

# Field-theoretic approach to large-scale structure formation

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# Field-theoretic approach to large-scale structure formation

## Veldtheoretische benadering van grootschalige structuurvorming

(met een samenvatting in het Nederlands)

### Proefschrift

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# Chapter 1

## Introduction

Theoretical cosmology aims at explaining the history of our universe. Spatial sections typically expand in time  $t$  with a scale factor  $a(t)$  whose relative growth is given by the Hubble rate

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \quad (1.1)$$

where the dot denotes the derivative with respect to time  $t$ . On very large scales the universe is modelled to be on average homogeneous and isotropic which is confirmed by looking at galaxy surveys as in figure 1.1 or at cosmic microwaves as in figure 1.2. Isotropy refers to the property that physics looks the same in all directions and homogeneity states that this true no matter from which point one is observing.



FIGURE 1.1: An image by the Hubble Space Telescope of galaxy clusters. Although galaxy clusters are the largest objects in the universe, blurring them out a little makes the universe look like a homogeneous and isotropic fluid. Source: ESA/Hubble, NASA.

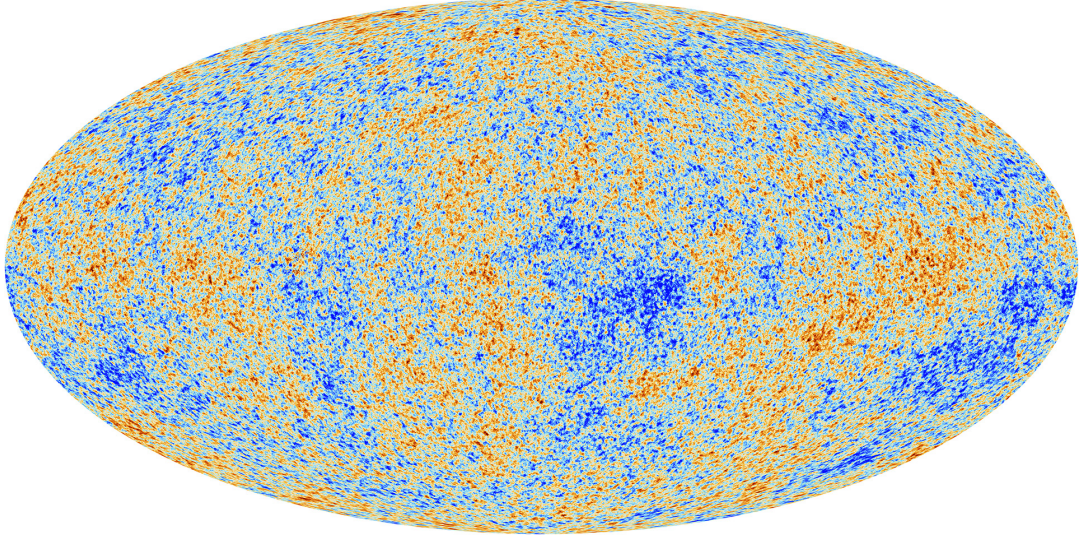


FIGURE 1.2: Anisotropies of the cosmic microwave background (CMB) as observed by the Planck satellite. This is a snapshot of small fluctuations of photons around a common background temperature of  $2.7\text{ K}$ . These photons now reach us from a time when the universe was 380000 years old where the universe became cool enough such that photons decoupled from the electron-proton plasma and were able to stream freely. Source: ESA and the Planck Collaboration.

In order to determine how exactly the universe expands in time, one has to solve the Einstein equations,<sup>1</sup>

$$G_{\mu\nu}[g_{\mu\nu}] = \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (1.2)$$

where the gravitational constant is denoted by  $G = 6.674 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$  and the speed of light is given by  $c = 2.997 \times 10^8 \text{ m} \cdot \text{s}^{-1}$  (which we will set to one from now on). The Einstein tensor  $G_{\mu\nu}$  on the left-hand-side of (1.2) measures space-time curvature and is expressible by second derivatives of the universe's metric  $g_{\mu\nu}$ , which in turn determines distances in space and time. The right-hand-side of the Einstein equation (1.2) tells us that the space-time curvature is sourced by the energy-momentum tensor  $T_{\mu\nu}$ . If we demand spatial homogeneity and isotropy, the energy-momentum tensor takes the form of a perfect fluid

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (1.3)$$

which is specified by its energy density  $\rho$ , its pressure  $p$  and its (dual) four-velocity  $u_\mu = g_{\mu\nu}u^\nu$ . The metric entering the Einstein tensor takes a diagonal form with the squared line element in spherical coordinates given by

