly one orbit O for which $P_{\pi|_N}(O)\mathcal{H} = \mathcal{H}$, and $P_{\pi|_N}(O)\mathcal{H} = 0$ for all other orbits, thus $P_{\pi|_N}$ is fully supported in a unique orbits O. ☐

Now, we describe how to obtain a system of imprimitivity of G. Let $(\pi, \mathcal{H})$ be an irreducible unitary representation of G, let O be the unique orbit on which $P_{\pi|_N}$ is supported and pick a $\chi \in O$. Denote by H_χ the stabilizer subgroup of χ and set $G_\chi = H_\chi \ltimes N$ as a subgroup of G. Define the canonical map $\iota : G \to \widehat{N}$ by $\iota(h, n) = h^{-1} \cdot \chi$.

Proposition 3.29 ([7])**.** *In the setting above, the canonical map ι descends to a well-defined bijection $\tilde{\iota} : G\backslash G_\chi \to O$ and induces a system of imprimitivity of G based on the coset space $G\backslash G_\chi$.*

Proof. We define $\tilde{\iota}(G_\chi(h, n)) = \iota(h, n)$. We first have to show that $\tilde{\iota}$ is well-defined. Note that $h_1^{-1}\cdot\chi = h_2^{-1}\cdot\chi$ if and only if $h_2 h_1^{-1} \in H_\chi$, hence it follows that $\tilde{\iota}(G_\chi(h_1, n_1)) = \tilde{\iota}(G_\chi(h_2, n_2))$ if $G_\chi(h_1, n_1) = G_\chi(h_2, n_2)$. Moreover, it implies that $\tilde{\iota}$ is injective. Note that $\mathrm{Im}(\tilde{\iota}) = H \cdot \chi = O$, so $\tilde{\iota}$ is surjective. This establishes the first part of Proposition 3.29. Next, we define $\tilde{\iota}^* P_\pi : \mathcal{P}(G_\chi\backslash G) \to end(\mathcal{H})$ by $\tilde{\iota}^* P_\pi(S) = P_\pi(\iota(S))$. By the previous lemma, we have that $\tilde{\iota}^* P_\pi(G_\chi\backslash G) = P_\pi(\tilde{\iota}(G_\chi\backslash G)) = P_\pi(O) = Id_{\mathcal{H}}$. Since $\tilde{\iota}$ is bijective, the other two properties of a projection valued measure follow almost directly. To conclude that $\tilde{\iota}^* P_\pi$ is a projection valued measure, we have to check that $\tilde{\iota}^* P_\pi$ satisfies (3.3.1).

We have to check that $\tilde{\iota}^* P_\pi(g \cdot S) = g \cdot \tilde{\iota}^* P_\pi(S) \cdot g^{-1}$. Writing out both sides of the equations, we have to check that $P_\pi(\tilde{\iota}(g \cdot S)) = g \cdot P_\pi(\tilde{\iota}(S)) \cdot g^{-1} = P_\pi(g \cdot \tilde{\iota}(S))$. This means we are done if we can show that $\tilde{\iota}(g \cdot S) = g \cdot \tilde{\iota}(S)$ for all $S \in \mathcal{P}(G_\chi\backslash G)$. Note that we have a natural action of G on N which is induced by the action of H on N. This yields that

$$\begin{aligned}\tilde{\iota}((h,n)\cdot G_\chi(h',n'))(N) =&\tilde{\iota}(G_\chi(h'n')(h,n)^{-1})(N) = (hh'^{-1}\cdot\chi)(N) = \chi(\phi_{h'h^{-1}}(N)) = \\ &(h'^{-1}\cdot\chi)(((h,n)^{-1}\cdot N)) = \tilde{\iota}(G_\chi(h',n'))((h,n)^{-1}\cdot N) = \\ &((h,n)\cdot\tilde{\iota}(G_\chi(h',n')))(N),\end{aligned}$$

so we conclude that $\tilde{\iota}(g\cdot S) = g\cdot\tilde{\iota}(S)$ for all $S\in\mathcal{P}(G_\chi\backslash G)$. It follows that $\tilde{\iota}^*P_\pi$ is a system of imprimitivity of G based on the coset space $G_\chi\backslash G$. □

By a similar approach as described in the proposition above, we can also construct an irreducible system of imprimitivtiy $(\pi|_H, \tau^*P_\pi)$ of H based on the coset space $H_\chi\backslash H$, where $\tau : H\to\widehat{N}$ is the canonical map. By the Imprimitivity Theorem, there exists an irreducible unitary representation (ξ, W) such that $\pi|_H\cong\mathrm{Ind}_{H_\chi}^H(\xi)$.

Now, we define the tensor product rerpresentation $(\xi\otimes\chi), W)$ of G_χ by $((\xi\otimes\chi)(h,n)w = \chi(n)\xi(h)w$. Similar to the classification carried out in [17], we have the following results.

Theorem 3.30 ([7]). *Keeping the notations and hypotheses above, we have that*

$$Ind_{G_\chi}^G(\xi\otimes\chi)\cong\pi.$$

Proof. Since $\pi|_H\cong\mathrm{Ind}_{H_\chi}^H(\xi)$, the Imprimitivity Theorem ensures that there exists an isometric H-equivariant isomorphism $T:\mathcal{H}\to\mathrm{Ind}_{H_\chi}^H(W)$. We can use the map T to realize π as a representation over $\mathrm{Ind}_{H_\chi}^H(W)$. Namely, define $\tilde{\pi} : G\to GL(\mathrm{Ind}_{H_\chi}^H(W))$ by $\tilde{\pi}(h,n)f = (T\circ\pi(h,n)\circ T^{-1})f$. Since T is H-equivariant, it follows that $\tilde{\pi}|_H\cong\mathrm{Ind}_{H_\chi}^H(\xi)$. Let O_χ denote the orbit of χ. To compute the N-action of $\tilde{\pi}$ remark first that

$$\begin{aligned}\pi(n) = &\sum_{\psi\in O_\chi}\psi(n)P_\pi(\{\psi\}) = \sum_{H_\chi h}(h^{-1}\cdot\chi)(n)P_\pi(\ \{h^{-1}\cdot\chi\}) = \\ &\sum_{H_\chi h}\chi(\phi_h(n))P_\pi(\tau(H_\chi h)) = \sum_{H_\chi h}\chi(\phi_h(n))\tau^*P_\pi(H_\chi h),\end{aligned}$$

hence

$$\begin{aligned}(\tilde{\pi}(n)f)(h') =&\sum_{H_\chi h}\chi(\phi_h(n))(T\circ\tau^*P_\pi(H_\chi h)\circ T^{-1})f(h') = \sum_{H_\chi h}\chi(\phi_h(n))P^\xi(H_\chi h)f(h') = \\ &\sum_{H_\chi h}\chi(\phi_h(n))P^\xi(H_\chi h)f(h') = \chi(\phi_{h'}(n))f(h'),\end{aligned}$$

We will now investigate $\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)$ and see how to realize this representation on $\mathrm{Ind}_{H_\chi}^H$. First, we define the canonical vector space isomorphism $S:\mathrm{Ind}_{G_\chi}^G(W)\to\mathrm{Ind}_{H_\chi}^H(W)$ by $(S\phi)(h) = \phi(h,e)$. Indeed, this map is an isomorphism since linearity and injectivity are clear. It is also surjective since $\mathrm{Ind}_{G_\chi}^G(W)$ and $\mathrm{Ind}_{H_\chi}^H(W)$ have equal dimensions. Since

$$S(h'\cdot\phi)(h) = (h'\cdot\phi)(h,e) = \phi(hh',e) = (S\phi)(hh',e) = h'\cdot(S\phi)(h),$$

the map S intertwines the action of the group H on both vector spaces. The map S is unitary, since

$$\begin{aligned}\langle\phi,\psi\rangle_{\mathrm{Ind}_{G_\chi}^G(W)} =& \frac{1}{|H_\chi||N|}\sum_{(h,n)\in G}\langle\phi(h,n),\psi(h,n)\rangle_W = \\ & \frac{1}{|H_\chi||N|}\sum_{(h,n)\in G}\langle\chi(n)\phi(h,e),\chi(n)\psi(h,e)\rangle_W = \\ & \frac{1}{|H_\chi|}\sum_{h\in H}\langle\phi(h,e),\psi(h,e)\rangle_W = \langle S\phi,S\psi\rangle_{\mathrm{Ind}_{H_\chi}^H(W)}.\end{aligned}$$

