SHARING TIME AMONG BRANCHES

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Abstract. In this paper we study Jointed Branching Time Structures equipped with a chrono-function that assigns times to moments. We develop a collection of examples of such structures in a systematic way. We use our systematic collection to provide examples and counter-examples about when a Jointed Branching Time Structure can or cannot be extended with a chrono-function of such-and-such a kind.

Two roads diverged in a yellow wood, And sorry I could not travel both.
Robert Frost

1. Introduction

If I had done this chore yesterday, then I could relax now. This sentence illustrates that we think of times as shared among alternative possible developments.

Branching Time is a persuasive model of alternative possible developments. However, if we want to be able to speak of shared time across branches, we should add some extra structure to the models. The obvious idea to handle shared time is to enrich branching time structures with a chrono-function, to wit a function that maps the moments of the structure to a linear time axis. We will study this idea in the present paper.

We note in passing that adding a chrono-function is a minimal measure to add shared time. We would, perhaps, like to think of shared times as *clock times*. However, that idea suggests that time-distances between moments can be compared and, thus, the presence of metric structure.

What properties should a chrono-function have? A natural idea is to demand that every history in our branching time structure is mapped isomorphically to our time axis. This gives us the notion of a *time-function*.[1] In this paper we will illustrate that this demand is rather strong. Some structures that would *prima facie* seem to be intended models of Branching Time Theory cannot have a time-function.

To get more focus on what chrono-functions are possible or not, we make the demands on a chrono-function weaker than the full demands on a time-function. We ask that *a chrono-function* is surjective and monotonic and sends the linear past of the current moment isomorphically to the past of the time of the current moment. In technical terms, we ask that a chrono-function is a surjective p-morphism w.r.t. the relation $>$.

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[1]The notion of *time-function* is introduced in Antje Rumberg's Thesis. The presence of a time-function is equivalent to the presence of instants in the sense of [BPX01, 7A.5].

We may add the further demand that, for any future time of the shared time of the current moment, there is at least one future moment that has that future time. A chrono-function satisfying this extra demand is an *optimistic chrono-function*. In technical terms: we demand that an optimistic chrono-function is a p-morphism both with respect to $<$ and to $>$.

We will also study one intermediate notion between *optimistic chrono-function* and *time-function*: a *strong chrono-function* is like a time-function for a set of histories that covers our given branching time structure but not necessarily for all histories. A set of histories that covers a branching time structure is called a *bundle*.

In this paper, we will do two things. We will carefully introduce all relevant notions and we will prove a number of basic theorems about these notions. Secondly, we develop a large collection of branching time structures that serve as examples. Strange as it may seem, the literature does not provide any carefully worked-out examples of this kind. Even some structures that I would consider as 'intended models' —like the structure $\mathsf{B}_{2^{\aleph_0},\omega}(\mathbb{R})$, that we will introduce in Section 3— have not been described with care. We will use these structures as examples of structures that do or do not have a time-function and as examples of structures that do or do not have characterizations of certain simple kinds. I hope, of course, that the store of carefully verified examples will have further use in future research beyond the limited aims of this paper.

The class of examples we develop has some universal meaning since every jointed branching time structure that carries a chrono-function can be embedded (in an appropriate sense) in one of our examples. We hope to work out this idea in a subsequent paper.

2. Branching Time

In this section, we introduce a number the basic notions surrounding branching time and prove some preliminary results.

2.1. **Jointed Branching Time Structures.**

In our paper our main emphasis is on *Jointed Branching Time Structures* or JBTS's. We first explain the notion of *Branching Time Structure*.

A *BTS* or *Branching Time Structure* $\mathcal{M}$ is a strict partial ordering $\langle M, < \rangle$. We call M the set of *moments* of the structure. We demand that $\mathcal{M}$ satisfies the following properties.

- $\mathsf{BT1}$ For all m, n and p, if $m < p$ and $n < p$, then $m \leqslant n$ or $n < m$. (Tree-likeness)
- $\mathsf{BT2}$ All pairs of moments have a lower bound. In other words, for all m and n, there is a p, such that $p \leqslant m$ and $p \leqslant n$. (Connectedness)
- $\mathsf{BT3}$ For all m there is an $n > m$. (Seriality)

There are alternative representations of BTS's, for example as Kamp frames.[2] See e.g. [Rey02].

A *history* h in a BTS $\mathcal{M}$ is a maximal linear subset of M. We will generally assume the Axiom of Choice: so every linear subset of M can be extended to a history. In particular, every pair of comparable points lies on a history. In a context where $\mathcal{M}$ is countable, the desired applications of Choice follow from countability.

[2] Strictly speaking Kamp frames correspond to bundled BTS's. The fact that every BTS can be extended to a bundled BTS uses the Axiom of Choice.

Open Question 2.1. It would be interesting to see what happens to Branching Time in a context without Choice. E.g., how would we think about models with no or very few histories? ❏

In this paper we will consider JBTS's that is BTS's with the stronger property of *jointedness*.

J All pairs of moments have a greatest lower bound. In other words, for all m and n, there is a p, such that $p \leqslant m$ and $p \leqslant n$ and, for all s with $s \leqslant m$ and $s \leqslant n$, we have $s \leqslant p$. (Jointedness)

In other words: a JBTS is a tree-like serial meet-semilattice (or: tree-like serial lower semilattice). Our restriction to JTBS's does not diminish the relevance of the paper to the study of BTS's in general, since our main aim is to build and study a store of examples. Examples are preserved from the more restricted to the wider class.

Remark 2.2. There is a metaphysical argument that says that, even if we grant that things may happen in deterministic stretches of branching time, we still have to admit that a branching counts as a happening and should be reflected by the presence of a moment —since all happenings are supposed to be at moments.

There is also a technical argument that says that we can always extend a BTS to a JBTS in a 'light' way by a construction that adds precisely the missing branching points and that is first-order. See Appendix A. ❏

From this point on the default structures that we are considering will be JBTS's.

Consider two histories h and h' in a JBTS. If h is not h', then the intersection of h and h' is linear, by BT1, and has a maximal element, by J. We call this maximal element *the branching point of* h *and* h'.

We define an equivalence relation $\sim_m$ on the $n > m$ as follows: $n \sim_m n'$ iff, for some $p > m$, we have $p \leqslant n$ and $p \leqslant n'$. We call the pair $\langle m, [n]_{\sim_m}\rangle$ a *direction*.

We define an equivalence relation on histories through m as follows: $h \equiv_m h'$ if, for some $n > m$, $n \in h \cap h'$. We will call the pair $\langle m, [h]_{\equiv_m}\rangle$ a *transition*.

Consider any $n > m$ and $n' > m$. Suppose $n \in h$ and $n' \in h'$. It is easy to see that $n \sim_m n'$ iff $h \equiv_m h'$. This means that, assuming the Axiom of Choice, there is a meaningful bijection between directions and transitions, where we map $[n]_{\sim_m}$ to $[h]_{\equiv_m}$, where h is any history containing n, and where we map any $h_{\equiv_m}$ to $[n]_{\sim_m}$, where n is any element in h with $n > m$.

By the above observation, we may conveniently confuse the direction $\langle m, [n]_{\sim_m}\rangle$. with the transition $\langle m, [h]_{\equiv_m}\rangle$, where $m < n \in h$. We note that the notion of direction has the advantage of also being meaningful in the Choice-free setting.

It is convenient to have the following operations on JBTS's available.

- $\mathcal{M}_0 \ast \mathcal{M}_1$ is the result of adding a copy of $\mathcal{M}_1$ above each history of $\mathcal{M}_0$. Formally the moments of $\mathcal{M}_0 \ast \mathcal{M}_1$ are of the form $\langle 0, \varnothing, m_0\rangle$ or $\langle 1, h, m_1\rangle$, where $m_i \in M_i$ and where h is a history of $\mathcal{M}_0$.

  The order on $\mathcal{M}_0 \ast \mathcal{M}_1$ is given by $\langle i, g, m\rangle < \langle i', g', m'\rangle$ iff ($i = i'$, $g = g'$ and $m <_{\mathcal{M}_i} m'$) or ($i = 0$ and $i' = 1$ and $m \in g'$).

  It is easy to see that $\mathcal{M}_0 \ast \mathcal{M}_1$ is indeed a JBTS.

- $\mathcal{M}_0 \veebar \mathcal{M}_1$ is the result of adding a bottom below the disjoint union of $\mathcal{M}_0$ and $\mathcal{M}_1$. Formally the elements of $\mathcal{M}_0 \veebar \mathcal{M}_1$ are of the form $\langle 0, 0\rangle$ or $\langle i+1, m_i\rangle$, where $m_i \in M_i$.

  The ordering on $\mathcal{M}_0 \veebar \mathcal{M}_1$ is given by $\langle i, m\rangle < \langle i', m'\rangle$ if ($i = 0$ and $i' \neq 0$) or ($i = i' \neq 0$ and $m <_{\mathcal{M}_{i-1}} m'$).

  It is easy to see that $\mathcal{M}_0 \veebar \mathcal{M}_1$ is indeed a JBTS.

2.2. **Chrono-Functions.** Let $\mathcal{M}$ be a JBTS and let $\mathcal{T}$ be a serial strict linear order.

Let T be the domain of $\mathcal{T}$. A function $\mathfrak{t} : M \to T$ is *a chrono-function* if it is a surjective p-morphism with respect to $>$. Thus, $\mathfrak{t}$ is strictly monotonic (forward property for $>$). Moreover, if $\mathfrak{t}(m) = t$ and $t' < t$, then, for some $m' < m$, we have $\mathfrak{t}(m') = t'$ (backward property for $>$). We will also write $\mathfrak{t} : \mathcal{M} \to \mathcal{T}$ in case $\mathfrak{t}$ is a chrono-function.

The following small insight will be used without mention in the rest of the paper.

Lemma 2.3. *Suppose* $\mathfrak{t} : \mathcal{M} \to \mathcal{T}$ *is a chrono-function. Then:*

a. The linear order on $\{n \mid n \leqslant m\}$ *is isomorphically mapped by* $\mathfrak{t}$ *to the linear order on* $\{t \mid t \leqslant \mathfrak{t}(m)\}$.

b. The $\mathfrak{t}$*-image* T_0 *of a history* h *is a serial initial segment of* $\mathcal{T}$. *The function* $\mathfrak{t} \restriction h$ *is an isomorphism between* (*the induced orders on*) h *and* T_0.

We omit the proof.

Example 2.4. It is easy to see that the following two JBTS's do not have a chrono-function: $\omega \veebar \mathbb{Q}$ and $\mathbb{Q} \ast (\omega \veebar (\mathbb{Q} \ast \omega))$. The second example has the nice property that all histories are isomorphic.

In Example 4.8, we will provide a JBTS without chrono-function in which all histories h, h' are isomorphic via an isomorphism f such that f is the identity on $h \cap h'$.

We note that our first example also illustrates that the property of having a chrono-function is not closed under $\veebar$. In Example 4.8 we will see that the property of having a chrono-function is not closed under $\ast$ either. ❏

It is sometimes useful to make the chrono-function part of the structure rather than something externally added. Thus, the following definition. A *chrono-structure* $\mathcal{C}$ is a tupel $\langle \mathcal{M}, \mathcal{T}, \mathfrak{t}\rangle$, where $\mathfrak{t} : \mathcal{M} \to \mathcal{T}$ is a chrono-function. Two chrono-structures $\mathcal{C}$ and $\mathcal{C}'$ are *isomorphic* via $\langle \phi, \psi\rangle$ if (i) ϕ is an isomorphism between $\mathcal{M}$ and $\mathcal{M}'$, (ii) ψ is an isomorphism between $\mathcal{T}$ and $\mathcal{T}'$, and (iii) $\mathfrak{t}'(\phi(m)) = \psi(\mathfrak{t}(m))$, for all $m \in M$.

A function $\mathfrak{t} : M \to T$ is *an optimistic chrono-function* if it is a surjective p-morphism both with respect to $<$ and to $>$. Thus, $\mathfrak{t}$ is surjective and strictly monotonic (forward property for $<$ and/or $>$). Moreover, we have:

- If $\mathfrak{t}(m) = t$ and $t < t'$, then, for some $m' > m$, we have $\mathfrak{t}(m') = t'$ (backward property for $<$)
- If $\mathfrak{t}(m) = t$ and $t' < t$, then, for some $m' < m$, we have $\mathfrak{t}(m') = t'$ (backward property for $>$)

We call a chrono-structure *optimistic* if its resident chrono-function is optimistic.

Example 2.5. The JBTS $\omega \veebar (\omega \times 2)$ has a chrono-function to $\omega \times 2$, but does not have an optimistic chrono-function. ❑

There may be many different optimistic chrono-functions on a given JBTS, as we will illustrate. They can even involve non-isomorphic time structures. See Example 4.9.

