Chapter 1
Seen Through Other Eyes—Opening Up New Vistas in Realistic Mathematics Education Through Visions and Experiences from Other Countries

Marja van den Heuvel-Panhuizen

Abstract This chapter is a synthesis of visions on and experiences with Realistic Mathematics Education (RME) described in the eighteen following chapters of this volume by forty-four authors from fifteen different countries. Through a process of synthesizing information from these chapters and combining and contrasting what the authors wrote about RME, a comprehensive image emerged of the theory and practice of RME, together with some new vistas. The chapter is structured around the following themes: making acquaintance with RME, narratives of first experiences with RME, highlighted outstanding features of RME, processes of implementation of RME and their challenges, adaptations of RME, criticisms of RME, and the flavours of RME that can be found in foreign curricula, textbooks, instructional materials, and teaching methods. Finally, to conclude the chapter, I reflect on new insights related to RME and directions for its further development that can be gained from this input from abroad.

Keywords Making acquaintance with Realistic Mathematics Education (RME) · Implementation and adaptation of RME · Challenges and criticisms of RME · Outstanding features of RME · Flavours of RME in foreign instructional material

1.1 Introduction

The story of what Realistic Mathematics Education (RME) is, how it came into existence and how it was developed further, has been described already by several people who are or were, in one way or another, part of the Dutch RME community. In this chapter this story is put under the spotlight again, but from the perspectives of people from abroad. The chapter tells how researchers and designers of mathematics

M. van den Heuvel-Panhuizen (✉)
Freudenthal Group, Faculty of Social and Behavioural Sciences & Freudenthal Institute, Faculty of Science, Utrecht University, Utrecht, the Netherlands
e-mail: m.vandenheuvel-panhuizen@uu.nl

Faculty of Education, Art and Culture, Nord University, Bodø, Norway
e-mail: m.vandenheuvel-panhuizen@nord.no

© The Author(s) 2020

M. van den Heuvel-Panhuizen (ed.), *International Reflections on the Netherlands Didactics of Mathematics*, ICME-13 Monographs,
https://doi.org/10.1007/978-3-030-20223-1_1

education, mathematics teacher educators, and mathematics teachers from fifteen countries outside the Netherlands, made acquaintance with RME, what they thought of it, what convinced them to adopt it, what aspects of RME they criticised, and what adaptations were required to incorporate RME in their own context. The visions and experiences explored in this chapter are based on Chaps. 2–19 of this volume in which forty-four authors tell their own RME story.

If one thing is unmistakably revealed in these chapters, it is in the first place that RME, although it may appear to be a well-defined unified theory of mathematics education, has many faces and should certainly not be considered a fixed and finished theory of mathematics education. Characteristic for RME is that there exists both internally, within the inner circle of RME developers at the Freudenthal Institute, and externally, including people in the Netherlands at other universities and institutions, differences in the interpretation and the appraisal of particular aspects of RME. The same applies to groups and persons in other countries who were inspired by RME. In addition to these concurrent differences, over time there have also been changes in focal points. For example, students' difficulties in learning mathematics was not really a theme that received special attention in the early years of RME. Only later, the development of a didactics for supporting low-achievers became an important issue, while in the last decade another move was made, but this time in favour of offering more learning opportunities to talented students. A further example of RME as a living theory is the rethinking of teaching and learning mathematics that was necessary when computer technology entered the classroom and provided teachers with new tools for organizing lessons and students with new ways of developing mathematical understanding. After all, in the time that the first ideas of RME were conceptualised there were, for example, no such things as online mini-games for fostering students' multiplicative reasoning ability. So, new didactical tools had to find their way into RME and these in turn opened new didactical approaches in RME.

Characteristic of RME are also the many people involved in its development and the mutual influences among these people. Teacher educators, school advisors, and textbook authors could always freely use RME tasks, ideas for lessons, models and strategies, and teaching-learning trajectories. Furthermore, this helping each other with good ideas also occurred in the opposite direction. RME designs have certainly also been inspired by ideas from teacher educators, school advisors, and textbook authors from outside the Freudenthal Institute. This reciprocal inspiration was also the case during all the joint projects the Dutch have carried out with people in other countries. There have always been exchanges of ideas and development in multiple directions. Bringing the visions and experiences from abroad together in this volume and in this chapter, and seeing the use of RME from different socio-cultural perspectives and educational systems can create new sources for reciprocal inspiration and opportunities for opening up new developments in RME.

1.2 Making Acquaintance with RME

1.2.1 Personal Encounters

Making acquaintance with RME was in most cases the result of a personal encounter at a gathering of mathematicians or mathematics educators somewhere in the world. For Wittmann (Chap. 4) this acquaintance took place in 1967 when he met Freudenthal who was one of the invited speakers at a colloquium held at the University of Erlangen in Germany. Wittmann had developed a strong aversion against the New Math movement and was very eager to speak with Freudenthal because of a paper Freudenthal wrote and published in 1963 in a German journal in which he explained that he saw mathematical activity, and not the learning of readymade axiomatics, as the crucial element of learning mathematics.

In Belgium, where New Math was introduced in the 1960s, an important meeting occurred in 1983 when proponents and opponents of New Math defended their positions. In this colloquium Freudenthal and Goddijn gave lectures about the Dutch approach to mathematics education. As is made clear by De Bock and his colleagues (Chaps. 3 and 11), in Belgium there was then, and even earlier, certainly interest in the RME approach, but after this meeting only some limited changes occurred in the programmes and in the formulation of the learning objectives. Yet for both of these small changes inspiration was found in the Dutch RME materials.

In 1983, Selter (Chap. 13) in Germany, while studying to become a primary school teacher, became aware of a paper by Treffers about teaching written multiplication and division by starting off with context problems containing large numbers. Students could solve these problems by using procedures of repeated addition and subtraction which gradually evolved into the more standard ways of written calculation. Reading this paper was a key event for Selter. He realised that this RME principle of progressive schematisation or progressive mathematisation was not only important for learning written calculation algorithms, but that it also could be considered a comprehensive, generally applicable principle for the organisation of mathematical learning or teaching processes.

