

Dynamic Toolbox for ETRINV

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Abstract

Recently, various natural algorithmic problems have been shown to be $\exists\mathbb{R}$ -complete. The reduction relied in many cases on the $\exists\mathbb{R}$ -completeness of the problem ETR-INV, which served as a useful intermediate problem. Often some strengthening and modification of ETR-INV was required. This led to a cluttered situation where no paper included all the previous details. Here, we give a streamlined exposition in a self-contained manner. We also explain and prove various universality results regarding ETR-INV.

These notes should not be seen as a research paper with new results. However, we do describe some refinements of earlier results which might be useful for future research. We plan to extend and update this exposition as seems fit.

1 Introduction

1.1 The complexity class $\exists\mathbb{R}$

The *first order theory of the reals* is a set of all true sentences involving real variables, universal and existential quantifiers, boolean and arithmetic operators, constants 0 and 1, parenthesis, equalities and inequalities, i.e., the alphabet is the set

$$\{x_1, x_2, \dots, \forall, \exists, \wedge, \vee, \neg, 0, 1, +, -, \cdot, (,), =, <, \leq\}.$$

A formula is called a *sentence* if it has no free variables, i.e., each variable present in the formula is bound by a quantifier. Note that using such formulas, we can easily express integer constants (using binary expansion) and powers. Each formula can be converted to a *prenex form*, which means that it starts with all the quantifiers and is followed by a quantifier-free formula. Such a transformation changes the length of the formula by at most a constant factor.

The *existential theory of the reals* is the set of all true sentences of the first-order theory of the reals in prenex form with existential quantifiers only, i.e., the sentences are of the form

$$(\exists x_1 \exists x_2 \dots \exists x_n) \Phi(x_1, x_2, \dots, x_n),$$

where $\Phi := \Phi(x_1, x_2, \dots, x_n)$ is a quantifier-free formula of the first-order theory of the reals with variables x_1, \dots, x_n . The problem ETR is the problem of deciding whether a given sentence of the above form is true, and we say that Φ is an *ETR formula*. We define

$$V(\Phi) := \{\mathbf{x} \in \mathbb{R}^n : \Phi(\mathbf{x})\}.$$

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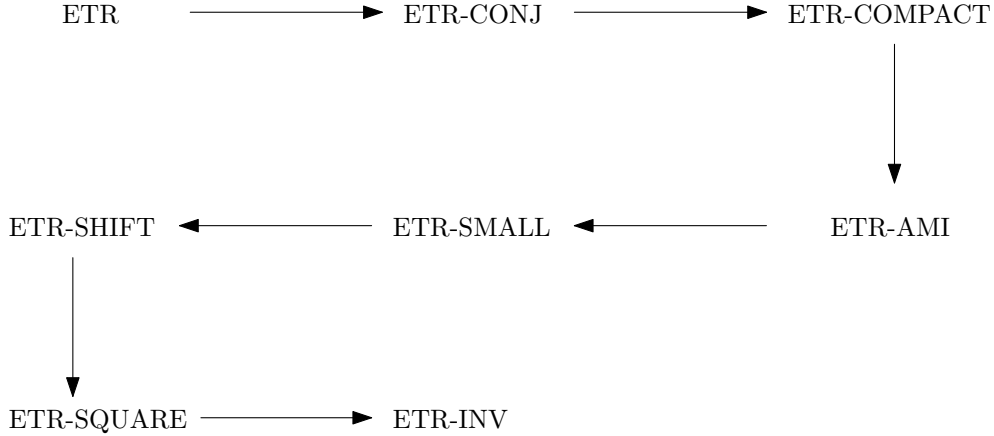


Figure 1: Overview of the problems described in these notes and how they are reduced to each other. Almost all reductions preserve rational equivalence and are linear.

Thus ETR is the problem of deciding if $V(\Phi)$ is non-empty. The complexity class $\exists\mathbb{R}$ consists of all problems that are reducible to ETR in polynomial time. It is currently known that

$$\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}.$$

It is not hard to see that the problem ETR is NP-hard, yielding the first inclusion. The containment $\exists\mathbb{R} \subseteq \text{PSPACE}$ is highly non-trivial, and it has first been established by Canny [5]. In order to compare the complexity classes NP and $\exists\mathbb{R}$, we suggest the reader to consider the following two problems. The problem of deciding whether a given polynomial equation $Q(x_1, \dots, x_n) = 0$ with integer coefficients has a solution with all variables restricted to $\{0, 1\}$ is easily seen to be NP-complete. On the other hand, if the variables are merely restricted to \mathbb{R} , the problem is $\exists\mathbb{R}$ -complete [10, Proposition 3.2].

The *description complexity* or simply *complexity* of an ETR formula Φ is the number of symbols in Φ , and is also denoted $|\Phi|$. The *complexity* of a semi-algebraic set S is the minimum complexity of a formula Φ such that $V(\Phi) = S$.

1.2 Contribution

Abrahamsen, Adamaszek and Miltzow [1] introduced the algorithmic problem ETR-INV and showed that it is $\exists\mathbb{R}$ -complete. This was one of the important conceptual steps to show that the Art Gallery Problem is $\exists\mathbb{R}$ -complete, for two reasons. First, all variables are conveniently bounded to the interval $[1/2, 2]$. Second, it is only necessary to encode inversion constraints ($x \cdot y = 1$) instead of the more general multiplication constraints ($x \cdot y = z$). See Section 8 for a precise definition.

In this exposition, we repeat the reduction from [1]. Some refinements are added that might turn out useful in future research [6, 7, 9]. Let us point out the advantages of this exposition.

Self-Containment and exhaustiveness. We explain all details in one coherent exposition. In particular those that are scattered around and pointed out in various papers. In particular, we also explain some parts that were folklore, and only cited in [1]. (However, we still make use of some facts from real algebraic geometry where we refer to other sources for a proof.)

Universality-type statements. Rather than just $\exists\mathbb{R}$ -hardness, we also want to make so-called *universality*-type statements, which we will explain in Section 1.5. For that, we have to point out some details that were not given in [1].

Running time analysis. We point out the running time of the reductions precisely. This can be helpful for so called fine-grained complexity and lower bounds based on the Exponential Time Hypothesis.

Linear extension. In our reductions, we transform one formula Φ to another Ψ . In generating Ψ , we replace old variables by new *replacement* variables and we add new *auxiliary* variables. Here, all the replacement variables are just shifted or scaled versions of the old variables, and the auxiliary variables are likewise determined by the old variables. We capture this by defining the notion of linear extensions in Section 1.5. This notion is useful for the so-called Picasso-Theorem [1] and Kempe’s universality theorem.

Dynamics. It has happened repeatedly that we needed some slightly stronger statement than was provided in the previous paper [1]. We expect this to happen in the future as well. This exposition is supposed to be dynamic and updated, so as to give a complete overview of the known techniques for making $\exists\mathbb{R}$ -hardness proofs by reductions from ETR-INV.

Modularity. Every reduction is stated as a separate lemma. We hope that in this way it is more convenient to reuse parts of the reductions in forthcoming papers.

Tiny range promise. Interestingly, all variables can be restricted to an arbitrarily small range within $[1/2, 2]$. This is an important observation for a forthcoming paper.

Reductions by diagrams. There were a large number of new variables and constraints introduced in the previous reduction to ETR-INV. It was straight-forward but tedious to check that those constraints work correctly and that each new variable was in the correct range. We introduce a new type of diagrams to express such reductions involving numerous constraints, which makes it much easier to read and check the reductions.

1.3 Main Result

To illustrate our findings, we point out one important theorem. Similar theorems can be derived from the reductions presented in this exposition. See Section 1.5 for the definition of rational equivalence and linear extension.

Theorem 1. *ETR-INV is $\exists\mathbb{R}$ -complete. Furthermore, for every instance Φ of ETR where $V(\Phi)$ is compact, there is an instance Ψ of ETR-INV such that $V(\Phi)$ and $V(\Psi)$ are rationally equivalent and $V(\Psi)$ is a linear extension of $V(\Phi)$.*

We give two corollaries here to the main theorem.

Corollary 2 (Algebraic consequence). *Let α be an algebraic number. Then there exists an instance Ψ of ETR-INV, such that Ψ has a solution when the variables are restricted to $\mathbb{Q}[\alpha]$, but no solution when the variables are restricted to a field that does not contain α .*

Proof. Let $p \in \mathbb{Z}[x]$ be a univariate polynomial with $p(\alpha) = 0$, $p \neq 0$. Furthermore, let $\alpha \in [a, b]$ be an interval such that α is the only root of p in $[a, b]$. Then $p(x) = 0$ and $a \leq x \leq b$, describe a compact semi-algebraic set V . By Theorem 1 there is an ETR-INV instance Ψ such that V and $V(\Psi)$ are rationally equivalent. \square

Corollary 3 (Torus). *There exists an instance Ψ of ETR-INV such that $V(\Psi)$ is homeomorphic to a torus.*

Proof. The equation

$$(x^2 + y^2 + z^2 + 5^2 - 1^2)^2 = 4 \cdot 5^2(x^2 + y^2)$$

describes a torus. This is a compact semi-algebraic set. By Theorem 1 exists an homeomorphic ETR-INV formula. \square

1.4 Related Work

We want to point out that many early parts of this reduction can be considered folklore. Already in 1876 Kempe [8] exposed ideas which are the core of our reduction in Section 4. Other important work was done by Shor [15] with his contribution to $\exists\mathbb{R}$ -hardness of stretchability. See also [11, 4, 14, 12].

