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Gibbs–non-Gibbs properties for n-vector lattice and mean-field models

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Abstract. We review some recent developments in the study of Gibbs and non-Gibbs properties of transformed n-vector lattice and mean-field models under various transformations. Also, some new results for the loss and recovery of the Gibbs property of planar rotor models during stochastic time evolution are presented.

1 Introduction

In the recent decade and a half there has been a lot of activity on the topic of non-Gibbsian measures. Most of the original studies were based on the question of whether renormalised Hamiltonians exist as properly defined objects, [4,15,16], with an emphasis on discrete-spin models. Another issue, which also arose in physics but somewhat later [30], was the following question: Apply a (stochastic) dynamics which converges to a system at a temperature T_1 to an initial state at temperature T_2 for a finite time. Is the resulting measure in this transient nonequilibrium regime a Gibbs measure? Could it be described in terms of an effective temperature (hopefully between initial and final one)? Again the first results [10] were for discrete spins. Afterwards more general dynamics and also unbounded spins were investigated in [2,24,27]. Although the work of [2,24] was about continuous spins, there remains something of a problem, in that for unbounded spins the notion of what one should call Gibbsianness for a "reasonable" interaction is less clear than in the compact case. Thus it turned out to be of interest to see how a model with compact but continuous spins behaves. Another extension of the original investigations was the investigation into the question of what the proper mean-field version of the Gibbs–non-Gibbs question might be. For this, see in particular [18,20,23]. This question has a particular charm for systems with a general local spin space.

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As there have recently been a number of reviews on other aspects of the non-Gibbsian problem [5,7,11,25,26], we here want to emphasize what has been found for n-vector spins. The results as yet are less complete than what is known for Ising or Potts spins, but it has also become clear that, although many things are similar, such systems have traits of their own which are somewhat different and require new ideas. We have mostly worked on transformations such as stochastic evolution, which does not rescale space, such as renormalization group transformations do. Note that in a statistical interpretation, such maps for discrete spins model imperfect observations, that is observations in which with some probability one makes a mistake, an interpretation which already was mentioned in [16]. For continuous spins, the probability of staying exactly at the initial value is zero, but for short times the map is close to the identity in the sense that the distribution of an evolved spin is concentrated on a set close to the initial value. We obtained conservation of Gibbsianness under stochastic evolutions when either the time is short, or when both initial and final temperature are high. We also found that loss of Gibbsianness occurs if the initial temperature is low, and the dynamics is an infinite-temperature one. If the initial system is in an external field, after a long time the measure can become Gibbsian again. In fact, here we extend the regime where such results can be proven.

Another question we could address is the discretization question. If one approximates a continuous model by a discrete one, is the approximation still a Gibbs measure, now for discrete spins? Morally, this question is somewhat related to renormalization-type questions, as in both cases some coarse-graining takes place, in which the transformed system only contains part of the initial information. It turns out that the transformed measure is Gibbsian, once the discretization is fine enough. All of these questions, the high-temperature and short-time Gibbsianness for stochastic evolutions, as well as the loss and recovery properties, can also be addressed in the mean-field setting, and we find that the results are similar as in the lattice case. Again, for transformations which in some sense are close enough to the identity, the transformed model is Gibbsian. Finally, one may ask which of our results depend on the fact that our local state space is a sphere and not just a compact space? The regularity results (preservation of Gibbsianness) do not, as they are based on absence of phase transitions. In fact, such extensions have been proved, for which we refer to the original papers. When it comes to a failure of Gibbsianness, an internal phase transition has to be exhibited. The mechanism of this is usually very model-dependent and this is where the intricacy but also the charm of the n-vector models lies.

2 Gibbsianness and non-Gibbsianness for n-vector lattice models

In this section we review some recent developments in the study of Gibbsianness and non-Gibbsianness for n-vector models subjected to various transformations.

The review is mainly based on the recent papers [8,9,21]. Before we plunge into details let us fix some definitions, notation and give some background from the theory of lattice spin systems.

2.1 Notation and definitions

For general information on Gibbs measures for lattice spins systems we refer the reader to [4,14]. In this review we will focus attention on models living on a d-dimensional lattice $\mathbb{Z}^d$ ($d \geq 1$). We will take $\mathbb{S}^n$, the n-dimensional sphere, as the single-site spin space equipped with a Borel probability measure α (the a priori measure). The measures we will study shall be given by Hamiltonians. The Hamiltonians in a finite volume $\Lambda \subset \mathbb{Z}^d$, with boundary condition ω outside Λ, will be given by

$$H_\Lambda^\omega(\sigma) = \sum_{A\,:\,A\cap\Lambda\neq\varnothing} \Phi_A(\sigma_\Lambda \omega_{\Lambda^c}), \tag{1}$$

where the interaction Φ is a translation-invariant family of functions $\Phi_A : (\mathbb{S}^n)^{\mathbb{Z}^d} \to \mathbb{R}$, with Φ_A depending only on the spins in the finite volume A. It satisfies the following absolute summability condition:

$$\|\Phi\| = \sum_{A\ni 0} \|\Phi_A\|_\infty < \infty. \tag{2}$$

The *Gibbs measures* for the interaction Φ are the measures μ on $(\mathbb{S}^n)^{\mathbb{Z}^d}$ whose finite-volume conditional distributions are given by

$$\mu_\Lambda(d\sigma_\Lambda|\omega_{\Lambda^c}) = \frac{\exp(-H_\Lambda^\omega(\sigma))\alpha^\Lambda(d\sigma_\Lambda)}{Z_\Lambda^\omega}, \tag{3}$$

where α^Λ is the product measure of α over the sites in Λ. Another, equivalent, way of defining a Gibbs measure was identified by Kozlov [19], via two properties of the family of conditional distributions $(\mu_\Lambda)_\Lambda$. These properties are *uniform nonnullness* and *quasilocality*. The latter property holds for a measure μ if for all continuous test functions $f : \mathbb{S}^n \to \mathbb{R}$, $\varepsilon > 0$, $i \in \mathbb{Z}^d$ and configurations η there exists a $\Lambda \ni i$ such that for all $\Gamma \supset \Lambda$ and pairs of configurations ω, ζ

$$\left|\mu(f(\sigma_i)|\eta_{\Lambda\setminus i}\omega_{\Gamma\setminus\Lambda}) - \mu(f(\sigma_i)|\eta_{\Lambda\setminus i}\zeta_{\Gamma\setminus\Lambda})\right| < \varepsilon. \tag{4}$$

A collection γ of everywhere defined conditional distributions $\gamma_\Lambda = \mu_\Lambda$ satisfying all the above conditions is referred to as a *Gibbsian specification*.

Now what can be said about the Gibbs properties of transformed Gibbsian n-vector models? In [8,9,21] the Gibbs properties of various transformations acting on n-vector models were investigated and we will review the results below.

2.2 Conservation of Gibbsianness under local transformations close to the identity

We discuss conservation of Gibbsianness for initial Gibbsian n-vector lattice models subjected to local transformations close to the identity. The discussion will mainly follow [9,21]. Though these two papers use different techniques, the results proved therein have some common ground and we will compare the advantages and disadvantages of both methods. We will mainly address conservation of Gibbsianness for transformed initial Gibbs measures in this subsection.

We start with a Gibbs measure μ of an n-vector model and apply local transformations to it. Examples of such local transformations are infinite-temperature diffusive dynamics (sitewise independent Brownian motions on spheres), fuzzification or discretization of the local spin space, etc. The natural question that comes to mind is whether such a transformed measure μ' is a Gibbs measure. For transformations close to identity the above question can be answered in the affirmative. This we make precise in the sequel by first stating a theorem which is the intersection of the results found in [9,21].

Theorem 2.1. *Suppose μ is the Gibbs measure for a translation-invariant interaction Φ on $(\mathbb{S}^1)^{\mathbb{Z}^d}$. Further, assume that Φ is twice continuously differentiable and of finite range. Let μ_t be the transformed (time-evolved) measure obtained by applying infinite-temperature diffusive dynamics to μ. Then for short times the time-evolved measure μ_t is a Gibbs measure.*

Theorem 2.1 can be proved either by using cluster expansion techniques as in [9] or by Dobrushin uniqueness techniques [21]. The results proved in these papers generalize the above theorem in different directions. In the following we will review some of the main issues discussed in them. Let us start with the approach of [9]. The advantage of using cluster expansion techniques is that we can prove short-time Gibbsianness for more general dynamics beyond the independent Brownian motion on the circles. In particular, one can handle a whole class of systems which are modeled via the solution $\sigma = (\sigma_i)_{i\in\mathbb{Z}^d}(t)$ of the following system of *interacting* stochastic differential equations:

$$\begin{cases} d\sigma_i(t) = -\nabla_i \frac{1}{2}\beta_1 H_i^d(\sigma(t))\,dt + dB_i^{\odot}(t), & t > 0, i \in \mathbb{Z}^d, \\ \sigma(0) \simeq \mu, & t = 0, \end{cases} \tag{5}$$

where $(B_i^{\odot}(t))_{i,t>0}$ denotes a family of independent Brownian motions moving on a circle, $\nabla_i = \frac{d}{d\sigma_i}$ and $\beta_1 \sim 1/T_1$ is the "dynamical" inverse temperature. We assumed that the "dynamical" Hamiltonian H^d is built from an absolute summable "dynamical" interaction which is again of finite range and at least twice continuously differentiable. Let $S(t)$ denote the semigroup of the dynamics defined in (5). Then one can prove that for all values of β_1 the time-evolved measure $\mu_t = \mu \circ S(t)$

is Gibbsian for short times. Note that the statement of Theorem 2.1 corresponds to the case where $\beta_1 = 0$. We note that the cluster expansion technology was heavily influenced by [2]. Extensions to different graphs are also immediate.

The proof in [9] makes also use of the fact that $\mathbb{S}^1 \simeq [0, 2\pi)$ where 0 and 2π are considered to be the same points. Consequently we can work on the real line and do not have to worry about more general compact manifolds $\mathbb{S}^n$. Although it is in principle possible to write a cluster expansion for $\mathbb{S}^n$ and we believe that short-time Gibbsianness for general interacting dynamics holds also in higher spin dimensions, this has not been done so far.

Next let us review the results in [21]. The Dobrushin uniqueness technique employed in that work applies to more general interactions on general n-spheres and also to more general graphs aside from $\mathbb{Z}^d$. Moreover, we expect that it provides better bounds for the Gibbsian regime than the cluster expansion approach does. On the other hand, no results for interacting dynamics have been obtained via this approach, although we believe in principle this should be possible.

