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Parameterized Complexity of Conflict-Free Graph Coloring

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Abstract. Given a graph G, a q-open neighborhood conflict-free coloring or q-ONCF-coloring is a vertex coloring $c\colon V(G) \rightarrow \{1, 2, \ldots, q\}$ such that for each vertex $v \in V(G)$ there is a vertex in $N(v)$ that is uniquely colored from the rest of the vertices in $N(v)$. When we replace $N(v)$ by the closed neighborhood $N[v]$, then we call such a coloring a q-closed neighborhood conflict-free coloring or simply q-CNCF-coloring. In this paper, we study the NP-hard decision questions of whether for a constant q an input graph has a q-ONCF-coloring or a q-CNCF-coloring. We will study these two problems in the parameterized setting. First of all, we study running time bounds on FPT-algorithms for these problems, when parameterized by treewidth. We improve the existing upper bounds, and also provide lower bounds on the running time under ETH and SETH. Secondly, we study the kernelization complexity of both problems, using vertex cover as the parameter. We show that both $(q \geq 2)$-ONCF-coloring and $(q \geq 3)$-CNCF-coloring cannot have polynomial kernels when parameterized by the size of a vertex cover unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$. On the other hand, we obtain a polynomial kernel for 2-CNCF-coloring parameterized by vertex cover. We conclude the study with some combinatorial results. Denote $\chi_{ON}(G)$ and $\chi_{CN}(G)$ to be the minimum number of colors required to ONCF-color and CNCF-color G, respectively. Upper bounds on $\chi_{CN}(G)$ with respect to structural parameters like minimum vertex cover size, minimum feedback vertex set size and treewidth are known. To the best of our knowledge only an upper bound on $\chi_{ON}(G)$ with respect to minimum vertex cover size was known. We provide tight bounds for $\chi_{ON}(G)$ with respect to minimum vertex cover size. Also, we provide the first upper bounds on $\chi_{ON}(G)$ with respect to minimum feedback vertex set size and treewidth.

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1 Introduction

Often, in frequency allocation problems for cellular networks, it is important to allot a unique frequency for each client, so that at least one frequency is unaffected by cancellation. Such problems can be theoretically formulated as a coloring problem on a set system, better known as conflict-free coloring [5]. Formally, given a set system $\mathcal{H} = (U, \mathcal{F})$, a q-conflict-free coloring $c\colon U \to \{1, 2, \dots, q\}$ is a function where for each set $f \in \mathcal{F}$, there is an element $v \in f$ such that for all $w \neq v \in f$, $c(v) \neq c(w)$. In other words, each set f has at least one element that is uniquely colored in the set. This variant of coloring has also been extensively studied for set systems induced by various geometric regions [1,8,14].

A natural step to study most coloring problems is to study them in graphs. Given a graph G, $V(G)$ denotes the set of n vertices of G while $E(G)$ denotes the set of m edges in G. A q-coloring of G, for $q \in \mathbb{N}$ is a function $c\colon V(G) \to \{1, 2, \dots, q\}$. The most well-studied coloring problem on graphs is proper-coloring. A q-coloring c is called a proper-coloring if for each edge $\{u, v\} \in E(G)$, $c(u) \neq c(v)$. In this paper, we study two specialized variants of q-conflict-free coloring on graphs, known as q-ONCF-coloring and q-CNCF-coloring, which are defined as follows.

Definition 1. *Given a graph G, a q-coloring $c\colon V(G) \to \{1, 2, \dots, q\}$ is called a q-ONCF-coloring, if for every vertex $v \in V(G)$, there is a vertex u in the open neighborhood $N(v)$ such that $c(u) \neq c(w)$ for all $w \neq u \in N(v)$. In other words, every open neighborhood in G has a uniquely colored vertex.*

Definition 2. *Given a graph G, a q-coloring $c\colon V(G) \to \{1, 2, \dots, q\}$ is called a q-CNCF-coloring, if for for every vertex $v \in V(G)$, there is a vertex u in the closed neighborhood $N[v]$ such that $c(u) \neq c(w)$ for all $w \neq u \in N[v]$. In other words, every closed neighborhood in G has a uniquely colored vertex.*

Observe that by the above definitions, the q-ONCF-coloring (or q-CNCF-coloring) problem is a special case of the conflict-free coloring of set systems. Given a graph G, we can associate it with the set system $\mathcal{H} = (V(G), \mathcal{F})$, where $\mathcal{F}$ consists of the sets given by open neighborhoods $N(v)$ (respectively, closed neighborhoods $N[v]$) for $v \in V(G)$. A q-ONCF-coloring (or q-CNCF-coloring) of G then corresponds to a q-conflict-free coloring of the associated set system.

