

The small-is-very-small principle

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The central result of this paper is *the small-is-very-small principle* for restricted sequential theories. The principle says roughly that whenever the given theory shows that a definable property has a small witness, i.e., a witness in a sufficiently small definable cut, then it shows that the property has a very small witness: i.e., a witness below a given standard number. Which cuts are sufficiently small will depend on the complexity of the formula defining the property. We draw various consequences from the central result. E.g., roughly speaking, (i) every restricted, recursively enumerable sequential theory has a finitely axiomatized extension that is conservative with respect to formulas of complexity $\leq n$; (ii) every sequential model has, for any n , an extension that is elementary for formulas of complexity $\leq n$, in which the intersection of all definable cuts is the natural numbers; (iii) we have reflection for Σ_2^0 -sentences with sufficiently small witness in any consistent restricted theory U ; (iv) suppose U is recursively enumerable and sequential. Suppose further that every recursively enumerable and sequential V that locally interprets U , globally interprets U . Then, U is mutually globally interpretable with a finitely axiomatized sequential theory. The paper contains some careful groundwork developing partial satisfaction predicates in sequential theories for the complexity measure *depth of quantifier alternations*.

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1 Introduction

Some proofs are like hollyhocks. If you are nice to them they give different flowers every year. This paper is about one such proof. We discovered it when searching for alternative, more syntactic, proofs of certain theorems by Friedman (discussed in [17]) and Krajíček [9]. The relevant theorem due to Friedman tells us that, if a finitely axiomatized, sequential, consistent theory A interprets a recursively enumerable theory U , then A interprets U faithfully. (Cf. § 2.4 for the definition of a sequential theory.) Krajíček's theorem tells us that a finitely axiomatized, sequential, consistent theory cannot prove its own inconsistency on arbitrarily small cuts. There is a close connection between these two theorems.

The quest for a syntactic proof succeeded and the results were reported in [20]. One advantage of having such a syntactic proof is clearly that it can be 'internalized' in the theories we study. We returned to the argument in a later paper [21], which contains improvements and, above all, a better theoretical framework. In our papers [26, 28], the argument is employed to prove results about provability logic and about degrees of interpretability, respectively.

The syntactic argument in question is a Rosser-style argument or, more specifically, a Friedman-Goldfarb-Harrington-style argument. It has all the mystery of a Rosser argument: even if every step is completely clear, it still retains a feeling of magic trickery.

1.1 Contents of the paper

In the present paper, we shall obtain more information from the Friedman-Goldfarb-Harrington-style argument discussed above. In previous work, the basic conclusion of the argument is that, given a consistent, finitely axiomatized, sequential theory A , there is an interpretation M of the basic arithmetic S_2^1 in A that is Σ_1^0 -sound. In the present paper, we extend our scope from *finitely axiomatized sequential theories* to *restricted sequential theories*—this means that we consider theories with axioms of complexity below a fixed finite bound. Secondly, we replace the Σ_1^0 -soundness by the more general small-is-very-small principle (SIVS).

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The improved results have a number of consequences. In § 4, we show that, for any n , every consistent, restricted, recursively enumerable, sequential theory has a finitely axiomatized extension that is conservative with respect to formulas of complexity $\leq n$. In § 5, we show that, for any n , every sequential model has an elementary extension with respect to formulas of complexity $\leq n$, such that the intersection of all definable cuts consists of the standard numbers. In § 6, we indicate how results concerning Σ_2^0 -soundness can be derived from our main theorem. Finally, in § 7, we prove a result in the structure of the combined degrees of local and global interpretability of recursively enumerable, sequential theories. We show that *if* a local degree contains a minimal global degree, *then* this global degree contains a finitely axiomatized theory. Thus, finite axiomatizability has a natural characterization, modulo global interpretability, in terms of the double degree structure.¹

§ 2 provides the necessary elementary facts. Unlike similar sections in other papers of mine, this section also contains something new. In [20], we provided groundwork for the development of partial satisfaction predicates for the complexity measure *depth of quantifier alternations*. Our present § 2.3 gives a much better treatment of the complexity measure than the one in [20]. § 2.5 develops the facts about sequential theories and partial satisfaction predicates in greater detail than previously available in the literature. Moreover, we provide careful estimates of the complexities yielded by the various constructions. On the one hand, these subsections contain ‘what we already knew’, on the other hand, as we found, even if you already know how things go, it can still be quite a puzzle to get all nuts and bolts at the precise places where they have to go. Of course, the present treatment is still not fully explicit, but we are further on the road.

§ 3 contains the central result of the paper. As the reader will see, after all is said and done, the central argument is amazingly simple. The work is in creating the setting in which the result can be comfortably stated.

2 Basic notions and facts

In the present section, we provide the basics needed for the rest of the paper. As pointed out in the introduction the development of partial satisfaction predicates is done in more detail here than elsewhere. For this reason this section may also turn out to be useful for subsequent work. Of course, the reader who wants to get on quickly to more exciting stuff could briefly look over the relevant subsections and, if needed, return to them later.

2.1 Theories

In this paper we shall study theories with finite signature. In most of our papers, theories are intensional objects equipped with a formula representing the axiom set. In the present paper, to the contrary, a theory is just a set of sentences of the given signature closed under deduction. This is because most of the results in the paper are extensional.

Also we do not have any constraints on the complexity of the axiom set of the theory. If a theory is finitely axiomatizable, *par abus de langage*, we use the variables like A and B for it, making the letters do double work: they both stand for the theory and for a single axiom. When we diverge from our general format this will always be explicitly mentioned. In the paper, we shall meet many concrete theories, to wit AS , PA^- , S_2^1 , EA , PRA , PA . We refer the reader to the textbooks [6, 8] for an introduction to these theories.

2.2 Translations and interpretations

We present the notion of *m-dimensional interpretation without parameters*. There are two extensions of this notion: we can consider piecewise interpretations and we can add parameters. We shall give some details on parameters in Appendix A. We shall not describe piecewise interpretations here.

Consider two signatures Σ and Θ . An m -dimensional translation $\tau : \Sigma \rightarrow \Theta$ is a quadruple $\langle \Sigma, \delta, \mathcal{F}, \Theta \rangle$, where $\delta(v_0, \dots, v_{m-1})$ is a Θ -formula and where, for any n -ary predicate P of Σ , $\mathcal{F}(P)$ is a Θ -formula $A(\vec{v}_0, \dots, \vec{v}_{n-1})$

¹ The author presented this result in a lecture for the Moscow Symposium on Logic, Algebra and Computation in 2006. However, he was not able to write down the proof, since he lacked the necessary groundwork on partial satisfaction. This groundwork is provided in § 2 of the present paper.

in the language of signature Θ , where $\vec{v}_i = v_{i0}, \dots, v_{i(m-1)}$. Both in the case of δ and A all free variables are among the variables shown. Moreover, if $i \neq j$ or $k \neq \ell$, then v_{ik} is syntactically different from $v_{j\ell}$. We shall often write P_τ for $\mathcal{F}(P)$.

We demand that we have $\vdash \mathcal{F}(P)(\vec{v}_0, \dots, \vec{v}_{n-1}) \rightarrow \bigwedge_{i < n} \delta(\vec{v}_i)$. Here \vdash is provability in predicate logic. This demand is inessential, but it is convenient to have.

We define B^τ as follows:

1. $(P(x_0, \dots, x_{n-1}))^\tau := \mathcal{F}(P)(\vec{x}_0, \dots, \vec{x}_{n-1})$.
2. $(\cdot)^\tau$ commutes with the propositional connectives.
3. $(\forall x A)^\tau := \forall \vec{x} (\delta(\vec{x}) \rightarrow A^\tau)$.
4. $(\exists x A)^\tau := \exists \vec{x} (\delta(\vec{x}) \wedge A^\tau)$.

There are two worries about this definition. First, what variables \vec{x}_i on the side of the translation A^τ correspond with x_i in the original formula A ? The second worry is that substitution of variables in δ and $\mathcal{F}(P)$ may cause variable-clashes. These worries are never important in practice: we choose ‘suitable’ sequences \vec{x} to correspond to variables x , and we avoid clashes by α -conversion. However, if we want to give precise definitions of translations and, e.g., of composition of translations, these problems come into play. The problems are clearly solvable in a systematic way, but this endeavor is beyond the scope of this paper.

We allow the identity predicate to be translated to a formula that is not identity. A translation τ is *direct*, if it is one-dimensional and if $\delta_\tau(x) := (x = x)$ and if it translates identity to identity. There are several important operations on translations.

1. We let id_Σ be the identity translation. We take $\delta_{\text{id}_\Sigma}(v) := v = v$ and $\mathcal{F}(P) := P(\vec{v})$.
2. We can compose translations. Suppose $\tau : \Sigma \rightarrow \Theta$ and $\nu : \Theta \rightarrow \Lambda$. Then $\nu \circ \tau$ or $\tau \nu$ is a translation from Σ to Λ . We define:

$$\delta_{\tau\nu}(\vec{v}_0, \dots, \vec{v}_{m_\tau-1}) := \bigwedge_{i < m_\tau} \delta_\nu(\vec{v}_i) \wedge (\delta_\tau(v_0, \dots, v_{m_\tau-1}))^\nu.$$

$$P_{\tau\nu}(\vec{v}_{0,0}, \dots, \vec{v}_{0,m_\tau-1}, \dots, \vec{v}_{n-1,0}, \dots, \vec{v}_{n-1,m_\tau-1}) := \bigwedge_{i < n, j < m_\tau} \delta_\nu(\vec{v}_{i,j}) \wedge (P(v_0, \dots, v_{n-1}))^\tau{}^\nu.$$

3. Let $\tau, \nu : \Sigma \rightarrow \Theta$ and let A be a sentence of signature Θ . We define the disjunctive translation $\sigma := \tau \langle A \rangle \nu : \Sigma \rightarrow \Theta$ as follows. We take $m_\sigma := \max(m_\tau, m_\nu)$. We write $\vec{v} \upharpoonright n$, for the restriction of \vec{v} to the first n variables, where $n \leq \text{length}(\vec{v})$.

$$\delta_\sigma(\vec{v}) := (A \wedge \delta_\tau(\vec{v} \upharpoonright m_\tau)) \vee (\neg A \wedge \delta_\nu(\vec{v} \upharpoonright m_\nu)).$$

$$P_\sigma(\vec{v}_0, \dots, \vec{v}_{n-1}) := (A \wedge P_\tau(\vec{v}_0 \upharpoonright m_\tau, \dots, \vec{v}_{n-1} \upharpoonright m_\tau)) \vee (\neg A \wedge P_\nu(\vec{v}_0 \upharpoonright m_\nu, \dots, \vec{v}_{n-1} \upharpoonright m_\nu))$$

Note that in the definition of $\tau \langle A \rangle \nu$ we used a padding mechanism. In case, e.g., $m_\tau < m_\nu$, the variables $v_{m_\tau}, \dots, v_{m_\nu-1}$ are used ‘vacuously’ when we have A . If we had piecewise interpretations, where domains are built up from pieces with possibly different dimensions, we could avoid padding by building the domain directly of disjoint pieces with different dimensions.

A translation relates signatures; an interpretation relates theories. An interpretation $K : U \rightarrow V$ is a triple $\langle U, \tau, V \rangle$, where U and V are theories and $\tau : \Sigma_U \rightarrow \Sigma_V$. We demand: for all theorems A of U , we have $V \vdash A^\tau$. Here are some further definitions.

1. $\text{ID}_U : U \rightarrow U$ is the interpretation $\langle U, \text{id}_{\Sigma_U}, U \rangle$.
2. Suppose $K : U \rightarrow V$ and $M : V \rightarrow W$. Then, $KM := M \circ K : U \rightarrow W$ is $\langle U, \tau_M \circ \tau_K, W \rangle$.
3. Suppose $K : U \rightarrow (V + A)$ and $M : U \rightarrow (V + \neg A)$. Then $K \langle A \rangle M : U \rightarrow V$ is the interpretation $\langle U, \tau_K \langle A \rangle \tau_M, V \rangle$. In an appropriate category $K \langle A \rangle M$ is a special case of a product.

A translation τ maps a model \mathcal{M} to an internal model $\tilde{\tau}(\mathcal{M})$ provided that $\mathcal{M} \models \exists \vec{x} \delta_\tau(\vec{x})$. Thus, an interpretation $K : U \rightarrow V$ gives us a mapping \tilde{K} from $\text{MOD}(V)$, the class of models of V , to $\text{MOD}(U)$, the class of models of U . If we build a category of theories and interpretations, usually MOD with $\text{MOD}(K) := \tilde{K}$ will be a contravariant functor.

We use $U \xrightarrow{K} V$ or $K : U \triangleleft V$ or $K : V \triangleright U$ as alternative notations for $K : U \rightarrow V$. The alternative notations \triangleleft and \triangleright are used in a context where we are interested in interpretability as a preorder or as a provability analogue. We write $U \triangleleft V$ and $V \triangleright U$, for “there is an interpretation $K : U \triangleleft V$ ”. We use $U \equiv V$, for $U \triangleleft V$ and $U \triangleright V$. The arrow notations are mostly used in a context where we are interested in a category of interpretations, but also simply when they improve readability. We write $U \triangleleft_{\text{loc}} V$ or $V \triangleright_{\text{loc}} U$ for the statement “for all finite subtheories U_0 of U , $U_0 \triangleleft V$ ”. We pronounce this as “ U is locally interpretable in V or V locally interprets U ”. We use \equiv_{loc} for the induced equivalence relation of $\triangleleft_{\text{loc}}$.

2.3 Complexity and restricted provability

Restricted provability plays an important role in the study of interpretability between sequential theories. An n -proof is a proof from axioms with Gödel number smaller or equal than n only involving formulas of complexity smaller or equal than n . To work conveniently with this notion, a good complexity measure is needed. Such a measure should satisfy three conditions.

- (i) Eliminating terms in favor of a relational formulation should raise the complexity only by a fixed standard number.
- (ii) Translation of a formula via the translation τ should raise the complexity of the formula by a fixed standard number depending only on τ .
- (iii) The tower of exponents involved in cut-elimination should be of height linear in the complexity of the formulas involved in the proof.

Such a good measure of complexity together with a verification of desideratum (iii) —a form of nesting degree of quantifier alternations— is supplied in the work of Gerhardy [4, 5]. A slightly different measure is provided by Buss in [2]. Buss also proves that (iii) is fulfilled for his measure. In fact, Buss proves a sharper result. He shows that the bound is $d + O(1)$ for d alternations. In the present paper, we shall follow Buss’s treatment.

We work over a signature Θ . The formula-classes we define are officially called $\Sigma_n^*(\Theta)$ and $\Pi_n^*(\Theta)$. However, we shall suppress the Θ when it is clear from the context. Let AT be the class of atomic formulas for Θ , extended with \top and \perp . We define:

$$\begin{aligned}\Sigma_0^* &:= \Pi_0^* := \emptyset \\ \Sigma_{n+1}^* &::= \text{AT} \mid \neg \Pi_{n+1}^* \mid (\Sigma_{n+1}^* \wedge \Sigma_{n+1}^*) \mid (\Sigma_{n+1}^* \vee \Sigma_{n+1}^*) \mid (\Pi_{n+1}^* \rightarrow \Sigma_{n+1}^*) \mid \exists v \Sigma_{n+1}^* \mid \forall v \Pi_n^* \\ \Pi_{n+1}^* &::= \text{AT} \mid \neg \Sigma_{n+1}^* \mid (\Pi_{n+1}^* \wedge \Pi_{n+1}^*) \mid (\Pi_{n+1}^* \vee \Pi_{n+1}^*) \mid (\Sigma_{n+1}^* \rightarrow \Pi_{n+1}^*) \mid \forall v \Pi_{n+1}^* \mid \exists v \Sigma_n^*\end{aligned}$$

Buss uses Σ_{n+1} and Π_{n+1} where we use Σ_{n+1}^* and Π_{n+1}^* . We employ the asterisk to avoid confusion with the usual complexity classes in the arithmetical hierarchy where bounded quantifiers also play a role. Secondly, we modified Buss’s inductive definition a bit in order to get unique generation histories. E.g., Buss adds Π_n^* to Σ_{n+1}^* instead of $\forall v \Pi_n^*$.² In addition our Σ_0^* and Π_0^* are empty, where Buss’s corresponding classes consist of the quantifier-free formulas. Cf. Figure 1 for an example depicting the parse-tree of a formula in Σ_4^* .

