Uniform interpolation and sequent calculi in modal logic

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Abstract

A method is presented that connects the existence of uniform interpolants to the existence of certain sequent calculi. This method is applied to several modal logics and is shown to cover known results from the literature, such as the existence of uniform interpolants for the modal logic K. The results imply that for modal logics $\mathsf{K4}$ and $\mathsf{S4}$, which are known not to have uniform interpolation, certain sequent calculi cannot exist.

Keywords: uniform interpolation, sequent calculus, modal logic, propositional quantifiers

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1 Introduction

Interpolation has been studied in a variety of settings since William Craig proved that classical predicate logic has interpolation in 1957. Interpolation is considered by many to be a "good" property for a logic to have because it indicates that the logic is well-behaved in a sense vaguely reminiscent to analycity: if an implication $\varphi \to \psi$ holds in the logic, then there is a χ in the common language of φ and ψ that *interpolates*, that is, such that $\varphi \to \chi$ and $\chi \to \psi$ hold. What the common language is depends on the logic one considers. In propositional logics it typically means that all atoms in χ occur in both φ and ψ.

In 1992 it was proved by Andrew Pitts that intuitionistic propositional logic IPC, which has interpolation, also satisfies the stronger property of *uniform interpolation*: given a formula φ and an atom p, there exist *uniform interpolants* $\forall p\varphi$ and $\exists p\varphi$ which are formulas (in the language of IPC) that do not contain p and such that for all ψ not containing p:

$$\vdash \varphi \to \psi \;\Leftrightarrow \vdash \exists p\varphi \to \psi \qquad \vdash \psi \to \varphi \;\Leftrightarrow \vdash \psi \to \forall p\varphi.$$

This is a strengthening of interpolation in which the interpolant only depends on the premiss (in the case of $\exists$) or the conclusion (in the case of $\forall$) of the given implication: $\forall p_1 \dots \forall p_n\varphi$ interpolates any $\psi \to \varphi$ and $\exists p_1 \dots \exists p_n\varphi$ interpolates

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any $\varphi \to \psi$ such that $p_1, \ldots, p_n$ do not occur in ψ. In case $\forall p_1 \ldots \forall p_n \varphi$ or $\exists p_1 \ldots \exists p_n \varphi$ contain additional variables not in the common language of φ and ψ, these have to be replaced by $\top$ or $\bot$ or a variable in the common language.

As the notation suggests, the fact that the uniform interpolants are definable in IPC also shows that the propositional quantifiers are definable in that logic. From the algebraic point of view the quantifiers are left and right adjoints of certain embeddings.

Around the time that Pitts obtained his result, Shavrukov (1993) proved, by completely different methods, that the modal logic GL has uniform interpolation. Since then, uniform interpolation has been established for various other logics, including the modal logics K and KT (Bílková, 2007; Visser, 1996a,b). Intriguingly, the modal logics $\mathsf{K4}$ and $\mathsf{S4}$ do not have uniform interpolation (Bílková, 2007; Ghilardi and Zawadowski, 1995). As there are only seven propositional intermediate logics with interpolation (Maxsimova, 1977), the number of intermediate logics with uniform interpolation is necessarily bounded by that number. Ghilardi and Zawadowski (2002) showed that there are exactly that many.

Whereas in the presence of a decent analytic sequent calculus, proofs of interpolation are often relatively straightforward, proofs of uniform interpolation are in general quite complex. Moreover, it is less clear in how far, if at all, proof systems such as sequent calculi can be of help in establishing the property, as there are logics with analytic sequent calculi that have uniform interpolation (K and GL) as well as logics with analytic sequent calculi that do not ($\mathsf{K4}$ and $\mathsf{S4}$).

In this paper our aim is twofold: to develop a method to extract uniform interpolants from sequent calculi and to prove, using this method, that logics without uniform interpolation lack certain calculi. For both aims it holds that the more general the calculi we consider are, the stronger the result. In this paper we restrict ourselves to classical propositional modal logics, but the method applies to intermediate logics as well. For such logics the method is more complicated though, since $\exists$ is not expressible in terms of $\forall$, whereas in the classical case one can just take $\neg\forall p \neg$ for $\exists p$. We treat intermediate logics in a seperate paper.

Because of the way in which we construct uniform interpolants from calculi, we achieve something else as well. Namely that of providing a modular approach to uniform interpolation, meaning that the relation between a particular rule in a calculus and the property of uniform interpolation of the whole calculus is clarified. Our method is different from but inspired by Pitts' ingenious syntactic method. Bílková (2007) used a similar method as Pitts for K, GL, KT and Grz. Most other proofs of uniform interpolation are of a semantical nature.

We isolate a certain type of propositional rules called *focussed rules* and a certain type of modal rules called *focussed modal rules* and prove that any logic with a terminating balanced sequent calculus consisting of focussed and focussed modal rules has uniform interpolation. Termination means that in no rule the premisses are more complex, in a certain ordering, than the conclusion. And a calculus is balanced if for certain combinations of left and right rules, either both rules belong to the calculus or both do not. This result then implies the well-known fact that classical propositional logic has uniform interpolation, and that so do K and KD. It also implies that $\mathsf{K4}$ and $\mathsf{S4}$ cannot have sequent calculi of the above kind. Although for $\mathsf{S4}$ this might be easy to infer in another way, for $\mathsf{K4}$ this seems to be a novel insight.

Furthermore, uniform interpolation is obtained for various other modal logics. The main interest in these results lies not so much in the logics involved, but rather in the illustration they provide of the flexibility of the method developed here. The calculi covered in this paper are not the only calculi to which our method applies, or so we conjecture. It seems likely that similar reasoning applies to other calculi for modal and intermediate logics. We chose, however, to first set up the general framework in this paper, mainly because we think it is of interest in itself and to separate it from the complexities that might be uncovered in applying it to other calculi than the ones treated here.

2 Logics and calculi

The logics we consider are modal propositional logics, formulated in a language $\mathcal{L}$ that contains constants $\top$ and $\bot$, propositional variables or atoms $p, q, r, \dots$ and the connectives $\wedge, \vee, \neg, \rightarrow$ and the modal operator $\Box$. We assume that all logics we consider are extensions of classical propositional logic CPC and satisfy the necessitation rule, but we do not assume them to be normal. The logics are given by consequence relations denoted by $\vdash$ or $\vdash_{\mathsf{L}}$. $\mathcal{F}$ denotes the set of formulas in $\mathcal{L}$ and $\mathcal{M}$ is the set of all finite multisets of formulas in $\mathcal{F}$. Given a set of atoms $\mathcal{P}$, $\mathcal{F}(\mathcal{P})$ denotes all formulas in $\mathcal{L}$ in which all atoms belong to $\mathcal{P}$. Language $\mathcal{L}^2$ is the extension of $\mathcal{L}$ with propositional quantifiers $\forall p$ and $\exists p$ for every atom p, and $\mathcal{F}^2$ is the set of formulas in that language. The set of formulas $\mathcal{F}^1$ consists of those formulas in $\mathcal{F}^2$ without nested quantifiers.

We mainly consider sequents, which are expressions $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas in $\mathcal{L}$ or $\mathcal{L}^2$ (depending on the context), that are interpreted as $I(\Gamma \Rightarrow \Delta) = (\bigwedge \Gamma \rightarrow \bigvee \Delta)$. We denote finite multisets by $\Gamma, \Pi, \Delta, \Sigma$. In a sequent, notation Π, Γ is short for $\Gamma \cup \Pi$. We also define (a for antecedent, s for succedent):

$$(\Gamma \Rightarrow \Delta)^a \equiv_{def} \Gamma \qquad (\Gamma \Rightarrow \Delta)^s \equiv_{def} \Delta.$$

When sequents are used in the setting of formulas, we often write S for $I(S)$, such as in $\vdash \bigvee_i (S_i \Rightarrow S)$, which thus means $\vdash \bigvee_i \big(I(S_i) \rightarrow I(S)\big)$. Multiplication of sequents is defined as

$$S_1 \cdot S_2 \equiv_{def} (S_1^a \cup S_2^a \Rightarrow S_1^s \cup S_2^s).$$

2.1 Rules and axioms

For a proper syntactic treatment of interpolation we need to make a distinction between the language and the meta-language. $\overline{\mathcal{L}}$ is a copy of $\mathcal{L}$ in which every atom p is replaced by $\overline{p}$. The set $\overline{\mathcal{F}}$ of formulas in this language is defined as usual. $\overline{\mathcal{M}}$ is an infinite set of symbols for *meta-multisets*, the elements we denote by $\overline{\Gamma}, \overline{\Pi}, \overline{\Delta}, \overline{\Sigma}$. A *meta-sequent* is an expression $X \Rightarrow Y$, where X and Y are multisets consisting of elements in $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$.