To be precise, one considers initial Gibbs measures for interactions Φ with finite triple norm, that is,

$$|||\Phi||| := \sup_{i \in \mathbb{Z}^d} \sum_{A: A \ni i} |A| \|\Phi_A\|_\infty < \infty. \tag{6}$$

Note that this summability condition implies the one in (2) and here we do not require that Φ is translation invariant. Initial Gibbs measures of such interactions were subjected to local (one-site) transformations given by $K(d\sigma_i, d\eta_i) = k(\sigma_i, \eta_i)\alpha(d\sigma_i)\alpha'(d\eta_i)$, with $\|\log k\|_\infty < \infty$. Here η represents the spin variable for the transformed system taking values in a compact separable metrizable space S', which now needs not be the same as $\mathbb{S}^n$.

In the language of Renormalization Group Transformations, one could think of the transformed system as the renormalized system obtained via the single-site renormalization map k. The map k can also be thought of as the transition kernel for an infinite-temperature dynamics, where the variable in the second slot will be the configuration of the system at some time after starting the dynamics from the configuration in the first slot of k. Sometimes we will refer to the time direction as the "vertical" direction.

Starting with an initial Gibbs measure μ for an interaction Φ with finite triple norm, in [21] it was studied to what extent the transformed measure

$$\mu'(d\eta) := \int_\Omega \mu(d\sigma) \prod_{i \in \mathbb{Z}^d} K(d\eta_i | \sigma_i)$$

will be Gibbsian. In the above we have set $\Omega = (\mathbb{S}^n)^{\mathbb{Z}^d}$. The study in [21] uses Dobrushin uniqueness techniques. The paper also provides continuity estimates for the single-site conditional distributions of the transformed system whenever it is

Gibbsian. To introduce these estimates the authors made use of a so-called "goodness matrix," which describes the spatial decay of the conditional distributions of the transformed measure.

In the sequel we will write i for $\{i\}$ and i^c for $\mathbb{Z}^d \setminus \{i\}$. In particular the following definition from [21] will be used.

Definition 2.2. Assume that d is a metric on $\mathbb{S}^n$ and $Q = (Q_{i,j})_{i,j\in\mathbb{Z}^d}$ is a nonnegative matrix with $\sup_{i\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} Q_{i,j} = \|Q\|_\infty < \infty$. A Gibbsian specification γ is said to be of goodness (Q, d) if the single-site parts γ_i satisfy the continuity estimates

$$\|\gamma_i(d\eta_i|\eta_{i^c}) - \gamma_i(d\eta_i|\bar{\eta}_{i^c})\| \le \sum_{j\in i^c} Q_{i,j} d(\eta_j, \bar{\eta}_j). \tag{7}$$

Here $\|\nu_1 - \nu_2\|$ is the variational distance between the measures ν_1 and ν_2.

The matrix Q controls the influence on the specification due to variations in the conditioning when we measure them in the metric d. The faster Q decays, the better, or "more Gibbsian," the system of conditional probabilities is. We note, without going into details, that a fast decay of Q also implies the existence of a fast decaying interaction potential, but in our view an estimate of the form (7) is more fundamental than a corresponding estimate on the potential.

We are restricting our attention to single-site γ_i's since all γ_Λ for finite Λ can be expressed by an explicit formula in terms of the γ_i's with $i \in \Lambda$. For the solution of this "reconstruction problem" see [12,14].

The Dobrushin interdependence matrix $C = (C_{ij})_{i,j\in\mathbb{Z}^d}$ of a Gibbsian specification γ [3,14], is the matrix with smallest matrix-elements for which the specification γ is of the goodness (C, d). Here d is the discrete metric on $\mathbb{S}^n$ given by $d(\eta_j, \eta'_j) = 1_{\eta_j\neq\eta'_j}$.

One says that γ satisfies the *Dobrushin uniqueness condition* whenever the Dobrushin constant $\sup_{i\in\mathbb{Z}^d} \sum_{j\in i^c} C_{ij} < 1$, and such a Gibbsian specification γ admits a unique Gibbs measure [3,14].

Let us now introduce some notation for our discussion on conservation of Gibbsianness for transforms of Gibbs measures. Set for each $i \in \mathbb{Z}^d$ $\alpha_{\eta_i}(d\sigma_i) := K(d\sigma_i|\eta_i)$, the a priori measures on the initial spin space which are obtained by conditioning on transformed spin configurations. We call

$$d'(\eta_i, \eta'_i) := \|\alpha_{\eta_i} - \alpha_{\eta'_i}\| \tag{8}$$

the posterior (pseudo-)metric associated to K on the transformed spin space S'. d' satisfies nonnegativity and the triangle inequality, but we may have $d'(\eta_i, \eta'_i) = 0$ for $\eta_i \neq \eta'_i$ (which happens, e.g., if σ_i and η_i are independent under K). For any

given Φ with finite triple norm write $\mathrm{std}_{i,j}(\Phi)$ for

$$\mathrm{std}_{i,j}(\Phi) := \sup_{\eta_i \in S'} \sup_{\substack{\zeta,\bar{\zeta}\in\Omega \\ \zeta_{j^c}=\bar{\zeta}_{j^c}}} \inf_{b\in\mathbb{R}} \left(\int \alpha_{\eta_i}(d\sigma_i) \big(H_i^{\zeta}(\sigma_i) - H_i^{\bar{\zeta}}(\sigma_i) - b\big)^2 \right)^{1/2}, \quad (9)$$

where $H_i^{\bar{\zeta}}(\sigma_i)$ is as in (1).

Consider the matrix

$$\bar{C}_{ij} := \frac{1}{2} \exp\Big(\sum_{A\supset\{i,j\}} \frac{\delta(\Phi_A)}{2} \Big) \mathrm{std}_{i,j}(\Phi). \quad (10)$$

Here we have denoted by $\delta(f)$, for f a real-valued observable (measurable function) on Ω, the oscillation of f given by

$$\delta(f) := \sup_{\omega\neq\xi} |f(\omega) - f(\xi)|.$$

The above quantity $\bar{C}_{ij}$, can be small if either the initial interaction Φ is weak or the measures α_{η_i} are close to delta measures. For example this is the case for short-time evolution of the initial Gibbs measure associated with Φ, as we will point out later. $\bar{C}$ is an upper bound on the Dobrushin matrices for the joint systems consisting of the initial and the transformed spins vertically coupled via the map k, and having fixed transformed configurations. A specification for this system is generated by Φ by replacing in equation (3) the a priori measure α by the α_{η_i}'s. The main tool used in [21] to show Gibbsianness of the transformed measure was the lack of phase transitions in the conditional joint system discussed above. This lack of phase transitions will follow if the Dobrushin constant of the matrix, $\bar{C}$, is strictly less than 1. More precisely the following theorem was proved ([21], Theorem 2.5).

Theorem 2.3. *Suppose that μ is a Gibbs measure associated with a lattice interaction Φ with finite triple norm. Suppose further that* $\sup_{i\in\mathbb{Z}^d} \sum_{j\in i^c} \bar{C}_{i,j} < 1$. *Then the transformed measure μ' is Gibbsian and the transformed Gibbsian specification γ' has goodness (Q, d'), where*

$$Q_{ij} = 4 \exp\Big(4 \sup_{i\in\mathbb{Z}^d} \sum_{A\ni i} \|\Phi_A\|_\infty \Big) \sum_{k\in i^c} \delta_k \Big(\sum_{A\supset\{i,k\}} \Phi_A \Big) \bar{D}_{kj} \quad (11)$$

with $\bar{D} = \sum_{n=0}^{\infty} \bar{C}^n$.

Thus the transformed measure μ' will be Gibbsian if either the initial interaction Φ is weak or the a priori measures α_{η_i} are close to delta measures. Furthermore, in the above theorem the goodness of the transformed specification is expressed in terms of the posterior metric d'. Can one have local transformations where this metric could be expressed in terms of more familiar metrics on $\mathbb{S}^n$? In what follows

we present two examples where the above question will have a positive answer. To do this we pay a price of putting further restrictions on the class of allowed interactions for the initial system.

Definition 2.4. Let us equip $\mathbb{S}^n$ with a metric d. Denote by $L_{ij} = L_{ij}(\Phi)$ the smallest constants such that the j-variation of the Hamiltonian H_i satisfies

$$\sup_{\substack{\zeta,\bar{\zeta} \\ \zeta_{j^c}=\bar{\zeta}_{j^c}}} \left|H_i^{\zeta}(\sigma_i) - H_i^{\bar{\zeta}}(\sigma_i) - \left(H_i^{\zeta}(a_i) - H_i^{\bar{\zeta}}(a_i)\right)\right| \le L_{ij} d(\sigma_i, a_i). \tag{12}$$

We say that Φ satisfies a Lipschitz property with constants $(L_{ij}(\Phi))_{i,j\in\mathbb{Z}^d\times\mathbb{Z}^d}$, if all these constants are finite.

For this class of interactions it is not hard to see from (9) that

$$\mathrm{std}_{i,j}(\Phi) \le L_{ij} \sup_{\eta_i\in S^{\prime}} \inf_{a_i\in\mathbb{S}^n} \left(\int \alpha_{\eta_i}(d\sigma_i) d(\sigma_i, a_i)^2\right)^{1/2}. \tag{13}$$

This follows from replacing the b in (9) by $H_i^{\zeta}(a_i) - H_i^{\bar{\zeta}}(a_i)$. Let us now see some concrete examples.

2.2.1 *Short-time Gibbsianness of n-vector lattice models under diffusive time-evolution.* To a Gibbs measure μ for a lattice interaction Φ we apply sitewise independent diffusive dynamics given by

$$K(d\sigma_i, d\eta_i) = K_t(d\sigma_i, d\eta_i) = k_t(\sigma_i, \eta_i)\alpha_0(d\sigma_i)\alpha_0(d\eta_i). \tag{14}$$

In the above α_0 is the equidistribution on $\mathbb{S}^n$ and k_t is the heat kernel on the sphere, that is,

$$(e^{\Delta t}\varphi)(\eta_i) = \int \alpha_o(d\sigma_i) k_t(\sigma_i, \eta_i)\varphi(\sigma_i), \tag{15}$$

where Δ is the Laplace–Beltrami operator on the sphere and φ is any test function. k_t is also called the *Gauss–Weierstrass kernel*. For more background on the heat-kernel on Riemannian manifolds, see the introduction of [1]. Let $\bar{C}(t)$ be the matrix with entries

$$\bar{C}_{i,j}(t) = \frac{L_{ij}}{\sqrt{2}}(1 - e^{-nt})^{1/2} \exp\Bigg(\sum_{A\supset\{i,j\}} \frac{\delta(\Phi_A)}{2}\Bigg). \tag{16}$$

With the above notation we have the following generalization of Theorem 2.7 of [21].