The extensional equivalence, for $n > 0$ of our definition to Buss’s is immediate from the following:

Fact 2.1 *The quantifier-free formulas are precisely $\Sigma_1^* \cap \Pi_1^*$. Moreover, we have $\Sigma_n^* \cup \Pi_n^* \subseteq \Sigma_{n+1}^* \cap \Pi_{n+1}^*$.*

The proof is by five simple inductions. We define:

$$\Delta_{n+1}^* ::= \text{AT} \mid \neg \Delta_{n+1}^* \mid (\Delta_{n+1}^* \wedge \Delta_{n+1}^*) \mid (\Delta_{n+1}^* \vee \Delta_{n+1}^*) \mid (\Delta_{n+1}^* \rightarrow \Delta_{n+1}^*) \mid \exists v \Sigma_n^* \mid \forall v \Pi_n^*.$$

We have the following theorem.

Theorem 2.2 $\Delta_{n+1}^* = \Sigma_{n+1}^* \cap \Pi_{n+1}^*$.

² We do this so that every formula enters Σ_{n+1}^* in at most one way. Otherwise, e.g., the same atomic formula could enter both via the atomic clause and the Π_n^* clause.

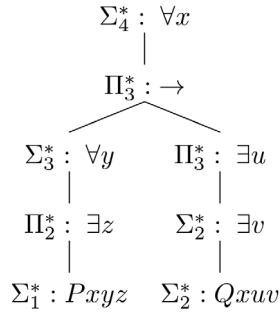


Fig. 1 The parse-tree of $\forall x (\forall y \exists z Pxyz \rightarrow \exists u \exists v Qxuv)$ as an element of Σ_4^* .

Proof. That $\Delta_{n+1}^* \subseteq \Sigma_{n+1}^* \cap \Pi_{n+1}^*$ is an easy induction based on Fact 2.1. We prove the converse by ordinary induction on formulas. The atomic case and the propositional cases are immediate. Suppose A in $\Sigma_{n+1}^* \cap \Pi_{n+1}^*$ has the form $\exists v B$. Then B must be in Σ_n^* . It follows that A is in Σ_n^* and, thus, that A is in Δ_{n+1}^* . \square

We want a complexity measure $\rho(A)$ such that $\rho(A)$ is the smallest n such that A is in Σ_n^* . This measure is very close to the measure that was employed in [20]. We recursively define this measure by taking $\rho := \rho_\exists$, where ρ_\exists is defined as follows:

$$\begin{aligned}
 \rho_\exists(A) &:= \rho_\forall(A) = 1, \text{ if } A \text{ is atomic} \\
 \rho_\exists(\neg B) &:= \rho_\forall(B) & \rho_\forall(\neg B) &:= \rho_\exists(B) \\
 \rho_\exists(B \wedge C) &:= \max(\rho_\exists(B), \rho_\exists(C)) & \rho_\forall(B \wedge C) &:= \max(\rho_\forall(B), \rho_\forall(C)) \\
 \rho_\exists(B \vee C) &:= \max(\rho_\exists(B), \rho_\exists(C)) & \rho_\forall(B \vee C) &:= \max(\rho_\forall(B), \rho_\forall(C)) \\
 \rho_\exists(B \rightarrow C) &:= \max(\rho_\forall(B), \rho_\exists(C)) & \rho_\forall(B \rightarrow C) &:= \max(\rho_\exists(B), \rho_\forall(C)) \\
 \rho_\exists(\exists v B) &:= \rho_\exists(B) & \rho_\forall(\exists v B) &:= \rho_\exists(B) + 1 \\
 \rho_\exists(\forall v B) &:= \rho_\forall(B) + 1 & \rho_\forall(\forall v B) &:= \rho_\forall(B) \\
 \rho(A) &:= \rho_\exists(A) & \rho_0(A) &:= \max(\rho_\exists(A), \rho_\forall(A)).
 \end{aligned}$$

We verify the basic facts about ρ .

Theorem 2.3 $\rho_\forall(A) \leq \rho_\exists(A) + 1$ and $\rho_\exists(A) \leq \rho_\forall(A) + 1$.

Proof. The proof is by induction on A . We treat the case that $A = \exists v B$. We have that $\rho_\forall(\exists v B) = \rho_\exists(B) + 1 = \rho_\exists(\exists v B) + 1$. Note that this does not use the induction hypothesis. \square

Theorem 2.4 $\Sigma_n^* = \{A \mid \rho_\exists(A) \leq n\}$ and $\Pi_n^* = \{A \mid \rho_\forall(A) \leq n\}$. It follows that, for $n > 0$, we have $\Delta_n^* = \{A \mid \rho_0(A) \leq n\}$.

Proof. We prove, by induction on n , that: $A \in \Sigma_n^*$ if and only if $\rho_\exists(A) \leq n$ and $A \in \Pi_n^*$ if and only if $\rho_\forall(A) \leq n$.

The case of 0 is clear. We prove by induction on the definition of Σ_{n+1}^* , that $A \in \Sigma_{n+1}^*$ if and only if $\rho_\exists(A) \leq n + 1$. The atomic case, the propositional cases and the existential case are clear. Suppose $A = \forall v B$. If $A \in \Sigma_{n+1}^*$, then B is in Π_n^* . By the induction hypothesis, $\rho_\forall(B) \leq n$, so $\rho_\exists(A) \leq n + 1$. If $\rho_\exists(A) \leq n + 1$, then $\rho_\forall(B) \leq n$. Hence, by the induction hypothesis, $B \in \Pi_n^*$, so $A \in \Sigma_{n+1}^*$. The case of Π_{n+1}^* is similar. \square

Let $\tau : \Sigma \rightarrow \Theta$ be a translation. We define $\rho^*(\tau)$ to be the maximum of $\rho_0(\delta_\tau)$ and the $\rho_0(P_\tau)$, for P in Σ . If K is an interpretation, then $\rho^*(K) := \rho^*(\tau_K)$.

Theorem 2.5 Let $\tau : \Sigma \rightarrow \Theta$. We have $\rho_{\exists}(A^{\tau}) \leq \rho_{\exists}(A) + \rho^*(\tau)$ and $\rho_{\forall}(A^{\tau}) \leq \rho_{\forall}(A) + \rho^*(\tau)$.

Proof. The proof is by induction on A . The case of the atoms is trivial.

We treat the case of implication and ρ_{\exists} . Suppose A is $B \rightarrow C$. We have

$$\begin{aligned} \rho_{\exists}(A^{\tau}) &= \max(\rho_{\forall}(B^{\tau}), \rho_{\exists}(C^{\tau})) \\ &\leq \max(\rho_{\forall}(B) + \rho^*(\tau), \rho_{\exists}(C) + \rho^*(\tau)) \\ &= \max(\rho_{\forall}(B), \rho_{\exists}(C)) + \rho^*(\tau) \\ &= \rho_{\exists}(A) + \rho^*(\tau). \end{aligned}$$

The other cases concerning the propositional connectives are similar.

We treat the case for universal quantification and ρ_{\exists} . Suppose A is $\forall v B$. We have

$$\begin{aligned} \rho_{\exists}(A^{\tau}) &= \rho_{\exists}(\forall \vec{v} (\delta_{\tau}(\vec{v}) \rightarrow B^{\tau})) \\ &= \rho_{\forall}(\delta_{\tau}(\vec{v}) \rightarrow B^{\tau}) + 1 \\ &= \max(\rho_{\exists}(\delta_{\tau}(\vec{v})), \rho_{\forall}(B^{\tau})) + 1 \\ &\leq \rho_{\forall}(B) + \rho^*(\tau) + 1 \\ &= \rho_{\exists}(A) + \rho^*(\tau). \end{aligned}$$

The remaining cases for the quantifiers are similar or easier. □

2.4 Sequential theories

The notion of sequentiality is due to Pudlák (cf., e.g., [6, 11, 13, 14]). To define sequentiality we use the auxiliary theory AS^+ (Adjunctive Set Theory with extras). The signature \mathfrak{A} of AS^+ consists of unary predicate symbols N and Z , binary predicate symbols \in , E , \leq , $<$, S , ternary predicate symbols A and M . Here E stands for an equivalence relation representing identity of N -numbers; Z gives us zero in N modulo E ; S gives us successor on N modulo E ; likewise for \leq , $<$, A (addition) and M (multiplication).

AS^+1 We have a set of axioms that provide a relative interpretation \mathcal{N} of S^1_2 in AS^+ , where N represents the natural numbers, E represents numerical identity, Z stands for zero modulo E , A stands for addition modulo E , and M stands for multiplication modulo E .

$AS^+2 \vdash \exists x \forall y y \notin x$,

$AS^+3 \vdash \forall x \exists y \forall z (z \in y \leftrightarrow z = x)$,

$AS^+4 \vdash \forall x, y \exists z \forall u (u \in z \leftrightarrow (u \in x \vee u \in y))$,

$AS^+5 \vdash \forall x, y \exists z \forall u (u \in z \leftrightarrow (u \in x \wedge u \in y))$,

$AS^+6 \vdash \forall x, y \exists z \forall u (u \in z \leftrightarrow (u \in x \wedge u \notin y))$.

An important point is that we do not demand extensionality for our sets. A many-sorted version of AS^+ would be somewhat more natural. We refrain from developing it in this way here to avoid the additional burden of working with interpretations between many-sorted theories. (Cf. [25] for information on the many-sorted case.)

A theory is *sequential* if and only if it interprets the theory AS^+ via a direct interpretation \mathcal{S} . We call such an \mathcal{S} a *sequence scheme*. It is possible to work with an even simpler base theory. The theory AS is given by the following axioms.

$AS1 \vdash \exists y \forall x x \notin y$,

$AS2 \vdash \forall x \forall y \exists z \forall u (u \in z \leftrightarrow (u \in y \vee u = x))$.

One can show that AS is mutually directly interpretable with AS^+ . For details concerning the bootstrap, cf., e.g., the textbook [6] and also [11, 22, 24, 25].

We could work in a somewhat richer class of theories, the *polysequential* theories [25]. Let us say that an interpretation is *m-direct*, if it is *m-dimensional*, if its domain consists of all *m*-tuples of the original domain, and if identity is interpreted as component-wise identity. A theory *U* is *m-sequential*, if there is an *m-direct* interpretation of AS in *U*. A theory is polysequential, if it is *m-sequential* for some $m \geq 1$. Note that if we want the AS⁺ format, the interpretation of the natural numbers should also be chosen to be *m-dimensional* for the given *m*. The development given in the present paper also works with minor adaptations in the polysequential case.

It is known that there are polysequential theories that are not sequential. However, we only have an artificial example. Every polysequential theory is polysequential without parameters, where a sequential theory may essentially need an interpretation with parameters to witness its sequentiality. (One raises the dimension to ‘eat up’ the parameters.) Polysequential theories are closed under bi-interpretability. Moreover, every polysequential theory is bi-interpretable with a sequential one.

2.5 Satisfaction & reflection

In this subsection, we develop partial satisfaction predicates for sequential theories with some care. We prove the corresponding partial reflection principles. This subsection is rather long because it provides many details. The impatient reader could choose to proceed to Theorem 2.15, since that is the main result of the subsection that we shall use in the rest of the paper.

Consider any signature Θ . We extend the signature \mathfrak{A} of AS⁺ in a disjoint way with Θ to, say, $\mathfrak{A} + \Theta$. Call the resulting theory (without any new axioms) AS⁺(Θ). We work towards the definition of partial satisfaction predicates. We provide a series of definitions illustrative of what we need to get off the ground.

$$\begin{aligned} \text{pair}(u, v, w) &:\Leftrightarrow \exists a \exists b (\forall c (c \in w \leftrightarrow (c = a \vee c = b)) \wedge \\ &\quad \forall d (d \in a \leftrightarrow d = u) \wedge \forall e (e \in b \leftrightarrow (e = u \vee e = v))). \end{aligned}$$

We can easily show that for all *u* and *v* there is a *w* such that pair(*u*, *v*, *w*) and, whenever pair(*u*, *v*, *w*) and pair(*u*′, *v*′, *w*), then *u* = *u*′ and *v* = *v*′. Note that there may be several *w* such that pair(*u*, *v*, *w*).

$$\begin{aligned} \text{Pair}(w) &:\Leftrightarrow \exists u \exists v \text{pair}(u, v, w). \\ \pi_0(w, u) &:\Leftrightarrow \exists v \text{pair}(u, v, w). \\ \pi_1(w, v) &:\Leftrightarrow \exists u \text{pair}(u, v, w). \\ \text{fun}(f) &:\Leftrightarrow \forall w \in f (\text{Pair}(w) \wedge \forall w' \in f \forall u (\pi_0(w, u) \wedge \pi_0(w', u) \rightarrow w = w')). \end{aligned}$$

Note that we adapt the notion of function to our non-extensional pairing. We demand that there is at most one witnessing pair for a given argument. This choice makes resetting a function on an argument where it is defined a simple operation: we subtract one pair and we add one.

$$\begin{aligned} \text{dom}(f, x) &:\Leftrightarrow \text{fun}(f) \wedge \exists w \in f \pi_0(w, x). \\ f(u) \approx v &:\Leftrightarrow \text{fun}(f) \wedge \exists w \in f \text{pair}(u, v, w). \\ \text{nfun}(\alpha) &:\Leftrightarrow \forall w \in \alpha (\text{Pair}(w) \wedge \forall u (\pi_0(w, u) \rightarrow \mathbf{N}(u)) \wedge \\ &\quad \forall w' \in \alpha \forall u \forall u' ((\pi_0(w, u) \wedge \pi_0(w', u') \wedge u \mathbf{E} u') \rightarrow w = w')). \end{aligned}$$

We note that nfun(α) implies fun(α).

$$\text{ndom}(\alpha, a) :\Leftrightarrow \text{nfun}(\alpha) \wedge \exists w \in \alpha \exists b (a \mathbf{E} b \wedge \pi_0(w, b)).$$

We fix a parameter x^* .

$$\alpha[a] \approx v :\Leftrightarrow \text{nfun}(\alpha) \wedge (\exists w \in \alpha \exists b (a \mathbf{E} b \wedge \text{pair}(b, v, w)) \vee (\neg \text{ndom}(\alpha, a) \wedge v = x^*)).$$

So outside of ndom we set the value of α to a default value. In this way we made it a total function on the natural numbers.³

$$\alpha \llbracket a \rrbracket \beta := \forall b (\text{ndom}(\beta, b) \leftrightarrow (\text{ndom}(\alpha, b) \vee b \in a)) \wedge \\ \forall b \forall x ((N(b) \wedge \neg a \in b) \rightarrow (\alpha[b] \approx x \leftrightarrow \beta[b] \approx x)).$$

$$\alpha[a : y] \approx \beta := \alpha \llbracket a \rrbracket \beta \wedge \beta[a] \approx y.$$

We shall use α, β to range over functions with numerical domains (elements of nfun). We note that $\text{AS}^+(\Theta)$ proves that the ‘operation’ $\alpha \mapsto \alpha[a : y]$ is total and that any output sequences are extensionally the same.

We employ a usual efficient coding of syntax in the interpretation \mathcal{N} of S_2^1 . We have shown above how to formulate things in order to cope with the fact that each number, set, pair, function and sequence can have several representatives. The definition of satisfaction would be completely unreadable if we tried to adhere to this high standard. Hence, we shall work more informally pretending, e.g., that each number has just one representative. An assignment will simply be a numerical function where we restrict our attention to the codes of variables in the domain.