A *substitution* σ is a map from $\overline{\mathcal{F}} \cup \overline{\mathcal{M}}$ to $\mathcal{F} \cup \mathcal{M}$ that commutes with the connectives and modal operator and such that $\sigma[\overline{\mathcal{F}}] \subseteq \mathcal{F}$ and $\sigma[\overline{\mathcal{M}}] \subseteq \mathcal{M}$. *Sub* is the set of all substitutions.

A *sequent calculus* is a set of *axioms* and *rules*, where the former are meta-sequents and the latter are expressions of the form

$$\frac{S_1 \quad S_2 \quad \dots \quad S_n}{S_0}\ \mathcal{R} \tag{1}$$

for some meta-sequents $S_0, S_1, \dots, S_n$. For any substitution σ, the inference

$$\frac{\sigma S_1 \quad \sigma S_2 \quad \dots \quad \sigma S_n}{\sigma S_0}\ \sigma\mathcal{R}$$

is an *instance* of $\mathcal{R}$. Throughout this paper we denote schematic rules by $\mathcal{R}$ and instances of rules by R. Sets of rules are denoted by $\mathcal{R}$, and $\mathcal{R}_{\mathsf{ins}}$ denotes the set of instances of rules in $\mathcal{R}$. We use the same symbols for axioms but will always indicate whether a symbol stands for a rule or an axiom.

As is often done implicitly in papers on sequent calculi, we will from now on confuse the meta-level with the object-level by omitting overscores and the word "meta", trusting that it will always be clear from the context (or does not matter) on which level we are. For example, an axiom such as $\overline{\Gamma}, \overline{p} \Rightarrow \overline{p}, \overline{\Delta}$ will simply be written as $\Gamma, p \Rightarrow p, \Delta$.

For $\mathcal{R}$ as in (1) and for any meta-sequent S, $\mathcal{R}(S)$ denotes the rule

$$\frac{S \cdot S_1 \quad S \cdot S_2 \quad \dots \quad S \cdot S_n}{S \cdot S_0}\ \mathcal{R}(S)$$

For instances R and sequents S, $R(S)$ is defined similarly.

A rule is *backwards applicable* to a sequent S when there is at least one instance of the rule with S as the conclusion. An instance of a rule is *backwards applicable* to S if its conclusion is S. A sequent is *free* if it is not the conclusion of any instance of any rule and it is not an instance of any axiom.

2.2 Focussed rules

A rule $\mathcal{R}$ is *focussed* if there are meta-sequents $S_1, \dots, S_n$, a meta-sequent S_0 consisting of exactly one meta-formula, and a meta-sequent $S = (\Gamma \Rightarrow \Delta)$ for two distinct meta-multisets Γ and Δ that do not occur in the S_i, such that $\mathcal{R}$ is the rule

$$\frac{S \cdot S_1 \quad S \cdot S_2 \quad \dots \quad S \cdot S_n}{S \cdot S_0}\ \mathcal{R}$$

The rule is a *right rule* in case S_0^a is empty and a *left rule* otherwise.

An axiom is *focussed* if it is either of the form $(\Gamma, r \Rightarrow r, \Delta)$ or $(\Gamma, \bot \Rightarrow \Delta)$ or $(\Gamma \Rightarrow \top, \Delta)$. A calculus is *focussed* if every rule and every axiom in it is focussed.

Typical focussed rules are the left and right rules of many Gentzen calculi. The right conjunction rule

$$\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta} \tag{2}$$

is clearly focussed, as one can take $(\Gamma \Rightarrow \Delta)$ for S, $(\Rightarrow \varphi \wedge \psi)$ for S_0 and $(\Rightarrow \varphi)$ and $(\Rightarrow \psi)$ for S_1 and S_2, respectively.

2.3 Terminating calculi

A calculus is defined to be a collection of axioms and rules. It is *terminating* if it is finite and for all sequents S and rules in the calculus there are at most finitely many instances of the rule backwards applicable to S, and there is a well-founded order $\prec$ on sequents such that in every rule of the calculus the premisses come before the conclusion in this order and all proper subsequents of a sequent come before that sequent.

A typical example of a rule that in general cannot belong to a terminating calculus is the cut rule, as there can be infinitely many instances of the rule with the same conclusion. We will see that many standard cut-free calculi for modal logic are terminating.

Given the order $\prec$ we define another order, the rank, on sequents in $\mathcal{L}^1$ as follows. The *rank* of S is n if there occurs a formula of the form $\forall p\varphi$ in S such that φ has complexity (number of connectives) n and for all other quantifier occurrences of $\forall q\psi$ in S, the complexity of ψ is not higher than that of φ.

2.4 Partitions

The method to construct uniform interpolants that we are going to develop uses partitions of sequents and rules. A *partition* of a sequent S is an ordered pair (S^r, S^i) (i for interpolant, r for rest) such that $S = S^r \cdot S^i$, where $=$ denotes equality on multisets. It is a *p-partition* if p does not occur in S^r. It is *trivial* if S^r is the empty sequent, and whence $S^i = S$.

A *partition* $\mathsf{pt}(\mathcal{R})$ of a rule

$$\frac{S_1 \quad \dots \quad S_n}{S_0}\ \mathcal{R}$$

is an expression of the form

$$\frac{(S_1^r, S_1^i) \quad \dots \quad (S_n^r, S_n^i)}{(S_0^r, S_0^i)}\ \mathsf{pt}(\mathcal{R})$$

where (S_j^r, S_j^i) is a partition of S_j. The partitions $(S_1^r, S_1^i), \dots, (S_n^r, S_n^i)$ of the premisses of $\mathcal{R}$ are said to $\mathcal{R}$-*correspond* to (S_0^r, S_0^i) under pt. For $\star \in \{i, r\}$ the rule $\mathcal{R}^\star$ is defined as

$$\frac{S_1^\star \quad \dots \quad S_n^\star}{S_0^\star}\ \mathcal{R}^\star$$

A *partition* of a calculus consists of a set of partitions of its rules, containing, for every rule $\mathcal{R}$ and every partition (S_0^r, S_0^i) of the conclusion of $\mathcal{R}$, a partition $\mathsf{pt}(\mathcal{R})$ of $\mathcal{R}$ with conclusion (S_0^r, S_0^i). A partition of a calculus is extended to instances of its rules by letting the partition of an instance $\sigma\mathcal{R}$ of rule $\mathcal{R}$ correspond to the partition of $\mathcal{R}$ by defining $\mathsf{pt}(\sigma\mathcal{R})$ to be

$$\frac{(\sigma S_1^r, \sigma S_1^i) \quad \dots \quad (\sigma S_n^r, \sigma S_n^i)}{(\sigma S_0^r, \sigma S_0^i)}\ \mathsf{pt}(\sigma\mathcal{R})$$

3 Uniform interpolants

Let $\mathcal{R}$ be a calculus. Recall that $\mathcal{R}_{\mathsf{ins}}$ denotes the set of instances of rules in $\mathcal{R}$. A *universal interpolant assignment* for $\mathcal{R}$, assigns, for every atom p, sequent S in $\mathcal{L}$ and for every R being either $\varnothing$ or an instance of a rule in $\mathcal{R}$ backwards applicable to S, to the expression $\forall\overset{R}{p}S$ a formula in $\mathcal{L}^2$ that is of lower rank than S. For free sequents S the interpolant assignment assigns formulas in $\mathcal{L}^2$ of lower rank than S to expressions $\forall\overset{\varnothing}{p}S$.

If S' is the sequent chosen for $\forall\overset{R}{p}S$, then we write $\forall\overset{R}{p}S \sim_0 S'$. We define an equivalence $\sim$ on formulas in $\mathcal{L}^2$ that is the smallest extension of $\sim_0$ that commutes with the connectives and modal operator and furthermore satisfies:

$$\forall pS \quad \sim \quad \begin{cases} \forall\overset{\varnothing}{p}S & \text{if } S \text{ is free} \\ \bigvee\{\forall\overset{R}{p}S \mid R \in \mathcal{R}_{\mathsf{ins}} \text{ backwards applicable to } S\} & \text{if } S \text{ is not free} \end{cases}$$

Observe that there could be more than one instance of a single rule R which has S as a conclusion, in which case every instance corresponds to a separate disjunct of the interpolant.

Given atoms $p_1 \dots p_n$, the *universal uniform interpolant of S with respect to* $p_1 \dots p_n$ is $\forall p_1 \forall p_2 \dots \forall p_n S$, which is abbreviated as $\forall p_1 \dots p_n S$. We will usually omit the word *universal*.

Remark 3.1 Recall that sequents consist of formulas in $\mathcal{L}$. Because of the definition of universal interpolant assignments, for every sequent S minimal in order $\prec$, $\forall pS$ is a formula in $\mathcal{L}$. Therefore one can show by induction along $\prec$ that for every sequent S and atoms $p_1 \dots p_n$ there exists a formula in $\mathcal{L}$ denoted by $\ulcorner \forall p_1 \dots p_n S \urcorner$ such that $\forall p_1 \dots p_n S \sim \ulcorner \forall p_1 \dots p_n S \urcorner$.

