On a modal logic of intuitionistic admissibility

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In [1], Mints constructed full semantics for the Lemmon [4] calculus fragment of S0.5 consisting of formulae without iterated modalities. More precisely, propositional formulae which are built from constants $\top$, $\bot$ and $\&$, $\vee$, $\neg$, $\Box$ were considered in [1]. Semantics in [1] is the following: the Lindenbaum algebra of classical logic is extended with $\Box$ s.t. $\Box A = \top$ iff A is a tautology, in any other case $\Box A = \bot$.

We will consider intuitionistic modal logic with semantics similar to the one that was suggested by Mints in [4].

We will consider propositional formulae which are constructed with $\&$, $\vee$, $\supset$, $\neg$. Let $\mathcal{F}$ be a Lindenbaum algebra of IPC. Let 1 be the largest element in $\mathcal{F}$ and 0 be the least element. And extend the operation:

$$\Box x = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

Let the obtained algebra be $\mathcal{F}^{\Box}$. Consider the logic $\mathcal{LF}^{\Box}$ of algebra $\mathcal{F}^{\Box}$, which consists of all formulae constructed with $\&$, $\vee$, $\supset$, $\neg$ and $\Box$, and that are true in $\mathcal{F}^{\Box}$.

In particular, the following formulae are true in $\mathcal{F}^{\Box}$:

$$(\Box\alpha \supset \alpha) \tag{1}$$

$$(\Box\alpha \supset \Box\Box\alpha) \tag{2}$$

$$(\Box(\alpha \supset \beta) \supset (\Box\alpha \supset \Box\beta)) \tag{3}$$

$$(\neg\Box\neg\Box\alpha \supset \Box\alpha) \tag{4}$$

$$(\Box(\alpha \vee \beta) \supset (\Box\alpha \vee \Box\beta)) \tag{5}$$

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Say, some formula $(\Box A \supset \Box B)$ is true in $\mathcal{LF}^{\Box}$, where A and B are propositional formulae. This means that as soon as we substitute propositional variables in A and obtained formula is true in the IPC, B will be also true under the same substitution. Therefore, the rule

$$\frac{A}{B}$$

is admissible in the IPC.

Consider the calculus $I_0^{\Box}$ with axioms (1)–(5) and inference rules: modus ponens: $\dfrac{x \quad x \supset y}{y}$; necessitation rule: $\dfrac{x}{\Box x}$.

Logic $\mathcal{LF}^{\Box}$ is an extension of logic $\mathcal{LI}_0^{\Box}$. Let

$$A_n = ((\alpha_1 \supset \beta_1)\& \ldots \&(\alpha_n \supset \beta_n)),$$

where $n = 1, 2, \ldots$. The following formulae are true in $\mathcal{F}^{\Box}$ but not derivable in $I_0^{\Box}$:

$$(\Box(A_n \supset (\gamma \vee \delta)) \supset \Box(\bigvee_{i=1}^{n}(A_n \supset \alpha_i) \vee (A_n \supset \gamma) \vee (A_n \supset \delta))).$$

Theorem 1. *The lattice of all extensions of logic $\mathcal{LI}_0^{\Box}$ is a pseudo boolean algebra, and all finitely axiomatizable extensions of $\mathcal{LI}_0^{\Box}$ form a pseudo-boolean sublattice of this lattice.*

The proof is the same as for the lattice of normal extensions of S4.

Theorem 2. *There exists an algorithm which for any formula $\Box A$ constructs deductively equivalent (in $I_0^{\Box}$ calculus) formula:*

$$\bigwedge_{i=1}^{n}(\Box A_i \supset \Box B_i)$$

where A_i, B_i are suitable formulae.

Therefore, any formula is deductively equivalent (i.e. mutually derivable in $I_0^{\Box}$) to a formula without iterated modalities.

In 1974, Kuznetsov formulated the problem (it is still unsolved) that, in terms of logic, $\mathcal{LF}^{\Box}$ is formulated as follows: Is it possible to describe the logic $\mathcal{LF}^\Box$ via a calculus? The similar question (also unsolved) formulated by Friedman [3] as problem 40, which in terms of logic $\mathcal{LF}^\Box$ can be formulated as: Is the logic $\mathcal{LF}^\Box$ decidable?

Note that if the logic $\mathcal{LF}^\Box$ can be given by a calculus, then it is decidable, since one can easily construct an algorithm which allows to check refutability of the formula $\Box A \supset \Box B$ in $\mathcal{F}^\Box$.

It is possible to construct the full algebraic semantics for the calculus of $I_0^\Box$. While the logic $\mathcal{LF}^\Box$ is not finitely approximable.

A partial solution to the Friedman problem is given by the following theorems.

Theorem 3. *If formula $\Box A \supset \Box B$ does not have negative occurrences of disjunction (positive occurrence of implication) and is valid in $\mathcal{LF}^\Box$, then formula $(A \supset B)$ is also valid in IPC.*

Thus, if formula $(\Box A \supset \Box B)$ satisfies conditions of theorem 3, then it is valid in $\mathcal{LF}^\Box$ iff when formula $(A \supset B)$ is valid in IPC.

Theorem 4. *There exists an algorithm which for any formula $(\Box A \supset \Box B)$ s.t. A is monotone in each variable, recognizes whether $(\Box A \supset \Box B)$ is valid in $\mathcal{LF}^\Box$.*

References

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