Notationally, let $\chi_{CF}(\mathcal{H})$ denote the minimum number of colors required for a conflict-free coloring of a set system $\mathcal{H}$. Similarly, we denote by $\chi_{ON}(G)$ and $\chi_{CN}(G)$ the minimum number of colors required for an ONCF-coloring and a CNCF-coloring of a graph G, respectively. The study of conflict-free coloring was initially restricted to combinatorial studies. This was first explored in [5] and [13]. Pach and Tardos [12] gave an upper bound of $\mathcal{O}(\sqrt{m})$ on $\chi_{CF}(\mathcal{H})$ for a set system $\mathcal{H} = (U, \mathcal{F})$ when the size of $\mathcal{F}$ is m. In [12], it was also shown that for a graph G with n vertices $\chi_{CN}(G) = \mathcal{O}(\log^2 n)$. This bound was shown to be tight in [7]. Similarly, [3] showed that $\chi_{ON}(G) = \Theta(\sqrt{n})$.

However, computing $\chi_{ON}(G)$ or $\chi_{CN}(G)$ is NP-hard. This is because deciding whether a 2-ONCF-coloring or a 2-CNCF-coloring of G exists is NP-hard [6]. This motivates the study of the following decision problems under the lens of parameterized complexity.

q-ONCF-Coloring
Input: A graph G.
Question: Is there a q-ONCF-coloring of G?

The q-CNCF-Coloring problem is defined analogously.

Note that because of the NP-hardness for q-ONCF-Coloring or q-CNCF-Coloring even when $q = 2$, the two problems are para-NP-hard under the natural parameter q. Thus, the problems were studied under structural parameters. Gargano and Rescigno [6] showed that both q-ONCF-Coloring and q-CNCF-Coloring have FPT algorithms when parameterized by (i) the size of a vertex cover of the input graph G, (ii) and the neighborhood diversity of the input graph. Gargano and Rescigno also mention that due to Courcelle's theorem, for a non-negative constant q, the two decision problems are FPT with the treewidth of the input graph as the parameter.

Our Results and Contributions. In this paper, we extend the parameterized study of the above two problems with respect to structural parameters. Our first objective is to provide both upper and lower bounds for FPT algorithms when using treewidth as the parameter (Sect. 3). We show that both q-ONCF-Coloring and q-CNCF-Coloring parameterized by treewidth t can be solved in time $(2q^2)^t n^{\mathcal{O}(1)}$. On the other hand, for $q \geq 3$, both problems cannot be solved in time $(q-\epsilon)^t n^{\mathcal{O}(1)}$ under Strong Exponential Time Hypothesis (SETH). For $q = 2$, both problems cannot be solved in time $2^{o(t)} n^{\mathcal{O}(1)}$ under Exponential Time Hypothesis (ETH).

We also study the polynomial kernelization question (Sect. 4). Observe that both q-ONCF-Coloring and q-CNCF-Coloring cannot have polynomial kernels under treewidth as the parameter, as there are straightforward AND-cross-compositions from each problem to itself.[1] Therefore, we will study the kernelization question by a larger parameter, namely the size of a vertex cover in the input graph. The kernelization complexity of the q-Coloring problem (asking for a proper-coloring of the input graph) is very well-studied for this parameter, the problem admits a kernel of size $\widetilde{\mathcal{O}}(k^{q-1})$ [10] which is known to be tight unless $\mathsf{NP} \subseteq \mathsf{coNP/poly}$ [9]. From this perspective however, q-CNCF-Coloring and q-ONCF-Coloring turn out to be much harder: q-CNCF-Coloring for $q \geq 3$ and q-ONCF-Coloring for $q \geq 2$ do not have polynomial kernels under

[1] This is true for a number of graph problems when parameterized by treewidth. For more information, see [4, Theorem 15.12] and the example given for Treewidth (parameterized by solution size) in [4, p. 534].

the standard complexity assumptions, when parameterized by the size of a vertex cover. Interestingly, 2-CNCF-Coloring parameterized by vertex cover size *does* have a polynomial kernel and we obtain an explicit polynomial compression for the problem. Although this does not lead to a polynomial kernel of reasonable size, we study a restricted version called 2-CNCF-Coloring-VC-Extension (Sect. 4.1) and show that this problem has a $\mathcal{O}(k^2 \log k)$ kernel where k is the vertex cover size. Therefore, 2-CNCF-Coloring behaves significantly differently from the other problems.

