Finite connected components in infinite directed and multiplex networks with arbitrary degree distributions

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This work presents exact expressions for size distributions of weak and multilayer connected components in two generalizations of the configuration model: networks with directed edges and multiplex networks with an arbitrary number of layers. The expressions are computable in a polynomial time and, under some restrictions, are tractable from the asymptotic theory point of view. If first partial moments of the degree distribution are finite, the size distribution for two-layer connected components in multiplex networks exhibits an exponent $-\frac{3}{2}$ in the critical regime, whereas the size distribution of weakly connected components in directed networks exhibits two critical exponents $-\frac{1}{2}$ and $-\frac{3}{2}$.

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I. INTRODUCTION

Many real world networks are well conceptualized when reduced to a graph, that is, a set of nodes that are connected with edges or links. This representation helps to uncover often the nontrivial role of the topology in the functioning of complex networks [1–4]. From a probabilistic perspective, many interesting network properties are well defined even when the total number of nodes approaches infinity. For instance, the degree distribution is a univariate function of a discrete argument that denotes the probability for a randomly chosen node to have a specific number of adjacent edges [5]. The notion of degree distribution is easy to adapt to various generalizations of simple graphs. When different types of edges are present or if edges are nonsymmetrical (directed network), the degree distribution denotes the joint probability for a randomly sampled node to have specific numbers of edges of each type [1].

Just as a degree distribution is attributed to a single instance of a network, one may reverse this association and talk about a class of networks that all match a given degree distribution. The class of such networks is known as the configuration model or generalized random graph [6–10]. In the configuration model, the connections between nodes are assigned at random with the only constraint that the degree distribution has to be preserved. This concept can naturally be extended to directed graphs, in which case the degree distribution is bivariate, counting incoming and outgoing edges [8,11], or to multiplex networks, where many types of edges exist and thus the degree distribution is multivariate [1,12–15].

A connected component is a set of nodes in which each node is connected to all other nodes with a path of finite or infinite length. Different notions of a path give rise to distinct definitions of connected components. Namely, if directed edges are present, in, out, weak, and strong components are distinguished [8]. As in multiplex networks, one may speak of a connected component that is solely contained within a single layer or a two-layer component having edges in both layers [12,16,17]. Even under the assumption of the thermodynamic limit, when the total number of nodes approaches infinity, the infinite network may contain connected components of finite size $n > 1$. Thus there are two key features that characterize sizes of connected components in configuration models: the size distribution of finite components and the size of the giant component. The size distribution is usually defined as the probability that a randomly sampled node belongs to a component of a specific size, while the size of the giant component is the probability that a randomly sampled node belongs to a component of size that scales linearly with the size of the whole system [8].

Considerable progress has been made in recovering both the size distribution and the size of the giant component that are associated with an arbitrary degree distribution in undirected single-layer configuration networks. Molloy and Reed [7] proposed a simple criterion to test the existence of the giant component. In Ref. [8], Newman *et al.* narrowed the problem of finding the size distribution down to a numerical solution of an implicit functional equation, which is followed by the generating function inversion. Somewhat later, a few cases were resolved analytically [9], and recently, the formal solution for the size distribution of connected components in undirected networks has been found by means of the Lagrange inversion [18,19]. Such a solution permits fast computation of exact numerical values and allows simple asymptotic analysis.

A smaller number of results, however, is available for *directed* and *multiplex* configuration models. In these cases the aforementioned functional equation remains the main bottleneck and is typically addressed numerically with the only exception of percolation studies. Some percolation criteria were obtained analytically both in directed networks (in and out percolation [8] and weak percolation [11]) and in multiplex networks (k-core percolation [13], weak percolation [16], a strong mutually connected component [12], and a giant connected component [20]). To date, few results are available on the *size distribution* of finite connected components in these configuration models.

The present paper applies the Lagrange inversion principle to find exact expressions for size distributions of connected components in two generalizations of the configuration model: directed configuration networks and multiplex configuration networks. First, a brief review of Good's multivariate generalization of the Lagrange inversion formula is given. Then the size distributions for in, out, and weak components in

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directed configuration networks are formulated in terms of convolution powers of the degree distribution. These results are complemented by a detailed asymptotic analysis that reveals the existence of two distinct critical exponents. In the next section, the general case of weak multilayer connected components (i.e., components that include edges from an arbitrary layer) is considered. A formal expression for the size distribution is constructed and the asymptotic analysis is provided for two-layer multiplex networks. Furthermore, the relation between these results and the existence of a two-layer giant component is studied by means of perturbation analysis within the critical window. Finally, the results for directed and multiplex networks are illustrated with a few examples in the last section.

II. LAGRANGE SERIES INVERSION

Suppose that $R(x)$, $A(x)$, and $F(x)$ are formal power series in x. Then, according to the Lagrange inversion formula [21], the implicit functional equation

$$A(x) = xR[A(x)] \tag{1}$$

has a unique solution $A(x)$. Instead of an expression for $A(x)$, the Lagrange inversion formula recovers a discrete function that is generated by $A(x)$. In fact, the equation yields a slightly more general result: For an arbitrary formal power series $F(x)$, the coefficients of power series $F[A(x)]$ at x^n read

$$[x^n]F[A(x)] = \frac{1}{n}[t^{n-1}]F'(t)R^n(t), \quad n > 0. \tag{2}$$

Here $[t^{n-1}]$ refers to the coefficient at t^{n-1} of the corresponding power series. In the context of configuration models, Eq. (2) proved to be useful when deriving a formal expression for the size distribution of connected components in undirected networks [18].

The Lagrange inversion was generalized to the case of multivariate series by Good [22]. Following the original notation from [21], the Lagrange-Good theorem in d dimensions reads as follows: If we let $\mathbf{R}(\mathbf{x}) = [R_1(\mathbf{x}), R_2(\mathbf{x}), \dots, R_d(\mathbf{x})]$ be a vector of formal power series in variables $\mathbf{x} = (x_1, x_2, \dots, x_d)$ and let $\mathbf{A}(\mathbf{x})$ be a vector of formal power series satisfying

$$\mathbf{A}_i(x_1, \dots, x_d) = x_i R_i(\mathbf{A}_1, \dots, \mathbf{A}_d), \quad i = 1, \dots, d, \tag{3}$$

then for any formal power series $F(\mathbf{x})$,

$$[\mathbf{x}^\mathbf{n}]F[\mathbf{A}(\mathbf{x})] = [\mathbf{t}^\mathbf{n}]F(\mathbf{t})\det[K(\mathbf{t})]\mathbf{R}^\mathbf{n}(\mathbf{t}), \quad \mathbf{n} \in \mathbb{N}^d, \tag{4}$$

where $K(\mathbf{t})$ is a matrix from $\mathbb{R}^{d\times d}$,

$$K(\mathbf{t})_{i,j} = \delta_{i,j} - \frac{t_i}{\mathbf{R}_i(\mathbf{t})}\frac{\partial \mathbf{R}_i}{\partial t_j}(\mathbf{t}), \quad i,j = 1, \dots, d,$$

and $\mathbf{t} = (t_1, \dots, t_d)$, $\mathbf{n} = (n_1, \dots, n_d)$, $\mathbf{x}^\mathbf{n} = [x_1^{n_1}, \dots, x_d^{n_d}]$, and $\mathbf{x}(\mathbf{y}) = [x_1(\mathbf{y}), \dots x_d(\mathbf{y})]$. Analogously to the one-dimensional case (2), the operator $[\mathbf{x}^\mathbf{n}]$ refers to the coefficient at $x_1^{n_1}, \dots, x_d^{n_d}$. In the case when $d = 1$, Eq. (4) simplifies to the Lagrange equation (2). Although the original formulation of the Lagrange-Good equation (4) does involve an inversion of a generating function (GF), the only reason the inversion is used is to perform the convolution. Where convenient, we will exploit this fact and write (2) without any reference to GFs at all by utilizing the convolution power notation $f(\mathbf{k})^{*n} = f(\mathbf{k})^{*n-1} * f(\mathbf{k})$ and $f(\mathbf{k})^{*0} := \delta(\mathbf{k})$, where the multidimensional convolution is defined as $d(\mathbf{n}) = f(\mathbf{k}) * g(\mathbf{k})$,

$$d(\mathbf{n}) = \sum_{\mathbf{j}+\mathbf{k}=\mathbf{n}} f(\mathbf{j})g(\mathbf{k}) = [t^\mathbf{k}]F(x)G(x). \tag{5}$$

Here $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, and $\mathbf{n}$ are d-dimensional vectors. The sum in Eq. (5) runs over all partitions of vector $\mathbf{n}$ into two summands $\mathbf{j}$ and $\mathbf{k}$ such that

$$j_i + k_i = n_i, \quad 0 \leqslant j_i, k_i \leqslant n_i, \quad i = 1, \dots, d.$$

In practice, numerical values of the convolution can be conveniently obtained with a fast Fourier transform (FFT). We will see now how the inversion equations (2) and (4) can be applied to find the size distributions for connected components in directed and multiplex networks that are defined by their degree distributions.