Similar as we did above, we define $\eta : G \to GL(\mathrm{Ind}_{H_\chi}^H(W))$ by $\eta(h,n) = S\circ\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)(h,n)\circ S^{-1}$. It follows directly that $\eta|_H = \mathrm{Ind}_{H_\chi}^H(W)$. As we did above, we compute the N-action of η on $\mathrm{Ind}_{H_\chi}^H(W)$. Note that for all $(h,n)\in G$ and all $f\in\mathrm{Ind}_{G_\chi}^G(W)$, we have that

$$(\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)(e,n)\cdot f)(h,e) = f(h,\phi_h(n)) = \chi(\phi_h(n))f(h,e),$$

hence

$$\eta(n)\phi(h) = (S\circ\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)(e,n)\circ S^{-1})\phi(h) = \chi(\phi_h(n))\phi(h).$$

It follows that $\eta|_N \cong \tilde{\pi}|_N$. This establishes the unitary equivalence $\pi\cong\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)$. □

Theorem 3.31 ([7]). *Let $\chi\in\widehat{N}$ and let ξ be an irreducible representation of H_χ. Then the induced representation $\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)$ is irreducible. Furthermore, if $\eta\in\widehat{N}$ and if ω is an irreducible representation of H_η such that $\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)\cong\mathrm{Ind}_{G_\eta}^G(\omega\otimes\eta)$, then there exists $g\in G$ such that $g\cdot\chi=\eta$ and $\xi\cong\omega\circ C_g$.*

Proof. The first part follows by the calculations performed in the proof of Theorem 3.30. We have seen that $\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)$ is completely determined (up to equivalence) by the projection valued measure P^ξ. Since ξ is irreducible, P^ξ is irreducible, too. By the Imprimitivity Theorem, $\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)$ is irreducible.

For the second part, assume that $\mathrm{Ind}_{G_\xi}^G(\xi\otimes\chi)\cong\mathrm{Ind}_{G_\eta}^G(\omega\otimes\eta)$. Then, their associated projection valued measures are isomorphic as well, hence they are supported in the same orbit. This means that there exists $(h,n)=g\in G$ such that $g\cdot\chi=\eta$. Next, we claim that $G_\eta = gG_\chi g^{-1}$. Indeed, $g_1\in G_\eta$ if and only if $g_1\cdot\eta=\eta$. But since $\eta=g\cdot\chi$, this is equivalent with stating that $g_1\cdot g\cdot\chi = g\cdot\chi$. So, we see that $g_1\in G_\eta$ if and only if $g^{-1}g_1g\in G_\chi$, from which it indeed follows that $G_\eta = gG_\chi g^{-1}$.

This relation between G_η and G_χ enables us to define a representation $g\cdot\xi$ of H_η by $g\cdot\xi(x) = (h^{-1}xh)$. We have to show that $g\cdot\xi\cong\omega$. Define the map $R:\mathrm{Ind}_{G_\chi}^G(\xi\otimes\chi)\to\mathrm{Ind}_{G_\eta}^G(g\cdot\xi\otimes\chi')$ by $R\phi(g')=\phi(gg'g^{-1})$. Then this map is well-defined since $g^{-1}g'g\in G_\chi$ for $g'\in G_\eta$. Also, it is unitary since

$$\langle\phi,\psi\rangle_{\mathrm{Ind}_{G_\chi}^G(W)} = \frac{1}{|G_\chi|}\sum_{g\in G}\langle\phi(g),\psi(g)\rangle_W = \frac{1}{|G_\eta|}\sum_{g'\in G}\langle\phi(gg'g^{-1}),\psi(gg'g^{-1})\rangle_W =$$

$$\frac{1}{|G_\eta|}\sum_{g\in G}\langle R\phi(g), R\psi(g)\rangle_W = \langle R\phi, R\psi\rangle_{\mathrm{Ind}_{G_\eta}^G(W)}.$$

It follows that R is unitary, hence an isometry thus injective. Since both spaces have equal dimensions, R is surjective, too and it establishes a unitary equivalence $\mathrm{Ind}_{G_\chi}^G(\xi \otimes \chi) \cong \mathrm{Ind}_{G_\eta}^G(g \cdot \xi \otimes \eta)$. Then it follows that $\mathrm{Ind}_{G_\eta}^G(g \cdot \xi \otimes \omega) \cong \mathrm{Ind}_{G_\eta}^G(\omega \otimes \eta)$. By Corollary 3.27, the equivalence of these representations induces an equivalence of their systems of imprimitivity based on $\widehat{N}$. The canonical map τ from Propostion 3.29 induces equivalent systems of imprimitivity based on the coset space $H\backslash H_\eta$. The Imprimitivity Theorem implies that $g \cdot \xi \cong \omega$, which concludes the proof. $\square$

It follows from Theorem 3.30 and Theorem 3.31 that we have classified all irreducible unitary representations of G. We can summarize this in the following corollary. Note how this statement equivalent to Theorem 3.17.

Corollary 3.32. *Let $G = H \ltimes N$, with N abelian. Let $\chi \in \widehat{N}$ and let ξ be an irreducible representation of the stabilizer group H_χ. Then we have the following*

i. The representation $Ind_{G_\chi}^G(\xi \otimes \chi)$ is irreducible.

ii. Each irreducible representation of G is isomorphic to $Ind_{G_\chi}^G(\xi\otimes\chi)$ for some $\chi \in \widehat{N}$ and some irreducible representation χ of H_χ.

iii. If $\eta \in \widehat{N}$ and ω is an irreducible representation of H_η such that $Ind_{G_\chi}^G(\xi\otimes\chi) \cong Ind_{G_\eta}^G(\eta\otimes\omega)$, then there exists $g \in G$ such that $g \cdot \chi = \eta$ and $\xi \cong \omega \circ C_g$.

Chapter 4

Wigner's classification, a qualitative discussion

In this section, we aim to give a qualitive discussion of Wigner's classification of elementary particles we will introduce the *Lorentz group* and use it to define the *Poincaré* group, the symmetry group of flat Minkowski spacetime. We will first study basic properties of these groups, and discuss the relation between elementary particles and irreducible projective representations of the Poincaré group. We will show how to apply Theorem 2.27 and the Mackey machine from Chapter 3 to give a first classification of elementary particles.

4.1 The Lorentz and the Poincaré group

Before we introduce the Lorentz group, we fix some notation. For $x = (x_1, ..., x_4) \in \mathbb{R}^4$, we write $x = (x_1, x')$ with $x' = (x_2, x_3, x_4)$. We define a bilinear form β on $\mathbb{R}^4$ by

$$\beta(x, y) = x_1 y_1 - \langle x', y' \rangle$$

We call $\mathbb{R}^4$ together with the form β flat Minkowski space.

Definition 4.1 (Lorentz group,[7]). *The Lorentz group $\mathcal{L} = O(1,3)$ is defined to be the subgroup of $GL(4,\mathbb{R})$ leaving the quadratic form $(x_1, x_2, x_3, x_4) \mapsto x_1^2 - x_2^2 - x_3^2 - x_4^2$ invariant, i.e. $\beta(Lx, Lx) = \beta(x, x)$ for all $L \in \mathcal{L}$.*

An equivalent characterization of $\mathcal{L}$ is given by

$$\mathcal{L} = \{A \in GL(4,\mathbb{R})\colon A^T J A = J\}, \tag{4.1.1}$$

where $J = \begin{pmatrix} 1 & 0 \\ 0 & -I_{3\times 3} \end{pmatrix}$.

We see from (4.1.1) that $\mathcal{L}$ is a closed subgroup from $GL(4,\mathbb{R})$ and therefore a Lie group itself. It is not compact since it is unbounded.

We now want to compute its Lie algebra $\mathfrak{l}$. We can rewrite (4.1.1) as $JA^TJ = A^{-1}$. Differentation of this relation implies that $X \in \mathfrak{l}$ if and only if $JX^TJ = -X$, so $\mathfrak{l} = \{X \in$

$\mathrm{End}(\mathbb{R}^4)\colon X^TJ = -JX\}$ and the bracket structure is given by the commutator of matrices. Far less trivial is that $\mathcal{L}$ is disconnected and has exactly 4 connected components. We refer to [7] to see the exact reasoning. We denote its component containing the identity by $SO(1,3)^\circ$. The form β induces a "metric" g on $\mathbb{R}^4$ which is given by

$$g(x,y) = \beta(x-y, x-y) = (x_1 - y_1)^2 - \sum_{i=2}^{4}(x_i - y_i)^2.$$

Not only is g invariant under transformations in $\mathcal{L}$, it is invariant under translations as well. This gives a motivation for the defintion of the Poincaré group.

Definition 4.2 ([7])**.** *The Poincaré group $\mathcal{P}$ is defined by $\mathcal{P} = O(1,3) \ltimes \mathbb{R}^4$, where $O(1,3)$ acts canonically on $\mathbb{R}^4$.*

Where the special Euclidean group $SE(n)$ was the full symmetry group of $\mathbb{R}^n$ as in Example 1.45, a similar argument shows that the Poincaré group is the full symmetry group of $\mathbb{R}^4$ endowed with the metric g.