Finally, we obtain a number of combinatorial results regarding ONCF-colorings of graphs. Denote by $\chi(G)$ the minimum q for which a q-proper-coloring for G exists. While $\chi_{CN}(G) \leq \chi(G)$, the same upper bound does not hold for $\chi_{ON}(G)$ [6]. For a graph G, let $\mathsf{vc}(G)$, $\mathsf{fvs}(G)$ and $\mathsf{tw}(G)$ denote the size of a minimum vertex cover, the size of a minimum feedback vertex set and the treewidth of G, respectively. From the known result that $\chi(G) \leq \mathsf{tw}(G) + 1 \leq \mathsf{fvs}(G) + 1 \leq \mathsf{vc}(G) + 1$, we could immediately obtain the fact that the same behavior holds for $\chi_{CN}(G)$. However, to show that $\chi_{ON}(G)$ behaves similarly more work needs to be done. To the best of our knowledge no upper bounds on $\chi_{ON}(G)$ with respect to $\mathsf{fvs}(G)$ and $\mathsf{tw}(G)$ were known, while a loose upper bound was provided with respect to $\mathsf{vc}(G)$ in [6]. We give a tight upper bound on $\chi_{ON}(G)$ with respect to $\mathsf{vc}(G)$ and also provide the first upper bounds on $\chi_{ON}(G)$ with respect to $\mathsf{fvs}(G)$ and $\mathsf{tw}(G)$ (Sect. 5).

Our main contributions in this work are structural results for the conflict-free coloring problem, which we believe gives more insight into the decision problems on graphs. Firstly, the gadgets we build for the ETH-based lower bounds could be useful for future lower bounds, but are also useful to understand difficult examples for conflict-free coloring which have not been known in abundance so far. We are able to reuse these gadgets in the constructions needed to prove the kernelization lower bounds. Secondly, our combinatorial results also give constructible conflict-free colorings of graphs and therefore provide more insight into conflict-free colored graphs. Finally, the kernelization dichotomy we obtain for q-ONCF-Coloring and q-CNCF-Coloring under vertex cover size as a parameter is a very surprising one.

2 Preliminaries

For a positive integer n, we denote the set $\{1, 2, \ldots, n\}$ in short with $[n]$. For a graph G, given a q-coloring $c\colon V(G) \to [q]$ and a subset $S \subseteq V(G)$, we denote by $c|_S$ the restriction of c to the subset S. For a graph G that is q-ONCF-colored by a coloring c, for a vertex $v \in V(G)$, suppose $w \in N(v)$ is such that $c(w) \neq c(w')$ for each $w' \neq w \in N(v)$; then $c(w)$ is referred to as the ONCF-color of v. Similarly, for a graph G that is q-CNCF-colored by a coloring c, for a vertex $v \in V(G)$, a unique color in $N[v]$ is referred to as the CNCF-color of v.

An *edge-star graph* is a generalization of a star graph where there is a central edge $\{u, v\}$ and all other vertices w have $N(w) = \{u, v\}$. A triangle is an example of an edge-star graph.

For statements marked with a star ($\star$), the (complete) proof can be found in the full version of the paper [2].

2.1 Parameterized Complexity

Let Σ be a finite alphabet. A parameterized problem $\mathcal{Q}$ is a subset of $\Sigma^* \times \mathbb{N}$.

Definition 3 (Kernelization). *Let $\mathcal{Q}, \mathcal{Q}'$ be two parameterized problems and let $h\colon \mathbb{N} \to \mathbb{N}$ be some computable function. A* generalized kernel *from $\mathcal{Q}$ to $\mathcal{Q}'$ of size $h(k)$ is an algorithm that given an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, outputs $(x', k') \in \Sigma^* \times \mathbb{N}$ in time poly$(|x| + k)$ such that (i) $(x, k) \in \mathcal{Q}$ if and only if $(x', k') \in \mathcal{Q}'$, and (ii) $|x'| \leq h(k)$ and $k' \leq h(k)$.*

The algorithm is a kernel *if $\mathcal{Q} = \mathcal{Q}'$. It is a* polynomial (generalized) kernel *if $h(k)$ is a polynomial in k.*

3 Algorithmic Results Parameterized by Treewidth

In this section, we state the algorithmic results obtained for the ONCF-Coloring and CNCF-Coloring problems parameterized by treewidth. On the algorithmic side, we have the following theorem.

Theorem 1 ($\star$). *q-ONCF-Coloring and q-CNCF-Coloring parameterized by treewidth t admits a $(2q^2)^t n^{\mathcal{O}(1)}$ time algorithm.*

We also obtain algorithmic lower bounds for the problems under standard assumptions.

Theorem 2 ($\star$). *The following algorithmic lower bounds can be obtained:*

1. *For $q \geq 3$, q-ONCF-Coloring or q-CNCF-Coloring parameterized by treewidth t cannot be solved in $(q - \varepsilon)^t n^{\mathcal{O}(1)}$ time, under SETH.*
2. *2-ONCF-Coloring or 2-CNCF-Coloring parameterized by treewidth t cannot be solved in $2^{o(t)} n^{\mathcal{O}(1)}$ time, under ETH.*

Due to paucity of space, the full proofs of the Theorems above have been omitted from this extended abstract. As a brief overview of our lower bound techniques, in the remainder of section we will show the running time lower bound on 2-ONCF-Coloring under ETH claimed in Theorem 2. The bound will be obtained by giving a reduction from 3-SAT, to give the reduction we will need the following type of gadget.

