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Short communication

On reconstruction of bandlimited signals from purely timing information

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ABSTRACT

The well known Logan's theorem asserts that, under some assumptions, a one octave bandpass signal is recoverable, modulo a multiplicative constant, from its zero crossings only. However, such recovery is numerically problematic and the theorem is not applicable to general bandlimited signals. In this paper, we demonstrate that the additional timing information sufficient for recovery of general band limited signals can be provided in the form of the zero crossings of several of its derivatives and propose novel numerically robust algorithms for such timing extraction and for signal reconstruction, both with high fidelity. We tested the proposed algorithms extensively, with both synthetic and audio signals. In particular, we numerically demonstrate that the timing of the zero crossings of a typical speech signal and of its first two derivatives are not sufficient for a numerically robust recovery of such a signal, but that a robust recovery becomes possible by adding the timing of the zero crossings of the third order derivative.

1. Introduction

The well know Logan's theorem [1] asserts that if a bandpass signal has no zeros of even multiplicity, never vanishes simultaneously with its Hilbert transform and is of bandwidth less than an octave, then such a signal is uniquely determined (up to a scaling factor) by its zero crossings (ZCs) only. However, signal recovery from ZCs is numerically problematic and, as detailed in [2], practical algorithms are either unstable or require additional information about the signal to stabilize the reconstruction, as it is done in [2–5].

In [6], we showed that for a robust recovery of signals satisfying the conditions of Logan's theorem, the timing of the ZCs of the signal and the ZCs of its first derivative are sufficient. In this paper, we empirically study when a general band limited (rather than bandpass) signal can be reconstructed from purely timing information in the form of ZCs of the signal and the ZCs of several of its derivatives. The contributions of this paper are as follows. **i)** We present a novel algorithm for the extraction of the ZCs of a BL signal and the ZCs of its derivatives, *up to a relatively high order*, which conventional approximations, such as splines, cannot achieve. **ii)** We present an algorithm for a highly accurate and numerically robust reconstruction of a BL signal from ZCs of the signal and ZCs of several of its higher order derivatives, for cases when information available suffices. **iii)** We empirically demonstrate that for long speech signals consisting of tens of thousands of samples, which contain several silent intervals between words, the ZCs of the signal and of its first two derivatives are insufficient for a robust reconstruction obtained by operating on data from sections of the signal of size of several hundred Nyquist rate sampling intervals. However, if in addition the ZCs of the third derivative are known, such a reconstruction becomes possible and is very robust, with an average signal to error ratio (SER) close to 50 dB, see Fig 2. To the best of our knowledge, the proposed algorithm achieves much higher reconstruction accuracy from purely timing information compared to the state of the art (e.g. [3]). The same observation applies to a wide variety of audio signals from the Google Audioset. **iv)** We also demonstrate a case of a synthetic signal where the ZCs of the signal and ZCs of its first six derivatives are insufficient for a robust reconstruction, but addition of the ZCs of its seventh derivative allows a robust reconstruction with a SER of 48 dB. **v)** Finally, motivated by the empirical results obtained, we argue that the failure of our reconstruction from the first six derivatives in the example from iv) is not due to a shortcoming of our method, but is intrinsically impossible, and we formulate a compelling conjecture on conditions for a reconstruction of a signal from non uniform samples of the outputs of several linearly independent filters applied to such a signal.

We note that i) and ii) above are relevant to research on event-driven sampling, which unlike the usual uniform sampling,

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encodes a signal using purely timing information, such as the zero crossings or level crossings instants of the signal. Extracting such timing information is useful in a number of applications; see, e.g., [3,7–11]. On the other hand, iii) and iv) represent an empirical investigation on the conditions when a general band limited signal is recoverable from purely timing data. The empirical results from iii) and iv) are used to lend credibility to our conjecture mentioned above, which has a clear theoretical significance and which, if true, can be seen as a generalization of the Papoulis' generalized sampling theorem to non-uniformly sampled filterbanks.

Our algorithms are based on chromatic approximations (CAs), briefly reviewed below, which possess a property not shared by the traditional methods based on splines [12] and other interpolation techniques: not only do the CAs provide extremely accurate approximations of BL signals, but also the derivatives of CAs are accurate approximations of the corresponding derivatives of the signal, up to relatively high orders. Next, we briefly review the basics of CAs relevant to this work; with extensive details found in [13,14].

