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Abstract

We consider spin–vorticity coupling—the generation of spin polarization by vorticity—in viscous two-dimensional electron systems with spin–orbit coupling. We first derive hydrodynamic equations for spin and momentum densities in which their mutual coupling is determined by the rotational viscosity. We then calculate the rotational viscosity microscopically in the limits of weak and strong spin–orbit coupling. We provide estimates that show that the spin–orbit coupling achieved in recent experiments is strong enough for the spin–vorticity coupling to be observed. On the one hand, this coupling provides a way to image viscous electron flows by imaging spin densities. On the other hand, we show that the spin polarization generated by spin–vorticity coupling in the hydrodynamic regime can, in principle, be much larger than that generated, e.g. by the spin Hall effect, in the diffusive regime.

1. Introduction

The field of spintronics is concerned with electric control of spin currents [1]. For the description of experimentally relevant systems it has, until very recently, been sufficient to consider their coupled spin-charge dynamics in the diffusive regime where the time scale for electron momentum scattering is fast compared to other time scales. The celebrated Valet–Fert theory for electron spin transport in magnetic multilayers [2] and the Dyakonov–Perel drift-diffusion theory for spin generation by the spin Hall effect [3], for example, fall within this paradigm.

Very recent experimental developments have brought about solid-state systems, such as ultra-clean encapsulated graphene, in which the momentum scattering time can be much longer than the time scale for electron–electron interactions [4–7]. In this so-called hydrodynamic regime, the electron momentum needs to be included as a hydrodynamic variable and the viscosity of the electron system cannot be neglected [8–17]. The finite electron viscosity leads to several physical consequences, such as a negative nonlocal resistance [4] and super-ballistic transport through point contacts (PCs) [7, 18]. These developments have spurred on a great deal of research, including proposals for measuring the Hall viscosity [19–21] and connections to strong-coupling predictions from string theory [22].

In a very different physical system, liquid Hg, spin-hydrodynamic generation, i.e. the generation of voltages from vorticity, was recently experimentally observed. [23]. Spin-hydrodynamic generation is believed to be a consequence of spin–vorticity coupling. Phenomenological theories of spin–vorticity coupling were developed early on [24] and have been applied to fluids consisting of particles with internal angular momentum such as ferrofluids [25], molecular nanofluids [26], and nematic liquid crystals [27]. In these phenomenological theories, the coupling between orbital angular momentum, i.e. vorticity of the fluid, and internal angular is governed by a dissipative coefficient, the so-called ‘rotational viscosity’. This type of viscosity has been estimated microscopically for classical systems (see e.g. [27]) and Hg [23], but not for viscous electrons in a crystal.

Motivated by the recent realization of solid-state systems hosting viscous electron fluids, we develop in this Letter the theory for spin–vorticity coupling in such systems. We derive the phenomenological equations

describing coupled spin and momentum diffusion, and compute the rotational viscosity microscopically. We apply our theory to viscous electron flow through a PC and show that the spin densities generated hydrodynamically can be much larger than the ones that are generated by the spin Hall effect in the diffusive transport regime. Our results may therefore stimulate experimental research towards novel ways of spin detection and generation.

2. Phenomenology

We consider two-dimensional (2D) electron systems with approximate translation invariance and approximate rotation invariance around the axis perpendicular to the plane (chosen to be the \hat{z} -direction). The conserved quantities of this system are energy, charge, linear momentum in the plane and angular momentum in the \hat{z} -direction. For brevity, we do not consider energy conservation explicitly and focus on momentum and angular momentum conservation. In the following, we adopt the discussion of [24, 28] and generalize it to include lack of Galilean invariance. The momentum density is denoted by $\mathbf{p}(\mathbf{r}, t)$ and is a 2D vector $\mathbf{p} = (p_x, p_y)$ in the \hat{x} – \hat{y} -plane with $\mathbf{r} = (x, y) = (r_x, r_y)$. The total angular momentum density in the \hat{z} -direction is the sum of orbital angular momentum density $\epsilon_{\alpha\beta} r_\alpha p_\beta$ and spin density $s(\mathbf{r}, t)$ (in the \hat{z} -direction). Here, $\epsilon_{\alpha\beta}$ is the 2D Levi-Civita tensor and summation over repeated indices $\alpha, \beta, \gamma, \delta \in \{x, y\}$ is implied. We denote with \mathbf{v} the conjugate variable to the momentum density, i.e. the velocity, whereas the spin chemical potential, commonly referred to as spin accumulation, μ_s is the conjugate variable to the spin density.