We will now discuss the physical application we had in mind from the beginnning: to provide a qualatative classification of elementary particles. Consider a particle in flat Minkowski space. As we discussed in Section 2.3, its state space should be $\mathbb{P}(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$. If two obervers $\mathcal{O}$ and $\mathcal{O}'$, related by a transformation $\Lambda \in \mathcal{P}$, perform a quantum-mechanical experiment, they measure a different $[\Phi_{\mathcal{O}}] \neq [\Phi_{\mathcal{O}'}]$. Since the laws of physics should be the same in each inertial frame, we expect these two states to be $\mathcal{P}$-equivarianty related, i.e. $[\Phi_{\mathcal{O}}] = T_\Lambda[\Phi_{\mathcal{O}'}]$ for some $T_\Lambda \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. This leads to a group homomorphism $\mathcal{P} \to \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$: a projective representation of $\mathcal{P}$. The key idea is to *identify* our particle in Minkowski space with a projective representation of the Poincaré group. If we make the (rather plausible) assumption that the whole system is built up of elementary particles, these elementary particles should exactly correspond to the *irreducible* projective representations of $\mathcal{P}$ [15].

This means that the question of classifying elementary particles is just a matter of classifying all irreducible projective representations of $\mathcal{P}$. Unfortunately, the group $\mathcal{P}$ is not connected, which has as consequence that studying its projective representations is a challenging task. It does describe interesting physical phenomena such as parity inversion, but we will not do this here. Instead, we will restrict ourselves to the identity component, $SO(1,3)^\circ \ltimes \mathbb{R}^4$. Since this group is connected, we can pass to its universal covering group, which happens to be $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$, as the following lemma shows.

Lemma 4.3. *The group $SL(2,\mathbb{C})$ is simply-connected and is a double cover for $SO(1,3)^\circ$.*

Proof. For the first part, we sketch the argument given in [7], but we do not give a full proof, since it makes use of the Cartan decomposition which we will not discuss here.

The fact that $SL(2,\mathbb{C})$ is simply-connected follows from the Cartan decomposition of $GL(2,\mathbb{C})$. We can express $GL(2,\mathbb{C})$ as

$$GL(2,\mathbb{C}) = U(2)\exp(S),$$

where $U(2)$ is the unitary group of degree 2 and S the space of Hermitian 2×2 matrices. Moreover, the map $U(2) \times S \to GL(2, \mathbb{C})$, $(u, s) \mapsto u \exp(s)$ is a diffeomorphism. Let $S_0 \subset S$ denote the space of traceless Hermitian matrices. Recall that for any matrix, we have that $\det(\exp(X)) = e^{\mathrm{Tr}(X)}$. Therefore, the map above restricts to a diffeomorphism $SU(2) \times S_0 \to SL(2, \mathbb{C})$. Since S_0 is simply-connected and $SU(2) \cong S^3$ is simply-connected, the same holds for $SL(2, \mathbb{C})$.

To prove the second part of the claim, we identify $\mathbb{R}^4$ with S by

$$(x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + x_4 & x_2 - ix_3 \\ x_2 + ix_3 & x_1 - x_4 \end{pmatrix}.$$

We denote the corresponding isomorphism $\mathbb{R}^4 \cong S$ by Ψ. Remark that $\det(\Psi(x)) = \beta(x, x)$. Now, we define $\Phi : SL(2, \mathbb{C}) \to SO(1, 3)^\circ$ by $\Phi(A)x = \Psi^{-1}(A\Psi(x)A^\dagger)$. First, we have to check well-definedness of the map Φ. First, remark that $(A\Psi(x)A^\dagger)^\dagger = A\Psi(x)^\dagger A^\dagger = A\Psi(x)A^\dagger$, so $A\Psi(x)A^\dagger$ is again Hermitian. Since $\det(A) = 1$, for all $A \in SL(2, \mathbb{C})$, we have that

$$\beta(\Phi(A)x, \Phi(A)x) = \det(A\Psi(x)A^\dagger) = \det(\Psi(x)) = \beta(x, x),$$

so $\Phi(A) \in SO(1, 3)^\circ$ and Φ is well-defined. Also, note that Φ is a Lie group homomorphism, since $\Phi(AB)x = \Phi(A)\Phi(B)x$ for all $x \in \mathbb{R}^4$. By Proposition 9.29 in [2], the map Φ is a covering map if we can prove that its kernel is discrete.

We remark that $\ker(\Phi) = \{A \in SL(2, \mathbb{C})\colon \Psi(x) = A\Psi(x)A^\dagger \,\text{for all}\, x \in \mathbb{R}^4\}$. If we take $x = (0, 0, 0, 1)$, we deduce that $AA^\dagger = Id$, hence $A^\dagger = A^{-1}$. Thus, we see that $\ker(\Phi) = \{A \in SL(2, \mathbb{C}) : A\Psi(x) = \Psi(x)A \,\text{for all}\, x \in \mathbb{R}^4\}$. In particular, all real diagonal matrices are contained in S. Note that A commutes with all real diagonal matrices if and only if $A = \pm\mathrm{Id}$. From this, we deduce that $\ker(\Phi) = \{\pm\mathrm{Id}\}$ which is discrete, hence Φ is a smooth covering map. $\square$

Lemma 4.3 has a few interesting consequences. First of all, we remark that $SL(2, \mathbb{C}) \ltimes \mathbb{R}^4$ is the universal covering group of $SO(1, 3)^\circ \ltimes \mathbb{R}^4$, as we promised before. Another direct consequence is that the Lie algebras $\mathfrak{so}(1, 3)$ and $\mathfrak{sl}(2, \mathbb{C})$ are isomorphic as Lie algebras. The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is well-known and is treated in most books on Lie algebras (for example, see [20]), and one of its key properties is that it is both simple and semisimple. This immediately implies that $\mathfrak{so}(1, 3)$ is both simple semisimple as well. This simplifies the task of finding all irreducible projective representations of $SO(1, 3)^\circ \ltimes \mathbb{R}^4$ greatly, since we can show that they are in bijective correspondence with irreducible unitary representations of $SL(2, \mathbb{C}) \ltimes \mathbb{R}^4$, as we will see in the next section.

4.2 Projective representations of the Poincaré group

The fundamental goal of this section is to set up a one-to-one correspondence between irreducible projective representations of $SO(1, 3)^\circ \ltimes \mathbb{R}^4$ and irreducible unitary representations of $SL(2, \mathbb{C}) \ltimes \mathbb{R}^4$. First, we will show a one-to-one correspondence between the irreducible unitary representations and the irreducible projective representations of $SL(2, \mathbb{C}) \ltimes \mathbb{R}^4$ and use this to set up a one-to-one correspondence between the irreducible projective representations of $SO(1,3)^\circ \ltimes \mathbb{R}^4$ and the irreducible unitary representations of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$.

We agree to write $\mathrm{Lie}(SL(2,\mathbb{C}) \ltimes \mathbb{R}^4) = \mathfrak{sl}(2,\mathbb{C}) \ltimes \mathbb{R}^4$, the semidirect sum of the Lie algebras. Recall from Theorem 2.27 that each projective representation of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$ lifts to a unitary representation if the second cohomology group of its Lie algebra in $\mathbb{R}$ vanishes. Thus, our first aim is to show that $H^2(\mathfrak{sl}(2,\mathbb{C}) \ltimes \mathbb{R}^4, \mathbb{R}) = 0$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let V be a finite dimensional $\mathfrak{g}$-module. We can use the module V to construct new $\mathfrak{g}$-modules. For example, its dual space V^* is a $\mathfrak{g}$-module as well, where the action of $\mathfrak{g}$ on V^* is given by $(X \cdot f)(v) = f(-Xv)$. Also, the tensor product $V \otimes V$ is a $\mathfrak{g}$-module, where the action of $\mathfrak{g}$ on $V \otimes V$ is given by $X(v \otimes v') = Xv \otimes v' + v \otimes Xv'$. This implies that the $\mathrm{Hom}(V, W)$, which is isomorphic to $V^* \otimes W$, is a $\mathfrak{g}$-module where $(X \cdot f)(v) = X \cdot f(v) - f(X \cdot v)$. Also, its quotient $\wedge^2 V$ is a $\mathfrak{g}$-module as well, where $X(v \wedge v') = Xv \wedge v' + v \wedge Xv'$. Hence, we can define the linear subspace

$$(\wedge^2 V)^{\mathfrak{g}} = \{v \in \wedge^2 V : \mathfrak{g}v = 0\}.$$

As usual, we identify $\wedge^2 V^*$ with the space of bilinear alternating maps $V \times V \to \mathbb{R}$. The following proposition will help significantly to show that $H^2(\mathfrak{sl}(2,\mathbb{C}) \ltimes \mathbb{R}^4, \mathbb{R}) = 0$.