Example 3.2 Suppose the calculus only contains the rule $\mathcal{R}$ for conjunction on the right as given in (2) and there exists an ordering in which sequents with more connectives come after sequents with less connectives. Let S be $(\Rightarrow \varphi_1 \wedge \psi_1, \varphi_2 \wedge \psi_2)$ and R_i stand for the instance of $\mathcal{R}$ with $\varphi_i \wedge \psi_i$ as the main formula. By the above definition,

$$\forall pS \sim \forall\overset{R_1}{p}S \vee \forall\overset{R_2}{p}S$$

Let $S_{\varphi_1} = (\Rightarrow \varphi_1, \varphi_2 \wedge \psi_2)$ and $S_{\psi_1} = (\Rightarrow \psi_1, \varphi_2 \wedge \psi_2)$, and similarly for S_{φ_2} and S_{ψ_2}. The standard interpolant assignment introduced below, assigns $\forall pS_{\varphi_1} \wedge \forall pS_{\psi_1}$ to $\forall\overset{R_1}{p}S$ and $\forall pS_{\varphi_2} \wedge \forall pS_{\psi_1}$ to $\forall\overset{R_2}{p}S$. This implies that

$$\forall pS \sim (\forall pS_{\varphi_1} \wedge \forall pS_{\psi_1}) \vee (\forall pS_{\varphi_2} \wedge \forall pS_{\psi_2}).$$

3.3 The interpolant properties

Given a formula φ, its *universal uniform interpolant with respect to* $p_1 \dots p_n$ is $\forall p_1 \dots p_n (\Rightarrow \varphi)$, which we write as $\forall p_1 \dots p_n \varphi$, and similarly for $\exists$. $\forall p\varphi$ and $\exists p\varphi$ are *uniform interpolants* if for any set of atoms $\mathcal{P}$ not containing p, the embedding of $\mathcal{F}(\mathcal{P})$ into $\mathcal{F}(\mathcal{P} \cup \{p\})$ has a right and a left adjoint: for all ψ not containing p and for all φ,

$$\vdash \psi \to \varphi \;\Leftrightarrow \vdash \psi \to \forall p\varphi \qquad \vdash \varphi \to \psi \;\Leftrightarrow \vdash \exists p\varphi \to \psi.$$

As we only consider classical logics we can take $\neg\forall\neg$ for $\exists$ and the above is equivalent to

$$\vdash \forall p\varphi \to \psi \qquad \vdash \psi \to \varphi \;\Rightarrow \vdash \psi \to \forall p\varphi. \tag{$\forall$}$$

As will be shown in Lemma 3.4, in the approach via sequents, ($\forall$) is replaced by the following requirements, the *interpolant properties*.

($\forall$l) for all p: $\vdash S^a, \forall p S \Rightarrow S^s$;

($\forall$r) for all p: $\vdash S^r \cdot (\Rightarrow \forall p S^i)$ if S is derivable and S^r does not contain p.

Property ($\forall$l) is the *independent* (from partitions) *interpolant property*, and ($\forall$r) is the *dependent interpolant property*. For every such property and any $R \in \mathcal{R}_{\mathsf{ins}} \cup \{\varnothing\}$, the R-*variant* is the result of replacing quantifier $\forall p$ in the property by $\overset{R}{\forall p}$. A partition (S^r, S^i) of S *satisfies* the interpolant properties if, in the case of the independent property, S satisfies it (in which case we also say that S satisfies it), and in case of the dependent property, it holds for that particular partition. A sequent *satisfies* a property if every possible partition of the sequent satisfies it. It is not hard to see that the following holds.

Lemma 3.4 If all sequents satisfy the interpolant properties, then L has uniform interpolation.

Proof We have to show that ($\forall$) holds. Consider $S = (\Rightarrow \varphi)$. Hence $\forall p\varphi = \forall p S$ by definition. By ($\forall$l), $\vdash I(\forall p\varphi \Rightarrow \varphi)$, that is, $\forall p\varphi \to \varphi$ is derivable.
Consider a ψ not containing p such that $\vdash \psi \to \varphi$. Let $S = (\psi \Rightarrow \varphi)$ and consider the partition (S^r, S^i), where $S^r = (\psi \Rightarrow)$ and $S^i = (\Rightarrow \varphi)$. Hence $\forall p\varphi = \forall p S^i$ by definition. And $\vdash (\psi \Rightarrow) \cdot (\Rightarrow \forall p\varphi)$ by ($\forall$r). Therefore $(\psi \to \forall p\varphi)$ is derivable. □

Fact 3.5 All free sequents satisfy the dependent interpolant properties.

3.6 Soundness

Given a calculus that contains a rule $\mathcal{R}$, an interpolant assignment is $\mathcal{R}$-*sound* or *sound with respect to* $\mathcal{R}$ if for every sequent S the following holds under the assumption that all sequents lower than S satisfy the interpolant properties.

(IP) For all instances R of $\mathcal{R}$ with conclusion S, S satisfies the R-variant of the independent interpolant property.

(DP) If S has a derivation which last inference is an instance of $\mathcal{R}$, then for every partition (S^r, S^i) of S such that S^i is not empty, the dependent interpolant property holds.

((IP) stands for *independent property*, (DP) for *dependent property*.)
An interpolant assignment is *sound with respect to free sequents* if for all free sequents S the following holds.

(FP) If all sequents lower than S satisfy the interpolant properties, then S satisfies the independent interpolant property.

We say that an interpolant assignment is *sound* for a given calculus if it is sound with respect to every rule of the calculus and with respect to free sequents.

Remark 3.7 Observe that the independent interpolant property holds for a sequent S if and only if the R-variant of the independent interpolant property holds for all R backwards applicable to S. If the R-variant of the dependent interpolant property holds for partition (S^r, S^i) for at least one R that is backwards applicable to S^i, then the property holds for (S^r, S^i).

The formulation of (FP) might suggest that one can always take $\bot$ for the interpolant assignment of free sequents since for this choice the independent interpolant property clearly holds. Free sequents do, however, also play a role in the dependent interpolant property, because the partition of a sequent might be such that S^i is a free sequent.

Lemma 3.8 If a logic L has a terminating calculus for which there exists a sound interpolant assignment, then all sequents satisfy the interpolant properties.

Proof We have to show that under the assumptions in the lemma, the interpolant properties hold for any partition (S^r, S^i) of any sequent S.

Let $\prec$ be the well-founded order on sequents such that in every rule of the calculus the premisses come before the conclusion in this order, and all proper subsequents of a sequent come before that sequent. We use induction to $\prec$ on S.

For sequents S minimal in the order, (IP) holds and (DP) holds if S is derivable. By Remark 3.7 this implies that the independent interpolant property holds for S. That the dependent interpolant property holds for (S^r, S^i) in case S is derivable follows immediately from (DP).

For the induction step assume that all sequents lower than S satisfy the interpolant properties. If S is free, then it satisfies the interpolant properties by (FP). Therefore assume it is not free. That the independent interpolant property holds for S follows from (IP) by Remark 3.7. That (S^r, S^i) satisfies the dependent property in case S is derivable follows immediately if S^i is empty (and thus $S^r = S$), and from (DP) and Remark 3.7 otherwise.

$\Box$

Theorem 3.9 If a logic $\mathsf{L} \supseteq \mathsf{CPC}$ has a terminating calculus for which there exists a sound interpolant assignment, then L has uniform interpolation.

Proof This follows from Lemmas 3.4 and Lemma 3.8. $\Box$

3.10 A sufficient condition for soundness

In this section a sufficient condition for the soundness of a rule is provided. For this we need the notion of an *interpol*, which is a set $\mathcal{S}$ of sequents such that for every $S \in \mathcal{S}$ there is a derivable sequent S_d and a partition (S_d^r, S_d^i) of S_d such that $S = S_d^r \cdot (\Rightarrow \forall p S_d^i)$. We write $\mathcal{S} \prec S'$ if $S_d \prec S'$ for all $S \in \mathcal{S}$.

Consider a calculus and let $\mathcal{R}$ denote a rule in the calculus. The following three properties of rules form, as we will see, the sufficient condition for a rule to be sound.

(IP$_\vdash$) $\vdash \bigwedge\{S_j \cdot (\forall p S_j \Rightarrow) \mid 1 \leq j \leq n\}$ implies $\vdash S_0 \cdot (\overset{R}{\forall p} S_0 \Rightarrow)$, for every instance $R = S_1 \dots S_n / S_0$ of $\mathcal{R}$.