Conservation of linear momentum yields

$$\frac{\partial p_\alpha(\mathbf{r}, t)}{\partial t} = -\frac{\partial \Pi_{\alpha\beta}(\mathbf{r}, t)}{\partial r_\beta}, \quad (1)$$

with $\Pi_{\alpha\beta}(\mathbf{r}, t)$ the stress tensor. Conservation of angular momentum in the z -direction is expressed as

$$\frac{\partial [\epsilon_{\alpha\beta} r_\alpha p_\beta(\mathbf{r}, t) + s(\mathbf{r}, t)]}{\partial t} = -\frac{\partial j_\alpha^J(\mathbf{r}, t)}{\partial r_\alpha}, \quad (2)$$

with $j_\alpha^J(\mathbf{r}, t)$ the α th component of the angular momentum current and in the above equations the summation is over both α and β . The equation for the spin density is found by subtracting the cross-product of \mathbf{r} with equation (1) from equation (2) and yields

$$\frac{\partial s(\mathbf{r}, t)}{\partial t} = -\frac{\partial j_\alpha^S(\mathbf{r}, t)}{\partial r_\alpha} - 2\Pi^a(\mathbf{r}, t), \quad (3)$$

with $\Pi^a(\mathbf{r}, t) = \epsilon_{\alpha\beta} \Pi_{\beta\alpha}(\mathbf{r}, t)/2$ the antisymmetric part of the stress tensor and $j_\alpha^S(\mathbf{r}, t) = j_\alpha^J(\mathbf{r}, t) - \epsilon_{\beta\gamma} r_\beta \Pi_{\gamma\alpha}(\mathbf{r}, t)$ the spin current.

A nonzero velocity and spin density increase the energy of the system. By symmetry, a nonzero velocity leads to a contribution $\rho_{\text{kin}} \mathbf{v}^2/2$ to the energy density. This expression defines the kinetic mass density ρ_{kin} , such that $\mathbf{p}(\mathbf{r}, t) = \rho_{\text{kin}} \mathbf{v}(\mathbf{r}, t)$ [29]. For the case that is of interest to us, i.e. 2D electrons with spin–orbit coupling, the kinetic mass density is not equal to the average mass density ρ because spin–orbit coupling breaks Galilean invariance. Similar considerations arise in the hydrodynamic description of phonons [30].

A nonzero spin density contributes $\chi_s \mu_s^2/2$ to the energy density, where χ_s is the static spin susceptibility, so that $s(\mathbf{r}, t) = \hbar \chi_s \mu_s(\mathbf{r}, t)$. These terms in the energy density lead to contributions to the entropy production from which relations between the fluxes (the spin current and antisymmetric part of the pressure tensor) and the forces (spin accumulation and velocity) are derived phenomenologically. In terms of $\mu_s(\mathbf{r}, t)$ and $\mathbf{v}(\mathbf{r}, t)$ we have for the antisymmetric part of the pressure tensor that [24]

$$\Pi^a(\mathbf{r}, t) = -\eta_r [\omega(\mathbf{r}, t) - 2\mu_s(\mathbf{r}, t)/\hbar], \quad (4)$$

with $\omega(\mathbf{r}, t) = \epsilon_{\alpha\beta} \partial v_\beta(\mathbf{r}, t)/\partial r_\alpha$ the vorticity and η_r the rotational viscosity. The above expression, after insertion into equation (3), shows that angular momentum is transferred, by spin–orbit coupling, between orbital and spin degrees of freedom until the antisymmetric part of the pressure tensor is zero. For the spin current we have that $j_\alpha^S(\mathbf{r}, t) = -\sigma_s \partial \mu_s(\mathbf{r}, t)/\partial r_\alpha = -D_s \partial s(\mathbf{r}, t)/\partial r_\alpha$ which defines the spin diffusion constant D_s and spin conductivity σ_s , which obey the Einstein relation $\sigma_s = \hbar D_s \chi_s$. Note that we are omitting an advective contribution $\sim v_\alpha s$ to the spin current as we restrict ourselves to the linear-response regime. Inserting these results for the fluxes into equation (3) and using equation (1) leads to