III. DIRECTED NETWORKS

In a directed network, the bivariate degree distribution $0 \leqslant u(k,l) \leqslant 1$ denotes the probability of choosing a node with $k \geqslant 0$ incoming edges and $l \geqslant 0$ outgoing edges uniformly at random. Partial moments of this distribution are given by

$$\mu_{ij} = \sum_{k,l=0}^{\infty} k^i l^j u(k,l). \tag{6}$$

Since $u(k,l)$ is normalized, $\mu_{00} = 1$, and since the expected numbers for incoming and outgoing edges must coincide, $\mu_{10} = \mu_{01} = \mu$. The directed degree distribution $u(k,l)$ has two corresponding excess distributions: $u_{\text{in}}(k,l) = \frac{k+1}{\mu}u(k+1,l)$ and $u_{\text{out}}(k,l) = \frac{l+1}{\mu}u(k,l+1)$. Throughout this section, capital letters are used to denote the corresponding bivariate GFs: $U(x,y)$, $U_{\text{in}}(x,y)$, and $U_{\text{out}}(x,y)$. Four types of connected components are distinguished in directed configuration models: in components, out components, weak component, and strong component (the latter always has an infinite size in the thermodynamic limit [8]).

A. Sizes of in and out components

The size distributions for both in components $h_{\text{in}}(n)$, as generated by $H_{\text{in}}(x)$, and out components $h_{\text{out}}(n)$, as generated by $H_{\text{out}}(x)$, can be found by solving the following systems of functional equations [8]:

$$\begin{aligned} H_{\text{out}}(x) &= xU[\tilde{H}_{\text{out}}(x), 1], \\ \tilde{H}_{\text{out}}(x) &= xU_{\text{out}}[\tilde{H}_{\text{out}}(x), 1] \end{aligned} \tag{7}$$

and

$$\begin{aligned} H_{\text{in}}(x) &= xU[1, \tilde{H}_{\text{in}}(x)], \\ \tilde{H}_{\text{in}}(x) &= xU_{\text{in}}[1, \tilde{H}_{\text{in}}(x)]. \end{aligned} \tag{8}$$

These equations are similar to those describing connected components in the undirected configuration network, and following a derivation similar to the one from Ref. [18], one immediately obtains formal solutions in terms of the convolution power of the degree distribution,

$$\begin{aligned} h_{\mathrm{in}}(n) &= \frac{\mu}{n-1}\tilde{u}_{\mathrm{in}}^{*n}(n-2), \quad n>1 \\ h_{\mathrm{out}}(n) &= \frac{\mu}{n-1}\tilde{u}_{\mathrm{out}}^{*n}(n-2), \quad n>1 \\ h_{\mathrm{in}}(1) &= h_{\mathrm{out}}(1) = u(0,0). \end{aligned} \tag{9}$$

Here $\tilde{u}_{\mathrm{in}}(k) = \sum_{l=0}^{\infty} u_{\mathrm{in}}(k,l)$ and $\tilde{u}_{\mathrm{out}}(l) = \sum_{k=0}^{\infty} u_{\mathrm{out}}(k,l)$.

B. Weakly connected components

The generating function for the size distribution of weak components $W(x)$ satisfies the following system of functional equations [11]:

$$\begin{aligned} W(x) &= xU[W_{\mathrm{out}}(x),W_{\mathrm{in}}(x)]; \\ W_{\mathrm{out}}(x) &= xU_{\mathrm{out}}[W_{\mathrm{out}}(x),W_{\mathrm{in}}(x)], \\ W_{\mathrm{in}}(x) &= xU_{\mathrm{in}}[W_{\mathrm{out}}(x),W_{\mathrm{in}}(x)]. \end{aligned} \tag{10}$$

To solve this system we apply the Lagrange-Good formalism (3). First, one should transform (10) to match the bivariate version ($d=2$) of Eq. (3). Consider three bivariate formal power series $A(x,y)$, $A_1(x,y)$, and $A_2(x,y)$ that take their diagonals from correspondingly $\frac{1}{x}W(x)$, $W_{\mathrm{out}}(x)$, and $W_{\mathrm{in}}(x)$, that is,

$$\begin{aligned} A(x,x) &= \frac{1}{x}W(x), \\ A_1(x,x) &= W_{\mathrm{out}}(x), \\ A_2(x,x) &= W_{\mathrm{in}}(x) \end{aligned} \tag{11}$$

for $|x|<1$ and $x\in\mathbb{C}$. Additionally, let $R_1(x,y) := U_{\mathrm{out}}(x,y)$ and $R_2(x,y) := U_{\mathrm{in}}(x,y)$. If the couple $A_1(x,y)$ and $A_2(x,y)$ satisfies condition (3) for all values of (x,y), then as a partial case ($x=y$), the weaker condition (10))′ is also satisfied. Furthermore, by assigning $F(x,y) := U(x,y)$ one obtains the expression for the coefficients of generating function $A(x,y)$: For $i,j \geqslant 0$,

$$\begin{aligned} a(i,j) &= [x^i y^j]A(x,y) = [x^i y^j]U[A_1(x,y),A_2(x,y)] \\ &= \left[t_1^i t_2^j\right]U(t_1,t_2)\det[K(t_1,t_2)]U_{\mathrm{out}}(t_1,t_2)^i U_{\mathrm{in}}(t_1,t_2)^j, \end{aligned} \tag{12}$$

which, when rewritten with the convolution power notation (5), become

$$a(i,j) = u(k,l) * u_{\mathrm{out}}(k,l)^{*i-1} * u_{\mathrm{in}}(k,l)^{*j-1} * d(k,l)|_{\substack{k=i,\\ l=j}} \tag{13}$$

where

$$\begin{aligned} d(k,l) &= [u_{\mathrm{out}}(k,l) - ku_{\mathrm{out}}(k,l)] * [u_{\mathrm{in}}(k,l) - lu_{\mathrm{in}}(k,l)] \\ &\quad - lu_{\mathrm{out}}(k,l) * ku_{\mathrm{in}}(k,l). \end{aligned} \tag{14}$$

Here $d(k,l)$ is chosen in such a way that it is generated by $U_{\mathrm{out}}(t_1,t_2)U_{\mathrm{in}}(t_1,t_2)\det[K(t_1,t_2)]$, the product that appears in Eq. (12). For this reason the convolution powers in Eq. (13) are diminished by one: $i-1$ and $j-1$. Now, on the one hand, $w(n+1)$ is generated by $\frac{1}{x}W(x) = A(x,x)$, while on the other, $x^i y^j|_{y=x} = x^{i+j}$ and thus the sum of all $a(i,j) = [x^j y^j]A(x,y)$ such that $i+j=n+1$ yields the values of $w(n+1)$. Therefore, the final expression for the size distribution of weak components is written out as a diagonal sum

$$w(n) = \begin{cases} \sum_{i=0}^{n-1} a(i,n-i-1), & n>1 \\ u(0,0), & n=1. \end{cases} \tag{15}$$

From the computational perspective, the most efficient way to evaluate Eq. (13) numerically is to apply the FFT algorithm to find the convolution powers. In this case, the computation of $w(n)$ requires $O(n^2 \log n)$ multiplicative operations.

Besides being suitable for numerical computations, expressions (9) and (15) can be further treated analytically to obtain the asymptotic behavior of size distributions $w(n)$, $h_{\mathrm{in}}(n)$, and $h_{\mathrm{out}}(n)$ in the large-n limit. That is, we will search for such $w_\infty(n)$ [or correspondingly $h_{\mathrm{in},\infty}(n)$ and $h_{\mathrm{out},\infty}(n)$] that

$$\frac{w(n)}{w_\infty(n)} \to 1, \quad n\to\infty. \tag{16}$$

In the context of asymptotic theory, we limit ourself to the case of finite first moments $\mu_{ij}<\infty$, $i+j\leqslant 3$. As will be shown further on, this assumption will allow us to utilize the standard central limit theorem and formulate the analytical expressions for the asymptotes as a function of solely the first partial moments of the degree distribution μ_{ij}, $i+j\leqslant 3$. To keep the derivation concise, we define the shorthand for the vectors of expected values and covariance matrices of $u(k,l)$, $\frac{k}{\mu_{10}}u(k,l)$, and $\frac{l}{\mu_{01}}u(k,l)$:

$$\begin{aligned} \mu_0 &= \begin{bmatrix}\mu_{10}\\ \mu_{01}\end{bmatrix}, \\ \Sigma_0 &= \begin{bmatrix}\mu_{20}-\mu_{10}^2 & \mu_{11}-\mu_{10}\mu_{01}\\ \mu_{11}-\mu_{10}\mu_{01} & \mu_{02}-\mu_{01}^2\end{bmatrix}; \\ \mu_1 &= \frac{1}{\mu_{10}}\begin{bmatrix}\mu_{20}\\ \mu_{11}\end{bmatrix}, \\ \Sigma_1 &= \frac{1}{\mu_{10}^2}\begin{bmatrix}\mu_{30}\mu_{10}-\mu_{20}^2 & \mu_{21}\mu_{10}-\mu_{11}\mu_{20}\\ \mu_{21}\mu_{10}-\mu_{11}\mu_{20} & \mu_{12}\mu_{10}-\mu_{11}^2\end{bmatrix}; \\ \mu_2 &= \frac{1}{\mu_{01}}\begin{bmatrix}\mu_{11}\\ \mu_{02}\end{bmatrix}, \\ \Sigma_2 &= \frac{1}{\mu_{01}^2}\begin{bmatrix}\mu_{21}\mu_{01}-\mu_{11}^2 & \mu_{12}\mu_{01}-\mu_{02}\mu_{11}\\ \mu_{12}\mu_{01}-\mu_{02}\mu_{11} & \mu_{03}\mu_{01}-\mu_{02}^2\end{bmatrix}. \end{aligned} \tag{17}$$

Note that in directed networks $\mu_{10}=\mu_{01}=\mu$.