Definition 4. *An* ONCF-gadget *is a gadget on ten vertices, as depicted in Fig. 1.*

The objective of this gadget is the following. The vertices $\{g_1, g_2, g_3, g_{10}\}$ in Fig. 1 will be the interaction points of the ONCF-gadget with the outside world. As will be proved in the following two lemmas, the gadget is designed so as to (i) disallow certain 2-ONCF-colorings and (ii) allow certain 2-ONCF-colorings on its interaction points.

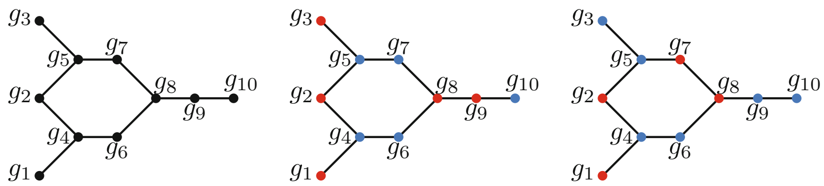

Fig. 1. The ONCF-gadget (left). Observe that if g_1, g_2, and g_3 are all red, then g_9 must also be red (middle), and if one of g_1, g_2, or g_3 is blue, then g_9 may be blue (right). (Color figure online)

Lemma 1 **($\star$).** *Let G be a ONCF-gadget with a coloring $c\colon V(G) \to \{\text{red}, \text{blue}\}$ such that for all $4 \le i \le 9$ the neighborhood of g_i is ONCF-colored by c. If $c(g_1) = c(g_2) = c(g_3) = \text{red}$, then $c(g_9) = \text{red}$.*

Lemma 2 **($\star$).** *Let G be a ONCF-gadget. Let $c'\colon \{g_1, g_2, g_3\} \to \{\text{red}, \text{blue}\}$ be a partial 2-ONCF-coloring of G. If there exists $i \in [3]$ such that $c'(g_i) = \text{blue}$, then c' can be extended to a coloring c satisfying*

1. *For every $4 \le i \le 9$, the neighborhood of vertex g_i is ONCF-colored by c (contains at most one red, or at most one blue vertex), and*
2. *$c(g_9) = \text{blue}$, $c(g_8) = \text{red}$, $c(g_4) = c(g_5) = \text{blue}$, and $c(g_{10}) = \text{blue}$.*

Now that we have introduced the necessary gadgets, we can prove the running time lower bound for 2-ONCF-Coloring.

Lemma 3 **($\star$).** 2-ONCF-Coloring *parameterized by treewidth t cannot be solved in $2^{o(t)} n^{\mathcal{O}(1)}$ time, under ETH.*

Proof (Proof sketch). We show this by giving a reduction from 3-SAT. Given an instance of 3-SAT with variables $x_1, \ldots, x_n$ and clauses $C_1, \ldots, C_m$, create a graph G as follows. Start by creating palette vertices R, R', and B, and edges $\{R, R'\}$ and $\{R', B\}$. For each variable $i \in [n]$, create vertices u_i, v_i, w_i and add edges $\{u_i, v_i\}$ and $\{v_i, w_i\}$. For each $j \in [m]$, add an ONCF-gadget G_j and connect g_{10} of this gadget to R. Add vertices s_j^1, s_j^2, and s_j^3 and connect s_j^b to g_b in G_j for $b \in [3]$. Let clause $C_j := (\ell_1, \ell_2, \ell_3)$. Now if $\ell_b = x_i$ for some $i \in [n], b \in [3]$, connect s_j^b to u_i. Similarly, if $\ell_b = \neg x_i$, connect s_j^b to w_i. This concludes the construction of G. The main idea towards showing that φ is satisfiable if and only if G is 2-ONCF-colorable is to let the situation where u_i is *red* and w_i is *blue* mean that the corresponding variable is set to *true*. A more detailed explanation can be found in the full version of this paper [2].