Theorem 2.5. *Suppose d is the Euclidean metric on $\mathbb{S}^n$ and Φ is an interaction for which there are finite constants $(L_i = L_i(\Phi))_{i\in\mathbb{Z}^d}$ such that*

$$\sup_{\omega\in\Omega} |H_i^{\omega}(\sigma_i) - H_i^{\omega}(a_i)| \le L_i d(\sigma_i, a_i). \tag{17}$$

Assume further that $\sup_{i\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}\bar{C}_{ij}(t) < 1$. *Then the transformed measure* μ_t *obtained from a Gibbs measure* μ *for* Φ *and* K_t *is Gibbsian and the specification for* μ_t *has goodness* $(\bar{Q}, d)$ *with*

$$\bar{Q}_{ij}(t) := \frac{1}{2}\min\left\{\sqrt{\frac{\pi}{t}}Q_{ij}(t), e^{4L_i} - 1\right\}. \tag{18}$$

Here $Q(t)$ *is defined in the same way as in* (11) *but has* $\bar{C}$ *replaced by* $\bar{C}(t)$.

Proof. The inequality (17) implies (12). But using (12) gives a better bound on the Dobrushin interdependence matrix. In (17) we keep all the interactions a given site i has with the rest of its environment, but in (12) only the interaction between i and a reference site j is kept. Note also that the entries of $\bar{C}(t)$ will be small if either the initial interaction is weak or t is small enough. This is a generalization of the corresponding Ising and planar rotor results found in [9,10] to more general interactions on any n-dimensional sphere, subjected to infinite-temperature diffusive dynamics.

The above theorem was proved in [21] for some special pair interaction. The proof there followed from three steps, namely: (1) an application of Theorem 2.3 to obtain continuity estimates in terms of the posterior metric d'; (2) a comparison result between d' and d (the Euclidean metric); see, for example, Proposition 2.8 of [21]; and (3) a telescoping argument over sites in the conditioning. The first two steps hold for any general interaction on the sphere.

In the third step one uses the continuity property (17) to proceed. In particular one replaces the constant c_j in inequality (100) of [21] by L_j. □

In the next subsection we consider another class of transformations which was studied in [21].

2.2.2 *Conservation of Gibbsianness for n-vector lattice models under discretizations* (*fine local approximations*). Consider a Gibbs measure μ for an interaction Φ satisfying (12). Furthermore, partition $\mathbb{S}^n$ into countably many pairwise disjoint subsets with nonzero α measure, indexed by elements in a countable set S'. Thus we have disjoint subsets S_i, such that $\mathbb{S}^n = \bigcup_{i\in S'} S_i$ and $\alpha(S_i) > 0$ for all $i \in S'$. For each such decomposition of $\mathbb{S}^n$ we define

$$\rho = \sup_{i\in S'} \operatorname{diam}(S_i),$$

where $\operatorname{diam}(S_i)$ is the diameter of the set S_i. We refer to ρ as the fineness of the decomposition. In this setup the conditional a priori measure is given by $\alpha_{\eta_i}(d\sigma_i) = \frac{\alpha_{|S_{\eta_i}}}{\alpha(S_{\eta_i})}$. The above decompositions of $\mathbb{S}^n$ define natural maps from the space of probability measures on Ω to probability measures on $(S')^{\mathbb{Z}^d}$. The question now is: "Which of these maps will lead to a Gibbsian image measure μ' upon

their application to the Gibbs measure μ?" This question is partially answered in Theorem 2.9 of [21], which we state below.

Theorem 2.6. *Suppose* Φ *is as above and*

$$\frac{\rho}{2} \sup_{i\in\mathbb{Z}^d} \sum_{j\in i^c} \exp\Big(\frac{1}{2} \sum_{A\supset\{i,j\}} \delta(\Phi_A)\Big) L_{ij} < 1. \tag{19}$$

Then for any Gibbs measure μ *of* Φ *the transformed measure* μ', *associated with the decomposition with fineness* ρ, *is a Gibbs measure for a Gibbsian specification* γ' *of goodness* (Q, d_0). *Here* d_0 *is the discrete metric on* S' *and* Q *is given in* (11) *with* $\bar{C}$ *given by*

$$\bar{C}_{ij} = \frac{\rho}{2} \exp\Big(\frac{1}{2} \sum_{A\supset\{i,j\}} \delta(\Phi_A)\Big) L_{ij}.$$

Observe from the above theorem that the quantity in (19) can be small if either the initial interaction is weak or the fineness ρ of the decomposition is small enough. Thus for any strength of the initial interaction, the transformed measures will remain Gibbsian if our decomposition is fine enough. We note that if we make a decomposition of the circle into equal intervals, the resulting model resembles a clock model. On could in fact also apply the theorem starting with discrete spins, such as a large-q clock model, but the advantage of considering continuous spin spaces is that the theorem can always be applied (in other words, there is *always* a fine enough decomposition).

Note also that such a discretization map has strong similarities with fuzzification maps such as have been studied for Potts models; see, for example, [17,28], in which one also decomposes the single-site spin space, into a smaller number of fuzzy spin values.

2.3 Large-time results: Conservation, loss and recovery of Gibbsianness

This section deals with what is known about conservation, loss and recovery of the Gibbs property in time-evolved Gibbsian measures of vector models on the lattice $\mathbb{Z}^d$. The conservation part will focus on large-time results, as the short-time results have been described in the previous section. We will concentrate here on the work done in [8,9]. Moreover we will present some new arguments which extend the results in [8].

In the previous section we defined Gibbs measures [see equation (3)] and furthermore we gave an equivalent description which we stated in equation (4). Let us now focus on the latter expression. In words it says that if a measure μ is Gibbsian, every configuration η is *good*, in the sense that for every η the measure is continuous w.r.t. a change in the conditioning. We referred to this property as the *quasilocality* property. But what does it mean for a measure μ to be *non*-Gibbsian?

Loss of Gibbsianness means essentially the failure of this quasilocality property. It is enough to find at least one point of essential discontinuity η^{spec} w.r.t. the conditioning, for example, a point satisfying

$$\sup_{\omega,\zeta} |\mu\big(f(\sigma_0)|\eta^{\text{spec}}_{\Lambda\setminus\{0\}}\omega_{\Gamma\setminus\Lambda}\big) - \mu\big(f(\sigma_0)|\eta^{\text{spec}}_{\Lambda\setminus\{0\}}\zeta_{\Gamma\setminus\Lambda}\big)| > \varepsilon \tag{20}$$

for some $\Gamma \supset \Lambda$ and continuous test function $f : \mathbb{S}^n \to \mathbb{R}$, uniformly in $\Lambda \subset \mathbb{Z}^d$, to prove that a measure is non-Gibbsian. Physically the failure of quasilocality means the following: The spin at the origin σ_0 is influenced by far away configurations $\omega_{\Gamma\setminus\Lambda}$ and $\zeta_{\Gamma\setminus\Lambda}$ even when the spins in between are frozen in the configuration $\eta^{\text{spec}}_{\Lambda\setminus\{0\}}$. For a measure which is Gibbsian, the spin σ_0 is shielded off from spins far away when intermediate spins are fixed. So there are no hidden fluctuations transmitting information from infinity to the origin. Typically, in the analysis one considers a double-layer system with the initial spin space in the first layer and the transformed system (or image-spin space) in the second layer. The question of the Gibbsianness of the measure on the second layer then can be shown to reduce to the question: Is it possible to end up in this particular configuration coming from one initial Gibbs measure only? It turns out that if the original spin system *conditioned* on a particular image spin configuration η^{spec} exhibits a phase transition, this implies for μ^{t_0} that this measure is not Gibbsian. The configuration η^{spec} is often called a *bad configuration*. We want to stress the difference between a phase transition of the initial system and a phase transition of the conditioned double-layer system. Even if the initial system exhibits a phase transition and the time-evolved measure at time t is a Gibbs measure, it means that conditioned on every possible image spin configuration η at time t, there is no phase transition for the conditioned system. In other words, for every possible η there is only one possible initial measure leading to this image spin configuration. Hence, a phase transition of the initial system does not necessarily imply non-Gibbsianness, nor does non-Gibbsianness imply a phase transition for the initial measure.

In the case of time-evolved XY-spins on the lattice $\mathbb{Z}^d$, in [8] and [9] some results about conservation, loss and recovery of Gibbsianness were proved. In [9] results are proven for conservation and loss of Gibbsianness during time-evolution. In particular, loss of Gibbsianness could be proven for zero initial external fields. The paper [8] deals with loss and recovery of Gibbsianness in a situation where there is a positive initial external field. As we already discussed in the previous section, the Gibbsian property is conserved for short times for all initial Gibbs measures evolving under diffusive dynamics consisting of Brownian motions on the circle, either with or without gradient Hamiltonian drifts, at all temperatures (for all values of T_1 and all values of the initial temperature T_2). Moreover, conservation for all times holds if the system starts with a high or infinite-temperature Gibbs measure and evolves under high or infinite-temperature dynamics ($T_1 \gg 1$).

Let us make the statement on the loss of Gibbsianness result from [9] more precise. Systems in $\mathbb{Z}^2$ are considered which start in a low-temperature initial measure

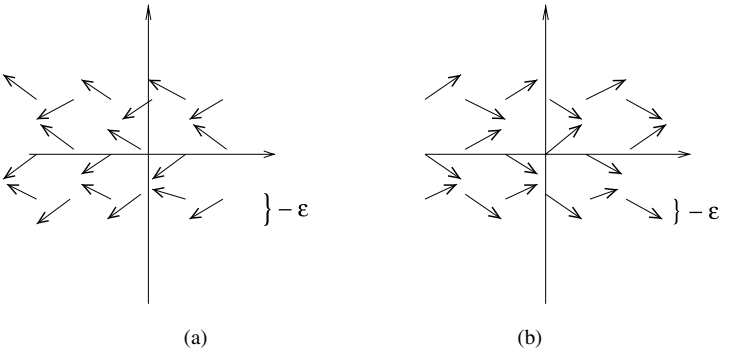

Figure 1 (a) *West-pointing ground state.* (b) *East-pointing ground state.*

with nearest neighbor interaction and zero external field,

$$\tilde{\varphi}_\Lambda(\sigma) = -J \sum_{i,j\in\Lambda: i\sim j} \cos(\sigma_i - \sigma_j)$$

and evolve under independent Brownian motion dynamics on the circle. Then there is a time interval, depending on the initial (inverse) temperature, such that the time-evolved measure is not Gibbsian. The idea of the proof is the following. Consider the double-layer measure and condition it on the alternating North–South configuration. The ground states of the conditioned system then are two symmetric configurations of spins pointing either to the East with a small correction $\pm\varepsilon_t$ or to the West with a small correction. Let us give a schematic picture. See Figure 1.