A set of histories H such that $\bigcup H = M$ is called a *bundle*. A function $\mathfrak{t} : M \to T$ is a *strong chrono-function* if there is a bundle H such that, for every $h \in H$, we have $\mathfrak{t} \upharpoonright h$ is an isomorphism between h and $\mathcal{T}$.[3] It is easy to see that every strong chrono-function is an optimistic chrono-function. If H is a bundle for $\mathcal{M}$, the pair $\langle \mathcal{M}, H \rangle$ is called a *bundled JBTS*. (The notion of bundled BTS was first introduced in [Bur79].)

A structure $\langle \mathcal{M}, H, \mathcal{T}, \mathfrak{t} \rangle$ is called a *bundled chrono-structure* if H is a bundle for $\mathcal{M}$, and $\mathfrak{t} : \mathcal{M} \to \mathcal{T}$ is a strong chrono-function witnessed by H.

In Appendix B, we will give an alternative representation, the Rumberg structures, of bundled chrono-structures, i.e. chrono-structures where the built-in chrono-function is strong.

We give a sufficient condition, for when an optimistic chrono-function is also a strong chrono-function.

Theorem 2.6. *Let $\mathfrak{t} : \mathcal{M} \to \mathcal{T}$ be an optimistic chrono-function. Suppose T, the domain of $\mathcal{T}$, contains a countable cofinal subset T_0. Then, $\mathfrak{t}$ is a strong chrono-function.*

Proof. We assume the conditions of the theorem. Consider an enumeration $t_0, t_1, \ldots$ of T_0. We construct a sequence $t_0^* < t_1^* < \ldots$ of elements of T_0 that is cofinal in T. We take $t_i^* := t_{j_i}$, where the j_i are specified as follows: $j_0 := 0$, and j_{i+1} is the smallest index j, such that $t_j > t_i^*$. Clearly $j_i < j_{i+1}$ and $t_i^* < t_{i+1}^*$.

Suppose $(t_i^*)_{i \in \omega}$ were not cofinal. Then, there would be an upper bound t of $(t_i^*)_{i \in \omega}$. Since, T_0 is cofinal, we can find such a t in T_0. Suppose $t = t_{i*}$. For some i, we have $j_{i+1} > i^*$. But, j_{i+1} is the smallest index j such that $t_j > t_{j_i}^*$. Moreover, $t_{i*} > t_{j_i}^*$. A contradiction.

Let m be any moment and consider $t_i^* > \mathfrak{t}(m)$. By the backward property for $<$, we find and $m_0 > m$ with $\mathfrak{t}(m_0) = t_i^*$. Repeating the use of the backward property for $<$, we construct a sequence $m_0 < m_1 < \ldots$ such that $\mathfrak{t}(m_j) = t_{i+j}^*$. By the monotonicity of $\mathfrak{t}$, the m_j cannot have an upper bound. We consider the set $h := \{n \mid \exists i \in \omega \; n < m_i\}$. Since h does not have an upper bound, h is a history. By the backward property for $>$, we find that the image of h is all of T.

Since, by the above reasoning, through every m we have a h that maps onto T, we can take as our bundle H the set of all h that map onto T. ❑

Example 4.9 provides a JBTS with two optimistic chrono-functions to non-isomorphic time structures of which one is not strong and the other is strong. Example 4.10 provides a JBTS with an optimistic chrono-function that does not admit a strong chrono-function.

[3] It is often convenient to confuse h *qua* set of moments with the ordering $\langle h, < \upharpoonright h \rangle$. We will make this our praxis.

Open Question 2.7. Is there a JBTS that has two strong chrono-functions to non-isomorphic linear time structures? ❑

A function $\mathsf{t} : M \to T$ is a *full* or *uniform* chrono-function or a *time-function* iff, for every history h, we have $\mathsf{t} \restriction h$ is an isomorphism between h and $\mathcal{T}$. A JBTS $\mathcal{M}$ has *uniform time* iff there is a uniform chrono-function on $\mathcal{M}$.

We will provide examples of JBTS's that have a strong chrono-function but not a time-function. See Example 4.7.

A JBTS $\mathcal{M}$ has *the iso-property* iff, any two histories h and h' are isomorphic via an isomorphism ϕ such that $\phi \restriction h \cap h' = \mathsf{id}_{h\cap h'}$. One might suspect that the iso-property implies the existence of a time-function. In Example 4.8, we show that there are JBTS's with the iso-property that do not even have a chrono-function.

The problem is the lack of coordination of the isomorphisms provided by the iso-property. If we add coordination, we do get the desired connection between the existence of such isomorphisms and the presence of a time-function. Let's say that $\mathcal{M}$ has the *strong iso-property* iff we have for each pair h, h' of histories an isomorphism $\phi_{hh'}$ that is the identity on the intersection of h and h', where the $\phi_{hh'}$ satisfy $\phi_{h''h''} \circ \phi_{hh'} = \phi_{hh''}$.

We note that it follows that we have a category with as objects the histories and with as single morphism from h to h' the function $\phi_{hh'} : h \to h'$.

Theorem 2.8. *The strong iso-property is equivalent to possession of a time-function.*

Proof. Consider $\mathcal{M}$ with the strong iso-property. We take any history h with its ordering as $\mathcal{T}$. We define $\mathsf{t} := \bigcup\{\phi_{h'h} \mid h' \text{ is a history of } \mathcal{M}\}$. Suppose m is both in h' and h'', then $\phi_{h'h''}(m) = m$, and, thus, $\phi_{h'h}(m) = \phi_{h''h}\phi_{h'h''}(m) = \phi_{h''h}(m)$. Thus, t is indeed a function.

Conversely suppose we have a time-function t. We define, for $m \in h$, $\phi_{hh'}(m) := (\mathsf{t} \restriction h')^{-1}\mathsf{t}(m)$. ❑

3. The Structures $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$

In this section, we describe the construction of a number of concrete JBTS's, to wit the structures $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$. Here $\kappa > 0$ is the cardinality of the transitions in each point and λ is a limit ordinal that gives us the length of a number of 'standard' histories. The structure $\mathcal{T}$ is a serial linear ordering. We demand that each $\alpha < \lambda$ can be embedded, qua order structure, in $\mathcal{T}$. The idea of the construction is as follows. The standard histories all will have a shared time $\mathcal{T}$. We call the elements t of $\mathcal{T}$ *times*. We generate the structure by starting with a history of order type $\mathcal{T}$. At each moment we add $\kappa - 1$ branches of histories of order type $\mathcal{T}$. We thereby generate a number of new moments. At each new moment we again add $\kappa - 1$ branches. We repeat the procedure over the ordinals. At limit stages we note that apart from the histories we explicitly created there are new histories. These new histories are mapped onto downwards closed and serial initial segments of $\mathcal{T}$. In case these histories are not already mapped to the whole $\mathcal{T}$, we add moments to complete their order type to $\mathcal{T}$. This creates new moments and we can add $\kappa - 1$ branches to these new moments again. We repeat the procedure λ times.

At the limit λ, new histories may come into existence that will not be completed. We call the histories that are explicitly created or completed *straight histories* and the histories that come into existence at the end *zigzag histories*.

We present one roads to implementation here. A second one using Rumberg structures is given in Appendix C.

Let $\kappa > 0$ and let λ be a limit ordinal. Let $\mathcal{T}$ be a serial linear order. We assume that each $\alpha < \lambda$ can be embedded in $\mathcal{T}$.[4]

In case κ is 1, we simply take $\mathsf{B}_{1,\lambda}(\mathcal{T})$ to be $\mathcal{T}$. Otherwise, we follow the construction below. So assume $\kappa > 1$. We define the structure $\mathcal{M}^\star := \mathsf{B}_{\kappa,\lambda}(\mathcal{T})$ as follows.

- The elements m of $M^\star$ are sequences of the form $t_0\alpha_0t_1\alpha_1\ldots t_\omega\alpha_\omega\ldots t_\gamma$, where the t_β are in $\mathcal{T}$ and the α_β are ordinals $< \kappa - 1$ and where $\gamma < \lambda$. We demand that $t_\beta < t_{\beta'}$ if $\beta < \beta'$.

  More formally, the m are functions on ordinals of the form $2i + 1$ or $\lambda' + 2i + 1$, where i is a natural number and $\lambda' < \lambda$ is a limit ordinal. These functions map elements of the form $2j$ or $\lambda'' + 2j$, where j is a natural number and λ'' is a limit $\leqslant \lambda'$ to times in such a way that the mapping is strictly monotonic. They map elements of the form $2j + 1$ or $\lambda'' + 2j + 1$ to ordinals $< \kappa - 1$.

  The times in our sequences, except the last one at γ, indicate subsequent branching points. The ordinals provide a label for the specific branch we are creating.[5] In other words, the ordinals provide the numerosity of the branching. Finally, t_γ tells us where we are after the last branching point.

  In case $\gamma = 0$, the sequences are simply the unit sequences t_0. These are notationally confused with the elements of $\mathcal{T}$. We write $|m| := \gamma$.
- $m \leqslant n$ iff $m = \mu t$ and $n = \mu t'\nu$ and $t \leqslant t'$.
  Here μ and ν are, possibly empty, sequences and t and t' are times.

We note that $m \leqslant n$ implies that $|m| \leqslant |n|$.

We will call sequences μ such that $\mu t \in M^\star$, for some large enough t, *worlds*. We will sometimes use w, w' to range over worlds. We will set $|w| := |wt|$.

Here is the heuristics that links our formal realisation to the intuitive story of our construction. The level β histories are given by worlds w with $|w| = \beta$. The elements of the history corresponding to w are the m of the form wt plus all elements $m' < wt$ for $wt \in M^\star$.

In case w is of the form $w't\alpha$, $w't$ gives the branching point and α tells us which of the many newly created branches we are looking at.

Theorem 3.1. *$\mathcal{M}^\star$ is a partial order.*

Proof. Suppose $m \leqslant n \leqslant p$. Suppose $m = \mu t$ and $n = \mu t'\nu$, where $t \leqslant t'$. In case ν is empty, we may suppose $p = \mu t''\pi$ and $t' \leqslant t''$. It follows that $t \leqslant t''$ and we are done. In case ν is not empty, it follows that $p = \mu t'\pi$ and we are done again.

Clearly $m \leqslant m$.

Suppose $m \leqslant n$ and $n \leqslant m$. In this case $|m| = |n|$. It follows that $m = \pi t$, $n = \pi t'$ and $t \leqslant t'$ and $t' \leqslant t$. Hence $t = t'$ and $m = n$. ❑

Theorem 3.2. *$\mathcal{M}^\star$ is a JBTS.*

[4]This demand could also be omitted. It's just that ordinals that cannot be embedded can play no role in the construction.

[5]We will speak as if the times and the ordinals are disjoint. It is clear that, using more words, this pretence can be eliminated.

Proof. We verify $\mathsf{BT1}$, to wit that our structure is tree-like. Suppose $m \leqslant p$ and $n \leqslant p$. Say $m = \mu t$ and $p = \mu t'\pi$ and $t \leqslant t'$ and, in addition, $n = \nu t''$ and $p = \nu t'''\pi'$ and $t'' \leqslant t'''$. Suppose $|\mu| = |\nu|$. In this case $\mu = \nu$, since both are initial subsequences of p. It follows that $m = \mu t$ and $n = \mu t''$. Since $t \leqslant t''$ or $t'' \leqslant t$, it follows that $m \leqslant n$ or $n \leqslant m$. Suppose that $|\nu| > |\mu|$. It follows that $\nu = \mu t'\nu'$ and, hence that $m < n$. Similarly, if $|\mu| > |\nu|$.

We verify $\mathsf{BT2}$, to wit that each two moments have an infimum. Consider any two moments m and n. Let w be the largest initial world that m and n have in common. (Note that w could be empty.)

We have: $m = wt\mu$ and $n = wt'\nu$. Let $u := \mathsf{min}(t, t')$. We claim that $p := wu$ is the desired infimum. Clearly, $p \leqslant m$ and $p \leqslant n$.

Suppose $r \leqslant m$ and $r \leqslant n$. Then, $r = w^*t^*$ and $m = w^*t^\circ\mu^\circ$ and $t^* \leqslant t^\circ$ and $n = w^*t^\diamond\nu^\diamond$ and $t^* \leqslant t^\diamond$. We find that w^* is an initial world both of m and of n. So, w^* must be an initial subworld of w.

Suppose $w^* = w$. In that case, $r = wt^*$, $m = wt^\circ\mu^\circ = wt\mu$, $n = wt^\diamond\nu^\diamond = wt'\nu$. It follows that $t^* \leqslant t^\circ = t$ and $t^* \leqslant t^\diamond = t'$. Ergo $t^* \leqslant u$ and $r = wt^* \leqslant wu = p$.

Next suppose that w^* is a strictly initial subworld of w, say $w = w^*\hat{t}\rho$. It follows that $m = w^*t^\circ\mu^\circ = w^*\hat{t}\rho t\mu$. So, $t^* \leqslant t^\circ = \hat{t}$. Thus, $r = w^*t^* < w^*\hat{t}\rho u = wu = p$.

We verify $\mathsf{BT3}$, to wit that our structure is serial. Consider $m = wt$. Let $t' > t$. Clearly, $m' := wt' > wt = m$. ❑

Our next order of business is to study the histories of $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$. We first study the histories h_w. Suppose w is a world. Remember that this means that, for some t, wt is in $M^\star$. We define: $h_w = \{m \mid \exists t\, (wt \in M^\star \text{ and } m \leqslant wt)\}$.