2. Chromatic derivatives and approximations

Let $\mathrm{w}(\omega)$ be any symmetric, piece-wise continuous, non-negative weight function in the frequency domain, supported on $[-\pi, \pi]$, such that $\int_{-\pi}^{\pi} \mathrm{w}(\omega) d\omega = 1$ and let $p_n(\omega)$ be the family of polynomials orthonormal with respect to $\mathrm{w}(\omega)$. *Chromatic derivatives* (CDs) associated with polynomials $p_n(\omega)$ are defined as $\mathcal{K}^n = (-j)^n p_n\left(j\,\frac{\mathrm{d}}{\mathrm{d}t}\right)$, where $p_n(j\,\mathrm{d}/\mathrm{d}t)$ are obtained from $p_n(\omega)$ by replacing powers ω^k with $j^k\,\mathrm{d}^k/\mathrm{d}t^k$; see [13, Eq. (3)], [14]. A well known fact that families of symmetric orthonormal polynomials satisfy a three term recurrence implies that the corresponding CDs satisfy the recurrence of the form $\mathcal{K}^{n+1} = \frac{1}{\gamma_n}\left(\frac{\mathrm{d}}{\mathrm{d}t} \circ \mathcal{K}^n\right) + \frac{\gamma_{n-1}}{\gamma_n}\mathcal{K}^{n-1}$ for some positive coefficients γ_n [13]. Typically, such CDs are obtained in practice by using either a least squares fit on sampled data or by using a front-end filter bank [14] which can also be implemented in the analog domain [15,16]. Let $B_0(t)$ be the inverse Fourier transform of the weight function $\mathrm{w}(\omega)$, i.e., $B_0(t) = \int_{-\infty}^{\infty} \mathrm{w}(\omega) \mathrm{e}^{j\omega t}\,\mathrm{d}\omega$ and let $B_n(t) = (-1)^n \mathcal{K}^n[B_0](t)$. *Chromatic expansion* of a BL signal $f(t)$ centered at $t = u$ is defined as $f(t) = \sum_{n=0}^{\infty} \mathcal{K}^n[f](u) B_n(t-u)$; its truncations lead to CAs of degree M of the form $f(t) \approx \mathrm{app}[f, M, u](t) := \sum_{n=0}^{M} \mathcal{K}^n[f](u) B_n(t-u)$; for an estimate of the error of such an approximation see [14, Theorem 2.1]. For most practical applications and in this paper, CDs correspond to the polynomials $p_n(\omega) = \sqrt{2n+1}\, P_n^L(\omega/\pi)$, orthonormal with respect to the constant weight function $\mathrm{w}(\omega) = \frac{1}{2\pi}$; they are obtained by normalizing and rescaling the Legendre polynomials $P_n^L(\omega)$. For such polynomials the recursion coefficients γ_n are given by $\gamma_n = \pi(n+1)/\sqrt{4(n+1)^2 - 1}$ and the expansion functions $B_n(t)$ are given by $B_n(t) = \sqrt{2n+1}\, j_n(\pi t)$, where $j_n(x)$ is the spherical Bessel function of the first kind and of order n; thus, in particular, $B_0(t) = \mathrm{sinc}(t)$. The recursion formula for operators $\mathcal{K}^n$ mentioned above implies that the derivatives $B_k^{(p)}(t)$ of the expansion functions $B_k(t)$ can be obtained by the following recursion

$$B_k^{(p)}(t) = \gamma_{k-1} B_{k-1}^{(p-1)}(t) - \gamma_k B_{k+1}^{(p-1)}(t). \tag{1}$$

Importantly, not only app[f, M, u](t) accurately approximates $f(t)$, but also its derivatives $\mathrm{app}^{(p)}[f, M, u](t) = \sum_{n=0}^{M} \mathcal{K}^n[f](u) B_n^{(p)}(t-u) \approx f^{(p)}(t)$ over an interval around u whose length depends on M and p; for a detailed estimates of the error of such an approximation see [14, Theorem 4.2]. Further, [14, Fig. 8] demonstrates that the CA of a BL signal is far superior to the Taylor's approximation (TA) which, unlike the CA, employs noise sensitive standard derivatives and monomials $t^n/n!$ that are not BL functions. Moreover, unlike the TA which is unbounded, CAs are bounded on the whole set of reals and converge both uniformly and in the sense of the corresponding norm, see [14, Section 4]. CAs are also superior to truncations of the Shannon expansion which are well known to have poor local fidelity due to extremely slow rate of convergence; see [17].

3. Algorithms for timing extraction and signal reconstruction

We now present two algorithms based on CAs by generalizing the specific algorithms we presented in [6, case study II] (for the bandpass case).