1.5 Universality-type theorems

A universality statement says that for every object of type A exists an object of type B such that A and B are equivalent in some sense C. In Theorem 1, we had compact semi-algebraic sets as objects of type A, ETR-INV formulas are objects of type B, and we preserve algorithmic, topological and algebraic properties. Given our reductions, it is possible to make such statements by replacing A, B and C by something else. We think that rational equivalence and compact semi-algebraic sets are good choices for two reasons. First, compact semi-algebraic sets are very versatile. Of course, it would be nicer to have general, rather than compact, semi-algebraic sets, but ETR-INV cannot encode any open set, so this is not conceivable. The second reason is that rational equivalence seems to preserve topological and algebraic properties in a very strong sense and we are not aware how to improve this further. In this paper, whenever, we refer to polynomials, we implicitly assume the polynomials to be multivariate with integer coefficients.

Definition 4 (Rational Equivalence). Consider two sets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ and a function $f : V \rightarrow W$. We say that f is a *homeomorphism* if it is continuous, invertible, and its inverse f^{-1} is continuous as well.

Write f as its components (f_1, \dots, f_m) , where $f_i : V \rightarrow \mathbb{R}$ for each $i \in \{1, \dots, m\}$. Then f is *rational* if each function f_i is the ratio of two polynomials, with integer coefficients.

The sets V and W are *rationally equivalent* if there exists a homeomorphism $f : V \rightarrow W$ such that both f and f^{-1} are rational functions. In that case, we write $V \simeq W$.

Be aware that the term linear extension, that we define below, has other meanings in other contexts.

Definition 5 (linear extension). Given two sets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$, we say that W is a *linear extension* of V if there is an orthogonal projection $\pi : W \rightarrow V$ and two vectors $a, b \in \mathbb{Q}^n$ such that the mapping

$$x \mapsto a \cdot \pi(x) + b$$

is a continuous bijection. In this case we write $V \leq_{\text{lin}} W$.

Remark 6. Rational equivalence linear extension are both transitive relations, i.e., if $V \simeq W$ and $W \simeq U$, then $V \simeq U$, and if $V \leq_{\text{lin}} W$ and $W \leq_{\text{lin}} U$, then $V \leq_{\text{lin}} U$.

Furthermore, for a compact set V , if $V \simeq W$ or $V \leq_{\text{lin}} W$, then W is compact as well.

Example 7. Let $V := [1, 2]$ and $W := \{(x, y) : x \in [1, 2], x = y^2\}$. Then W is not a linear extension of V , and V and W are neither rationally equivalent, as W has two connected components.

1.6 Naming the Variables

We typically denote a new variable with a multi-character symbol such as $\llbracket v \rrbracket$. Here, v is an expression involving already known quantities, and $\llbracket v \rrbracket$ should be thought of as a placeholder with that value. In the case that v is a constant, e.g., $v = 2$ such that the variable is $\llbracket 2 \rrbracket$, then we say that $\llbracket v \rrbracket$ is a *constant variable*. By *constructing* a constant variable $\llbracket v \rrbracket$ in an ETR formula Φ , we mean introducing variables and constraints to Φ such that it follows that in every solution to Φ , we have $\llbracket v \rrbracket = v$.

The expression v of a variable $\llbracket v \rrbracket$ can also involve other parameters such as for instance $\llbracket x^2 + 2x + 1 \rrbracket$. In that case $\llbracket x^2 + 2x + 1 \rrbracket$ should be thought of as a variable holding the value $x^2 + 2x + 1$, where x is another variable or a parameter of the problem. It should follow from the assumptions or constraints introduced in the concrete case that $\llbracket x^2 + 2x + 1 \rrbracket$ indeed has the value $x^2 + 2x + 1$ in any solution to Φ .

2 Reduction to Conjunctive Form

Definition 8. An ETR-CONJ formula $\Phi := \Phi(x_1, \dots, x_n)$ is a conjunction $C_1 \wedge \dots \wedge C_m$, where $m \geq 0$ and each C_i is of one of the two forms

$$x \geq 0, \quad p(y_1, \dots, y_l) = 0$$

for $x, y_1, \dots, y_l \in \{x_1, \dots, x_n\}$ and p is a polynomial.

Note that since there are no strict inequalities in a formula Φ in ETR-CONJ, the set $V(\Phi)$ is closed.

We show how to reduce a general ETR formula to an ETR-CONJ formula. The reduction preserves rational equivalence and runs in linear time. A similar reduction has been described by Schaefer and Štefankovič [14].

Lemma A. *Given an ETR formula Φ , we can in $O(|\Phi|)$ time compute an ETR-CONJ formula Ψ such that $V(\Phi) \simeq V(\Psi)$ and $V(\Phi) \leq_{\text{lin}} V(\Psi)$.*

Proof. We start with an ETR formula Φ and modify it repeatedly to attain an ETR-CONJ formula Ψ . Each modification leads to an equivalent formula. Our modifications can be summarized in four steps. (1) Delete “ \neg ”. (2) Delete “ $>$ ”. (3) Move “ \geq ” to variables only. (4) Delete “ \vee ”. In the rest of this proof p and q denote polynomials.

Step (1): Here, we merely “pull” every negation \neg in front of every atomic predicate. For instance $\neg(A \vee B \vee C)$ becomes $(\neg A \wedge \neg B \wedge \neg C)$. To see that this can be done in linear time, note that the length of Φ is at least the number of atomic predicates. In the end of this process, every atomic predicate is preceded by either a negation or not. It may be that \wedge and \vee symbols are swapped, but we count both as one symbol.

Thereafter each atomic predicate preceded by \neg is replaced as follows:

$$\begin{aligned}\neg(q > 0) &\mapsto -q \geq 0 \\ \neg(q = 0) &\mapsto (q > 0) \vee (-q > 0) \\ \neg(q \geq 0) &\mapsto -q > 0\end{aligned}$$

Those replacements are done repeatedly until there are no occurrence of “ \neg ” left in the formula.

Step (2): We replace each inequality as follows:

$$q > 0 \mapsto (q \cdot z - 1 = 0) \dots \wedge z \geq 0.$$

The dots indicate that the predicate $z \geq 0$ does not immediately follow after $(q \cdot z - 1 = 0)$, but will be adjoined at the end of the new formula. Furthermore, z denotes a new variable. Those replacements are done repeatedly till there are no occurrence of “ $>$ ” left in the formula.

Step (3): We replace all atomic predicates of the form $q \geq 0$ by the predicate $q - z = 0$ and adjoin a new predicate $z \geq 0$ at the the end of the formula. Again z denotes a new variable.

Step (4): We delete disjunctions as follows. It will also be necessary to replace some conjunctions. Let Φ be the formula after Step (1)–(3). Suppose that there is a disjunction somewhere in Φ , and write it as $\Phi_1 \vee \Phi_2$ for two sub-formulas Φ_1 and Φ_2 . Note that $\Phi_1 \vee \Phi_2$ might just be a small part of Φ – there will in general be more of Φ to the right and left of this part.

We want to reduce each of Φ_i to a single polynomial equation, as follows. Note that since we have already performed Step (1)–(3), there are no inequalities in Φ_i . Suppose that Φ_i is not already a single polynomial equation. Then there must somewhere in Φ_i be either (i) a disjunction $p = 0 \vee q = 0$ or (ii) a conjunction $p = 0 \wedge q = 0$. We now explain how to reduce each of these cases to a simpler case.

- **Case (i):** We make the replacement

$$p = 0 \vee q = 0 \mapsto p \cdot q = 0.$$

- **Case (ii):** We make the replacement

$$p = 0 \wedge q = 0 \mapsto x \cdot x + y \cdot y = 0 \dots \wedge p - x = 0 \wedge q - y = 0.$$

Here, x and y are new variables. As in Step (2), the part following the dots is appended at the end of the complete formula Φ .

Eventually, we have reduced each Φ_i to a single polynomial equation. Thus the original disjunction $\Phi_1 \vee \Phi_2$ has the form as in Case (i), and we apply the replacement rule described there.

At first, it might seem easier in Case (ii) to replace $p = 0 \wedge q = 0$ by $p \cdot p + q \cdot q = 0$. However, we want our reduction to be linear and the simplified step could, if done repeatedly, lead to very long formulas.

With the replacement rules we have suggested, each iteration reduces the number of disjunctions and conjunctions by one and increases the length of the formula by at most a constant. Those replacements are done repeatedly till there are no disjunctions left in the formula.

This reduction takes linear time and the final formula Ψ is in conjunctive form. We need to describe a rational function

$$f : V(\Phi) \rightarrow V(\Psi).$$

Note that Ψ has all the original variables x_1, \dots, x_n of Φ plus some additional variables, which we denote by z_1, \dots, z_k . If $z \in \{z_1, \dots, z_k\}$ is introduced in step (2), it is assigned the value

$z = \frac{1}{q}$ and if it is introduced in step (3) or (4), it is assigned the value $z = q$ for some polynomial q . This defines f . Assume that Ψ has the free variables $x_1, \dots, x_n, z_1, \dots, z_k$, where z_1, \dots, z_k are the variables introduced by the reduction. Then

$$f^{-1} : (x_1, \dots, x_n, z_1, \dots, z_k) \mapsto (x_1, \dots, x_n).$$

Thus f and f^{-1} are rational bijective functions. Thus f is a homeomorphism. The description of f^{-1} implies that $V(\Psi)$ is a linear extension of $V(\Phi)$. □

Remark 9. Note that the standard way to remove strict inequalities is

$$q > 0 \mapsto q \cdot y \cdot y - 1 = 0.$$

However, this implies that $y = \pm\sqrt{1/q}$. This transformation has two issues. First, the number of solutions in a sense doubles, as the sign of y is not fixed. Second, irrational solutions are introduced where before may have been only rational solutions.