$$\begin{aligned}
\frac{\partial s(\mathbf{r}, t)}{\partial t} &= D_s \nabla^2 s(\mathbf{r}, t) \\
&+ 2\eta_r \left[\omega(\mathbf{r}, t) - \frac{2s(\mathbf{r}, t)}{\hbar^2 \chi_s} \right] - \frac{s(\mathbf{r}, t)}{\tau_{sr}}; \\
\rho_{\text{kin}} \frac{\partial v_\alpha(\mathbf{r}, t)}{\partial t} &= -\frac{e\rho E_\alpha}{m} + \nu \rho_{\text{kin}} \nabla^2 v_\alpha(\mathbf{r}, t) \\
&+ \eta_r \epsilon_{\alpha\beta} \frac{\partial}{\partial r_\beta} \left[\omega(\mathbf{r}, t) - \frac{2s(\mathbf{r}, t)}{\hbar^2 \chi_s} \right] - \frac{\rho_{\text{kin}} v_\alpha(\mathbf{r}, t)}{\tau_{mr}}.
\end{aligned} \tag{5}$$

In the above we have assumed the linear-response regime and introduced the kinematic viscosity ν using that the symmetric part of the stress tensor is given by $\Pi_{\alpha\beta} = \nu \rho_{\text{kin}} \partial v_\alpha / \partial r_\beta$. Furthermore, we have added spin and momentum relaxation terms, parameterized by the phenomenological time scales τ_{sr} and τ_{mr} , respectively. We have also included an electric field \mathbf{E} (the electron has charge $-e$).

Equations (5) are the main phenomenological equations for spin density and velocity. The term proportional to η_r in the first equation describes generation of spin accumulation in response to vorticity, e.g. spin–vorticity coupling, and arises because of spin–orbit coupling. See [31] for an example of spin dynamics in the hydrodynamic regime without spin–orbit coupling.

In the steady state the hydrodynamic equations are characterized by three length scales. The first is a length scale that results from the spin–vorticity coupling equal to $\ell_{sv} = \sqrt{D_s \hbar^2 \chi_s / (2\eta_r)}$, which is the characteristic length over which the orbital and spin angular momentum equilibrate. Furthermore, we have the spin diffusion length $\ell_{sr} = \sqrt{D_s \tau_{sr}}$ that determines the length scales for relaxation of spin due to impurities, and the momentum diffusion length $\ell_{mr} = \sqrt{\nu \tau_{mr}}$. The most interesting regime, which occurs in the limit of strong spin–orbit coupling relative to momentum and spin relaxation, is the one where ℓ_{sv} is the shortest length scale. In this case the spin density locally follows the vorticity, which is determined by the electron flow.

3. Application

We consider electron flow through a PC [7, 18] driven by a voltage V . Taking $\tau_{mr}, \tau_{sr} \rightarrow \infty$ we have from [18] for the velocity distribution at the PC that

$$v_y(x) = -\frac{\pi\rho eV}{4m\nu\rho_{\text{kin}}} \sqrt{\left(\frac{w}{2}\right)^2 - x^2}, \tag{6}$$

where the flow is in the y -direction and w is the PC width. From equation (5), in the limit $\ell_{sv} \ll w$ the steady-state spin density generated at the PC by spin–vorticity coupling in the hydrodynamic regime is then

$$\frac{s(x)}{\hbar^2 \chi_s j^c} = -\frac{m}{\pi e w \rho} \frac{4x}{\sqrt{(w/2)^2 - x^2}}, \tag{7}$$

where $j^c = -e\rho \int dx v_y(x) / (mw)$ is the average current density.