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right], \quad (1.4)$$

where  $d\Omega^2$  is the squared line element on the 2-sphere and  $\kappa$  is equal to  $-1, 0, +1$

<sup>1</sup>For readers not familiar with the concepts of differential geometry, we give a short introduction in appendix 1.C.



for a spatially open, flat or closed universe, respectively. The particular metric (1.4) is the most general metric for a homogeneous and isotropic universe and tells us that spatial distances vary in time according to the scale factor  $a(t)$ . It is referred to as the FLRW-metric, standing for Friedmann, Lemaître, Robertson and Walker. Inserting the energy-momentum (1.3) and the metric (1.4) into the Einstein equations (1.2) yields the Friedmann equations

$$H^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}, \quad (1.5)$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3}(\rho + 3p). \quad (1.6)$$

Moreover, the Einstein equations (1.2) imply that energy and momentum are locally conserved,

$$\nabla_\mu T^{\mu\nu} = 0. \quad (1.7)$$

This leads to the conservation equation for the perfect fluid,

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (1.8)$$

which can also be obtained by combining the two Friedmann equations (1.5) and (1.6). The Friedmann and energy conservation equation determine the average expansion of the homogeneous and isotropic universe and form the first building block of cosmological history. We will now give a very brief introduction to this history where current models favour a spatially flat universe with  $\kappa$  equal to zero. For motivations and detailed explanations, we refer the reader to [1–4]. Many parameters and constraints of cosmological models can be derived from data of the Planck space observatory [5] which serves as our main reference on experimental verification of those models.

## 1.1 A brief history of the homogeneous universe

The fundamental starting point of the universe at time  $t = 0$  is speculative. Getting an idea of what could have happened is not only complicated by a lack of observational access, but also by a lack of theoretical foundations, since quantum gravity, if it is based on the Einstein equations, breaks (perturbatively) down on energies close to the reduced Planck mass  $M_P = 2.435 \times 10^{18}$  GeV. We can understand this naively by looking at the Einstein equations (1.2) and note that they are not scale invariant. Indeed, the gravitational constant is dimensionful and unexpected phenomena may happen if one approaches the energy scale related to it. This energy scale is by definition the reduced Planck mass

$$M_P = \sqrt{\frac{\hbar}{8\pi G}}, \quad (1.9)$$

where  $\hbar = 6.582 \times 10^{-16}$  eV  $\cdot$  s is the reduced Planck constant. Thus, a theory of quantum gravity based on the Einstein equations is an effective theory which (at least perturbatively) breaks down on energy scales close to the Planck scale.

Fortunately, we can skip this era of uncertainty and study the history of the universe from times  $t \sim 10^{-36}s$  and onwards which are relevant for observations. This early era is marked by an exponential expansion of the universe called inflation where  $a(t) \propto e^{Ht}$  with the Hubble rate being approximately constant. The idea behind inflation is broadly speaking to explain why the universe is so isotropic and homogeneous. The most common set of models, subsumed as single-field inflationary models, postulate a slowly decaying condensate  $\bar{\phi}(t)$  of a field called inflaton which is a priori not related to any standard model field. Since the condensate is very slowly varying in time, the energy density is almost constant and so is the Hubble rate which leads to a quasi-exponential growth of the scale factor. The slow decay is realized by letting the inflaton role down a flat potential  $V(\bar{\phi})$ . For this to happen one needs the following slow-roll conditions

$$\dot{\bar{\phi}}^2 \ll |V(\bar{\phi})|, \quad |\ddot{\bar{\phi}}| \ll H|\dot{\bar{\phi}}|, \quad (1.10)$$

which can conveniently be quantified in terms of the principal slow-roll parameter

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} \approx \frac{M_P^2}{2} \left( \frac{V'(\bar{\phi})}{V(\bar{\phi})} \right)^2 \ll 1, \quad (1.11)$$

and it's relative change in time

$$\varepsilon_2 \equiv \frac{\dot{\varepsilon}}{H\varepsilon} \approx 4\varepsilon - 2M_P^2 \frac{V''(\bar{\phi})}{V(\bar{\phi})} \ll 1. \quad (1.12)$$