3 Reduction to Compact Semi-Algebraic Sets

Definition 10. In the problem ETR-COMPACT, we are given an ETR-CONJ formula Φ with the promise that $V(\Phi)$ is compact. The goal is to decide if $V(\Phi)$ is non-empty.

In this section, we describe a reduction from ETR-CONJ to ETR-COMPACT. We need a tool from real algebraic geometry. The following corollary has been pointed out by Schaefer and Štefankovič [14] in a simplified form.

Corollary 11 (Basu, Roy [3] Theorem 2). *Let Φ be an instance of ETR of complexity $L \geq 4$ such that $V(\Phi)$ is a non-empty subset of \mathbb{R}^n . Let B be the set of points in \mathbb{R}^n at distance at most $2^{L^{8n}} = 2^{2^{8n \log L}}$ from the origin. Then $B \cap V(\Phi) \neq \emptyset$.*

Lemma B. *Given an ETR-CONJ formula $\Phi := \Phi(x_1, \dots, x_n)$, we can in $O(|\Phi| + n \log |\Phi|)$ time create an ETR-CONJ formula Ψ such that $V(\Psi)$ is compact and $V(\Phi) \neq \emptyset \Leftrightarrow V(\Psi) \neq \emptyset$. In other words, there is a reduction from ETR-CONJ to ETR-COMPACT in near-linear time.*

Proof. Let an instance Φ of ETR-CONJ be given and define $k := \lceil 8n \log L \rceil$. To make an equivalent formula Ψ such that $V(\Psi)$ is compact, we start by including all the variables and constraints of Φ in Ψ . We then construct a large constant variable $\llbracket 2^{2^k} \rrbracket$ using *exponentiation by squaring*.

$$\begin{aligned} \llbracket 2^{2^0} \rrbracket - 1 - 1 &= 0 \\ \llbracket 2^{2^1} \rrbracket - \llbracket 2^{2^0} \rrbracket \cdot \llbracket 2^{2^0} \rrbracket &= 0 \\ &\vdots \\ \llbracket 2^{2^k} \rrbracket - \llbracket 2^{2^{k-1}} \rrbracket \cdot \llbracket 2^{2^{k-1}} \rrbracket &= 0 \end{aligned}$$

For each variable x of Φ , we now introduce the variables $\llbracket x - 2^{2^k} \rrbracket$ and $\llbracket x - 2^{2^k} - x \rrbracket$ and the constraints

$$\begin{aligned} \llbracket x + 2^{2^k} \rrbracket - x - \llbracket 2^{2^k} \rrbracket &= 0 \\ \llbracket x + 2^{2^k} \rrbracket &\geq 0 \\ \llbracket 2^{2^k} - x \rrbracket - \llbracket 2^{2^k} \rrbracket + x &= 0 \\ \llbracket 2^{2^k} - x \rrbracket &\geq 0. \end{aligned}$$

Note that this corresponds to introducing the constraint $-2^{2^k} \leq x \leq 2^{2^k}$ in Ψ .

It now follows by Corollary 11 that

$$V(\Phi) \neq \emptyset \Leftrightarrow V(\Psi) \neq \emptyset.$$

Note that $V(\Psi)$ is compact since Ψ contains no strict inequalities and each variable is bounded. This finishes the proof. \square

Remark 12. Unfortunately, we do not have $V(\Phi) \simeq V(\Psi)$ in the above reduction. That is not possible as it would imply, together with Lemma A, that an open subset of \mathbb{R}^n is homeomorphic to a compact set. We can also not hope for the reduction to yield a linear extension, as a bounded set cannot be a linear extension of an unbounded one.

4 Reduction to ETR-AMI

ETR-AMI is an abbreviation for **E**xistential **T**heory of the **R**eals with **A**ddition, **M**ultiplication, and **I**nequalities.

Definition 13. An ETR-AMI formula $\Phi := \Phi(x_1, \dots, x_n)$ is a conjunction $C_1 \wedge \dots \wedge C_m$, where $m \geq 0$ and each C_i is a constraint of one of the forms

$$x + y = z, \quad x \cdot y = z, \quad x \geq 0, \quad x = 1$$

for $x, y, z \in \{x_1, \dots, x_n\}$.

Lemma C (ETR-AMI Reduction). *Given an instance of ETR-COMPACT defined by a formula Φ , we can in $O(|\Phi|)$ time construct an ETR-AMI formula Ψ such that $V(\Phi) \simeq V(\Psi)$ and $V(\Phi) \leq_{lin} V(\Psi)$.*

Proof. Recall that Φ is a conjunction of atomic formulas of the form $p = 0$ for a polynomial p and $x \geq 0$ for a variable x . Each polynomial p may contain minuses, zeros, and ones. The reduction has four steps. In each step, we make changes to Φ . In the end, Φ has become a formula Ψ with the desired properties. In step (1)–(3), we remove unwanted ones, zeros and minuses by replacing them by constants. In step (4), we eliminate complicated polynomials.

Step (1): We introduce the constant variable $\llbracket 1 \rrbracket$ and the constraint $\llbracket 1 \rrbracket = 1$ to Φ . We then replace all appearances of 1 with $\llbracket 1 \rrbracket$ in the atomic formulas of the form $p = 0$.

Step (2): We introduce the constant variable $\llbracket 0 \rrbracket$ and the constraint $\llbracket 1 \rrbracket + \llbracket 0 \rrbracket = \llbracket 1 \rrbracket$ to Φ . We then replace all appearances of 0 with $\llbracket 0 \rrbracket$ except in the constraints of the form $x \geq 0$.

Step (3): We introduce the constant variable $\llbracket -1 \rrbracket$ and the constraint $\llbracket 1 \rrbracket + \llbracket -1 \rrbracket = \llbracket 0 \rrbracket$ to Φ . We then replace all appearances of minus with a multiplication by $\llbracket -1 \rrbracket$ in Φ .

Step (4): We replace bottom up every occurrence of multiplication and addition by a new variable and an extra addition or multiplication constraint, which will be adjoined at the end of the formula. Here are two examples of such replacements:

$$\begin{aligned} x_1 + x_2 \cdot x_4 + x_5 + x_6 = \llbracket 0 \rrbracket &\mapsto x_1 + z_1 + x_5 + x_6 = \llbracket 0 \rrbracket \dots \wedge z_1 = x_2 \cdot x_4 \\ x_1 + z_1 + x_5 + x_6 = \llbracket 0 \rrbracket &\mapsto z_2 + x_5 + x_6 = \llbracket 0 \rrbracket \dots \wedge z_2 = x_1 + z_1 \end{aligned}$$

In this way every atomic predicate is eventually transformed to atomic predicates of ETR-AMI or is of the form $x = \llbracket 0 \rrbracket$. In the latter case, we replace $x = \llbracket 0 \rrbracket$ by $x + \llbracket 0 \rrbracket = \llbracket 0 \rrbracket$.

To see that the reduction is linear, note that every replacement adds a constant to the length of the formula. Furthermore, at most linearly many replacements will be done.

Let us show that this reduction preserves rational equivalence and linear extension. This is trivial for steps (1)–(3), as these just introduce constants in order to rewrite polynomials without using zeros, ones, and minuses. In Step (4), we repeatedly make one of two types of steps, replacing either a multiplication or an addition. Thus it is sufficient to show that one such step preserves all of those properties. Consider a step where we go from Φ_1 to Φ_2 and Φ_1 has the variables x_1, \dots, x_n and Φ_2 has the variables x_1, \dots, x_n, z , with $z = x_i \odot x_j$. Here \odot is either multiplication or addition. This defines f as

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, x_i \odot x_j),$$

and f^{-1} is defined by

$$(x_1, \dots, x_n, z) \mapsto (x_1, \dots, x_n).$$

Both functions are rational and bijective, and f^{-1} is an orthogonal projection. This implies both rational equivalence and linear extension between $V(\Phi)$ and $V(\Psi)$. \square

5 Reduction to ETR-SMALL

Let $\delta \in (0, 1)$ be given. The definition of ETR-SMALL depends on δ , but we will suppress δ in the notation to keep it simpler.

Definition 14. An *ETR-SMALL formula* $\Phi := \Phi(x_1, \dots, x_n)$ is a conjunction $C_1 \wedge \dots \wedge C_m$, where $m \geq 0$ and each C_i is a constraint of one of the forms

$$x + y = z, \quad x \cdot y = z, \quad x \geq 0, \quad x = \delta$$

for $x, y, z \in \{x_1, \dots, x_n\}$. We define $\delta(\Phi) := \delta$.

In the *ETR-SMALL problem*, we are given an ETR-SMALL formula Φ and promised that $V(\Phi) \subset [-\delta, \delta]^n$. The goal is to decide whether $V(\Phi) \neq \emptyset$.

We are going to present a reduction from the problem ETR-AMI to ETR-SMALL. As a preparation, we present another tool from real algebraic geometry. Schaefer [13] made the following simplification of a result from [3], which we will use. More refined statements can be found in [3].