A *sat-sequence* is a triple of the form $\langle i, \alpha, A \rangle$, where: i is $+$ or $-$ (coded as, say, 1 and 0), α is an assignment and A is a formula. An *a-sat-sequence* is a sat-sequence $\langle i, \alpha, A \rangle$, where A is Σ_a^* , if $i = +$ and A is Π_a^* , if $i = -$. We call the virtual class⁴ of all *a-sat* sequences \mathcal{K}_a . A set X is *good* if, for all numbers b the virtual class $X \cap \mathcal{K}_b$ exists as a set. We have the following lemma.

Lemma 2.6 ($\text{AS}^+(\Theta)$). *The good sets are closed under the empty sets, singletons, union, intersection and subtraction and are downwards closed with respect to the subset ordering.*

Proof. We leave the easy proof to the reader. □

We count two sat-sequences $\langle i, \alpha, A \rangle$ and $\langle j, \beta, B \rangle$ as *extensionally equal* if (i) i and j are E-equal, (ii) α and β have the same functional behaviour on the natural numbers N , and (iii) A and B are E-equal. We say that a sequence σ is *of the form* $\langle i, \alpha, A \rangle$ if it is extensionally equal to a sequence τ with $\tau_0 = i$, $\tau_1 = \alpha$ and $\tau_2 = A$.

We define $n + 1$ -adequacy and sat_n by external recursion on n . We define

$$\text{sat}_0(+, \alpha, A) := \perp \text{ and } \text{sat}_0(-, \alpha, A) := \top.$$

We define $\text{sat}_{n+1}(i, \alpha, A)$ if and only if, for some $n + 1$ -adequate set X , we have $\langle i, \alpha, A \rangle \in X$. A set X is $n + 1$ -adequate, if it is good, if its elements are sat-sequences, and if it satisfies the following clauses:

- (a) If a sequence of the form $\langle +, \beta, P(v_0, v_1) \rangle$ is in X , then $P(\beta(v_0), \beta(v_1))$.⁵ Similarly, for other atomic formulas including \top and \perp .
- (b) If a sequence of the form $\langle -, \beta, P(v_0, v_1) \rangle$ is in X , then $\neg P(\beta(v_0), \beta(v_1))$. Similarly, for other atomic formulas including \top and \perp .
- (c) If a sequence of the form $\langle +, \beta, \neg B \rangle$ is in X , then a sequence of the form $\langle -, \beta, B \rangle$ is in X .
- (d) If a sequence of the form $\langle -, \beta, \neg B \rangle$ is in X , then a sequence of the form $\langle +, \beta, B \rangle$ is in X .
- (e) If a sequence of the form $\langle +, \beta, (B \wedge C) \rangle$ is in X , then sequences of the form $\langle +, \beta, B \rangle$ and $\langle +, \beta, C \rangle$ are in X .
- (f) If a sequence of the form $\langle -, \beta, (B \wedge C) \rangle$ is in X , then a sequence of the form $\langle -, \beta, B \rangle$ is in X or a sequence of the form $\langle -, \beta, C \rangle$ is in X .
- (g) If a sequence of the form $\langle +, \beta, (B \vee C) \rangle$ is in X , then a sequence of the form $\langle +, \beta, B \rangle$ is in X or a sequence of the form $\langle +, \beta, C \rangle$ is in X .

³ The need for a parameter is regrettable but in a sequential theory there need not be definable elements. Of course, we could set the value at a outside the ndom of α to α , but that would mean that when we reset our function we would change all the values outside the ndom too. One way to eliminate the parameter would be to make the default value an extra part of the data for the function. Then, a reset would keep the default value intact. The option of working with partial functions is certainly feasible. However, e.g., clause (e) of the definition of adequate set would be more complicated.

⁴ A *virtual class* in a theory is a class given by a formula that need not be represented by an object of the domain. E.g., V and L and On in the context of ZF are virtual classes.

⁵ Our variables are coded as numbers. Conceivably not all numbers code variables. The values of β on non-variables are simply *don't care*.

- (h) If a sequence of the form $\langle -, \beta, (B \vee C) \rangle$ is in X , then sequences of the form $\langle -, \beta, B \rangle$ and $\langle -, \beta, C \rangle$ are in X .
- (i) If a sequence of the form $\langle +, \beta, (B \rightarrow C) \rangle$ is in X , then a sequence of the form $\langle -, \beta, B \rangle$ is in X or a sequence of the form $\langle +, \beta, C \rangle$ is in X .
- (j) If a sequence of the form $\langle -, \beta, (B \rightarrow C) \rangle$ is in X , then sequences of the form $\langle +, \beta, B \rangle$ and $\langle -, \beta, C \rangle$ are in X .
- (k) If a sequence of the form $\langle +, \beta, \exists v B \rangle$ is in X , then, for some γ with $\beta \Vdash \gamma$, a sequence of the form $\langle +, \gamma, B \rangle$ is in X .
- (l) If a sequence of the form $\langle -, \beta, \exists v B \rangle$ is in X , then $\neg \text{sat}_n(+, \beta, \exists v B)$.
- (m) If a sequence of the form $\langle +, \beta, \forall v B \rangle$ is in X , then $\neg \text{sat}_n(-, \beta, \forall v B)$.
- (n) If a sequence of the form $\langle -, \beta, \forall v B \rangle$ is in X , then, for some γ with $\beta \Vdash \gamma$, a sequence of the form $\langle -, \gamma, B \rangle$ is in X .

Note that if σ and τ are extensionally equal and if X is n -adequate and σ is in X , then the result of replacing σ in X by τ is again n -adequate. We shall often write $\alpha \models_n^i A$ for: $\text{sat}_n(i, \alpha, A)$. The relation sat_n , when restricted to \mathcal{K}_n , will have a number of desirable properties. We write sat_n^* for $\text{sat}_n \cap \mathcal{K}_n$. We note that we have implicitly given a formula $\Phi_0(\mathcal{X}, i, \alpha, A)$, where \mathcal{X} is a second order variable with $\text{sat}_{n+1}(i, \alpha, A) = \Phi_0(\text{sat}_n, i, \alpha, A)$. Thus, for some fixed standard c_0 , we have $\rho(\text{sat}_{n+1}(u, v, w)) = \rho(\text{sat}_n(u, v, w)) + c_0$. We may conclude that $\rho(\text{sat}_n(u, v, w)) = c_0 n + c_1$, for some fixed standard number c_1 .

We note that \mathcal{X} occurs twice in the formula $\phi_0(\mathcal{X}, k, \alpha, A)$ described above. This has no effect on the growth of the complexity of the formula sat_n but it makes the *number of symbols* of the formula sat_n grow exponentially in n . For the purposes of this paper, this is good enough. However, a slightly more careful rewrite of our definition reduces the number of occurrences of \mathcal{X} to one. As a consequence, we can get the number of symbols of sat_n linear in n . So, the code of sat_n will be bounded by a polynomial in n , assuming we use an efficient Gödel numbering. Cf. [15, 16] for a discussion of writing formulas in an efficient way.

Our next step is to verify in $\text{AS}^+(\Theta)$ some good properties of $n + 1$ -adequacy and sat_n . We first show that n -adequacy is preserved under certain operations.

Theorem 2.7 *The n -adequate sets are closed under unions and under intersection with the virtual class of a -sat-sequences, for any a .*

Proof. Closure under unions is immediate given that good sets are closed under unions. Closure under restriction to a -sat-sequences is immediate by the definition of *good* and the fact that our formula classes are closed under subformulas. \square

We prove a theorem connecting sat_k^* and sat_n^* , for $k < n$. We remind the reader that \mathcal{K}_k is the virtual class of all k -sat-sequences and that $\text{sat}_n^* = \text{sat}_n \cap \mathcal{K}_n$.

Theorem 2.8 ($\text{AS}^+(\Theta)$) *Suppose $k < n$. Then, $\text{sat}_k^* = \text{sat}_n^* \cap \mathcal{K}_k$.*

Proof. The proof is by external induction on n . The case that $n = 0$ is trivial. Suppose X is an $n + 1$ -adequate set. Let $k < n + 1$ and let $Y := X \cap \mathcal{K}_k$. We note that Y is a set by Theorem 2.7. We claim that Y is k -adequate. It is clear that Y satisfies all clauses for a k -adequate set automatically except (l) and (m). Let us zoom in on (l). Suppose a sequence of the form $\langle -, \beta, \exists v B \rangle$ is in Y . From this it follows that $k \neq 0$, since \mathcal{K}_0 is empty. Since X is $n + 1$ -adequate, it follows that: $\neg \text{sat}_n(+, \beta, \exists v B)$. By the induction hypothesis, we find that $\neg \text{sat}_{k-1}(+, \beta, \exists v B)$. Hence, the clause for k -adequacy is fulfilled. Clause (m) is similar.

Conversely, suppose Z is k -adequate. Let $W := Z \cap \mathcal{K}_k$. The argument that W is also $n + 1$ -adequate, is analogous to the argument above. \square

We note that in the proof of Theorem 2.8, we could as well do the induction on k . This observation is important in case we study models with a full satisfaction predicate. In this context, we can replace n by a non-standard number and still get our result for standard k .

In the following theorem we prove the commutation conditions for sat_{n+1} .

Theorem 2.9 ($AS^+(\Theta)$) *We have*

- (a) $\beta \models_{n+1}^+ P(v_0, v_1)$ if and only if $P(\beta(v_0), \beta(v_1))$, and similarly for the other atomic formulas including \top and \perp ,
- (b) $\beta \models_{n+1}^- P(v_0, v_1)$ if and only if $\neg P(\beta(v_0), \beta(v_1))$, and similarly for the other atomic formulas including \top and \perp ,
- (c) $\beta \models_{n+1}^+ \neg B$ if and only if $\beta \models_{n+1}^- B$,
- (d) $\beta \models_{n+1}^- \neg B$ if and only if $\beta \models_{n+1}^+ B$,
- (e) $\beta \models_{n+1}^+ B \wedge C$ if and only if $\beta \models_{n+1}^+ B$ and $\beta \models_{n+1}^+ C$,
- (f) $\beta \models_{n+1}^- B \wedge C$ if and only if $\beta \models_{n+1}^- B$ or $\beta \models_{n+1}^- C$,
- (g) $\beta \models_{n+1}^+ B \vee C$ if and only if $\beta \models_{n+1}^+ B$ or $\beta \models_{n+1}^+ C$,
- (h) $\beta \models_{n+1}^- B \vee C$ if and only if $\beta \models_{n+1}^- B$ and $\beta \models_{n+1}^- C$,
- (i) $\beta \models_{n+1}^+ B \rightarrow C$ if and only if $\beta \models_{n+1}^- B$ or $\beta \models_{n+1}^+ C$,
- (j) $\beta \models_{n+1}^- B \rightarrow C$ if and only if $\beta \models_{n+1}^+ B$ and $\beta \models_{n+1}^- C$,
- (k) $\beta \models_{n+1}^+ \exists v B$ if and only if, for some γ with $\beta \Vdash v \gamma$, we have $\gamma \models_{n+1}^+ B$,
- (l) $\beta \models_{n+1}^- \exists v B$ if and only if $\beta \not\models_n^+ \exists v B$,
- (m) $\beta \models_{n+1}^+ \forall v B$ if and only if $\beta \not\models_n^- \forall v B$,
- (n) $\beta \models_{n+1}^- \forall v B$ if and only if, for some γ with $\beta \Vdash v \gamma$, we have $\gamma \models_{n+1}^- B$.

Proof. We shall treat the illustrative clauses (e), (k) and (l). In the first two cases the left-to-right direction is trivial.

Ad (e). Suppose X witnesses that $\beta \models_{n+1}^+ B$ and Y witnesses that $\beta \models_{n+1}^+ C$. Let σ be a triple of the form $\langle \beta, +, (B \wedge C) \rangle$. Let Z be a union of X and Y and a singleton with element σ . It is immediate that Z witnesses $\beta \models_{n+1}^+ B \wedge C$.

Ad (k). Suppose that $\beta \Vdash v \gamma$ and that X witnesses that $\gamma \models_{n+1}^+ B$. Let σ be of the form $\langle \beta, +, \exists v B \rangle$. Let Y be a union of X and a singleton with element σ . Then Y witnesses $\beta \models_{n+1}^+ \exists v B$.

Ad (l). Suppose $\beta \not\models_n^+ \exists v B$. Let σ be a triple of the form $\langle \beta, -, \exists v B \rangle$. Let X be a singleton with element σ . Then X witnesses $\beta \models_{n+1}^- \exists v B$. Conversely, if Y witnesses $\beta \models_{n+1}^- \exists v B$, then we must have $\neg \text{sat}_n(\beta, +, \exists v B)$. \square

We note that the commutation conditions are inherited by sat_{n+1}^* , provided that the formulas in the conditions belong to the right classes.

The commutation conditions proven in Theorem 2.9 are not yet full commutation conditions. The defect is in the clauses (l) and (m). Let us zoom in on (m): $\beta \models_{n+1}^+ \forall v B$ if and only if $\beta \not\models_n^- \forall v B$. The right-hand-side is equivalent to: for all γ with $\beta \Vdash v \gamma$, we have $\gamma \not\models_n^- B$. To get the desired commutation condition, we would like to move from $\gamma \not\models_n^- B$ to $\gamma \models_{n+1}^+ B$. We have seen in Theorem 2.8 that to make our predicates behave in expected ways, it is better to consider the formulas in their ‘intended range’. So what if $\forall v B$ is in Σ_{n+1}^* ? In this case B must be in Π_n^* and hence in Δ_{n+1}^* . Thus, by Theorem 2.8, it is sufficient if we have: whenever C is in Δ_{n+1}^* , then $\alpha \not\models_{n+1}^- C$ if and only if $\alpha \models_{n+1}^+ C$? To prove this we need induction, which we do not have available in $AS^+(\Theta)$. The solution is to move to a cut.

To realize this idea, we define a second measure of complexity v (depth of connectives) as follows: $v(A) := 0$ if A is atomic, $v(\neg A) := v(\exists v A) := v(\forall v A) := v(A) + 1$ and $v(A \circ B) := \max(v(A), v(B)) + 1$, where \circ is a binary propositional connective. Let $\Gamma_x := \{A \mid v(A) \leq x\}$.

We define J_{n+1}^\dagger as the virtual class of all numbers x such that, for all α and for all $C \in \Delta_{n+1}^* \cap \Gamma_x$, we have $\alpha \not\models_{n+1}^- C$ if and only if $\alpha \models_{n+1}^+ C$. We have the following theorem.

Theorem 2.10 ($AS^+(\Theta)$) *The class J_{n+1}^\dagger contains 0, is closed under successor and is downwards closed with respect to \leq .*

Proof. Downwards closure is immediate. By definition, we have $\alpha \not\models_{n+1}^- C$ if and only if $\alpha \models_{n+1}^+ C$ if C is an atom. So 0 is in J_{n+1}^\dagger . Suppose x is in J_{n+1}^\dagger . We shall show that $x + 1$ is in J_{n+1}^\dagger , i.e., for all $C \in \Delta_{n+1}^* \cap \Gamma_{x+1}$, we have $\alpha \not\models_{n+1}^- C$ if and only if $\alpha \models_{n+1}^+ C$.