Note that the graph induced by $V(G) \setminus (\{R, R', B\} \cup \{u_i, v_i, w_i \mid i \in [n]\})$ is a disjoint union of ONCF-gadgets for which every g_b for $b \in [3]$ has an additional degree-1 vertex attached to it. It is easy to see that every connected component of this graph has treewidth two. Thus, G has treewidth at most $3n + 5$. This implies that a 3-SAT formula ϕ on n variables and m clauses is reduced to a graph G with treewidth at most $3n + 5$. Since 3-SAT cannot be solved in $2^{o(n)} n^{\mathcal{O}(1)}$

time under ETH, this also implies that 2-CNCF-Coloring parameterized by treewidth t cannot be solved in $2^{o(t)}n^{\mathcal{O}(1)}$ time, under ETH. □

Note that a reduction from 3-SAT to 2-ONCF-Coloring was given in Theorem 2 of [6]. However, that reduction led to a quadratic blow-up in the input size. Hence, the need for the alternative reduction given above.

4 Kernelization

In this section, we will study the kernelizability of the ONCF- and CNCF-coloring problems, when parameterized by the size of a vertex cover. We prove the following two theorems to obtain a dichotomy on the kernelization question.

Theorem 3 ($\star$)**.** *q-ONCF-Coloring for $q \geq 2$ and q-CNCF-Coloring for $q \geq 3$, parameterized by vertex cover size do not have polynomial kernels, unless* NP $\subseteq$ coNP/poly.

Theorem 4. 2-CNCF-Coloring *parameterized by vertex cover size k has a generalized kernel of size $\mathcal{O}(k^{10})$.*

Note that by using an NP-completeness reduction, this results in a polynomial kernel for 2-CNCF-Coloring parameterized by vertex cover size. We also obtain an $\mathcal{O}(k^2 \log k)$ kernel for an extension problem of 2-CNCF-Coloring and this is described in Sect. 4.1.

In the remainder of this section we will prove Theorem 4, by obtaining a polynomial generalized kernel for 2-CNCF-Coloring parameterized by vertex cover size. This result is in contrast to the kernelization results we obtain for q-CNCF-Coloring for $q \geq 3$ as well as q-ONCF-Coloring for $q \geq 2$. We will start by transforming an instance of 2-CNCF-Coloring to an equivalent instance of another problem, namely d-Polynomial root CSP. We will then carefully rephrase the d-Polynomial root CSP instance such that it uses only a limited number of variables, such that we can use a known kernelization result for d-Polynomial root CSP to obtain our desired compression. We start by introducing the relevant definitions.

Define d-Polynomial root CSP over a field F as follows [11].

> d-Polynomial root CSP
> **Input:** A list L of polynomial equalities over variables $V = \{x_1, \ldots, x_n\}$. An equality is of the form $f(x_1, \ldots, x_n) = 0$, where f is a multivariate polynomial over F of degree at most d.
> **Question:** Does there exist an assignment of the variables $\tau\colon V \to \{0, 1\}$ satisfying all equalities (over F) in L?

A field F is said to be *efficient* if both the field operations and Gaussian elimination can be done in polynomial time in the size of a reasonable input encoding. In particular, $\mathbb{Q}$ is an efficient field by this definition. The following theorem was shown by Jansen and Pieterse.

Theorem 5 **([11, Theorem 5]).** *There is a polynomial-time algorithm that, given an instance (L, V) of d-*Polynomial root CSP *over an efficient field F, outputs an equivalent instance (L', V) with at most $n^d + 1$ constraints such that $L' \subseteq L$.*

Using the theorem introduced above, we can now prove Theorem 4.

Proof (Proof of Theorem 4). Given an input instance G with vertex cover S of size k, we start by preprocessing G. For each set $X \subseteq S$ with $|X| \leq 2$, mark 3 vertices in $v \in G \setminus S$ with $N(v) = X$ (if there do not exist 3 such vertices, simply mark all). Let $S' \subseteq V(G) \setminus S$ be the set of all marked vertices. Remove all $w \in V(G) \setminus (S \cup S')$ with $deg(w) \leq 2$ from G. Let the resulting graph be G'.

Claim 1. *G' is 2-CNCF-colorable if and only if G is 2-CNCF-colorable.*

Proof. In one direction, suppose G' has a 2-CNCF coloring c using colors $\{r, b\}$. Consider a vertex $w \in V(G) \setminus V(G')$. Let $X_w \subseteq S$ be the neighborhood of w. Note that $|X_w|$ is at most 2. Consider $N(X_w) \cap S'$. Since w was deleted, there are 3 vertices in $N(X_w) \cap S'$. Consider the color from $\{r, b\}$ that appears in majority on the vertices of $N(X_w) \cap S'$. If we color w with the same color, it is easy to verify that this extension of c to G is a 2-CNCF coloring of G.