Proposition 4.4 ([7])**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra and let V be a finite dimensional $\mathfrak{g}$-module satisfying $(\wedge^2 V^*)^{\mathfrak{g}} = 0$. Then $H^2(\mathfrak{g} \ltimes V, \mathbb{R}) = 0$.*

In the proof of Proposition 4.4, we will use the following result.

Theorem 4.5 (Weyl's theorem on complete reducibility,[21])**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then every finite dimensional $\mathfrak{g}$-module is completely reducible.*

We postpone the proof of this theorem to the Appendix. If V is a finite dimensional $\mathfrak{g}$-module with submodule $\mathfrak{a}$, it follows from Theorem 4.5 that each short exact sequence

$$0 \to \mathfrak{a} \to \mathfrak{b} \xrightarrow{p} \mathfrak{c} \to 0$$

of $\mathfrak{g}$-modules is trivial, i.e. there exists a $\mathfrak{g}$-module homomorphism $f : \mathfrak{c} \to \mathfrak{b}$ such that $p \circ f = \mathrm{Id}_{\mathfrak{c}}$. This is the key to prove Proposition 4.4.

Proof. We follow [7] for this proof. We set $\mathfrak{s} = \mathfrak{g} \ltimes V$. Let $\omega : \mathfrak{s} \times \mathfrak{s} \to \mathbb{R}$ be a cocycle. We have to prove that ω is exact. It is readily seen that $\omega_{\mathfrak{g}} = \omega|_{\mathfrak{g}\times\mathfrak{g}}$ is a cocycle on $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, $\omega_{\mathfrak{g}}$ is exact, hence there exists a linear map $\lambda : \mathfrak{g} \to \mathbb{R}$ such that $\omega_{\mathfrak{g}}(X, Y) = \lambda([X, Y])$. We can extend λ to a linear map $\tilde{\lambda} : \mathfrak{s} \to \mathbb{R}$ by setting $\tilde{\lambda}(X, v) = \lambda(X)$. We define $\omega_1 : \mathfrak{s} \times \mathfrak{s} \to \mathbb{R}$ by

$$\omega_1((X, v), (Y, v')) = \omega((X, v), (Y, v')) - \tilde{\lambda}[(X, v), (Y, v')].$$

We can observe directly that $\omega_1 - \omega$ is exact, hence $[\omega_1] = [\omega]$ in $H^2(\mathfrak{s}, \mathbb{R})$. Note that the restriction of ω_1 to $\mathfrak{g} \times \mathfrak{g}$ is trivial. Now, We define the linear map $\rho(X) : \mathbb{R} \oplus V \to \mathbb{R} \oplus V$ by

$$\rho(X)(t, v) = (\omega_1((X, 0), (0, v)), Xv).$$

Then ρ is a Lie algebra representation since ω_1 is a cocycle. Moreover, the short sequence

$$0 \to \mathbb{R} \overset{\iota}{\hookrightarrow} \mathbb{R} \oplus V \xrightarrow{\mathrm{Pr}_V} V \to 0$$

is exact. By Weyl's theorem, this sequence is trivial. This means that there exists a Lie algebra homomorphism $\phi : V \to \mathbb{R} \oplus V$. We can write $\phi = (\mu, \mathrm{Id}_V)$ with $\mu \in V^*$. Since ϕ is a Lie algebra homomorphism, it follows that $\mu(Xv) = \omega_1((X, 0), (0, v))$. We extend μ to a linear map $\tilde{\mu}\mathfrak{s} \to \mathbb{R}$ by putting it to zero on $\mathfrak{g}$. Then $\omega_2 : \mathfrak{s} \times \mathfrak{s} \to \mathbb{R}$ defined by

$$\omega_2((X, v), (Y, v')) = \omega((X, v), (Y, v')) - \tilde{\lambda}[(X, v), (Y, v')] - \tilde{\mu}[(X, v), (Y, v')]$$

vanishes on $\mathfrak{g} \times \mathfrak{s}$, hence is complete determined by its restriction $\omega_2|_{V\times V}$. By closedness of ω_2, we have that

$$0 = \omega_2((X, 0), (0, [v, w])) = \omega_2|_{V\times V}(Xv, w) + \omega_2|_{V\times V}(v, Xw)$$

Thus, we see that $\omega_2|_{V\times V} \in (\wedge^2 V^*)^{\mathfrak{g}}$, hence $\omega_2 = \omega_2|_{V\times V} = 0$. We infer that

$$\omega((X, v), (Y, v')) = (\tilde{\lambda} + \tilde{\mu})[(X, v), (Y, v')],$$

hence ω is exact. This proves the claim. $\square$

Proposition 4.4 has the following corollary, which we are looking for.

Corollary 4.6 ([7])**.** $H^2(\mathfrak{so}(1, 3) \ltimes \mathbb{R}^4, \mathbb{R}) = 0$.

Proof. By the Proposition 4.4, we have to check that each $\mathfrak{so}(1, 3)$-invariant alternating bilinear map $\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ is zero. Let $\omega : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be such a map. By nondegeneratness of the bilinear form β, there exists a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\omega(x, y) = \beta(Tx, y) \text{ for all } x, y \in \mathbb{R}^4.$$

Since ω and β are $\mathfrak{so}(1, 3)$-invariant, the map T commutes with the $\mathfrak{so}(1, 3)$ action, hence $T = C\,\mathrm{Id}$ for some $C \in \mathbb{R}$. By linearity of β, it follows that $\omega = C\beta$. Since ω is alternating, we get that

$$0 = \omega(e_1, e_1) = C\beta(e_1, e_1).$$

Since $\beta(e_1, e_1) \neq 0$, it follows that $C = 0$ so $\omega = 0$. By Proposition 4.4, we conclude that $H^2(\mathfrak{so}(1, 3) \ltimes \mathbb{R}^4, \mathbb{R}) = 0$. $\square$

Since we had already established that $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(1, 3)$ as Lie algebras, we see that $H^2(\mathfrak{sl}(2, \mathbb{C}) \ltimes \mathbb{R}^4, \mathbb{R}) = 0$. Now, we can establish our first equivalence.

Theorem 4.7 ([7])**.** *Let $\mathcal{H}$ be a complex Hilbert space. Every projective representation ρ : $SL(2, \mathbb{C}) \ltimes \mathbb{R}^4 \to Aut(\mathbb{P}(\mathcal{H}))$ lifts to a unique unitary representation $\tilde{\rho} : SL(2, \mathbb{C}) \ltimes \mathbb{R}^4 \to \mathcal{U}(\mathcal{H})$, which is irreducible if and only if ρ is irreducible.*

Proof. The existence of the lift $\tilde{\rho}$ follows from Theorem 2.27 and Corollary 4.6. Assume that $\tilde{\pi}$ is a second lifting. By what we established before, there exists a unique map $\phi : SL(2,\mathbb{C}) \ltimes \mathbb{R}^4 \to \widetilde{\mathbb{T}}$ such that $\tilde{\pi}(x) = \tilde{\rho}(x)\phi(x)$ for all x. Let $\bar{\phi}$ be the restriction of ϕ to $SL(2,\mathbb{C})$. Then we have that $d\overline{\phi}_e : \mathfrak{sl}(2,\mathbb{C}) \to \mathbb{R}$ is a Lie algebra homomorphism. Since $\mathfrak{sl}(2,\mathbb{C})$ is simple, it contains no non-zero abelian ideals. Since $\ker(d\overline{\phi}_e)$ is an abelian ideal, it follows that $\ker(d\overline{\phi}_e) = \{0\}$, so $\bar{\phi} = \mathrm{Id}$, as $SL(2,\mathbb{C})$ is connected. Note that the set $\ker(d\phi_e)\cap\mathbb{R}^4$ is $SO(1,3)^\circ$-invariant, it must be all of $\mathbb{R}^4$. Therefore, we have that $\phi = \mathrm{Id}$, hence $\tilde{\rho} = \tilde{\pi}$. Note that if $\mathcal{H}$ is a closed invariant subspace for $\tilde{\rho}$, it follows from the construction of $\mathbb{P}(\mathcal{H})$ that $\mathbb{P}(V)$ is a closed invariant subsapce for ρ. The converse holds by a similar argument. □

This is the first correspondence we wished to establish. The following theorem shows the second.