Further from home, in China, the introduction to RME happened through Freudenthal's book *Mathematics as an Educational Task*. As described by Sun and He (Chap. 10), it was Jiang who read this book in 1985, which gave him a new perspective on understanding mathematics education. Next, this was followed by a face-to-face meeting of Jiang's former student Wang with Freudenthal at the CIEAEM conference in London in 1986. This meeting is considered the start of a new era of exchange in mathematics education between China and the Netherlands.

Also, in many other countries the exchange and collaboration with the Dutch started with personal meetings. For example, in Argentina (Chap. 9), it was Rosenberg who in 1984 came to the Netherlands to specialise in the didactics of mathematics at Utrecht University. This stay was followed by a return visit by De Lange and Schoemaker who introduced RME to professors at the University of Buenos Aires and the National University of Tucumán.

The long-lasting cooperation in mathematics education between the Netherlands and the United States begun when Romberg, who was involved in the development of the NCTM Standards, invited De Lange to the National Center for Research in Mathematical Sciences Education (NCRMSE) at the University of Wisconsin-Madison in the spring of 1988. In their chapter, Webb and Peck (Chap. 2) do not attempt to conceal that it was a beneficial development that these two mathematics educators on opposite sides of the Atlantic with a passion for reforming mathematics teaching and learning, have become colleagues and partners. In the 1990s Romberg also brought about a connection with Puerto Rico (Chap. 16) by proposing López-Fernández to collaborate with him and De Lange on the development of Spanish versions of the materials of the textbook series *Mathematics in Context* (MiC) that NCRMSE was developing together with the Dutch.

The 1990s were busy times. Apart from the activities with and in the United States and Puerto Rico, in 1994 RME also affected Indonesia when Sembiring from the Institut Teknologi Bandung saw De Lange presenting a keynote about RME at the ICMI conference in Shanghai. As is explained by Zulkardi, Putri, and Wijaya (Chap. 18), Sembiring was a representative of the government of Indonesia. He was inspired by the presentation and asked De Lange whether he could help Indonesia to reform the approach to teaching and learning school mathematics that was influenced by New Math. His first job would be to persuade the Indonesian government that RME is the right approach to reforming mathematics education. Four years later De Lange agreed to take on this task.

1.2.2 Narratives of First RME Experiences

When describing acquaintance with RME, very often the narratives that came to the fore are reflecting the thrilling and emotional feelings that arose when one became aware what RME means. In the United States, for Peck (Chap. 2), who was introduced to RME during his second year as a high school mathematics teacher, this breakthrough moment came when he saw an RME task in which hot dogs and lemonade were ordered in two different compositions and only the total price of each of the orders was given. The assignment for the students was to find out what one hot dog and one lemonade cost. He acknowledged that until that moment, he had always used Gaussian elimination to solve systems of equations, yet he never had understood why it worked. Now he found himself drawn to the context and combined the orders of the food in various ways to make new combinations, eventually eliminating the hot dogs. At this very moment it was clear for him that this context was not just a dressing-up for formal mathematics, but begged to be mathematised. In Peck's own words: "I finally understood elimination! I was hooked. It was clear to me that RME was a powerful tool for didactical design."

In Israel, Arcavi (Chap. 6) had a similar experience. Whereas he had always enjoyed the highly procedural and rule-oriented mathematics that he was offered in school, especially in algebra in which he liked the ingenuity of transforming expressions and inventing particular rules, his acquaintance with RME provided him with a broader view of mathematics. In his university studies, he always experienced mathematical modelling as an application of an already known piece of pure mathematics. It was a real eye-opener for him that RME inverted the order and that a real-world phenomenon could and should be a springboard for mathematisation. Also, RME allowed him to look with new eyes at his initial fondness for the procedural. It led him to consider that the procedural and the conceptual should be deeply interwoven. This new insight formed the roots of his work on sense making with symbols and with images.

For Abrahamson (Chap. 14), working both in Israel and the United States, the moment that—in his own words—was about to change everything, was when he found a paper published in 1979 by the RME designers and researchers Van den Brink and Streefland. In this paper they described and analysed a conversation between a father (Streefland himself) and his eight-year old son about a poster showing a man and a whale, in which the size of the whale compared to that of the man was exaggerated to make it more sensational. The questions addressed to the child and the analysis of the answers revealed that the child clearly realised that the ratio between the man and the whale was wrong. While Abrahamson was searching in vain in cognitive psychology literature for a grounding of his own ideas on children's early development of multiplicative concepts based on sensorimotor experiences, he was very happy to find this observation and the way the Dutch didacticians interpreted the observation and revealed the boy's thinking.

In the chapter about RME-based work in Argentina, Zolkower, Bressan, Pérez, and Gallego (Chap. 9) show that getting acquainted with RME can indeed change one's view on mathematics and mathematics teaching. A teacher student did not leave any doubt about this when testifying: "My relationship to mathematics changed a lot. It used to be very hard for me. I would often get frustrated… I used to hate it. But this year, I think because of how we approached it in this class, focusing on learning and understanding, it changed completely my view of this subject." A similar voice came from a teacher involved in one of the study groups organised in Argentina: "From the start, what intrigued us the most about RME is how it opens up the classroom doors to common sense, imagination, desire to learn, and the mathematising potential of our students."