Theorem 3.3. *h_w is a history.*

Proof. Consider any $m, n \in h_w$. Suppose $m \leqslant wt$ and $n \leqslant wt'$. Let $t^* := \mathsf{max}(t, t')$. We have: $m \leqslant wt^*$ and $n \leqslant wt^*$. By tree-likeness, we find that m and n are comparable. So h_w is linear.

Next we show that h_w is a maximal linear suborder. Consider any p that is comparable with all $m \in h_w$. If $p \leqslant m$, for some $m \in h_w$, we are immediately done, since, obviously, h_w is downward closed. Suppose $p \geqslant m$, for $m = wt \in h_w$. Thus, $p = wt'\mu$, where $t' \geqslant t$. If μ is empty, $p \in h_w$ and we are done. If μ is not empty, consider any $t^\circ > t'$. We find that wt° is incomparable with p, contradicting our assumption that p is comparable with all elements of h_w. ❑

Theorem 3.4. *If $h_w = h_{w'}$, then $w = w'$.*

Proof. Suppose $h_w = h_{w'}$. Suppose $wt \in M^\star$ and $w't' \in M^\star$. Since both are in h_w, they are comparable. Suppose e.g. $wt \leqslant w't'$. Then $w't' = wt^*\rho$, where $t \leqslant t^*$. If ρ is empty, we have $w = w'$ and we are done. Suppose ρ is not empty. Let $t^\circ > t^*$. We find that wt° is incomparable to $w't'$. Moreover $wt^\circ \in h_w$. A contradiction. The case that $w't' \leqslant wt$ is similar. ❑

A *superworld* Ω is a sequence $t_0\alpha_0 \dots t_\beta\alpha_\beta \dots$ of length $\lambda' \leqslant \lambda$. As before we demand the t_β to be strictly ascending and the α_β to be $< \kappa - 1$. We ask that either the length of Ω is λ or that the times in Ω are cofinal in $\mathcal{T}$. In case the length of Ω is λ we say that Ω is an *orderly* superworld. In case the times of Ω are cofinal in $\mathcal{T}$ we speak of a *timely* superworld. We note that superworlds can be both orderly and timely. We write $|\Omega|$ for the length of Ω.

We define h_Ω as the set of n such that $n \leqslant m$, for some m that is an initial subsequence of Ω.

Theorem 3.5. *h_Ω is a history.*

Proof. Consider n and n' in h_Ω. We have $n \leqslant m$ and $n' \leqslant m'$ where m and m' are initial in Ω. Clearly, m and m' are comparable. Let m^* be the largest of the two. Then, $n \leqslant m^*$ and $n' \leqslant m^*$. Ergo, by tree-likeness, $n \leqslant n'$ or $n' \leqslant n$.

We prove the maximality of h_Ω. Consider any n not in h_Ω. Suppose, n is comparable with every moment in h_Ω. If $n \leqslant m \in h_\Omega$, we are done, since h_Ω is clearly downwards closed. Suppose all $m \in h_\Omega$ are $< n$. It follows that for every m that is initial in Ω, $m = wt$ and $n = wt\nu$. Hence Ω is itself initial in m. But this is impossible by our assumptions on superworlds. ❑

Theorem 3.6. $h_w \neq h_\Omega$

Proof. Suppose $h_w = h_\Omega$. Consider $wt \in M^\star$. Then, $wt \in h_w$ and, hence in h_Ω. It follows that some $wt'\rho$ is initial in Ω and $t \leqslant t'$. Since the length of Ω must be a limit, it follows that we can assume that ρ is not empty. Let $t^* > t'$. It follows that $wt^* \in h_w$ and $wt'\rho \in h_\Omega$ are incomparable. A contradiction. ❑

Theorem 3.7. *If $h_\Omega = h_{\Omega'}$, then $\Omega = \Omega'$.*

Proof. Suppose $h_\Omega = h_{\Omega'}$. Let m and m' be initial in Ω and Ω' of length β. Since m and m' are in the same history, they must be comparable. Say $m \leqslant m'$. So we have $m = wt$ and $m' = wt'$. So Ω and Ω' have the same intial world of length β. It follows that Ω and Ω' coincide below $\mathsf{min}(|\Omega|, |\Omega'|)$. However, by the definition of a superworld there can be no n extending a superworld and *a fortiori* no superworld can strictly extend another one. So, $\Omega = \Omega'$. ❑

Theorem 3.8. *Every history in $\mathcal{M}^\star$ has the form h_w or h_Ω.*

Proof. Suppose h is a history in $\mathcal{M}^\star$. There are two possibilities: there is an m of greatest length in h or there is not.

We consider the first case. Suppose wt is a moment of greatest length in h. We prove that $h_w \subseteq h$. Since h_w is a history, it follows that $h_w = h$. It is sufficient to show that wt' is in h for any t' such that $wt' \in M^\star$.

Consider any t' such that $wt' \in M^\star$ and any $m \in h$. If $|m| = |wt|$, then $m = wt''$, since m is comparable to wt. Hence, m is also comparable with wt'. If $|m| < |wt|$ then we must have $m < wt$ and hence $m = w't^*$ and $wt = w't^\circ\rho t$, and $t^* \leqslant t^\circ$. It follows that $m = w't^* \leqslant w't^\circ\rho t' = wt'$. So, for all m in h, we find that wt' is comparable with m. Ergo, $wt' \in h$.

Suppose the set of lengths of elements of h is serial. Consider wt and $w't'$ in h and suppose $w't'$ is longer than h. It then follows that w is a strict initial subsequence of w'. Let Ω be the minimal sequence that has all w with $wt \in h$ as initial subsequences. It is clear that, for no t, Ωt can be in M^*, since otherwise Ωt would majorise the elements of h, contradicting the maximality of h. So Ω is a superworld. Now consider any $m \in h$. Let m' be a longer element of h. We have $m \leqslant m'$ and, thus, $m = wt$ and $m' = wt'\alpha t''\rho$ in h, where $t \leqslant t'$. Ergo, wt' is initial in Ω. Thus, since $m = wt \leqslant wt'$, we find $m \in h_\Omega$. We may conclude that $h = h_\Omega$. ❑

Finally, we show that every $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$ is equipped with a strong chrono-function.

Theorem 3.9. *Consider $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$. Let $\mathfrak{t} : wt \mapsto t$. Then $\mathfrak{t}$ is a strong chrono-function.*

Proof. It is easy to see that $H := \{h_w \mid h \text{ is a world of } \mathcal{M}^\star\}$ covers $M^\star$ and that h_w is isomorphically mapped by $\mathfrak{t}$ to $\mathcal{T}$. ❑

In Example 4.9, we show that on $\mathsf{B}_{\aleph_1,\omega}(\mathbb{Q})$ there is an optimistic chrono-function to $\mathbb{Q}\times\omega_1$ that is not strong.[6] Thus, the standardly delivered chrono-function needs not be the only possible one on $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$.

Since with $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$ we can associate a standard time, a standard chrono-function and a standard bundle, we have the following obvious further definitions.

- $\mathsf{C}_{\kappa,\lambda}(\mathcal{T})$ is the chrono-structure $\langle \mathsf{B}_{\kappa,\lambda}(\mathcal{T}), \mathcal{T}, \mathfrak{t}\rangle$, where $\mathfrak{t} : wt \mapsto t$.
- $\mathsf{BB}_{\kappa,\lambda}(\mathcal{T})$ is the bundled JBTS $\langle \mathsf{B}_{\kappa,\lambda}(\mathcal{T}), H\rangle$, where $H = \{h_w \mid w \text{ is a world of } \mathsf{B}_{\kappa,\lambda}(\mathcal{T})\}$.
- $\mathsf{BC}_{\kappa,\lambda}(\mathcal{T})$ is the bundled chrono-structure $\langle \mathsf{B}_{\kappa,\lambda}(\mathcal{T}), H, \mathcal{T}, \mathfrak{t}\rangle$, where $H = \{h_w \mid w \text{ is a world of } \mathsf{B}_{\kappa,\lambda}(\mathcal{T})\}$ and $\mathfrak{t} : wt \mapsto t$.

4. Chrono-functions on the $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$

In this section we will study chrono-functions on the $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$.

4.1. **Examples with Time Function.**

Let us first consider examples of structures where the standard strong chrono-function is indeed a time-function.

We define $\|\mathcal{T}\|$ as the supremum of the ordinals that can be embedded in $\mathcal{T}$.[7] We note that $\|\mathcal{T}\|$ must be a limit ordinal $\leqslant \mathsf{card}^+(\mathcal{T})$. In case $\|\mathcal{T}\|$ itself can be embedded in $\mathcal{T}$ the embedding must be cofinal.

Example 4.1. If λ is any limit ordinal, then $\|\lambda\| = \lambda$.

$\|\mathbb{Q}\| = \|\mathbb{R}\| = \|\mathbb{Q}\times\omega_1\| = \|\mathbb{R}\times\omega_1\| = \omega_1$. We note that ω_1 cannot be embedded in $\mathbb{Q}$, but it can be embedded in $\mathbb{Q}\times\omega_1$. ❑

We note that our constraint on λ in the definition of $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$ can now be simply formulated as: $\lambda \leqslant \|\mathcal{T}\|$.

Suppose $\lambda = \|\mathcal{T}\|$. In that case every orderly superworld in $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$, that is every superworld of length λ, is timely. So, it follows that all superworlds are timely. From this, we may conclude that all histories are isomorphically mapped by $\mathfrak{t} : wt \mapsto t$ on $\mathcal{T}$. In other words: $\mathfrak{t}$ is a time-function. We summarize this modest insight in a theorem.

Theorem 4.2. *In $\mathsf{B}_{\kappa,\|\mathcal{T}\|}(\mathcal{T})$ all superworlds are timely and, thus, $\mathfrak{t} : wt \mapsto t$ is a time-function.*

Example 4.3. Consider $\mathsf{B}_{\kappa,\lambda}(\lambda)$. Since $\|\lambda\| = \lambda$, we find that $\mathfrak{t} : w\alpha \mapsto \alpha$ is a time-function.

In this case, we have even more: let $\mathfrak{t}_0 : \mathsf{B}_{\kappa,\lambda}(\lambda) \to \mathcal{T}_0$ be any chrono-function on $\mathsf{B}_{\kappa,\lambda}(\lambda)$. One can easily show that $\phi := \mathfrak{t}_0 \circ \mathfrak{t}^{-1}$ is an isomorphism between λ and

[6] The structure $\mathbb{Q}\times\omega_1$ is the obvious ordering of ω_1 copies of $\mathbb{Q}$.

[7] It is more usual to consider $\mathsf{o}(\mathcal{T})$ which is the smallest ordinal that can not be embedded in $\mathcal{T}$. We note that either $\mathsf{o}(\mathcal{T}) = \|\mathcal{T}\| + 1$ or $\mathsf{o}(\mathcal{T}) = \|\mathcal{T}\|$ depending on the question whether there is a largest ordinal which can be embedded in $\mathcal{T}$.

$\mathcal{T}_0$. We just show that ϕ is functional. Suppose $\mathsf{t}(m) = \mathsf{t}(m') = \alpha$. It follows that both $\{n \mid n < m\}$ and $\{n' \mid n' < m'\}$ are isomorphic to α. Then $\{t \mid t <_0 \mathsf{t}_0(m)\}$ and $\{t' \mid t' <_0 \mathsf{t}_0(m')\}$ are also isomorphic to α. By the rigidity of ordinals, it follows that $\mathsf{t}_0(m) = \mathsf{t}_0(m')$.

In fact, by the above argument, even more generally, any $\mathcal{M}$ that has a chrono-function $\mathsf{t}^* : \mathcal{M} \to \lambda$ has at most one chrono-function (modulo isomorphic copies of λ). We note however that we do not generally have that t^* is a time-function as is witnessed by the standard chrono-function on $\mathsf{B}_{2,\omega}(\omega^2)$: the history given by the orderly superworld 00102030... maps onto ω and not ω^2. ❑

Example 4.4. Consider the structures $\mathcal{M}_0 := \mathsf{B}_{\kappa,\omega_1}(\mathbb{Q})$, $\mathcal{M}_1 := \mathsf{B}_{\kappa,\omega_1}(\mathbb{Q} \times \omega_1)$, $\mathcal{M}_2 := \mathsf{B}_{\kappa,\omega_1}(\mathbb{R})$ and $\mathcal{M}_3 := \mathsf{B}_{\kappa,\omega_1}(\mathbb{R} \times \omega_1)$. Since we have $\|\mathbb{Q}\| = \|\mathbb{Q} \times \omega_1\| = \|\mathbb{R}\| = \|\mathbb{R} \times \omega_1\| = \omega_1$, it follows that the standard chrono-functions on these structures are time-functions.