(DPB$_\vdash$) For every instance $S_1 \dots S_n / S_0$ of $\mathcal{R}$ and every p–partition (S_0^r, S_0^i) of S_0 such that $\mathcal{R}$ is backwards applicable to S_0^i, there exists a partition of $\mathcal{R}$ such that $\vdash \bigwedge\{S_j^r \cdot (\Rightarrow \forall p S_j^i) \mid 1 \leq j \leq n\}$ implies $\vdash S_0^r \cdot (\Rightarrow \overset{R}{\forall p} S_0^i)$, where R is an instance of $\mathcal{R}$ with conclusion S_0^i.

(DPN$_\vdash$) For every instance $S_1 \dots S_n / S_0$ of $\mathcal{R}$ and every p–partition (S_0^r, S_0^i) of S_0 such that S_0^i is not empty and $\mathcal{R}$ is not backwards applicable to S_0^i, there exists an interpol $\mathcal{S} \prec S_0$, such that $\vdash \bigwedge \mathcal{S}$ implies $\vdash S_0^r \cdot (\Rightarrow \forall p S_0^i)$.

(B stands for *backwards applicable*, N for *not backwards applicable.*)

Lemma 3.11 In any terminating calculus, any rule $\mathcal{R}$ that satisfies (IP$_\vdash$), (DPB$_\vdash$) and (DPN$_\vdash$) is $\mathcal{R}$-sound.

Proof For any instance $R = (S_1 \dots S_n / S_0)$ of $\mathcal{R}$, we have to show that under the assumption that all sequents $\prec$-below S_0 satisfy the interpolant properties, for all p-partitions (S_0^r, S_0^i) of S_0:

(IP) S_0 satisfies the R-variant of the independent interpolant property;

(DP) if S_0 is derivable by a derivation which last inference is an instance of $\mathcal{R}$, then for every partition (S_0^r, S_0^i) such that S_0^i is not empty, the dependent interpolant property holds.

(IP) By assumption, $\vdash S_j^a, \forall p S_j \Rightarrow S_j^s$ for all $1 \leq j \leq n$. It now follows from (IP$_\vdash$) that $\vdash S_0^a, \overset{R}{\forall p} S_0 \Rightarrow S_0^s$.
(DP) If $\mathcal{R}$ is backwards applicable to S_0^i we conclude from (DPB$_\vdash$), the fact that the calculus is terminating and the assumption, that sequent $S_0^r \cdot (\Rightarrow \overset{R}{\forall p} S_0^i)$ is derivable. Thus so is $S_0^r \cdot (\Rightarrow \forall p S_0^i)$ by Remark 3.7. If $\mathcal{R}$ is not backwards applicable to S_0^i, there exists, by (DPN$_\vdash$), a set $\mathcal{S}$ of sequents of the form $S^r \cdot (\Rightarrow \forall p S^i)$ for some $S \prec S_0$, such that $\mathcal{S} \vdash S_0^r \cdot (\Rightarrow \forall p S_0^i)$. As $S \prec S_0$, it satisfies the interpolant properties. This implies that all sequents in $\mathcal{S}$ are derivable. Hence so is $S_0^r \cdot (\Rightarrow \forall p S_0^i)$. □
Lemmas 3.4, 3.8, and 3.11 lead to the following theorem.

Theorem 3.12 If a logic $\mathsf{L} \supseteq \mathsf{CPC}$ has a terminating calculus in which all rules satisfy (IP$_\vdash$), (DPB$_\vdash$) and (DPN$_\vdash$), then L has uniform interpolation.

4 Standard assignments for propositional logic

In this section we define a specific universal interpolant assignment for propositional logic inspired by Pitts' method that will be proved to be sound for focussed axioms and rules.
For an instance

$$\frac{S_1 \quad S_2 \quad \dots \quad S_n}{S_0} \; R$$

of a n-premiss rule the *standard universal interpolant assignment* is

$$\forall^R_p S_0 \sim_0 \forall p S_1 \wedge \forall p S_2 \wedge \ldots \wedge \forall p S_n.$$

If all instances of a schematic rule R have a standard interpolant assignment, then the interpolant assignment for R is *standard*.

For the focussed axiom $\Gamma, r \Rightarrow r, \Delta$ the *standard interpolant assignment* is defined as follows, depending on whether in the instance the main atom is p or an atom q different from p:

$$\forall^R_p(\Gamma, q \Rightarrow q, \Delta) \sim_0 \forall p(\Gamma \Rightarrow \Delta) \vee q \qquad \forall^R_p(\Gamma, p \Rightarrow p, \Delta) \sim_0 \top.$$

For the focussed axioms $(\Gamma, \bot \Rightarrow \Delta)$ and $(\Gamma \Rightarrow \top, \Delta)$ the *standard interpolant assignment* is defined as

$$\forall^R_p(\Gamma, \bot \Rightarrow \Delta) \sim_0 \top \qquad \forall^R_p(\Gamma \Rightarrow \top, \Delta) \sim_0 \top.$$

We fix a partitioning of instances R of focussed rules $\mathcal{R}$ with the property that if $\mathcal{R}$ is backwards applicable to $(S \cdot S_0)^i$, then R^i is an instance of $\mathcal{R}$ and if $\mathcal{R}$ is not backwards applicable to $(S \cdot S_0)^i$, then R^r is an instance of $\mathcal{R}$. Therefore we define a partition $\mathsf{pt}(R)$

$$\frac{((S \cdot S_1)^r, (S \cdot S_1)^i) \quad \ldots \quad ((S \cdot S_n)^r, (S \cdot S_n)^i)}{((S \cdot S_0)^r, (S \cdot S_0)^i)}\ \mathsf{pt}(R)$$

of an instance $(S \cdot S_1), \ldots, (S \cdot S_n)/(S \cdot S_0)$ of a focussed rule to be *standard* if for all $1 \leq j \leq n$:

$$(S \cdot S_j)^i = \begin{cases} (S^i \cdot S_j) & \text{if } (S \cdot S_0)^i = S^i \cdot S_0 \text{ for some } S^i \subseteq S \\ (S \cdot S_0)^i & \text{otherwise.} \end{cases}$$

$(S \cdot S_j)^r$ is defined to be the unique sequent such that $((S \cdot S_j)^r, (S \cdot S_j)^i)$ is a partition of $(S \cdot S_j)$. If the first case in the definition of $(S \cdot S_j)^i$ does not apply, then as S_0 contains exactly one formula, $S_0 \subseteq (S \cdot S_0)^r$ and $(S \cdot S_0)^i \subseteq S$. Therefore

$$(S \cdot S_j)^r = \begin{cases} (S \cdot S_0)^r & \text{if } (S \cdot S_0)^i = S^i \cdot S_0 \text{ for some } S^i \subseteq S \\ (S^r \cdot S_j) & \text{if } (S \cdot S_0)^r = S^r \cdot S_0 \text{ for some } S^r \subseteq S. \end{cases}$$

Example 4.1 Consider the following instance R of the rule $\mathcal{R}$ for conjunction on the right:

$$\frac{\Gamma \Rightarrow \varphi_1, \Delta \quad \Gamma \Rightarrow \varphi_2, \Delta}{\Gamma \Rightarrow \varphi_1 \wedge \varphi_2, \Delta}$$

Thus $S_0 = (\Rightarrow \varphi_1 \wedge \varphi_2)$, $S_j = (\Rightarrow \varphi_j)$ for $j = 1, 2$ and $S = (\Gamma \Rightarrow \Delta)$. We will describe two standard partitions of the rule, one corresponding to the first case in the definition above and the other to the second. First, consider partition (S_0^r, S_0^i) of the conclusion S_0, where $(S \cdot S_j)^i = (\Rightarrow \varphi_1 \wedge \varphi_2)$. From $S_0 \subseteq (S \cdot S_j)^i$ it follows that $(S \cdot S_j)^i = (\Rightarrow \varphi_j)$ for $j = 1, 2$ and $S_j^r = S$ for $j = 0, 1, 2$.

Second, assume $\Delta \neq \varnothing$ and the formula in S_0 does not occur in Δ, and consider partition (S_0^r, S_0^i) of S_0, where $(S \cdot S_0)^i = (\Rightarrow \Delta)$. Thus $(S \cdot S_0)^r = ((\Gamma \Rightarrow) \cdot S_0)$. As $S_0 \not\subseteq (S \cdot S_0)^i$, $(S \cdot S_j)^i = S^i$ for $j = 0, 1, 2$, $(S \cdot S_0)^r = (\Gamma \Rightarrow \varphi_1 \wedge \varphi_2)$, and $(S \cdot S_j)^r = (\Gamma \Rightarrow \varphi_j)$ for $j = 1, 2$.

Observe that in the first case R^i is an instance of $\mathcal{R}$, and in the second case R^r is. Generalizing this gives the following lemma.