For the Manchester Metropolitan University group visiting the Netherlands some ten years ago, what they saw in classrooms came as a revelation. According to Dickinson, Eade, Gough, Hough, and Solomon (Chap. 19), they were not just struck by the confidence with which the Dutch students gave correct answers, but also by the variety of justifications the students gave for them. For example, when comparing the size of fractions some used an appropriate whole number (a mediating quantity, as suggested by Streefland) to argue that 3/4 of 60 was larger than 2/3 of 60. Others used a percentage or a decimal argument or compared the fractions with a whole one, arguing that 3/4 needs only an extra 1/4 to make it up to a whole one and is therefore the larger. The English visitors supposed that such methods would not be available to students in their country at that time. A further characteristic of RME which the Manchester group said gave them a new way of thinking about how to teach mathematics, was the slow route to formal mathematics as explained by the iceberg model developed by Boswinkel and her colleagues. Influenced by RME, they began to define mathematical progress differently in two ways. As well as recognising that progress could be defined through the progressive formalisation of models, they also changed their view of the use of contexts as an aid for abstraction. While earlier their idea was to take the context away in order to work on more formal mathematics, after learning about RME, they saw that adding more contexts could also help students. In their own words the group from Manchester formulated it even better than it was ever done within RME itself: "[A]llowing students to see the 'sameness' of different situations, was actually a far more powerful route to abstraction."

1.2.3 Outstanding Features of RME

As described by Sun and He (Chap. 10), to steer a reform movement and make decisions about how to prepare students for society, and especially how to foster students' creativity, having clearly formulated goals is not enough. Also, theoretical power on which one can rely to guide concrete practice towards these goals is necessary. RME is considered to have contributed to generating such a theory for mathematics education in China. In addition, for Chinese mathematics educators it is seen as an outstanding feature of RME, that, in line with a famous Chinese saying, it keeps pace with the times. It is continuously open to new developments and innovations according to the ever-changing society and accumulated experiences of people. Only when this applies to a theory, can it have lasting vitality and the power to extend without limit in both theoretical and applicable aspects. This is very much appreciated in RME.

Wittmann (Chap. 4) was particularly attracted to the ideas Freudenthal and his colleagues at IOWO (Institute for the Development of Mathematics Education) had about research: they did not regard themselves as researchers, but as producers of instruction, as engineers in the educational field. Another important feature of RME for Wittmann was its focus on mathematics as a field of knowledge, though later RME became, as he sees this, too much focused on application. Wittmann also appreciated the genetic view on teaching and learning. He is, like Freudenthal, against the idea of didactical transpositions in which the higher levels of mathematics for mathematicians are converted into lower levels of mathematics for teaching mathematics. Also, the shift away from the strong fixation on standard algorithms towards various ways of calculating based on arithmetical laws was something he valued in RME. All in all, Wittmann has high regard for the contribution Freudenthal and his IOWO colleagues have delivered to mathematics education as a research domain with didactical analysis of the subject matter as the most important source for designing learning environments and curricula.

In other chapters further aspects of RME are highlighted as rewarding. When talking about the United States, Webb and Peck (Chap. 2) emphasise that RME has recast people's mathematical experience as one that should be meaningful, relevant and accessible. According to Niss (Chap. 17) it was the fact that students' individual conceptions and experiences have to be respected and are taken as points of departure for teaching and learning that made RME resonate with Danish mathematics educators so much. This student-centred approach of RME and its great attention to students' personal developments, as expressed in a paper by Freudenthal published in 1971, also received much praise from Abrahamson, Zolkower and Stone in their RME project at Berkeley (Chap. 14). The idea of connecting the teaching of mathematics to fostering youth independence and empowerment was considered as a great vision.

1.3 Processes of Implementation of RME

Getting to know about RME by meeting a knowledgeable person or reading a mind-altering book or paper is one thing, but what it is really about is how this first encounter continues. After a few pioneers in a country were introduced to RME, often a process followed in which the ideas were shared and many people became involved. For example, in England (Chap. 19), over the past ten years a number of projects developing classroom approaches based on RME, working with teachers and their students, have been carried out. In total over 40 schools, 80 teachers and 2000 students took part in these projects.

In Indonesia (Chap. 18) the coverage of RME-related projects and initiatives was more nationwide. Here, after a period of intensive exchange of Dutch and Indonesian staff and particularly by having master and PhD students coming to the Netherlands, several projects were set up to develop Pendidikan Matematika Realistik Indonesia (PMRI), an Indonesian adaptation of the RME approach to teaching mathematics. In addition, an RME-inspired master and an RME-inspired PhD program were also created, as well as courses for teachers, conferences, a website and a national and local centres for PMRI.

The implementation process in Argentina encompassed from the beginning a high degree of teacher involvement. According to Zolkower and her colleagues (Chap. 9), rather than applying the principles of RME top down as dogmas and using RME instructional materials as ready-made recipes, the Patagonian Group of Mathematics Didactics (GPDM) was engaged in the processes of design, try-outs, reflection, revision, new try-outs, through which they reinvented RME. These processes took place in spiral movements in which the participants interconnected their own mathematising activities with those of students in Grades K–12 and with those used in teacher preparation courses.

In other countries as well, there was a strong demand for developing ownership with the RME approach and getting to grips with this way of teaching. As Hernández-Rodríguez, López-Fernández, Quintero-Rivera, and Velázquez-Estrella (Chap. 16) reported, in Puerto Rico the need to have teachers participate 'as students' in working out together the details of the Spanish versions of the MiC units was recognised immediately. Such sessions were followed by detailed discussions around the mathematics addressed in the units and reflections on the use of paradigmatic situations and, above all, on finding ways to integrate the new materials in the mainstream curriculum and in the Puerto Rican culture.

The process of using RME in the United States, described by Webb and Peck (Chap. 2), also reflects a remarkable epistemological consistency between the characteristics of RME and how it was put into practice. In the same vein as in RME where students' active involvement in the learning process is considered as crucial, and the design of instructional materials is considered as engineering and tinkering, they characterise the past twenty years in which RME in the United States was piloted, disseminated, and integrated into mathematics resources as teacher-centred. In this process, signified as "from tinkering to systematic innovation", the focus was on reconsidering how students learn mathematics by having teachers re-experience mathematics through the lens of progressive formalisation and related didactic approaches. The teachers involved—who were often dedicated, volunteer teachers who wanted to take risks—collaborated with researchers to develop and improve RME lesson sequences and curricula and have become instructional leaders who facilitated professional development on RME.