Note that we do not have orderly superworlds in $\mathcal{M}_0$ and $\mathcal{M}_2$, but that we do have them in $\mathcal{M}_1$ and $\mathcal{M}_3$. ❑

4.2. **Examples without Time-function.** We consider an arbitrary structure $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$.

We start with a useful insight. We remind the reader of the notion of the cofinality of a linear order. We define $\mathsf{cf}(\mathcal{T})$, the cofinality of $\mathcal{T}$, as the smallest ordinal α such that there is a monotonic cofinal mapping from α into $\mathcal{T}$. We note that $\mathsf{cf}(\mathcal{T})$ will always be an initial ordinal, i.e. an ordinal we use to represent a cardinal. Moreover, $\mathsf{cf}(\mathcal{T}) \leqslant \mathsf{card}(\mathcal{T})$.

Suppose $\lambda = \mathsf{cf}(\mathcal{T})$. In that case every timely superworld must be orderly. So every superworld is orderly. We lay down this insight in a modest theorem.

Theorem 4.5. *Every superworld in $\mathsf{B}_{\kappa,\mathsf{cf}(\mathcal{T})}(\mathcal{T})$ is orderly.*

The following theorem gives us information about the possible ranges of time functions on histories.

Theorem 4.6. *Consider $\mathcal{M} := \mathsf{B}_{\kappa,\lambda}(\mathcal{T})$ with $\kappa \geqslant 2$. Let $\widetilde{\mathsf{t}}$ be any optimistic chrono-function from $\mathcal{M}$ to $\widetilde{\mathcal{T}}$. Let $\widetilde{t}_0 < \widetilde{t}_1 \ldots$ be an ascending sequence of $\widetilde{T}$-times of length λ. Let $\widetilde{T}_0 := \{\widetilde{t} \in \widetilde{T} \mid \exists \alpha < \lambda \ \widetilde{t} \leqslant \widetilde{t}_\alpha\}$. Then, there is a history h such that $\widetilde{\mathsf{t}}(h)$ is an initial subset of $\widetilde{T}_0$.*

In case $\lambda = \mathsf{cf}(\mathcal{T})$, then we can always find h such that the image of h is precisely $\widetilde{T}_0$.

Proof. We proceed as follows. We construct a sequence $m_0 < m_1 < \ldots < m_\alpha < \ldots$ such that $\widetilde{\mathsf{t}}(m_\alpha) = \widetilde{t}_\alpha$ and that $|m_0| < |m_1| < \ldots$ The promised history h will be $\{m \in M \mid \exists \alpha \, m \leqslant m_\alpha\}$.

Let $\mathsf{t} : wt \mapsto t$ be the standard chrono-function on $\mathcal{M}$.

- We take m_0 arbitrary such hat $\widetilde{\mathsf{t}}(m_0) = \widetilde{t}_0$. Such an m_0 exists by the surjectivity of $\widetilde{\mathsf{t}}$.
- Suppose we have found m_α. Consider any $t > \mathsf{t}(m_\alpha)$ and let $n := m_\alpha 0 t$.
  In case $\widetilde{\mathsf{t}}(n) \leqslant \widetilde{t}_{\alpha+1}$, there is, by the optimism of $\widetilde{\mathsf{t}}$, an $m \geqslant n$ such that $\widetilde{\mathsf{t}}(m) = \widetilde{t}_{\alpha+1}$. We take $m_{\alpha+1} := m$. We note that $m_{\alpha+1}$ is of the form $m_\alpha 0 p$.

In case $\tilde{\mathfrak{t}}(n) > \tilde{t}_{\alpha+1}$, there is an $m < n$ such that $\tilde{\mathfrak{t}}(m) = \tilde{t}_{\alpha+1}$. We note that m_α and m, being both $< n$, must be comparable. By the monotonicity of $\tilde{\mathfrak{t}}$, it follows that $m_\alpha < m < n$. We take $m_{\alpha+1} := m$. Clearly, $m_{\alpha+1}$ must be of the form $m_\alpha 0 t'$.

- Suppose we have a sequence $m_0, \ldots$ of length λ' for $\lambda' < \lambda$. There are two possibilities: either there is an n above all the m_α in our sequence at this point or there is not.

  In the first case, we have two possibilities. Either (a) $\tilde{\mathfrak{t}}(n) \leqslant \tilde{t}_{\lambda'}$ or (b) $\tilde{\mathfrak{t}}(m) > \tilde{t}_{\lambda'}$. In case (a), we can find an m with $n \leqslant m$ and $\tilde{\mathfrak{t}}(m) = \tilde{t}_{\lambda'}$. We set $m_{\lambda'} := m$. In case (b), there is an $m < n$ with $\tilde{\mathfrak{t}}(m) = \tilde{t}_{\lambda'}$. By the monotonicity of $\tilde{\mathfrak{t}}$, we find that m is bigger than the m_α. So, we can take $m_{\lambda'} := m$.

  In the second case the sequence of m_α gives us a superworld Ω of length λ'. It follows that h_Ω is mapped isomorphically to an initial segment of $\tilde{T}_0$. If this obtains our procedure halts.
- If the second case of the previous item never applies, the m_α deliver an orderly superworld Ω and h_Ω is mapped isomorphically onto $\tilde{T}_0$.

Suppose $\lambda = \mathsf{cf}(\mathcal{T})$ and suppose our procedure would break off at $\lambda' < \lambda$. Consider the superworld Ω given by the m_α, for $\alpha < \lambda'$. In case this superworld would be orderly, the mapping $F : \alpha \mapsto |m_\alpha|$ would be cofinal from λ' to λ. Let G be a cofinal mapping from λ to $\mathcal{T}$. It follows that $G \circ F$ is cofinal from λ' to $\mathcal{T}$. A contradiction. So, Ω has to be essentially timely. But this is again impossible since the $|m_\alpha|$ for $\alpha < \lambda'$ are not cofinal in $\mathcal{T}$. ❑

We can use the above theorem to generate many examples of structures with a strong chrono-function but without time-fuction. We just give two interesting ones.

Example 4.7. Let $\mathcal{M}_0 := \mathsf{B}_{\kappa,\omega}(\mathbb{Q})$ for $\kappa \geqslant 2$ and let $\mathsf{t}^* : \mathcal{M}_0 \to \mathcal{T}^*$ be an optimistic chrono-function.[8] Let T_0^* be any non-cofinal cut in T^*, i.e. (i) T_0^* is downwards closed in T^*, (ii) T_0^* is serial and (iii) for some t^* all elements of T_0^* are below t^*.

Consider any m^* in $\mathcal{M}_0$ such that $\mathsf{t}^*(m^*) = t^*$. Since $\{n \mid n < m^*\}$ is isomorphic to $\mathbb{Q}$, we may conclude that $\{t \mid t < t^*\}$ is isomorphic to $\mathbb{Q}$. Hence, T_0^* is isomorphic to $\mathbb{Q}$. Thus, for some ω-sequence $t_0 < t_1 < \ldots$, we have $T_0^* = \{t \mid \exists i \in \omega \;\; t < t_i\}$. We note that $\mathsf{cf}(\mathbb{Q}) = \omega$. Hence, by Theorem 4.6, we can find a history h in $\mathcal{M}_0$ such that the t^*-image of h is precisely T_0^*.

Inspection of the above argument shows that the same considerations work for $\mathcal{M}_1 := \mathsf{B}_{\kappa,\omega}(\mathbb{R})$, for $\kappa \geqslant 2$.

Since t^* was arbitrary, we may conclude that neither $\mathcal{M}_0$ nor $\mathcal{M}_1$ admit a time-function. ❑

In Appendix D we present a sufficient condition for having no time function.

4.3. **The Iso-Property.** We remind the reader that a JBTS $\mathcal{M}$ has *the iso-property* iff any two histories h and h' are isomorphic via an isomorphism ϕ such that $\phi \restriction h \cap h' = \mathsf{id}_{h \cap h'}$. Clearly, our examples $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$ and $\mathsf{B}_{\kappa,\omega}(\mathbb{R})$, for $\kappa \geqslant 2$

[8]It is not difficult to see that, modulo isomorphism, $\mathcal{T}^*$ is either $\mathbb{Q}$ or $\mathbb{Q} \times \omega_1$.

in Example 4.7, do have the iso-property, but do not have a time-function. In Example 4.8, we will show that we can do better. We provide an example with the iso-property that does not even have a chrono-function.

Example 4.8. We provide an example of a JBTS $\mathcal{M}$ with the iso-property that does not have a chrono-function. $\mathcal{M}$ is obtained by taking $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$, for $\kappa \geqslant 2$ and by attaching sticks of order type ω above each history. Thus, $\mathcal{M} = \mathsf{B}_{\kappa,\omega}(\mathbb{Q}) \ast \omega$.

We give an alternative representation of $\mathcal{M}$. We consider the order $\mathbb{Q} + \omega$. We identify the first component with the rationals and designate the elements of the second component as $\widehat{0}$, $\widehat{1}$, ... We let '$\widehat{n}$' range over the second component. So $q < \widehat{n}$, for all q and $\widehat{n}$. We let '$x, x' \ldots$' range over all the elements of $\mathbb{Q} + \omega$.

The elements of $\mathcal{M}$ can be represented as the elements of $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$ plus elements of the form $w\widehat{n}$ and $\Omega\widehat{n}$, where w is a world of $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$ and Ω is a superworld of $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$. Thus, the worlds W of our new structure are the worlds and the superworlds of $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$. As usual, we define $m \leqslant m'$ iff $m = Wx$ and $n = Wx'\nu$ and $x \leqslant x'$.

We can think of the $\mathbb{Q}$-times as earthly times and the ω-times as heavenly times that succeed the times in the earthly vale. Of course, in heaven, everything is perfect and there is no point to branching.

Since all branching points are in the part that corresponds with $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$, it follows that all parts of histories strictly above a branching point have order type $\mathbb{Q} + \omega$. Thus, $\mathcal{M}$ has the iso-property.

Suppose we had a chrono-function $\mathfrak{t} : \mathcal{M} \to \mathcal{T}$. The order type of the image of $\{m \mid m \leqslant W\widehat{1}\}$ will be $\mathbb{Q} + 2$. So $\mathcal{T}$ will have an initial segment $\mathbb{Q} + 2$. We note that all initial segments of order type $\mathbb{Q} + 2$ must be identical, so all moments $W\widehat{1}$ are mapped to the time $\mathfrak{t}(\widehat{1})$. From this it is easy to see that $\mathfrak{t}(W\widehat{n}) = \mathfrak{t}(\widehat{n})$. If follows that all histories are mapped onto $\mathcal{T}$ and that $\mathcal{T}$ has order type $\mathbb{Q} + \omega$. Thus, we have a chrono-function to $\mathbb{Q} + \omega$. Now every history h of $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$ is equal to $\{m \mid m < W\widehat{0}\}$, for some W. Thus, every history of $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$ is mapped isomorphically onto the $\mathbb{Q}$-part of our time axis $\mathbb{Q} + \omega$.

We can now define a time function $\mathfrak{t}^* : \mathsf{B}_{\kappa,\omega}(\mathbb{Q}) \to \mathbb{Q}$ by setting $\mathfrak{t}^*(m) := \mathfrak{t}(m)$. But this is impossible by the considerations of Example 4.7.

We note that our example also shows that the property of having a chrono-function is not closed under $\ast$. ❑

4.4. **An Optimistic Chrono-function without Strength & Different Shared Time Structures on one JBTS.** In this subsection, we provide an example of a structure equipped with an optimistic chrono-function that is not strong. Since, the same structure also has a strong chrono-function to a non-isomorphic time structure, the example also illustrates that the same JBTS can have optimistic chrono-functions to non-isomorphic time structures.

Example 4.9. We consider $\mathcal{M} := \mathsf{B}_{\aleph_1,\omega}(\mathbb{Q})$. For any world $w = q_0\alpha_0 \ldots q_{\ell-1}\alpha_{\ell-1}$, we define: $\lceil w \rceil := 1 + \sum_{i<\ell} \alpha_i$.

With every world w, we associate an isomorphism Φ_w from h_w to $\mathbb{Q} \times \lceil w \rceil$. We will arrange this in such a way that if $w' = wq\alpha$, then, for all $m \leqslant wq$, we have $\Phi_{w'}(m) = \Phi_w(m)$. We can do this as follows. We set $\Phi_\varepsilon(q) := q$. Suppose we have obtained Φ_w. Let $w' = wq\alpha$. We set $\Phi_{w'}(m) := \Phi_w(m)$ if $m \leqslant wq$. Clearly, the sets

$X := \{q' \in \mathbb{Q} \mid q < q'\}$ and $A := \{a \in \mathbb{Q} \times \lceil w' \rceil \mid \Phi_w(wq) < a\}$ are linear countable dense orderings without end points. Thus, there is an isomorphism ϕ from X to A. We take $\Phi_{w'}(w'q') := \phi(q')$. It is clear that the Φ_w have the desired properties.

We define $\mathsf{t}^*(wq) := \Phi_w(wq)$. We claim that t^* is an optimistic chrono-function from $\mathcal{M}$ to $\widehat{\mathbb{Q}} := \mathbb{Q} \times \omega_1$.