Lemma 4.2 For any instance $R = \big((S \cdot S_1) \dots (S \cdot S_n)/(S \cdot S_0)\big)$ of a focussed rule $\mathcal{R}$ with a standard partition, either $S_0 \subseteq (S \cdot S_0)^i$ and R^i is an instance of $\mathcal{R}$ or $S_0 \subseteq (S \cdot S_0)^r$ and R^r is an instance of $\mathcal{R}$.

Lemma 4.3 For any focussed rule $\mathcal{R}$ in a terminating calculus and any standard interpolant assignment, $\mathcal{R}$ satisfies ($\text{IP}_\vdash$), ($\text{DPB}_\vdash$) and ($\text{DPN}_\vdash$).

Proof Without loss of generality we assume that $\mathcal{R}$ is a two premiss rule and consider an instance $R = (S_1 \ S_2/S_0)$. Thus $S_1, S_2 \prec S_0$.
($\text{IP}_\vdash$) It suffices to prove that

$$\{S_j \cdot (\forall p S_j \Rightarrow) \mid j = 1,2\} \vdash S_0 \cdot (\overset{R}{\forall p} S_0 \Rightarrow).$$

Clearly, $\{S_1, S_2\} \vdash S_0$, as $\mathcal{R}$ belongs to the calculus. As $\mathcal{R}$ is focussed, for any S we have $\{S \cdot S_1, S \cdot S_2\} \vdash S \cdot S_0$. When taking $(\overset{R}{\forall p} S_0 \Rightarrow)$ for S, this gives $\{S_1 \cdot (\overset{R}{\forall p} S_0 \Rightarrow), S_2 \cdot (\overset{R}{\forall p} S_0 \Rightarrow)\} \vdash S_0 \cdot (\overset{R}{\forall p} S_0 \Rightarrow)$. As the interpolant assignment is standard, $\overset{R}{\forall p} S_0 = \forall p S_1 \wedge \forall p S_2$, and whence $S_j \cdot (\forall p S_j \Rightarrow)$ derives $S_j \cdot (\overset{R}{\forall p} S_0 \Rightarrow)$ for $j = 1, 2$. This implies that what we had to show.
($\text{DPB}_\vdash$) Consider a p–partition (S_0^r, S_0^i) of S_0 such that $\mathcal{R}$ is backwards applicable to S_0^i. It suffices to show that for some instance R of $\mathcal{R}$ with conclusion S_0^i there is some partition of $\mathcal{R}$ such that

$$\{S_j^r \cdot (\Rightarrow \forall p S_j^i) \mid j = 1,2\} \vdash S_0^r \cdot (\Rightarrow \overset{R}{\forall p} S_0^i).$$

Consider the standard partition of $\mathcal{R}$. As $\mathcal{R}$ is backwards applicable to S_0^i, $S_0 \subseteq S_0^i$. Therefore $S_0^r = S_1^r = S_2^r$. By Lemma 4.2, R^i is an instance of $\mathcal{R}$. As the interpolant assignment is standard, it follows that $\overset{R}{\forall p} S_0^i = \forall p S_1^i \wedge \forall p S_2^i$, which implies what we had to show.
($\text{DPN}_\vdash$) It suffices to show that for all p–partitions (S_0^r, S_0^i) of S_0 such that S_0^i is not empty and $\mathcal{R}$ is not backwards applicable to S_0^i, there exists an interpol $\mathcal{S}$ such that $\mathcal{S} \vdash S_0^r \cdot (\Rightarrow \forall p S_0^i)$. Recall that $\mathcal{S}$ being an interpol means that it is a set of sequents of the form $S^r \cdot (\Rightarrow \forall p S^i)$ for some $S \prec S_0$. As $\mathcal{R}$ is not backwards applicable to S_0^i, R^r is an instance of $\mathcal{R}$ by Lemma 4.2, using the standard partition of $\mathcal{R}$. Moreover $S_0^i = S_1^i = S_2^i$ in this case. As $\mathcal{R}$ is focussed, $\{S \cdot S_1^r, S \cdot S_2^r\}/S \cdot S_0^r$ is an instance of $\mathcal{R}$ for any S. In particular, for S being $(\Rightarrow \forall p S_0^i)$. Therefore $S \cdot S_1^r, S \cdot S_2^r \vdash S \cdot S_0^r$. Since $\forall p S_0^i = \forall p S_1^i = \forall p S_2^i$, we can take $\{S \cdot S_1^r, S \cdot S_2^r\}$ for $\mathcal{S}$ and are done. □

Lemma 4.4 For any focussed rule or focussed axiom $\mathcal{R}$ in a terminating calculus and any interpolant assignment that is standard for $\mathcal{R}$, the interpolant assignment is $\mathcal{R}$-sound.

Proof For rules the lemma follows from Lemmas 3.11 and 4.3. For axioms we consider an instance of a focussed axiom $\mathcal{R}$. This instance R is equal to a sequent S. We have to show that under the assumption that all sequents lower than S satisfy the interpolant properties, the following holds.

(IP) S satisfies the R-variant of the independent interpolant property.

(DP) For every partition (S^r, S^i) such that S^i is not empty, the dependent interpolant property holds.

(IP) holds because S is derivable. For (DP), consider a partition (S^r, S^i) of S. If S^i is a proper subsequent of S the interpolant properties hold by assumption. In particular, the dependent property holds for the partition $(\varnothing, S^i)$ of S^i. Thus $\vdash (\Rightarrow \forall p S^i)$, and therefore $\vdash S^r \cdot (\Rightarrow \forall p S^i)$. If $S^i = S$, we distinguish by cases. If $\mathcal{R}$=L$\bot$ or $\mathcal{R}$=R$\top$, then (DP) trivially holds. If $\mathcal{R}=At$ and $S = (\Gamma, r \Rightarrow r, \Delta)$, $\forall\!\overset{R}{p} S$ is $r \vee \forall p(\Gamma \Rightarrow \Delta)$ or $\top$, depending on whether $r = p$ or $r \neq p$. In the last case the dependent interpolant property clearly holds. By the induction hypothesis the dependent interpolant property holds for all partitions of $(\Gamma \Rightarrow \Delta)$. In particular $\forall p(\Gamma \Rightarrow \Delta)$ is derivable. Hence so is $\forall\!\overset{R}{p} S$. This shows that the dependent property holds for $(\varnothing, S)$. □

Theorem 4.5 A logic $\mathsf{L} \supseteq \mathsf{CPC}$ with a terminating calculus with a universal interpolant assignment that is sound with respect to free sequents and all axioms and rules that are not focussed and that is standard with respect to all axioms and rules that are focussed, has uniform interpolation.

Proof By Theorem 3.9 it suffices to prove that the interpolant assignment is sound with respect to focussed rules and axioms, which is proved in Lemma 4.4. □

Corollary 4.6 A logic $\mathsf{L} \supseteq \mathsf{CPC}$ with a terminating calculus in which all rules and axioms are focussed has uniform interpolation.

4.7 Classical logic

To establish that classical propositional logic has uniform interpolation is not hard. More generally, it is known that if a logic has interpolation and is locally tabular[1], properties that indeed hold for CPC, then it has uniform interpolation. However, with the methods developed in the previous sections one can also easily infer that CPC has uniform interpolation from the existence of a terminating calculus for the logic in which all axioms and rules are focussed. The propositional part, G3p, of the calculus G3 from (Troelstra and Schwichtenberg, 1996) which is given in Figure 4.7 has these properties. It is terminating (using the natural ordering given by the number of symbols in a sequent) and the axioms and rules are clearly focussed. The calculus has no structural rules, but they are admissible in it, as is the cut rule.

[1]For any finite set of variables there are only finitely many nonequivalent formulas in those variables.

$$\Gamma, p \Rightarrow p, \Delta \quad \text{At} \qquad\qquad \Gamma, \bot \Rightarrow \Delta \quad L\bot$$

$$\frac{\Gamma\varphi \Rightarrow \varphi \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi\varphi, \Delta}\ \text{R}\wedge \qquad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta}\ \text{L}\wedge$$

$$\frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}\ \text{R}\vee \qquad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta}\ \text{L}\vee$$

$$\frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \to \psi, \Delta}\ \text{R}\to \qquad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta}\ \text{L}\to$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \neg\varphi, \Delta}\ \text{R}\neg \qquad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \neg\varphi \Rightarrow \Delta}\ \text{L}\neg$$

Figure 1: The Gentzen calculus G3p

Theorem 4.8 Classical propositional logic has uniform interpolation.

Proof By Corollary 4.6 and the fact that the terminating calculus G3p is sound and complete for CPC. □

Note that for the above result one cannot use the propositional part of Gentzen's LK or other calculi that contain the Cut Rule, as it is not clear whether such calculi are terminating.