In South Africa, as is indicated by Julie and Gierdien (Chap. 5), teachers were also considered as major role-players in collaboration with university-based mathematics educators, mathematicians and mathematics curriculum advisors when using RME to improve mathematics education. For the development of local instructional theories, it was essential that there was some alignment with the operative school mathematics curriculum. This is linked to the issue of immediacy in the sense that the appropriation of a teaching innovation by teachers is highly driven by their sense of the direct applicability of the ideas distributed by the innovation for their practice.

Whereas in some countries projects with teachers to apply RME or adaptations thereof in classrooms were started immediately, in China there was first much exchange between representatives of RME and Chinese mathematics educators through lectures. At the beginning the discussions about RME remained more at a theoretical level and there was no direct connection between RME theory and what occurred in Chinese classroom practice. Therefore, for example, the idea of 'free productions' was hard to be understood. It was difficult to imagine how to use it in the Chinese educational context. In contrast, 'mathematisation under the guidance of the teacher' was easier to understand because it was closer to the situation in China. This idea did not only affirm students' primary role of learning mathematics, but also emphasises the importance of teacher guidance during the process of mathematisation. As a result, this idea was quickly accepted and supported by the Chinese audience. As Sun and He (Chap. 10) concluded, knowing how RME was concretised in textbook design and classroom instruction was very necessary for understanding the essence of RME. Many examples mentioned in the lectures have become classical cases used in China for mathematics teachers' professional development. By analysing and reflecting on these cases, many Chinese mathematics teachers gain a better understanding of RME and try to change their former teaching practice of direct transmission.

The attitude of thoroughly studying RME sources was also characteristic for Korea. Lee, Chong, Na, and Park (Chap. 15) in their chapter give many examples of Korean mathematics educators who discussed RME ideas. These discussions already started in 1980 with a critical paper by Woo in which he refuted Freudenthal's criticism on Piaget's point of view. A few years later, Woo changed his mind and suggested mathematics teachers in Korea to focus more on mathematical thinking rather than on the mathematical content itself and taking as a guideline for this Freudenthal's didactical phenomenology. Many doctoral studies followed in which didactical phenomenological analyses were carried out on mathematical concepts such as function, negative number, and proportion. Moreover, researchers reflected on the difficulties underlying the Korean instruction methods of such concepts and proposed instruction methods that were more desirable.

1.4 Challenges in Implementing RME

Like in the Netherlands where moving from mechanistic mathematics teaching to an RME approach meant a break with the regular practice, also in other countries where initiatives were taken aimed at implementing RME this implied a paradigm shift in the teaching of mathematics and coping with the challenges that come with this new approach. That such a paradigm shift in the teachers' mindset is necessary for adopting the RME model was explicitly mentioned by Kaur, Wong, and Govindani (Chap. 7) when discussing differences between the Singapore approach in textbooks to teach equations and the approach in the RME-based textbook series MiC. Although in Singapore a drastic change into teaching methods that promote mathematical reasoning and communication might not be necessary, because they are already used in Singapore classrooms, taking up the RME approach would still require a turn in teachers' thinking on how mathematics learning takes place: 'from content to application' should be transformed to 'content through application'.

To activate and reshape mathematics education in Korea inspired by RME necessitated that several problems connected to the traditional mathematics education had to be overcome. According to Lee and her colleagues (Chap. 15) these problems were students' low understanding of mathematical concepts, the focus on blind memorisation of mathematical rules, procedures, and algorithms, and the existence of a poor connection between school mathematics and out-of-school mathematics and a teacher-centred style of mathematics teaching. The challenge the Korean textbook developers faced was to find and develop appropriate contexts through which students can experience that mathematics is a human activity existing near to them, can learn the principles and concepts of mathematics naturally through their own activities, and can improve their interest in and gain a positive attitude towards mathematics. Feedback from teachers who worked with RME-inspired materials revealed on the one hand that through the contexts the students indeed came to various strategies and they learned to communicate in their own words showing that they fully understood what they were doing instead of using only formal mathematical terms. On the other hand the teachers indicated that teaching in this way was very demanding in terms of class preparation and the continuous care and observations of students. In addition, teachers were concerned about the connection to the overall curriculum and how the students would fare in the usual mathematics classes in subsequent grades.

Since in Puerto Rico also there is a large difference between the principles and design methods used in the development of the Puerto Rican curriculum and those used in the development of RME, the paradigm change required for implementing RME there was a big challenge to overcome as well. Therefore, according to Hernández-Rodríguez and his colleagues (Chap. 16), a major balancing act had to be completed, on one side promoting teachers' inventiveness on how to work with the RME-based materials while on the other side following the official curriculum.

When discussing the development of an RME approach in England, Dickinson and his colleagues (Chap. 19) highlighted that the differences between the Dutch and English education system and the effect of the English system on teachers' and students' experiences and expectations have presented them with considerable challenges. Teachers in England are very aware of the pressure to move towards formal mathematics as quickly as possible. Therefore, they are anxious to see students acquiring formal procedures, and teachers may intervene and demonstrate the formal procedure after only one contextual problem. In RME, the process to working at a formal level may involve many lessons, and may even be spread out over a number of years, thus enabling students to gain conceptual understanding of how the procedure works, where it might be used, and how it connects to other areas of mathematics. However, in England it is often expected that performing a mathematical procedure can and should be achieved within one or two lessons. In addition, moving from the faster rote-learned alternatives to slow learning may also encounter resistance from students. This can also occur in response to the challenge that an RME-based classroom culture presents to students when they have to explain their thinking and make connections, ask questions and generally take more risks than in the case of simply 'learning the rules'.