The case for atomic C follows by previous reasoning. Suppose, e.g., that $C := (D \rightarrow E)$ is in $\Delta_{n+1}^* \cap \Gamma_{x+1}$. Then D and E are both in $\Delta_{n+1}^* \cap \Gamma_x$. By the fact that x is in J_{n+1}^\dagger we find that

$$\begin{aligned}
\alpha \models_{n+1}^+ (D \rightarrow E) &\leftrightarrow (\alpha \models_{n+1}^- D) \text{ or } (\alpha \models_{n+1}^+ E) \\
&\leftrightarrow (\alpha \not\models_{n+1}^+ D) \text{ or } (\alpha \not\models_{n+1}^- E) \\
&\leftrightarrow \neg((\alpha \models_{n+1}^+ D) \text{ and } (\alpha \models_{n+1}^- E)) \\
&\leftrightarrow \neg \alpha \models_{n+1}^- (D \rightarrow E).
\end{aligned}$$

The other unary and binary propositional connectives are similar. Now suppose C is of the form $\exists v D$ and $C \in \Delta_{n+1}^* \cap \Gamma_{x+1}$. Since C is in Π_{n+1}^* , we must have $D \in \Sigma_n^*$. It follows that $n \neq 0$. We have, by Theorem 2.8 that

$$\begin{aligned}
\alpha \models_{n+1}^+ \exists v D &\leftrightarrow \alpha \models_n^+ \exists v D \\
&\leftrightarrow \alpha \not\models_{n+1}^- \exists v D.
\end{aligned}$$

Curiously, this step does not use the fact that x is in J_{n+1}^\dagger . The case of \forall is similar. \square

Let $\Gamma_{J_{n+1}^\dagger} := \bigcup_{x \in J_{n+1}^\dagger} \Gamma_x$. We have found that, for any C in $\Delta_{n+1}^* \cap \Gamma_{J_{n+1}^\dagger}$, we have $\alpha \not\models_{n+1}^- C$ if and only if $\alpha \models_{n+1}^+ C$. Hence, we also have the full Tarskian commutation clauses for these C . This means that, whenever C in $\Delta_{n+1}^* \cap \Gamma_{J_{n+1}^\dagger}$ and $C = (D \wedge E)$, then $\alpha \models_{n+1}^+ C$ if and only if $(\alpha \models_{n+1}^+ D \text{ and } \alpha \models_{n+1}^+ E)$, and, similarly, for the other connectives. Here we note that the quantifiers over C , D and E are theory-internal.

We note that J_{n+1}^\dagger is $\Phi_1(n+1, \text{sat}_{n+1}, x)$, for a standard formula $\Phi_1(y, \mathcal{X}, x)$. Thus the ρ -complexity of J_{n+1}^\dagger is linear in n where the relevant linear term is of the form $c_0 n + c_2$.

We still miss an important ingredient. Let $\langle i, \alpha, A \rangle$ be a sat-sequence. Suppose α and β assign the same values to the free variables in A . Do we have $\alpha \models_n^i A$ if and only if $\beta \models_n^i A$? To prove such a thing we need induction. We would like to have even more than this, since we want to check the validity of the inference rules for the quantifiers. We define the property Q_{n+1} as follows. The formula A has the property Q_{n+1} if the following holds. Consider any sat-sequence $\langle i, \alpha, A \rangle$. Suppose w is free for v in A . Let B be (of the form) $A[v := w]$. Suppose further that the functions $\alpha[u] = \beta[u]$ for all free variables u of A , except possibly v , and that $\beta[w] = \alpha[v]$. Then, $\alpha \models_{n+1}^i A$ if and only if $\beta \models_{n+1}^i B$.

We allow that v and w are equal and that v does not occur in A . Both degenerate cases tell us that $Q_{n+1}(A)$ implies that $\alpha \models_{n+1}^i A$ if and only if $\beta \models_{n+1}^i A$, whenever α and β agree on the free variables of A .

We can now proceed in two ways to construct a cut that gives us the desired property for the formulas of ν -complexity in the cut. One way does not involve the Σ_n^* and the Π_n^* and one way does involve them. The second way yields a more efficient construction of the cut. For completeness, we explore both ways.

We first address the first way. We define $J_0^\circ := \mathbb{N}$ and $J_{n+1}^\circ := \{x \in J_n^\circ \mid \Gamma_x \subseteq Q_{n+1}\}$. We note that the definition of J_{n+1}° is of the form $\Phi_2(\text{sat}_{n+1}, J_n^\circ, x)$ for a fixed $\Phi_2(\mathcal{X}, \mathcal{Y}, x)$. So, $\rho(J_{n+1}^\circ) = \max(\rho(\text{sat}_{n+1}), \rho(J_n^\circ)) + c_3$, for a fixed standard c_3 . It follows that $\rho(J_n^\circ)$ is estimated by some linear term $c_4 n + c_5$.

Theorem 2.11 ($\text{AS}^+(\Theta)$) *The virtual class J_n° is closed under 0, successor, and is downwardly closed with respect to \leq .*

Proof. Closure under 0 and downwards closure are trivial. We prove closure under successor by induction on n . The case of J_0° is trivial. Suppose that J_n° is closed under successor. Consider x in J_{n+1}° . Let C and D be in Γ_x .

Let A be of the form $(C \wedge D)$. Suppose w is free for v in A . Let B be (of the form) $A[v := w]$. Suppose further that α and β assign the same values to the free variables of A except v and $\beta[w] = \alpha[v]$. Clearly B is of the form $E \wedge F$, where E is of the form $C[v := w]$ and F is of the form $D[v := w]$. We have

$$\begin{aligned}
\alpha \models_{n+1}^+ C \wedge D &\leftrightarrow \alpha \models_{n+1}^+ C \text{ and } \alpha \models_{n+1}^+ D \\
&\leftrightarrow \beta \models_{n+1}^+ E \text{ and } \beta \models_{n+1}^+ F \\
&\leftrightarrow \beta \models_{n+1}^+ E \wedge F.
\end{aligned}$$

Similarly for the \models^- -case. The other propositional cases are similar.

We treat the case of the existential quantifier, the case of the universal quantifier being similar. Let A be of the form $\exists z C$. Suppose w is free for v in A . Let B be (of the form) $A[v := w]$. Suppose further that α and β assign the same values to the free variables of A except v and $\beta[w] = \alpha[v]$.

We first address the \models^+ -case. The argument splits into two subcases. First we have the case that z is (of the form) v , we find that B is of the form A . Hence, replacing z by v , we have

$$\begin{aligned} \alpha \models_{n+1}^+ \exists v C &\leftrightarrow \exists \gamma (\alpha \llbracket v \rrbracket \gamma \text{ and } \gamma \models_{n+1}^+ C) \\ &\leftrightarrow \exists \delta (\beta \llbracket v \rrbracket \delta \text{ and } \delta \models_{n+1}^+ C) \\ &\leftrightarrow \beta \models_{n+1}^+ \exists v C. \end{aligned}$$

E.g., in the left-to-right direction of the second step we can take δ of the form $\beta[v : \gamma(v)]$. We can use the fact that C has v -complexity x and $x \in J_{n+1}^\circ$. The property Q_{n+1} is applied with v in the role of both v and w .

Next we have the case that z and w are different variables. Let D be of the form $C[v : w]$. So B is of the form $\exists z D$. We have

$$\begin{aligned} \alpha \models_{n+1}^+ \exists z C &\leftrightarrow \exists \gamma (\alpha \llbracket z \rrbracket \gamma \text{ and } \gamma \models_{n+1}^+ C) \\ &\leftrightarrow \exists \delta (\beta \llbracket z \rrbracket \delta \text{ and } \delta \models_{n+1}^+ D) \\ &\leftrightarrow \beta \models_{n+1}^+ \exists z D. \end{aligned}$$

E.g., in the left-to-right direction of the second step we can again take δ of the form $\beta[z : \gamma(z)]$.

Finally we address the \models^- -case. We have

$$\begin{aligned} \alpha \models_{n+1}^- \exists v C &\leftrightarrow \neg \alpha \models_n^+ \exists v C \\ &\leftrightarrow \neg \beta \models_n^+ \exists v D \\ &\leftrightarrow \beta \models_{n+1}^- \exists v D. \end{aligned}$$

Here we use the fact that $x \in J_n^\circ$, so that also $x + 1 \in J_n^\circ$. □

We turn to the second approach. We define $J_{n+1}^* := \{x \in \mathbb{N} \mid (\Gamma_x \cap \Delta_{n+1}^*) \subseteq Q_{n+1}\}$, and have the following result.

Theorem 2.12 ($AS^+(\Theta)$) *The virtual class J_{n+1}^* is closed under 0, successor and is downwards closed with respect to \leq .*

Proof. The cases of closure under 0 and downwards closure are trivial. Suppose x is in J_{n+1}^* . The cases of the propositional connectives use the same argument as we saw in the proof of theorem 2.11. We turn to the case of the existential quantifier, the case of the universal quantifier being dual. Suppose C is in $\Gamma_x \cap \Delta_{n+1}^*$. The case of \models^+ is again the same as we saw in the proof of Theorem 2.11. We consider the case of \models^- . Suppose $\exists v C$ is in Δ_{n+1}^* . In this case $\exists v C$ must be in Σ_n^* . We have

$$\begin{aligned} \alpha \models_{n+1}^- \exists v C &\leftrightarrow \neg \alpha \models_n^+ \exists v C \\ &\leftrightarrow \neg \alpha \models_{n+1}^+ \exists v C \\ &\leftrightarrow \neg \beta \models_{n+1}^+ \exists v D \\ &\leftrightarrow \neg \beta \models_n^+ \exists v D \\ &\leftrightarrow \beta \models_{n+1}^- \exists v D. \end{aligned}$$

The first and the fifth step use the commutation conditions for \exists . The second and the fourth step use Theorem 2.8. The third step uses the previous case for \models^+ . □

We note that the definition of J_{n+1}^* is of the form $\Phi_3(n, \text{sat}_n)$. So, its ρ_0 -complexity is estimated by a linear term of the form $c_0n + c_6$. Here the use of J_{n+1}^* has an advantage over J_n° , since construction of the J_m° gives us a linear complexity but conceivably with a higher constant as coefficient of n .

Let us take stock of what we accomplished. We have defined virtual classes J_{n+1}^\dagger that are closed under 0 and S and that are downwards closed such that for all formulas A in $\Gamma_{J_{n+1}^\dagger} \cap \Delta_{n+1}^*$, we have, for all α , that $\alpha \models_{n+1}^+ A$ if and only if $\alpha \not\models_{n+1}^- A$. Here, as before, $\Gamma_{J_{n+1}^\dagger} := \bigcup_{x \in J_{n+1}^\dagger} \Gamma_x$.

Also, we have developed virtual classes J_{n+1}° and J_{n+1}^* such that all A in $\Gamma_{J_{n+1}^\circ}$, and, similarly, all A in $\Gamma_{J_{n+1}^*} \cap \Delta_{n+1}^*$ have the property Q_{n+1} defined above.

So, if we take J_{n+1}^\dagger either $J_{n+1}^\dagger \cap J_{n+1}^\circ$ or $J_{n+1}^\dagger \cap J_{n+1}^*$, then J_{n+1}^\dagger is progressive and all elements of $\Xi_{n+1} := \Gamma_{J_{n+1}^\dagger} \cap \Delta_{n+1}^*$ have both good properties. Let us choose for J^* in the definition of Ξ_{n+1} , so that its ρ -complexity is estimated by $c_0n + c_7$.

We summarize the result in a theorem. We put $\Xi_0 := \emptyset$.

Theorem 2.13 ($AS^+(\Theta)$) *We have full commutation of $\text{sat}(+, \cdot, \cdot)$ for the Ξ_n -formulas. Moreover, the Ξ_n -formulas have the property Q_n .*

We choose, as our proof system, the system for natural deduction in sequent style as given in [18, § 2.1.4]. An n -proof* is a proof only involving Ξ_n -formulas. We write $\mathcal{A} \supset_n A$ for the formalization of $\mathcal{A} \vdash_n A$, where \mathcal{A} codes a finite set of formulas and \vdash_n is provability in predicate logic where we restrict ourselves to n -proof*s. We choose to code the set of formulas in the natural numbers. This is a bit unnatural since $AS^+(\Theta)$ contains sets as first-class citizens. However, if we code sets of formulas in the sets provided by $AS^+(\Theta)$ directly we do not know, e.g., that $\text{ass}(p)$ the set of assumptions of a proof p is a set. Of course, this problem can be evaded by shortening N in such a way that any set coded in the natural numbers maps to first-class set. If the reader prefers this other road, we think it is sufficiently clear how to adapt the results below to this alternative approach.

We write $p : \mathcal{A} \supset_n A$ for: p is the code of an n -proof* witnessing $\mathcal{A} \supset_n A$. We write $\Lambda_{n,y}$ for the class of n -proofs p where the number of steps of p is $\leq y$. On the semantical side, we define, for $\mathcal{A} \cup \{A\} \subseteq \Xi_n$:

$$\mathcal{A} \models_n A \leftrightarrow \forall \alpha (\forall A' \in \mathcal{A} \text{ sat}_n(+, \alpha, A') \rightarrow \text{sat}_n(+, \alpha, A)).$$

We work in $AS^+(\Theta)$. We write $\text{ass}(p)$ for the assumption set of (proof code) p . Let Y_n be the class of y such that, for all $p \in \Lambda_{n,y}$, if $p : \text{ass}(p) \supset_n A$, then $\text{ass}(p) \models_n A$.

Theorem 2.14 ($AS^+(\Theta)$) *The virtual class Y_n is downwards closed under \leq , contains 0, and is closed under successor.*

Proof. The case $n = 0$ is trivial, so we assume $n > 0$. Downwards closure under \leq is trivial. We show that Y_n is progressive. By Theorem 2.13, the propositional cases are immediate. We shall treat the introduction and the elimination rule of the universal quantifier. This follows mainly the usual text book proof. For the convenience of the reader, we repeat the property Q_n :

The formula C has the property Q_n if the following holds. Consider any sat-sequence $\langle i, \alpha, C \rangle$. Suppose u is free for z in C . Suppose further that $\alpha[a] = \beta[a]$ for all free variables a of A except z and that $\beta[u] = \alpha[z]$. Then, $\alpha \models_n^i C$ if and only if $\beta \models_n^i C[z := u]$.

We treat the case of universal generalization. Let w be substitutable for v in A and suppose w does not occur freely in the elements of $\mathcal{A} \cup \{A\}$. Suppose we have $(\dagger) \mathcal{A} \models_n A[v := w]$. We show that $\mathcal{A} \models_n \forall v A$.

Consider any α and suppose $\alpha \models_n^+ A'$, for all A' in \mathcal{A} . We want to show that $\alpha \models_n^+ \forall v A$. Let d be any element. It is clearly sufficient to show that $\alpha[v : d] \models_n^+ A$.

We first note that $\alpha[w : d]$ and α are the same on the free variables of A' in \mathcal{A} . So, by Q_n in a degenerate case, we find that $\alpha[w : d] \models_n^+ A'$, for all A' in \mathcal{A} . By (\dagger) , we find $\alpha[w : d] \models_n^+ A[v := w]$. We note that $\alpha[w : d]$ and $\alpha[v : d]$ assign the same values to all free variables u of A , except possibly v . This uses that w does not occur in A . Moreover, $\alpha[w : d][w] = \alpha[v : d][v]$. So, by $Q_n(A)$, we find $\alpha[v : d] \models_n^+ A$ if and only if $\alpha[w : d] \models A[v := w]$. Thus, we may conclude $\alpha[v : d] \models_n^+ A$ as desired.

We treat the case of universal instantiation. Suppose $(\ddagger) \mathcal{A} \models \forall v A$. We want to conclude $\mathcal{A} \models A[v := w]$. Suppose, for all $A' \in \mathcal{A}$, we have $\alpha \models_n^+ A'$. It follows that $(\$) \alpha \models_n^+ \forall v A$. We want to conclude that $\alpha \models A[v := w]$. From $(\$)$, we have $\alpha[v := \alpha[w]] \models^+ A$. We note that α and $\alpha[v := \alpha[w]]$ assign the same values to all free variables of A except possibly v . Moreover $\alpha[v := \alpha[w]][v] = \alpha[w]$. By $Q_n(A)$, we may conclude that $\alpha \models_{n+1}^+ A[v := w]$. \square

Inspecting the construction of Y_n , for $n > 0$, we see that it is of the form $\Phi_4(\underline{n}, \text{sat}_n, J_n^\dagger)$, where $\Phi_4(x, \mathcal{X}, \mathcal{Y})$ is a fixed formula. The case $n = 0$ is trivial. Thus, $\rho(Y_n)$ is estimated by $c_0n + c_8$.