In the reverse direction, suppose G has a 2-CNCF coloring c using colors $\{r, b\}$. We describe a new coloring c' for G as follows. Consider a subset $X \subseteq S$ of size at most 2 and let N be the set of vertices in $G \setminus S$ that have X as their neighborhood. If $|N| > 3$ and $N \setminus S'$ has a vertex w that is uniquely colored in the set N, then we arbitrarily choose a vertex $w' \in N \cap S'$. We define $c'(w') = c(w)$ and $c'(w) = c(w')$. All other vertices have the same color in c and c'. It is easy to verify that c' is also a 2-CNCF coloring of G and the restriction of c' to G' is a 2-CNCF coloring of G'. ⌟

We continue by creating an instance of 2-Polynomial root CSP that is satisfiable if and only if G' is 2-CNCF-colorable. Let $V := \{r_v, b_v \mid v \in V(G)\}$ be the variable set. We create L over $\mathbb{Q}$ as follows.

1. For each $v \in V(G')$, add the constraint $r_v + b_v - 1 = 0$ to L.
2. For all $v \in V(G')$, add the constraint $(-1+\sum_{u \in N[v]} r_v) \cdot (-1+\sum_{u \in N[v]} b_v) = 0$.
3. For each $v \in V(G') \setminus (S \cup S')$ of degree $d_v = |N(v)|$ add the constraint

$$(\sum_{u \in N(v)} r_u)(-1 + \sum_{u \in N(v)} r_u)(-(d_v - 1) + \sum_{u \in N(v)} r_u)(-d_v + \sum_{u \in N(v)} r_u) = 0.$$

   Note that such a constraint is a quadratic polynomial.

Intuitively, the first constraint ensures that every vertex is either *red* or *blue*. The second constraint ensures that in the closed neighborhood of every vertex, exactly one vertex is *red* or exactly one is *blue*. The third constraint is seemingly redundant, saying that the open neighborhood of every vertex outside the vertex cover does not have two *red* or two *blue* vertices, which is clearly forbidden. The requirement for these last constraints is made clear in the proof of Claim 4.

We show that this results in an instance that is equivalent to the original input instance, in the following sense.

Claim 2 ($\star$). *(L, V) is a yes-instance of* 2-Polynomial root CSP *if and only if G' is 2-CNCF-colorable.*

Clearly, $|V| = 2n$ if n is the number of vertices of G'. We will now show how to modify L, such that it uses only variables for the vertices in $S \cup S'$. To this end, we introduce the following function. For $v \notin (S \cup S')$, let $f_v(V) := g\big(\sum_{u \in N(v)} r_u, |N(v)|\big)$, where

$$g(x, N) = -\frac{(N-x)(x-1)(N-2(x+1))}{N(N-2)}.$$

Note that for any fixed $N > 2$, $g(x, N)$ describes a degree-3 polynomial in x over $\mathbb{Q}$. The following is easy to verify.

Observation 1. $g(0, N) = g(N-1, N) = 1$*, and* $g(N, N) = g(1, N) = 0$ *for all* $N \in \mathbb{Z} \setminus \{0, 2\}$.

Observe that f_v only uses variables defined for vertices that are in S. As such, let $V' := \{r_v, b_v \mid v \in S\} \cup \{r_v, b_v \mid v \in S'\}$, and let L' be equal to L with every occurrence of r_v for $v \notin (S \cup S')$ substituted by f_v and every occurrence of b_v for $v \notin (S \cup S')$ substituted by $(1 - f_v(V))$.

Claim 3 ($\star$). *If $\tau \colon V \to \{0, 1\}$ is a satisfying assignment for (L, V), then $\tau|_{V'}$ is a satisfying assignment for (L', V').*

The next claim shows the equivalence between (L', V') and (L, V).

Claim 4 ($\star$). *If $\tau \colon V' \to \{0, 1\}$ is a satisfying assignment for (L', V'), then there exists a satisfying assignment $\tau' \colon V \to \{0, 1\}$ for (L, V) such that $\tau'|_{V'} = \tau$.*

Using the method described above, we obtain an instance (L, V) of 2-Polynomial root CSP such that (L, V) has a satisfying assignment if and only if G is 2-CNCF-colorable by Claims 1 and 2. Then we obtain an instance (L', V') such that (L', V') is satisfiable if and only if (L, V) is satisfiable by Claims 3 and 4. As such, (L', V') is a yes-instance if and only if G is 2-CNCF-colorable and it suffices to give a kernel for (L', V'). Observe that $|V'| = \mathcal{O}(k^2)$.

We start by partitioning L' into three sets L'_S, L'_1 and L'_2. Let L'_S contain all equalities created for a vertex $v \in S$. Let L'_1 contain all equations that contain at least one of the variables in $\{r_v, b_v \mid v \in S'\}$ and let L_2 contain the remaining equalities. Observe that $|L'_S| = k$ by definition. Furthermore, the polynomials in L'_1 have degree at most 2, as they were created for vertices in $V(G') \setminus S$, and these are not connected. As such, we use Theorem 5 to obtain $L''_1 \subseteq L'_1$ such that $|L''_1| = \mathcal{O}((k^2)^2) = \mathcal{O}(k^4)$ and any boolean assignment satisfying all equalities in L''_1 satisfies all equalities in L'_1.