We first show the monotonicity of t^* (which is the forward property for both $<$ and $>$). Suppose $m < m'$.

If both m and m' have the same length we have, for some w, q, q', that $m = wq$, $m' = wq'$, $q < q'$. Hence, $\mathsf{t}^*(m) = \Phi_w(wq) < \Phi_w(wq') = \mathsf{t}^*(m')$.

Suppose $m = wq$ and $m' = wq'_0\gamma_0 \dots q'_i\gamma_i$, where $q \leqslant q'_0$. In this case, by our construction:

$$\Phi_w(m) = \Phi_{wq'_0\gamma_0}(m) = \dots = \Phi_{wq'_0\gamma_0 \dots q'_i\gamma_i}(m) = \Phi_{w'}(m).$$

Hence. $\mathsf{t}^*(m) = \Phi_w(m) = \Phi_{w'}(m) < \Phi_{w'}(m') = \mathsf{t}^*(m')$.

We turn to the backward property for $<$. Suppose $t < t'$ and $\mathsf{t}^*(m) = t$. Suppose $t' \in \mathbb{Q} \times \alpha$. Then clearly t' is in the $\Phi_{m\alpha}$-image of some $m\alpha q'$, since $\lceil m\alpha \rceil \geqslant \alpha$. So $m < m\alpha q' =: m'$ and $\mathsf{t}^*(m') = \Phi_{m\alpha}(m') = t'$.

Finally, we consider the backward property for $>$. Suppose $t' < t$ and $\mathsf{t}^*(m) = t$. Let $m = wq$. Then, $\Phi_w(wq) = t$. We can find an $m' < m$ such that $\Phi_w(m') = t'$. Suppose $m' = w'q'$. As before we can show that:

$$\mathsf{t}^*(m') = \Phi_{w'}(m') = \Phi_w(m') < \Phi_w(m) = \mathsf{t}^*(m).$$

Thus t^* is an optimistic chrono-function for $\mathcal{M}^\star$ and $\widehat{\mathbb{Q}}$. We note that t^* fails to be strong in the most radical way: every history of $\mathcal{M}^\star$ is isomorphic to $\mathbb{Q}$, so, no history is isomorphic to $\widehat{\mathbb{Q}}$. ❏

Example 4.10. It is easy to see, using the construction of Example 4.9, that $\mathcal{M}^\star := \mathsf{B}_{\aleph_1,\omega}(\mathbb{Q}) \veebar \widehat{Q}$ has an optimistic chrono-function to $\widehat{Q}^+$, which is $\widehat{Q}$ with a root added. However, whenever $\mathsf{t} : \mathcal{M}^\star \to \mathcal{T}$ is a chrono-function, $\mathcal{T}$ has to be $\widehat{Q}^+$ (modulo isomorphism). Since all histories of $\mathcal{M}^\star$ but one are countable, $\mathcal{M}^\star$ does not admit a strong chrono-function.

Alternatively, we could consider $\mathcal{M}^\circ := \mathbb{Q} \ast (\mathsf{B}_{\aleph_1,\omega}(\mathbb{Q}) \veebar \widehat{Q})$ with time structure $\widehat{Q}$. ❏

In Example 4.9, we have seen a JBTS with optimistic chrono-functions to different linear time structures. This makes one wonder what the relation is between the different time structures associated with a given JBTS $\mathcal{M}$. We have, at present, only a modest insight to offer.

We define $\mathcal{T}_0 \asymp \mathcal{T}_1$ by: there is a relation R between T_0 and T_1 such that (i) R is a (non-empty) bisimulation with respect to $<$ and (ii) whenever t_0Rt_1, there is a $\phi \subseteq R$ such that ϕ is an isomorphism between the $\{t' \mid t' \leqslant t_0\}$ and the $\{t'' \mid t'' \leqslant t_1\}$. It is easy to see that $\asymp$ is an equivalence relation between linear orderings.

Theorem 4.11. *Suppose $\mathcal{T}_0 \asymp \mathcal{T}_1$. Then, $\|\mathcal{T}_0\| = \|\mathcal{T}_1\|$.*

Proof. Suppose $R : \mathcal{T}_0 \asymp \mathcal{T}_1$. Let $\lambda_0 := \|\mathsf{t}_0\|$ and $\lambda_1 := \|\mathsf{t}_1\|$. Suppose e.g. $\lambda_0 < \lambda_1$. Then, $\lambda_0 + 1$ can be embedded in $\mathcal{T}_1$, say via ϕ. Let $t_1 := \mathsf{t}_1(\phi(\lambda_0))$ and suppose t_0Rt_1. Then $\{t \mid t \leqslant t_0\}$ is isomorphic to $\{t' \mid t' \leqslant t_1\}$. But then $\lambda_0 + 1$ can be embedded in $\{t \mid t \leqslant t_0\}$ and, hence in T_0. A contradiction. Similarly, for $\lambda_1 < \lambda_0$. Hence, $\lambda_0 = \lambda_1$. ❑

Theorem 4.12. *Suppose* $\mathsf{t}_0 : \mathcal{M} \to \mathcal{T}_0$ *and* $\mathsf{t}_1 : \mathcal{M} \to \mathcal{T}_1$ *are optimistic chrono-functions. Then* $\mathcal{T}_0 \asymp \mathcal{T}_1$.

Proof. We take $R := \mathsf{t}_1 \circ \mathsf{t}_0^{-1}$. ❑

We note that:

- $\mathcal{T} \asymp \mathbb{Q}$ iff $\mathcal{T} = \mathbb{Q}$ or $\mathcal{T} = \widehat{\mathbb{Q}}$.
- $\mathcal{T} \asymp \mathbb{R}$ iff $\mathcal{T} = \mathbb{R}$ or $\mathcal{T} = \widehat{\mathbb{R}}$.

5. The BQ_κ-property

In this section we show that the familiar result that $\mathbb{Q}$ is the unique linear dense ordering without end points generalizes to certain JBTS's.

Suppose $\kappa \in \{1, \ldots \aleph_0\}$. A JBTS has the *$BQ_\kappa$-property* if:

$\mathsf{BQ}_\kappa 1$ For all m, there is an $n < m$.

$\mathsf{BQ}_\kappa 2$ For all m, n with $m < n$, there is a p with $m < p < n$. (Density)

$\mathsf{BQ}_\kappa 3$ At every point we have precisely κ directions. This means:

- There are $n_i > m$, for $0 \leqslant i < \kappa$, such that $\mathsf{inf}(n_i, n_j) = m$, for $i < j < \kappa$. (Existence of at least κ transitions at m)
- In case κ is finite: suppose $n_i > m$, for $0 \leqslant i \leqslant \kappa$. Then, for some $i < j \leqslant \kappa$, we have $m < \mathsf{inf}(n_i, n_j)$.[9] (Existence of at most κ transitions at m)

$\mathsf{BQ}_\kappa 4$ The domain of moments is countable.[10]

We first show that the properties $\mathsf{B}\mathbb{Q}_\kappa$ are inhabited.

Theorem 5.1. $\mathsf{B}_{\kappa,\omega}(\mathbb{Q})$ *has the BQ_κ-property.*

Proof. In case $\kappa = 1$, we have $\mathsf{B}_{1,\omega}(\mathbb{Q}) = \mathbb{Q}$. In that case our result is easy. We assume $2 \leqslant \kappa \leqslant \aleph_0$. Let $\mathcal{M}^\star := \mathsf{B}_{\kappa,\omega}(\mathbb{Q})$.

We verify $\mathsf{BQ}_\kappa 1$. Consider any $m = qu$ in $M^\star$. We can take $n := q - 1 < qu = m$.

We verify $\mathsf{BQ}_\kappa 2$ (density). Suppose $m < n$. So $m = wq$ and either $n = wq'$ and $q < q'$ or $n = wq'iq''u$ and $q \leqslant q'$. In the first case, we take $p := w\frac{(q+q')}{2}$ and we have $m = wq < w\frac{(q+q')}{2} < wq' = n$. In the second case, we take $p := wq'i\frac{(q'+q'')}{2}$ and we have: $m = wq < wq'i\frac{(q'+q'')}{2} < wq'iq''u$.

We check $\mathsf{BQ}_\kappa 3$, to wit that at each moment there are precisely κ transitions. We treat the case that κ is finite, the infinite case being easier.

Consider any moment $m = wq$. We consider the moments $m_0 = w(q + 1)$, $m_1 = wq0(q+1), \ldots, m_{\kappa_1} = wq(\kappa-2)(q+1)$. It is easy to see that $m = \mathsf{inf}(m_i, m_j)$ for $i \neq j$.

Now suppose $n > m$. In case $n = wq'\nu$ and $q' > q$, then $\mathsf{inf}(m_0, n) = wq^*$, where $q^* = \mathsf{min}(q + 1, q')$, and, so $m < wq^*$. In case $n = wq\nu$, then ν must be

[9] In case κ is infinite, we should have a condition about uncountably many n_i. However, since the upper bound in this case is also handled by the condition that the domain is countable, we will refrain from formulating it.

[10] Example 5.6 provides two non-isomorphic JBTS's satisfying all conditions of the BQ_2-property except countability.

non-empty. So $n = wqiq'\nu'$, where $i < \kappa - 1$. We have $\mathsf{inf}(m_i, n) = wqiq^*$, where $q^* = \mathsf{min}(q+1, q')$. So $\mathsf{inf}(m_i, n) > m$.

Ad $\mathsf{BQ}_\kappa 4$: the countability of our domain is immediate. ❑

We show that any two JBTS's $\mathcal{M}$ and $\mathcal{M}'$ with the BQ_κ-property are isomorphic. Let $\mathcal{M}$ and $\mathcal{M}'$ be JBTS's and let $\vec{h}$ and $\vec{h}'$ be finite sequences of histories of length ℓ in $\mathcal{M}$, respectively $\mathcal{M}'$. We write $\Phi : \vec{h} \xrightarrow{\mathsf{iso}} \vec{h}'$ iff Φ is an isomorphism between $\bigcup_{i<\ell} h_i$ ordered by $<$ and $\bigcup_{i<\ell} h'_i$ ordered by $<'$ with the additional property that each h'_i is the Φ-image of h_i.

Lemma 5.2. *Let $\kappa, \kappa' \in \{1, \ldots \aleph_0\}$ with $\kappa \leqslant \kappa'$. We consider a JBTS $\mathcal{M}$ with the BQ_κ-property and a JBTS $\mathcal{M}'$ with the $BQ_{\kappa'}$-property. Let $\Phi : \vec{h} \xrightarrow{\mathsf{iso}} \vec{h}'$. Consider any history g in $\mathcal{M}$. Then, we can find a history g' in $\mathcal{M}'$ such that there is a $\Psi \supseteq \Phi$ such that $\Psi : \vec{h}g \xrightarrow{\mathsf{iso}} \vec{h}'g'$.*

Proof. We work under the assumptions of the lemma. In case g is h_i, for some $i < \ell$, we can take $g' := h'_i$ and $\Psi := \Phi$. Suppose g is not one of the h_i. Let m_i be the branching point of g and h_i. Let m^* be the maximum of the m_i. Suppose h_j contains m^*. Clearly, h_j cannot have the same direction as g in m^*, by the maximality of m^*. It follows that the number of directions of the h_j in m^* is $< \kappa$. It is easily seen that h_i and h_j have the same direction in m^* iff $h'_i := \Phi(h_i)$ and $h'_j := \Phi(h_j)$ have the same direction in $\Phi(m^*)$. It follows that in $\Phi(m^*)$ there is a direction $\mathfrak{d}'$ that is not the direction of one of the h'_i. We choose g' through $\Phi(m^*)$ in the direction $\mathfrak{d}'$. The elements of g above m^* are not in $\bigcup_{i<\ell} h_i$ and have the order type of $\mathbb{Q}$. The elements of g' above $\Phi(m^*)$ are not in $\bigcup_{i<\ell} h'_i$ and have also the order type of $\mathbb{Q}$. Let θ be an isomorphism between the elements h above m^* and the elements h' above $\Phi(m^*)$. It is easily seen that $\Psi := \Phi \cup \theta$ satisfies the claim of the theorem. ❑

Theorem 5.3. *Let $\kappa, \kappa' \in \{1, \ldots \aleph_0\}$ with $\kappa \leqslant \kappa'$. We consider a JBTS $\mathcal{M}$ with the BQ_κ-property and a JBTS $\mathcal{M}'$ with the $BQ_{\kappa'}$-property. There is an isomorphic embedding of Φ of $\mathcal{M}$ into $\mathcal{M}'$.*

Proof. Let $m_0, m_1, \ldots$ be an enumeration of M and let $h_0, h_1, \ldots$ be a sequence of histories such that $m_i \in h_i$. Consider any history h'_0 of $\mathcal{M}'$. We can find a $\Phi_0 : h_0 \xrightarrow{\mathsf{iso}} h'_0$, since both histories are isomorphic to $\mathbb{Q}$. We now apply Lemma 5.2 to Φ_0, h_1. This produces a history h'_1 and $\Phi_1 : h_0 h_1 \xrightarrow{\mathsf{iso}} h'_0 h'_1$. We again apply Lemma 5.2 to Φ_1, h_2, etc. The union of the Φ_i is the promised embedding Φ. ❑

Theorem 5.4. *Let $\kappa \in \{1, \ldots \aleph_0\}$. We consider JBTS's $\mathcal{M}$ and $\mathcal{M}'$ with the BQ_κ-property. Then, $\mathcal{M}$ is isomorphic to $\mathcal{M}'$.*

Proof. Let $m_0, m_1, \ldots$ be an enumeration of M and let $m'_0, m'_1, \ldots$ be an enumeration of M'. Let $h_0, h_1, \ldots$ be a sequence of histories such that $m_i \in h_i$ and let $h'_0, h'_1, \ldots$ be a sequence of histories such that $m'_i \in h'_i$. We construct our isomorphism in stages.