Theorem 4.8 ([7])**.** *Every irreducible unitary representation of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$ in a complex Hilbert space $\mathcal{H}$ induces an irreducible projective representation of $SO(1,3)^\circ \ltimes \mathbb{R}^4$ in $\mathbb{P}(\mathcal{H})$. Moreover, every such a representation comes from a untiary representation of $SL(2,\mathbb{C})$.*

Proof. Recall that the kernel of the covering map $\Phi \times \mathrm{Id} : SL(2,\mathbb{C}) \ltimes \mathbb{R}^4 \to SO(1,3)^\circ \ltimes \mathbb{R}^4$ has kernel $\{(\pm\mathrm{Id}, 0)\}$, which happens to be the centre of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$. Thus, if π is an irreducible unitary representation $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$, the subgroup $\ker(\Phi \times \mathrm{Id})$ acts by scalars. Thus, π induces an irreducible projective representation $\tilde{\pi}$ of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$ which is trivial on $\ker(\Phi \times \mathrm{Id})$, hence factors through an irreducible projective representation of $SO(1,3)^\circ \ltimes \mathbb{R}^4$.

Conversely, assume that $\tilde{\rho}$ is an irreducible projective representation of $SO(1,3)^\circ \ltimes \mathbb{R}^4$. Then $\tilde{\rho} \circ (\Phi \times \mathrm{Id})$ is an irreducible projective representation of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$. By the previous theorem, this has a unique lift to an irreducible unitary representation of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$. □

Now that we know that we can lift each irreducible projective representation to an irreducible unitary representation of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$, we can classify them according to Corollary 3.32.

4.3 Wigner's Classification

In this section, we classify the unitary irreducible representations of $SL(2,\mathbb{C}) \ltimes \mathbb{R}^4$, hence the irreducible projective representations of $SO(1,3)^\circ \ltimes \mathbb{R}^4$. By the Mackey machine, these are classified by two parameters: a representative χ of an orbit in $\widehat{\mathbb{R}^4}$ and an irreducible representation of the stabilizer of χ under the $SL(2,\mathbb{C})$ action. We follow [7] for this classification.

First, we remark that the continuous unitary characters of $\mathbb{R}^4$ are exactly the continuous homomorphisms $\mathbb{R}^4 \to S^1$. By the nondegenerateness of β, we can identify $\mathbb{R}^4$ with its character group by the isomorphism

$$v \mapsto \xi_v, \text{ where } \xi_v(x) = e^{i\beta(v,x)}.$$

Indeed, ξ_v is a homomorphism for all $v \in \mathbb{R}^4$. By linearity of β, we have that $\xi_v(x + y) = e^{i\beta(v,x+y)} = e^{i\beta(v,x)}e^{i\beta(v,y)} = \xi_v(x)\xi_v(y)$. Also, this map intertwines the $SL(2,\mathbb{C})$ action induced by the covering map Φ on $\mathbb{R}^4$ with the action of $SL(2,\mathbb{C})$ on the character group. Indeed, since β is $SO(1,3)^\circ$ invariant, we have for $A \in SL(2,\mathbb{C})$ that

$$(A \cdot \xi_v)(x) = \xi_v(\Phi(A)^{-1}x) = e^{i\beta(v,\Phi(A)^{-1}x)} = e^{i\beta(\Phi(A)v,x)} = \xi_{\Phi(A)v}(x).$$

It follows that $v \in \mathbb{R}^4$ is $SL(2,\mathbb{C})$ invariant if and only if ξ_v is, too. Indeed, $A \cdot \xi_v = \xi_v$ if and only if $v = \Phi(A)v$. So instead of computing the orbits of the $SL(2,\mathbb{C})$-action in the more abstract space $\widehat{\mathbb{R}^4}$, we can compute the orbits of the induced $SL(2,\mathbb{C})$ action on $\mathbb{R}^4$. As a preparation to finding these orbits, define

$$Y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that $YJ = -JY$, so $Y \in \mathfrak{l}$, the Lie algebra of the Lorentz group. As a linear space, define $\mathfrak{a} = \mathbb{R}Y$. Then a computation shows that

$$\exp(\mathfrak{a}) = \exp(tY) = \begin{pmatrix} \cosh(t) & 0 & 0 & \sinh(t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(t) & 0 & 0 & \cosh(t) \end{pmatrix}$$

is a closed subgroup of $SO(1,3)^\circ$, since it contains the identity matrix.

For $c \in \mathbb{R}$, define the sets

$$X_c = \{x \in \mathbb{R}^4 \colon \beta(x,x) = c\}.$$

By invariance of β, the sets X_c are $SL(2,\mathbb{C})$ invariant. Since this group is connected, its orbits must be connected as well. This provides us with the following families of orbits.

For $c = m^2 > 0$, we have that $x_1^2 - x_2^2 - x_3^2 - x_4^2 = m^2$, hence all $x \in X_c$ satisfy $x_1 \neq 0$. The sets X_{m^2} form a family of two-sheeted hyperboloids, distinguished by the sign of x_1. For fixed m^2, the two sets $X^{\pm}_{m^2} = \{x \in X_{m^2} \colon x_1 \in \mathbb{R}^{\pm}\}$ are disjoint and both connected. Therefore, to check that these sets form two families of orbits, it suffices to check that $SO(1,3)^\circ$ acts transitively on both (the orbits cannot be bigger by connectedness). We prove this only for $X^+_{m^2}$, since $X^-_{m^2}$ works similar.

Let $v = (m,0,0,0) \in X^+_{m^2}$. It follows that $\exp(tY)v = m(\cosh(t),0,0,\sinh(t))$. Let $w \in X^+_{m^2}$, it follows that $w_1 \geq v_1$, since v_1 is minimal. Therefore, there exists $t \in \mathbb{R}$ such that $w_1 = \cosh(t)v_1$. Since $w \in X^+_{m^2}$, we know that $w_2^2 + w_3^2 + w_4^2 = m^2\sinh^2(t)$. Therefore, there exists a rotation $R \in SO(3)$ such that $R(0,0,m\sinh(t)) = (w_2,w_3,w_4)$. Embedding $SO(3)$ in the lower right corner of $SO(1,3)^\circ$, we get that $R\exp(tY)v = w$. We conclude that $X^+_{m^2}$ is an orbit. The same holds for $X^-_{m^2}$, so we have two families of orbits.

For $c = 0$, we get that $X_0 = \{x \in \mathbb{R}^4 \colon x_1^2 = x_2^2 + x_3^2 + x_4^2\}$. Clearly, $\{0\}$ is a $SL(2,\mathbb{C})$-orbit as this is a fixed point of the action. Then $X_0 \setminus \{0\}$ has again two connected components, depending on the sign of x_1, which we denote by X_0^+ and X_0^-. If $x_1 > 0$, consider

$v = (1, 0, 0, 1) \in X_0^+$. Note that $\exp(tY)v = e^t v$. If $w \in X_0^+$, there exists $t \in \mathbb{R}$ such that $e^t v_1 = w_1$. Then it follows that

$$e^{2t}(v_2^2 + v_3^2 + v_4^2) = e^{2t} v_1^2 = w_1^2 = w_2^2 + w_3^2 + w_4^2,$$

so there exists a rotation $R \in SO(3)$ such that $e^t R(v_2, v_3, v_4) = (w_1, w_2, w_3)$. Embedding this R in the lower right corner of $SO(1,3)^\circ$, it follows that $R \circ \exp(tY)v = w$, hence X_0^+ is an orbit. The same holds for X_0^-, so this gives us three other orbits.

Now, we consider $c = -m^2 < 0$. Since this level set is connected, we expect the existence of only one family of orbits. Indeed, let $v = (0, 0, 0, m) \in X_c$. Then $\exp(tY)v = m(\sinh(t), 0, 0, \cosh(t))$. If $w \in X_c$, there exists $t \in \mathbb{R}$ such that $w_1 = \sinh(t)m$. Since $w \in X_c$, we know that $w_2^2 + w_3^2 + w_4^2 = m^2 \cosh^2(t)$. Therefore, we know that there exists a rotation $R \in SO(3)$ such that $R(0, 0, m\cosh(t)) = (w_2, w_3, w_4)$. We embed R in $SO(1,3)^\circ$ and see that $R \circ \exp(tY)v = w$. This gives us the last family of orbits.

Now, we come to the computation of the stabilizer of each of the orbits. As in the proof of Lemma 4.3, we identify elements in $\mathbb{R}^4$ with a Hermitian matrix.

First, consider $c = m^2 > 0$. For $X_{m^2}^+$, we take the representative $(m, 0, 0, 0)$. As we saw in Lemma 4.3, the stabilizer condition implies that

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = A \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} A^\dagger = AA^\dagger \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix},$$

hence $AA^\dagger = \mathrm{Id}$. It follows that A is unitary and since $A \in SL(2, \mathbb{C})$, we know that $\det(A)$. Combining these two facts yields that the stabilizer for the orbits $X_{m^2}^+$ is the group $SU(2)$. The same discussion holds for the orbits $X_{m^2}^-$.