Similarly, we observe that L_2' by definition contains none of the variables in $\{r_v, b_v \mid v \in S'\}$, implying that the equations in L_2' are equations over only k variables. Since the polynomials in L_2' have degree at most 6, we can apply Theorem 5 to obtain $L_2'' \subseteq L_2'$ such that $|L_2''| \leq \mathcal{O}(k^6)$ and any assignment satisfying all equations in L_2'' satisfies all equalities in L_2'.

We now define $L'' := L_1'' \cup L_2'' \cup L_S'$, and the output of our polynomial generalized kernel will be (L'', V'). The correctness of the procedure is proven above, it remains to bound the number of bits needed to store instance (L'', V').

By this definition, $|L''| \leq \mathcal{O}(k^6)$. To represent a single constraint, it is sufficient to store the coefficients for each variable in V'. The storage space needed for a single coefficient is $\mathcal{O}(\log(n))$, as the coefficients are bounded by a polynomial in n. Thereby, (L'', V') can be stored in $\mathcal{O}(k^6 \cdot k^2 \log n)$ bits. To bound this in terms of k, we observe that it is easy to solve 2-CNCF-Coloring in time $\mathcal{O}(2^{k^2} \cdot \text{poly(n)})$. This is done by guessing the coloring of S, extending this coloring to the entire graph (observe $G \setminus S$ has no vertices of degree less than three) and verifying whether this results in a CNCF-coloring. Therefore, we can assume that $\log(n) \leq k^2$, as otherwise we can solve the 2-CNCF-Coloring problem in $\mathcal{O}(2^{k^2}\text{poly}(n))$ time, which is then polynomial in n. Thereby we conclude that (L'', V') can be stored in $\mathcal{O}(k^{10})$ bits. □

4.1 Kernelization Bounds for Conflict-Free Coloring Extension

We furthermore provide kernelization bounds for the following extension problems.

q-CNCF-Coloring-VC-Extension **Input:** A graph G with vertex cover S and partial q-coloring $c\colon S \to [q]$. **Question:** Does there exist a q-CNCF-coloring of G that extends c?

We define q-ONCF-Coloring-VC-Extension analogously.

We obtain the following kernelization results when parameterized by vertex cover size, thereby classifying the situations where the extension problem has a polynomial kernel. The extension problem turns out to have a polynomial kernel in the same case as the normal problem. However, we manage to give a significantly smaller kernel. Observe that the kernelization result is non-trivial, since 2-CNCF-Coloring-VC-Extension is NP-hard ($\star$).

Theorem 6 **($\star$).** *The following results hold.*

1. 2-CNCF-Coloring-VC-Extension *has a kernel with $\mathcal{O}(k^2)$ vertices and edges that can be stored in $\mathcal{O}(k^2 \log k)$ bits. Here k is the size of the input vertex cover S.*
2. q-CNCF-Coloring-VC-Extension *for any $q \geq 3$, and* 2-ONCF-Coloring-VC-Extension *parameterized by the size of a vertex cover do not have a polynomial kernel, unless* NP $\subseteq$ coNP/poly.

5 Combinatorial Bounds

Given a graph G, it is easy to prove that $\chi_{\mathsf{CN}}(G) \leq \chi(G)$. However, there are examples that negate the existence of such bounds with respect to χ_{ON} [6]. In this section, we prove combinatorial bounds for χ_{ON} with respect to common graph parameters like treewidth, feedback vertex set and vertex cover.

First, note that if G is a graph with isolated vertices then the graph can have no ONCF-coloring. Therefore, in all the arguments below we assume that G does not have any isolated vertices. We obtain the following result. Recall that for a graph G, $\mathsf{vc}(G)$, $\mathsf{fvs}(G)$ and $\mathsf{tw}(G)$ denote the size of a minimum vertex cover, the size of a minimum feedback vertex set and the treewidth of G, respectively.

Theorem 7 **(⋆).** *Given a connected graph G,*

1. $\chi_{\mathsf{ON}}(G) \leq 2\mathsf{tw}(G) + 1$,
2. $\chi_{\mathsf{ON}}(G) \leq \mathsf{fvs}(G) + 3$,
3. $\chi_{\mathsf{ON}}(G) \leq \mathsf{vc}(G) + 1$. *Furthermore, if G is not a star graph or an edge-star graph, then* $\chi_{\mathsf{ON}}(G) \leq \mathsf{vc}(G)$.