- **stage 0:** We set $\widetilde{h}_0 := h_0$ and $\widetilde{h}'_0 := h'_0$ and we pick a $\Phi_0 : \widetilde{h}_0 \xrightarrow{\mathsf{iso}} \widetilde{h}'_0$.
- **stage $2i+1$:** We have constructed $\Phi_{2i} : \widetilde{h}_0 \cdots \widetilde{h}_{2i} \xrightarrow{\mathsf{iso}} \widetilde{h}'_0 \cdots \widetilde{h}'_{2i}$. We now apply Lemma 5.2 to Φ_{2i} and $\widetilde{h}_{2i+1} := h_{i+1}$. This produces a $\widetilde{h}'_{2i+1}$ and a $\Phi_{2i+1} : \widetilde{h}_0 \cdots \widetilde{h}_{2i+1} \xrightarrow{\mathsf{iso}} \widetilde{h}'_0 \cdots \widetilde{h}'_{2i+1}$.

stage $2i+2$**:** We have constructed $\Phi_{2i+1} : \widetilde{h}_0 \cdots \widetilde{h}_{2i+1} \xrightarrow{\text{iso}} \widetilde{h}'_0 \cdots \widetilde{h}'_{2i+1}$. We now apply Lemma 5.2 to Φ^{-1}_{2i+1} and $\widetilde{h}'_{2i+2} := h'_{i+1}$. This produces a $\widetilde{h}_{2i+2}$ and a $\Phi^{-1}_{2i+2} : \widetilde{h}'_0 \cdots \widetilde{h}'_{2i+2} \xrightarrow{\text{iso}} \widetilde{h}_0 \cdots \widetilde{h}_{2i+2}$. Inverting again we get Φ_{n+2}.

It is easy to see that the union of the Φ_i is the desired isomorphism. ❑

The proof of Theorem 5.4 contains more information than is contained in the statement of the theorem. The next theorem makes a bit of that extra information visible.

Theorem 5.5. *Let* $\kappa \in \{1, \ldots \aleph_0\}$*. We consider a JBTS* $\mathcal{M}$ *with the* BQ_κ*-property. Let* h *and* h' *be histories in* $\mathcal{M}$*. Then there is an automorphism of* $\mathcal{M}$ *that interchanges* h *and* h'*.*

Proof. By a minor adaptation of the proof of Theorem 5.4. ❑

It is good to reflect a moment on the meaning of Theorem 5.5. Consider $\mathcal{M}^\star := \mathsf{B}_{2,\omega}(\mathbb{Q})$. Let $\mathfrak{t} : wt \mapsto t$ be the standard chrono-function on $\mathcal{M}^\star$. Let $\Omega := 00\frac{1}{2}0\frac{3}{4}\ldots$. We write ε for the empty sequence. Now h_ε is a straight history that is mapped by $\mathfrak{t}$ to all of $\mathbb{Q}$. On the other hand, $\mathfrak{t}$ maps h_Ω to $\{q \mid q < 1\}$. Let Φ be an automorphism of $\mathcal{M}^\star$ that interchanges h_ε and h_Ω. Let $\mathfrak{t}^\star := \mathfrak{t} \circ \Phi$. Then $\mathfrak{t}^\star$ maps h_Ω to all of $\mathbb{Q}$, and $\mathfrak{t}^\star$ maps h_ε to $\{q \mid q < 1\}$. Thus, we see that in $\mathcal{M}^\star$ concepts like straight history and zigzag history, world and superworld, standard chrono-function are all *implementation artifacts*. They have no meaning in terms of the order type of $\mathcal{M}^\star$. For the eyes of the order type all histories are equal. Only, in our implementation, they are not *created* equal.

Example 5.6. We show that countability is essential for uniqueness. We provide an example of two JBTS's of cardinality $2^{\aleph_0}$ in which all histories are isomorphic to $\mathbb{Q}$ and such that in each point we have precisely two transitions.

Let $\kappa \in \{2, \ldots \infty\}$ and let $\mathcal{M}_0 := \mathsf{B}_{\kappa,\omega\times 2}(\mathbb{Q})$. We take $\mathcal{M}_1$ to be the substructure of $\mathcal{M}_0$ where all moments of length $\geqslant \omega$ are constrained to begin with 0. Now it is clear that in $\mathcal{M}_0$ there are $2^{\aleph_0}$ moments above every moment, but in $\mathcal{M}_1$ there are only $\aleph_0$ moments above, say, 1. So, the two models cannot be isomorphic. ❑

Example 5.7. What happens if we stipulate that we have $2^{\aleph_0}$ moments and $2^{\aleph_0}$ transitions at every point and that every history is isomorphic to $\mathbb{Q}$? We show that there are non-isomorphic structures satisfying these conditions.

We take $\mathcal{M}_0 := \mathsf{B}_{2^{\aleph_0},\omega\times 2}(\mathbb{Q})$ and $\mathcal{M}_1 := \mathsf{B}_{2^{\aleph_0},\omega}(\mathbb{Q})$. It is easy to see that these structures satisfy our conditions.

Suppose we have an isomorphism Φ between $\mathcal{M}_0$ and $\mathcal{M}_1$. We define sequence $m_0 < m_1 < \ldots$ of finite sequences in $\mathcal{M}_0$ such that $|\Phi(m_0)| < |\Phi(m_1)| < \ldots$ It follows that the $\Phi(m_i)$ have no upperbound in $\mathcal{M}_1$. This is impossible since, in $\mathcal{M}_0$ there is an element m_ω above the m_i. However there can be no $\Phi(m_\omega)$ above the $\Phi(m_i)$.

We take $m_0 := 0$. Suppose we have defined $m_i = w_i q_i$. Consider $m_i0(q_i+1)$ and $m_i1(q_i+1)$. Since these moments are in different directions in m_i, their Φ-images should also have different directions in $\Phi(m_i)$. It follows that at least one of $\Phi(m_i0(q_i+1))$ and $\Phi(m_i1(q_i+1))$ is longer than $\Phi(m_i)$. We take $m_{i+1} := m_i0(q_i+1)$, if $\Phi(m_i0(q_i+1))$ is longer than $\Phi(m_i)$. We take $m_{i+1} := m_i1(q_i+1)$ otherwise. ❑

Open Question 5.8. One would suspect that there is an analogous development for $\mathbb{R}$ substituted for $\mathbb{Q}$ of the results of this section. I have not tried to develop this. It would be interesting to see how it works out. ❑

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Appendix A. Adding Branching Points

In this Appendix, we show how to transform a BTS into a JBTS in a way that is as light as possible since (i) we only add branching points when needed and (ii) the construction is essentially first order. The extending JBTS can be viewed as an internal model of the BTS.

One obvious idea is to add new moments to a structure is to take all linear initial segments as the new moments. However, this construction is somewhat heavy. It is not first order and may, for example, raise the cardinality of our structure. For this reason, we use a pair construction.

Let a BTS $\mathcal{M}$ be given. We construct a JBTS $\widetilde{\mathcal{M}}$ as follows. The moments of $\widetilde{\mathcal{M}}$ will be equivalence classes of pairs of moments of the old BTS. We first describe the pairs and their preorder.

For $m \in M$, let $I_m := \{m' \mid m' \leqslant m\}$. We note that I_m is linearly ordered by $<$ and downwards closed under $<$. We define:

- $\langle m, m' \rangle \leqslant \langle n, n' \rangle$ iff $I_m \cap I_{m'} \subseteq I_n \cap I_{n'}$.

Clearly, $\leqslant$ is transitive and reflexive. Our JBTS will consist of the pairs modulo the induced equivalence relation of $\leqslant$, ordered by $\leqslant$ lifted to the equivalence classes. We check the properties of a JBTS for the structure so obtained.

BT1. Suppose $\langle m, m' \rangle \leqslant \langle p, p' \rangle$ and $\langle n, n' \rangle \leqslant \langle p, p' \rangle$. We find that $I_m \cap I_{m'}$ and $I_n \cap I_{n'}$ are downwards closed sub-orderings of the linear order $I_p \cap I_{p'}$. So, $I_m \cap I_{m'} \subseteq I_n \cap I_{n'}$ or $I_n \cap I_{n'} \subseteq I_m \cap I_{m'}$. Hence $\langle m, m' \rangle \leqslant \langle n, n' \rangle$ or $\langle n, n' \rangle \leqslant \langle m, m' \rangle$.

J. Consider $\langle m, m' \rangle$ and $\langle n, n' \rangle$. Consider $I_m \cap I_{m'}$, $I_m \cap I_n$ and $I_m \cap I_{n'}$. These orderings are downwards closed sub-orderings of the linear ordering I_m. It follows that one of them is the $\subseteq$-smallest, say it is $I_m \cap I_k$ for $k \in \{m', n, n'\}$. We claim that $\langle m, k \rangle$ is an infimum of $\langle m, m' \rangle$ and $\langle n, n' \rangle$. We note that $I_m \cap I_k = I_m \cap I_{m'} \cap I_n \cap I_{n'}$. Hence,

$$\begin{aligned}
\langle p, p' \rangle \leqslant \langle m, m' \rangle \text{ and } \langle p, p' \rangle \leqslant \langle n, n' \rangle \quad &\Leftrightarrow \quad I_p \cap I_{p'} \subseteq I_m \cap I_{m'} \text{ and } I_p \cap I_{p'} \subseteq I_n \cap I_{n'} \\
&\Leftrightarrow \quad I_p \cap I_{p'} \subseteq I_m \cap I_{m'} \cap I_n \cap I_{n'} \\
&\Leftrightarrow \quad I_p \cap I_{p'} \subseteq I_m \cap I_k \\
&\Leftrightarrow \quad \langle p, p' \rangle \leqslant \langle m, k \rangle
\end{aligned}$$

BT3. Consider any $\langle m, m' \rangle$. By the seriality of $\mathcal{M}$, there is an $n > m$. We clearly have $\langle m, m' \rangle \leqslant \langle m, m \rangle < \langle n, n \rangle$.

We specify the construction of dividing out explicitly. Let $\sim$ be the induced equivalence relation of $\leqslant$. The moments of $\widetilde{\mathcal{M}}$ will be $[m, m'] := [\langle m, m' \rangle]_\sim$. We note that

$$\begin{aligned} [m, m] < [n, n] \quad &\Leftrightarrow \quad I_m \subsetneq I_n \\ &\Leftrightarrow \quad m < n \end{aligned}$$

So, $m \mapsto [m, m]$ is an embedding of $\mathcal{M}$ into $\widetilde{\mathcal{M}}$. Of course, this embedding usually will not have the backwards property for $>$. We note that $[m, m] = [n, p]$ iff $m = \mathsf{inf}(n, p)$. Thus, elements of the form $[n, p]$ that are not equal to a $[m, m]$ are added because n and p have no infimum in the original structure.

Inspection of our construction shows that our extension procedure specifies an interpretation of the theory of JBTS's in the theory of BTS's. Thus, our construction does, in a sense, not use extrinsic means.

Appendix B. Rumberg Structures

A *Rumberg structure* or RS $\mathcal{R}$ is given by $\langle W, T, <, \approx \rangle$, where:

i. W is a non-empty set of worlds.
ii. $\mathcal{T} := \langle T, < \rangle$ is a serial strict linear order.
iii. $\approx$ is a relation on $W \times T \times W$. We write $w \approx_t w'$. We demand:
   a. $\approx_t$ is an equivalence relation on W.
   b. If $w \approx_t w'$ and $t' < t$, then $w \approx_{t'} w'$.
   c. If $w \neq w'$, then there is a maximal t such that $w \approx_t w'$.

Thus, a Rumberg Structure is a $T \times W$-frame in the sense of [Tho84] with seriality en jointedness added.

Let $\mathcal{R}$ be a RS. We write $[w]_t$ for the $\approx_t$ equivalence class of w. We construct a bundled chrono-structure $\mathcal{C} = \langle \mathcal{M}, H, \mathcal{T}, \mathfrak{rum} \rangle := \mathsf{Rum}(\mathcal{R})$ as follows.

- The moments in M are all elements $\langle t, [w]_t \rangle$.
- $\langle t, [w]_t \rangle < \langle t', [w']_{t'} \rangle$ iff $t < t'$ and $w \approx_t w'$.
- The elements of H are the h_w for $w \in W$, where $h_w = \{\langle t, [w]_t \rangle \mid t \in T\}$.
- $\mathfrak{rum}(\langle t, [w]_t \rangle) = t$.