5 Standard assignment for modal logics

In this section we extend the method developed above to modal logic by extending the class of rules to which Corollary 4.6 applies. We use the convention that $\Box S = (\{\Box\varphi \mid \varphi \in S^a\} \Rightarrow \{\Box\psi \mid \psi \in S^s\})$, implying that $\Box(\Gamma \Rightarrow) = (\Box\Gamma \Rightarrow)$ and $\Box(\Rightarrow \Delta) = (\Rightarrow \Box\Delta)$. A rule $\mathcal{R}$ is a *focussed modal rule* if it is of the form

$$\frac{\Box S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}\ \mathcal{R} \tag{3}$$

for certain sequents S_0, S_1, S_2 such that S_2^a, S_2^s both consist of a meta-multiset, S_1^a, S_1^s both consist of at most one element, which is a meta-multiset, and S_0^a, S_0^s both consist of zero or one meta-multiset and zero or one meta-atom. Moreover, S_0 contains exactly one meta-atom and no meta-multiset occurs in $S_2 \cdot S_1 \cdot S_0$ more than once. Thus S_1 may be empty, but S_2 and S_0 may not. Note that S_2 is of the form $\Lambda \Rightarrow \Theta$ for two meta-multiset symbols Λ and Θ that do not occur in S_1 or S_0.

Up to the renaming of meta-symbols there are eight possibilities for S_0 (L,l (R,r) for Left (Right)):

$$S_0 = \begin{cases} \Gamma, p \Rightarrow \Delta & \mathcal{R}_{\mathsf{LlR}} \quad \Gamma \Rightarrow p, \Delta \quad \mathcal{R}_{\mathsf{LrR}} \\ \Gamma, p \Rightarrow & \mathcal{R}_{\mathsf{Ll}} \quad \Gamma \Rightarrow p \quad \mathcal{R}_{\mathsf{Lr}} \\ p \Rightarrow \Delta & \mathcal{R}_{\mathsf{lR}} \quad \Rightarrow p, \Delta \quad \mathcal{R}_{\mathsf{rR}} \\ p \Rightarrow & \mathcal{R}_{\mathsf{l}} \quad \Rightarrow p \quad \mathcal{R}_{\mathsf{r}} \end{cases}$$

The subscripts are considered to be sets, but for brevity the accolades and commas have been left out, such as in $\mathcal{R}_{\mathsf{lR}}$, which is short for $\mathcal{R}_{\{\mathsf{l},\mathsf{R}\}}$. For S_1

there are four possibilities, at least up to renaming of meta-multiset symbols, as S_1^a and S_1^s may or may not be empty. The terminology for the 32 focussed modal rules is as follows. Consider a focussed modal rule as in (3) and let $\mathcal{R}_{\mathsf{x}}$ be the name in the list for S_0 above. Then the name of the rule, depending on the form of S_1, is given in the following table (E (W) standing for East (West)).

$$S_1 = \begin{cases} (\Rightarrow) & \mathcal{R}_{\mathsf{x}} \\ (\Pi \Rightarrow) & \mathcal{R}_{\mathsf{Wx}} \\ (\Rightarrow \Sigma) & \mathcal{R}_{\mathsf{xE}} \\ (\Pi \Rightarrow \Sigma) & \mathcal{R}_{\mathsf{WxE}} \end{cases}$$

For example, $\mathcal{R}_{\mathsf{WlR}}$ and $\mathcal{R}_{\mathsf{Lr}}$ denote the following rules, where the latter is the well-known rule for K.

$$\frac{\Box\Pi, p \Rightarrow \Delta}{\Lambda, \Box\Pi, \Box p \Rightarrow \Box\Delta, \Theta}\ \mathcal{R}_{\mathsf{WlR}} \qquad \frac{\Gamma \Rightarrow p}{\Lambda, \Box\Gamma \Rightarrow \Box p, \Theta}\ \mathcal{R}_{\mathsf{Lr}}$$

$\mathcal{R}_{\mathsf{Lr}}$ is also denoted by $\mathcal{R}_{\mathsf{K}}$, and a calculus containing that rule is called *normal*. For subsets $\mathsf{x}, \mathsf{y} \subseteq \{\mathsf{W}, \mathsf{L}, \mathsf{l}, \mathsf{r}, \mathsf{R}, \mathsf{E}\}$, $\mathcal{R}_{\mathsf{x}}$ is an y-rule if all elements of y occur in x. Given a calculus, rule $\mathcal{R}_{\mathsf{x}}$ is *left covered* if the calculus contains an Lx-rule or an $\mathsf{lRx}\backslash\{\mathsf{r}\}$-rule. A rule $\mathcal{R}_{\mathsf{x}}$ is *right covered* if the calculus contains an Rx-rule or an $\mathsf{rLx}\backslash\{\mathsf{l}\}$-rule. A calculus is *balanced* if every r-rule is left covered and every l-rule is right covered.

The *standard assignment* for every instance $R = (\Box S_1 \cdot S_0 / S_2 \cdot \Box S_1 \cdot \Box S_0)$ of a focussed modal rule $\mathcal{R}$ is defined as follows.

$$\forall \overset{R}{p}(S_2 \cdot \Box S_1 \cdot \Box S_0) \sim_0 \begin{cases} \Box\forall p(\Box S_1 \cdot S_0) & \text{if } \mathcal{R} \text{ is an r-rule} \\ \neg\Box\neg\forall p(\Box S_1 \cdot S_0) & \text{if } \mathcal{R} \text{ is an l-rule.} \end{cases}$$

Given a calculus, free sequents of the form $S \cdot (\Box\Gamma \Rightarrow)$ with nonempty Γ, receive the following *standard assignment*.

$$\forall p(S \cdot (\Box\Gamma \Rightarrow)) \sim_0 (\Rightarrow \neg\Box\neg\forall p(\Gamma \Rightarrow)).$$

Free sequents of the form $S \cdot (\Rightarrow \Box\Delta)$, with nonempty Δ, receive the following *standard assignment*.

$$\forall p(S \cdot (\Rightarrow \Box\Delta)) \sim_0 (\Rightarrow \Box\forall p(\Rightarrow \Delta)).$$

For all free sequents S that have not received an interpolant under this convention, $\forall pS$ is defined to be $\bot$.

For this to be well-defined we have to assume that no nonempty sequent of the form $(\Box\Gamma \Rightarrow \Box\Delta)$ (that is, $(\Box\Gamma \Rightarrow) \cdot (\Rightarrow \Box\Delta)$) where Γ and Δ are not empty, is free. Observe that this holds for all modal logics considered here.

As for focussed rules we fix a partitioning of instances of focussed modal rules, for which we have to address a small technical detail first. Given the conclusion of a focussed modal rule $S = (S_2 \cdot \Box S_1 \cdot \Box S_0)$, we assume that for every formula in S it is indicated, although we will not explicitly do so, whether it belongs to S_0, S_1 or S_2. In this way, for any partition (S^i, S^r) of S, $j = 0, 1, 2$ and $x \in \{i, r\}$, there exist unique sequents $S_j^x \subseteq S_j$ such that $S^x = S_2^x \cdot \Box S_1^x \cdot \Box S_0^x$. Given this assumption, a partition

$$\frac{((\Box S_1 \cdot S_0)^r, (\Box S_1 \cdot S_0)^i)}{((S_2 \cdot \Box S_1 \cdot \Box S_0)^r, (S_2 \cdot \Box S_1 \cdot \Box S_0)^i)}\ \mathsf{pt}(R)$$

of an instance $R = (\Box S_1 \cdot S_0 / S_2 \cdot \Box S_1 \cdot \Box S_0)$ of a focussed modal rule is *standard* if for $x \in \{i, r\}$:

$$(\Box S_1 \cdot S_0)^x = \Box S_1^x \cdot S_0^x.$$

Lemma 5.1 For any instance R of any focussed modal rule $\mathcal{R}$ with a standard partition, either R^i or R^r (or both) is an instance of $\mathcal{R}$.

Proof In case $\mathcal{R}$ is an r-rule, either S_0^{is} or S_0^{rs} is not empty. In the first case, R^i is an instance of $\mathcal{R}$ and in the second case R^r is. The case of an l-rule is similar. $\Box$

In the following lemmas,

$$\frac{\Box S_1 \cdot S_0}{S_2 \cdot \Box S_1 \cdot \Box S_0}\ R$$

will always be an instance of a focussed modal rule $\mathcal{R}$, and $S_u = \Box S_1 \cdot S_0$ denotes the upper sequent and $S_l = S_2 \cdot \Box S_1 \cdot \Box S_0$ the lower sequent of R.

Lemma 5.2 If $R = (S_u/S_l)$ is an instance of a focussed modal r-rule $\mathcal{R}$ in a balanced calculus, then $\vdash S_u \cdot (\forall p S_u \Rightarrow)$ implies $\vdash S_l \cdot (\Box \forall p S_u \Rightarrow)$.