We now have a refined result involving separate restrictions on ρ on v and on the length of the proofs. For other applications this refinement may be useful, however, in the present paper, we shall simply demand, in the case that $n > 0$, that our proofs are in a cut $\mathfrak{S}_n(\Theta)$ that is obtained by taking the intersection of J_n^\dagger and Y_n and shortening to obtain downwards closure and closure under $0, S, +, \times$ and ω_1 . Since the shortening procedure only adds a standardly finite depth to the input formula $J_n^\dagger \cap Y_n$, the formula $\mathfrak{S}_n(\Theta)$ will have complexity estimated by $c_0n + c_9$. Moreover, when p is in $\mathfrak{S}_n(\Theta)$, then ipso facto its length is in Y_n and its v -complexity is in J_n^\dagger .

We write $[\mathcal{A} \vdash_n A]$ for provability in predicate logic involving only Δ_n^* -formulas. We write $[\mathcal{A} \vdash_n^J A]$ when the witness for $[\mathcal{A} \vdash_n A]$ is constrained to the cut J . We write $\square_{\Theta, n} A$ for $[\emptyset \vdash_n A]$, and $\square_{\Theta, n}^J A$ for $[\emptyset \vdash_n^J A]$. For sentences A , we shall write $\text{true}_{\Theta, n}(A)$ for $\forall \alpha \text{ sat}_n(+, \alpha, A)$.

Theorem 2.15 *We can find an ω_1 -cut $\mathfrak{S}_n(\Theta)$ such that $\rho(\mathfrak{S}_n(\Theta))$ is of order $c_0n + c_9$ and such that*

$$\text{AS}^+(\Theta) \vdash \forall \mathcal{A}, A ([\mathcal{A} \vdash_n^{\mathfrak{S}_n(\Theta)} A] \rightarrow \mathcal{A} \models_n A).$$

As a special case, we have $\text{AS}^+(\Theta) \vdash \forall A \in \text{sent}^{\mathfrak{S}_n(\Theta)} (\square_{\Theta, n}^{\mathfrak{S}_n(\Theta)} A \rightarrow \text{true}_{\Theta, n}(A))$.

Results in the spirit of Theorem 2.15 were well known around 1985. Cf., e.g., [14, Theorem 3.1]. A direct precursor of the present theorem is [20, Fact 2.4.5]. The main contribution of the present result is its precise statement—including the insight that the complexity of the cut involved is linear in the complexity bound on the proof—and its detailed proof.

3 Small-is-very-small principles

In this section we present the central argument of this paper. It is a simple Rosser argument. The bulk of the work has already been done in creating the setting for the result. We choose to give the pure argument in Theorem 3.1 rather than proceed immediately to the somewhat more complicated Theorem 3.2. The more complicated version is needed for application in model theory.

First some preliminaries and notations, in order to avoid too heavy notational machinery.

We shall work in sequential theories U of signature Θ with sequence scheme \mathcal{S} . So, $\mathcal{S} : \text{AS}^+ \xrightarrow{\text{dir}} U$. We can lift \mathcal{S} to a direct interpretation $\mathcal{S}_\Theta : \text{AS}^+(\Theta) \xrightarrow{\text{dir}} U$ by translating Θ identically.⁶

We remind the reader that $\mathcal{N} : \mathcal{S}_2^1 \rightarrow \text{AS}^+$. We shall write $N := \mathcal{S} \circ \mathcal{N} : \mathcal{S}_2^1 \rightarrow U$. So, e.g., $\delta_N = \mathbb{N}^{\mathcal{S}}$. We write \mathfrak{S}_n for $(\mathfrak{S}_n(\Theta))^{\mathcal{S}_\Theta}$. When we write numerals \underline{n} these are always numerals with respect to N . We note that the numerals really are eliminated using the term elimination algorithm. However, this elimination just gives an overhead of 1 in ρ_0 -complexity.

Let η be a Σ_1^b -formula defining a set of axioms. We write \square_η for provability from the axioms in η . We write \square_η^J for the result of restricting the witnesses for \square_η to J . We write $\square_{\eta, n}$ for the result of restricting the formulas in a witnessing proof to Δ_n^* .⁷ Formulas like $\square_{\eta, n}^J$ have the obvious meanings. We suppress the information about the signature Θ , which should be clear from the context. In case $\eta = (x = \ulcorner A \urcorner)$, we write \square_A for \square_η .⁸

We shall employ witness comparison notation:

$$\exists x \in \delta_N A_0(x) \leq \exists y \in \delta_N B_0(y) \quad \text{if and only if} \quad \exists x \in \delta_N (A_0(x) \wedge \forall y <^N x \neg B_0(y)).$$

$$\exists x \in \delta_N A_0(x) < \exists y \in \delta_N B_0(y) \quad \text{if and only if} \quad \exists x \in \delta_N (A_0(x) \wedge \forall y \leq^N x \neg B_0(y)).$$

⁶ In case \mathcal{S} would be a sequence scheme for a polysequential we would need a slight adaptation.

⁷ In previous papers, we also used this notation to signal that the codes of the axioms were constrained to be $\leq n$. In this paper this extra demand is not made.

⁸ Clearly, this introduces an ambiguity. E.g., does \square_\top mean provability from all sentences or from the axiom \top ? In the first case, \top is viewed as a sentence of the form $\alpha(x)$ with zero occurrences of x . In the second case, \top is viewed as representing $x = \ulcorner \top \urcorner$. However, what we intend will be always clear from the context.

Theorem 3.1 *Let A be a finitely axiomatized sequential theory in a language with signature Θ with sequence scheme \mathcal{S} . Consider any sentence B in the language of A of the form $B := \exists x \in \delta_N B_0(x)$. Let $n := \max(\rho_0(A), \rho_0(B) + \mathfrak{c}_{10}, \rho_0(\mathcal{S}) + \mathfrak{c}_{10})$. Here \mathfrak{c}_{10} is a fixed finite constant that does not depend on A , B and \mathcal{S} . We shall determine \mathfrak{c}_{10} below.*

Suppose $A \vdash \exists x \in \mathfrak{S}_n B_0(x)$. Then, for some k , we have $A \vdash \exists x \leq^N \underline{k} B_0(x)$, or, equivalently, $A \vdash \bigvee_{q \leq k} B_0(q)$.

Proof. We work under the conditions of the theorem. Using the Gödel Fixed Point Lemma, we find a sentence R such that $A \vdash R \leftrightarrow B \leq \square_{A, \underline{n}}^N R$. We need that $\rho_0(R) \leq n$.

Under the usual Gödel construction, R is of the following form:

$$\exists x (\delta_N(x) \wedge B_0(x) \wedge \exists z (\delta_N(z) \wedge \text{sub}^N(\ell, \underline{\ell}, z) \wedge \forall y (y <^N x \rightarrow \neg \text{proof}_{A, \underline{n}}^N(y, z)))).$$

Here the numerals are to be interpreted in N . The formula proof represents a standard Gödel coding of the proof predicate. The number ℓ codes a suitable formula provided by the Gödel construction.

Thus, $\rho_0(R)$ is estimated by $\max(\rho_0(\text{sub}), \rho_0(\text{proof})) + \max(\rho_0(\mathcal{S}), \rho_0(B_0)) + 3$. Here the $+3$ is due to the additional quantifiers. We note that, if we unravel the numerals \underline{n} wide scope, we even just need $+2$. So, we can take $\mathfrak{c}_{10} := \max(\rho_0(\text{sub}), \rho_0(\text{proof})) + 3$.

Reason in A . We have $\exists x \in \mathfrak{S}_n B_0(x)$. In case $\neg \square_{A, \underline{n}}^N R$, we have R . Suppose $\square_{A, \underline{n}}^N R$. By reflection, as guaranteed by Theorem 2.15, we find R . So, in both cases, we may conclude that R . We stop reasoning in A and return to the meta-theory.

We have shown (i) $A \vdash R$. By cut-elimination, we find: $A \vdash_n R$. Hence, (ii) for some k , we find $A \vdash \text{proof}_{A, \underline{n}}^N(\underline{k}, R)$. Combining (i) and (ii), we may conclude that $A \vdash \exists x \leq^N \underline{k} B_0(x)$, or, equivalently, $A \vdash \bigvee_{q \leq k} B_0(q)$. \square

We note that, due to the use of cut-elimination, we need the totality of superexponentiation in the meta-theory. Such theorems usually leave watered-down traces in weaker metatheories. We do not explore such possibilities in the present paper.

The above argument has some analogies with Friedman's beautiful proof that, in a constructive setting, the disjunction property implies the existence property [3]. We analyzed this argument in [27], having the benefit of many perceptive remarks by Jeřábek. One surprising aspect of the above proof is that the minimization principle is not used. Joosten pointed out to us in conversation that the closely related Friedman-Goldfarb-Harrington Theorem also can be proven without using minimization.

For our model theoretic applications we need a variant of Theorem 3.1 that adds domain constants. We allow for the domain constants the exceptional position that they are real constants rather than unary predicates posing as constants.

Theorem 3.2 *Consider a finite set of domain constants \mathcal{C} . Let A_0 be any finitely axiomatized sequential theory with signature Θ and sequence scheme \mathcal{S} . Let $A_1 := A_1(\vec{c})$ be any sentence in the language with signature $\Theta + \mathcal{C}$. Let $A := A_0 \wedge A_1$.*

Consider any sentence $B(\vec{c})$ in the language of signature $\Theta + \mathcal{C}$ of the form $B(\vec{c}) := \exists x \in \delta_N B_0(x, \vec{c})$. Let $n := \max(\rho_0(A), \rho_0(B) + \mathfrak{c}_{10}, \rho_0(\mathcal{S}) + \mathfrak{c}_{10})$.

Suppose $A(\vec{c}) \vdash \exists x \in \mathfrak{S}_n B_0(x, \vec{c})$.⁹ Then, for some k , we have that $A(\vec{c}) \vdash \exists x \leq^N \underline{k} B_0(x, \vec{c})$, or, equivalently, $A(\vec{c}) \vdash \bigvee_{q \leq k} B_0(q, \vec{c})$.

Proof. We work under the conditions of the theorem. We find a sentence $R(\vec{c})$ such that

$$A(\vec{c}) \vdash R(\vec{c}) \leftrightarrow B(\vec{c}) \leq \square_{A(\vec{c}), \underline{n}}^N R(\vec{c}).$$

Reason in $A(\vec{c})$. In case $\neg \square_{A(\vec{c}), \underline{n}}^N R(\vec{c})$, we have $R(\vec{c})$. Suppose $\square_{A(\vec{c}), \underline{n}}^N R(\vec{c})$. Replacing the extra constants \vec{c} , in $A(\vec{c})$ and $R(\vec{c})$ by fresh variables \vec{v} , we get $[A(\vec{v}) \vdash_n R(\vec{v})]$. Hence, $A(\vec{v}) \models_n R(\vec{v})$. Since A and R are standard¹⁰ and since we have $A(\vec{c})$, we find $R(\vec{c})$.

⁹ The fact that the constants in \mathcal{C} do not occur in $\mathfrak{S}_n = \mathfrak{S}_n^{\mathcal{S}}(\Theta)$ is the whole point of the refined result.

¹⁰ We note that, since we have the commutation conditions, we have the Tarski biconditionals at the standard level.

Thus, we have shown (i) $A(\vec{c}) \vdash R(\vec{c})$. By cut-elimination, we find: $A(\vec{c}) \vdash_n R(\vec{c})$. Hence, (ii) for some k , we have $A(\vec{c}) \vdash \text{proof}_{A(\vec{c}),n}^N(k, R(\vec{c}))$. Combining (i) and (ii), we get $A(\vec{c}) \vdash \exists x \leq^N k B_0(x, \vec{c})$, or, equivalently, $A(\vec{c}) \vdash \bigvee_{q \leq k} B_0(q, \vec{c})$. \square

We call a theory U *restricted* if, for some m all its axioms are in Δ_m^* .

Theorem 3.3 *Suppose A_0 is a finitely axiomatized sequential theory in signature Θ with sequence scheme \mathcal{S} . Let m be any number such that $m \geq \rho_0(A_0)$. Let \mathcal{C} be a set of domain constants: \mathcal{C} is allowed to have any cardinality. Let U be a restricted theory bounded by m in the language of signature $\Theta + \mathcal{C}$ extending A_0 . The theory U may have any complexity. Consider any sentence B in the language of signature $\Theta + \mathcal{C}$ of the form $B := \exists x \in \delta_N B_0(x)$. Let $n := \max(m, \rho_0(B) + c_{10}, \rho_0(\mathcal{S}) + c_{10})$. Suppose $U \vdash \exists x \in \mathfrak{S}_n B_0(x)$. Then, for some k , we have $U \vdash \exists x \leq^N k B_0(x)$, or, equivalently, $U \vdash \bigvee_{q \leq k} B_0(q)$.*

Proof. The theorem is immediate from Theorem 3.2, using compactness. \square

It is of course trivial to take the contraposition of Theorem 3.3. However this contraposition has some heuristic value. So we state it here as a separate theorem.

Theorem 3.4 *Suppose A_0 is a finitely axiomatized sequential theory in signature Θ with sequence scheme \mathcal{S} . Let m be any number such that $m \geq \rho_0(A_0)$. Let \mathcal{C} be a set of domain constants: \mathcal{C} is allowed to have any cardinality. Let U be a restricted theory bounded by m in the language of signature $\Theta + \mathcal{C}$ extending A_0 . The theory U may have any complexity. Consider any formula $C(x)$. Let $n := \max(m, \rho_0(C) + c_{10}, \rho_0(\mathcal{S}) + c_{10})$. If the theory $U + \{C(q) \mid q \in \omega\}$ is consistent, then the theory $U + \forall x \in \mathfrak{S}_n C(x)$ is consistent.*

Proof. We apply Theorem 3.3 to $\exists x \in \delta_N \neg C(x)$ and take the contraposition. \square

4 A conservativity result

We can use the machinery we built up to prove a Lindström-style result on conservative extensions.

Suppose U is a restricted, sequential, recursively enumerable theory with sequence scheme \mathcal{S} . Let p be a bound for the complexity of the axioms of U . By Craig's trick, we can give a Σ_1^b -axiomatization of U . Say the Σ_1^b -formula representing the axioms is η . Suppose A_0 is a finite subtheory of U such that \mathcal{S} makes A_0 sequential. Clearly U can be axiomatized by $A_0 + \{\eta^N(\underline{q}) \rightarrow \text{true}_p(\underline{q}) \mid q \in \omega\}$. This representation of the axiom set leads immediately to the following theorem.

Theorem 4.1 *Suppose U is a restricted, sequential, recursively enumerable theory. Consider any number m . Then there is a finitely axiomatized sequential theory A in the same language that extends U and is Δ_m^* -conservative over U .*

Proof. By our observations there is a finitely axiomatized sequential theory A_0 and a formula $B(x)$ such that U can be axiomatized as $A_0 + \{B(\underline{q}) \mid q \in \omega\}$. Let $n := \max(\rho_0(A_0), \rho_0(B) + c_{10}, m + c_{10}, \rho_0(\mathcal{S}) + c_{10})$.

We take $A := A_0 + \forall x \in \mathfrak{S}_n B(x)$. We note that A is a finitely axiomatized extension of U . Consider any $C \in \Delta_m^*$. Suppose $U \not\vdash C$. Then, the theory $A_0 + \{(B(\underline{q}) \wedge \neg C) \mid q \in \omega\}$ is consistent. We may conclude that $A_0 + \forall x \in \mathfrak{S}_n (B(x) \wedge \neg C)$ is consistent. In other words, we find $A \not\vdash C$. \square

We have the following corollary.