Corollary 15 ([3]). *If a bounded semi-algebraic set in \mathbb{R}^n has complexity at most $L \geq 5n$, then all its points have distance at most $2^{2^{L+5}}$ from the origin.*

Lemma D (ETR-SMALL Reduction). *Given an ETR-AMI formula Φ such that $V(\Phi)$ is compact, we can in $O(|\Phi|)$ time construct an instance of ETR-SMALL defined by a formula Ψ such that $V(\Phi) \simeq V(\Psi)$ and $V(\Phi) \leq_{lin} V(\Psi)$.*

Proof. Let Φ be an instance of ETR-AMI with n variables x_1, \dots, x_n . We construct an instance Ψ of ETR-SMALL.

We set $\varepsilon := \delta \cdot 2^{-2^{L+5}}$, where $L := |\Phi|$. In Ψ , we first define a constant variable $\llbracket \varepsilon \rrbracket$. This is obtained by exponentiation by squaring, using $O(L)$ new constant variables and constraints. We first define $\llbracket \delta \rrbracket$, $\llbracket 0 \rrbracket$, and $\llbracket \delta \cdot 2^{-2^0} \rrbracket$ by the equations

$$\begin{aligned} \llbracket \delta \rrbracket &= \delta \\ \llbracket 0 \rrbracket + \llbracket \delta \rrbracket &= \llbracket \delta \rrbracket \\ \llbracket \delta \cdot 2^{-2^0} \rrbracket + \llbracket \delta \cdot 2^{-2^0} \rrbracket &= \llbracket \delta \rrbracket. \end{aligned}$$

We then use the following equations for all $i \in \{0, \dots, L+4\}$,

$$\begin{aligned} \llbracket \delta \cdot 2^{-2^i} \rrbracket \cdot \llbracket \delta \cdot 2^{-2^i} \rrbracket &= \llbracket \delta^2 \cdot 2^{-2^{i+1}} \rrbracket \\ \llbracket \delta \cdot 2^{-2^{i+1}} \rrbracket \cdot \llbracket \delta \rrbracket &= \llbracket \delta^2 \cdot 2^{-2^{i+1}} \rrbracket. \end{aligned}$$

Finally, we define $\llbracket \varepsilon \rrbracket$ by the constraint $\llbracket \varepsilon \rrbracket + \llbracket 0 \rrbracket = \llbracket \delta \cdot 2^{-2^{L+5}} \rrbracket$.

In Ψ , we use the variables $\llbracket \varepsilon x_1 \rrbracket, \dots, \llbracket \varepsilon x_n \rrbracket$ instead of x_1, \dots, x_n . An equation of Φ of the form $x = 1$ is transformed to the equation $\llbracket \varepsilon x \rrbracket + \llbracket 0 \rrbracket = \llbracket \varepsilon \rrbracket$ in Ψ . An equation of Φ of the form $x + y = z$ is transformed to the equation $\llbracket \varepsilon x \rrbracket + \llbracket \varepsilon y \rrbracket = \llbracket \varepsilon z \rrbracket$ of Ψ . For an equation of Φ of the form $x \cdot y = z$, we also introduce a variable $\llbracket \varepsilon^2 z \rrbracket$ of Ψ and the equations

$$\begin{aligned} \llbracket \varepsilon x \rrbracket \cdot \llbracket \varepsilon y \rrbracket &= \llbracket \varepsilon^2 z \rrbracket \\ \llbracket \varepsilon \rrbracket \cdot \llbracket \varepsilon z \rrbracket &= \llbracket \varepsilon^2 z \rrbracket. \end{aligned}$$

At last, constraints of the form $x \geq 0$ become $\llbracket \varepsilon x \rrbracket \geq 0$.

We now describe a function $f : V(\Phi) \rightarrow V(\Psi)$ in order to show that Ψ has the properties stated in the lemma. Let $\mathbf{x} := (x_1, \dots, x_n) \in V(\Phi)$. In order to define f , it suffices to specify the values of the variables of Ψ depending on \mathbf{x} . For all the constant variables $\llbracket c \rrbracket$, we define $\llbracket c \rrbracket := c$. Note that these are all *rational* constants. For all $i \in \{1, \dots, n\}$, we now define $\llbracket \varepsilon x_i \rrbracket := \varepsilon x_i$ and (when $\llbracket \varepsilon^2 x_i \rrbracket$ appears in Ψ) $\llbracket \varepsilon^2 x_i \rrbracket = \varepsilon^2 x_i$. Since \mathbf{x} is a solution to Φ , it follows from the constraints of Ψ that these assignments are a solution to Ψ .

We need to verify that Ψ defines an ETR-SMALL problem, i.e., that Ψ satisfies the promise that $V(\Psi) \subset [-\delta, \delta]^m$, where m is the number of variables of Ψ . To this end, consider an assignment of the variables of Ψ that satisfies all the constraints. Note first that the constant variables are non-negative and at most δ . For the other variables, we consider the inverse f^{-1} , which is given by the assignment $x_i := \llbracket \varepsilon x_i \rrbracket / \varepsilon$ for all $i \in \{1, \dots, n\}$. It follows that this yields a solution to Φ . Since $V(\Phi)$ is compact, it follows from Corollary 15 that $|\llbracket \varepsilon x_i \rrbracket / \varepsilon| \leq 2^{2^{L+5}}$. Hence $|\llbracket \varepsilon x_i \rrbracket| \leq \varepsilon \cdot 2^{2^{L+5}} = \delta \cdot 2^{-2^{L+5}} \cdot 2^{2^{L+5}} = \delta$. Similarly, when $\llbracket \varepsilon^2 x_i \rrbracket$ is a variable of Ψ , we get $|\llbracket \varepsilon^2 x_i \rrbracket| \leq \varepsilon \cdot \delta < \delta$.

By the definitions of f and f^{-1} , we have now established that $V(\Phi) \simeq V(\Psi)$ and $V(\Phi) \leq_{\text{lin}} V(\Psi)$. The length of Ψ is $O(L)$ longer than the length of Φ , and Ψ can thus be computed in $O(|\Phi|)$ time. \square

6 Reduction to ETR-SHIFT

Let $\delta > 0$ be given. The definition of ETR-SHIFT depends on δ , but we will suppress δ in the notation to keep it simpler.

Definition 16. An *ETR-SHIFT* formula $\Phi := \Phi(x_1, \dots, x_n)$ is a conjunction

$$\left(\bigwedge_{i=1}^n 1/2 \leq x_i \leq 2 \right) \wedge \left(\bigwedge_{i=1}^m C_i \right),$$

where $m \geq 0$ and each C_i is of one of the forms

$$x + y = z, \quad x \cdot y = z$$

for $x, y, z \in \{x_1, \dots, x_n\}$.

An instance $\mathcal{I} := [\Phi, (I(x_1), \dots, I(x_n))]$ of the *ETR-SHIFT problem* is an ETR-SHIFT formula Φ and, for each variable $x \in \{x_1, \dots, x_n\}$, an interval $I(x) \subseteq [1/2, 2]$ such that $|I(x)| \leq 2\delta$. For every *multiplication constraint* $x \cdot y = z$, we have $I(x) \subset [1 - \delta, 1 + \delta]$ and $I(y) \subset [1 - \delta, 1 + \delta]$. Define $\delta(\mathcal{I}) := \delta$ and $\Phi(\mathcal{I}) := \Phi$.

We are promised that $V(\Phi) \subset I(x_1) \times \dots \times I(x_n)$. The goal is to decide whether $V(\Phi) \neq \emptyset$.

We will now present a reduction from the problem ETR-SMALL to ETR-SHIFT. The following technical lemma is a handy tool to show that all variables x of the constructed ETR-SHIFT problem are in the ranges $I(x)$ we are going to specify.

Lemma 17. Let $g(x, y) := \frac{p(x, y)}{q(x, y)}$ be a rational function such that

$$\begin{aligned} p(x, y) &:= a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6, \quad \text{and} \\ q(x, y) &:= b_1 x^2 + b_2 xy + b_3 y^2 + b_4 x + b_5 y + b_6, \end{aligned}$$

where $b_6 > 0$. Let $\alpha := |b_1| + \dots + |b_5|$ and $\beta := |b_1| + \dots + |b_5|$.

Then for all $x, y \in [-\delta, \delta]$, where $\delta \in [0, 1]$, we have

$$g(x, y) \in \left[\frac{-\alpha\delta + a_6}{\beta\delta + b_6}, \frac{\alpha\delta + a_6}{-\beta\delta + b_6} \right]. \quad (1)$$

In particular,

(a) if $q(x, y) = b_6 = 1$ and $a_1, \dots, a_5 \in [0, c]$ for some $c \geq 0$, then

$$g(x, y) \in [a_6 - 5c\delta, a_6 + 5c\delta],$$

and

(b) if $a_1, \dots, a_5, b_1, \dots, b_5 \in [-c, c]$ for some $c \geq 0$ and $\delta \leq \frac{\varepsilon b_6^2}{5c(a_6 + (1+\varepsilon)b_6)}$ for a given $\varepsilon > 0$, then

$$g(x, y) \in [a_6/b_6 - \varepsilon, a_6/b_6 + \varepsilon].$$

Proof. We bound each term in each polynomial from below and above and get

$$\begin{aligned} p(x, y) &\in [-(|a_1| + |a_2| + |a_3|)\delta^2 - (|a_4| + |a_5|)\delta + a_6, \\ &\quad + (|a_1| + |a_2| + |a_3|)\delta^2 + (|a_4| + |a_5|)\delta + a_6], \quad \text{and} \\ q(x, y) &\in [-(|b_1| + |b_2| + |b_3|)\delta^2 - (|b_4| + |b_5|)\delta + b_6, \\ &\quad + (|b_1| + |b_2| + |b_3|)\delta^2 + (|b_4| + |b_5|)\delta + b_6]. \end{aligned}$$

The bounds (1) now follows as $\delta \in [0, 1]$ so that $\delta^2 \leq \delta$.