Proof As the calculus is balanced, $\mathcal{R} = \mathcal{R}_{\mathsf{xr}}$ is covered by a left rule, meaning that the calculus contains a rule $\mathcal{R}'$ that is an Lxr-rule or an lRx-rule. If the calculus does not contain an Lxr-rule, this implies that $\mathcal{R}$ is not an L-rule, and thus S_0^a is empty. Therefore in both cases, since $\mathcal{R}$ can be applied to S_u, $\mathcal{R}'$ can be applied to $S_u \cdot (\forall p S_u \Rightarrow)$ to obtain $S_l \cdot (\Box \forall p S_u \Rightarrow)$. $\Box$

Lemma 5.3 If $R = (S_u/S_l)$ is an instance of a focussed modal l-rule $\mathcal{R}$ in a balanced calculus, then $\vdash S_u \cdot (\forall p S_u \Rightarrow)$ implies $\vdash S_l \cdot (\neg \Box \neg \forall p S_u \Rightarrow)$.

Proof Clearly, $S_u \cdot (\forall p S_u \Rightarrow) \vdash S_u \cdot (\Rightarrow \neg \forall p S_u)$. As the calculus is balanced, $\mathcal{R} = \mathcal{R}_{\mathsf{lx}}$ is covered by a right rule, meaning that the calculus contains a rule $\mathcal{R}'$ that is an lxR-rule or an Lrx-rule. Since $\mathcal{R}$ can be applied to S_u, $\mathcal{R}'$ can be applied to $S_u \cdot (\Rightarrow \neg \forall p S_u)$ to obtain $S_l \cdot (\Rightarrow \Box \neg \forall p S_u)$. This last sequent derives the desired $S_l \cdot (\neg \Box \neg \forall p S_u \Rightarrow)$. $\Box$

Lemma 5.4 If a calculus is balanced, then ($\mathrm{IP}_\vdash$) holds for all its rules.

Proof We have to show that

$$\begin{array}{lll} \vdash S_u \cdot (\forall p S_u \Rightarrow) \text{ implies } \vdash S_l \cdot (\Box \forall p S_u \Rightarrow) & \text{if } \mathcal{R} \text{ is an } \mathsf{r}\text{-rule.} \\ \vdash S_u \cdot (\forall p S_u \Rightarrow) \text{ implies } \vdash S_l \cdot (\neg \Box \neg \forall p S_u \Rightarrow) & \text{if } \mathcal{R} \text{ is an } \mathsf{l}\text{-rule.} \end{array}$$

This follows from Lemma 5.2 and Lemma 5.3. $\Box$

Lemma 5.5 ($\mathrm{DPB}_\vdash$) holds for all focussed modal rules in a calculus.

Proof Assume that $\mathcal{R}$ is backwards applicable to S_l^i and consider its standard partition. Thus $\Box \forall p S_u^i$ is a disjunct of $\forall \overset{R}{p} S_l^i$ if $\mathcal{R}$ is an r-rule and $\neg \Box \neg \forall p S_u^i$ is a disjunct of $\forall \overset{R}{p} S_l^i$ if $\mathcal{R}$ is an l-rule. Therefore it suffices to show:

$$\begin{array}{lll} \vdash S_u^r \cdot (\Rightarrow \forall p S_u^i) \text{ implies } \vdash S_l^r \cdot (\Rightarrow \Box \forall p S_u^i) & \text{if } \mathcal{R} \text{ is an } \mathsf{r}\text{-rule.} \\ \vdash S_u^r \cdot (\Rightarrow \forall p S_u^i) \text{ implies } \vdash S_l^r \cdot (\Rightarrow \neg \Box \neg \forall p S_u^i) & \text{if } \mathcal{R} \text{ is an } \mathsf{l}\text{-rule.} \end{array}$$

If $\mathcal{R}$ is an r-rule, then it is an R-rule or S_0^{rs} is empty. In both cases $\mathcal{R}$ can be applied to $S_u^r \cdot (\Rightarrow \forall p S_u^i)$ to obtain $S_l^r \cdot (\Rightarrow \Box\forall p S_u^i)$.
If $\mathcal{R}$ is an l-rule, then it is an L-rule or S_0^{ra} is empty. In both cases $\mathcal{R}$ can be applied to $S_u^r \cdot (\neg\forall p S_u^i \Rightarrow)$ to obtain $S_l^r \cdot (\Box\neg\forall p S_u^i \Rightarrow)$, which clearly implies $S_l^r \cdot (\Rightarrow \neg\Box\neg\forall p S_u^i)$. □

Lemma 5.6 If a balanced calculus consists of focussed and focussed modal rules only, then (DPN$_\vdash$) holds.

Proof Consider a partition (S_l^r, S_l^i) of S_l and assume that a focussed modal rule $\mathcal{R}$ is not backwards applicable to S_l^i (whence $S_l^i \neq S_l$) and S_l^i is not empty. This means that either $S_0^{is} = S_1^{is} = \varnothing$ (in case $\mathcal{R}$ is an r-rule) or $S_0^{ia} = S_1^{ia} = \varnothing$ (in case $\mathcal{R}$ is an l-rule). We have to show that for some interpol $\mathcal{S} \prec S_l$,

$$\vdash \mathcal{S} \text{ implies } \vdash S_l^r \cdot (\Rightarrow \forall p S_l^i). \tag{4}$$

First we treat the case that S_l^i is not free. Thus S_l^i is the conclusion of an instance

$$\frac{T_1 \quad \dots \quad T_n}{S_l^i}\ R'$$

or some rule $\mathcal{R}'$ in the calculus. In case $\mathcal{R}'$ is a focussed modal rule, $T_1, \dots, T_n/S_l$ is an instance of the rule as well. By Lemma 5.5 there is a partition of R' such that S_l is partitioned as (S_l^r, S_l^i) and

$$\vdash \bigwedge\{T_j^r \cdot (\Rightarrow \forall p T_j^i) \mid 1 \leq j \leq n\} \text{ implies } \vdash S_l^r \cdot (\Rightarrow \forall p S_l^i).$$

The set on the left is an interpol $\mathcal{S} \prec S_l$ for which (4) holds.
In case $\mathcal{R}'$ is a focussed rule, $R'(S_l^r)$ is an instance of the rule with conclusion S_l. By Lemma 4.3 there is a partition of $R'(S_l^r)$ such that S_l is partitioned as (S_l^r, S_l^i) and

$$\vdash \bigwedge\{(S_l^r \cdot T_j)^r \cdot (\Rightarrow \forall p(S_l^r \cdot T_j)^i) \mid 1 \leq j \leq n\} \text{ implies } \vdash S_l^r \cdot (\Rightarrow \forall p S_l^i).$$

holds. The set on the left is an interpol $\mathcal{S} \prec S_l$ for which (4) holds. This completes the case that S_l^i is not free. In the case that S_l^i is free, we treat the case that $\mathcal{R}$ is an r–rule, the case of the l–rule being similar.
$\mathcal{R} = \mathcal{R}_{\mathsf{xr}}$: S_l^{is} is empty and S_l^{rs} is not, since $\mathcal{R}$ is backwards applicable to S_l and not to S_l^i. Therefore S_l^i is of the form $S \cdot (\Box\Gamma \Rightarrow)$ for some $S \subseteq S_2$ and nonempty set Γ. If $\mathcal{R}$ is neither a W–rule nor an L–rule, then $S_l^i \subseteq S_2$, and therefore the derivability of $S_l = S_l^r \cdot S_l^i$ implies the derivability of $S_l^r \cdot (\Rightarrow \forall p S_l^i)$ right away. In the other cases we reason as follows.
In case $\mathcal{R}$ is a W–rule we define a sequent S by putting $S^i = (\Box\Gamma \Rightarrow)$ and letting S^r be such that an application of $\mathcal{R}$ to $S = S^r \cdot S^i$ gives S_l. For example, if $S_l^r = (\Box\Pi \Rightarrow \Box\varphi, \Box\Sigma)$ in case $\mathcal{R}$ is a WrR-rule, then S^r would be $(\Box\Pi \Rightarrow \varphi, \Sigma)$. Let $\mathcal{S}$ be the interpol consisting of $S^r \cdot (\Rightarrow \forall p S^i)$, and assume that $\vdash \bigwedge \mathcal{S}$. Note that $\mathcal{S} \prec S_l$. We have to prove that $\vdash S_l^r \cdot (\Rightarrow \forall p S_l^i)$. As $\forall p S_l^i = \forall p S^i$ and $\forall p S^i = \neg\Box\neg\forall p(\Gamma \Rightarrow)$, the derivability of the sequents in $\mathcal{S}$ implies that $\vdash S^r \cdot (\Box\neg\forall p(\Gamma \Rightarrow) \Rightarrow)$. An application of $\mathcal{R}$ gives $\vdash S_l^r \cdot (\Box\neg\forall p(\Gamma \Rightarrow) \Rightarrow)$, which implies the desired.