Corollary 4.2 *Suppose U is a restricted, sequential, recursively enumerable theory. Suppose further that D is a finite extension of U such that $U \not\vdash D$. Then, there is a finite extension D' of U , such that $D \vdash D'$ but $D' \not\vdash D$.*

Proof. Let m be a ρ_0 -bound on U and on D . Let A be the sentence promised in Theorem 4.1 for Δ_m^* . Let $D' := D \vee A$. Clearly $D \vdash D'$ and $D' \vdash U$. Suppose $D' \vdash D$. Then, it follows that $A \vdash D$, contradicting the Δ_m^* -conservativity of A over U . \square

Since, as is well-known, the finitely axiomatized sequential theories in the signature of U are dense with respect to \neg , it follows that we can add to the statement of the Corollary that $U \not\vdash D'$: in case the D' provided by the theorem would happen to axiomatize U , we simply replace it by a D'' strictly between the original D' and D .

Here is one more corollary.

Corollary 4.3 Consider any finitely axiomatized, sequential theory A in signature Θ . Suppose that for some class of Θ -sentences Ω we have a definable predicate TRUE such that, for any Ω -sentence B , we have $A \vdash B \leftrightarrow \text{TRUE}(\ulcorner B \urcorner)$. Let X be any recursively enumerable set of Ω -sentences. Then there is a finite extension A^+ of $A + X$ such that A^+ is Ω -conservative over $A + X$.

Proof. Clearly, the theory $A + \{\text{TRUE}(B) \mid B \in X\}$ is restricted. We apply Theorem 4.1 taking $m := \rho_0(\text{TRUE}(x)) + 1$. \square

We give two examples of applications of the result.

Example 4.4 Let U be any recursively enumerable extension of PA. Then, there is a finite extension A of ACA_0 such that the arithmetical consequences of A are precisely the consequences of U . Similarly for the pair ZF and GB. (This result was previously proven by Van Wesep in his paper [19].)

Example 4.5 By Parsons's result $\text{I}\Sigma_1$ is Π_2 -conservative over PRA.¹¹ Since, over EA, we have Σ_m -truth predicates, it follows that, for every m , we have a finite extension A_m of PRA that is Σ_m -conservative. We can easily arrange that these extensions become strictly weaker when m grows.

We refer the reader to [12] for a number of results in the same niche using a different methodology.

5 Standardness regained

Finiteness is Predicate Logic's nemesis. However hard Predicate Logic tries, there is no way it can pin down the set of standard numbers. What happens when we invert the question? *Are there theories that interpret some basic arithmetic that do not have models in which the standard numbers are interpretable?* The answer is a resounding *yes*. E.g., $\text{PA} + \text{incon}(\text{PA})$ has no models in which the standard numbers are interpretable. More generally, consider any recursively enumerable consistent theory U with signature Θ . Suppose the signature of arithmetic is Ξ . Then, the theory $U + \{(\bigwedge (S_2^1)^\tau \rightarrow \text{incon}^\tau(U)) \mid \tau : \Xi \rightarrow \Theta\}$ is consistent and does not have any models that have an internal model isomorphic to the standard numbers.¹²

The situation changes when we put some restriction on the complexity of the axioms of the theory. The classical work concerning this idea is the beautiful paper by McAloon [10]. McAloon shows that arithmetical theories with axioms of restricted complexity *that are consistent with PA* always have a model in which the standard integers are definable. McAloon's work was further extended by Adamowicz, Cerdón-Franco, and Lara-Martín [1].

Our aim in this paper is to find an analogue of McAloon's Theorem that works for all sequential theories. We prove a result that is more general in scope but, at the same time, substantially weaker in its statement. We show that any consistent restricted sequential theory U has a model in which the *intersection of all definable cuts* is isomorphic to the standard natural numbers. This intersection is not generally itself definable in the model. We shall show that *the intersection of all definable cuts* is a good notion that, for sequential theories, is not dependent on the original choice of the interpretation of number theory.

5.1 The intersection of all definable cuts

In this subsection, we establish that *the intersection of all definable cuts* is a good notion. A *sequential model* is a model such that the theory of the model is sequential. Consider a sequential model \mathcal{M} . Let \mathcal{N} be an \mathcal{M} -internal model satisfying S_2^1 . Let $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ be the intersection of all \mathcal{M} -definable \mathcal{N} -cuts in \mathcal{M} .

Now consider two \mathcal{M} -internal models \mathcal{N} and \mathcal{N}' satisfying S_2^1 . By a result of Pudlák [14], there is an \mathcal{M} -definable isomorphism \mathcal{F} between an \mathcal{M} -definable cut \mathcal{I} of \mathcal{N} and an \mathcal{M} -definable cut \mathcal{I}' of \mathcal{N}' . It is easily seen that \mathcal{F} restricted to $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ is an isomorphism between $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ and $\mathcal{J}_{\mathcal{M},\mathcal{N}'}$.

Suppose \mathcal{G} and \mathcal{H} are two \mathcal{M} -definable partial functions between \mathcal{N} and \mathcal{N}' such that the restrictions of \mathcal{G} and \mathcal{H} to $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ commute with zero and successor. Then it is easy to see that \mathcal{G} and \mathcal{H} are extensionally equal

¹¹ We consider a version of PRA in the original arithmetical language here.

¹² We assume that the axioms of identity are part of the axiomatization of S_2^1 .

isomorphisms between $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ and $\mathcal{J}_{\mathcal{M},\mathcal{N}'}$. Thus, in a sense, there is a unique definable isomorphism between $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ and $\mathcal{J}_{\mathcal{M},\mathcal{N}'}$. The above observations justify the notation $\mathcal{J}_{\mathcal{M}}$ for $\mathcal{J}_{\mathcal{M},\mathcal{N}}$ modulo isomorphism.

We note that, in the definition of $\mathcal{J}_{\mathcal{M}}$ it does not matter whether we allow parameters in the definition of the cuts. Every cut with parameters has a parameter-free shortening. Suppose \mathcal{I} is an \mathcal{N} -cut that is given by $I(x, \vec{b})$. Then, $I^*(x) := \forall \vec{y} (\text{cut}(\{z \mid I(z, \vec{y})\}) \rightarrow I(x, \vec{y}))$ defines a cut \mathcal{I}^* that is a shortening of \mathcal{I} .

What happens if the sequence scheme \mathcal{S} itself involves parameters? In [25], it is shown that these parameters can be eliminated by raising the dimension of the interpretation. Since the standard development of an interpretation of S_2^1 in a sequential theory does not involve parameters, it follows that even in a sequential theory with a sequence scheme involving parameters, there is an interpretation of S_2^1 that is parameter-free! However, the cost of this fact is that there may be no such interpretation that is one-dimensional.

Before going on with the main line of our story, we want to give some basic facts about $\mathcal{J}_{\mathcal{M}}$ in order to place it in perspective. We write con_n for the notion of consistency corresponding to provability where all formulas in the proof are restricted to formulas of ρ -complexity n .¹³

Theorem 5.1 *Suppose \mathcal{M} is a sequential model. We have that $\mathcal{J}_{\mathcal{M}} \models \text{EA} + \text{B}\Sigma_1 + \{\text{con}_n(A) \mid n \in \omega \text{ and } \mathcal{M} \models A\}$.*

Proof. Suppose N is, as before, the interpretation given by the sequence scheme \mathcal{S} that defines an internal S_2^1 -model of \mathcal{M} . We shall consider $\mathcal{J}_{\mathcal{M}}$ as the intersection of all N -cuts. We again write \mathfrak{I}_n for $(\mathfrak{I}_n(\Theta))^{\mathcal{S}_\Theta}$. Since there is an N -cut J such that on J we have $W := \text{I}\Delta_0 + \Omega_1 + \text{B}\Sigma_1$ and since this property is downwards preserved, we find that $\mathcal{J}_{\mathcal{M}} \models W$. Suppose $\mathcal{M} \models A$. Without loss of generality, we can assume that $n \geq \rho_0(A)$. Clearly, we have $\mathcal{M} \models \text{con}_n^{\mathfrak{I}_n}(A)$. Hence, by downwards persistence, $\mathcal{J}_{\mathcal{M}} \models \text{con}_n(A)$.

Finally, consider any a in $\mathcal{J}_{\mathcal{M}}$. Consider any N -cut I . There is an N -cut I' such that for every b in I' , we have 2^b is in I . Since a is in I' , it follows that 2^a is in I . Since I was arbitrary, we have 2^a is in $\mathcal{J}_{\mathcal{M}}$. \square

Let $\mathfrak{I}_U := \text{Th}(\{\mathcal{J}_{\mathcal{M}} \mid \mathcal{M} \models U\})$, and let $\mathfrak{U}_U := S_2^1 + \{\text{con}_n(U \upharpoonright n) \mid n \in \omega\}$. Here con_n refers to consistency for provability in which all formulas in a proof are restricted to ρ -complexity n and $U \upharpoonright n$ is the theory axiomatized by the axioms of U whose Gödel numbers are $\leq n$. Then, we have the following corollary.

Corollary 5.2 $\mathfrak{I}_U \vdash \text{EA} + \text{B}\Sigma_1 + \mathfrak{U}_U$.

The next theorem is a kind of overspill principle.

Theorem 5.3 *Suppose \mathcal{M} is a sequential model and M defines an internal S_2^1 -model of \mathcal{M} . We treat $\mathcal{J}_{\mathcal{M}}$ as the intersection of all M -cuts. Let $B(x)$ be any formula. We have that for all b in $\mathcal{J}_{\mathcal{M}}$, $\mathcal{M} \models B(b)$ if and only if for some M -cut J , $\mathcal{M} \models \forall x \in J B(x)$.*

Proof. The right-to-left direction is trivial. Suppose for all b in $\mathcal{J}_{\mathcal{M}}$, we have $\mathcal{M} \models B(b)$. Let $X := \{x \in \delta_{\mathcal{M}} \mid \forall y <^M x B(y)\}$. If X is closed under successor, then we can shorten X to an M -cut J for which we have $\forall x \in J B(x)$, and we are done. Otherwise, there is a c such that $\neg B(c) \wedge \forall y < c B(y)$. Since c cannot be in $\mathcal{J}_{\mathcal{M}}$, it follows that there is a cut J below c . \square

Our overspill principle immediately gives the following result.

Theorem 5.4 *Suppose \mathcal{M} is a sequential model and M defines an internal S_2^1 -model of \mathcal{M} . Let P be a Π_1 -formula. Then, $\mathcal{J}_{\mathcal{M}} \models P$ if and only if, for some M -cut J , $\mathcal{M} \models P^J$.*

Corollary 5.5 *Let P be a Π_1 -formula. Then, $\mathfrak{I}_U \vdash P$ if and only if, for some $M : S_2^1 \triangleleft U$, we have $U \vdash P^M$.*

Proof. The proof of the corollary is by a simple compactness argument. \square

Open Question 5.6 Any further information on \mathfrak{I}_U would be interesting. E.g., what are the possible complexities of \mathfrak{I}_U for recursively enumerable sequential theories U ?

¹³ Elsewhere we use con^n for the n times iterated con and con^M for the relativization of con to the interpretation M .

5.2 ω -models

Before proceeding, we briefly reflect on the notion of ω -model. The common practice is to say, e.g., that \mathcal{M} is an ω -model of ZF if the von Neumann numbers of \mathcal{M} are (order-)isomorphic to ω . Of course, there are other interpretations M of arithmetic in ZF. However, we have the feature that if the M -numbers are isomorphic to ω , then so are the von Neumann numbers—but not *vice versa*. If we consider GB instead of ZF we do not know whether this feature is preserved. It is conceivable that, in some model, a definable cut of the von Neumann numbers is isomorphic to ω and the von Neumann numbers are not.

It seems to us that the proper codification of the common practice would be to say that an ω -model is not strictly a model but a pair $\langle \mathcal{M}, M \rangle$ of a model and an interpretation M of a suitable arithmetic in \mathcal{M} such that $\tilde{M}(\mathcal{M})$ is isomorphic to ω .

Of course there is the option of existentially quantifying out the choice of the interpretation of arithmetic. Let us say that \mathcal{M} is an e - ω -model, if for some M , $\langle \mathcal{M}, M \rangle$ is an ω -model.

Finally, in the sequential case, there is a third option. We define: a sequential model \mathcal{M} is an *i*- ω -model if $\mathcal{J}_{\mathcal{M}}$ is isomorphic to the standard numbers. (The letter *i* stands for “intersection”.) In other words, \mathcal{M} is an *i*- ω -model if, for some interpretation M of S_2^1 , for every non-standard element a , there is an \mathcal{M} -definable M -cut I such that $I < a$. We have the following property of *i*- ω -models.

Theorem 5.7 *Suppose \mathcal{M} is a sequential i- ω -model and $M : S_2^1 \triangleleft \mathcal{M}$. Let X be a parametrically definable class of M -numbers. Suppose $\omega \subseteq X$. (We confuse the standard part of M with ω .) Then there is a \mathcal{M} -definable M -cut J such that $J \subseteq X$.*

Proof. Suppose $\omega \subseteq X$. Consider the class $Y := \{a \in N \mid \forall b \leq a \ b \in X\}$. In case Y is closed under successor we can shorten it to a definable cut, and we are done. In case Y is not closed under successor, there is an a_0 such that $\forall b \leq a_0 \ b \in X$ but $Sa_0 \notin X$. By our assumption $\omega < a_0$. Hence there must be a definable cut I with $\omega \leq I < a_0$. So, $I \subseteq X$. \square

5.3 The main result

If we are content with the countable case, our main result is a simple application of the Omitting Types Theorem. We first give this easier proof.

Theorem 5.8 *Let U be a consistent restricted sequential theory. Here U may be of any complexity. We allow countably many constants in U . Then, U has a model \mathcal{M} in which $\mathcal{J}_{\mathcal{M}}$ is isomorphic to the standard natural numbers.*

Proof. We fix a sequence scheme S for U . We work with the interpretation N of S_2^1 provided by this scheme. Suppose that in all countable U -models the type $\mathcal{T}(x) := \{x \neq \underline{n} \mid n \in \omega\} \cup \{x \in \mathfrak{S}_n \mid n \in \omega\}$ is realized. Then, by the Omitting Types Theorem, there is a formula $A(x)$, such that, for a fresh constant c , we have (i) $U + A(c)$ is consistent and (ii) $U + A(c) \vdash c \neq \underline{n}$, for each $n \in \omega$, and (iii) $U + A(c) \vdash c \in \mathfrak{S}_n$, for each $n \in \omega$.

We apply Theorem 3.4 to (i) and (ii) obtaining that, for some n^* , the theory $U + A(c) + \forall x \in \mathfrak{S}_{n^*} \ c \neq x$ is consistent. However, this directly contradicts (i) and (iii).

We may conclude that there is a countable model \mathcal{M} in which $\mathcal{T}(x)$ is omitted. Clearly, this tells us that $\mathcal{J}_{\mathcal{M}}$ is isomorphic to the standard numbers. \square

We proceed to prove the stronger version of our theorem where the restriction to countability is lifted. We first prove a lemma.

Lemma 5.9 *Let \mathcal{M} be any sequential model of signature Θ with domain M . Let S be a sequence scheme for \mathcal{M} . As usual, N is the interpretation of S_2^1 given by the sequence scheme. Let k be any number. We note that the ρ_0 -complexity of the axioms of AS^+ is a fixed number, say s . So the sequentiality of \mathcal{M} is witnessed by the satisfaction of a sentence D of complexity below $s + \rho_0(S)$. Let $n := \max(k, s + \rho_0(S), 1 + \epsilon_{10}, \rho_0(S) + \epsilon_{10})$. Then, \mathcal{M} has a sequential Δ_k^* -elementary extension \mathcal{K} with sequence scheme S such that $\mathfrak{S}_n^{\mathcal{K}} \cap M = \omega$.*

Proof. Without loss of generality we may assume that $k \geq s + \rho_0(S)$, so that $\Gamma := \text{Th}_{\Delta_k^*(M)}(\mathcal{M})$ is sequential. We claim that $\Gamma^* := \Gamma + \{\mathfrak{S}_n < m \mid \mathcal{M} \models m \in \delta_N \wedge \omega <^N m\}$ is consistent (for n as given in the statement

of the lemma). If not, then we have $\Gamma \vdash m_0 \in \mathfrak{S}_n \vee \dots \vee m_{\ell-1} \in \mathfrak{S}_n$, for some nonstandard $m_0, \dots, m_{\ell-1}$ in M . Let m be the minimum of the m_i . We find $\Gamma \vdash m \in \mathfrak{S}_n$. On the other hand, the theory $\Gamma + 0 < m, 1 < m, \dots$ is consistent. Hence, by Theorem 3.4, we have that $\Gamma + \mathfrak{S}_n < m$ is consistent. A contradiction.