C. Asymptotes for in and out components

In the case of in and out components the asymptotic analysis coincides with the one performed in the case of the undirected network and has been covered elsewhere; for instance, compare Eq. (9) to Eq. (8) in Ref. [18]. Taking this into the account, we can immediately proceed with expressions for the asymptotes

$$h_{\mathrm{in},\infty}(n) = C_{1,1} e^{-C_{1,2} n} n^{-3/2},$$
$$C_{1,1} = \frac{\mu^2}{\sqrt{2\pi\left(\mu\mu_{30} - \mu_{20}^2\right)}},$$
$$C_{1,2} = \frac{(\mu_{20} - 2\mu)^2}{2\left(\mu\mu_{30} - \mu_{20}^2\right)}; \tag{18}$$

$$h_{\mathrm{out},\infty}(n) = C_{2,1} e^{-C_{2,2} n} n^{-3/2},$$
$$C_{2,1} = \frac{\mu^2}{\sqrt{2\pi\left(\mu\mu_{03} - \mu_{02}^2\right)}},$$
$$C_{2,2} = \frac{(\mu_{02} - 2\mu)^2}{2\left(\mu\mu_{03} - \mu_{02}^2\right)} \tag{19}$$

and refer the reader to Ref. [18] for the derivation. One can see that, depending on the values of the moments, the asymptotes (18) and (19) switch between exponential and algebraic decays. The algebraic asymptote exhibits slope $-\frac{3}{2}$, which implies that in this case the size distributions feature infinite expected values. According to Eqs. (18) and (19), the algebraic asymptote emerges when $\mu_{20} - 2\mu = 0$ for in components and $\mu_{02} - 2\mu = 0$ for out components, both of which coincide with the critical point for the existence of the corresponding giant components [11].

D. Asymptote for weakly connected components

The asymptotic analysis for the size distribution of weak components is conceptually different from the previous case: Unlike in Eq. (9), the expression for the size distribution (13)–(15) contains the complete bivariate degree distribution and therefore cannot be treated analogously to the case of undirected networks.

We start by replacing the generating function appearing on the right-hand side of Eq. (12) with a characteristic function by introducing a change of variables $t_1 = e^{i\omega_1}$ and $t_2 = e^{i\omega_2}$:

$$\phi_a(\omega_1,\omega_2) = U(e^{i\omega_1},e^{i\omega_2})\det[K(e^{i\omega_1},e^{i\omega_2})] \times U_{\mathrm{out}}(e^{i\omega_1},e^{i\omega_2})^i U_{\mathrm{in}}(e^{i\omega_1},e^{i\omega_2})^j. \tag{20}$$

Here the complex unity is defined as $i^2 = -1$; it should not be confused with the parameters i,j. By setting $\phi(\omega_1,\omega_2) := U(e^{i\omega_1},e^{i\omega_2})$ and expanding U_{in}, U_{out}, and K according to their definitions one obtains

$$\begin{aligned}\phi_a(\omega_1,\omega_2) = e^{-i(i\omega_2 + j\omega_1)}\phi(\omega_1,\omega_2)\Bigg(&\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^j\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^i + \frac{1}{j}i\frac{\partial}{\partial\omega_2}\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^j\\ &\times\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^i + \frac{1}{i}\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^j i\frac{\partial}{\partial\omega_1}\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^i - \frac{1}{ij}\frac{\partial}{\partial\omega_2}\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^j\\ &\times\frac{\partial}{\partial\omega_1}\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^i + \frac{1}{ij}\frac{\partial}{\partial\omega_1}\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^j\frac{\partial}{\partial\omega_2}\left[-\frac{i}{\mu}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^i\Bigg).\end{aligned} \tag{21}$$

Having $\phi_a(\omega_1,\omega_2)$ in this format allows us to apply the central limit theorem, which guarantees the pointwise convergence of the following limits:

$$\begin{aligned}\lim_{j\to\infty}\left|\left(-\frac{i}{\mu}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right)^j - \phi_g(\omega,j\mu_1,j\Sigma_1)\right| = 0,\\ \lim_{i\to\infty}\left|\left(-\frac{i}{\mu}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right)^i - \phi_g(\omega,i\mu_2,i\Sigma_2)\right| = 0.\end{aligned} \tag{22}$$

Here $\phi_g(\omega,\mu,\Sigma) = e^{i\mu^\top\omega - (1/2)\omega^\top\Sigma\omega}$, $\omega = (\omega_1,\omega_2)^\top$ denotes the characteristic function for the bivariate Gaussian-distributed random variable, and μ_1, μ_2, Σ_1, and Σ_2 are as defined in Eq. (17). Now, after substituting the limiting functions from (22) into (21), evaluating the partial derivatives, and using the symmetry of matrices Σ_1 and Σ_2, one obtains

$$\begin{aligned}\phi_{a,\infty}(\omega_1,\omega_2) = \exp\{i[j\mu_1 + i\mu_2 - (j,i) + \mu_0]\omega - \tfrac{1}{2}\omega^\top(j\Sigma_1 + i\Sigma_2)\omega\}\{I(\mu_1 - \mu_2) + \mu_1^\top D\mu_2\\ + [I(\Sigma_1 - \Sigma_2) + \mu_1^\top D\Sigma_2 - \mu_2^\top D\Sigma_1]\omega - \omega^\top\Sigma_2 D\Sigma_1\omega\},\end{aligned} \tag{23}$$

where

$$D = \begin{bmatrix} 0 & -1\\ 1 & 0\end{bmatrix}, \quad I = \begin{bmatrix} 1\\ -1\end{bmatrix}.$$

Note that Eq. (23) does not contain Σ_0, which becomes negligible in the limit of large i and j. After applying the inverse Fourier transform, (23) becomes

$$a_\infty(i,j) = C(x)\frac{e^{-(1/2)x^\top\Sigma^{-1}x}}{2\pi\sqrt{\det(\Sigma)}}, \tag{24}$$

where $x = (i,j) + (j,i) - (j\mu_1^\top + i\mu_2^\top) + \mu_0$ and

$$C(x) = C_0 + C_1\frac{x}{n} + x^\top C_2\frac{x}{n^2},\quad C_0 = I(\mu_1 - \mu_2) + \mu_1^\top D\mu_2,\quad C_1 = [I(\Sigma_1 - \Sigma_2) + \mu_1^\top D\Sigma_2 - \mu_2^\top D\Sigma_1]\left(\frac{1}{n}\Sigma\right)^{-1},$$

$$C_2 = -\left(\frac{1}{n}\Sigma\right)^{-1}\Sigma_2 D\Sigma_1\left(\frac{1}{n}\Sigma\right)^{-1},\quad \Sigma = j\Sigma_1 + i\Sigma_2.$$

By introducing new variables

$$n = i + j,\quad z = \frac{i-j}{i+j}, \tag{25}$$

one obtains $x = n(az + b + \mu_0/n)$ and

$$\Sigma = n(Az + B), \tag{26}$$

where

$$\begin{aligned} a &= \frac{\mu_1 - \mu_2}{2},\quad b = 1 - \frac{\mu_1 + \mu_2}{2},\\ A &= \frac{\Sigma_2 - \Sigma_1}{2},\quad B = \frac{\Sigma_1 + \Sigma_2}{2}. \end{aligned} \tag{27}$$