It is easy to see that our definition of the ordering is independent of the chosen representatives.

Theorem B.1 (Rumberg). $\mathsf{Rum}(\mathcal{R})$ *is a bundled chrono-structure.*

Proof. It is easy to see that $<$ is a serial strict ordering.

We prove Jointedness. Consider $m = \langle t, [w]_t \rangle$ and $m' = \langle t', [w']_{t'} \rangle$. Let t^* be maximal such that $w \approx_{t*} w'$. Let t° be the minimum of t, t' and t^*. We claim that $m^\circ := \langle t^\circ, [w]_{t^\circ} \rangle = \mathsf{inf}(m, m')$.

Since $t^\circ \leqslant t^*$, we find $w \approx_{t^\circ} w'$. Hence m° is smaller or equal to m, m' and $m^* := \langle t^*, [w]_{t*} \rangle$. Consider $n = \langle u, [v]_u \rangle$ and suppose $n \leqslant m$ and $n \leqslant m'$. Then $u \leqslant t$ and $u \leqslant t'$. Moreover, $v \approx_u w$ and $v \approx_u w'$. Hence, $w \approx_u w'$ and, thus, $u \leqslant t^*$. We may conclude that $u \leqslant t^\circ$. It follows that $n = \langle u, [v]_u \rangle \leqslant \langle t^\circ, [w]_{t^\circ} \rangle = m^\circ$.

We verify tree-likeness. Consider $m_0 = \langle t_0, [w_0]_{t_0} \rangle$, $m_1 = \langle t_1, [w_1]_{t_1} \rangle$ and $m = \langle t, [w]_t \rangle$. Suppose $m_0 \leqslant m$ and $m_1 \leqslant m$. Then, $t_0 \leqslant t$, $t_1 \leqslant t$, $w_0 \approx_{t_0} w$,

$w_1 \approx_{t_1} w$. Suppose $t_0 \leqslant t_1$. Then, $w_0 \approx_{t_0} w \approx_{t_0} w_1$. So, $w_0 \approx_{t_0} w_1$, and, hence, $m_0 = \langle t_0, [w_0]_{t_0}\rangle \leqslant \langle t_1, [w_1]_{t_1}\rangle = m_1$. The case that $t_1 \leqslant t_0$ is similar.

We show that h_w is a history that is isomorphic to $\mathcal{T}$. It is immediate that h_w is linear and isomorphic to $\mathcal{T}$. Suppose $n = \langle t, [v]_t\rangle$ is comparable with all elements of h_w. Then, n must be comparable with $\langle t, [w]_t\rangle$. But then it follows that $w \approx_t v$, and, thus, that $n = \langle t, [w]_t\rangle \in h_w$.

We note that, in $\mathsf{Rum}(\mathcal{R})$, the union of the h_w is M. Moreover the mapping $\mathfrak{rum} : \langle t, [w]_t\rangle \mapsto t$ has the property that $\mathfrak{rum} \restriction h_w$ is an isomorphism between h_w and $\mathcal{T}$. Thus, $\mathfrak{rum}$ is a strong chrono-function. ❑

We define a function ruM from bundled chrono-structures to Rumberg structures as follows. Let a bundled chrono-structure $\mathcal{C} = \langle \mathcal{M}, H, \mathcal{T}, \mathfrak{t}\rangle$ be given. We define an RS $\mathcal{R} = \langle W, T, <, \approx\rangle$ as follows.

- $W := H$,
- T is the domain of $\mathcal{T}$, and $<$ is the ordering of $\mathcal{T}$.
- $h \approx_t h'$ iff, for some $m \in h \cap h'$, we have $\mathfrak{t}(m) = t$.

Theorem B.2. *Suppose $\mathcal{C}$ be a bundled chrono-structure. The structure* $\mathsf{muR}(\mathcal{C})$ *is a Rumberg structure.*

Proof. Let $\mathcal{C}$ be a bundled chrono-structure. Let $\mathsf{muR}(\mathcal{C})$ be as described.

Clearly, $\approx_t$ is reflexive and symmetric. Suppose $h \approx_t h'$ and $h' \approx_t h''$. Then, we find $m \in h \cap h'$ and $m' \in h' \cap h''$ with $\mathfrak{t}(m) = t = \mathfrak{t}(m')$. Since, both m and m' are in h' it follows that $m = m'$. Hence $m \in h \cap h''$ and, thus, $h \approx_t h''$.

Suppose $h \approx_t h'$ and $t' < t$. Suppose $m \in h \cap h'$ with $\mathfrak{t}(m) = t$. Then there is an $m' < m$ with $\mathfrak{t}(m') = t'$ since $\mathfrak{t} \restriction h$ is an isomorphism. We also have $m' \in h \cap h'$, so we are done.

Finally, let m be the branching point of h and h'. Suppose $\mathfrak{t}(m) = t$. Then, t is the maximal time such that $h \approx_t h'$. ❑

Theorem B.3. muR *and* Rum *are inverses modulo isomorphism.*

Proof. Let us start with a Rumberg structure $\mathcal{R}$. The worlds of $\mathsf{muR}(\mathsf{Rum}(\mathcal{R}))$ are the histories h_w of $\mathsf{Rum}(\mathcal{R})$. Our isomorphism will be given by the mapping $\Phi : w \mapsto h_w = \{\langle t, [w]_t\rangle \mid t \in T\}$. Moreover T and $<$ are the same for $\mathcal{R}$ and $\mathsf{muR}(\mathsf{Rum}(\mathcal{R}))$. It is easy to see that Φ is a bijection. Finally, we have:

$$\begin{aligned} h_w \approx_t h_{w'} &\Leftrightarrow \exists \mu \in h_w \cap h_{w'}\, \mathfrak{rum}(\mu) = t \\ &\Leftrightarrow \exists w^*\, \langle t, [w^*]_t\rangle \in h_w \cap h'_w \\ &\Leftrightarrow \langle t, [w]_t\rangle = \langle t, [w']_t\rangle \\ &\Leftrightarrow w \approx_t w' \end{aligned}$$

Let us consider a bundled chrono-structure $\mathcal{C}$. The moments of $\mathsf{Rum}(\mathsf{MuR}(\mathcal{C}))$ are of the form $\langle \mathfrak{t}(m), [h]_{\mathfrak{t}(m)}\rangle$, for $m \in h$. We note that $[h]_{\mathfrak{t}(m)}$ is the set of $h' \in H$ such that $m \in h'$, so $[h]_{\mathfrak{t}(m)}$ is independent of the specific choice of a h with the desired property. Our isomorphism is based on the mapping $\Psi : m \mapsto \langle \mathfrak{t}(m), [h]_{\mathfrak{t}(m)}\rangle$, for $m \in h$. It is easy to see that Φ is a bijection. Finally, suppose $m < m'$. Let $m' \in h \in H$. Then $m \in h$. It follows that

$$\Phi(m) = \langle \mathfrak{t}(m), [h]_{\mathfrak{t}(m)}\rangle < \langle \mathfrak{t}(m'), [h]_{\mathfrak{t}(m')}\rangle = \Phi(m'). \quad ❑$$

Appendix C. The Rumberg Road to $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$

We have shown that $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$ has a strong chrono-function. Thus, it is a Rumberg JBTS. In this appendix, we directly construct a RS corresponding to $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$.

Let $\kappa > 1$, λ and $\mathcal{T}$ be given. We assume that each $\alpha < \lambda$ can be embedded in $\mathcal{T}$. We specify a RS $\mathcal{R}^\star$.

- Worlds are sequences with one of two forms:
  A. $t_0\alpha_0t_1\alpha_1 \ldots t_\omega\alpha_\omega \ldots t_\gamma\alpha_\gamma$
  B. $t_0\alpha_0t_1\alpha_1 \ldots t_\omega\alpha_\omega \ldots$
  (so the length of the sequence is a limit ordinal)

  Here the t_β are times, where $t_\beta < t_{\beta'}$, if $\beta < \beta'$, and the α_β are strictly below $\kappa - 1$. We demand that the length of our sequence is $< \lambda$ and, in case (B), that the times in the sequence have an upper bound.

  We note that our worlds are just the worlds of the previous subsection.
- We define $\langle w\rangle_t$ as the longest initial subworld of w, in which only times strictly below t occur. We define $w \approx_t w'$ iff $\langle w\rangle_t = \langle w'\rangle_t$.

We note that $\langle\langle w\rangle_t\rangle_{t'} = \langle w\rangle_{t'}$ if $t' \leqslant t$. Hence. we have $w \approx_t \langle w\rangle_t$. Thus, we can view $\langle w\rangle_t$ as designated element of $[w]_t$.

Before proceeding, we give a different characterisation of $\approx_t$ that is useful to have.

Theorem C.1. *Let w^* be the maximal shared initial subworld of w and w' (this might be empty). To avoid laborious case splitting we use a virtual element ∞ that is supposed to exceed all times. We have:*

$$w \approx_t w' \text{ iff } w\infty = w^*t'\theta \text{ and } w'\infty = w^*t''\theta' \text{ and } t \leqslant \mathsf{min}(t', t'').$$

(t unlike t', t'' is not allowed the value ∞.)

Proof. Suppose $w \approx_t w'$. Let $w^\circ := \langle w\rangle_t = \langle w'\rangle_t$. Let w^* be the maximal shared initial subworld of w and w'. It follows that w° is an initial subworld of w^*. Suppose $w\infty = w^*t'\theta$ and $w'\infty = w^*t''\theta'$. It is now immediate that t must be $\leqslant t'$ and $\leqslant t''$.

Conversely, suppose $w\infty = w^*t'\theta$ and $w'\infty = w^*t''\theta'$ and $t \leqslant \mathsf{min}(t', t'')$. It follows that $\langle w\rangle_t = \langle w^*\rangle_t = \langle w'\rangle_t$. ❑

Theorem C.2. *$\mathcal{R}^\star$ is an RS.*

Proof. Trivially, $\approx_t$ is an equivalence relation.

Suppose $w \approx_t w'$ and $t' < t$. Then,

$$\langle w\rangle_{t'} = \langle\langle w\rangle_t\rangle_{t'} = \langle\langle w'\rangle_t\rangle_{t'} = \langle w'\rangle_{t'}.$$

Suppose $w \neq w'$. We want to show that there is a maximal t such that $w \approx_t w'$. Let w^* be the maximal shared initial subworld of w and w'. We have $w\infty = w^*t'\theta$ and $w'\infty = w^*t''\theta'$. Since $w \neq w'$ one of t', t'' is not ∞. So, by Theorem C.1, the desired maximal t is $\mathsf{min}(t', t'')$. ❑

Form our RS we can now construct the JBTS $\mathsf{Rum}(\mathcal{R}^\star)$.

Theorem C.3. $\mathsf{Rum}(\mathcal{R}^\star)$ *is isomorphic to* $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$.

Proof. Our isomorphism is given by the mapping $\Theta : \langle t, [w]_t \rangle \mapsto \langle w \rangle_t t$. To avoid confusion, locally, we write $\sqsubseteq$ for the ordering in $\mathsf{Rum}(\mathcal{R}^\star)$ and $\preceq$ for the ordering in $\mathsf{B}_{\kappa,\lambda}(\mathcal{T})$. We have

$$(\dagger) \quad \langle t, [w]_t \rangle \sqsubseteq \langle t', [w']_{t'} \rangle \text{ iff } t \leqslant t' \text{ and } \langle w \rangle_t = \langle w' \rangle_t.$$

In case $\langle w' \rangle_{t'} = \langle w' \rangle_t$, the right hand side of (†) is equivalent to

$$\Theta(\langle t, [w]_t \rangle) = \langle w \rangle_t t \preceq \langle w' \rangle_t t' = \langle w' \rangle_{t'} t' = \Theta(\langle t', [w]_{t'} \rangle).$$

In case $\langle w' \rangle_{t'}$ is strictly longer than $\langle w \rangle_t$, we have $\langle w' \rangle'_{t'} t' = \langle w \rangle_t t'' \alpha\theta$. Here $t'' \geqslant t$. So the right hand side of (†) is equivalent to

$$\Theta(\langle t, [w]_t \rangle) = \langle w \rangle_t t \preceq \langle w \rangle_t t'' \alpha\theta = \langle w' \rangle_{t'} t' = \Theta(\langle t', [w]_{t'} \rangle). \qquad ❑$$

Appendix D. A Sufficient Condition for Not Having a Time-function

We present a result concerning *not having a time-function.*

We say that a JBTS is *seriously branching* if whenever $m < m'$ there is a branching point m° such that $m \leqslant m^\circ < m'$. In other words, whenever $m < m'$, there is an $n > m$ such that n is incomparable with m'.