Let \mathcal{K} be a model of Γ^* . Clearly, in \mathcal{K} , we have that \mathfrak{S}_n is below all non-standard elements inherited from \mathcal{M} (but, of course, not necessarily below new non-standard elements). Also \mathcal{K} is, by construction, a Δ_n^* -elementary extension. Finally, since we have chosen $n \geq s + \rho_0(\mathcal{S})$, the model \mathcal{K} is again sequential with the same sequence scheme. \square

With the lemma in hand, we can now prove the promised theorem using a limit construction.

Theorem 5.10 *Let \mathcal{M} be any sequential model. Then, for any k , \mathcal{M} has a Δ_k^* -elementary extension \mathcal{K} in which $\mathcal{J}_{\mathcal{K}}$ is (isomorphic to) ω .*

Proof. Let \mathcal{S} be a sequence scheme for \mathcal{M} . We work with the numbers N provided by this scheme. Let s be as in Lemma 5.9. We define

$$n_0 := \max(k, s + \rho_0(\mathcal{S}));$$

$$n_{j+1} := \max(\rho_0(\mathfrak{S}_j) + 1, s + \rho_0(\mathcal{S}), 1 + c_{10}, \rho_0(\mathcal{S}) + c_{10}).$$

(We note that $\rho_0(\mathfrak{S}_j) \approx c_0 j + c_9$ and that for $j > 0$, we have $n_{j+1} := \rho_0(\mathfrak{S}_j) + 1$.) We construct a chain of models \mathcal{M}_i . Let $\mathcal{M}_0 = \mathcal{M}$. Suppose we have constructed \mathcal{M}_j . We now take as \mathcal{M}_{j+1} a model that is a $\Delta_{n_{j+1}}^*$ -elementary extension of \mathcal{M}_j such that $\mathfrak{S}_{n_{j+1}}^{\mathcal{M}_{j+1}} \cap \mathcal{M}_j = \omega$.

Let \mathcal{K} be the limit of $(\mathcal{M}_i)_{i \in \omega}$. Consider any non-standard element a in $\tilde{N}(\mathcal{K})$. We have to show that there is a \mathcal{K} -definable cut below it. Suppose a occurs in \mathcal{M}_j . We have, by the construction of our sequence, that $(\dagger) \mathfrak{S}_{n_j}^{\mathcal{M}_{j+1}} < a$. By the fact that all \mathcal{M}_s , with $s > j + 1$ are $\Delta_{n_{j+1}}^*$ -elementary extensions of \mathcal{M}_{j+1} , it follows that (\dagger) is preserved to the limit: $\mathfrak{S}_{n_{j+1}}^{\mathcal{K}} < a$. \square

From Theorem 5.10, we have immediately the desired strengthening of Theorem 5.8.

Theorem 5.11 *Let U be a consistent restricted sequential theory. Here U may be of any complexity. We allow a number of constants in U of any cardinality. Then U has a model \mathcal{M} in which $\mathcal{J}_{\mathcal{M}}$ is isomorphic to the standard natural numbers.*

Two remarks are in place:

Firstly, from Theorem 5.11, we retrace our steps and derive a less explicit form of Theorem 3.4. Let U be a restricted sequential theory and consider any $C(x)$. Suppose $V := U + \{C(\underline{n}) \mid n \in \omega\}$ is consistent. Let \mathcal{M} be a model of V in which $\mathcal{J}_{\mathcal{M}}$ is isomorphic to the standard natural numbers. By Theorem 5.7, there is a definable \mathcal{M} -cut J so that, in \mathcal{M} , we have $\forall x \in J C(x)$.

Now J is a definable cut in \mathcal{M} , but it need not automatically be a cut in U . There is a standard trick to remedy that. We define $J^\circ := J(\text{cut}(J))N$. Clearly, J° is a definable cut in U . Moreover, in the context of \mathcal{M} , the cuts J and J° coincide. By the above considerations, it follows that $U + \forall x \in J^\circ Cx$ is consistent, where J° is a U -definable cut.

For the second remark, consider any model \mathcal{M} . We define $\text{DEF}(\mathcal{M})$ as the class of (parametrically) definable classes of \mathcal{M} . We define $\text{DEF}^-(\mathcal{M})$ as the class of classes over \mathcal{M} that are definable without parameters. Also, $\text{DEF}_n(\mathcal{M})$ is the class of (parametrically) definable n -ary relations and, similarly, for the parameter-free case.

It would seem that Theorem 5.10 gives us information about possible sequential models $\langle \mathcal{M}, \text{DEF}(\mathcal{M}) \rangle$, since $\mathcal{J}_{\mathcal{M}}$ is definable in $\langle \mathcal{M}, \text{DEF}(\mathcal{M}) \rangle$. However, this is not so, since we have a much stronger result for the models $\langle \mathcal{M}, \text{DEF}(\mathcal{M}) \rangle$, where \mathcal{M} is sequential.

We assume that \mathcal{M} has finite signature, where we allow an infinity of constants. In each model $\langle \mathcal{M}, \text{DEF}(\mathcal{M}) \rangle$, where \mathcal{M} satisfies these demands, the natural numbers are definable. The argument is simple. Let $\text{comm}_x(X)$ mean that X satisfies the commutation conditions for satisfaction of Δ_x^* -formulas in $\mathcal{J}_{\mathcal{M}}$. Consider, in the model $\langle \mathcal{M}, \text{DEF}(\mathcal{M}) \rangle$ the class $Y := \{x \in N \mid \exists X \text{comm}_x(X)\}$. Clearly, each standard x is in Y . If a non-standard number b would be in Y , the defining formula for the witnessing X would violate Tarski's Theorem of the undefinability of truth for \mathcal{M} .¹⁴

¹⁴ Enayat tells us that the basic idea of this argument is originally due to Mostowski.

If the sequence scheme for \mathcal{M} is parameter-free, then the same argument works for $\langle \mathcal{M}, \text{DEF}^-(\mathcal{M}) \rangle$. If the sequence scheme contains parameters, we can make the argument work for $\langle \mathcal{M}, \text{DEF}_n^-(\mathcal{M}) \rangle$, for sufficiently large n .

Finally, note that if \mathcal{M} is a non-standard model of Peano Arithmetic, then $\mathcal{J}_{\mathcal{M}}$ is simply isomorphic to \mathcal{M} itself. Thus, adding $\mathcal{J}_{\mathcal{M}}$ (viewed as intersection of all cuts on the identical interpretation of S_2^1) to \mathcal{M} does not increase the expressiveness of the language. This consideration shows that adding the definable sets can be more expressive than adding $\mathcal{J}_{\mathcal{M}}$ (as intersection of the cuts for a given interpretation of S_2^1).

6 Reflection

If we apply Theorem 3.3 to a formula of a special form, we get a reflection principle.

Theorem 6.1 *Consider any consistent, restricted, sequential theory U with sequence scheme S . Let N be the interpretation of the numbers provided by S . Let m_0 be the bound for U and let m_1 be any number. Let $n := \max(m_0, m_1 + \rho_0(S) + c_{10})$. Then, for every Σ_2 -sentence C of the form $C = \exists x C_0(x)$, where C_0 is Π_1 and $\rho_0(C) \leq m_1$, we have: if $U \vdash \exists x \in \mathfrak{S}_n C_0^N(x)$, then C is true.*

Proof. Under the assumptions of the theorem, we suppose $U \vdash \exists x \in \mathfrak{S}_n C_0^N(x)$. By Theorem 3.3, there is a k such that $U \vdash \bigvee_{q \leq k} C_0^N(q)$. Suppose C is false. Then, for each $q \leq k$, we have $\neg C_0(q)$. Hence, by Σ_1 -completeness, for each $q \leq k$, we have $U \vdash \neg C_0^N(q)$. It follows that $U \vdash \perp$. *Quod non.* \square

We note that the above proof uses Σ_1^0 -collection in the metalanguage.

If we take the formulas still simpler, we can improve the above result. We fix a logarithmic S_2^1 -cut \tilde{J} . There is a Σ_1 -truth predicate, say True , for Σ_1 -sentences, such that, for any Σ_1 -sentence S , we have $S_2^1 \vdash \text{True}(S) \rightarrow S$ and $S_2^1 \vdash S^{\tilde{J}} \rightarrow \text{True}(S)$. (Cf., e.g., [6, Part V, Chapter 5b] for details.)

Theorem 6.2 *Consider any consistent, restricted, sequential theory U with sequence scheme S and bound m . Let $n := \max(m, \rho_0(\text{True}(x)) + \rho_0(S) + c_{10} + 1)$. For all Σ_1 -sentences S , we have: if $U \vdash S^{\tilde{J}_{\mathfrak{S}_n}}$, then S is true.*

Proof. Suppose $U \vdash S^{\tilde{J}_{\mathfrak{S}_n}}$. Then, we have $U \vdash \text{True}^{\mathfrak{S}_n}(S)$. By Theorem 6.1, this implies $\text{True}(S)$ is true, whence S is true. \square

7 Degrees of interpretability

In this section, we apply our results to study the joint degree structure of local and global interpretability for recursively enumerable sequential theories. The main result of this section is a characterization of finite axiomatizability in terms of the double degree structure. We shall study the degree structures as partial pre-orderings.

7.1 The basic idea

The degree structures we are interested in are *the degrees of global interpretability of recursively enumerable sequential theories* Glob_{seq} and *the degrees of local interpretability of recursively enumerable sequential theories* Loc_{seq} . It is well known that both structures are distributive lattices. We have the obvious projection functor π from Glob_{seq} onto Loc_{seq} .

Let Fin be the property of global degrees of *containing a finitely axiomatized theory*. What we want to show is that, if we start with the pair Glob_{seq} and Loc_{seq} and with the projection π , then we can define Fin (using a first-order formula).

The basic idea is simple. Zoom in on a local degree of U . This degree contains a distributive lattice, say, \mathcal{L} of global degrees. The lattice \mathcal{L} has a maximum, to wit \mathcal{U}_U . Does it have a minimum? Well, if there is a finitely axiomatized theory U_0 in \mathcal{L} , then its global degree shall automatically be the minimum. We shall see that (i) not in all cases a minimum degree of \mathcal{L} exists and (ii) if such a minimum exists it contains a finitely axiomatizable

theory. In other words, the mapping φ with $\varphi(U) \triangleleft_{\text{glob}} V \Leftrightarrow U \triangleleft_{\text{loc}} \pi(V)$ is partial. However, if it has a value, this value contains a finitely axiomatized theory. So, we can define: $\text{Fin}(U)$ if and only if, for all V , we have $U \triangleleft_{\text{glob}} V$ if and only if $\pi(U) \triangleleft_{\text{loc}} \pi(V)$.

Open Question 7.1 Can we define Fin in Glob_{seq} alone?

In the context of local degrees of arbitrary theories with arbitrarily large signatures, Mycielski, Pudlák and Stern characterize loc-finite as the same as *compact* in terms of the $\triangleleft_{\text{loc}}$ -ordering. This will not work in our context of global interpretability and recursively enumerable sequential theories. We briefly give the argument that, in our context, every non-minimal globally finite degree is non-compact.

Let A be any \triangleleft -non-minimal, finitely axiomatized, sequential theory. Let B_i be an enumeration of all finitely axiomatized sequential theories. Let $C_0 := S_2^1$. We note that, by our assumption, $C_0 \not\triangleleft A$. Let $C_{n+1} := B_n$ if $C_n \not\triangleleft B_n \triangleleft A$ and $C_{n+1} := C_n$ otherwise. Clearly, $A \triangleright C_i$, for all i . Consider any sequential recursively enumerable theory U such that $U \triangleright C_i$, for all i . Without loss of generality, we may assume that the signatures of U and A are disjoint. We easily see that the theory U^* axiomatized by $\{(D \vee A) \mid D \text{ is an axiom of } U\}$ is the infimum in the degrees of global interpretability of U and A . So $C_i \triangleleft U^* \triangleleft A$. Suppose $U^* \not\triangleleft A$. In this case, by [29, Theorem 5.3], we can find a finitely axiomatized sequential B such that $U^* \triangleleft B \triangleleft A$. Let $B = B_j$. Since $C_j \triangleleft U^* \triangleleft B \triangleleft A$, we shall have $C_{j+1} := B_j$. A contradiction. It follows that $U^* \equiv A$. We may conclude that $A \triangleleft U$. So, A is the supremum of the C_i . Clearly, A cannot be the supremum of a finite number of the C_i . Thus, A is not compact.

7.2 Preliminaries

We consider a recursively enumerable theory U . By Craig's Theorem, we can give U a Σ_1^b -definable axiomatization X . Let this axiomatization be given by a Σ_1^b -formula η . We define $U \upharpoonright n$ as the theory axiomatized by the axioms of U , as given by η that are $\leq n$.

An important functor is \bar{U} . We remind the reader that: $\bar{U}_U := S_2^1 + \{\text{con}_n(U \upharpoonright n) \mid n \in \omega\}$. We note that \bar{U}_U is extensionally independent of the choice of η .

One can show that \bar{U} is the right adjoint of π : $U \triangleleft_{\text{glob}} \bar{U}_V \Leftrightarrow \pi(U) \triangleleft_{\text{loc}} V$. Cf., e.g., [23, 29].

A theory is *glob-finite* iff it is mutually globally interpretable with a finitely axiomatized theory. A theory is *loc-finite* iff it is mutually locally interpretable with a finitely axiomatized theory. If we enrich Glob_{seq} with a predicate Fin for the globally finite degrees, we have a first-order definition of Loc_{seq} over this structure as follows:

$$U \triangleleft_{\text{loc}} V \Leftrightarrow \forall A \in \text{Fin} (A \triangleleft_{\text{glob}} U \Rightarrow A \triangleleft_{\text{glob}} V).$$

Here is a first basic insight.

Theorem 7.2 *The theory U is a -finite, for $a \in \{\text{glob}, \text{loc}\}$, if and only if $U \equiv_a U \upharpoonright n$, for some n .*

Proof. Suppose $U \equiv_a V$, where V is finitely axiomatized. Clearly, $U \upharpoonright n \triangleright V$, for some n . We have that $U \supseteq U \upharpoonright n \triangleright V \triangleright_a U$; and we are done. \square

7.3 Some examples

Before formulating and proving our main result, we briefly pause to provide a few examples.

Example 7.3 The theories $\text{I}\Delta_0$ and $S_2 = \text{I}\Delta_0 + \Omega_1$ are examples of theories of which the finite axiomatizability is an open problem, but which are, by an argument of Alex Wilkie, *glob-finite*. They are, e.g., mutually interpretable with the finitely axiomatized sequential theory AS and with the finitely axiomatized sequential theory PA^- .¹⁵

We show that any global, recursively enumerable, sequential degree contains an element that is not finitely axiomatizable. Let α represent an axiom-set. We write $\Box_\alpha^0 B := B$, $\Box_\alpha^{n+1} B := \Box_\alpha \Box_\alpha^n B$ and $\text{con}^n(\alpha) := \neg \Box_\alpha^n \perp$.