Under this change of variables, $\frac{1}{n}\Sigma = (Az + B)$ is independent of n and, consequently, so are C_0, C_1, and C_2. Furthermore, the exponential function from (24) can be now rewritten as a univariate Gaussian function in z,

$$\begin{aligned} \frac{e^{-(1/2)x^\top(\Sigma)^{-1}x}}{2\pi\sqrt{\det(\Sigma)}} &= \frac{\exp\left[-\frac{1}{2}n^2(az+b+\mu_0/n)^\top[n(Az+B)]^{-1}\left(az+b+\frac{\mu_0}{n}\right)\right]}{2\pi\sqrt{\det[n(Az+B)]}}\\ &= \frac{\exp\left[-\frac{1}{2}(az+b+\mu_0/n)^\top\left(\frac{Az+B}{n}\right)^{-1}\left(az+b+\frac{\mu_0}{n}\right)\right]}{2\pi\sqrt{\det[n(Az+B)]}}\\ &= \frac{\exp\left[-\left(z+\frac{\mu_0^\top S_n^{-1}a/n + a^\top S_n^{-1}b}{a^\top S_n^{-1}a}\right)^2\Big/2(a^\top S_n^{-1}a)^{-1}\right]}{\sqrt{2\pi\left(a^\top S_n^{-1}a\right)^{-1}}}C_b(n,z)\\ &= C_b(n)\mathcal{N}\left(z, -\frac{\mu_0^\top S_n^{-1}a/n + a^\top S_n^{-1}b}{a^\top S_n^{-1}a}, a^\top S_n^{-1}a\right), \end{aligned} \tag{28}$$

where $S_n = \frac{Az+B}{n}$ and

$$C_b(n) = \frac{\exp\left[-\frac{1}{2}\left((b+\mu_0/n)^\top S_n^{-1}(b+\mu_0/n) - \frac{(\mu_0^\top S_n^{-1}a/n + a^\top S_n^{-1}b)^2}{a^\top S_n^{-1}a}\right)\right]}{n^2\sqrt{2\pi\,\det(S_n)a^\top S_n^{-1}a}}$$

and $|z| \leqslant 1$. At the limit $n \to \infty$ the variance of this Gaussian function vanishes as $O(n^{-1})$ and the expected value remains bounded. Indeed, for a fixed z such that S_n^{-1} exists,

$$a^\top S_n^{-1}a = O(n^{-1}),\quad \frac{\mu_0^\top S_n^{-1}a/n + a^\top S_n^{-1}b}{a^\top S_n^{-1}a} = \frac{a^\top(Az+B)^{-1}b}{a^\top(Az+B)^{-1}a} + O(n^{-1}),$$

so the Gaussian function itself tends to the Dirac delta function $\delta(z + \frac{a^\top(Az+B)^{-1}b}{a^\top(Az+B)^{-1}a})$. Recall that, according to (15), the size distribution is defined as a sum of the diagonal elements

$$w_\infty(n+1) = \sum_{i+j=n} a_\infty(i,j) = \sum_{k=1}^{n} a_\infty(i,j)\Big|_{\substack{i+j=n,\\ j=(i-j)k/n}}.$$

This sum can be viewed as an estimator for an integral

$$w_\infty(n+1) = \frac{1}{n}\int_{-1}^{1}\delta\left(z + \frac{a^\top(Az+B)^{-1}b}{a^\top(Az+B)^{-1}a}\right)C_a\left[n\left(az+b+\frac{\mu_0}{n}\right)\right]C_b(n,z)dz \tag{29}$$

such that $\lim_{n\to\infty}|w(n) - w_\infty(n)| = 0$. The δ function under the integral is nonzero only at $z = r_k$, where r_k are roots of the nonlinear equation

$$a^\top(Az+B)^{-1}az + a^\top(Az+B)^{-1}b = 0. \tag{30}$$

Since A and B are symmetric matrices from $\mathbb{R}^{2\times 2}$, the matrix equation (30) simplifies to

$$a^\top \mathrm{adj}(A)az^2 + [a^\top \mathrm{adj}(B)a + a^\top \mathrm{adj}(A)b]z + a^\top \mathrm{adj}(B)b = 0 \tag{31}$$

for such z that $\det(Az + B) \neq 0$. Here $\mathrm{adj}(A) := D^\top AD$ is the adjugate matrix of A. Depending on the value of the leading coefficient, Eq. (31) is either a linear equation, if $a^\top \mathrm{adj}(A)a = 0$, and has one root

$$r_1 = -\frac{a^\top \mathrm{adj}(B)a + a^\top \mathrm{adj}(A)b}{2a^\top \mathrm{adj}(A)a}$$

or a quadratic equation [$a^\top \mathrm{adj}(A)a \neq 0$] having at most two distinct roots

$$r_1 = \frac{-a^\top \mathrm{adj}(B)a - a^\top \mathrm{adj}(A)b - \sqrt{d}}{2a^\top \mathrm{adj}(A)a},$$
$$r_2 = \frac{-a^\top \mathrm{adj}(B)a - a^\top \mathrm{adj}(A)b + \sqrt{d}}{2a^\top \mathrm{adj}(A)a},$$

where

$$d = [a^\top \mathrm{adj}(B)a + a^\top \mathrm{adj}(A)b]^2 - 4a^\top \mathrm{adj}(A)aa^\top \mathrm{adj}(B)b.$$

Suppose that there is only one real root $r_1 \in [-1,1]$, which automatically implies that the other root either does not exist or is greater than 1 in its absolute value. As a convolution with the δ function, the integral in (29) is simply an evaluation at a point,

$$w_\infty(n+1) = \frac{1}{n} C_a\left[n\left(az + b + \frac{\mu_0}{n}\right)\right] C_b(n,z)\bigg|_{z=r1}.$$

After expanding $C_a(x)$ and $C_b(n,z)$ according to their definitions and some basic algebraic transformations the latter expression becomes

$$w_\infty(n+1) = L_0(L_1 n^{-3/2} + L_2 n^{-5/2}) e^{-(E_1 n + E_0 + E_{-1} n^{-1})} \tag{32}$$

and is exhaustively defined by the definitions (17) and (27) and the following list of constants:

$$L_0 = 2^{-3/2}[\pi \det(S) a^\top S^{-1} a]^{-1/2},$$
$$L_1 = C_1\mu_0 + (r_1 a + b)^\top (C_2 + C_2^\top)\mu_0,$$
$$L_2 = \mu_0^\top C_2 \mu_0, \quad E_1 = \frac{a^\top S^{-1}(ab^\top - ba^\top)S^{-1}b}{2a^\top S^{-1} a}, \tag{33}$$
$$E_0 = \frac{a^\top S^{-1}(ab^\top - ba^\top)S^{-1}\mu_0}{a^\top S^{-1} a},$$
$$E_{\text{-}1} = \frac{a^\top S^{-1}(a\mu_0^\top - \mu_0 a^\top)S^{-1}\mu_0}{2a^\top S^{-1} a}, \quad S = Ar_1 + B.$$

Note that in the derivation of (32) the terms containing $n^{-0.5}$ cancel out.

If $E_1 \neq 0$, the asymptote (32) decays exponentially fast; conversely, $E_1 = 0$ is a sufficient and necessary condition for the asymptote to decay as an algebraic function. The latter condition is equivalent to

$$ab^\top - ba^\top = 0,$$

which after expansion according to the definitions (17) and (27) simplifies to

$$2\mu\mu_{11} - \mu\mu_{02} - \mu\mu_{20} + \mu_{02}\mu_{20} - \mu_{11}^2 = 0.$$

This expression coincides with the definition of the critical point for the weak giant component [11].

E. Degenerate case of excess degree distribution

Degeneration to the univariate case degree distributions $u(k,l) = 0\,(k > 0)$ or $u(k,l) = 0\,(l > 0)$ presents little interest as no connected components with size greater than 1 can be formed. However, the asymptotic analysis for the case when one of the bivariate *excess* distributions is degenerate, $u_{\text{in}}(k,l) = 0\,(k > 0)$ or $u_{\text{out}}(k,l) = 0\,(l > 0)$, requires separate attention. Without loss of generality, suppose

$$u_{\text{in}}(k,l) = 0, \quad k > 0. \tag{34}$$

Then, on the one hand, the covariance matrix Σ_1 is singular, and if $\det(\Sigma_2) \neq 0$, the determinant

$$\det(Az + B) = \tfrac{1}{2}\det[(\Sigma_1 - \Sigma_2)z + \Sigma_1 + \Sigma_2] = 0$$

only if $z = 1$. On the other hand, $z = 1$ is the only root of the quadratic equation (31) from the interval of interest, $z \in [-1,1]$. Consequently, Eq. (29) fails to provide a valid description of the asymptote since $(Az + B)^{-1}$ does not exist at z and one must seek an alternative route to perform the asymptotic analysis.

Qualitatively, the condition (34) means that there is at most one incoming edge per node. In view of the fact that the topology is locally treelike, which is characteristic of finite components in configuration models, each finite component has exactly *one* node with no incoming edges: the root node. Evidently, in this case, the asymmetry of the edges forces the connected components to be globally asymmetric as well: There is exactly one node per component with ingoing degree 0 and the whole component can be explored by starting at the root node and following exclusively outgoing edges. We will now exploit the presence of such a global directionality in order to perform an asymptotic analysis for component sizes.