The potential of the conditioned model is of C-type (nearest neighbor, invariant under reflection in vertical and horizontal direction and invariant under translation), so that the corresponding measure is reflection positive. Using a percolation argument for low-energy clusters (clusters of vertices connected by low-energy bonds) from [13], in two dimensions one proves that at low temperatures there exist two distinct extremal Gibbs measures for the thus-conditioned system. This implies that the conditioned double-layer system undergoes a phase transition via discrete symmetry breaking and therefore the time-evolved measure is not Gibbsian. This phase transition is called of "spin-flop" type. Let us make a remark on one special feature of this result. In the Ising spin case (see [10]), where one also finds at zero external field loss of Gibbsianness, the initial system itself is already not unique. The XY spin model, however, does not have a first-order phase transition in two dimensions. So even though starting from a unique Gibbs measure, there is a time interval where Gibbsianness is lost.

We also mention that the result can be extended to $\mathbb{Z}^3$ and arbitrary large times. In that case the initial measure is not unique, and there is long-range order for any strength of the alternating magnetic field, including zero.

Unfortunately, the techniques which are used rely on the reflection positivity property of the measure, and therefore cannot be applied to a system which evolves

with high-temperature dynamics, since then the conditioned measure is not reflection positive any more. Also for higher-component spins the proof breaks down.

For discrete spins, the authors in [10] prove that loss of Gibbsianness appears also for high-temperature dynamics; for rotor spins we believe the same is true but this has not yet been proven. By some Pirogov–Sinai type arguments one might hope to extend the above result to high-temperature dynamics. But this seems a technically hard question.

In the presence of an initial external field h loss and also recovery results were obtained in [8]. Similar to the situation for discrete spins in [10], one finds that also in the presence of a small initial external field there can be a time interval (t_0, t_1) where Gibbsianness is lost in $d \geq 3$ dimensions. Moreover there exists a time t_2 such that for all $t \geq t_2$, the time-evolved measure is again a Gibbs measure. This re-entrance result was obtained for strong initial external fields in $d \geq 2$ lattice dimensions.

Intuitively, for an intermediate time interval, the strength of the initial field is compensated by the induced field coming from the time-evolution. This compensation makes the system behave like in a modified zero-field situation. The system looks like a zero-field system plus some rest terms which have a discrete symmetry instead of a continuous one. For low enough initial temperatures there is a time interval where this symmetry is broken for the conditioned double-layer system, and therefore Gibbsianness is lost. After some time the influence of the time-induced fields decreases and the system follows the initial field again which brings it back to the Gibbsian regime. Thus the presence of the initial external field is responsible for the recovery of the Gibbsian property.

The proof in [9] is similar to the one in [8]. One considers a double-layer system and conditions on spins pointing all southwards. Then the two ground states of the conditioned system are again symmetric, pointing either to the East or to the West. We present a schematic picture of the ground states (Figure 2).

Since the interaction of the conditioned system is also of C-type, that is, among other properties invariant under reflections, one can use again the percolation of low-energy clusters argument of Georgii; see [13].

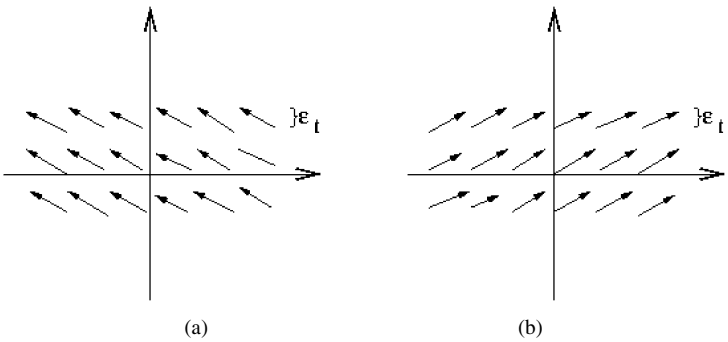

Figure 2 (a) *East-pointing ground state.* (b) *West-pointing ground state.*

The proof for the *loss* of Gibbsianness works only for $d \geq 3$ and for small initial fields.

In the following we propose an argument for a loss of Gibbsianness result which works for a general initial field h already in a two-dimensional lattice. Moreover we will prove a recovery result valid for all strengths of the initial field at low enough temperatures.

Proposition 2.7. *Let h be given. For β big enough, there exists a time interval $(t_0(\beta, h), t_1(\beta, h))$ such that for $t_0 < t < t_1$, the time-evolved measure μ^t is not Gibbsian.*

Proof. Let us consider the double-layer joint measure at time 0 and time t simultaneously. The dynamical Hamiltonian $\mathbf{H}^t_\beta(\sigma, \eta)$ (the inverse temperature is in this case absorbed into the definition of the Hamiltonian) is formally given by

$$-\mathbf{H}^t_\beta(\sigma, \eta) = -\beta \tilde{H}(\sigma) + \sum_{i \in \mathbb{Z}^2} \log(p^\odot_t(\sigma_i, \eta_i)),$$

where $\sigma, \eta \in [0, 2\pi)^{\mathbb{Z}^2}$, $p^\odot_t(\sigma_i, \eta_i)$ is the transition kernel on the circle and the initial Hamiltonian $\tilde{H}(\sigma)$ is formally given by

$$-\tilde{H}(\sigma) = J \sum_{i \sim k} \cos(\sigma_i - \sigma_k) + h \sum_i \cos(\sigma_i),$$

while $p^\odot_t$ equals

$$p^\odot_t(\sigma_i, \eta_i) = \frac{1}{\sqrt{2\pi t}} \sum_{n \in \mathbb{Z}} e^{-(\sigma_i - \eta_i + 2\pi n)^2/(2t)}.$$

We condition the system at time t to point down alternatingly with a small correction ε (which we will specify later) to the East, resp. with a small correction to the West, that is,

$$\eta^{\text{spec}}_{i,\varepsilon} = \begin{cases} \pi + \varepsilon, & \text{if } |i| = i_1 + i_2 \text{ is even,} \\ \pi - \varepsilon, & \text{else.} \end{cases}$$

Let us look at the corresponding dynamical Hamiltonian. It can be written as a sum over all nearest neighbor pairs of the pair interaction potential

$$\begin{aligned} \mathbf{H}^t_{\beta,\{i,i+1\}}(\sigma, \eta^{\text{spec}}_\varepsilon) &= \Phi^{t,\varepsilon}_\beta(\sigma_i, \sigma_{i+1}) \\ &= -\beta J \cos(\sigma_i - \sigma_{i+1}) - \beta \frac{h}{4}\big(\cos(\sigma_i) + \cos(\sigma_{i+1})\big) \\ &\quad - \frac{1}{4}\log\big(p^\odot_t(\sigma_i, \pi + \varepsilon)\big) - \frac{1}{4}\log\big(p^\odot_t(\sigma_{i+1}, \pi - \varepsilon)\big), \end{aligned}$$

where h, β, t, ε are parameters (at J fixed), and σ_i, σ_{i+1} denote the values of neighboring spins. The single-site terms coming from the dynamics play the role of dynamical alternating external fields. Let us rewrite these terms as

$$\log\big(p_t^{\odot}(\sigma_i, \pi+\varepsilon)\big) = -\log\big(\sqrt{2\pi t}\big) - \frac{(\sigma_i - (\pi+\varepsilon))^2}{2t} + \log\Bigg(1 + \sum_{n\in\mathbb{Z}\setminus 0} e^{-(\sigma_i-(\pi+\varepsilon)+2n\pi)^2/(2t)+(\sigma_i-(\pi+\varepsilon))^2/(2t)}\Bigg)$$

and similarly for the second one. We choose t of order $\mathcal{O}(1/\beta h)$, or more precisely in such a way that

$$\beta h\big(\cos(\sigma_i) + \cos(\sigma_{i+1})\big) = 2\log\big(\sqrt{2\pi t}\big) + \frac{(\sigma_i - (\pi+\varepsilon))^2}{2t} + \frac{(\sigma_{i+1} - (\pi-\varepsilon))^2}{2t} + o_t(\sigma_i^4, \sigma_{i+1}^4),$$

where $o_t(\sigma_i^4, \sigma_{i+1}^4)$ is an error term going to 0 for t small. Call

$$R_t(\sigma_i, \pi+\varepsilon) := \log\Bigg(1 + \sum_{n\in Z\setminus 0} e^{-(\sigma_i-(\pi+\varepsilon)+2n\pi)^2/(2t)+(\sigma_i-(\pi+\varepsilon))^2/(2t)}\Bigg)$$

and similarly for $R_t(\sigma_{i+1}, \pi-\varepsilon)$. Observe that the pair interaction potential is equal to

$$\Phi_\beta^{t,\varepsilon}(\sigma_i, \sigma_{i+1}) = -\beta J\cos(\sigma_i - \sigma_{i+1}) - \tfrac{1}{4}R_t(\sigma_i, \pi+\varepsilon) - \tfrac{1}{4}R_t(\sigma_{i+1}, \pi-\varepsilon) + o_t(\sigma_i^4, \sigma_{i+1}^4).$$

We end up with a ferromagnetic system with alternating dynamical external fields of strength $\mathcal{O}(\varepsilon/t)$ coming from the terms $R_t(\sigma_i, \pi+\varepsilon)$ which effectively point in a direction perpendicular to the original fields. Let us assume $\varepsilon/t \ll \beta J$. Then the strength of the fields is in fact of order $\mathcal{O}(\varepsilon h/J)$ and the direction alternates, pointing almost to the East or almost to the West, while the strength is relatively weak compared to the nearest-neighbor interaction. We will be able to show that the spin-flop mechanism causes a phase transition to occur.

