On the recognition of admissibility of some rules in intuitionistic logic

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April 1, 2015

A rule is called a modus rule [1] if it corresponds to the following sequent:

$$A_1, \ldots, A_n \to B \tag{1}$$

and it allows to derive the result of a substitution in B from the results of the same substitution in $A_1, \ldots, A_n$. Let

$$C_n = \bigwedge_{i=1}^{n} (p_i \supset q_i),$$

where p_i, q_i are propositional variables, $n = 1, 2, \ldots$.

The rules r_n corresponding to the sequents:

$$(C_n \supset (s \vee t)) \to (\bigvee_{i=1}^{n} (C_n \supset p_j) \vee (C_n \supset s) \vee (C_n \supset t)),$$

are admissible in the propositional intuitionistic logic (IPC). These rules are generalisations of Mints' rule [1, 2].

We say that formula B occurs in formula A positively (negatively) if $\vdash (B \supset C)$ implies $\vdash (A \supset A')$ (respectively $\vdash (A' \supset A)$) where A' is obtained from A by replacing this occurrence of B by C. An occurrence of a variable in a formula is monotone if all its occurrences are of the "same sign".

Theorem 1. *There exists an algorithm which, given a rule corresponding to* (1) *such that every variable of the formula* $A_1 \wedge \ldots \wedge A_n$ *occurs monotonely, determines whether this rule is admissible in* IPC.

*Originally appeared as: А. И. Циткин, "О Проверке Допустимости Некоторых Правил Интуиционистской Логики", Пятая всесоюзная конференция по математической логике (1979), Новосибирск, p. 162. Translation from the Russian by Kristina Gogoladze.

All admissible modus rules of IPC satisfying the above condition are derivable from the rule system $\{r_n \mid n = 1, 2, \ldots\}$. Let r be an admissible rule corresponding to the sequent (1).

Corollary 2 (comp. [2]). *If a formula $(A_1 \wedge \ldots \wedge A_n)$ does not contain positive occurrences of disjunction (negative occurrence of implication), then r is derivable in* IPC*, i.e.*

$$\vdash (A_1, \ldots, A_n) \supset B$$

References

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