Let us compare equation (7) with the spin density generated by the spin Hall effect in the diffusive limit. In the latter case, the spin accumulation is determined by $\partial^2 \mu_s / \partial x^2 = \mu_s / \ell_{sr}^2$, which follows from equations (5) in the limit $\ell_{sr} \ll \ell_{sv}$, together with the expression $j_y^s = -\sigma_s \partial \mu_x / \partial x + \theta_{\text{SH}} \hbar j_y^c / (2e)$ for the spin current. Here $j_y^c = \sigma_e E_y$ is the diffusive charge current through the PC, with $\sigma_e = e^2 \rho^2 \tau_{mr} / (m^2 \rho_{\text{kin}})$ the electrical conductivity and θ_{SH} the spin Hall angle. Using the boundary conditions $j^s(-w/2) = j^s(w/2) = 0$, we find for the spin density in the diffusive limit that

$$\frac{s_{\text{diff}}(x)}{\hbar^2 \chi_s j_y^c} = \frac{\theta_{\text{SH}} \ell_{sr}}{2e\sigma_s} \operatorname{sech}\left(\frac{w}{2\ell_{sr}}\right) \sinh\left(\frac{x}{\ell_{sr}}\right). \tag{8}$$

A crucial difference is thus that for diffusive spin transport and when $w \gg \ell_{sr}$, the spin density is only nonzero within a distance $\sim \ell_{sr}$ away from the edges of the PC, while when $w \gg \ell_{sv}$ and in the hydrodynamic limit, the spin density (see equation (7)) is nonzero everywhere (except at $x = 0$ where it vanishes by symmetry).

In both hydrodynamic and diffusive limits, the maximum spin density occurs at the edges. In the hydrodynamic limit the spin density formally diverges as $|x| \rightarrow w/2$, since the vorticity that results from the velocity in equation (6) diverges in the same limit. This divergence is, however, unphysical, as there will be a microscopic length scale ℓ_{edge} over which the velocity goes to zero near the edge of the sample, resulting in a maximum spin density of $|s(\pm w/2)| / (\hbar^2 \chi_s j^c) \sim m / (e\rho \ell_{\text{edge}})$ near the edges of the sample. We expect the latter to be much larger than the maximum spin density $|s_{\text{diff}}(\pm w/2)| / (\hbar^2 \chi_s j^c) \sim m^2 \theta_{\text{SH}} \ell_{sr} / (e \hbar \rho \tau_{mr})$ generated by the spin Hall effect in the diffusive regime (where we estimated $\sigma_s \sim \hbar \rho \tau_{mr} / m^2$), because

$\hbar\tau_{\text{mr}}/(m\theta_{\text{SH}}\ell_{\text{sr}}) \sim \ell_{\text{mr}}/(\theta_{\text{SH}}k_{\text{F}}\ell_{\text{sr}})$ is expected to be much larger than the microscopic length scale ℓ_{edge} . Here, k_{F} is the Fermi wave number.

4. Microscopic theory

We proceed by calculating the rotational viscosity microscopically, following argumentation similar to that developed for the drift-diffusion equation in [32]. We consider the limit that $\tau_{\text{sr}}, \tau_{\text{mr}} \rightarrow \infty$. To facilitate a linear-response calculation, we couple the spin density to a Zeeman magnetic field in the z -direction. This adds a term $-b(\mathbf{r}, t)s(\mathbf{r}, t)$ to the energy density, where $b(\mathbf{r}, t)$ is proportional to the Zeeman magnetic field and has the relevant prefactors absorbed. Note that for the purpose of developing the microscopic calculation we consider this magnetic field to couple to the spin only. We also put the electric field to zero. With these modifications, the hydrodynamic equations become [28]

$$\begin{aligned} \frac{\partial s(\mathbf{r}, t)}{\partial t} &= D_s \nabla^2 [s(\mathbf{r}, t) - \hbar^2 \chi_s b(\mathbf{r}, t)] \\ &+ 2\eta_r \left[\omega(\mathbf{r}, t) - \frac{2s(\mathbf{r}, t)}{\hbar^2 \chi_s} + 2b(\mathbf{r}, t) \right]; \\ \rho_{\text{kin}} \frac{\partial v_\alpha(\mathbf{r}, t)}{\partial t} &= \nu \rho_{\text{kin}} \nabla^2 v_\alpha(\mathbf{r}, t) \\ &+ \eta_r \epsilon_{\alpha\beta} \frac{\partial}{\partial r_\beta} \left[\omega(\mathbf{r}, t) - \frac{2s(\mathbf{r}, t)}{\hbar^2 \chi_s} + 2b(\mathbf{r}, t) \right], \end{aligned} \quad (9)$$

whereas equation (4) becomes

$$\Pi^a(\mathbf{r}, t) = -\eta_r [\omega(\mathbf{r}, t) - 2\mu_s(\mathbf{r}, t)/\hbar + 2b(\mathbf{r}, t)]. \quad (10)$$