In order to understand the phases of such a Hamiltonian we will look at first at its ground states. We want that $\Phi_\beta^{t,\varepsilon}(\sigma_i, \sigma_{i+1})$ is minimal at $(\delta_t, -\delta_t)$ and $(\pi + \delta_t', \pi - \delta_t')$, so the ground states point almost to the North, namely in North-East and North-West direction, or almost to the South. In general one of them is a local minimum and one is a global one. One determines δ_t and δ_t' in such a way that the configurations $(\delta_t, -\delta_t)$ and $(\pi+\delta_t', \pi-\delta_t')$ are the only minima. To make them both global, we specify $\varepsilon = \varepsilon(h,t)$ such that the following equation is true:

$$\begin{aligned} 0 &= \Phi_\beta^{t,\varepsilon}(\delta_t, -\delta_t) - \Phi_\beta^{t,\varepsilon}(\pi+\delta_t', \pi-\delta_t') \\ &= -\tfrac{1}{4}\big(R_t(\delta_t, \pi+\varepsilon) - R_t(\delta_t', \varepsilon) + R_t(-\delta_t, \pi-\varepsilon) - R_t(-\delta_t', \varepsilon)\big). \end{aligned}$$

In contrast to the zero-field situation, the spin flip between σ_i and $\pi - \sigma_i$ is not a symmetry of the Hamiltonian anymore. Indeed, for the particular choice of the time t and ε two ground states occur which are not related by any symmetry. As we described above, the conditioning more or less cancels in the direction of the original field, and one ends up with a model having alternating single-site terms (external fields), which are pointing in directions which are almost perpendicular to those of the original fields. The coexistence of the ground states can, for low enough temperature T, be extended to coexistence of two Gibbs measures, now not related by any spin-flip symmetry, for a slightly different choice of $\varepsilon = \varepsilon(h, t)$. As the choice of the "bad" conditioning configuration which contains the $\varepsilon(h, t)$ can be continuously varied, we can deduce the existence of a time-interval of non-Gibbsianness. □

We remark that, unlike the two previous cases where one could use the reflection positivity property of the measure (as well as the spin-flip symmetry), in this case unfortunately we cannot. The interaction is not invariant under lattice reflections, so the measure is not reflection positive. We have to use other techniques. We will use a general contour argument from [29]. Let us recall the statement.

Theorem 2.8 (Theorem 6 from [29]). *Let $S = [0, 1] \subset \mathbb{R}$, and let $\Psi(s_1, s_2, u_1, \ldots, u_{N-1})$ be an (N-1)-parameter family of potentials, defined for $u = (u_1, \ldots, u_{N-1}) \in \mathbb{R}^{N-1}$, varying in a neighborhood of 0 in $\mathbb{R}^{N-1}$. Assuming the family $\Psi(s_1, s_2, u)$ satisfies the following conditions*:

1. *the function $\Psi(s_1, s_2, u)$ is smooth in all its variables*,
2. *for $u = 0$ the function $\Psi(s_1, s_2, 0)$ has N absolute minima at points situated on the diagonal of the square $S \times S$, that is*,

$$\Psi(m_i, m_i, 0) = 0 \qquad \text{for all } i,$$
$$\Psi(s_1, s_2, 0) > 0 \qquad \text{for all } (s_1, s_2) \neq (m_i, m_i),$$

3. *at the minima (m_i, m_i) the second differential of the function $\Psi(s_1, s_2, 0)$ is strictly positive and moreover*

$$\left| \frac{d^2\Psi}{ds_1\, ds_2}\Big|_{s_1=s_2=m_i} \right| < \eta \frac{d^2\Psi}{ds_1^2}\Big|_{s_1=s_2=m_i},$$

   where η is a sufficiently small constant,
4. *at points (m_i, m_i), the differentials of $\Psi(s_1, s_2, u)$ at $u = 0$ are nonzero*.

THEN there exists a point $u_0 = u_0(\beta)$ such that for the system described by the potential $\Psi(s_1, s_2, u_0)$ there exist at least N distinct limit Gibbs distributions.

Proof. We want to apply the above theorem. For the assumptions to be satisfied we have to transform and shift our potential Φ_β, which allows us to generalize the

statement about translation-invariant potentials to a statement which also applies to periodic ones. We will define our new potential Ψ on $S^2 \times S^2$ instead of $S \times S$ as required in the assumptions. This does not affect the proof in any essential way. Our spin space $\mathbb{S}^1$ is isomorphic to $[0, 1]$ by the isomorphism $\sigma \mapsto \sigma/2\pi$, where 0 and 1 are considered to be the same points. We abbreviate $\sigma' := \sigma/2\pi$. Let u be a smooth function around a small neighborhood of 0 in $\mathbb{R}$ and let $m := \inf_{\sigma,\zeta}\{\Phi^{t,\varepsilon}_\beta(\sigma, \zeta)\}$. We define the new potential $\Psi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1, \sigma'_2, \zeta'_2, u)$ as being a sufficiently smooth function of all its variables. Furthermore let the differentials of $\Psi^{t,\varepsilon}_\beta(m_i, m_i, u)$ at $u = 0$ be nonzero. For $u = 0$ the potential is given by

$$\Psi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1, \sigma'_2, \zeta'_2, 0) := \Phi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1) + \Phi^{t,\varepsilon}_\beta(\sigma'_2, \zeta'_2) - 2m; \tag{21}$$

note that it is physically equivalent to $\Phi^{t,\varepsilon}_\beta$. Then obviously $\Psi^{t,\varepsilon}$ inherits the two minima from $\Phi^{t,\varepsilon}_\beta$ which we call m_1 and m_2. The second assumption is satisfied by the definition of $\Psi^{t,\varepsilon}$. Let us further examine the determinant of the Hessian matrix to check the third condition. We call $\mathrm{Hess}(\Psi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1, \sigma'_2, \zeta'_2, 0))$ the Hessian matrix for the function $\Psi^{t,\varepsilon}_\beta$. Then one observes that for the determinant of the Hessian

$$\begin{aligned}
&\det(\mathrm{Hess}(\Psi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1, \sigma'_2, \zeta'_2, 0))) \\
&\quad = \det(\mathrm{Hess}(\Phi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1)))\det(\mathrm{Hess}(\Phi^{t,\varepsilon}_\beta(\sigma'_2, \zeta'_2))) \\
&\quad = \big(\det(\mathrm{Hess}(\Phi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1)))\big)^2
\end{aligned}$$

which is strictly positive at the minimal points (m_1, m_1) and (m_2, m_2) for the parameters t and ε chosen above and β big enough. Then using the theorem we deduce that for a sufficiently large β there exists a u_0 such that for the system described by $\Psi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1, \sigma'_2, \zeta'_2, u_0)$ there exist at least two distinct Gibbs measures. Since $\Psi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1, \sigma'_2, \zeta'_2, u_0)$ and $\Psi^{t,\varepsilon}_\beta(\sigma'_1, \zeta'_1, \sigma'_2, \zeta'_2, 0)$ are physically equivalent this follows also for our potential $\Phi^{t,\varepsilon}_\beta$. □

Let us now present a recovery result which will be valid for all strengths of the initial field at sufficiently low temperatures.

Proposition 2.9. *Let h be given, then for t large enough and β large enough, for example of order $\mathcal{O}(e^{t^2})$, there is a unique constrained first-layer measure uniformly in the conditionings on the second layer. Thus the evolved measure is Gibbsian.*

Proof. We want to prove that for large enough times the constrained first-layer measure is unique, uniformly in the conditionings on the second layer. To prove this we want to use Theorem 7 from [29] which is a Pirogov–Sinai type argument for continuous models with one ground state. Let us cite their Theorem 7. □

Theorem 2.10 (Theorem 7 from [29]). *Let $S = [-1, +1] \subset \mathbb{R}$ and let us consider the lattice $\mathbb{Z}^d$. Suppose that the function $\Psi(s_1, s_2)$ is smooth in a neighborhood of $(0, 0)$ and on $S \times S$ achieves an absolute minimum at $(0, 0)$. Let us also suppose that $\Psi(0, 0) = 0$. Moreover let the expansion of $\Psi(s_1, s_2)$ in a neighborhood of $(0, 0)$ have the form*

$$\Psi(s_1, s_2) = s_1^2 + 2\eta s_1 s_2 + s_2^2 + \mathcal{O}(s_1^3 + s_2^3), \tag{22}$$

where η is a small (positive OR negative constant).

THEN there exists a temperature $\beta_0 = \beta_0(\Psi, d)$ such that for $\beta > \beta_0$ there exists a unique limit Gibbs distribution which depends analytically on β.

Proof. S is the state space of the spins and Ψ is the potential on the product space $S \times S$. So all we have to do is again rewrite our potential and prove the assumptions given in the theorem. Our original potential without approximation is given by

$$\begin{aligned}\Phi_\beta(\sigma_i, \zeta_{i+1}) = &-\beta J \cos(\sigma_i - \zeta_{i+1}) - \frac{\beta h}{4}\big(\cos(\sigma_i) + \cos(\zeta_{i+1})\big) \\ &- \tfrac{1}{4}\big(\log(p_t^{\odot}(\sigma_i, \eta_i)) - \log(p_t^{\odot}(\zeta_{i+1}, \eta_{i+1}))\big).\end{aligned}$$

It is defined including the inverse temperature β, which does not pose a problem. For t large enough the unique minimum of Φ_β is equal to $(0, 0)$. Let us rescale the potential $\Phi_\beta(\sigma_i, \zeta_{i+1})$ by $\sigma' : \sigma \mapsto \sigma/2\pi$ and consider the isomorphism $[0, 2\pi]/2\pi \simeq [-1, 1]$ where -1 and 1 are considered to be the same points. Moreover we subtract a constant from the potential to ensure that $\Phi_\beta(0, 0) = 0$, that is,

$$\begin{aligned}\Phi'_\beta(\sigma_i, \zeta_{i+1}) = &-\beta J \cos(\sigma'_i - \zeta'_{i+1}) - \frac{\beta h}{4}\big(\cos(\sigma'_i) + \cos(\zeta'_{i+1})\big) \\ &- \frac{1}{4}\log(p_t^{\odot}(\sigma'_i, \eta'_i)) - \frac{1}{4}\log(p_t^{\odot}(\zeta'_{i+1}, \eta'_{i+1})) \\ &+ \beta J + \beta h/2 + \frac{1}{4}\log(p_t^{\odot}(0, \eta'_i)) + \frac{1}{4}\log(p_t^{\odot}(0, \eta'_{i+1})).\end{aligned}$$