¹⁵ Cf. [25] for some historical notes on the mutual interpretability of Q and AS . The fact that Q interprets $\text{I}\Delta_0$ and $\text{I}\Delta_0 + \Omega_1$ is due to Wilkie. The sequentiality of PA^- is proved in [7].

	loc-finite	glob-finite	fin. axiom.
GB	+	+	+
$I\Delta_0, S_2$	+	+	?
GB°	+	+	—
\mathcal{U}_{GB}	+	—	—
PA	—	—	—

Fig. 2 Separating Sequential Local and Global Finiteness

Theorem 7.4 Consider any consistent, sequential, recursively enumerable theory U . Then there is a $U^\circ \equiv U$ such that U° is sequential and recursively enumerable and not finitely axiomatizable.

Proof. Consider a consistent, sequential and recursively enumerable theory U . In case U is not finitely axiomatizable, we are done, taking $U^\circ := U$. Suppose U is finitely axiomatizable, say by a single sentence A . Par abus de langage, we write A also for the theory axiomatized by $x = \ulcorner A \urcorner$.

By Theorem 6.2, we can find $M : S_2^1 \triangleleft A$ for which A is Σ_1^0 -sound. We consider the theory $U^\circ := A + \{(\text{con}(A) \rightarrow \text{con}^{n+1}(A))^M \mid n \in \omega\}$. We have, by Feferman's version of the Second Incompleteness Theorem, $A \subseteq U^\circ \subseteq (A + \text{incon}^M(A)) \triangleleft A$. So $U^\circ \equiv A$. Suppose U° were finitely axiomatizable. Then, we would have, for some $n > 0$, $A + (\text{con}(A) \rightarrow \text{con}^n(A))^M \vdash (\text{con}(A) \rightarrow \text{con}^{n+1}(A))^M$. Hence $A \vdash (\Box_A^{n+1} \perp \rightarrow \Box_A^n \perp)^M$. So, by Löb's Theorem, $A \vdash (\Box_A^n \perp)^M$, contradicting the Σ_1^0 -soundness of A with respect to M . \square

Here is a sufficient condition for failure to be loc-finite. We define $\mathcal{U}_U^+ := S_2^1 + \{\text{con}(U \upharpoonright n) \mid n \in \omega\}$. Here $U \upharpoonright n$ is defined with respect to a chosen Σ_1^b -formula η that represents the axiom set of U . We call U *strongly loc-reflexive* if $U \triangleright \mathcal{U}_U^+$. We note that, e.g., PRA is an example of a strongly loc-reflexive theory.

Theorem 7.5 Suppose that U is strongly loc-reflexive. Then, U is not loc-finite.

Proof. Suppose $U \triangleright_{\text{loc}} \mathcal{U}_U^+$. Suppose, to obtain a contradiction, that U is loc-finite. Then, for some i , we have $U \upharpoonright i \triangleright_{\text{loc}} U$. Moreover, $U \triangleright (S_2^1 + \text{con}(U \upharpoonright i))$. So, $U \upharpoonright i \triangleright (S_2^1 + \text{con}(U \upharpoonright i))$, contradicting the second incompleteness theorem. \square

7.4 Characterizations

The following characterization of loc-finiteness may look like an *obscurum per obscurius*. However, it is a central tool in what follows.

Theorem 7.6 Suppose U is sequential. The variable I will range over S_2^1 -definable cuts. The following are equivalent:

1. U is loc-finite.
2. $\exists i \forall j \geq i \exists I \ S_2^1 + \text{con}_i(U \upharpoonright i) \vdash \text{con}_j^I(U \upharpoonright j)$.
3. $\exists i \forall j \geq i \exists I \ S_2^1 + \text{con}_j(U \upharpoonright j) \vdash \text{con}_{j+1}^I(U \upharpoonright (j+1))$.

Proof. (1) \Rightarrow (2). Suppose U is loc-finite. Then, $U \upharpoonright i \triangleright_{\text{loc}} U$, for some i . Consider any $j \geq i$. For some M , we have $M : U \upharpoonright i \triangleright U \upharpoonright j$. Using M , we can transform a $U \upharpoonright j$, j -inconsistency proof into an $U \upharpoonright i$, k -inconsistency proof, for a sufficiently large k . Thus, we have $S_2^1 \vdash \text{con}_k(U \upharpoonright i) \rightarrow \text{con}_j(U \upharpoonright j)$. On the other hand, for some I , $S_2^1 + \text{con}_i(U \upharpoonright i) \vdash \text{con}_k^I(U \upharpoonright i)$. This last step can be seen, e.g., from the fact that a cut-elimination that transforms a k -proof to an i -proof is multi-exponential (cf. [2]).¹⁶ It suffices to take as I an appropriate multi-logarithmic cut. We may conclude that $S_2^1 + \text{con}_i(U \upharpoonright i) \vdash \text{con}_j^I(U \upharpoonright j)$.

¹⁶ Alternatively, we can prove the step by a combination of the Interpretation Existence Lemma [29] and the local reflexivity of $U \upharpoonright i$ that is a direct consequence of Theorem 2.15 of the present paper.

(2) \Rightarrow (3) and (3) \Rightarrow (2) are trivial.

For (2) \Rightarrow (1), one shows that, for i as promised, $S_2^1 + \text{con}_i(U \upharpoonright i)$ is mutually locally interpretable with U . \square

We note that in the above proof, the precise point where sequentiality is used, is the insight that U interprets $S_2^1 + \text{con}_i(U \upharpoonright i)$. Thus, only (2) \Rightarrow (1) depends on sequentiality.¹⁷

By a result of Wilkie and Paris [30] (cf. also [23]), we have, for Σ_1 -sentences P and Q , that $\text{EA} + P \vdash Q$ if and only if, for some S_2^1 -cut I , we have $S_2^1 + P \vdash Q^I$. Hence, it follows that:

Corollary 7.7 *Suppose U is sequential. The following are equivalent:*

1. U is loc-finite.
2. $\exists i \forall j \geq i \text{ EA} + \text{con}_i(U) \vdash \text{con}_j(U)$.
3. $\exists i \forall j \geq i \text{ EA} + \text{con}_j(U) \vdash \text{con}_{j+1}(U)$.

7.5 The main theorem

Consider the partial preorder, say \mathcal{L} , of sequential degrees of global interpretability contained in a given degree of local sequential interpretability. Note that \mathcal{L} is closed under suprema and infima. Suppose our given local degree is not loc-finite. Then, \mathcal{L} does not have a minimum. This insight is formulated in the following theorem.

Theorem 7.8 *Suppose that U is a recursively enumerable, sequential theory that is not loc-finite. Then, there is a theory \tilde{U} , such that $\tilde{U} \equiv_{\text{loc}} U$, but $\tilde{U} \not\equiv_{\text{glob}} U$.*

Proof. Suppose that U is recursively enumerable, sequential and not loc-finite. Let the signature of U be Θ and let a sequence scheme for U be \mathcal{S} . As usual N is the standard interpretation of S_2^1 in U given by \mathcal{S} . We write \mathfrak{S}_i for $\mathfrak{S}_i^{\mathcal{S}}(\Theta)$. The complexities of the \mathfrak{S}_i are estimated by $c_0i + c_9 + \rho_0(\mathcal{S})$. Taking $c_{11} := c_9 + \rho_0(\mathcal{S})$, our estimate becomes: $c_0i + c_{11}$. By the choice of the \mathfrak{S}_i we have: $U \vdash \text{con}_i^{\mathfrak{S}_i}(U \upharpoonright i)$.

Consider S_2^1 . Say the signature of S_2^1 is Ξ and a sequence scheme that interprets the numbers identically is \mathcal{T} . We write \mathcal{I}_i for $\mathfrak{S}_i^{\mathcal{T}}(\Xi)$. The complexities of the \mathcal{I}_i are estimated by $c_0i + c_9 + \rho_0(\mathcal{T})$. Taking $c_{12} := c_9 + \rho_0(\mathcal{T})$, our estimate becomes $c_0i + c_{12}$.

We define the theory \tilde{U} as follows: let $F(i) := (c_0 + 1)i + 1$.¹⁸

$$\tilde{U} := S_2^1 + \{\text{con}_i^{\mathcal{I}_{F(i)}}(U \upharpoonright i) \mid i \in \omega\}.$$

Clearly, $\tilde{U} \equiv_{\text{loc}} U$.

Suppose, to obtain a contradiction, that, for some K , we have $K : \tilde{U} \triangleright U$. By Pudlák's theorem [14], there is a \tilde{U} -cut J of N and a \tilde{U} -cut J' of $K \circ N$ and a \tilde{U} -definable isomorphism G between J and J' . We define $J_i := G^{-1}[\mathfrak{S}_i^K \cap J']$. We clearly have that J_i is a \tilde{U} -cut and $\tilde{U} \vdash \text{con}_i^{J_i}(U)$. We note that

$$x \in J_i \Leftrightarrow \exists y (Gxy \wedge (y \in \mathfrak{S}_i^K \wedge y \in J')),$$

so that $\rho_0(J_i)$ is estimated by a linear term of the form $c_0i + c_{13}$. (Here c_{13} is dependent on $\rho_0(K)$.)

Consider any s . We shall make s more specific in the run of the argument. We have that $\tilde{U} \vdash \text{con}_{s+1}^{J_{s+1}}(U \upharpoonright (s+1))$. Hence, by compactness, for some $p \geq s$:

$$S_2^1 + \{\text{con}_i^{\mathcal{I}_{F(i)}}(U \upharpoonright i) \mid i \leq s\} + \{\text{con}_j^{\mathcal{I}_{F(j)}}(U \upharpoonright j) \mid s < j \leq p\} \vdash \text{con}_{s+1}^{J_{s+1}}(U \upharpoonright (s+1)) \quad (1)$$

Thus, it follows that:

$$S_2^1 + \text{con}_s(U \upharpoonright s) + \text{incon}_{s+1}^{J_{s+1}}(U \upharpoonright (s+1)) \vdash \text{incon}_p^{\mathcal{I}_{F(s+1)}}(U \upharpoonright p) \quad (2)$$

Since U is not loc-finite, we have, by Theorem 7.6(3), that $\forall i \exists j > i \forall I \ S_2^1 + \text{con}_i(U \upharpoonright j) \not\vdash \text{con}_{j+1}^I(U \upharpoonright (j+1))$. So, for arbitrarily large s 's, the theory $A_s := S_2^1 + \text{con}_s(U \upharpoonright s) + \text{incon}_{s+1}^{J_{s+1}}(U \upharpoonright (s+1))$ is consistent. We note that $\rho_0(A_s)$ is estimated by $c_0s + c_{14}$ for a suitable c_{14} .

¹⁷ This is not true anymore when we employ the argument of Footnote 16.

¹⁸ It is a sport to take the choice of F as sharp as possible. The argument below becomes a bit more relaxed if we take $F(i) := i^2 + 1$ and just keep track of linear dependencies.

Consider any s for which A_s is consistent. We remind the reader of Theorem 6.1. Applied to the case at hand this tells us the following. Let $n := \max(\rho_0(A_s), c_{15})$ (where c_{15} is a fixed constant). Then, if $A_s \vdash \text{incon}_p^{\mathcal{I}_n}(U \upharpoonright p)$, then $\text{incon}_p(U \upharpoonright p)$ is true. Since we assumed that U is consistent, it follows that $A_s \not\vdash \text{incon}_p^{\mathcal{I}_n}(U \upharpoonright p)$.

We note that n is estimated by $c_0s + c_{16}$ for a suitable c_{16} . We may choose s large enough so that $F(s+1) > c_0s + c_{16}$ and such that A_s is consistent. It follows that $\mathcal{I}_{F(s+1)}$ is a subcut of \mathcal{I}_n . Since, by Equation (2), $A_s \vdash \text{incon}_p^{\mathcal{I}_{F(s+1)}}(U \upharpoonright p)$, it follows that $A_s \vdash \text{incon}_p^{\mathcal{I}_n}(U \upharpoonright p)$. A contradiction. \square

Note that Theorem 7.8 implies that any local sequential degree that is not loc-finite contains an infinity of global degrees.

Open Question 7.9 Every element of Loc_{seq} contains an extension of S_2^1 (to wit an element of the form \mathcal{U}_U). Does every element of Glob_{seq} contain an extension of S_2^1 ?

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Appendix A: Parameters

In general, interpretations are allowed to have parameters. We shall briefly sketch how to add parameters to our framework. We first define a translation with parameters. The parameters of the translation are given by a fixed sequence of variables \vec{w} that we keep apart from all other variables. A translation is defined as before, but for the fact that now the variables \vec{w} are allowed to occur in the domain-formula and in the translations of the predicate symbols in addition to the variables that correspond to the argument places. Officially, we represent a translation $\tau_{\vec{w}}$ with parameters \vec{w} as a quintuple $\langle \Sigma, \delta, \vec{w}, F, \Theta \rangle$. The parameter sequence may be empty: in this case our interpretation is parameter-free.

An interpretation with parameters $K : U \rightarrow V$ is a quadruple $\langle U, \pi, E, \tau_{\vec{w}}, V \rangle$, where $\tau_{\vec{w}} : \Sigma_U \rightarrow \Sigma_V$ is a translation and π is a V -formula containing at most \vec{w} free. The formula π represents the parameter domain. E.g., if we interpret the hyperbolic plane in the Euclidean plane via the Poincaré interpretation, we need two distinct points to define a circular disk. These points are parameters of the construction, the parameter domain is $\pi(w_0, w_1) = (w_0 \neq w_1)$. (For this specific example, we can also find a parameter-free interpretation.) The formula E represents an equivalence relation on the parameter domain. In practice this is always pointwise identity for parameter sequences, but for reasons of theory one must admit other equivalence relations too. We demand: $\vdash \delta_{\tau_{\vec{w}}}(\vec{v}) \rightarrow \pi(\vec{w})$, $\vdash P_{\tau_{\vec{w}}}(\vec{v}_0, \dots, \vec{v}_{n-1}) \rightarrow \pi(\vec{w})$, $V \vdash \exists \vec{w} \pi(\vec{w})$, $V \vdash E(\vec{w}, \vec{z}) \rightarrow (\pi(\vec{w}) \wedge \pi(\vec{z}))$, that V proves that E represents an equivalence relation on the sequences forming the parameter domain, $\vdash E(\vec{w}, \vec{z}) \rightarrow \forall \vec{x} (\delta_{\tau_{\vec{w}}}(\vec{x}) \leftrightarrow \delta_{\tau_{\vec{z}}}(\vec{x}))$, $\vdash E(\vec{w}, \vec{z}) \rightarrow \forall \vec{x}_0, \dots, \vec{x}_{n-1} (P_{\tau_{\vec{w}}}(\vec{x}_0, \dots, \vec{x}_{n-1}) \leftrightarrow P_{\tau_{\vec{z}}}(\vec{x}_0, \dots, \vec{x}_{n-1}))$, and for all U -axioms A , $V \vdash \forall \vec{w} (\pi(\vec{w}) \rightarrow A^{\tau_{\vec{w}}})$.

We can lift the various operations in the obvious way. Note that the parameter domain of $N := M \circ K$ and the corresponding equivalence relation should be

$$\pi_N(\vec{w}, \vec{u}_0, \dots, \vec{u}_{k-1}) := \pi_M(\vec{w}) \wedge \bigwedge_{i < k} \delta_{\tau_M}(\vec{w}, \vec{u}_i) \wedge (\pi_K(\vec{u}))^{\tau_M, \vec{w}}.$$

$$E_N(\vec{w}, \vec{u}_0, \dots, \vec{u}_{k-1}, \vec{z}, \vec{v}_0, \dots, \vec{v}_{k-1}) := E_M(\vec{w}, \vec{z}) \wedge \bigwedge_{i < k} \delta_{\tau_M}(\vec{w}, \vec{u}_i) \wedge \bigwedge_{i < k} \delta_{\tau_M}(\vec{w}, \vec{v}_i) \wedge (E_K(\vec{u}, \vec{v}))^{\tau_M, \vec{w}}.$$