One of the challenges that reform movements can be faced with is related to political issues. This is clearly the case in England where, as shown by Dickinson and his colleagues (Chap. 19), politically driven accountability pressures result in increasingly frequent assessment and a rigorous inspection regime, offering little scope for modifying education. Conversely, in the Cayman Islands (see also Chap. 19), although influenced by the British tradition, a less strict system provided their colleague Eade with more opportunities to work on developing an RME approach. Even so, he has concerns about the future: "There is still a long way to go and there is still a danger that, if the political/educational climate changes, then it would be very easy to destroy the fragile advances that have been made".

Such a political climate change also happened in Puerto Rico which had great consequences for the implementation of RME. At a particular point when Hernández-Rodríguez and his colleagues (Chap. 16) experienced that all the elements pointed to the possibility that the Puerto Rican version of MiC could become the spearhead of mathematics education in Puerto Rico, they suffered a real setback. Although there was public policy support for using this textbook series, the educational materials were developed, training was given and there was an entire infrastructure to disseminate the materials, the scaling up did not occur. An important factor in this was that there was a change in Puerto Rico's governing party and consequently a change in the Puerto Rico Department of Education. Given that the new staff responsible was not as enthusiastic about this new approach to teaching mathematics, the necessary funds for carrying out the dissemination were not allocated.

A completely different situation was the case in South Africa. As set out by Julie and Gierdien (Chap. 5), here the new political system rather opened up opportunities for RME. In fact, RME was introduced in South Africa during a period when curriculum changes were introduced to fit the educational ideals of the 'new' South Africa, including fostering learner-centredness and non-authoritarian ways of working classrooms. This kind of learning and teaching that was desired by the first democratically elected government in the country is exactly where RME stands for.

1.5 Adaptations of RME

Implementing RME, being inspired by RME and coping with the challenges that come with this new approach evidently require that adaptations are made to RME, in order to make it workable in a country's educational context and system. In South Africa, despite the common grounds in general ideas about mathematics education such adaptations were, according to Julie and Gierdien (Chap. 5), necessary because of the tension between the content of the RME-based materials and the 'legitimate school mathematics', that is, the mathematics that is valued in high-stakes examinations. There is a strong demand for proximity of the used RME resources. Teachers wanted to be assured of the immediate relevance of innovations to their current responsibilities and accountabilities with respect to the curriculum and accompanying activities such as examinations. As a result of this requirement it happened that RME-based modules were not disseminated further after trying them out in the classroom. Such was the case with a module on vision geometry. This particular module was chosen due to problems students in South Africa have with geometry and because the topic of vision geometry was quite in line with the RME perspective to provide students with activities where they can experience mathematics. However, even though the activities in this module were found enjoyable and not above the abilities of the students, after a few trials and notwithstanding some revisions to make it closer to the curriculum, it was not further used.

Adaptations were also necessary in the RME-based materials developed for elementary school in Puerto Rico. Hernández-Rodríguez and his colleagues (Chap. 16) illustrated this by describing what happened to the topic of written algorithms. As prescribed by official requirements, the materials had to be aligned to the Puerto Rican mathematics standards released in 2000 and 2007, which include that students learn the digit-based algorithms for addition and subtraction of natural numbers very early in elementary school. This was a fixed standard that the mathematics educators involved in the reform had to take into account, even though they were aware that research has shown that direct exposure to these algorithms can lead to serious conceptual errors related to the order of magnitude and the decimal representations of numbers. Postponing the teaching of digit-based algorithms like in the Netherlands was not possible. In Puerto Rico, if these algorithms are not present in the arithmetic lessons for the second grade, teachers and the official educational system will not accept such lessons as adequate for teaching. Therefore, the Puerto Rican team followed the standards, but presented the algorithms in such a way in the materials that their teaching was made more meaningful, which indeed significantly improved the students' understanding of them as was revealed in follow-up research. Another adaptation stemmed from the teachers' wish for didactical material full of interesting and concise contexts, but avoiding general and open-ended tasks. Since findings from previous pilot testing showed that MiC material requires students to do extensive reading, tasks which had much text had to be avoided because it kept teachers from using this material. Instead there was a need for a more piecemeal approach.

In Korea, as Lee and her colleagues (Chap. 15) reported, despite the challenges connected to RME, the teachers were rather positive about it, because they think mathematics instruction based on RME can change students' attitude to mathematics to a positive stance by providing them natural situations and activities that can encourage them to actively participate through diverse thoughts and communications. However, a strong suggestion came from the teachers to shorten the process of mathematisation and include repetitive exercises to make RME workable for the Korean educational context.

The work of Selter and Walter (Chap. 13) in Germany stressed adapting the RME principle of progressive mathematisation by including mathematics conferences. These conferences are meant to stimulate and organise exchanges amongst the students that will promote learning and by developing so-called "mathematics language tools", with the purpose to provide students with an instrument for further developing their ability to verbalise the description and justification of mathematical facts.

As described by De Bock, Van Dooren, and Verschaffel (Chap. 3), the Belgian approach to mathematics education was undoubtedly inspired by the Dutch RME model, which is, for example, reflected by the fact that the general objectives for primary school mathematics in Belgium are almost copies of those that were formulated by Treffers and colleagues in the late 1980s. Nevertheless, Belgian mathematics education is not considered to be RME. References to 'realistic' are purposely avoided and instead expressions are used such as 'meaningful situations', which indicates that other choices have been made in mathematics education. An illustration of this is that in Belgium, in contrast with the Netherlands, attention is paid first to standard arithmetical procedures, and more flexible procedures are only taught afterwards.

1.6 Criticisms of RME and Dissenting Views

Apart from all kinds of adaptations necessary to make a reform inspired by RME in accordance with a country's educational regulations and classroom culture, adaptations can also stem from dissenting views on mathematics education or from disapproval about RME. The RME ideas did not travel around the world without meeting criticism.

The main point of criticism echoed in the chapters is that RME, which strength it is to connect mathematics to the real world, is attaching too much weight to horizontal mathematisation. Concerns about this are expressed seriously by Wittmann (Chap. 4). Of course, he can understand that Freudenthal and his IOWO colleagues in the early days of RME wanted to establish a distinct counterpart to New Math and therefore put a lot of emphasis on applications, but he is more in favour of a balanced approach. Therefore, he welcomed that under the flag of RME recently publications have appeared again, such as from Kindt and De Moor, that are extremely interesting in terms of the mathematical structures they address.