We call $f^t(\sigma'_i, \eta'_i) := \frac{1}{4}\log(p_t^{\odot}(0, \eta_i^{\eta})) - \frac{1}{4}\log(p_t^{\odot}(\sigma'_i, \eta'_i))$ and write the above potential as

$$\begin{aligned}\Phi'_\beta(\sigma'_i, \zeta'_{i+1}) = &-\beta J[\cos(\sigma'_i - \zeta'_{i+1}) - 1] - \frac{\beta h}{4}[\cos(\sigma'_i) - 1 + \cos(\zeta'_{i+1}) - 1] \\ &+ f^t(\sigma'_i, \zeta'_i) + f^t(\zeta'_{i+1}, \eta'_{i+1}).\end{aligned}$$

Around the absolute minimum $(0, 0)$, we have the following expansion of $\Phi'_\beta(\sigma'_i, \zeta'_{i+1})$, using the abbreviation $c(J, h) := \frac{4J+h}{4(2\pi)^2}$:

$$\begin{aligned}\Phi'_\beta(\sigma'_i, \zeta'_{i+1}) = &\beta c(J, h)(\sigma'_i)^2 + \beta\left(\frac{-2J}{(2\pi)^2}\right)\sigma'_i\zeta'_{i+1} + \beta c(J, h)(\zeta'_{i+1})^2 \\ &+ \mathcal{O}\big((\sigma'_i)^4 + (\zeta'_{i+1})^4\big) + o_{i,i+1}(e^{-t}).\end{aligned}$$

We clearly see that the expansion gives us for t large enough, at least bigger than $\log(c(J,h))$, the desired quadratic form required for Theorem 7. □

3 Gibbsianness of n-vector mean-field models and their transforms

Mean-field models are spin systems whose distribution is permutation-invariant. In [18,20,23] the Gibbs properties of various mean-field models (with finitely many spin values) were investigated when subjected to various transformations. In the recent study in [22], extensions to more general mean-field models with compact Polish spaces as their single-site spin space are discussed. We describe these results in this section, restricting to the case where the spins take values on a sphere. Let us start by recalling the general notion of mean-field models and what it takes for them to be Gibbsian.

3.1 General mean-field models and mean-field Gibbsianness

We now present the definition of general mean-field models and mean-field Gibbsianness for such models for n-vector spins [22]. We write $V_N = \{1, 2, \ldots, N\}$ for the volume at size N.

Definition 3.1. For each $N \in \mathbb{N}$, let μ_N be a probability measure on the space $(\mathbb{S}^n)^N$.

1. We refer to the sequence of the probability measures $(\mu_N)_{N\in\mathbb{N}}$ as a *mean-field model* if the μ_N's are permutation invariant.
2. A mean-field model $(\mu_N)_{N\in\mathbb{N}}$ is said to be Gibbsian if the following holds:
   (a) The limit of conditional probabilities
   $$\gamma_1(dx_1|\lambda) := \lim_{N\uparrow\infty} \mu_N(dx_1|x^N_{V_N\setminus 1}), \tag{23}$$
   exists for any sequence $x^N_{V_N\setminus 1} = (x^N_i)_{i\in V_N\setminus 1}$ of conditionings for which the empirical distribution converges weakly as N tends to infinity, $\lambda = \lim_{N\uparrow\infty} \frac{1}{N}\sum_{i=2}^N \delta_{x_i^N}$.
   (b) The function $\lambda \mapsto \gamma_1(\cdot|\lambda)$ is weakly continuous.

Above, Gibbsianness of mean-field models is defined in terms of the asymptotic behavior of a sequence of measures instead of a single limiting measure. This is in contrast to the lattice case where we just investigated the single infinite-volume measure. The results one would get by only looking at the infinite-volume limit measures for mean-field systems would provide a lot less, and in some sense misleading, information. Indeed, such limit measures are either product measures, and thus trivially Gibbsian, or nontrivial mixtures of product measures and thus highly non-Gibbsian (see for this fact [6]).

The notion of Gibbsianness given in Definition 3.1 is equivalent to the one considered in [20,23] for the corresponding Curie–Weiss model (for which of course one has a simpler single-site spin space and measure). This is the case since the distribution of a binary random variable is uniquely determined by its mean. Hence for the Curie–Weiss model conditioning on the empirical averages gives the same information as conditioning on empirical measures. For the rest of this section the term "Gibbsian" should be taken in the sense provided by Definition 3.1.

3.1.1 *Mean-field interactions*. In the above we have defined general mean-field models. We are now going to prescribe a class of mean-field models given via some potential functionals defined on the space of measures on the single-site spin space introduced in [22]. In the following we have denoted by $\mathcal{M}(\mathbb{S}^n)$ and $\mathcal{M}_+(\mathbb{S}^n)$ the spaces of finite signed measures and finite measures on $\mathbb{S}^n$ respectively.

Definition 3.2. We shall refer to a map $\Phi : \mathcal{M}_+(\mathbb{S}^n) \to \mathbb{R}$ as a proper mean-field interaction if:

1. it is weakly continuous,
2. it satisfies the uniform directional differentiability condition, meaning that, for each $\nu \in \mathcal{M}_+(\mathbb{S}^n)$ the derivative $\Phi^{(1)}(\nu, \mu)$ at ν in direction μ exists and we have
$$\Phi(\nu + \mu) - \Phi(\nu) - \Phi^{(1)}(\nu, \mu) = r(\mu) \tag{24}$$
with $\lim_{t\to 0^+} \frac{r(t\mu)}{t} = 0$ uniformly in $\mu \in \mathcal{M}(\mathbb{S}^n)$ for which $\nu + t\mu \in \mathcal{M}_+(\mathbb{S}^n)$, for $t \in [0, 1]$ and
3. $\Phi^{(1)}(\nu, \mu)$ is a continuous function of ν.

Standard examples are the quadratic interactions for the q-state mean-field Potts and the Curie–Weiss model (which are defined on the product spaces of finite single-site spin spaces, instead of n-spheres, with symmetric a priori measure) and which are respectively given by

$$\Phi^P(\nu) = -\frac{1}{2}\sum_{i=1}^{q} \nu(i)^2 \quad \text{and} \quad \Phi^{\mathrm{CW}}(\nu) = -\frac{1}{2} m(\nu)^2, \tag{25}$$

where $m(\nu)$ is the mean of the measure ν.

For each mean-field interaction Φ and each $N \in \mathbb{N}$ we define the *finite-volume Hamiltonian* H_N [a real-valued function on the product space $(\mathbb{S}^n)^N$] as

$$H_N(\sigma_{V_N}) := N\Phi(L_N(\sigma_{V_N})), \tag{26}$$

where $L_N(\sigma_{V_N}) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\sigma_i}$ is the empirical measure of the configuration σ_{V_N}. Observe from the permutation invariance of the empirical measures that H_N is permutation invariant. With this notation the sequence of probability measures $\mu_{\beta,N}$

associated with the finite-volume Hamiltonians H_N at inverse temperature β given by

$$\mu_{\beta,N}(d\sigma_{V_N}) := \frac{e^{-\beta H_N(\sigma_{V_N})}\alpha^{\otimes N}(d\tilde{\sigma}_{V_N})}{\int_{(\mathbb{S}^n)^N} e^{-\beta H_N(\tilde{\sigma}_{V_N})}\alpha^{\otimes N}(d\tilde{\sigma}_{V_N})} \tag{27}$$

is a mean-field model (on $\mathbb{S}^n$ associated with Φ and the a priori measure α). In the above we have used $\otimes$ to denote the product measure, which is the product of single-site measures α. In the following, unless otherwise stated, the inverse temperature β will be absorbed into the interaction Φ and we write μ_N instead of $\mu_{\beta,N}$. We will, with abuse of notation, write μ_N for the sequence $(\mu_N)_{N\in\mathbb{N}}$. It is show in Proposition 2.4 of [22] that the mean-field models obtained in this way are Gibbsian. Now we finished the discussion on Gibbsianness for general n-vector mean-field models, we can turn our attention to discussing Gibbs properties of transforms of Gibbsian n-vector mean-field models.

3.2 Gibbsianness of transformed n-vector mean-field models

We now review the notion of Gibbsianness for transformed Gibbsian n-vector mean-field models as found in [22]. We take S' as the single-site spin space for the transformed system, which we also assume to be a compact complete separable metrizable space. Further, we let α' be some appropriately chosen a priori probability measure on S'. Now we take $K(d\sigma_i, d\eta_i)$ as the joint a priori probability measure on $\mathbb{S}^n \times S'$ given by

$$K(d\sigma_i, d\eta_i) := k(\sigma_i, \eta_i)\alpha(d\sigma_i)\alpha'(d\eta_i) \in \mathcal{P}(\mathbb{S}^n \times S'), \tag{28}$$

where

$$k : \mathbb{S}^n \times S' \to (0, \infty)$$

just as we had before for the corresponding transformed lattice spin models. Now the question of interest is the following. Starting with an initial sequence of mean-field Gibbs measures μ_N, associated to a fixed general mean-field interaction Φ, will the transformed sequence of measures μ'_N with $(\alpha')^N$ density

$$\frac{d\mu'_N}{d(\alpha')^N}(d\eta) = \int_{(\mathbb{S}^n)^N} \prod_{i=1,\ldots,N} k(\sigma_i, \eta_i)\mu_N(d\sigma) \tag{29}$$

be Gibbsian? In other words, will the transformed single-site kernel (a) exist, and (b) be a continuous function of the empirical measures of the conditionings?

It is shown in Theorem 3.10 of [22] that the transformed mean-field model μ'_N will remain Gibbsian if a certain constrained potential function has a unique global minimizer, uniformly over the domain of the constraint variable. The ideology behind this theorem is the same as in the lattice: absence of hidden phase transition

in the initial system, constrained to be mapped to a fixed configuration in the transformed system implies Gibbsianity for the transformed system. In the mean-field situation estimates can be made explicitly. To see something concrete, the authors in [22] focused attention on mean-field interactions Φ of the form

$$\Phi(\nu) = F(\nu[g_1], \dots, \nu[g_l]), \tag{30}$$

where g_i are some fixed bounded nonconstant real-valued measurable functions defined on $\mathbb{S}^n$, $l \geq 1$ and $F : \mathbb{R}^l \to \mathbb{R}$ is some twice continuously differentiable function. In the above we have denoted by $m_i = \nu[g_i]$ the expectation of g_i with respect to ν. We further set $F_j(m) = \frac{\partial}{\partial m_j} F(m)$ and $F_{ju}(m) = \frac{\partial^2}{\partial m_j \, \partial m_u} F(m)$.