For part (a), note that $\beta = 0$ and $\alpha \in [0, 5c]$. For part (b), we get that

$$g(x, y) \in \left[\frac{-5c\delta + a_6}{5c\delta + b_6}, \frac{5c\delta + a_6}{-5c\delta + b_6} \right].$$

One can then check that if $\delta \leq \frac{\varepsilon b_6^2}{5c(a_6 + (1+\varepsilon)b_6)}$, that range is contained in $[a_6/b_6 - \varepsilon, a_6/b_6 + \varepsilon]$. \square

Lemma E (ETR-SHIFT Reduction). *Let $\delta_2 \in (0, 1/4)$ be given, and let $\delta_1 \leq \delta_2/5$ such that $\delta_1 = 2^{-l}$ for $l \in \mathbb{N}$. Consider an instance of the ETR-SMALL problem, defined by a formula Φ_1 such that $\delta(\Phi_1) = \delta_1$. We can in $O(|\Phi_1| + l)$ time compute an instance \mathcal{I}_2 of the ETR-SHIFT problem with $\delta(\mathcal{I}_2) = \delta_2$ and formula $\Phi_2 := \Phi(\mathcal{I}_2)$ such that $V(\Phi_1) \simeq V(\Phi_2)$ and $V(\Phi_1) \leq_{lin} V(\Phi_2)$.*

Proof. In the following, we specify the variables and constraints we add to Φ_2 . Define $\Delta := 1 - \delta_1$. As a first step, we construct constant variables $\llbracket c \rrbracket$ for each of $c \in \{1/2, 3/4, 1, 3/2\}$, as follows. We first use the constraint $\llbracket 1 \rrbracket \cdot \llbracket 1 \rrbracket = \llbracket 1 \rrbracket$. Note that the solutions to this are $\llbracket 1 \rrbracket \in \{0, 1\}$, but since we are restricted to $[1/2, 2]$, we conclude that $\llbracket 1 \rrbracket = 1$. We observe that $\llbracket 1 \rrbracket$ is in the promised range $[1 - \delta_2, 1 + \delta_2]$ of variables involved in multiplication constraints.

We can then construct the other constants as follows.

$$\begin{aligned}\llbracket 1/2 \rrbracket + \llbracket 1/2 \rrbracket &= \llbracket 1 \rrbracket \\ \llbracket 1 \rrbracket + \llbracket 1/2 \rrbracket &= \llbracket 3/2 \rrbracket \\ \llbracket 3/4 \rrbracket + \llbracket 3/4 \rrbracket &= \llbracket 3/2 \rrbracket\end{aligned}$$

We now show how to construct a constant variable $\llbracket \Delta \rrbracket$. To this end, we construct constant variables $\llbracket 1 - 2^{-i} \rrbracket$ for $i \in \{1, \dots, l\}$, so that $\llbracket \Delta \rrbracket$ is a synonym for $\llbracket 1 - 2^{-l} \rrbracket$. For the base case $i = 1$, note that this is just the already constructed $\llbracket 1/2 \rrbracket$. Suppose inductively that we have constructed the constant variable $\llbracket 1 - 2^{-i} \rrbracket$. In order to construct $\llbracket 1 - 2^{-(i+1)} \rrbracket$, we proceed as follows.

$$\begin{aligned}\llbracket 1 - 2^{-i} \rrbracket + \llbracket 1 \rrbracket &= \llbracket 2 - 2^{-i} \rrbracket \\ \llbracket 1 - 2^{-(i+1)} \rrbracket + \llbracket 1 - 2^{-(i+1)} \rrbracket &= \llbracket 2 - 2^{-i} \rrbracket\end{aligned}$$

Thus we can generate a variable with the value $\llbracket \Delta \rrbracket$ in $O(l)$ steps.

For each of the constant variables $\llbracket c \rrbracket$ thus created, we define $I(\llbracket c \rrbracket) := [c - \delta_2, c + \delta_2] \cap [1/2, 2]$. Note that it follows from the constraints that in any solution to Φ_2 , we must have $\llbracket c \rrbracket = c$.

For each each variable $x \in [-\delta_1, \delta_1]$ of Φ_1 , we make a corresponding variable $\llbracket x + 1 \rrbracket$ of Φ_2 . As we shall see, for every solution $\mathbf{x} := (x_1, \dots, x_n)$ of Φ_1 , there will be a corresponding solution to Φ_2 with $\llbracket x_i + 1 \rrbracket = x_i + 1$, and vice versa.

For each variable x of Φ_1 , we construct the variables $\llbracket x + 3/2 \rrbracket$, $\llbracket x + 3/4 \rrbracket$, and $\llbracket x + \Delta \rrbracket$ as follows.

$$\begin{aligned}\llbracket x + 1 \rrbracket + \llbracket 1/2 \rrbracket &= \llbracket x + 3/2 \rrbracket \\ \llbracket x + 3/4 \rrbracket + \llbracket 3/4 \rrbracket &= \llbracket x + 3/2 \rrbracket \\ \llbracket x + 3/4 \rrbracket + \llbracket \Delta \rrbracket &= \llbracket x + 3/4 + \Delta \rrbracket \\ \llbracket x + \Delta \rrbracket + \llbracket 3/4 \rrbracket &= \llbracket x + 3/4 + \Delta \rrbracket.\end{aligned}$$

For each of these of the form $\llbracket x + b \rrbracket$, $b \in \{3/4, \Delta, 3/2\}$, it holds that if $\llbracket x + 1 \rrbracket = x + 1$, then $\llbracket x + b \rrbracket = x + b$.

We now go through the constraints of Φ_1 and create equivalent constraints in Φ_2 . For each equation $x = \delta_1$ of Φ_1 , we add the equation $\llbracket x + \Delta \rrbracket = 1$ to Φ_2 . The equation implies that if $\llbracket x + 1 \rrbracket = x + 1$, then $x = \delta_1$.

For each equation $x + y = z$ of Φ_1 , we add

$$\llbracket x + 3/4 \rrbracket + \llbracket y + 3/4 \rrbracket = \llbracket z + 3/2 \rrbracket \tag{2}$$

to Φ_2 . This equation implies that if $\llbracket x + 1 \rrbracket = x + 1$ and $\llbracket y + 1 \rrbracket = y + 1$, then $\llbracket z + 1 \rrbracket = x + y + 1$.

For each equation $x \cdot y = z$ of Φ_1 , we have the following set of equations in Φ_2 .

$$\begin{aligned} \llbracket x + 1 \rrbracket \cdot \llbracket y + 1 \rrbracket &= \llbracket xy + x + y + 1 \rrbracket \\ \llbracket xy + x + y + 1 \rrbracket + \llbracket 1/2 \rrbracket &= \llbracket xy + x + y + 3/2 \rrbracket \\ \llbracket xy + x + 3/4 \rrbracket + \llbracket y + 3/4 \rrbracket &= \llbracket xy + x + y + 3/2 \rrbracket \\ \llbracket xy + x + 3/4 \rrbracket + \llbracket 3/4 \rrbracket &= \llbracket xy + x + 3/2 \rrbracket \\ \llbracket z + 3/4 \rrbracket + \llbracket x + 3/4 \rrbracket &= \llbracket xy + x + 3/2 \rrbracket \end{aligned}$$

These equations imply that if $\llbracket x + 1 \rrbracket = x + 1$ and $\llbracket y + 1 \rrbracket = y + 1$, then $\llbracket z + 1 \rrbracket = xy + 1$.

At last, for each constraint $x \geq 0$ of Φ_1 , we introduce the variable $\llbracket x + 1/2 \rrbracket$ of Φ_2 and the equation $\llbracket x + 1/2 \rrbracket + \llbracket 1/2 \rrbracket = \llbracket x + 1 \rrbracket$. The constraint $\llbracket x + 1/2 \rrbracket \geq 1/2$, which holds for all variables of Φ_2 by definition of ETR-SHIFT, then corresponds to $x \geq 0$.

Note that each of the introduced variables has the form $\llbracket p(x, y) \rrbracket$, where $p(x, y)$ is a polynomial of degree at most 2 and with constant term $c := p(0, 0) \in \{1/2, 3/4, 1, 3/2\}$. We now define $I(\llbracket p(x, y) \rrbracket) := [c - \delta_2, c + \delta_2] \cap [1/2, 2]$.

The construction of Φ_2 is now finished, and we need to check that it has the claimed properties. Let the variables of Φ_1 be x_1, \dots, x_n and the variables of Φ_2 be

$$\llbracket x_1 + 1 \rrbracket, \dots, \llbracket x_n + 1 \rrbracket, \llbracket y_1 \rrbracket, \dots, \llbracket y_m \rrbracket.$$

For each variable $\llbracket y_i \rrbracket$, $i \in \{1, \dots, m\}$, the expression y_i is a polynomial of degree at most two in two variables among x_1, \dots, x_n (this includes the case that y_i is a constant). Consider any solution $\mathbf{x} := (x_1, \dots, x_n) \in [-\delta_1, \delta_1]^n$ to Φ_1 . We get a corresponding solution $f(\mathbf{x})$ to Φ_2 as follows. We set $\llbracket x_i + 1 \rrbracket := x_i + 1$ for every $i \in \{1, \dots, n\}$. For every $i \in \{1, \dots, m\}$, y_i is a (possibly constant) polynomial in two variables among x_1, \dots, x_n , and we assign $\llbracket y_i \rrbracket$ the value we get when evaluating this polynomial.