Now, suppose $c = 0$. We have three different orbits under the $SL(2, \mathbb{C})$. The stabilizer of $\{0\}$ is the full group $SL(2, \mathbb{C})$. For X_0^+, we take the representative $v = (m, 0, 0, m)$. The stabilizer condition reads

$$\begin{pmatrix} 2m & 0 \\ 0 & 0 \end{pmatrix} = A \begin{pmatrix} 2m & 0 \\ 0 & 0 \end{pmatrix} A^\dagger = \begin{pmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 2m & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix} =$$

$$2m \begin{pmatrix} |a_{11}|^2 & a_{11} a_{21}^* \\ a_{11}^* a_{21} & |a_{21}|^2 \end{pmatrix},$$

from which we deduce that $|a_{11}| = 1$, hence $a_{11} = e^{i\varphi}$ for some $\varphi \in \mathbb{R}$, and $a_{21} = 0$. From the condition that $A \in SL(2, \mathbb{C})$, we deduce that $a_{22} = a_{11}^{-1} = e^{-i\varphi}$ and a_{12} can be any complex number. Thus, the stabilizer group is given by

$$\left\{ \begin{pmatrix} e^{i\varphi} & a \\ 0 & e^{-i\varphi} \end{pmatrix} : \varphi \in \mathbb{R}, a \in \mathbb{C} \right\},$$

which is isomorphic to the group $SO(2) \ltimes \mathbb{R}^2$. The same holds for X_0^-.

Now, suppose that $c = -m^2 < 0$. Remember that we had one orbit, and we take $(0, 0, m, 0)$ as representative (the advantage is that our conjugated matrix is now off-diagonal). The stabilizer condition implies that

$$A \begin{pmatrix} 0 & -im \\ im & 0 \end{pmatrix} A^\dagger = \begin{pmatrix} 0 & -im \\ im & 0 \end{pmatrix}$$

To get conditions on A, we apply the following trick. Note that $\det(A) = a_{11}a_{22} - a_{12}a_{21} = 1$. Then we see that

$$A \begin{pmatrix} 0 & -im \\ im & 0 \end{pmatrix} A^T = \begin{pmatrix} 0 & (a_{21}a_{12} - a_{11}a_{22})im \\ (a_{11}a_{22} - a_{12}a_{21})im & 0 \end{pmatrix} = \begin{pmatrix} 0 & -im \\ im & 0 \end{pmatrix}$$

Since both A and $\begin{pmatrix} 0 & -im \\ im & 0 \end{pmatrix}$ are invertible, it follows that $A^T = A^\dagger$. It follows that all coefficents of A are real, hence $A \in SL(2, \mathbb{R})$.

Now that we have classified the stabilizer subgroups, the last step in the classification procedure is the analysis of the irreducible representations of these stabilizer groups.

We start with $m^2 > 0$. As we have just computed, the stabilizer of the orbits is the group $SU(2)$, which is a compact and connected Lie group as it is isomorphic to S^3 . It follows that all its irreducible representations are finite dimensional [3].

We will now construct all its irreducible representations. Denote by $P(\mathbb{C}^2)$ the space of polynomials $p : \mathbb{C}^2 \to \mathbb{C}$. We can write $g \in SU(2)$ as $g = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ with $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|^2 + |\beta^2| = 1$. We define a representation of π of $SU(2)$ on $P(\mathbb{C}^2)$ by

$$\pi(g)p(z_1, z_2) = p(g^{-1}(z_1, z_2)) = p(\bar{\alpha}z_1 + \bar{\beta}z_2, -\beta z_1 + \alpha z_2).$$

This is indeed a representation since $\pi(gh)p(z) = p(h^{-1}g^{-1}z) = \pi(h)p(g^{-1}z) = \pi(g)\pi(h)p(z)$. For each n, the space $P_n(\mathbb{C}^2)$ of homogeneous polynomials is an invariant subspace for π. We denote by π_n the restriction of π to $P_n(\mathbb{C}^2)$. The following lemma helps significantly with showing that each π_n is irreducible.

Lemma 4.9 ([3])**.** *Let (π, V) be a finite dimensional representation of a Lie group G. If π is unitarizable and if $Hom_G(V, V) = \mathbb{C}Id$, then π is irreducible.*

Proof. This is Lemma 20.27 in [3]. □

Now, we can prove irreducibility of the π_n.

Proposition 4.10 ([3])**.** *Let π_n be as above. Then π_n is irreducible for each n.*

Proof. We follow [3]. Since $SU(2)$ is compact, a similar argument as in Example 2.21, but this time involving an integral with respect to an invariant measure, shows that π_n is unitarizable. In view of the previous lemma, it suffcies that each $SU(2)$-equivariant homomorphism $P_n(\mathbb{C}^2) \to P_n(\mathbb{C}^2)$ is a scalar multiple of the identity. For $0 \leq k \leq n$, define

$p_k(z_1, z_2) = z_1^{n-k} z_2^k$. Then the polynomials $\{p_k \colon 0 \leq k \leq n\}$ form a basis for $P_n(\mathbb{C}^2)$. We define two closed subgroups R and T of $SU(2)$ by

$$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\} \qquad R = \left\{ \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} : \phi \in \mathbb{R} \right\}.$$

For $t_\theta \in T$, we see for each k that

$$\pi_n \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} (p_k(z_1, z_2)) = p_k(e^{-i\theta} z_1, e^{i\theta} z_2) = e^{i\theta(2k-n)} z_1^{n-k} z_2^k = e^{i\theta(2k-n)} p_k(z_1, z_2).$$

It follows that all p_k are eigenvectors of all $\pi_n(t)$ with $t \in T$. Let $A \in \mathrm{Hom}_{SU(2)}(P_n(\mathbb{C}^2), P_n(\mathbb{C}^2)$. Since A is $SU(2)$-equivariant, A and $\pi_n(t)$ commute for all $t \in T$, and thus they preserve each others eigenspaces. It follows that A leaves all the eigenspaces $\mathbb{C}p_k$ invariant, thus there exist $\lambda_k \in \mathbb{C}$ such that $Ap_k = \lambda_k p_k$ for each $0 \leq k \leq n$. Denote by V_0 the eigenspace of A corresponding to λ_0. We are done if we can prove that $V_0 = P_n(\mathbb{C}^2)$, since this indeed implies that A is scalar multiplication by λ_0.

The eigenspace V_0 is $SU(2)$ invariant. Indeed, we have for all $P \in V_0$ that that

$$A(\pi_n(g)P) = \pi_n(g)(AP) = \lambda_0 \pi_n(g)P,$$

so $\pi_n(g)P \in V_0$ for all $g \in SU(2)$. Also, it contains p_0 since

$$Ap_0 = A\left(\pi_n \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\right) e^{in\theta} p_0 = e^{in\theta} \pi_n \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} Ap_0 = \lambda_0 p_0.$$

It follows that $\pi_n(r)p_0 \in V_0$ for all $r \in R$. A computation using Newton's binomium shows that

$$\pi_n \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} p_0(z_1, z_2) = (\cos(\phi) z_1 + \sin(\phi) z_2)^n = \sum_{j=0}^{n} \binom{j}{n} \cos^{n-j}(\phi) \sin^j(\phi) p_j(z_1, z_2).$$

In particular, it follows that

$$\lambda_0 \sum_{j=0}^{n} \binom{j}{n} \cos^{n-j}(\phi) \sin^j(\phi) p_j = \sum_{j=0}^{n} \binom{j}{n} \cos^{n-j}(\phi) \sin^j(\phi) \lambda_j p_j$$

for all $\phi \in \mathbb{R}$, hence we have for all $\phi \in \mathbb{R}$ that

$$\sum_{j=0}^{n} \binom{j}{n} \cos^{n-j}(\phi) \sin^j(\phi)(\lambda_j - \lambda_0) p_j = 0.$$

Since the p_j form a basis of $P_n(\mathbb{C}^2)$, we conclude that $\lambda_j = \lambda_0$ for all j. We conclude that $V_0 = P_n(\mathbb{C}^2)$, so π_n is irreducible. This concludes the proof. $\square$

The proof that each irreducible representation of $SU(2)$ is actually isomorphic to one of the π_n is too involved to present here. It uses the Peter-Weyl theorem and several results on Fourier theory of abelian groups. We refer to Chapter 28 in [3].

We return to the the classification for $c = m^2 > 0$. Without loss of generality, we suppose that $m > 0$. We call the parameter m the *mass* of the particle. Having classified the irreducible representations of $SU(2)$, we see that we can realize them on $P_n(\mathbb{C}^2) \cong \mathbb{C}^{n+1}$, thus they are classified by their dimension. If we set $s \in \frac{1}{2}\mathbb{N}$, the representations are thus realized over $\mathbb{C}^{2s+1}$, hence they are classified by the number s. We call this number s the *spin* of the particle. Examples are the electron ($s = \frac{1}{2}$) and Higgs-boson ($s = 1$).