For the Belgian mathematics educators in Flanders, RME could also have been more in balance. This is in line with how they view their own approach to mathematics education. As De Bock, Deprez, and Janssens (Chap. 11) explain, Flemish mathematics education in secondary education is so balanced because it resulted from multiple influences. It contains elements of the more traditional approach, which focuses on calculation drill and algebraic techniques, as well as of more structural elements, which focuses on a logical organisation of content and on proof and argumentation, and elements from RME, which undoubtedly enriched Flemish mathematics education, but which never led to the implementation of an orthodox version of the RME model.

With respect to Belgium primary school mathematics, De Bock, Van Dooren, and Verschaffel (Chap. 3) report even more explicit in criticising regarding particular features of RME; or more precisely expressed: features of which it is assumed they belong to RME. RME is criticised for disregarding the mechanistic aspects of learning, the lack of guidance of the construction of knowledge, the excessive freedom that is given to students to construct their own solution methods, the limited attention for the process of de-contextualising, and finally the insufficient recognition of the value of mathematics as a cultural product. Indeed, for some of these issues, such as neglecting the mechanistic aspects of learning and not viewing mathematics as a cultural product, RME can be criticised for not considering them as spearheads principles of RME. However, for other issues this is certainly not the case. The assumed lack of guidance is the opposite of what RME stands for. The excessive freedom that is supposedly given to students to construct their own solution methods is a wrong interpretation of the RME aim to break with the mechanistic approach of solving particular types of problems always in the same manner, but instead stimulate students to choose a solution strategy that suits the problem the students have to solve.

Having said this, it is unmistakably true that RME is often viewed in the wrong way and that these prejudices are often expressed, and often not in such a professional way as is done in the chapters of this volume. Of course, on the one hand a first reaction may be to rectify these misunderstandings, but on the other hand they also offer RME a mirror to look at itself and see which pitfalls there are when promoting RME and its guiding principles. Thus, even when these statements about RME are not fully true, RME should take them into account. What is true in any case is that Flanders outperforms the Netherlands in international comparisons—whatever value one attaches to these.

The critical remarks of the German mathematics educators Selter and Walter (Chap. 13) correspond to those by Wittmann. Their critique is about the limited interpretation of what is meant by context. According to them pure numerical contexts can also be quite meaningful for students. Moreover, numbers can also be realistic. Here again a statement is voiced as critique while it corroborates completely with the RME point of view. RME did always work with a broad conception of context. Yet later in the chapter of Selter and Walter, their critical remark becomes more distinct when it turns out that their main message is that, although they found in several RME publications that attention is paid to vertical mathematisation and that mathematics is regarded as a context of its own, they think that RME could possibly highlight these aspects more strongly.

Interesting in this respect is that while the message from the mathematics educators in Germany and Belgium is that RME should move more towards mathematics as a context of its own and vertical mathematisation, for Arcavi (Chap. 6) in Israel, RME was a kind of wake-up call to move in the other direction: from highly procedural and rule-oriented mathematics to using the real world as a springboard for mathematisation. RME gave him a broader view in the other direction.

For Niss (Chap. 17), discussing the Danish perspective, the point seems not to be the direction—moving more to this side or to that side of the spectrum. His point is the difference in emphasis in the meaning of 'realistic' in Denmark and in the Netherlands. In the RME interpretation, 'real' and 'realistic' incline to refer to students' experiential or emotional worlds and not necessarily to reality in the external world. In RME, fantasy stories or games are considered real and realistic if they are so to the students. This is in contrast with the Danish position which tends to emphasise the external objective reality of the surroundings in which students live such as family, friends, school, the local, national or global community, and scholarly and scientific fields or areas of practice. In RME, 'realistic' includes both problems based on real world situations and problems that students can experience as real. The latter relates to 'realistic' in the meaning of 'realising'; making a situation 'real' for oneself. Maybe within RME, this second meaning is too much emphasised in order to escape from the paralysing extreme requirement of authenticity that is often attributed to RME, and to make room for problems with powerful contexts that can become a model for developing mathematical concepts. Perhaps RME's focus is too much on contexts that lend themselves particularly well for evolving into a model that can be used for solving other problems or for eliciting helpful strategies, instead of on really complex daily life situations that require modelling and where mathematics has to be used to solve them. This different approach to 'realistic' is also reflected in the Danish view on the RME concept of horizontal and vertical mathematisation. According to Niss, the distinction between these two ways of mathematisation never got a foot in the door in Danish mathematics education, because in Denmark modelling involving the extra-mathematical domain, and internal mathematical transformations and processes are considered as very different. In the words of Niss, RME means "modelling for the sake of mathematics (learning)" while "[t]he Danish position tends to put emphasis on the reverse goal, namely mathematics (learning) for the sake of modelling." Although these differences, as Niss acknowledged, are not fundamental, but lie rather in priorities and emphases, it might be fruitful for RME to explore its further development more in this latter direction. Actually, the point to take away from Niss' chapter is again that there should be more balance in RME.

1.7 RME Flavours in Foreign Curricula, Textbooks, Instructional Materials, and Teaching Methods

Despite the fact that there is criticism and that at some points other choices are made, in many of the countries that made acquaintance with RME, ideas, principles and designs that have been developed in the Netherlands can be recognised in the countries' curricula, textbooks, instructional materials, and teaching methods.

The conclusion of Lee and colleagues (Chap. 15) is that in Korea, RME has become one of the major perspectives on mathematics education which has been widely discussed and applied by mathematics educators and mathematics teachers to reform Korean mathematics education over the past 35 years. The careful studies of the RME theory and the MiC textbook series that have been carried out in Korea have exercised a concrete influence on the mathematics curriculum and the textbook development since 2000, both implicitly and explicitly. In particular progressive mathematisation is considered as a potential perspective that would improve and complement Korean mathematics education. Therefore, the changes in the 2015 Mathematics Curriculum intended, for example, to implement the approach of progressive mathematisation for the concept of function.