In order to show that this yields a solution to Φ_2 , we consider the constraint (2) introduced to Φ_2 due to an addition $x + y = z$ of Φ_1 . The other constraints can be verified in a similar way. Due to the construction of $\llbracket x + 3/4 \rrbracket$, it follows from $\llbracket x + 1 \rrbracket := x + 1$ that $\llbracket x + 3/4 \rrbracket = x + 3/4$, and similarly that $\llbracket y + 3/4 \rrbracket = y + 3/4$ and $\llbracket z + 3/2 \rrbracket = z + 3/2$. Hence we have

$$\llbracket x + 3/4 \rrbracket + \llbracket y + 3/4 \rrbracket = x + 3/4 + y + 3/4 = z + 3/2 = \llbracket z + 3/2 \rrbracket,$$

so indeed the constraint is satisfied.

Note that the inverse of f is

$$f^{-1} : (\llbracket x_1 + 1 \rrbracket, \dots, \llbracket x_n + 1 \rrbracket, \llbracket y_1 \rrbracket, \dots, \llbracket y_m \rrbracket) \mapsto (\llbracket x_1 + 1 \rrbracket - 1, \dots, \llbracket x_n + 1 \rrbracket - 1).$$

We now show that f^{-1} is a map from $V(\Phi_2)$ to $V(\Phi_1)$, i.e., that given any solution to Φ_2 , f^{-1} yields a solution to Φ_1 . Consider a constraint of Φ_1 of the form $x + y = z$. We then have

$$\begin{aligned} x + y &= \llbracket x + 1 \rrbracket - 1 + \llbracket y + 1 \rrbracket - 1 = \llbracket x + 3/4 \rrbracket + 1/4 - 1 + \llbracket y + 3/4 \rrbracket + 1/4 - 1 \\ &= \llbracket z + 3/2 \rrbracket - 3/2 = z. \end{aligned}$$

In a similar way, the other constraints of Φ_1 can be shown to hold due to the constraints of Φ_2 .

It follows that f is a rational homeomorphism so $V(\Phi_1) \simeq V(\Phi_2)$, and since f^{-1} merely subtracts 1 from some variables, we also have $V(\Phi_1) \leq_{\text{lin}} V(\Phi_2)$.

At last, we need to verify that Φ_2 satisfies the promise that in every solution, each variable $\llbracket p(x, y) \rrbracket$ is in the interval $I(\llbracket p(x, y) \rrbracket)$. Here, $p(x, y)$ is a polynomial of degree at most 2 and with constant term $c := p(0, 0) \in \{1/2, 3/4, 1, 3/2\}$. By the map f^{-1} , we get a solution to Φ_1 by the assignments $x := \llbracket x + 1 \rrbracket - 1$ and $y := \llbracket y + 1 \rrbracket - 1$ (and similarly for the remaining

variables of Φ_1). It then follows from the constraints of Φ_2 that $\llbracket p(x, y) \rrbracket = p(x, y)$. By the promise of Φ_1 , we get that $x, y \in [-\delta_1, \delta_1]$. The coefficients of the non-constant terms of $p(x, y)$ are all either 0 or 1. We therefore get by Lemma 17 that since $x, y \in [-\delta_1, \delta_1]$, then $p(x, y) \in [c - 5\delta_1, c + 5\delta_1] \subset [c - \delta_2, c + \delta_2]$.

Recall that $I(\llbracket p(x, y) \rrbracket) := [c - \delta_2, c + \delta_2] \cap [1/2, 2]$ and that $\delta_2 < 1/4$. With the exception of the case $c = 1/2$, we therefore have that $I(\llbracket p(x, y) \rrbracket) = [c - \delta_2, c + \delta_2]$, so it follows that $\llbracket p(x, y) \rrbracket \in I(\llbracket p(x, y) \rrbracket)$. Note that the case $c = 1/2$ only occurs when $p(x, y) = x + 1/2$ and $I(\llbracket x + 1/2 \rrbracket) = [1/2, 1/2 + \delta_2]$. In this case, there is a constraint $x \geq 0$ in Φ_1 . Hence $x \in [0, \delta_1]$, so that $\llbracket x + 1/2 \rrbracket = x + 1/2 \in [1/2, 1/2 + \delta_1] \subset [1/2, 1/2 + \delta_2] = I(\llbracket x + 1/2 \rrbracket)$.

In order to construct $\llbracket \Delta \rrbracket$ in Φ_2 , we introduce $O(l)$ variables and constraints. For each of the $O(|\Phi_1|)$ variables and constraints of Φ_1 , we make a constant number of variables and constraints in Φ_2 . It thus follows that the running time is $O(|\Phi_1| + l)$. This completes the proof. \square

7 Reduction to ETR-SQUARE

In this and the following section, we show that the problem ETR-SHIFT remains essentially equally hard even when we only allow more specialized types of multiplications in our formulas. In this section, we require every multiplication to be a *squaring* of the form $x^2 = y$, and in the following section, we only allow *inversion* of the form $x \cdot y = 1$. The result that these two restricted types of constraints preserve the full expressibility of ETR-SHIFT is related to the result of Aho et al. [2, Section 8.2] that squaring and taking reciprocals of integers require work proportional to that of multiplication.

Let $\delta > 0$ be given. The definition of ETR-SQUARE depends on δ , but we will suppress δ in the notation to keep it simpler.

Definition 18. An *ETR-SQUARE formula* $\Phi := \Phi(x_1, \dots, x_n)$ is a conjunction

$$\left(\bigwedge_{i=1}^n 1/2 \leq x_i \leq 2 \right) \wedge \left(\bigwedge_{i=1}^m C_i \right),$$

where $m \geq 0$ and each C_i is of one of the forms

$$x + y = z, \quad x^2 = y$$

for $x, y, z \in \{x_1, \dots, x_n\}$.

An instance $\mathcal{I} = [\Phi, (I(x_1), \dots, I(x_n))]$ of the *ETR-SQUARE problem* is an ETR-SQUARE formula Φ and, for each variable $x \in \{x_1, \dots, x_n\}$, an interval $I(x) \subseteq [1/2, 2]$ such that $|I(x)| \leq 2\delta$. For every *squaring constraint* $x^2 = y$, we have $I(x) \subset [1 - \delta, 1 + \delta]$. Define $\delta(\mathcal{I}) := \delta$ and $\Phi(\mathcal{I}) := \Phi$.

We are promised that $V(\Phi) \subset I(x_1) \times \dots \times I(x_n)$. The goal is to decide whether $V(\Phi) \neq \emptyset$.

Below, we present a reduction from the problem ETR-SHIFT to ETR-SQUARE.

Lemma F (ETR-SQUARE Reduction). *Let $\delta_2 \in (0, 1/4)$ be given, and let $\delta_1 := \delta_2/10$. Consider an instance \mathcal{I}_1 of the ETR-SHIFT problem such that $\delta(\mathcal{I}_1) = \delta_1$, and let $\Phi_1 := \Phi(\mathcal{I}_1)$. We can in $O(|\Phi_1|)$ time compute an instance \mathcal{I}_2 of the ETR-SQUARE problem with $\delta(\mathcal{I}_2) = \delta_2$ and formula $\Phi_2 := \Phi(\mathcal{I}_2)$ such that $V(\Phi_1) \simeq V(\Phi_2)$ and $V(\Phi_1) \leq_{lin} V(\Phi_2)$.*

Proof. As in the proof of Lemma E, we first construct constant variables $\llbracket b \rrbracket$ for each of $b \in \{1/2, 3/4, 1, 3/2\}$. The only difference is that we now construct $\llbracket 1 \rrbracket$ using the constraint $\llbracket 1 \rrbracket^2 = \llbracket 1 \rrbracket$. The other constants are then constructed in exactly the same way as before.

We include each variable x of Φ_1 in Φ_2 as well, and reuse the interval $I(x)$ from \mathcal{I}_1 in \mathcal{I}_2 . We also reuse all constraints from Φ_1 of the form $a + b = c$ in Φ_2 , but we have to do something different for the constraints $a \cdot b = c$. The idea is very simple and was also used by Aho et al. [2, Section 8.2], and can be expressed by the equations

$$\begin{aligned}x &= (a + b)/2 \\y &= (a - b)/2 \\u &= x^2 \\v &= y^2 \\c &= u - v = a \cdot b.\end{aligned}$$

If there were no range constraints, we could just replace each multiplication $a \cdot b = c$ of Φ_1 by equations as above (after rewriting the subtractions as additions, etc.). However, in our situation all intermediate variables w need to be in a range $I(w) \subset [1/2, 2]$ in any solution. While this makes the description more involved, careful shifting will work for us.