The hydrodynamic equations are solved after Fourier transforming to frequency ω and momentum \mathbf{q} , which yields for the antisymmetric part of the pressure tensor that

$$\Pi^a(q, \omega) = \frac{2\hbar^2 \rho_{\text{kin}} \chi_s \eta_r \omega (i\nu q^2 + \omega) b(q, \omega)}{D(q, \omega)}, \quad (11)$$

with

$$\begin{aligned} D(q, \omega) &= 4\eta_r \rho_{\text{kin}} \hbar^2 (\nu q^2 - i\omega) \\ &- \chi_s (D_s q^2 - i\omega) ((\eta_r - \nu \rho_{\text{kin}}) q^2 + i\rho_{\text{kin}} \omega). \end{aligned} \quad (12)$$

Hence, the antisymmetric part of the pressure tensor has the static limit $\Pi^a(q, \omega \rightarrow 0) = 0$, and the dynamic limit

$$\Pi^a(q \rightarrow 0, \omega) = \frac{2i\hbar^2 \chi_s \eta_r \omega b(q \rightarrow 0, \omega)}{4\eta_r - i\hbar^2 \chi_s \omega}. \quad (13)$$

Using standard imaginary-time linear-response theory [33], we find that

$$\langle \hat{\Pi}^a(\mathbf{r}, \tau) \rangle = \frac{1}{\hbar} \int_0^{\hbar\beta} \int d\mathbf{r}' K(\mathbf{r} - \mathbf{r}'; \tau - \tau') b(\mathbf{r}', \tau'), \quad (14)$$

where the imaginary-time correlation function $K(\mathbf{r} - \mathbf{r}'; \tau - \tau') \equiv \langle \hat{\Pi}^a(\mathbf{r}, \tau) \hat{s}(\mathbf{r}', \tau') \rangle_0$, and the expectation value $\langle \dots \rangle_0$ is taken at equilibrium. In this expression, the spin density operator in imaginary time τ is $\hat{s}(\mathbf{r}, \tau) = \hbar [\hat{\psi}_\uparrow^\dagger(\mathbf{r}, \tau) \hat{\psi}_\uparrow(\mathbf{r}, \tau) - \hat{\psi}_\downarrow^\dagger(\mathbf{r}, \tau) \hat{\psi}_\downarrow(\mathbf{r}, \tau)]/2$, where the dependence on τ of the electron creation $[\hat{\psi}_\sigma^\dagger(\mathbf{r})]$ and annihilation $[\hat{\psi}_\sigma(\mathbf{r})]$ operators indicates their corresponding Heisenberg evolution in imaginary time ranging from 0 to $\hbar\beta$, with $\beta = 1/(k_{\text{B}}T)$ the inverse thermal energy. The operator expression $\hat{\Pi}^a(\mathbf{r}, \tau)$ for the antisymmetric part of the pressure tensor depends on the microscopic hamiltonian and is determined by the operator version of equation (3) in imaginary time. Using this latter equation at zero wavevector, however, we do not need the microscopic expression for the antisymmetric part of the pressure tensor as we use equation (3) to express it in terms of the spin density. Namely, at zero wavevector, i.e. for the homogeneous situation, we have from equation (3) in imaginary time that

$$\frac{\partial \hat{s}(\mathbf{r}, t)}{\partial \tau} = 2\hat{\Pi}^a(\mathbf{r}, \tau). \quad (15)$$

From this, we find $K(\mathbf{r} - \mathbf{r}'; \tau - \tau') = \langle (\partial \hat{s}(\mathbf{r}, \tau) / \partial \tau) \hat{s}(\mathbf{r}', \tau') \rangle_0 / 2$, which, after Fourier transforming, yields $K(q \rightarrow 0; i\omega_n) \propto \omega_n \chi_s(i\omega_n)$, where $\chi_s(i\omega_n)$ is the imaginary-time spin-spin response function at zero wavevector (see equation (19) below), and where the zero-wavevector limit has to be taken because equation (15) can only be used in this limit. Moreover, $\omega_n = 2\pi n / (\hbar\beta)$ is a bosonic Matsubara frequency. After a Wick

rotation we have the usual retarded response functions $K^{(+)}(q, \omega) = K(q; i\omega_n \rightarrow \omega + i0^+)$ and $\chi_s^{(+)}(\omega) = \chi_s(i\omega_n \rightarrow \omega + i0^+)$, and find $K^{(+)}(q \rightarrow 0; \omega) = -i\hbar^2\omega\chi_s^{(+)}(\omega)/2$. With this result inserted, the Fourier-transformed version of equation (14) yields

$$\langle \hat{\Pi}^a \rangle(q \rightarrow 0, \omega) = \frac{i\hbar^2\omega\chi_s^{(+)}(\omega)b(q \rightarrow 0, \omega)}{2}. \quad (16)$$