Although in Argentina, as explained by Zolkower and her colleagues (Chap. 9), the design activities of the Patagonian Group of Mathematics Didactics (GPDM) did not use the RME-based materials of the textbook series MiC and the RME-based project Mathematics in the City as ready-made recipes, many RME designs such as the bus context, the percentage bar and the double number line appeared in Argentinean materials. Also, the described way of teaching is quite in line with RME, reflecting the approach of progressive mathematising, the use of tools and contexts to support this mathematisation process, the idea of guided reinvention, dealing with heterogeneous classrooms, the relevance of reflection, and making room for students' productions and constructions and using them in their teaching. As a result of the many seminars and teaching experiments throughout Argentina and the invitations many GPDM members got to lead teacher-training seminars, offer thematic workshops, present at research conferences, and elaborate or evaluate curriculum documents and instructional materials, the GPDM has become an important referent on RME within Spanish speaking South America and in this way exerts its influence on mathematics education.

Following the report of Selter and Walter (Chap. 13), nowadays mathematisation is seen in Germany also as a guiding principle within the didactics of mathematics for primary school. Moreover, similarities with RME can be recognised in the basic keystones formulated for mathematics education as well. In particular they can be found in considering learning as a (re)constructive activity facilitated by reflection on one's own thought processes and those of others, in viewing teaching as guiding students from their informal, context-bound methods to formal mathematics, and in offering students opportunities for communication and cooperation in small group work or whole-class discussion. However, Scherer's (Chap. 8) concern is that this approach to teaching mathematics does not apply to the German practice of teaching special needs students. Inspired by RME, she thinks that low achievers in mathematics should be offered opportunities to show what they are able to do. Through her studies she collects evidence that low achievers can also benefit from an open approach and are able to choose their own strategies, make use of structures and relations, find patterns and show creative and effective work.

Regarding their experiences in England, Dickinson and his colleagues (Chap. 19) report that although they cannot claim that RME has been implemented fully in schools, they are quite sure that it is the case for many of its principles. The mathematics departments with which they have worked are now far more likely to use models such as the ratio table and the empty number line, and to use contexts throughout a topic, for example, to use the context of a sandwich for teaching fractions, which eventually becomes a model for the formal comparison of fractions. Also, teachers are more apt to invoke visualisations and imagery in their lessons. In addition, there seems to be a slight move in schools to delay the journey to more formal mathematics, and embracing progressive formalisation.

Belgium, which has its own balanced approach to mathematics education resulting from multiple influences, also has elements of RME. An example given by De Bock and colleagues (Chap. 11) for secondary school is related to the teaching of derivates which was inspired by the Dutch HEWET materials in which the derivative was distilled from different real-world contexts in which (rate of) change had to be measured. In addition also a number of RME-inspired didactical innovations have ended up in the Belgian secondary school programme. Of these, perhaps the most important one is the role given to modelling and applications. Furthermore, more attention is given to (guided) self-discovery and active learning processes in the teaching and learning of mathematics; instead of only confronting students with 'end products' of mathematical activity. As discussed earlier, in the Flemish post-New-Math curricula and standards for the primary level (Chap. 3) much can also be recognised from the Dutch RME model, but at the same time valuable elements of the strong Belgian tradition in developing calculation skills and some New Math accents can be found.

According to Ponte and Brocardo (Chap. 12), in Portugal, RME has clearly influenced the mathematics curriculum for elementary school, notably in the topic of numbers and operations. For example, like in RME, much importance is attached to delaying the introduction of the standard algorithms and progressively developing more high-level abbreviated strategies and coming to generalisation and formalisation. Further, the influence of RME is also reflected in using the context of tasks as a starting point and source for modelling, the use of representations and models such as the empty number line, and the emphasis on the flexible use of mental calculation strategies. Here, similar to the RME teaching-learning trajectory for calculating with whole number, stringing strategies (with movements along the counting row), splitting strategies (processing the numbers based on the ten's structure) and varying strategies (based on arithmetic properties) are taught. Research groups in Portugal also make frequent references to key ideas of RME and use the method of didactical phenomenology to explore in depth a mathematical topic with great attention to everyday situations in which such a topic can be traced.

In the United States, Webb and Peck (Chap. 2) estimated the influence of RME on mathematics education as significant. The use of context and models has affected state and national curricula, including the recent *Common core standards for school mathematics*. Models such as the empty number line, percentage bar and ratio table are now common elements in instructional materials and assessments. Moreover, teachers continue to incorporate RME instructional principles into their classrooms and strive to find meaningful ways to engage students in the human activity of mathematising. However, the design principles that give the models such power—didactical phenomenology, emergent modelling and progressive formalisation, and guided reinvention—are often unknown to teachers and thus are incorporated only sparingly.

RME left its fingerprints in China as well. As explained by Sun and He (Chap. 10), from the 2001 Curriculum Standards document it is evident that the design of the standards was influenced by RME, because many keywords and expressions which echo the basic characteristics of RME had never appeared in similar official documents before 2001. Moreover, after this curriculum reform, the basic structure in most textbooks series used in primary and secondary mathematics started with a context problem, followed by a series of questions to lead students to what they are supposed to learn. This way of structuring textbooks was to a great extent inspired by RME. In addition, there was also a change in content. For example, geometry in traditional primary school textbooks involves measurement, including the definition of area and volume with the main focus on calculation, while after the reform in line with the RME approach the important concept of space was also included in mathematics textbooks.

For Denmark, according to Niss (Chap. 17) it is clear that RME in its broadest sense has had an impact on Danish mathematics education, but there are, as discussed earlier, also differences with respect to the meaning of 'realistic' and the role of mathematical modelling. What was, however, in any case an inspiration for several Danish mathematics educators was the method of design research as integrating research and development.