Let $w_0(n)$ denote the probability that a component associated with the root node has size n. It is n times more likely to randomly select any other node than the root from a given component. Therefore,

$$w(n) = \frac{1}{C} n w_0(n),$$

where the normalization constant C is the expected component size. The condition on u_{in}, as given in Eq. (34), can be rewritten as a condition on $u(k,l)$, that is, $u(k,l) = 0\ (k > 1)$. Let us introduce some auxiliary notation

$$\mu_0' = \sum_{l=0}^{\infty} u(0,l), \quad \mu_1' = \sum_{l=0}^{\infty} u(1,l),$$
$$u_0(l) = \frac{u(0,l)}{\mu_0'}, \quad u_1(l) = \frac{u(1,l)}{\mu_1'},$$
$$\mu_{0j}' = \sum_{l=0}^{\infty} l^j u_0(l), \quad \mu_{1j}' = \sum_{l=0}^{\infty} l^j u_1(l),$$

where $j = 0,1,2$. We will go through a derivation similar to Eq. (10) and construct a set of univariate equations for $w_0(n)$,

$$W_0(x) = xU_0[W_{01}(x)],$$
$$W_{01}(x) = xU_1[W_{01}(x)], \tag{35}$$

where $W_0(x)$, $U_0(x)$, and $U_1(x)$ are generating functions for, respectively, $w_0(n)$, $u_0(l)$, and $u_1(l)$. By solving (35) for $C = W_0(1)$ one obtains the expected component size

$$C = 1 + \frac{\mu'_{01}}{1-\mu'_{11}}.$$

Furthermore, applying the Lagrangian inversion to Eq. (35) gives the formal solution

$$w_0(n) = \frac{1}{n-1}[ku_0(k) * u_1^{*n-1}(k)](n-2),$$

which leads to the following asymptote for large n:

$$w_\infty(n) = L_0 n^{-1/2} e^{-E_0 - nE_1}, \tag{36}$$

where

$$L_0 = \frac{\mu'_{01}(\mu'_{11}-1)}{(\mu'_{11}-\mu'_{01}-1)\sqrt{2\pi\left(\mu'_{12}-\mu'^2_{11}\right)}},$$

$$E_0 = \frac{(\mu'_{11}-1)(\mu'_{01}+\mu'_{02}-\mu'_{01}\mu'_{11})}{\mu'_{01}\left(\mu'_{12}-\mu'^2_{11}\right)},$$

$$E_1 = \frac{(\mu'_{11}-1)^2}{2\left(\mu'_{12}-\mu'^2_{11}\right)}.$$

In contrast to the nondegenerate asymptote (32), which has the leading exponent $-\frac{3}{2}$, the leading exponent in the degenerate asymptote (36) is $-\frac{1}{2}$. Nevertheless, a pure algebraic asymptote $n^{-1/2}$ cannot be observed under the condition of finite moments μ'_{03} and μ'_{12}. Indeed, $E_1 \to 0$ also implies that $\mu'_{11} - 1 \to 0$ and, consequently, the prefactor L_0 vanishes as well.

IV. MULTIPLEX NETWORKS

A. General case of an arbitrary number of layers

This section considers the multiplex configuration model: a generalization of the configuration model in which undirected edges are partitioned into subsets commonly referred to as types, layers, or colors [1]. In multiplex networks each edge belongs to one of many layers. Figure 1 illustrates an instance of a three-layer multiplex network with ten nodes. There are multiple ways to define a path in multiplex networks. A multilayer path, or simply a path in this section of the paper, is a path that combines edges from arbitrary layers. This definition of a path gives rise to a definition of multilayer connected components as sets of nodes connected together with the path.

Suppose that each edge belongs to a layer from $\Omega = \{1,\dots,N\}$. We update the definition of the degree distribution to be a multivariate function $u(k_1,\dots,k_N)$, $k_i \in \mathbb{N}_0$, that denotes the probability of randomly choosing a node with k_i adjacent edges from layer i. As before, the degree distribution is normalized $\sum_{k_1,\dots,k_N} u(k_1,\dots,k_n) = 1$. The excess degree distribution associated with the ith layer is defined as $u_i(k_1,\dots,k_N) := \frac{k_i+1}{\mu_i} u_i(k_1,\dots,k_i+1,\dots,k_N)$,

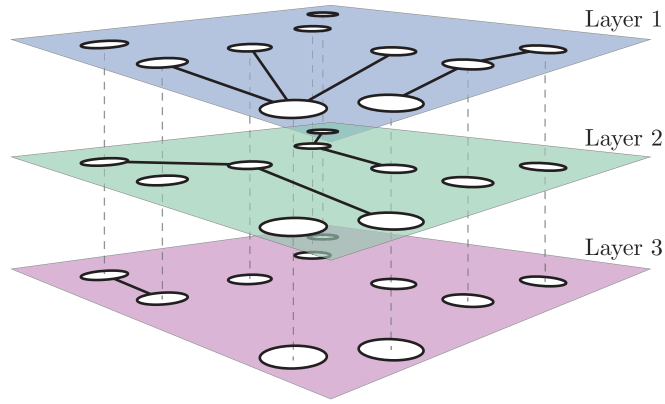

FIG. 1. Example of multiplex network with three layers. Each edge belongs to only one layer, whereas each node has one copy in each layer. Even though in each separate layer the nodes can be partitioned into different sets of connected components, the network is fully connected in the weak sense, when the connectivity information from all layers is combined.

where $\mu_i = \sum_{k_1,\dots,k_N}(k_i+1)u_i(k_1,\dots,k_i+1,\dots,k_N)$ is the expected degree for ith-layer edges. Let $w(n)$ denote the size distribution for multilayer components. By following similar considerations as in Sec. III, one derives functional equations for the GF of the size distribution $W(n)$:

$$\begin{aligned} W(x) &= xU[W_1(x),\dots,W_N(x)],\\ W_1(x) &= xU_1[W_1(x),\dots,W_N(x)],\\ &\vdots\\ W_N(x) &= xU_N[W_1(x),\dots,W_N(x)], \end{aligned} \tag{37}$$

where the uppercase notation $U(x_1,\dots,x_N)$ and $U_i(x_1,\dots,x_N)$, $i=1,\dots,N$, is used to denote multivariate generating functions of the corresponding distributions. The system of functional equations (37) is a special case of (3) and thus can be solved by applying the Lagrange-Good formula. Indeed, let $W_i(x)$ define the diagonals of $\mathbf{A}_i(\mathbf{x})$, that is, $\mathbf{A}_i(x,\dots,x) := W_i(x)$, $|x| < 1$ and $x \in \mathbb{C}$, for $i = 1,\dots,N$. Additionally, let $\mathbf{R}(\mathbf{x}) = [U_1(\mathbf{x}),\dots,U_1(\mathbf{x})]$ and $F(\mathbf{x}) = U(\mathbf{x})$. Then the Lagrange-Good formula yields the expression for $a(k_1,\dots,k_N)$ that is generated by $A(x_1,\dots,x_N) = \frac{1}{x}W(x)$. The values for $w(n)$ can then be recovered using the relation

$$w(n) = \sum_{\substack{k_1+\dots+k_N=n-1\\ k_i\geqslant 0}} a(k_1,\dots,k_N)$$

and the complete equation for the component size distribution in the multilayered configuration network reads, for $n > 1$,

$$w(n) = \sum_{\substack{k_1+\dots+k_N=n-1\\ k_i\geqslant 0}} u(\mathbf{k}) * \det_*[D(\mathbf{k})] * u_1(\mathbf{k})^{*k_1} * \dots * u_N(\mathbf{k})^{*k_N}, \tag{38}$$

where

$$D(\mathbf{k})_{i,j} = \delta_{i,j} - [k_j u_i(\mathbf{k})] * u_i(\mathbf{k})^{*(-1)}, \quad i,j = 1,\dots,N,$$

and $\det_*[D(\mathbf{k})]$ refers to the determinant of matrix D computed with the multiplication replaced by the convolution, for example,

$$\det_*\begin{bmatrix} a(\mathbf{k}) & b(\mathbf{k}) \\ c(\mathbf{k}) & d(\mathbf{k}) \end{bmatrix} = a(\mathbf{k}) * d(\mathbf{k}) - b(\mathbf{k}) * c(\mathbf{k}).$$