Comparing this with the phenomenological result in equation (13), we find that

$$\frac{1}{\eta_r} = -\left(\frac{2}{\hbar\chi_s}\right)^2 \lim_{\omega \rightarrow 0} \frac{\text{Im}[\chi_s^{(+)}(\omega)]}{\omega}, \quad (17)$$

which is a convenient starting point for calculating η_r microscopically.

The above result for η_r can also be understood by considering the phenomenological equation (5) in the homogeneous limit, while taking $\tau_{sr}, \tau_{mr} \rightarrow \infty$. Then, the term proportional to the rotational viscosity takes the form of a spin-relaxation term. This shows that, in the absence of other spin relaxation mechanisms (i.e. when $\tau_{sr} \rightarrow \infty$) the rotational viscosity governs the spin relaxation. This makes sense because the only way for spin angular momentum to decay in a translation and rotation invariant electron fluid is by transferring angular momentum to the motion of the electron fluid itself, which then manifests as vorticity.

As a representative example, we compute the rotational viscosity using standard linear-response techniques for a 2D electron gas with Rashba spin-orbit coupling, which has the following Hamiltonian [34]:

$$\hat{\mathcal{H}} = \int d\mathbf{r} \sum_{\sigma \in \{\uparrow, \downarrow\}} \hat{\psi}_\sigma^\dagger(\mathbf{r}) \left[-\frac{\hbar^2 \nabla^2}{2m} + \lambda \hbar \hat{z} \cdot \left(\frac{\nabla}{i} \times \boldsymbol{\tau} \right) \right] \hat{\psi}_\sigma(\mathbf{r}), \quad (18)$$

where $\boldsymbol{\tau}$ is a vector of Pauli matrices. The unit vector in the \hat{z} -direction is denoted by \hat{z} . The constant λ parametrizes the strength of spin-orbit interactions. We have for the imaginary-time spin-spin response function

$$\chi_s(i\omega_n) = \frac{1}{\hbar} \int d\mathbf{r} \int_0^{\hbar\beta} d\tau \langle \hat{s}(\mathbf{r}, \tau) \hat{s}(\mathbf{r}, 0) \rangle_0 e^{i\omega_n \tau}. \quad (19)$$

Neglecting vertex corrections due to interactions, this is worked out to yield

$$\chi_s(i\omega_n) = -\frac{1}{4\hbar V} \sum_{\mathbf{k}} \sum_{\delta \neq \delta'} \int d\hbar\omega d\hbar\omega' A_\delta(k, \omega) A_{\delta'}(k, \omega') \times \left[\frac{N(\hbar\omega) - N(\hbar\omega')}{\omega - \omega' + i\omega_n} \right], \quad (20)$$

with $N(\hbar\omega) = [e^{\beta(\hbar\omega - \mu)} + 1]^{-1}$ the Fermi-Dirac distribution function at chemical potential μ . The spectral functions $A_\delta(k, \omega)$ are labeled by the Rashba spin-orbit-split band index $\delta = \pm$. We incorporate electron-electron interactions into the spectral function by taking them equal to Lorentzians broadened by the electron collision time τ_{ee} (this corresponds to dressing bare propagator lines in the spin bubble in equation (19) by self-energy insertions), i.e.