(A) Timing extraction. To extract highly accurate ZCs of a BL signal $f(t)$ and its derivatives $f^{(p)}(t)$ up to order $p_{\max}$, we partition the signal into 50% overlapping (rectangular) windows W_k of length $L_k = 128$ Nyquist rate sampling intervals (NRIs) and use the samples $f(t_i)$ within each window W_k to obtain a (Tychonov L^2) regularized least square fit (LSF) approximation of $f(t)$, by using a "generic" approximation $\mathrm{app}(X_0, \dots, X_M) = \sum_{j=0}^{M} X_j B_j(t-u)$ with variables X_j in place of the unknown CDs and finding the values X_j^* of these variables which minimize the expression

$$\sigma(X_0, \dots, X_M) = \sum_{i=1}^{N_s} \left(\sum_{j=0}^{M} X_j B_j(t_i - u_k) - f(t_i) \right)^2 + \rho \sum_{j=0}^{M} X_j^2, \tag{2}$$

where ρ is a small regularization factor, N_s is the number of samples within each window and M is chosen as described in [6], i.e., $M = 14 + \lfloor 1.67L \rfloor$ where L is the length of the span of the samples in NRIs; thus, for $L = 128$ NRI we obtain $N = 227$. Such a LSF is solved by the standard Matlab LSF solver and is very tolerant with respect to the choice of the regularization constant ρ, which can be set to 10^{-14}, see [6, Fig 2]. To make such LSF more accurate and robust the signal is first twice upsampled; see [6, Fig. 1(b)]. Once the values $X_1^*, \dots, X_M^*$ of the variables $X_1, \dots, X_M$ are found, we obtain CA of the signal over W_k as $\mathrm{app}[f, M, u_k](t) = \sum_{j=0}^{M} X_j^* B_j(t-u_k)$, where u_k is the center of window W_k.

The length of the sub-window $w_k \subset W_k$ over which the derivatives of a CA of orders up to $p_{\max}$, obtained through such a LSF, are accurate approximations of the corresponding derivatives of the signal depends on the length of the window W_k, on the oversampling factor, and on $p_{\max}$, and is hard to estimate analytically. Thus, we generated synthetic signals as linear combinations of 1024 sinusoids with random amplitudes, frequencies and phases, normalized such signals to a unit RMS and then sampled them at twice the Nyquist rate on an interval of length of 128 NRIs. We then applied a LSF as described above and compared the values of the derivatives of orders up to $p_{\max} = 7$ of the resulting CA with the analytically computed derivatives of the original signal. The absolute values of the errors of approximation were averaged over 10^6 runs and are presented in Fig. 1. These plots show a remarkable feature of CAs; i.e., the higher order derivatives of the CA are accurate approximations of the corresponding derivatives of the signal over the central sub-interval of length more than half of the length of W_k, with an error smaller than 10^{-4}.

Thus, continuing with our algorithm, we approximate the ZCs of the signal $f(t)$ and of its derivatives $f^{(p)}(t)$ of orders up to 7, contained in the central sub-intervals $w_k = [u_k - 32, u_k + 32] \subset W_k$ of length $l_k = 64$ NRI, with the ZCs of app[f, M, u_k](t) and $\mathrm{app}^{(p)}$[f, M, u_k](t), respectively. This is accomplished by looking for ZCs which are within each interval $[t_m, t_{m+1}]$, where t_m are the Nyquist rate instants contained within w_k, in the following way. We use (1) to evaluate derivatives $B_j^{(p)}(t_m + 1/2)$ up to order 15 and then replace functions $B_j(t-u_k)$ in the formula for app[f, M, u_k](t) with their TAs of degree $N = 15$, centered at the midpoint $t_m + 1/2$ of the interval $[t_m, t_{m+1}]$. As shown in [13], for functions associated

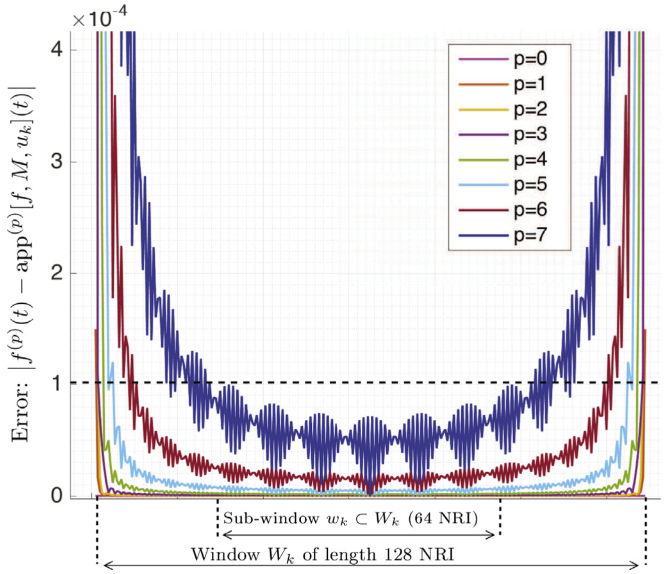

Fig. 1. The error between higher order derivatives of a CA and $f^{(p)}(t)$ for $0 \leq p \leq 7$ demonstrating high local fidelity of CAs.