Let $a \cdot b = c$ be a multiplication constraint in Φ_1 . Let us rename the variables as $\llbracket x + 1 \rrbracket := a$, $\llbracket y + 1 \rrbracket := b$, and $\llbracket xy + x + y + 1 \rrbracket := c$, so that $a \cdot b = c$ becomes

$$\llbracket x + 1 \rrbracket \cdot \llbracket y + 1 \rrbracket = \llbracket xy + x + y + 1 \rrbracket. \quad (\dagger)$$

Consider two numbers $x, y \in \mathbb{R}$ and the two conditions

$$\llbracket x + 1 \rrbracket = x + 1 \quad \text{and} \quad \llbracket y + 1 \rrbracket = y + 1, \quad (\star)$$

$$\llbracket xy + x + y + 1 \rrbracket = xy + x + y + 1. \quad (\star\star)$$

We claim that (\dagger) is equivalent to

$$(\star) \text{ implies } (\star\star). \quad (\ddagger)$$

To show this claim, suppose first that (\dagger) holds. Define $x := \llbracket x + 1 \rrbracket - 1$ and $y := \llbracket y + 1 \rrbracket - 1$. Then $\llbracket xy + x + y + 1 \rrbracket = \llbracket x + 1 \rrbracket \cdot \llbracket y + 1 \rrbracket = (x + 1) \cdot (y + 1) = xy + x + y + 1$, so that $(\star\star)$ holds. Hence we have (\ddagger) . On the other hand, suppose that (\ddagger) holds, and define $x := \llbracket x + 1 \rrbracket - 1$ and $y := \llbracket y + 1 \rrbracket - 1$ so that (\star) holds. Our assumption implies that $(\star\star)$ holds, and we thus have $\llbracket xy + x + y + 1 \rrbracket = xy + x + y + 1 = (x + 1) \cdot (y + 1) = \llbracket x + 1 \rrbracket \cdot \llbracket y + 1 \rrbracket$, so that (\dagger) holds. Our aim is therefore to make constraints in Φ_2 that ensure (\ddagger) .

For every variable $\llbracket q \rrbracket$ of Φ_2 , we can construct $\llbracket q/2 \rrbracket$ using the constraint $\llbracket q/2 \rrbracket + \llbracket q/2 \rrbracket = \llbracket q \rrbracket$. Similarly, we can construct $\llbracket 2q \rrbracket$ by $\llbracket q \rrbracket + \llbracket q \rrbracket = \llbracket 2q \rrbracket$.

The construction of $\llbracket xy + x + y + 1 \rrbracket$ is shown in Figure 2. The diagram should be understood in the following way. We start with the original variables $\llbracket x + 1 \rrbracket$ and $\llbracket y + 1 \rrbracket$. Each arrow is labeled with the operation that leads to the new variable. It is straightforward to check that the construction ensures condition (\ddagger) .

Note that each of the constructed auxilliary variables has the form $\llbracket p(x, y) \rrbracket$, where $p(x, y)$ is a second degree polynomial with constant term $c := p(0, 0) \in \{3/4, 1, 3/2\}$. We define $I(\llbracket p(x, y) \rrbracket) := [c - \delta_2, c + \delta_2]$. This finishes the construction of \mathcal{I}_2 . Note that since $\delta_2 \leq 1/4$ and $c \in \{3/4, 1, 3/2\}$, we get that $I(\llbracket p(x, y) \rrbracket) \subset [1/2, 2]$, as required.

We now verify that \mathcal{I}_2 has the claimed properties. Consider a solution $\mathbf{x} \in V(\Phi_1)$. We point out an equivalent solution to Φ_2 . For all the variables of Φ_2 that also appear in Φ_1 , we use the same value as in the solution \mathbf{x} . Each auxilliary variable has the form $\llbracket p(x, y) \rrbracket$, where $p(x, y)$ is a second degree polynomial, and Φ_1 contains the multiplication constraint (\dagger) . We then get from the promise of Φ_1 that $\llbracket 1 + x \rrbracket = 1 + x$ and $\llbracket 1 + y \rrbracket = 1 + y$ for $x, y \in [-\delta_1, \delta_1]$. From the

construction shown in Figure 2, it follows that in order to get a solution to Φ_2 , we must define $\llbracket p(x, y) \rrbracket := p(x, y)$. Recall that the constant term $c := p(0, 0)$ satisfies $c \in \{3/4, 1, 3/2\}$, and note that all the coefficients of the non-constant terms of $p(x, y)$ are in the interval $[0, 2]$. We then get from Lemma 17 that $\llbracket p(x, y) \rrbracket \in [c - 10\delta_1, c + 10\delta_1] = [c - \delta_2, c + \delta_2] = I(\llbracket p(x, y) \rrbracket) \subset [1/2, 2]$. The variables are thus in the range $[1/2, 2]$, so we have described a solution to Φ_2 .

Similarly, we see that any solution to Φ_2 corresponds to a solution to Φ_1 . Using the promise of \mathcal{I}_1 and Lemma 17 as above, we then also confirm the promise of \mathcal{I}_2 that each variable u of Φ_2 is in $I(u)$.

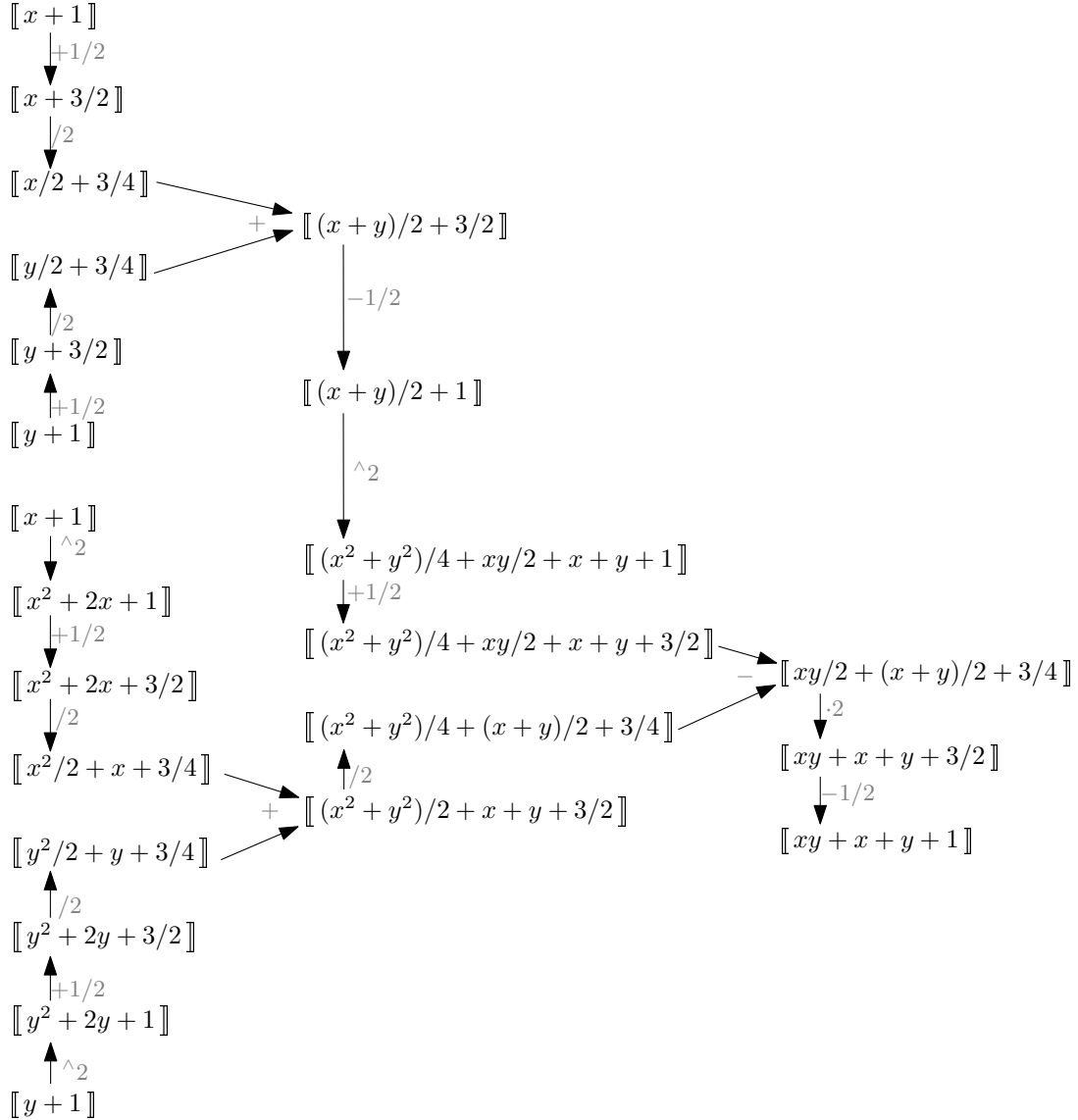


Figure 2: The construction of $\llbracket xy + x + y + 1 \rrbracket$ from $\llbracket x + 1 \rrbracket$ and $\llbracket y + 1 \rrbracket$. Squaring a variable is denoted by $\wedge 2$.

The correspondance described implies that $V(\Phi_1) \simeq V(\Phi_2)$ and $V(\Phi_1) \leq_{\text{lin}} V(\Phi_2)$. For each constraint of Φ_1 , we introduce $O(1)$ variables and constraints of Φ_2 , so the reduction takes $O(|\Phi_1|)$ time. \square

8 Reduction to ETR-INV

Let $\delta > 0$ be given. The definition of ETR-INV depends on δ , but we will suppress δ in the notation to keep it simpler.

Definition 19. An *ETR-INV formula* $\Phi = \Phi(x_1, \dots, x_n)$ is a conjunction

$$\left(\bigwedge_{i=1}^n 1/2 \leq x_i \leq 2 \right) \wedge \left(\bigwedge_{i=1}^m C_i \right),$$

where $m \geq 0$ and each C_i is of one of the forms

$$x + y = z, \quad x \cdot y = 1$$

for $x, y, z \in \{x_1, \dots, x_n\}$.