$$A_\delta(k, \omega) = \frac{\hbar}{2\pi\tau_{ee}} \frac{1}{[\hbar\omega - \hbar\omega_\delta(k)]^2 + \left(\frac{\hbar}{2\tau_{ee}}\right)^2}, \quad (21)$$

where $\hbar\omega_\delta(k) = \hbar^2 k^2/2m + \delta\hbar\lambda k$ is the Rashba band dispersion. Inserting equation (21) into equation (20) and performing a Wick rotation yields

$$\eta_r = \frac{4\pi^2 \hbar^4 \chi_s^2}{m\tau_{ee}} \left[2\pi + \frac{8\left(\frac{\mu\tau_{ee}}{\hbar}\right)}{1 + 4\left(\frac{\mu\tau_{ee}}{\hbar}\right)^2} + 4 \tan^{-1} \left(\frac{2\mu\tau_{ee}}{\hbar} \right) \right], \quad (22)$$

where we took $\lambda \rightarrow 0$. In the limit $\mu\tau_{ee}/\hbar \gg 1$, we have $\eta_r = \pi\hbar^4\chi_s^2/(m\tau_{ee})$.

Since we have neglected vertex corrections, the result in equation (22) does not vanish in the $\lambda \rightarrow 0$ limit and is strictly speaking only valid when spin-orbit coupling is so strong that the spin-vorticity coupling is limited by electron-electron interactions, i.e. when $\lambda k_F \tau_{ee} \gg 1$. In the opposite limit, where the bottleneck for spin relaxation is the spin-orbit coupling, we perform a Fermi's Golden Rule calculation to determine the decay rate of a spin polarization to second order in the strength of the spin-orbit interactions. Similar calculations were carried out by Bogulawski in the non-degenerate limit [35] and by Bir *et al* for electron-hole spin-flip scattering [36]. This gives at low temperatures that

$$\eta_r = -\frac{\pi\hbar}{8} \int \frac{d\mathbf{k}}{(2\pi)^2} A^2(k, \mu) (\lambda\hbar k)^2, \quad (23)$$

where $A(k, \mu)$ is the spectral function obtained from equation (21) by replacing $\hbar\omega_\delta(k) \rightarrow \hbar^2 k^2/2m$. Carrying out the remaining integral gives

$$\eta_r = \frac{m\lambda^2}{2\hbar} \left[1 + \pi \left(\frac{\mu\tau_{ee}}{\hbar} \right) + 2 \left(\frac{\mu\tau_{ee}}{\hbar} \right) \tan^{-1} \left(\frac{2\mu\tau_{ee}}{\hbar} \right) \right], \quad (24)$$

which indeed vanishes as $\lambda \rightarrow 0$. When $\mu\tau_{ee}/\hbar \gg 1$, we have that $\hbar\eta_r \sim (\lambda k_F)(\lambda k_F \tau_{ee})$, showing the dependence on the small parameter $\lambda k_F \tau_{ee} \ll 1$ explicitly. Interestingly, since the kinematic viscosity $\nu \propto \tau_{ee}$, we have that the rotational viscosity $\eta_r \propto 1/\nu$ in the limit of strong spin–orbit coupling and $\eta_r \propto \nu$ in the limit of weak spin–orbit coupling, with a maximum rotational viscosity when $\lambda k_F \tau_{ee} \sim 1$. This is reminiscent of the Dyakonov–Perel result for spin relaxation [37] with the momentum scattering time replaced by the electron–electron scattering time, and is also expected on dimensional grounds. This also shows that in the degenerate limit rotational viscosity is mostly determined by the strength of the interactions and spin–orbit coupling.

For completeness, we also compute the kinetic mass density for the Rashba model in the noninteracting limit. We then have that the momentum density is given by

$$\mathbf{p} = \sum_\delta \int \frac{d\mathbf{k}}{(2\pi)^2} \hbar \mathbf{k} A_\delta(k, \omega) N(\hbar\omega - \mu - \hbar \mathbf{k} \cdot \mathbf{v}), \quad (25)$$

which yields $\mathbf{p} = \rho_{\text{kin}} \mathbf{v}$ for $\mathbf{v} \rightarrow \mathbf{0}$, with the kinetic mass density $\rho_{\text{kin}} = \rho[1 + \lambda^2 m/2\mu + \mathcal{O}(\lambda^4)]$. The correction, i.e. the second term between the brackets, is typically small since one almost always has that $\lambda^2 m/2\mu \ll 1$.