If $\mathcal{R}$ is an L–rule but not a W–rule, then we define a sequent S by putting $S^i = (\Gamma \Rightarrow)$ and letting S^r be such that an application of $\mathcal{R}$ to $S = S^r \cdot S^i$ gives S_l. Thus $\forall p S_l^i = \neg\Box\neg\forall p S^i$. Let $\mathcal{S} \prec S_l$ be the interpol consisting of $S^r \cdot (\Rightarrow \forall p S^i)$, and assume that $\vdash \bigwedge \mathcal{S}$. Therefore $\vdash S^r \cdot (\neg\forall p S^i \Rightarrow)$. An application of $\mathcal{R}$ gives $\vdash S_l^r \cdot (\Box\neg\forall p S^i \Rightarrow)$, which implies $\vdash S_l^r \cdot (\Rightarrow \forall p S_l^i)$. $\Box$

Lemma 5.7 If a terminating calculus consists of focussed and focussed modal rules only and contains an Lr-rule or an lR-rule, then all free sequents satisfy (FP).

Proof Let S be a free sequent and assume that all sequents $\prec$-lower than S satisfy the interpolant properties. We have to show that S satisfies the independent interpolant property, that is, that $S^a, \forall p S \Rightarrow S^s$ is derivable. In case $\forall p S$ is equivalent to falsum, this clearly holds. If both S^a and S^s contain a boxed formula, then any Lr-rule or lR-rule is backwards applicable to S, and so S is not free. For example, if $S = (\Box\varphi \Rightarrow \Box\psi, \chi)$ and an Lr-rule $\Box S_1 \cdot S_0 / S_2 \cdot \Box S_1 \cdot \Box S_0$ is present, one could take $(\varphi \Rightarrow \psi)$ for S_0, the empty sequent for S_1, and $(\Rightarrow \chi)$ for S_2. Thus we conclude that because S is free only the following two cases can occur.

If $S = S' \cdot (\Box\Gamma \Rightarrow)$, then $\forall p S$ is $(\Rightarrow \neg\Box\neg\forall p(\Gamma \Rightarrow))$. Write χ for $\forall p(\Gamma \Rightarrow)$. As S is free, the calculus cannot contain an lR-rule. For if so, for any $\varphi \in \Gamma$, $(\varphi \Rightarrow)/S$ is an instance of any lR-rule, contradicting that S is free. Therefore the calculus contains an Lr-rule. Since $(\Gamma, \chi \Rightarrow)$ is derivable by assumption, an application of an Lr-rule to $(\Gamma \Rightarrow \neg\chi)$ shows that $S' \cdot (\Box\Gamma \Rightarrow \Box\neg\chi)$ is derivable. An application of left negation implies that what we had to show.

If $S = S' \cdot (\Rightarrow \Box\Delta)$, then $\forall p S$ is $(\Rightarrow \Box\forall(\Rightarrow \Delta)$. Let χ denote $\forall p(\Rightarrow \Delta)$. Thus $(\chi \Rightarrow \Delta)$ is derivable by assumption. As S is free, the calculus cannot contain an Lr-rule. Thus the calculus contains an lR-rule, which can be applied to $(\chi \Rightarrow \Delta)$ to prove the derivability of $S' \cdot (\Box\chi \Rightarrow \Box\Delta)$, which is what had to be shown. $\Box$

Theorem 5.8 Any balanced terminating calculus that consists of focussed and focussed modal rules and contains an Lr-rule or an lR-rule, is sound with respect to the standard interpolant assingment.

Proof By Lemma 4.4 it suffices to show that the calculus is sound with respect to free sequents and focussed modal rules. For free sequents S, we have to show that (FP) holds, which has been proved in Lemma 5.7. For focussed modal rules it suffices by Lemmas 3.11 to prove that they satisfy ($\text{IP}_\vdash$), ($\text{DPB}_\vdash$) and ($\text{DPN}_\vdash$). This has been proven in Lemmas 5.4, 5.5 and 5.6. $\Box$

Theorem 5.9 A logic $\mathsf{L} \supseteq \mathsf{CPC}$ with a balanced terminating calculus that consists of focussed and focussed modal rules and axioms and contains an Lr-rule or an lR-rule, has uniform interpolation.

Since normal modal logics contain $\mathcal{R}_{\mathsf{K}} = \mathcal{R}_{\mathsf{Lr}}$ the above theorem implies the following.

Corollary 5.10 Any normal modal logic with a balanced terminating calculus that consists of focussed and focussed modal rules and axioms, has uniform interpolation.

6 Applications

In order to apply Theorem 5.9 it first has to be established that many standard calculi for modal logics are terminating. For this we use the following *weight* function on modal formula from Bílková (2007): $\varphi \prec \psi \equiv_{def} w(\varphi) < w(\psi)$, where

$$w(p) = w(\bot) = 1$$
$$w(\varphi \vee \psi) = w(\varphi) + w(\psi) + 1$$
$$w(\Box\varphi) = w(\varphi) + 1.$$

The weight $w(\Gamma)$ of a multiset Γ is the sum of the weights of the formula occurrences in Γ. It is easy to see that calculi consisting of G3p plus some focussed modal rules are terminating with respect to this order.

For the application of Theorem 5.9 the next issue that needs to be addressed is which of the calculi that consist of G3p plus some focussed modal rules are balanced. Because of the requirements of the theorem we are interested in calculi that contain an Lr-rule or an lR-rule.

There are 32 calculi that contain exactly one focussed modal rule. For such calculi it is easy to establish whether they are balanced. Therefore the following observation follows easily from Theorem 5.9

Theorem 6.1 Every calculus that consists of G3p plus a single rule, which is an Lr-rule or an lR-rule, has uniform interpolation.

For calculi that contain more than one focussed modal rule the situation is more complex. Here we restrict ourselves to calculi with two focussed modal rules. We treat the case that the calculus contains an Lr-rule, as the other case is by symmetry analogous. First assume the calculus contains $\mathcal{R}_{\mathsf{Lr}}$, that is, $\mathcal{R}_{\mathsf{K}}$. Then every other r-rule can be added to the calculus to obtain a balanced calculus, as the rule will be left covered by $\mathcal{R}_{\mathsf{K}}$. Of course, some of such rules are superfluous in the presence of $\mathcal{R}_{\mathsf{K}}$, such as $\mathcal{R}_{\mathsf{r}}$, but some are not, $\mathcal{R}_{\mathsf{rE}}$ being an example. For the l-rules, any rule that is right covered by itself can be added to obtain a balanced calculus. But also one of the rules $\mathcal{R}_{\mathsf{l}}$ or $\mathcal{R}_{\mathsf{Ll}}$ can be added, as they are also right covered by $\mathcal{R}_{\mathsf{K}}$.

Theorem 6.2 Every calculus that consists of G3p extended by $\mathcal{R}_{\mathsf{K}}$ plus a focussed modal rule that is an r-rule, lR-rule or equal to $\mathcal{R}_{\mathsf{l}}$ or $\mathcal{R}_{\mathsf{Ll}}$, has uniform interpolation.

Similar results can be obtained for other Lr-rules, as well as for calculi with more than two focussed modal rules.

The terminating calculus G3K for the modal logic K consists of G3p extended by the rule $\mathcal{R}_{\mathsf{K}} = \mathcal{R}_{\mathsf{Lr}}$ below, and similarly for the calculus G3D. G3KD stands for the calculus G3K extended by $\mathcal{R}_{\mathsf{D}} = \mathcal{R}_{\mathsf{Ll}}$.

$$\frac{\Gamma \Rightarrow \varphi}{\Gamma', \Box\Gamma \Rightarrow \Box\varphi, \Delta}\ \mathcal{R}_{\mathsf{K}} \qquad \frac{\Gamma, \varphi \Rightarrow}{\Gamma', \Box\Gamma, \Box\varphi \Rightarrow \Delta}\ \mathcal{R}_{\mathsf{D}}$$

As a corollary of Theorem 6.2 we obtain the following results by Bílková (2007).

Theorem 6.3 (Bílková, 2007) K and KD have uniform interpolation.

S4 is known to have no uniform interpolation (Ghilardi and Zawadowski, 1995), and the same holds for K4 (Bílková, 2007). This leads to the following observation.

Theorem 6.4 Neither K4 nor S4 is sound and complete with respect to a balanced terminating extension of G3K by focussed rules and focussed modal rules.

Observe that K4 does have a calculus consisting of focussed (modal) rules and axioms and the cut rule, namely, G3K extended by Cut and $\mathcal{R}_{\mathsf{Wr}}$. This calculus, however, is not balanced, as $\mathcal{R}_{\mathsf{Wr}}$ is not left covered.

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