Additionally, we assume that $g = (g_1, \dots, g_l)$ is a Lipschitz function from $\mathbb{S}^n$ to $\mathbb{R}^l$, with Lipschitz-norm

$$\|g\|_{d,2} = \sup_{\sigma_i \neq \bar{\sigma}_i} \frac{\|g(\sigma_i) - g(\bar{\sigma}_i)\|_2}{d(\sigma_i, \bar{\sigma}_i)}, \tag{31}$$

where d is the metric on $\mathbb{S}^n$. We also denote by $\delta(g)$ the sum of the oscillations of the components of g, that is,

$$\delta(g) = \sum_{j=1}^{l} \sup_{\sigma_i, \bar{\sigma}_i \in \mathbb{S}^n} |g_j(\sigma_i) - g_j(\bar{\sigma}_i)|. \tag{32}$$

For any g satisfying the above conditions we set

$$D_g = \overline{\{\nu[g] : \nu \in \mathcal{P}(\mathbb{S}^n)\}}. \tag{33}$$

Note that D_g is a compact subset of $\mathbb{R}^l$ by the boundedness of g. In the sequel we will write $\|\partial^2 F\|_{\max,\infty}$ for the supremum of the matrix max-norm of the Hessian $\partial^2 F$, that is,

$$\begin{aligned} \|\partial^2 F\|_{\max,\infty} &= \sup_{m \in D_g} \|\partial^2 F(m)\|_{\max}, \qquad \text{where} \\ \|\partial^2 F(m)\|_{\max} &= \max_{1 \leq i,j \leq l} |F_{ij}(m)|. \end{aligned} \tag{34}$$

Furthermore, we also set

$$\delta_{F,g} = \sup_{m \in D_g} \sup_{\sigma_i, \bar{\sigma}_i \in \mathbb{S}^n} \left| \sum_{j=1}^{l} F_j(m) \big(g_j(\sigma_i) - g_j(\bar{\sigma}_i) \big) \right|. \tag{35}$$

With the above interaction, it is proved in [22] that the transformed system associated to any $K(d\sigma_i, \eta_i) = k(\sigma_i, \eta_i)\alpha(d\sigma_i)\alpha'(d\eta_i)$ will remain Gibbsian if a certain quantity is small. Before we make the above result more precise, some more notation is in order. We set

$$\begin{aligned} \operatorname{std}_\alpha(k) &:= \sup_{\eta_i \in S'} \inf_{a_i \in \mathbb{S}^n} \left(\int_{\mathbb{S}^n} d^2(\sigma_i, a_i) k(\sigma_i, \eta_i) \alpha(d\sigma_i) \right)^{1/2} \quad \text{and} \\ C(F, g) &:= 2\|\partial^2 F\|_{\max,\infty} \delta(g) \|g\|_{d,2} \exp\left(\frac{\delta_{F,g}}{2} \right). \end{aligned} \tag{36}$$

With this notation we have the following theorem.

Theorem 3.3. *Consider the transformed system μ'_N associated to the initial mean-field model μ_N (given by the interaction Φ satisfying the above conditions) and joint a priori measure K described above. Suppose that*

$$C(F, g)\,\mathrm{std}_\alpha(k) < 1. \tag{37}$$

Then:

1. *the transformed system is Gibbsian and*
2. *the single-site kernel γ'_1 of the transformed system satisfies the continuity estimate*

$$\|\gamma'_1(\cdot|\nu'_1) - \gamma'_1(\cdot|\nu'_2)\| \le C(F, g)^2\,\mathrm{std}_\alpha(k)\,\mathrm{std}_\alpha\,\|\nu'_1 - \nu'_2\|, \tag{38}$$

where $\mathrm{std}_\alpha = \mathrm{std}_\alpha(1)$ *and* $\|\nu'_1 - \nu'_2\|$ *is the variational distance between* ν'_1 *and* ν'_2.

The above theorem is found in [22] as Theorem 4.3. The smallness of the quantity $C(F, g)\,\mathrm{std}_\alpha(k)$ may come from two sources; namely:

(1) the smallness of $C(F, g)$, arising from the weakness of the interaction Φ among the components of the initial system and

(2) the smallness of $\mathrm{std}_\alpha(k)$, coming from the good concentration property of the conditional measures $\alpha_{\eta_1}(d\sigma_1) := k(\sigma_1, \eta_1)\alpha(d\sigma_1)$, uniformly in $\eta_1 \in S'$.

Thus we could start with a very strong interaction, but if the measures $\alpha_{\eta_1}(d\sigma_1)$ are close to delta measures then the transformed system will be Gibbsian. An advantage of this result is that it provides explicit continuity estimates for γ'_1 whenever the transformed system is Gibbsian, which were lacking in all the results before. However, it has the drawback that the estimates it provides for the regions in parameter space where the transformed system is Gibbsian might not be sharp, as techniques employed in [18] and [23] do provide.

We now review two examples discussed in [22], which are reminiscent of some of the results found in [18,23].

3.2.1 *Short-time Gibbsianness of n-vector mean-field models under diffusive time-evolution.* Here we present the result found in [22] but for general mean-field interactions Φ given in terms of F and g. We study the Gibbs properties of the transformed (time-evolved) system $\mu'_{t,N}$ obtained upon application of infinite-temperature diffusive dynamics to the initial Gibbsian mean-field model μ_N, associated with Φ. In this set-up $S' = \mathbb{S}^n$. The joint single-site a priori measure K is then given as in (14) of Section 2.2.1. The following theorem is the result about the short-time conservation of Gibbsianness for the time-evolved system $\mu'_{t,N}$.

Theorem 3.4. *Suppose we have* $\sqrt{2}C(F,g)(1-e^{-nt})^{1/2} < 1$, *then the time-evolved system* $\mu'_{t,N}$ *will be Gibbsian and* $\gamma'_{1,t}$, *the single-site kernel for* $\mu'_{t,N}$, *has the continuity estimate*

$$\|\gamma'_{1,t}(\cdot|\nu'_1) - \gamma'_{1,t}(\cdot|\nu'_2)\| \le 2C(F,g)^2(1-e^{-nt})^{1/2}\|\nu'_1 - \nu'_2\|. \tag{39}$$

Observe from Theorem 3.4 that the time-evolved measure will be Gibbsian whenever either the initial interaction is weak or t is small enough. The above result was only stated in [22] for the corresponding Curie–Weiss model. We present this case below. For the Curie–Weiss rotator model the interaction for the initial system is given by

$$\Phi(\nu) = F(\nu[\sigma_i^1], \dots, \nu[\sigma_i^{n+1}]) = -\frac{\beta \sum_{j=1}^{n+1} \nu[\sigma_i^j]^2}{2}, \tag{40}$$

where $g_j(\sigma_i) = \sigma_i^j$ is the jth coordinate of the point $\sigma_i \in \mathbb{S}^n$ and $l = n+1$. As a corollary to Theorem 3.4 we have the following short-time Gibbsianness result for the Curie–Weiss rotator model under diffusive time evolution.

Corollary 3.5. *Suppose we have* $4\sqrt{2}\beta(n+1)e^{\beta}(1-e^{-nt})^{1/2} < 1$, *then the time-evolved system* $\mu'_{t,N}$ *will be Gibbsian and* $\gamma'_{1,t}$, *the single-site kernel for* $\mu'_{t,N}$ *has the continuity estimate*

$$\|\gamma'_{1,t}(\cdot|\nu'_1) - \gamma'_{1,t}(\cdot|\nu'_2)\| \le 32\beta^2(n+1)^2e^{2\beta}(1-e^{-nt})^{1/2}\|\nu'_1 - \nu'_2\|. \tag{41}$$

Corollary 3.5 is found in [22] as Lemma 5.1. This corollary is reminiscent of the result in Theorem 2.2 of [23], where the Curie–Weiss model under independent spin-flip dynamics was studied. It is shown therein that if β is small enough (weak initial interaction), then the time-evolved system will be Gibbsian forever, but if β is large, then the time-evolved system will only be Gibbsian for short times.

3.2.2 *Conservation of Gibbsianness for n-vector mean-field models under fine local approximations*. Consider general F and g as above, and decompose $\mathbb{S}^n$ into countably many pairwise disjoint subsets (countries) as in Section 2.2.2 above.

Then with this notation it follows from Theorem 3.3 that

Proposition 3.6. *If the quantity* $\rho C(F,g) < 1$, *then the transformed system is Gibbsian and the single-site kernel* γ'_1 *satisfies the continuity estimate*

$$\|\gamma'_1(\cdot|\nu'_1) - \gamma'_1(\cdot|\nu'_2)\| \le \rho C(F,g)^2 \operatorname{std}_\alpha \|\nu'_1 - \nu'_2\|. \tag{42}$$

The above proposition can be found in Lemma 5.2 of [22]. Thus the transformed system μ'_N will be Gibbsian if either the initial interaction Φ is weak or the local coarse-grainings (i.e., the S_i) are fine enough. In other words: If we have initial Gibbsian mean-field system with spins living on the sphere and we partition the

sphere into countries, representing each country by a distinct point in S', then the resulting transformed system will be Gibbsian if the countries are small enough.

Let us mention in this context the result of Theorem 1.2 of [18] for the corresponding fuzzy Potts mean-field model. In that paper it was shown that the transformed system will be Gibbsian at all temperatures whenever the sets of points contracted into single points by the fuzzy map have cardinality at most 2.

3.3 Mean-field rotators in nonvanishing external magnetic field: Loss and recovery of Gibbsianness

In this section we specialize to the quadratic mean-field rotator model on the circle, where we focus now on the interesting case $h \neq 0$. In fact, although we do not treat the simpler case $h = 0$ here, one can in a very similar way prove loss of Gibbsianness, again just as in the lattice situation.

We start with the measure

$$\begin{aligned}&\mu_{\beta,h,N}(d\sigma_1,\ldots,d\sigma_N)\\ &\quad=\frac{1}{Z_{\beta,h,N}}\exp\bigl(N\beta m(\sigma_1,\ldots,\sigma_N)^2+N\beta h\cdot m(\sigma_1,\ldots,\sigma_N)\bigr)\prod_{i=1}^{N}\alpha(d\sigma_i),\end{aligned}\tag{43}$$

where

$$m(\sigma_1,\ldots,\sigma_N)=\frac{1}{N}\sum_{i=1}^{n}\sigma_i$$

is a vector-sum in $\mathbb{R}^2$ and $\alpha(d\sigma_i)$ is the equidistribution. We take a time-evolution with the transition kernels $p_t(\sigma_i,\eta_i)$ describing Brownian motion on the circle, as above.