B. Two-layer multiplex network

Suppose $N = 2$, that is to say, each edge belongs to either layer 1 or layer 2. In this case, the degree distribution $d(k,l)$ is the probability of randomly selecting a node that bears k edges in layer 1 and l edges in layer 2. Where it leads to no confusion, we will reuse the notation from the preceding section. For instance, the shorthand notation for the moments and for the vectors of expected values and covariance matrices are as given in (6) and (17), respectively. The total probability is normalized $\mu_{00} = 1$, but the expected numbers of edges in each layer need not be the same:

$$\mu_{10} \neq \mu_{01}.$$

The two-dimensional version of (37) reads

$$\begin{aligned} W(x) &= xU[W_1(x), W_2(x)], \\ W_1(x) &= xU_1[W_1(x), W_2(x)], \\ W_2(x) &= xU_2[W_1(x), W_2(x)], \end{aligned} \tag{39}$$

where $U(x,y)$, $U_1(x,y)$, and $U_2(x,y)$ denote the corresponding generating functions for degree and excess distributions, and $W(x)$ is the generating function for the size distribution of two-layer connected components. The only structural difference between the equation for directed networks (10) and the equation for two-layered network (39) is the order of arguments in the degree distribution GFs, which indicates the presence or absence of symmetric edges (compare $W_{\text{in}}(x) = xU_{\text{in}}[W_{\text{out}}(x), W_{\text{in}}(x)]$ against $W_1(x) = xU_1[W_1(x), W_2(x)]$). By setting $N = 2$ in (38) one obtains the formal solution to (39),

$$w(n) = \sum_{i=0}^{n-1} a(i, n-i-1), \quad n > 1, \tag{40}$$

where, for $i,j \geqslant 0$,

$$a(i,j) = u(k,l) * u_1(k,l)^{*(i-1)} * u_2(k,l)^{*(j-1)} * d(k,l)|_{\substack{k=i \\ l=j}} \tag{41}$$

and

$$\begin{aligned} d(k,l) = {} & [u_1(k,l) - ku_1(k,l)] * [u_2(k,l) - lu_2(k,l)] \\ & - lu_1(k,l) * ku_2(k,l). \end{aligned} \tag{42}$$

We will now see how the asymptotic theory from Sec. III D can be recast to fit the case of the two-layer multiplex networks.

C. Asymptotic analysis for a bilayer network

Let μ_1 and μ_2 denote expected values and Σ_1 and Σ_2 covariance matrices of $\frac{k}{\mu_{10}}u(k,l)$ and $\frac{l}{\mu_{01}}u(k,l)$, as given in the definition (17). The characteristic function for the right-hand side of Eq. (41) reads

$$\begin{aligned} \phi_a(\omega_1,\omega_2) = {} & e^{-i(i\omega_1 + j\omega_2)}\phi(\omega_1,\omega_2)\Bigg(\left[-\frac{i}{\mu_1}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^i\left[-\frac{i}{\mu_2}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^j + \frac{i}{j}\frac{\partial}{\partial\omega_2}\left[-\frac{i}{\mu_1}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^i \\ & \times\left[-\frac{i}{\mu_2}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^j + \frac{i}{i}\left[-\frac{i}{\mu_1}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^i\frac{\partial}{\partial\omega_1}\left[-\frac{i}{\mu_2}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^j - \frac{1}{ij}\frac{\partial}{\partial\omega_2}\left[-\frac{i}{\mu_1}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^i \\ & \times\frac{\partial}{\partial\omega_1}\left[-\frac{i}{\mu_2}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^j + \frac{1}{ij}\frac{\partial}{\partial\omega_1}\left[-\frac{i}{\mu_1}\frac{\partial}{\partial\omega_1}\phi(\omega_1,\omega_2)\right]^i\frac{\partial}{\partial\omega_2}\left[-\frac{i}{\mu_2}\frac{\partial}{\partial\omega_2}\phi(\omega_1,\omega_2)\right]^j\Bigg). \end{aligned} \tag{43}$$

In the large-n limit, the latter approaches

$$a_\infty(i,j) = C(x)\frac{e^{-(1/2)x^\top\Sigma^{-1}x}}{2\pi\sqrt{\det(\Sigma)}}, \tag{44}$$

where $x = 2(i,j) - (i\mu_1^\top + j\mu_2^\top) + \mu_0$ and

$$\begin{aligned} C(x) &= C_0 + C_1\frac{x}{n} + \frac{x^\top}{n}C_2\frac{x}{n}, \quad C_0 = 4 - 2(I_1\mu_1 + I_2\mu_2) - \mu_1 D\mu_2, \\ C_1 &= [\mu_2^\top D\Sigma_1 - \mu_1^\top D\Sigma_2 - 2(I_1^\top\Sigma_1 + I_2^\top\Sigma_2)]\left(\frac{1}{n}\Sigma\right)^{-1}, \\ C_2 &= -\left(\frac{1}{n}\Sigma\right)^{-1}\Sigma_1 D\Sigma_2\left(\frac{1}{n}\Sigma\right)^{-1}, \quad \Sigma = i\Sigma_1 + j\Sigma_2. \end{aligned} \tag{45}$$

By applying the change of variables (25), one obtains $x = n(az + b + \frac{\mu_0}{n})$ and $\Sigma = n(Az + B)$, where

$$a = I_1 - I_2 + \frac{\mu_1 - \mu_2}{2},\quad b = 1 - \frac{\mu_1 + \mu_2}{2},\quad A = \frac{\Sigma_1 - \Sigma_2}{2},\quad B = \frac{\Sigma_1 + \Sigma_2}{2}. \tag{46}$$

Now the coefficients C_0, C_1, C_2, and $\frac{1}{n}\Sigma = Az + B$ are independent of n and Eq. (44) is identical to (24) up to the definitions of the constants a, b, A, B, C_0, C_1, and C_2 that are given above. Therefore, we can readily use the asymptote (32) also in the case of a two-layer network. It is enough to redefine the constants according to definitions (45) and (46) and take $z = r_1$, where $r_1 \in [-1,1]$ denotes the root of Eq. (31). As before, the condition $ab^\top - ba^\top = 0$ indicates the emergence of the algebraic decay $n^{-3/2}$ in the sizes of connected components. When the latter equality is expanded according to the definitions (46) and (17), one obtains the criterion in terms of degree distribution moments

$$G(u) := \mu_{11}^2 - (\mu_{20} - 2\mu_{10})(\mu_{02} - 2\mu_{01}) = 0. \tag{47}$$

As in the case of directed networks, the degenerate excess degree distribution $u_1(k,l) = 0\,(k > 0)$ renders the asymptotic analysis not applicable. Nevertheless, the degenerate case is equivalent to a one-layer network with coupled nodes that has a univariate degree distribution $d(l) = d(0,l) + \frac{1}{2}d(1,l)$ $(l = 0, \ldots)$. The asymptotic theory for monolayer components has been covered in Ref. [18] and, unlike in the case of directed networks, no new asymptotic modes emerge when the excess distribution is degenerate.

D. Criticality in two-layer multiplex networks

When a configuration network is two layered, one may speak of a connected component contained within a specific layer, i.e., a path comprised solely of edges from one layer, or a multilayer (weak) connected component that emerges from a combination of two layers, i.e., both types of edges may appear in the path. No matter what type of connected components is considered, the asymptote of the component size distribution exhibits either exponential or algebraic decays.

When focusing on single-layer connected components, for instance, in layer 1, the condition $\mu_{20} - 2\mu_{10} = 0$ signifies the critical regime of the corresponding size distribution. Furthermore, a giant component exists within this layer if and only if $\mu_{20} - 2\mu_{10} > 0$. The existence of a giant component within a single layer is a strong condition: It automatically implies the existence of the weak two-layer giant component. More importantly, the two-layer giant component can also exist even when there are no single-layer giant components.

When two-layered connected components are considered, the criterion (47) gives the condition for the algebraic decay in the component size distribution. It is important to note that one should consider this inequality only together with the validity conditions of the asymptotic theory, $\mu_{ij} < \infty$, $i + j \leqslant 3$, and the existence of the root of Eq. (30), $|r_1| \leqslant 1$. For instance, unlike in the case of a single-layer network, one cannot associate the existence of the two-layer giant component solely with the sign of the left-hand side of Eq. (47). For a simple counterexample, set $\mu_{11} = 0$. Then the left-hand side of Eq. (47) is positive if and only if one layer contains a giant component and the other does not. When both layers contain a giant component simultaneously (or both layers contain no giant component), the sign is negative.