Here, we only give a proof sketch of the result with respect to $\mathsf{vc}(G)$ and relegate the other two combinatorial results to the full version of the paper. The next lemma bounds the value of $\chi_{\mathsf{ON}}(G)$ for graphs with a vertex cover of size k. In particular, we improve the bound given by Gargano and Rescigno [6, Lemma 4], who showed that $\chi_{\mathsf{ON}}(G) \leq 2k + 1$.

Lemma 4 **(⋆).** *Let G be a connected graph with $\mathsf{vc}(G) = k$. Then $\chi_{\mathsf{ON}}(G) \leq k+1$. Furthermore, if G is not a star graph or an edge-star graph, then $\chi_{\mathsf{ON}}(G) \leq k$.*

Proof (Proof sketch). See Fig. 2 for a sketch of the colorings described in the proof. We start by proving the bounds for the case where G is not a star and not an edge-star. Let S be a minimum vertex cover of G and let k be the size of S. We do a case distinction on the size and connectedness of S.

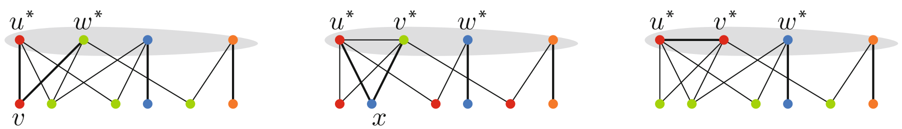

Fig. 2. (left) A coloring of the graph when all vertices in $G[S]$ are isolated. (middle) The case where $G[S]$ contains an edge and the endpoints have a common neighbor. (right) The case where $G[S]$ contains an edge and the endpoints have no common neighbors.

($k = 2$ and S connected) First, we prove the bounds for $k = 2$ and $G[S]$ is an edge $\{u^*, v^*\}$. Note that G is not an edge-star graph. Therefore at least one of u^* or v^* have neighbors with degree exactly 1 in $G \setminus S$. As shown in the full proof, it is possible to ONCF-color such a graph with 2 colors, namely r and b.

($G[S]$ disconnected or $k \geq 3$) We now prove the bounds for $k = 2$ and $G[S]$ is disconnected, and $k \geq 3$. We consider a number of cases.

(Suppose $G[S]$ contains a connected component C of size at least three.) Let $v^* \in C$ be a vertex such that $G[C \setminus \{v\}]$ remains connected. We color the vertices in G as follows. For every vertex $u \in S$, let $c(u) := c_u$. For every vertex $u \in S$ that is isolated in $G[S]$, pick an arbitrary neighbor $v \notin S$ and (re)color v such that $c(v) := c_u$. Notice that a vertex v in $G \setminus S$ may be picked multiple times as the candidate for an arbitrary neighbor for an isolated vertex in S, and in this case the color of this vertex v is set to the last color it is assigned. For every vertex v that is not yet colored, let $c(v) := c_{v^*}$. It can be shown that c is the required coloring.

(Suppose $G[S]$ only contains connected components of size one.) Note that $|S| > 1$. Start by letting $c(v) := c_v$ for every vertex $v \in S$. Since G is connected, there exists $v \notin S$ such that $|N(v)| \geq 2$. Pick two vertices $u^*, w^* \in N(v)$ with $u^* \neq w^*$. Let $c(v) := c_{u^*}$. For every vertex $u \in S \setminus \{u^*, w^*\}$ pick an arbitrary neighbor $v \notin S$ and recolor v to c_u. Color the vertices that remained uncolored by this procedure with c_{w^*}. It can be shown that c is the required coloring.

(Otherwise.) In this case $G[S]$ has size at least 3, contains multiple connected components, and at least one such component has size two. This leads to two further cases, that have been analyzed in the complete proof in the full version of this paper.

If G is not a star and not an edge-star, we are in one of the cases above. Otherwise, it is easy to observe that stars have a vertex cover of size one and can always be colored with two colors, and edge-stars can be colored with three colors while having a minimum vertex cover size of two. □

Observe that the bounds of Lemma 4 are tight. First, a star graph requires 2 colors and has vertex cover size 1 while an edge-star graph requires 3 colors and has vertex cover size 2. On the other hand, given an $q \geq 3$, taking the complete graph K_q and subdividing each edge once results in a graph that requires q colors [6] for an ONCF-coloring and has a vertex cover of size q.

6 Open Problems

The study in this paper leads to some interesting open questions. In this paper we only exhibit a generalized kernel of size $\mathcal{O}(k^{10})$ for 2-CNCF-Coloring and it remains to resolve the size of tight polynomial kernels for the problem. On the combinatorial side, with respect to minimum vertex cover, we obtain tight upper bounds on $\chi_{ON}(G)$. It would be interesting to obtain corresponding tight bounds for $\chi_{ON}(G)$ with respect to feedback vertex set and treewidth.

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