We are interested in the Gibbsian character of the time-evolved measures

$$\mu_{\beta,h,t,N}(d\eta_1,\ldots,d\eta_N)=\int\mu_{\beta,h,N}(d\sigma_1,\ldots,d\sigma_N)\prod_{i=1}^{N}p_t(\sigma_i,d\eta_i)$$

in the sense of continuity of limiting conditional kernels, as described above. The virtue of mean-field models is that we can describe the limiting kernels explicitly. By this we mean a description in terms of a minimization problem of an explicit expression. This has been done in the general setup of site-wise independent transformations in [22]. For the present case we get the following concrete results.

Proposition 3.7. *The limiting kernels $\gamma'_{1,\beta,h,t}(d\eta_1|\lambda)$ of the time-evolved mean-field models $\mu_{\beta,h,t,N}$ are given by the formula*

$$\gamma'_{1,\beta,h,t}(d\eta_1|\lambda)=\frac{\int e^{\beta\sigma_1\cdot(m^*(\beta,h,t,\lambda)+h)}p_t(\sigma_1,d\eta_1)\alpha(d\sigma_1)}{\int e^{\beta\sigma_1\cdot(m^*(\beta,h,t,\lambda)+h)}\alpha(d\sigma_1)}\tag{44}$$

for all choices of the (nonnegative) parameters β, h, t and the conditioning λ (in the probability measures on the circle), for which the minimizer (in the closed unit disk)

$$m^*(\beta, h, t, \lambda) = \operatorname{argmin}\{m \mapsto F(m; \beta, h, t, \lambda)\}$$

is unique with

$$F(m; \beta, h, t, \lambda) = \beta \frac{|m|^2}{2} - \int \lambda(d\eta_1) \log \int e^{\beta\bar{\sigma}\cdot(m+h)} p_t(\eta_1, d\bar{\sigma}). \tag{45}$$

We do not give a proof of (44) here (which can be deduced from the general result in [22]), but we briefly sketch a heuristics which explains what happens: Note first that $F(m; \beta, h, t, \lambda)$ denotes the rate function (up to an additive constant) of the initial model, constrained to have an empirical distribution λ in the transformed model. Conditioning the empirical distribution of the transformed spins outside the site 1 to λ produces a quenched system involving the initial spins which acquires the magnetization $m^*(\beta, h, t, \lambda)$. The propagation of the corresponding distribution of σ_1 to η_1 with the kernel p_t gives the desired conditional probability distribution $\gamma'_{1,\beta,h,t}(d\eta_1|\lambda)$.

3.3.1 *Gibbsianness at large times.* Compare the rate function (45) to the well-known rate-function of the initial model (43) given by

$$F_0(m; \beta, h) = \beta \frac{|m|^2}{2} - \log \int e^{\beta\bar{\sigma}\cdot(m+h)} \alpha(d\bar{\sigma}). \tag{46}$$

The map $m \mapsto F_0(m; \beta, h)$ has a unique minimizer $m^*(\beta, h)$, if $h \neq 0$ is arbitrary, pointing in the direction of h.

The input to understand the large time-behavior is the fact that the kernel $p_t(\eta_i, d\sigma_i)$ converges to the equidistribution $\alpha(d\sigma_i)$, uniformly in η_i.

From this we see that, at fixed β, h, the functions $m \mapsto F(m; \beta, h, t, \lambda)$ converge to $m \mapsto F_0(m; \beta, h)$, uniformly in λ. The same holds for higher derivatives w.r.t. m. These statements imply that, for t sufficiently large, for all choices of λ there is only one minimizer $m^*(\beta, h, t, \lambda)$. Looking at the linear appearance of the measure λ in (45), we see that $m^*(\beta, h, t, \lambda)$ changes continuously under a variation of λ.

By the form of (44) this implies Gibbsianness.

3.3.2 *Non-Gibbsianness at intermediate times.* To prove non-Gibbsianness at the parameter-triple (β, h, t) we use the formula (44) for the limiting kernels for those quadruples (β, h, t, λ) where they are well defined and, for fixed (β, h, t) we show that there exists a $\lambda = \lambda^{\text{spec}}$ at which the limiting kernels are not continuous.

To do so, it suffices to exhibit a one-parameter trajectory $\varepsilon \mapsto \lambda_\varepsilon$ which is continuous in the weak topology s.t.:

1. $F(m;\beta,h,t,\lambda_\varepsilon)$ has unique minimizers for ε in a neighborhood of $\varepsilon^{\text{spec}}$, excluding $\varepsilon^{\text{spec}}$;
2. $\lim_{\varepsilon\uparrow\varepsilon^{\text{spec}}} m^*(\beta,h,t,\lambda_\varepsilon) \neq \lim_{\varepsilon\downarrow\varepsilon^{\text{spec}}} m^*(\beta,h,t,\lambda_\varepsilon)$.

So far, the reasoning is general. Now, to create a phase transition in the constrained model, a suitable choice of λ which is able to balance the influence of the external magnetic field h has to be found. We choose conditionings of the type

$$\lambda_\varepsilon = \tfrac{1}{2}\delta_{e(\pi+\varepsilon)} + \tfrac{1}{2}\delta_{e(\pi-\varepsilon)}, \tag{47}$$

where $e(\theta)$ denotes the vector on the circle corresponding to the angle θ. This conditioning mimics the choice of conditionings on $\mathbb{Z}^2$ obtained by putting $e(\pi+\varepsilon)$ on the even sublattice and $e(\pi-\varepsilon)$ on the odd sublattice.

Proposition 3.8. *Let $h=\bar{h}e(0)\neq 0$ be given. For β large enough there exists a time interval such that for any t in this interval there exists an $\varepsilon(\beta,\bar{h},t)$ for which $m\mapsto F(m;\beta,h,t,\lambda_\varepsilon)$ has two different global minimizers of the form $m=ue(0)$, pointing in the direction or in the opposite direction of h.*

We provide an explanation of this phenomenon. Let us look at the rate-function for magnetization-values pointing in the direction of h, in the conditioning λ_ε which reads

$$\begin{aligned} F(ue(0);\beta,h,t,\lambda_\varepsilon) = \beta\frac{u^2}{2} &- \frac{1}{2}\log\int e^{\beta\cos\theta(u+\bar{h})}q_t\big(\theta-(\pi+\varepsilon)\big)\,d\theta \\ &- \frac{1}{2}\log\int e^{\beta\cos\theta(u+\bar{h})}q_t\big(\theta-(\pi-\varepsilon)\big)\,d\theta \end{aligned}$$

with the diffusion kernel on the sphere written in angular coordinates θ as

$$q_t(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} e^{-k^2t}\cos(k\theta).$$

For fixed parameters $\beta,\bar{h}$, we use the new magnetization variable $U=\beta(u+\bar{h})$ to rewrite

$$F(ue(0);\beta,h,t,\lambda_\varepsilon) - \frac{\beta\bar{h}^2}{2} = \frac{U^2}{2\beta} - U\bar{h} - L(U;\varepsilon,t), \tag{48}$$

where

$$L(U;\varepsilon,t) = \frac{1}{2}\log\int e^{U\cos(\theta+(\pi+\varepsilon))}q_t(\theta)\,d\theta + \frac{1}{2}\log\int e^{U\cos(\theta+(\pi-\varepsilon))}q_t(\theta)\,d\theta.$$

The second term on the l.h.s. of (48) is an unimportant constant. This choice of parameters is handy because we have separated their influence, and moreover, two of them are appearing only linearly in our four-parameter family.

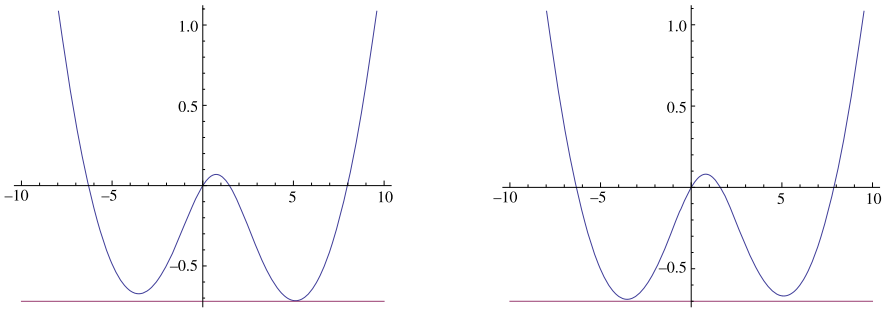

Figure 3 (a) $\beta = 5, \bar{h} = 0.16, \varepsilon = 0.4, t = 1$. (b) $\beta = 5, \bar{h} = 0.16, \varepsilon = 0.3, t = 1$.

Let us explain how a balance between ε and $\bar{h}$ can be used to create a situation of a pair of different equal depth-minimizers, without going into the details of the analysis of the regions in parameter-space where this can be done.

For this heuristic argument, let us fix the ε first. We note that $U \mapsto L(U; \varepsilon, t)$ is convex, so $\frac{U^2}{2\beta} - L(U; \varepsilon, t)$ has a chance to have more than one local minimum, for good choices of β, ε, t. Having found such a situation, a tuning of the $\bar{h}$ will result in a tilting of the rate-function which can create a situation where this pair has an equal depth in the full function $\frac{U^2}{2\beta} - U\bar{h} - L(U; \varepsilon, t)$. The mechanism described provides us with a curve in the space of ε and $\bar{h}$ where the two minima have equal depth. Now, fixing a value of $\bar{h}$ and varying the ε across this curve, yields a jump of the global minimizer which implies non-Gibbsianness. (Similarly, due to the continuity of the above expression in the variables t, h and ε, at a given h slightly varying the time t gives a slightly varying ε for which two equal-depth minimizers occur.) In Figure 3, showing the plot of $U \mapsto G(U; \beta^{-1}, \bar{h}, \varepsilon, t)$ we see this mechanism at work.

It is clear from the above diagram that for $\beta = 5, \bar{h} = 0.16$, and$t = 1$ there is a choice of ε^* such that $F(ue(0); 5, 0.16, 1, \lambda_{\varepsilon *})$ has two global minimizers. Numerically we find $\varepsilon^* \in (0.33481860, 0.33481863)$. Hence at such values for β, h and t, the transformed system will be non-Gibbsian.

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