An instance $\mathcal{I} = [\Phi, (I(x_1), \dots, I(x_n))]$ of the *ETR-INV problem* is an ETR-INV formula Φ and, for each variable $x \in \{x_1, \dots, x_n\}$, an interval $I(x) \subseteq [1/2, 2]$ such that $|I(x)| \leq 2\delta$. Define $\delta(\mathcal{I}) := \delta$ and $\Phi(\mathcal{I}) := \Phi$.

We are promised that $V(\Phi) \subset I(x_1) \times \dots \times I(x_n)$. The goal is to decide whether $V(\Phi) \neq \emptyset$.

We will now present a reduction from the problem ETR-SQUARE to ETR-INV.

Lemma G (ETR-INV Reduction). *Let $\delta_2 \in (0, 1/6)$ be given, and let $\delta_1 := \delta_2/1800$. Consider an instance \mathcal{I}_1 of the ETR-SQUARE problem such that $\delta(\mathcal{I}_1) = \delta_1$, and let $\Phi_1 := \Phi(\mathcal{I}_1)$. We can in $O(|\Phi_1|)$ time compute an instance \mathcal{I}_2 of the ETR-INV problem with $\delta(\mathcal{I}_2) = \delta_2$ and formula $\Phi_2 := \Phi(\mathcal{I}_2)$ such that $V(\Phi_1) \simeq V(\Phi_2)$ and $V(\Phi_1) \leq_{lin} V(\Phi_2)$.*

Proof. As in the proof of Lemma E, we first construct constant variables $\llbracket b \rrbracket$ for each of $b \in \{1/2, 3/4, 1, 3/2\}$. The only difference is that we now construct $\llbracket 1 \rrbracket$ using the constraint $\llbracket 1 \rrbracket \cdot \llbracket 1 \rrbracket = 1$. It follows from this constraint that $\llbracket 1 \rrbracket \in \{-1, 1\}$, and since $\llbracket 1 \rrbracket \in [1/2, 2]$, we must have $\llbracket 1 \rrbracket = 1$ in every valid solution. The other constants are then constructed in exactly the same way as before. For this reduction we also need the constant variable $\llbracket 2/3 \rrbracket$ which is constructed as $\llbracket 2/3 \rrbracket \cdot \llbracket 3/2 \rrbracket = 1$.

We include each variable x of Φ_1 in Φ_2 as well, and reuse the interval $I(x)$ from \mathcal{I}_1 in \mathcal{I}_2 . We also reuse all constraints from Φ_1 of the form $a + b = c$ in Φ_2 , but we have to do something different for the squaring constraints $a^2 = b$. In Φ_2 , we rename the variables as $\llbracket x + 1 \rrbracket := a$ and $\llbracket x^2 + 2x + 1 \rrbracket := b$, so that $a^2 = b$ becomes

$$\llbracket x + 1 \rrbracket^2 = \llbracket x^2 + 2x + 1 \rrbracket. \quad (\dagger)$$

Consider a number $x \in \mathbb{R}$ and the two conditions

$$\llbracket x + 1 \rrbracket = x + 1, \quad (\star)$$

$$\llbracket x^2 + 2x + 1 \rrbracket = x^2 + 2x + 1. \quad (\star\star)$$

As in the proof of Lemma F, one can prove that (\dagger) is equivalent to

$$(\star) \text{ implies } (\star\star). \quad (\ddagger)$$

Our aim is therefore to make constraints in Φ_2 that ensure (\ddagger) .

In the same way as described in Section 6 and Section 7, we can add values, subtract values, half and double variables. Figure 3 shows the construction of $\llbracket x^2 + 2x + 1 \rrbracket$ using these tricks as well as inversions. It is straightforward to check that the construction ensures condition (\ddagger) .

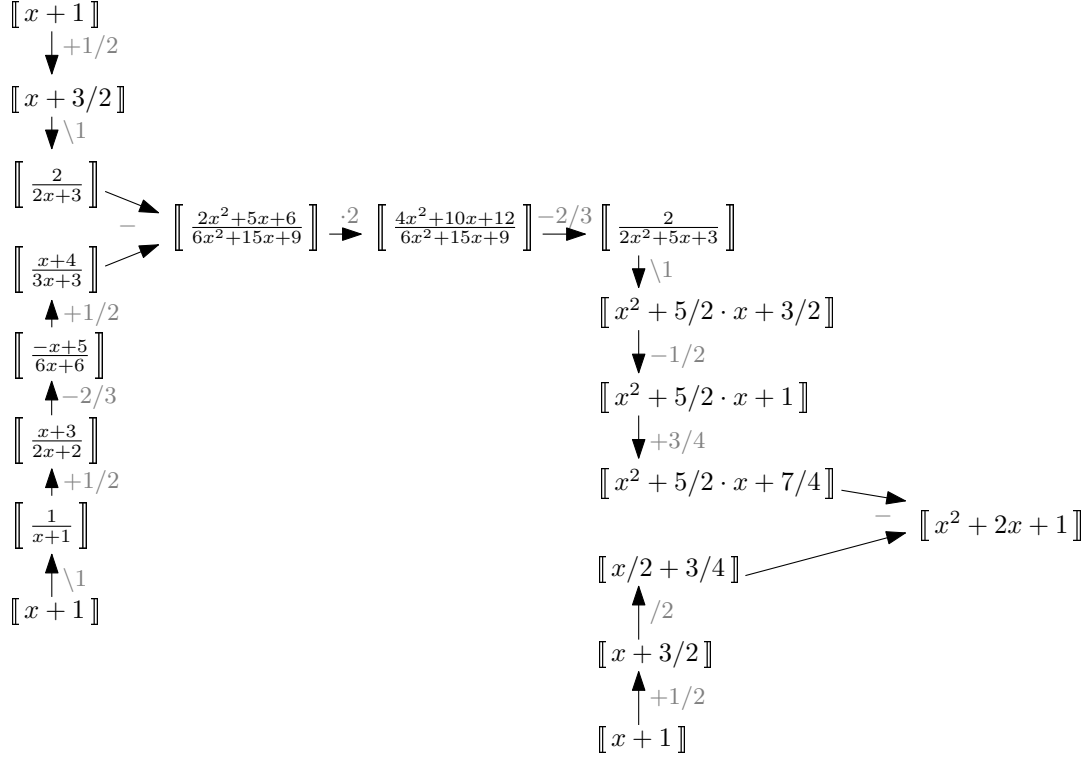


Figure 3: The sequence above enforces $y = x^2$, using only addition and inversion. Inversion is denoted by $\setminus 1$.

For all the variables x of Φ_2 that also appear in Φ_1 , we define the interval $I(x)$ in \mathcal{I}_2 as it is in \mathcal{I}_1 . Each of the auxilliary variables has the form $\left\llbracket \frac{p(x)}{q(x)} \right\rrbracket$, where $p(x)$ and $q(x)$ are polynomials of degree 2. We define $c := \frac{p(0)}{q(0)}$ and note that $c \in [2/3, 7/4]$. We define $I\left(\left\llbracket \frac{p(x)}{q(x)} \right\rrbracket\right) := [c - \delta_2, c + \delta_2]$. This finishes the construction of \mathcal{I}_2 . Note that since $\delta_2 \leq 1/6$ and $c \in [2/3, 7/4]$, we get that $I\left(\left\llbracket \frac{p(x)}{q(x)} \right\rrbracket\right) \subset [1/2, 2]$.

We now verify that \mathcal{I}_2 has the claimed properties. Consider a solution $\mathbf{x} \in V(\Phi_1)$. For all the variables of Φ_2 that also appear in Φ_1 , we use the same value as in the solution \mathbf{x} . Each auxilliary variable has the form $\left\llbracket \frac{p(x)}{q(x)} \right\rrbracket$, where $p(x)$ and $q(x)$ are polynomials of degree 2, and Φ_1 contains the squaring constraint (\dagger) . We then get from the promise of Φ_1 that $\llbracket 1+x \rrbracket = 1+x$ for $x \in [-\delta_1, \delta_1]$. From the construction shown in Figure 3, it follows in order to get a solution to Φ_2 , we must define $\left\llbracket \frac{p(x)}{q(x)} \right\rrbracket := \frac{p(x)}{q(x)}$. We are going to apply case (b) of Lemma 17 to show that this solution stays in the required range $[1/2, 2]$. The coefficients of the non-constant terms of $p(x)$ and $q(x)$ are in the range $[-1, 15]$. Denote by a_6 and b_6 the constant terms of $p(x)$ and $q(x)$, respectively. We observe that $b_6^2 \geq 1$ and $a_6 + (1 + \delta_2)b_6 \leq 6 + 2 \cdot 9 = 24$. We therefore get that since $\delta_1 := \delta_2/1800 = \frac{\delta_2}{5 \cdot 15 \cdot 24} < \frac{\delta_2 b_6^2}{5 \cdot 15 (a_6 + (1 + \delta_2)b_6)}$, then $\left\llbracket \frac{p(x)}{q(x)} \right\rrbracket \in [c - \delta_2, c + \delta_2] \subset [1/2, 2]$.

Similarly, we see that any solution to Φ_2 corresponds to a solution to Φ_1 . Using the promise of \mathcal{I}_1 and Lemma 17 as above, we then also confirm the promise of \mathcal{I}_2 that each variable u of Φ_2 is in $I(u)$.

The correspondance described implies that $V(\Phi_1) \simeq V(\Phi_2)$ and $V(\Phi_1) \leq_{\text{lim}} V(\Phi_2)$. For each constraint of Φ_1 , we introduce $O(1)$ variables and constraints of Φ_2 , so the reduction takes $O(|\Phi_1|)$ time. \square

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