Now let us consider a critical degree distribution $u_c(k,l)$ such that $G(u_c) = 0$ and $\mu_{11} > 0$. Assume that there are no single-layer giant components, that is to say, $2\mu_{10} - \mu_{20}$ and $2\mu_{01} - \mu_{02}$ are positive quantities. The upper bounds on these quantities can be obtained from the Cauchy-Schwarz inequality $\mu_{11}^2 \leqslant \mu_{20}\mu_{02}$. The latter, when combined with the condition $G(u_c) = 0$, yields

$$0 < 2\mu_{10} - \mu_{20} \leqslant \frac{\mu_{10}\mu_{02}}{\mu_{01}}, \quad 0 < 2\mu_{01} - \mu_{02} \leqslant \frac{\mu_{01}\mu_{20}}{\mu_{10}}. \tag{48}$$

Since there are no isolated nodes $u(k,l) = 0$ $(k,l = 0)$ the sum of expected numbers of edges is bounded below with

$$\mu_{10} + \mu_{01} \geqslant 1. \tag{49}$$

Additionally, one obtains the following bounds from the monotonicity of the moments:

$$\mu_{20} < \mu_{10}^2, \quad \mu_{02} < \mu_{01}^2. \tag{50}$$

Let us perturb the expected number of edges μ_{10} by uniformly adding (or removing) a small number of edges $d\alpha$ in the first layer. Due to this perturbation, the degree distribution varies as $du(k,l) = [u(k-1,l) - u(k,l)]d\alpha$. The perturbation conserves the total probability $\sum_{k,l=0}^{\infty} du(k,l) = 0$, whereas the expected number of edges indeed varies as

$$d\mu_{10} = \sum_{k,l=0}^{\infty} k[u(k-1,l) - u(k,l)]d\alpha = \sum_{k,l=0}^{\infty} [(k+1)u(k,l) - ku(k,l)]d\alpha = d\alpha.$$

After expanding variations $d\mu_{11} = \mu_{01}d\alpha$ and $d\mu_{20} = (2\mu_{10} + 1)d\alpha$ in a similar fashion, we write the Gâteaux derivative

$$\begin{aligned}\frac{d}{d\alpha}G(u_c) &= \lim_{d\alpha\to 0} \frac{G(u_c + du) - G(u_c)}{d\alpha}\\ &= (\mu_{11} + \mu_{01}d\alpha)^2 - [\mu_{20} + (2\mu_{10} + 1)d\alpha\\ &\quad - 2(\mu_{10} + d\alpha)](\mu_{02} - 2\mu_{01})\\ &= (2\mu_{01} - \mu_{02})(2\mu_{10} - 1) + 2\mu_{01}\mu_{11}.\end{aligned}$$

We will now show that $\frac{d}{d\alpha}G(u_c) > 0$ by considering two cases. First, let $2\mu_{10} - 1 \geqslant 0$; then $2\mu_{01} - \mu_{02} > 0$ according to (48) and consequently $\frac{d}{d\alpha}G(u_c) > 0$. Second, let us assume that the opposite is true, $0 < \mu_{10} < \frac{1}{2}$: By expressing μ_{11} from (47) and plugging it into $\frac{d}{d\alpha}G(u_c) > 0$ one obtains

$$\mu_{01}(2\mu_{10} - \mu_{20}) - \frac{1}{\mu_{01}}(2\mu_{01} - \mu_{02})(1 - 2\mu_{10})^2 > 0. \tag{51}$$

Combining the lower bound $\mu_{01} \geqslant 1 - \mu_{10} = \frac{1}{2}$ [as follows from (49)] and the upper bound on μ_{20} [as given in (50)], the first term in (51) is bounded from below with $\mu_{01}(2\mu_{10} - \mu_{20}) \geqslant \frac{1}{2}(2\mu_{10} - \mu_{10}^2)$. The lower bound for the second term of (51) follows from sequentially applying (48) and (50):

$$-\frac{1}{\mu_{01}}(2\mu_{01} - \mu_{02})(1 - 2\mu_{10})^2 \geqslant -\frac{\mu_{20}}{\mu_{10}}(1 - 2\mu_{10})^2 \geqslant -\mu_{10}(1 - 2\mu_{10})^2 \quad (52)$$

so that

$$\mu_{01}(2\mu_{10} - \mu_{20}) - \frac{1}{\mu_{01}}(2\mu_{01} - \mu_{02})(1 - 2\mu_{10})^2 \geqslant \frac{1}{2}\left(2\mu_{10} - \mu_{10}^2\right) - \mu_{10}(1 - 2\mu_{10})^2 = \frac{1}{2}(7 - 8\mu_{10})\mu_{10}^2 > \frac{3}{2}\mu_{10}^2 > 0. \quad (53)$$

The fact that $\frac{d}{d\alpha}G(u_c) > 0$ means that perturbing the configuration network at the critical regime $G(u) = 0$ by a uniform addition of new edges forces the value of $G(u)$ to become positive. The opposite is also true: Uniform removal of existing edges at the critical regime forces values of $G(u)$ to become negative. Similar derivation holds for the perturbation in the second layer.

Finally, suppose one modifies $u(k,l)$ in such a manner that the expected numbers of edges μ_{10} and μ_{01} remain constant whereas the second moments vary. Such a perturbation of the degree distribution causes rewiring of the network but keeps the expected numbers of edges in each layer the same. Consider a function $f(\mu_{20},\mu_{02}) := \mu_{11}^2 - (\mu_{20} - 2\mu_{10})(\mu_{02} - 2\mu_{01}) = G(u_c)$. As follows from the lower bounds (48), both components of the gradient vector

$$\nabla f(\mu_{20},\mu_{02}) = (2\mu_{01} - \mu_{02}, 2\mu_{10} - \mu_{20})$$

are positive. This fact confirms that rewiring that moves edges within a single layer toward the nodes with higher degree forces the value of $G(u)$ to become positive. The total action of the simultaneous rewiring in two layers is defined by $\text{sgn}[(2\mu_{01} - \mu_{02}, 2\mu_{10} - \mu_{20})^\top(\partial\mu_{20}, \partial\mu_{02})]$.

According to the asymptote (32), if $G(u) = 0$, the size distribution decays algebraically with exponent $-\frac{3}{2}$ and therefore the expected component size diverges. On the one hand, perturbations of the network by inflection with new edges or a rewiring that moves edges to nodes with larger degree do not reduce the size of the largest component; on the other hand, after such a perturbation $G(u) \neq 0$, the component size distribution switches to the exponential decay and features a finite expected component size. The deficit in expected component size, which due to the nature of the perturbation could have only increased if all connected components were finite, is attributed to the emergence of the giant two-layer component.

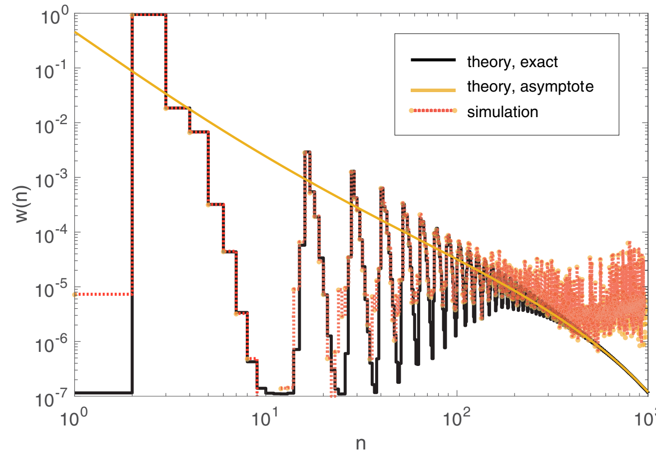

FIG. 2. An oscillatory example of the size distribution of connected components in the two-layer configuration model as predicted by the analytical expression (40) (black solid line) is compared against the data obtained from simulations (scattered points linked with a dashed line that indicates the trend). Low probabilities are naturally underrepresented in simulated data due to the limited size of the Monte Carlo sample. The theory, as given by Eqs. (32) and (45), predicts the asymptote with a transient slope $-\frac{3}{2}$ (yellow solid line).

V. DISCUSSION AND CONCLUSIONS

The main results of this study are the formal expressions for the size distributions of connected components in directed and multiplex networks. These expressions involve the convolution power and, in practice, can be evaluated exactly with a FFT algorithm with the requirement of $O(n^2 \log n)$ multiplicative operations in the case of directed networks and $O(n^N \log n)$—in the case of multiplex networks with $1 < N < n$ layers. These expressions are very general and do not rely upon any restrictions on the degree distribution itself. The supporting code is accessible at the GitHub repository [23]. Unlike the

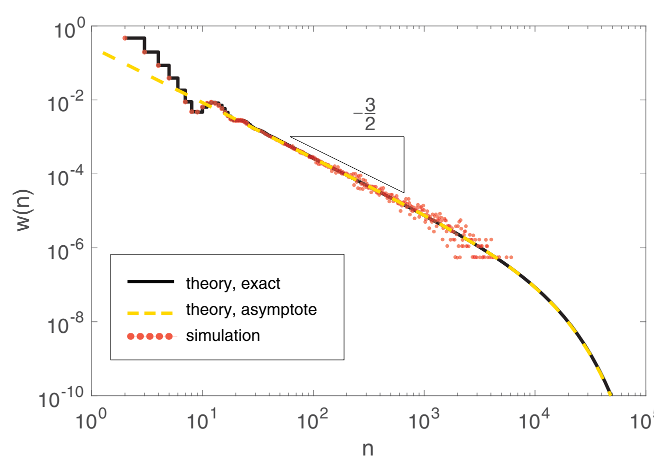

FIG. 3. An example of the size distribution of weakly connected components in a directed configuration model as predicted by the analytical expression (15) (solid line) is compared against the data obtained from simulations (scattered points). The theory, as given by Eq. (32), predicts the asymptote with a transient slope $-\frac{3}{2}$ (dashed line).