For $c = 0$, we have computed that the stabilizer was the group $SO(2) \ltimes \mathbb{R}^2$. Of course, we can again apply the Mackey machine to this group. The $SO(2)$-orbits in $\mathbb{R}^2$ are circles around the origin with radius $\rho \geq 0$. For $\rho > 0$, the stabilizer is trivial, while $\rho = 0$, the stabilizer is the group $SO(2)$. Note that $SO(2) \cong S^1$. Thus, it suffices to classify the irreducible unitary representations of S^1. By Schur's lemma, they are all one-dimensional. Given a continuous representation $\rho : S^1 \to \mathbb{C}^\times$, it has a compact, hence bounded, image. In particular, the image must be contained in S^1, hence the representations are continuous homomorphisms $\rho : S^1 \to S^1$. This means that we get a continuous homomorhpism $\mathbb{R} \to S^1$ given by $x \mapsto \rho(e^{ix})$. By covering theory, there must exist $c \in \mathbb{R}$ such that $\rho(e^{ix}) = e^{icx}$ [8]. Since $1 = e^{2\pi i c}$, it follows that $c \in \mathbb{Z}$. This means that $\rho(z) = z^n$ for some $n \in \mathbb{Z}$ and in particular, the list

$$\rho_n : S^1 \to S^1,\ \rho_n(z) = z^n \text{ for } n \in \mathbb{Z}$$

gives a classification of all irreducible representations of S^1, hence of $SO(2)$.

We return to the classification for $c = 0$. We call these particles the *massless* particles. If we set $s \in \mathbb{Z}$, we can classify these representations by the number s, whose absolute value we again call the *spin* of the particle. Examples are the photons ($s = 1$) and the hypothesized graviton ($s = 2$).

In order to give a more exhaustive list of elementary particles, one would have to study the irreducible unitary representations of the groups $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})$, both simple and noncompact Lie groups. The analysis of these representations goes far beyond the aim of this thesis, as they are all infinite dimensional, except from the trivial representation. The classification of these representations was first carried out by Bargmann in 1947 and was the birth of the study of representation theory of noncompact semisimple Lie groups, which is dominated by the work of Harish-Chandra [7].

Chapter 5

Appendix: Weyl's theorem on complete reducibility

This chapter is devoted to prove Weyl's theorem on complete reducibility, Theorem 4.5. The proof we present here is purely algebraic, although the original proof by Weyl was of a more analytic nature. For this proof we mainly follow [21].

Theorem 5.1 (Weyl's Theorem on complete reducibility)**.** *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and let $\rho : \mathfrak{g} \to End(V)$ be a finite dimensional representation of $\mathfrak{g}$. Then ρ is completely reducible.*

Our first aim in the proof of Theorem 5.1 is to define the *Casimir element* of the representation. To do this, we need the following lemma.

Lemma 5.2. *Let V be a finite dimensional vector space and let $\beta : V \times V \to \mathbb{R}$ be a nondegenerate symmetric bilinear form. If $\{X_1, ..., X_n\}$ is a basis of V, there exists a basis $\{X^1, ..., X^n\}$ of V such that $\beta(X_i, X^j) = \delta_{ij}$.*

Proof. The map $\Phi : v \mapsto \beta(\cdot, v)$ is an isomorphism $V \to V^*$. If $\{X_1, ..., X_n\}$ is a basis of V, there exists a dual basis $\{X_1^*, ..., X_n^*\}$ of V^* such that $X_i^*(X_j) = \delta_{ij}$. Define $X^i = \Phi^{-1}(X_i^*)$. Then $\{X^1, ..., X^n\}$ is a basis of V since Φ is an isomorphism and we have that

$$\beta(X_i, X^j) = \Phi(X^j)(X_i) = \Phi(\Phi^{-1}(X_j^*))(X_i) = X_j^*(X_i) = \delta_{ij},$$

so $\{X^1, ..., X^n\}$ is the required basis. $\square$

If $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a finite dimensional injective representation of a semisimple Lie algebra $\mathfrak{g}$, the bilinear form $B_\rho : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ given by

$$B_\rho(X, Y) = \mathrm{Tr}(\rho(X) \circ \rho(Y))$$

is nondegerenate. Remark that if ρ is the adjoint representation, B_ρ is nothing more than the Killing form of the Lie algebra. With this form B_ρ, we can define the Casimir element of the representation.

Definition 5.3 (Casimir element). *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and let $\rho : \mathfrak{g} \to End(V)$ be a finite dimensional injective Lie algebra representation. If $\{X_1, ..., X_n\}$ is a basis of $\mathfrak{g}$ and if $\{X^1, ..., X^n\}$ is the dual basis with respect to B_ρ, we define the Casimir element C_ρ of the representation by*

$$C_\rho = \sum_{i=1}^{n} \rho(X_i)\rho(X^i).$$

First, we show that the Casimir element C_ρ is independent of the choice of basis. If $\{Z_1, ..., Z_n\}$ is another basis of $\mathfrak{g}$ with dual basis $\{Z^1, ..., Z^n\}$ with respect to B_ρ, we can expres $Z_i = \sum_{j=1}^{n} a_{ij} X_j$ and $Z^k = \sum_{l=1}^{n} b_{kl} X^l$. If we set A to be the matrix with coefficients a_{ij} and B the matrix with coefficients b_{kl}, we see that

$$\delta_{ik} = B_\rho(Z_i, Z^k) = \sum_{j=1}^{n}\sum_{l=1}^{n} b_{kl} a_{ij} B_\rho(X_j, X^l) = \sum_{j=1}^{n}\sum_{l=1}^{n} a_{ij} b_{kl} \delta_{jl} = \sum_{j=1}^{n} a_{ij} b_{kj},$$

so $B^T \cdot A = \mathrm{Id}$. But this means that

$$\sum_{i=1}^{n} \rho(Z_i)\rho(Z^i) = \sum_{j,l=1}^{n}\sum_{i=1}^{n} a_{ij} b_{il} \rho(X_j)\rho(X^l) = \sum_{j,l=1}^{n} \delta_{jl} \rho(X_j)\rho(X^l) = \sum_{j=1}^{n} \rho(X_j)\rho(X^j),$$

so the Casimir element is independent on the choice of basis. The Casimir element has several nice properties. One of them is that it commutes with the $\mathfrak{g}$-action.

Lemma 5.4. *Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra and let $\rho : \mathfrak{g} \to End(V)$ be a finite dimensional injective representation. The Casimir element C_ρ commutes with $\rho(X)$ for all $X \in \mathfrak{g}$.*

Proof. Let $\{X_1, ..., X_n\}$ be a basis of $\mathfrak{g}$ and let $\{X^1, ..., X^n\}$ be the dual basis with respect to B_ρ. For $X \in \mathfrak{g}$, we can write for all $1 \leq i \leq n$ that

$$[X, X_i] = \sum_{j=1}^{n} a_{ij} X_j,$$
$$[X, X^i] = \sum_{j=1}^{n} b_{ij} X^j.$$

Then we see that a_{ik} and b_{ik} for all i and k are related by

$$a_{ik} = \sum_{j=1}^{n} a_{ij}\delta_{jk} = \sum_{j=1}^{n} a_{ij} B_\rho(X_j, X^k) = B_\rho(\sum_{j=1}^{n} a_{ij} X_j, X^k) = B_\rho([X, X_i], X^k) =$$
$$- B_\rho(X_i, [X, X^k]) = -B_\rho(X_i, \sum_{j=1}^{n} b_{kj} X^j) = -\sum_{j=1}^{n} b_{kj}\delta_{ij} = -b_{ki}.$$

Then we see that

$$\begin{aligned}
[\rho(X), C_\rho] &= \sum_{i=1}^{n} [\rho(X), \rho(X_i)\rho(X^i)] = \sum_{i=1}^{n} \big([\rho(X), \rho(X_i)]\rho(X^i) + \rho(X_i)[\rho(X), \rho(X^i)] \big) \\
&= \sum_{i=1}^{n} \big(\rho([X, X_i])\rho(X^i) + \rho(X_i)\rho([X, X^i]) \big) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\rho(X_j)\rho(X^i) + b_{ij}\rho(X_i)\rho(X^j) = 0.
\end{aligned}$$

□

It follows by Schur's lemma that if the representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is injective and irreducible, $C_\rho = C \cdot \mathrm{Id}$ for some $C \in \mathbb{R}$. On the one hand, this means that $\mathrm{Tr}(C_\rho) = C \cdot \dim(V)$. On the other hand, we see that

$$\mathrm{Tr}(C_\rho) = \sum_{i=1}^{n} \mathrm{Tr}(\rho(X_i)\rho(X^i)) = \sum_{i=1}^{n} B_\rho(X_i, X^i) = \sum_{i=1}^{n} \delta_{ii} = n = \dim(\mathfrak{g}).$$

We conclude that $C_\rho = \dfrac{\dim(\mathfrak{g})}{\dim(V)}\mathrm{Id}$.