Lemma D.1. *Let $\mathcal{M}$ be seriously branching and let $\mathfrak{t} : \mathcal{M} \to \mathcal{T}$ be an optimistic chrono-function. Suppose $\mathfrak{t}(m) = t$ and $t < t'$. Let p be an arbitrary moment. Then, there is an $m' > m$ with $\mathfrak{t}(m') = t'$ and $m' \nleqslant p$.*

Proof. We assume the conditions of the lemma. We have an $m^* > m$ such that $\mathfrak{t}(m^*) = t'$. If $m^* \nleqslant p$, we take $m' := m^*$, and we are done. Otherwise, we consider an $n > m$ such that n is incomparable with m^*. By the backward property of $<$ and $>$, we can find an m' comparable with n such that $\mathfrak{t}(m') = t'$. In case $m' \geqslant n$ certainly $m' > m$. If $m' \leqslant n$, by tree-likeness, $m' \leqslant m$ or $m < m'$. However, $m' \leqslant m$ is impossible, since $\mathfrak{t}$ is monotonic. So, in all cases $m < m'$. Is it possible that $m' \leqslant p$? In that case, we would have $m^* \leqslant m'$ or $m' \leqslant m^*$. Since $\mathfrak{t}(m^*) = t' = \mathfrak{t}(m')$, by monotonicity, we must have $m^* = m'$. But this contradicts the fact that m' is comparable with n. ❑

Theorem D.2. *Let $\mathcal{M}$ be a seriously branching JBTS and let $\mathfrak{t} : \mathcal{M} \to \mathcal{T}$ be an optimistic chrono-function. Let λ be any limit ordinal. Suppose further that there is a set of moments P of cardinality $\mathsf{card}(\lambda)$ that is cofinal in $\mathcal{M}$, i.e., for any m, there is a $p \in P$ with $m \leqslant p$.*

Let σ be be an ascending sequence $t_0 < t_1 < t_2 \ldots < t_\alpha < \ldots$ in $\mathcal{T}$ of length λ. Let T_σ be the set t such that $t \leqslant t_\alpha$, for some $\alpha < \lambda$. Then, there is a history h in $\mathcal{M}$ such that the $\mathfrak{t}$-image of h is initial in T_σ.

Proof. We assume the conditions of the theorem. Let $p_0, p_1, \ldots p_\alpha \ldots$ be an enumeration of P of length λ. We define an ascending sequence $m_0 < m_1 < \ldots < m_\alpha < \ldots$ such that $\mathfrak{t}(m_\alpha) = t_\alpha$ and $m_{\alpha+1} \nleqslant p_\alpha$. This sequence has as length a limit ordinal λ' with $\lambda' \leqslant \lambda$. Moreover, $h := \{n \mid \exists \alpha < \lambda' \;\; n \leqslant m_\alpha\}$ will be history.

We first find m_0 with $\mathfrak{t}(m_0) = t_0$. Suppose we have already found m_α. Then we can find a desired $m_{\alpha+1}$ with $\mathfrak{t}(m_{\alpha+1}) = t_{\alpha+1}$, $m_\alpha < m_{\alpha+1}$ and $m_{\alpha+1} \nleqslant p_\alpha$ using Lemma D.1. Suppose $\lambda^* < \lambda$ is a limit and we have defined our sequence for all $\alpha < \lambda^*$. There are two possibilities. Either the m_α, for $\alpha < \lambda^*$, have an upper bound n or they have not. If not, we set $\lambda' := \lambda^*$ and our construction stops. It is immediate that h is a history. Suppose there is an upper bound n and let $\mathfrak{t}(n) = u$.

In case $u \leqslant t_{\lambda*}$, there is an $m \geqslant n$ such that $\mathfrak{t}(m) = t_{\lambda*}$. We set $m_{\lambda*} := m$. Otherwise there is an $m < n$ such that $\mathfrak{t}(m) = t_{\lambda*}$. By tree-likeness, we find that m is comparable with all m_α with $\alpha < \lambda^*$. By the monotonicity of $\mathfrak{t}$, we find that m is an upper bound of the m_α with $\alpha < \lambda^*$. Thus, we can take $m_{\lambda*} := m$.

In case our procedure does not halt before λ, we show that h is a history as follows. Suppose $n > m_\alpha$, for all $\alpha < \lambda$. Since P is cofinal in $\mathcal{M}$, we have, for some p_α in P, that $p_\alpha \geqslant n$. Hence $p_\alpha \geqslant n \geqslant m_{\alpha+1}$. Quod non. ❑

Theorem D.3. *Suppose $\mathcal{M}$ is a seriously branching JBTS. Suppose there is a cofinal set P such that $\mathsf{card}(P) < \|h\|$, for some history h. (Here we consider $\mathsf{card}(P)$ as an initial limit ordinal.) Then, $\mathcal{M}$ has no time function.*

Proof. Suppose $\mathcal{M}$ is seriously branching. Suppose further that, for some history h and for some cofinal set P, we have $\mathsf{card}(P) < \|h\|$. To obtain a contradiction, let's assume that there is a time function $\mathfrak{t} : \mathcal{M} \to \mathcal{T}$. Then h is isomorphic to $\mathcal{T}$, and thus $\mathsf{card}(P) < \|\mathcal{T}\|$. Thus, we can embed $\mathsf{card}(P)$, qua ordinal, non-cofinally in $\mathcal{T}$, say by $\mathfrak{e}$. By Theorem D.2, we find that there is a history h^* that is embedded by $\mathfrak{t}$ in the downward closure of the $\mathfrak{e}$-image of $\mathsf{card}(P)$. A contradiction. ❑

Theorems D.2 and D.3 have limited value since the cardinality constraint on P is somewhat restrictive. For example, if the cardinality of the directions in points of our structure is too high, there will not be such a set P. The following development brings some relief.

Let $\mathcal{M}$ and $\mathcal{N}$ be JBTS's. A *t-morphism* $\mathfrak{T} : \mathcal{M} \to \mathcal{N}$ is an order preserving mapping from M to N that maps histories onto histories. Thus, for any history g in $\mathcal{M}$, the $\mathfrak{T}$-image of g is an $\mathcal{N}$-history h. Moreover, $\mathfrak{T} \restriction h$ will be an isomorphism between g and h. We note that a time-function is just a special case of a t-morphism. We have the following trivial insight.

Theorem D.4. *The JBTS's in combination with the t-morphisms form a category.*

A JBTS $\mathcal{M}$ is *$\leqslant 2$-branching* if the cardinality of the set of directions at each moment is $\leqslant 2$. Now consider any seriously branching $\mathcal{M}$. If an $\mathcal{N}$ that is seriously branching and $\leqslant 2$-branching can be embedded, say via $\mathfrak{T}$, in $\mathcal{M}$, and if $\mathfrak{t}$ is a time-function on $\mathcal{M}$, then $\mathfrak{t} \circ \mathfrak{T}$ is a time-function on $\mathcal{N}$. Hence, if $\mathcal{N}$ cannot have a time function than neither can $\mathcal{M}$. The following theorem guarantees a lot of seriously branching and $\leqslant 2$ branching t-sub-BTS's for a seriously branching JBTS.

Theorem D.5. *Every JBTS $\mathcal{M}$ has a maximal $\leqslant 2$-branching sub-BTS $\mathcal{M}^*$. The identical embedding of $\mathcal{M}^*$ into $\mathcal{M}$ is a t-morphism. In other words, every history of $\mathcal{M}^*$ is a history of $\mathcal{M}$. If $\mathcal{M}$ is seriously branching, then so is $\mathcal{M}^*$.*

Proof. Consider any JBTS $\mathcal{M}$. Let $\mathfrak{M}$ be the class of all subsets M' of M such that (i) M' is closed under $\mathcal{M}$-infima and (ii) for any m, n_0, n_1, n_2 in M', if $m < n_0$, $m < n_1$, $m < n_2$, then there is a $p \in M'$, such that $m < p$ and (($p \leqslant n_0$ and $p \leqslant n_1$) or ($p \leqslant n_0$ and $p \leqslant n_2$) or ($p \leqslant n_1$ and $p \leqslant n_2$)). We employ the ordering $\subseteq$ on $\mathfrak{M}$.

(A) Consider any M' in $\mathfrak{M}$. Let h be a maximal linear sub-set of M'. Note that h may have a maximum, since M' need not be serial. Suppose m in M is strictly above all elements of h. We claim that $M' \cup \{m\}$ is in $\mathfrak{M}$.

We check (i). The only interesting case is the infimum n of m and m', for $m' \in M'$. Suppose $n \leqslant p$, for some $p \in h$. It follows that $n = \mathsf{inf}(p, m')$ and, hence

$n \in M'$. Suppose $n \nleq p$ for all $p \in h$. Since $n \leqslant m$ and m majorizes h, it follows that $m' \geqslant n > p$, for all $p \in h$, contradicting the maximality of h.

We check (ii). We note that we cannot have $m < q \in M'$, by the maximality of h. So, there is just one interesting case to consider. Suppose $m' < m$, $m' < n_0$, $m' < n_1$, for $m' \in M'$, $n_0 \in M'$, $n_1 \in M'$. As is easily seen $m' \in h$. Moreover m' cannot be maximal in h, since $m' < n_0 \in M'$. Let $m' < m'' \in h$. We may apply (ii) for M' to m', m'', n_0, n_1 to find a $p \in M'$ such that $m' < p$ and $((p \leqslant m'' < m$ and $p \leqslant n_0)$ or $(p \leqslant m'' < m$ and $p \leqslant n_1)$ or $(p \leqslant n_0$ and $p \leqslant n_1))$. So, we are done.

(B) Consider any M' in $\mathfrak{M}$. Suppose $m < m'$ for $m' \in M'$. We claim that $M' \cup \{m\}$ is in $\mathfrak{M}$.

We check (i). The only interesting case is the infimum of m and $n \in M'$. Consider the infimum p of m' and n. Clearly, p is in M' and is comparable with m. If $p \leqslant m$, then p is the infimum of m and n. If $m < p$, then $m < n$ and hence m is the infimum of m and n.

We check (ii). There are two interesting cases.

(a) Suppose $m'' \in M'$, $n_0 \in M'$, $n_1 \in M'$ and $m'' < m$, $m'' < n_0$, $m'' < n_1$. It follows that $m'' < m'$, we apply (ii) for M' to m'', m', n_0, n_1. We find a $p \in M'$ such that $m'' < p$ and $((p \leqslant m'$ and $p \leqslant n_0)$ or $(p \leqslant m'$ and $p \leqslant n_1)$ or $(p \leqslant n_0$ and $p \leqslant n_1))$. If the third disjunct holds we can choose p as our witness. Suppose one the the first two disjuncts holds. It follows that $p \leqslant m'$. Hence, p must be comparable with m. In case $p \leqslant m$, we can choose p as our witness of (ii) for $M' \cup \{m\}$. Otherwise, we can choose m itself.

(b) Suppose n_0, n_1, n_2 are in M' and $m < n_0$, $m < n_1$, $m < n_2$. It follows that $m < \mathsf{inf}(n_0, n_1, n_2) =: p \in M'$. We can now take p as the witness for (ii) in $M' \cup \{m\}$.

(C) Suppose $m' \in M'$, $m'' \in M'$ and $m' < m''$. Suppose m' is a branching point in M, but not in M'. It follows that there is an $m \in M$ with $m' < m$ and $m' = \mathsf{inf}(m, m'')$. We claim that $M' \cup \{m\}$ is in $\mathfrak{M}$.

We check (i). The only interesting case is the infimum p of m and n for $n \in M'$. We find that p is comparable to m'. In case $p \leqslant m'$, clearly p is the infimum of m' and n, so $p \in M'$. In case $p > m'$, then m' is the infimum of m'' and p and hence of m'' and n. Moreover, $n > m'$. This is impossible since m' was not a branching point in M'.

We check (ii). The only interesting case is as follows. Suppose $m^\circ \in M'$, $n_0 \in M'$, $n_1 \in M'$ and $m^\circ < m$, $m^\circ < n_0$, $m^\circ < n_1$. We find that m° is comparable to m'. In case $m^\circ < m'$, we can apply (ii) for M' to m°, m', n_0 and n_1. Suppose $m' \leqslant m^\circ$. In this case we must have $m' = m^\circ$ since otherwise m' would be a branching point in M' as witnessed by m'' and m°. Again using the fact that m' is not a branching point in M' we find a $p > m' = m^\circ$ such that $p \leqslant n_0$ and $p \leqslant n_1$.

It is easy to see that the union of a chain of elements of $\mathfrak{M}$ is again in $\mathfrak{M}$. Thus, we may apply Zorn's Lemma to obtain a maximal element M^* of $\mathfrak{M}$.

By (A) it follows that M^* is serial: if there were a maximal element m^*, then $h := \{n \in M^* \mid n \leqslant m^*\}$ would be a maximal linear ordering in M^*. So we could add an element above h to obtain a larger set in $\mathfrak{M}$. Quod non. Thus, M^* is a $\leqslant 2$-branching JBTS.

By (B) it follows that M^* is downward closed in M. Also, by (A) and downward closure, any history h in M^* is a history in M.

Finally, suppose M is seriously branching. By (C) any m in M^* that is a branching point in M is also a branching point in M^*. It follows that M^* is also seriously branching. ❑

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