1.8 A Reflection to Conclude

My aim with this chapter was to bring together visions on RME and experiences with it from outside the Dutch circle of RME as they are laid down in the remaining eighteen chapters of this volume covering fifteen countries. Of course, this is not a random sample of countries. The chapters have been written by people who are supporters of RME or who have at least an interest in RME, but despite this the authors did not really display a prejudice towards RME in the sense that they were expressing that RME is the one and only way of teaching mathematics. They did not hold back when airing criticism, and did not mince their words when writing about what they think of RME. The merits of RME were recognised very well, but so were blind spots and unbalanced aspects.

Reviewing all that RME has set in motion it is hard to avoid the conclusion that, since its conception at the end of the 1960s, RME has gained a designated place in the theories of teaching and learning of mathematics. A significant moment of its recognition, as reported by Webb and Peck (Chap. 2), came in 1999 when the RME-based textbook series *Mathematics in Contexts* was described in the seminal book *How People Learn* as an example of a new approach to teaching mathematics that supports learning with understanding. Furthermore, the dispersion to so many and diverse countries worldwide, including countries in western, eastern, northern and southern regions, as well as the different socio-cultural contexts and educational systems which were receptive to RME ideas, can be considered as an illustration of both its robustness and its flexibility. The way mathematics educators in other countries see RME, how they made and make it work, how they talk about it, has let RME rise above a particular personal preference of teaching mathematics. RME has become a multifaceted approach to mathematics education with a joint ownership of many.

This engagement from abroad is very essential to keep RME a living theory. Visions and experiences of others can open our eyes to possible improvement. In this way the following chapters can also provide an impetus for sharpening and revising particular aspects of RME. For example, inspired by Arcavi (Chap. 6) we might elaborate more on the connections between the conceptual and the procedural, and on linking the different representations of mathematical entities. Furthermore, the experiences in England described by Dickinson and his colleagues (Chap. 19) give grounds for reconsidering the RME focus on slow learning and investigating whether there is also room for having quicker routes to formalisation without playing down the fundamental principle of progressive schematisation or mathematisation, as is also suggested by Puerto Rico (Chap. 16) and Korea (Chap. 15).

Creating more space for formal mathematics was a message that could be heard regularly. For Wittmann (Chap. 4) this touches the fundaments of the basis that Freudenthal and his IOWO colleagues have laid for RME. In his eyes, more attention should be paid to vertical mathematisation and to mathematical structures and thinking. A preference for a more balanced approach to mathematics education is, for example, also the sound that is heard from Belgium as expressed by De Bock and colleagues (Chaps. 3 and 11).

A further point that Dickinson and his colleagues (Chap. 19) brought to the fore, which is interesting for further exploration, is that a more formal, abstract level of understanding cannot only be reached by taking away the context, but also by adding more contexts. The latter would allow students to see the 'sameness' of different situations and this might also provide a route to abstraction. Although stemming from a different point of departure, namely his critique on the limited meaning of 'realistic' in RME, Niss (Chap. 17) pointed in a way to the same argument of bringing in more context. According to him, RME tends to insufficiently emphasise the external objective reality of students' surroundings, and the modelling of reality in RME is especially meant for the purpose of the learning of mathematics. This contrasts with the approach in Denmark where the focus is rather on the reverse, namely on learning mathematics to model a problem situation and solve it. Again, this is a suggestion to RME not to concentrate merely on ingenious contexts that can evolve into models intended to serve as a didactical aid for learning mathematics. In fact, this (once more) means bringing more balance between the context as a source and the context as a domain of application.

A last issue that struck me was that of all the characteristics of RME, there was one that was mentioned only sparsely as being relevant when a country was inspired by RME. This is the idea of didactical phenomenology or mathe-didactical analysis as a foundation for developing and researching mathematics education. Webb and Peck (Chap. 2) also noticed in their chapter that while certain RME models are widely used by teachers in the United States, most of them are not familiar with, for example, the idea of didactical phenomenology. Based on what is written in the chapters, this RME idea is not used as widely. Yet there are three prominent exceptions. The first one is Wittmann (Chap. 4) for whom the didactical analysis of the subject matter is the most important source for designing learning environments and curricula. Therefore, he thinks that Freudenthal's book *Didactical Phenomenology of Mathematical Structures* is of overriding importance. The second exception is the use of RME in Portugal. In the chapter of Ponte and Brocardo (Chap. 12) it is clearly shown that didactical phenomenology is considered an important RME idea that is present in several of their research studies. Using this idea means that a given mathematical topic is explored in depth, with great attention to everyday situations in which it can be traced. The third exception was found in Korea. As reported by Lee and colleagues (Chap. 15), in Korea, from the introduction to RME on, the perspective of Freudenthal's didactical phenomenology was taken on board and didactical phenomenological analyses were carried out on mathematical concepts which, among other things, influenced the adoption of progressive mathematisation.

Finally, a warning and an expression of hope. Although all chapters in this volume show RME as a vivid and promising theory with a lot of potential, there are also some concerns about its further development. Wittmann (Chap. 4) is worried about loosening the engagement in mathematics and Niss (Chap. 17) is wondering whether the changes of the Freudenthal Institute, including the split in organisational structure, will undermine the contributions of the Dutch to the further development of mathematics education. These concerns should be a wake-up call to all who wish to make more of the potential of RME. Now that the RME fire is kindled in so many countries, fuelled by the common goal of making mathematics accessible, meaningful, and relevant for all students, I hope that we can keep the essential flame alive and elaborate on it.

Open Access This chapter is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits use, duplication, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, a link is provided to the Creative Commons license and any changes made are indicated.

The images or other third party material in this chapter are included in the work's Creative Commons license, unless indicated otherwise in the credit line; if such material is not included in the work's Creative Commons license and the respective action is not permitted by statutory regulation, users will need to obtain permission from the license holder to duplicate, adapt or reproduce the material.