5. Estimates

Next, we estimate the spin–vorticity coupling for graphene with proximity-induced spin–orbit coupling. We consider the degenerate limit where lack of Galileian invariance and thermal effects may be ignored [38, 39]. We take $\lambda\hbar k_F$ to be on the order of 1 meV [40]. Furthermore, we take $\tau_{ee} \sim 100$ fs [4]. We thus have that $\lambda\hbar k_F$ is about one order of magnitude smaller than \hbar/τ_{ee} and use the weak spin–orbit coupling expression in equation (23). Evaluating equation (23) for a linear dispersion $\hbar v_F k$, where $v_F \sim 10^6$ m s^{−1} is the graphene Fermi velocity, we find that

$$\eta_r \sim \frac{(\lambda\hbar k_F)^2}{\hbar v_F^2} \left(\frac{\mu\tau_{ee}}{\hbar} \right), \quad (26)$$

using $\mu\tau_{ee} \gg \hbar$. We estimate the corresponding inverse time scale as

$$\frac{\eta_r}{\hbar^2 \chi_s} \sim \frac{(\lambda\hbar k_F)^2}{\hbar^3 \chi_s v_F^2} \left(\frac{\mu\tau_{ee}}{\hbar} \right) \sim 100 \text{ GHz}, \quad (27)$$

where we took $\mu\tau_{ee}/\hbar \sim 10$, and estimated the spin susceptibility as $\chi_s \sim D(\mu)$, with the density of states at the Fermi level $D(\mu) \sim \sqrt{n_e}/(\hbar v_F)$, and the electron number density $n_e \sim 10^{12}$ cm^{−2} [4].

To estimate the corresponding length scale ℓ_{sv} , we assume that spin diffusion is in the hydrodynamic regime determined by electron–electron interactions that lead to spin drag [41]. We then have for the spin diffusion constant that $D_s \sim \hbar\rho\tau_{ee}/(m^2\chi_s)$. The spin–vorticity length scale is then $\ell_{sv} \sim v_F \hbar \sqrt{\tau_{ee}\chi_s}/\eta_r \sim 1$ μm. This is the same order of magnitude as the momentum relaxation length scale ℓ_{mr} [4], so that the rotational viscosity appears to be high enough to lead to observable spin–vorticity coupling. Moreover, the limit where $\ell_{sv} < \ell_{mr}$ seems to be within experimental reach. Note that in the regime of weak spin–orbit coupling we have for the spin relaxation the Dyakonov–Perel result that $1/\tau_{sr} \propto \tau_{mr}$ [37], which yields that in the hydrodynamic regime we have $\ell_{sr} \sim \ell_{sv} \sqrt{\tau_{ee}/\tau_{mr}} \gg \ell_{sv}$.

A simple interpretation of the spin–vorticity coupling is that the electron spins are polarized by an effective magnetic field $\hbar\omega(\mathbf{r}, t)/\mu_B$, with μ_B the Bohr magneton, in the frame that rotates with the electron flow vorticity. We estimate the vorticity $\omega \sim v/\ell_{mr}$ using $\ell_{mr} \sim 0.1$ – 1 μm, and a drift velocity of $v \sim 100$ m s^{−1} [4], which yields a substantial effective magnetic field of 1–10 mT.

6. Discussion and conclusions

We have developed the theory for spin–vorticity coupling in viscous electron fluids, both phenomenologically and microscopically, and we have estimated that the proximity-induced spin–orbit coupling in graphene is large enough for observable effects. As an example, we predict a large spin polarization induced by spin–hydrodynamic generation in a PC. This large spin density may e.g. be observed optically [42] or via nitrogen-

vacancy center magnetometry [43, 44]. The imaged spin density would provide a fingerprint of the vorticity of the electron flow.

An interesting direction for future research is generalization of the phenomenological and microscopic derivation to other spin–orbit couplings, including, in particular, also the effects of violation of translational and rotational invariance beyond the phenomenological relaxation terms that we included here. One example would be that of Weyl semi-metals that naturally have sizeable spin–orbit coupling and have also been reported to be able to reach the hydrodynamic regime [45]. Other candidates are bismuthene [46] and stanene [47] that combine strong spin–orbit coupling with high mobility. Further interesting directions of research include incorporating effects of a magnetic field and computation of the rotational viscosity in the regime where spin–orbit interactions and electron–electron interactions are comparable in magnitude. In this regime, the crossover from weak-to-strong spin–orbit coupling takes place, whereas inclusion of momentum-relaxing scattering would lead to a crossover from the spin–vorticity coupling to the spin Hall effect.

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