with the Legendre polynomials $\|B_n(t)\| = 1$ for all n; thus, using the Lagrange formula for the error of the Taylor expansion of order N, we obtain that there exists $\xi \in [t_m, t_{m+1}]$ such that the error of approximation $\mathrm{err}_{\mathrm{T}}(t)$ over the unit interval $[t_m, t_{m+1}]$ is bounded by

$$
\begin{aligned}
|\mathrm{err}_{\mathrm{T}}(t)| &\leq \frac{|B_n^{(N+1)}(\xi)|\;|t-(t_m+1/2)|^{N+1}}{(N+1)!} \\
&\leq \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi}|\omega^{N+1}\widehat{B_n}(\omega)|\,\mathrm{d}\omega}{2^{N+1}(N+1)!} \\
&\leq \frac{1}{2^{N+1}(N+1)!}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\omega^{2(N+1)}\,\mathrm{d}\omega\right)^{1/2}\|B_n\| \\
&= \frac{1}{2^{N+1}(N+1)!}\,\frac{\pi^{N+1}}{\sqrt{2N+3}}.
\end{aligned}
$$

For $N = 15$ we obtain $\mathrm{err}_{\mathrm{T}}(t) < 1.15 \times 10^{-11}$; this ensures that, if we replace B_j in $\mathrm{app}[f, M, u_k](t) = \sum_{j=0}^{M} X_j^* B_j(t-u_k)$ by the corresponding TAs of order 15, we obtain a polynomial approximation $\mathrm{app}^*[f,\ M,\ u_k](t)$ highly accurate in the interval $[t_m, t_{m+1}]$. We now find the real roots x^* of the algebraic equation $\mathrm{app}^*[f, M, u_k](x) = 0$ which are within the interval $[t_m, t_{m+1})$; the corresponding ZCs are then of the form $t_m + 1/2 + x^*$. To obtain ZCs of $f^{(p)}(t)$, we use (1) to replace the derivatives of the basis functions by linear combinations of the basis functions which we again approximate by their TAs over each of the intervals $[t_m, t_{m+1}]$, again obtaining algebraic equation satisfied by the ZCs of $f^{(p)}(t)$. The above procedure can be performed in parallel for all windows W_k.

(B) Signal reconstruction. To reconstruct the signal from such timing information, we partition the ZCs of both the signal and its derivatives up to order $p_{\max}$ into partially overlapping processing windows W_n of length $L_w = 256$ NRIs, each containing N_n^p many ZCs of $f^{(p)}(t)$, for $0 \leq p \leq p_{max}$. Overlap between two consecutive processing windows is $3L_w/4 = 192$ NRIs. Let u_n be the center of the processing window W_n containing ZCs $\left\{t_i^p : 1 \leq i \leq N_n^p, 1 \leq p \leq p_{\max}\right\}$. We now obtain an approximation of the CDs of a waveform which has the required ZCs, by solving the following regularized LSF:

$$
\text{minimize } \sigma(X_1,\ldots,X_M) = \rho\sum_{j=0}^{M}X_j^2 + \sum_{i=1}^{N_n^0}\left(\sum_{j=0}^{M}X_jB_j(t_i^0-u_n)\right)^2 + \sum_{p=1}^{p_{\max}}\sum_{i=1}^{N_n^p}\left(\sum_{j=0}^{M}X_jB_j^{(p)}(t_i^p-u_n)\right)^2 + \left(\sum_{j=0}^{M}X_jB_j\left(\frac{t_1^0+t_2^0}{2}-u_n\right)-1\right)^2;
$$

the fourth term ensures that the value of the waveform at the midpoint between the first two ZCs is equal to 1 thus ensuring that the solution to the above LSF is non-trivial. Once the values $X_{1,n}^*, \ldots, X_{M,n}^*$ of variables $X_1, \ldots X_M$ are found for each window W_n, we obtain waveforms which are piece-wise CAs of the signal, modulo a scaling factor, centered at u_n, as $\mathrm{app}[f, M, u_n](t) = \sum_{j=0}^{M} X_{j,n}^* B_j(t-u_n)$. Thus obtained piece-wise CAs are then evaluated within the central sub intervals $w_n = [u_n - L_w/4, u_n + L_w/4) \subset W_n$ at the Nyquist rate to obtain sequences r_n, each containing 128 samples, with r_n and r_{n+1} overlapping over 64 sampling instants. These sequences are now recursively rescaled and concatenated, as described in detail in [6, case study II].