With these properties of the Casimir element, we can present a proof of Weyl's theorem on complete irreducibility.

Proof. Remark that we can assume without loss of generality that the representation ρ is injective (if it is not, we can quotient out its kernel without losing semisimplicity and (ir)reducibility.) It suffices to show that for each $\mathfrak{g}$-submodule W of V, there exists a $\mathfrak{g}$-submodule X of V such that $V = X \oplus W$. First, we deal with the case that W is of codimension 1.

Here, we have two consider two subcases. First, we assume that W is irreducible. Since the representation ρ is injective, the Casimir element C_ρ acts on W by scalar multiplication. It follows directly that $W \cap \ker(C_\rho) = \{0\}$. Since W is a $\mathfrak{g}$-submodule, V/W is a one-dimensional $\mathfrak{g}$-module, hence C_ρ acts trivial on V/W since $\mathfrak{g}$ is semisimple (as $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$). We infer that $V/W \subseteq \ker(C_\rho)$, hence $\dim(\ker(\rho)) \geq 1$. Since $W \cap \ker(C_\rho) = \{0\}$, it follows that $V = W \oplus \ker(C_\rho)$. Since C_ρ commutes with the $\mathfrak{g}$-action, $\ker(C_\rho)$ is a $\mathfrak{g}$-module and we are done.

Now, we assume that W is a reducible $\mathfrak{g}$-submodule of codimension 1. We do induction on $\dim(W)$. Certainly, if $\dim(W) = 0$, the statement holds. Now let $Z \subset W$ be a proper $\mathfrak{g}$-submodule. Then W/Z is a submodule of V/Z with codimension 1. Since $\dim(W/Z) < \dim(W)$, the induction hypothesis is valid for the pair $(W/Z, V/Z)$, so there exists a $\mathfrak{g}$-submodule Y such that $Z \subset Y \subset V$ and

$$V/Z = W/Z \oplus Y/Z. \tag{5.0.1}$$

Since $\dim(Y) - \dim(Z) = 1$ and $\dim(Z) < \dim(W)$, the induction hypothesis is also valid for the pair (Z, Y). Therefore, there exists a $\mathfrak{g}$-submodule $X \subset Y$ such that

$$Y = Z \oplus X. \tag{5.0.2}$$

We will show that $V = W \oplus X$. First, we remark that (5.0.1) and (5.0.2) imply that

$$\dim(V) = \dim(W) + \dim(Y) - \dim(Z) = \dim(W) + \dim(X).$$

Now we are done if we show that $W \cap X = \{0\}$. Since $X \subset Y$, it follows that $W \cap X \subset W \cap Y$. It follows from (5.0.1) that $W \cap Y \subset Z$. It follows from (5.0.2) that

$$W \cap X \subset W \cap Y \subset Z,$$

so

$$W \cap X = (W \cap X) \cap Z = W(X \cap Z) = W \cap \{0\} = \{0\}.$$

We conclude that $W \cap X = \{0\}$ hence

$$V = W \oplus X,$$

and we are done. The result follows by induction.

Now, we assume that W is an arbitrary $\mathfrak{g}$-submodule. We look for a section for the canonical injection $\iota : V \overset{W}{\hookrightarrow}$, i.e. a $\mathfrak{g}$-linear map $f_0 : V \to W$ such that $f_0|_W = \mathrm{Id}_W$, since this would imply that $W \cong V/\ker(f_0)$, hence $V = W \oplus \ker(f_0)$. We would like to reduce this case to the first case, so we introduce the induced $\mathfrak{g}$-module $\mathrm{Hom}(V, W)$ of linear maps $f : V \to W$, where the $\mathfrak{g}$-action is given by

$$(X \cdot f)(v) = X \cdot f(v) - f(X \cdot v).$$

The idea is to consider the $\mathfrak{g}$-submodule

$$\mathcal{V} = \{f \in \mathrm{Hom}(V, W) \colon f|_W = \lambda \mathrm{Id}_W \colon \lambda \in \mathbb{R}\}$$

and its codimension 1 $\mathfrak{g}$-submodule

$$\mathcal{W} = \{f \in \mathrm{Hom}(V, W) \colon f|_W = 0\}.$$

Note that $X \cdot \mathcal{V} \subset \mathcal{W}$, since

$$(X \cdot f)(w) = X \cdot f(w) - f(X \cdot w) = X \cdot (\lambda w) - \lambda X \cdot w = 0.$$

By the previous cases, there exists a one dimensional $\mathfrak{g}$-submodule $\mathcal{U} \subset \mathcal{V}$ such that

$$\mathcal{V} = \mathcal{W} \oplus \mathcal{U}.$$

Since $\mathcal{U}$ is one-dimensional, we can write $\mathcal{U} = \mathbb{C} \cdot f_0$ with $f_0 \in \mathcal{V}$. Without loss of generality, we can assume $f_0|_W = \mathrm{Id}_W$. Since $\mathfrak{g}$ acts trivial on $\mathcal{U}$ as $\mathcal{U}$ is one-dimensional, it follows that $(X \cdot f_0)(v) = f_0(X \cdot v) - X \cdot f_0(v) = 0$, so $f_0(X \cdot v) = X \cdot f_0(v)$. We infer that $\ker(f_0)$ is $\mathfrak{g}$-submodule of V. Since $W = V/\ker(f_0)$ and $\ker(f_0)$ is a $\mathfrak{g}$-module, we have that

$$V = W \oplus \ker(f_0).$$

This proves Weyl's theorem on complete irreducibility. $\square$

Bibliography

[1] Armstrong, M.A. (1988) *Groups and Symmetry*, 1st edition. Springer-Verlag, New York.

[2] Lee, J.M. (2013) *Introduction to Smooth Manifolds*, 2nd edition. Springer-Verlag, New York.

[3] Van den Ban, E.P. (2010) *Lecture Notes on Lie Groups*, Utrecht University. Retrieved from `https://www.staff.science.uu.nl/~ban00101/lecnotes/lie2010.pdf`, (Last accessed on 6-1-2020).

[4] Duistermaat, J.J., Kolk, J.A.C. (2000) *Lie Groups*, 1st edition, Springer-Verlag, Berlin Heidelberg.

[5] Varadarajan, V.S. (1984) *Lie groups, Lie algebras and Their Representations*, 1st edition, Springer-Verlag, New York.

[6] Serre, J.P. (1992) *Lie Groups and Lie Algebras*, 2nd edition, Springer-Verlag, Berlin Heidelberg

[7] Van den Ban, E.P. (2004) *Representation theory and applications in classical quantum mechanics*, Utrecht University. Retrieved from `https://www.staff.science.uu.nl/~ban00101/lecnotes/repq.pdf` , (Last accessed on 6-1-2020).

[8] Hatcher, A.E. (2002) *Algebraic Topology*, Cambridge University Press.

[9] Fulton W., Harris, J. (2004) *Representation Theory: A First Course*, 1st edition, Springer-Verlag, New York.

[10] James, G. and Liebeck, M. (2012) *Representations and Characters of groups*, Cambridge University Press.

[11] Schaefer, H.H. (1999) *Topological Vector Spaces*, 2nd edition, Springer-Verlag, New York.

[12] Rynne, B. and Youngson, M.A. (2008) *Linear Functional Analysis*, 2nd edition, Springer-Verlag London.

[13] Varadarajan, V.S. (1989) *An Introduction to Harmonic Analysis on Semisimple Lie Groups*, Cambridge University Press.

[14] Bourbaki, N. (1989) *Lie Groups and Lie Algebras*, 1st edition, Springer-Verlag Berlin.

[15] Wigner, E.P. (1939) *On Unitary Representations of the Inhomogenous Lorentz Group*, Annals of Mathematics, 40 (1): 149-204.

[16] Montgomery, D. and Zippin, L. (1995) *Topological transformations of groups*, Bulletin of the American Mathematical Society, 48 (6): 448-452.

[17] Serre, J.P. (1977) *Linear representations of finite groups*, Springer-Verlag, New york.

[18] Mackey, G.W. (1955) *The theory of Group Representations*, Department of Mathematics, University of Chicago.

[19] Newton, J. (2016) *Group Representation Master Material: Induced Representations*, Retrieved from `https://nms.kcl.ac.uk/james.newton/M3P12/induced.pdf`, (Last accessed 6-1-2020).

[20] Humphreys, J. (1972) *Introduction to Lie Algebras and Representation Theory*, 1st edition, Springer-Verlag, New York.

[21] Wehler, J. *Lie Algebras*, Retrieved from `http://www.mathematik.uni-muenchen.de/~wehler/SS2018_LieAlgebrasScript.pdf`, (Last accessed 6-1-2020).