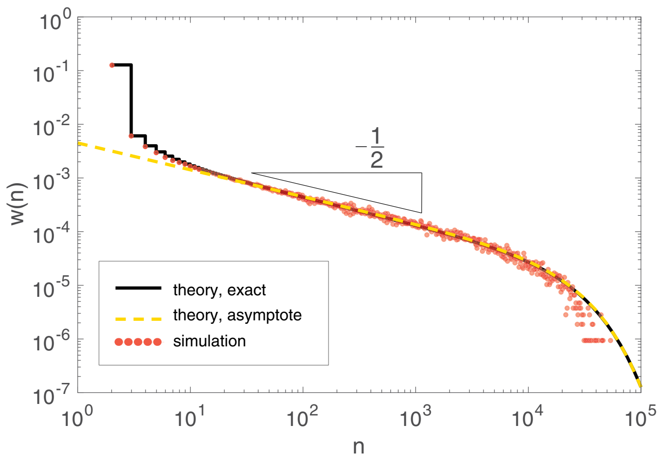

FIG. 4. An example of the size distribution of weakly connected components in a directed configuration model as predicted by the analytical expression (15) (solid line) is compared against the data obtained from simulations (scattered points). The theory, as given by Eq. (36), predicts the asymptote with a transient slope $-\frac{1}{2}$ (dashed line).

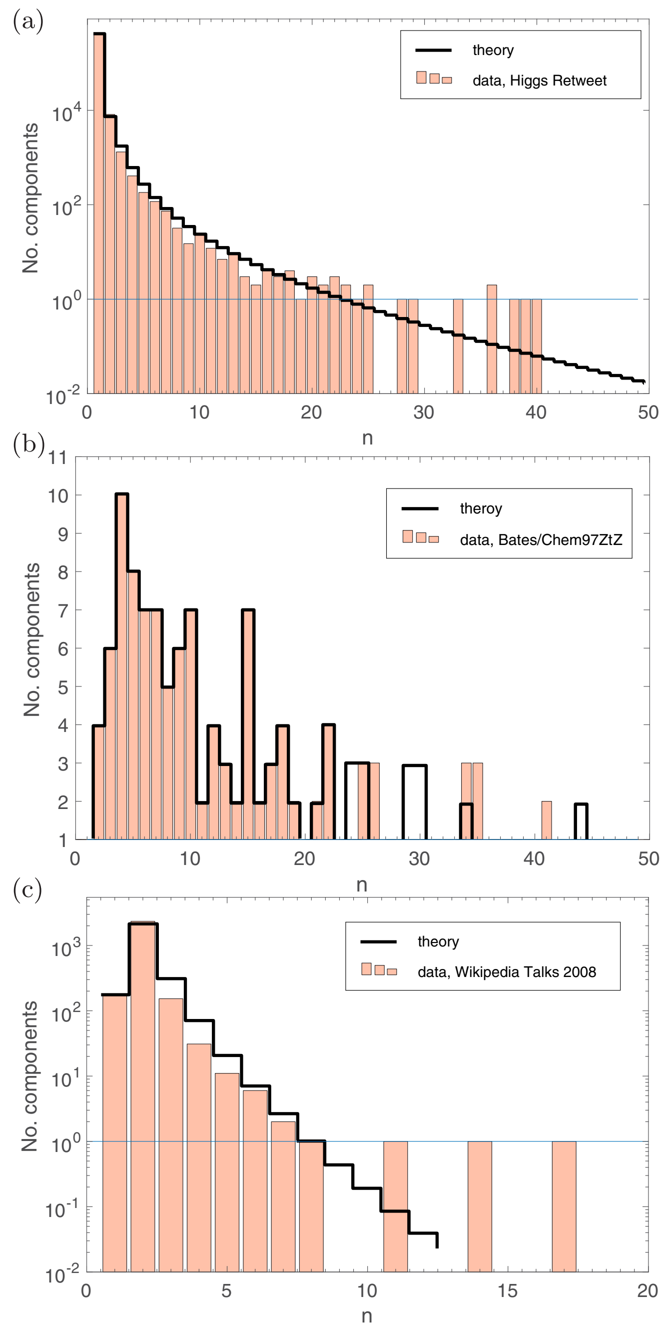

FIG. 5. Comparisons of respective theoretical size distributions of finite components against the empirical data. Three cases of directed networks containing N nodes in total are considered: (a) the network of retweets in the Higgs-Twitter data set, $N = 425\,008$ [24]; (b) the graph of the sparse statistical matrix Chem97ZtZ, $N = 2\,541$ [25]; and (c) the network of communications on Wikipedia until January 2008, $N = 2\,394\,385$ [26].

fixed point formulations (10) and (39), the formal expressions for component size distributions are tractable from an asymptotic theory point of view. The asymptotic analysis for weak and multilayer connected components resulted in simple analytical expressions that, under certain conditions, feature a self-similar behavior. The asymptotic theory, however, does rely on a few assumptions that to a certain extent limit the space of applicable degree distributions. First, we assume finiteness of partial moments, $\mu_{ij} < \infty$, $i + j \leqslant 3$; second, we rely upon the existence of the real root of Eq. (30) such that $|r_1| \leqslant 1$. Finally, there is a practical restriction that arises if one aims to utilize the asymptote as an approximation for the size distribution itself: The best approximation accuracy is gained when the network is in the critical window $|ar_1 + b| = \epsilon$, where ϵ is infinitesimal.

A few examples of size distributions of connected components and analytical asymptotes corresponding to them are given in Figs. 2–4. Figure 2 compares our theory against simulated data for the case of a two-layer network with the degree distribution given by

$$u(k,l) = 0.9782e^{-5[(k-1)^2+l^2)]} + 0.002e^{-10[(k-9)^2+(l-3)^2]}.$$

This example was selected to demonstrate the possibility of an oscillatory behavior arising in the size distribution of connected components. It can be noted that the theory, as given by Eq. (40), accurately predicts the nontrivial oscillations present in the data. For large n, the theoretical predictions in this example converge to the asymptote, as given by Eq. (32).

Figure 3 features the size distribution of connected components in a directed network featuring a nondegenerate degree distribution,

$$u(k,l) = 0.5167e^{-k^2-l^2} + 0.0052e^{-2.5[(k-4)^2+(l-4)^2]},$$

whereas Fig. 4 features the results obtained for a degenerate degree distribution,

$$u(0,k) = 0.9073e^{-2.266k}, \quad k \geqslant 0$$
$$u(1,k) = 0.9073e^{-0.7k}, \quad k \geqslant 0$$
$$u(l,k) = 0, \quad l > 1,\ k \geqslant 0.$$

As in the previous example, both Figs. 3 and 4 compare the theoretical size distribution, as given by Eq. (15), to the simulated data. In both figures, the theoretical predictions and the data converge to the asymptotes for large n. In the case of the nondegenerate degree distribution, the asymptote features a transient slope $-\frac{3}{2}$, as predicted by Eq. (32). However, in the case of the degenerate degree distribution the transient slope of the asymptote is $-\frac{1}{2}$, which is in accord with Eq. (36). The latter observation is a surprising result. This is evidence that a configuration model with a light-tailed degree distribution may feature an exponent distinct from $-\frac{3}{2}$. Importantly, in the multiplex configuration network with two layers such an anomaly is not present. When the degree distribution is light tailed, both nondegenerate directed networks and two-layer networks feature a leading exponent $-\frac{3}{2}$ in the critical regime, which is also the case in undirected networks.

A comparison of the theory against a few examples of empirical data is given in Fig. 5. This figure presents theoretical size distributions of weakly connected components normalized to the number of nodes and compares them to empirical component count distributions extracted from various data sets of directed networks.

In undirected single-layer configuration networks, a heavy tail in the size distribution is observed when $\mu_2 - 2\mu_1 = 0$, where μ_2 and μ_1 are the moments of the univariate degree distribution. Furthermore, when the equality sign in this criterion is replaced by an inequality sign, $\mu_2 - 2\mu_1 > 0$, one obtains the criterion for giant component existence. A similar inequality criterion can be constructed for directed networks: Also in this case the condition for a heavy tail in the size distribution relates to the giant component existence [11]. However, this principle breaks down already in multiplex networks that consist of as few as two layers. The sign of the left-hand side of the criticality condition (47) cannot be directly associated with the existence of the giant two-layer component. Nevertheless, as was argued in Sec. IV D, if the equality (47) fails to hold due to a small perturbation in the expected numbers of edges μ_{10} and μ_{01} (or rewiring caused by an increase of second moments μ_{20} and μ_{02}) at the critical regime, one can still associate the sign of the left-hand side in Eq. (47) with the giant component existence. This association is guaranteed to be valid within the critical window.

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