4. Experimental results

We now experimentally validate our CA based timing extraction and reconstruction algorithms with synthetic and speech/audio signals.

(A) Synthetic signals: Here, we demonstrate the requirement of ZCs of higher order derivatives for a robust reconstruction. For this example, the input signal is a twice oversampled linear combination of 1024 sinusoids with equally spaced frequencies in $[-\pi, \pi]$, randomly generated phases, and amplitudes tapered by a Gaussian with zero mean and variance of 0.138, producing 1024 samples in total. The variance has been chosen so that resulting signal has a total density of the ZCs of the signal and its first 6 derivatives (considered together) equal to 0.9969 ZCs per NRI and for such a signal, with $p_{\max} = 6$, our reconstruction fails, producing a signal with the same ZCs, but substantially different to the original signal. However, addition of ZCs of the derivative of order seven of such a signal brings the total density of all ZCs taken together to 1.3051 ZCs per NRI and our algorithm reconstructs the same signal with SER exceeding 47 dB. We claim that the failure of our algorithm to reconstruct the signal with $p_{\max} = 6$ is not due to a shortcoming of our method, but that such a reconstruction is intrinsically impossible, because the mean density of the data points provided by the first 6 derivatives is lower than the Landau mean density [18] of non uniform samples of a BL signal which is necessary to uniquely determine such a signal. Obviously, Landau's theorem is not directly applicable here, so we formulate the following compelling conjecture which would generalize the Landau theorem in a manner similar to how the Papoulis theorem [19] generalizes the Shannon sampling theorem: *Assume that we are given non uniform samples of the outputs of several linearly independent filters applied to an arbitrary bandlimited signal. Then such a signal is uniquely determined by these samples just in case their mean total density exceeds the Landau mean density of one sample per Nyquist rate interval.*

(B) Audio signals: The standard TIMIT database consists of 0–8 kHz signals (https://catalog.ldc.upenn.edu/LDC93S1) which are sampled at 16 kHz, consisting on average of about 64,000 samples. We twice upsampled such signals and normalized them to unity RMS. To handle the silent periods between words, a white noise dither signal of small RMS (about 0.08) is added prior to obtaining timing data and is removed after the reconstruction. Our

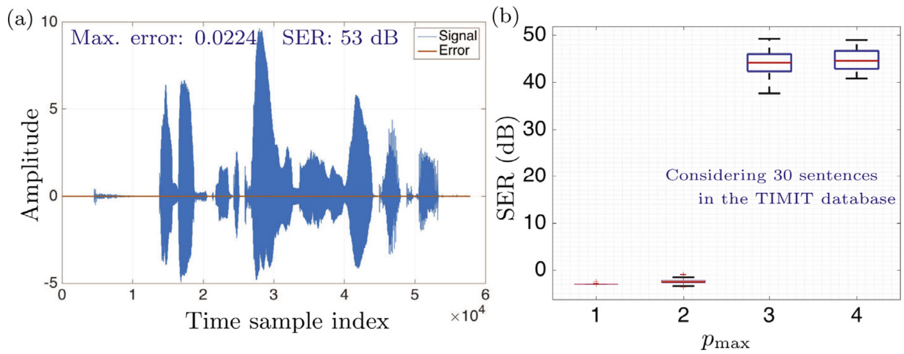

Fig. 2. (a) Reconstruction of an entire speech signal from ZCs of the signal and ZCs of its first 3 derivatives. (b) Reconstruction accuracy vs $p_{\max}$ considering 30 speech signals.

experiments[1] reveal that an accurate reconstruction would require $p_{\max} = 3$ (see Fig. 2(a)), while $p_{\max} = 2$ proved to be insufficient[2] for a robust reconstruction, due to insufficient total mean density of the ZCs of the signal and its first two derivatives which was often barely above the Landau density, as well as in occasional large gaps between consecutive ZCs (> 4 NRIs). Our extensive simulations with more than a hundred test cases verify that for robust reconstruction of long speech signals having silent periods, our CA based reconstruction algorithm requires a mean density of ZCs of about 1.5 ZCs per NRI. Fig. 2(b) shows the reconstruction accuracy vs. $p_{\max}$ across 30 speech waveforms from the TIMIT database. The same observations apply to various sound files from the Google Audioset.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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