Inflation ends with the process of reheating in which the inflaton condensate decays via rapid oscillations into other relativistic particles which commence the era of radiation lasting from  $t \sim 10^{-32}s$  to  $t \sim 10^4$  yrs. The energy density of radiation is equal to three times the radiative pressure. According to the energy conservation equation (1.8) it scales as  $a^{-4}$  which results in a much slower expanding universe with a scale factor  $\propto t^{1/2}$ . Subsequently, the universe cools down and massive, non-relativistic particles eventually dominate the evolution. Since their energy density scales like  $a^{-3}$ , the expansion speeds up again with a scale factor  $\propto t^{2/3}$ . According to the late-time  $\Lambda$ CDM concordance model, the main part of this non-relativistic matter is attributed to cold dark matter (CDM) which in contrast to baryonic matter interacts mainly gravitationally. Finally, the universe today is, according to the  $\Lambda$ CDM model, dominated by a dark (unknown) form of energy  $\Lambda$  that behaves similar to the inflationary condensate although its origin is - similar to that of dark matter - yet unknown.

## 1.2 Origin of galaxies and CMB fluctuations

The expansion history of the universe that we just sketched is an average expansion also referred to as background evolution since it is the same at every point in space. We know, however, that the universe does not exactly look the same everywhere if we think for example about galaxies and voids. So we have to augment the previous picture by perturbing all matter, radiation and gravitational fields around the homogeneous background with small inhomogeneities (at least

on large scales) which is referred to as cosmological perturbation theory [6–9]. These field perturbations in cosmology have a stochastic character such that their correlations are still homogeneous and isotropic which takes account of the fact that galaxies and other constituents of the universe are on average distributed without a preferred direction no matter from which point one is observing them from.

Where do these perturbations come from? According to the single-field inflationary scenario, all matter originates from the inflaton condensate and so will its perturbations. During inflation we have the following decomposition of fields into background and perturbation,

$$\phi(t, \vec{x}) = \bar{\phi}(t) + \varphi(t, \vec{x}), \quad (1.13)$$

$$g_{\mu\nu}(t, \vec{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t, \vec{x}), \quad (1.14)$$

where  $\bar{\phi}$  is the homogeneous inflaton condensate with inhomogeneous perturbation  $\varphi$ . Likewise,  $\bar{g}_{\mu\nu}$  is the FLRW-metric during inflation with perturbation  $\delta g_{\mu\nu}$ .

General relativity contains gauge degrees of freedom which allow for describing one and the same physical process from several coordinate systems. The physical degrees of freedom per space-time point of the coupled inflaton/metric system are counted as one scalar (spin 0) and two tensorial (spin 2) degrees of freedom which add up to three. The scalar part stems from the inflaton since it would not be present if there was no matter. The two tensorial degrees of freedom are only the traceless and transverse parts of the metric which are associated to gravitational waves or gravitons with left-handed or right-handed polarization. However, their effect in inflation is much smaller than the one of scalar perturbations which is why we will neglect them here. In any case, since the metric is symmetric in its indices, we conclude that we have  $10 + 1 - 3 = 8$  redundant variables. Part of these redundant variables may be eliminated by fixing a coordinate system. One can achieve this by picking four new coordinates for every point in space-time as a function of the four old coordinates,

$$\tilde{x}^\mu = \tilde{x}^\mu(x^\alpha). \quad (1.15)$$

Scalar fields such as the inflaton and rank two tensors such as the metric then change according to

$$\tilde{\phi}(\tilde{x}^\alpha) = \phi(x^\alpha) \quad (1.16)$$

$$\tilde{g}_{\mu\nu}(\tilde{x}^\sigma) = g_{\alpha\beta}(x^\sigma) \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu}. \quad (1.17)$$

If we demand that any coordinate transformation leaves the background unchanged (the background is strictly speaking an unphysical reference system), the perturbations (1.13) and (1.14) change as

$$\Delta\varphi(x^\alpha) = \tilde{\varphi}(x^\alpha) - \varphi(x^\alpha), \quad (1.18)$$

$$\Delta\delta g_{\mu\nu}(x^\alpha) = \delta\tilde{g}_{\mu\nu}(x^\alpha) - \delta g_{\mu\nu}(x^\alpha). \quad (1.19)$$

Hence, no matter which coordinate system we have started with, we can choose four new coordinates and eliminate four gauge degrees of freedom by applying the transformations (1.18) and (1.19) within a perturbative scheme. We can decompose these transformations into a shift in time which allows us to remove one scalar degree of freedom and a three-dimensional shift in space which decomposes into a longitudinal (scalar) component and two transverse (vectorial) components. Thus, four of eight redundant degrees of freedom will be removed. The four remaining redundant variables will be constrained via the Einstein equations such that they do not follow an independent dynamical evolution. Since they are related to the physical degrees of freedom via constraint equations, they are in some sense also physical. However, since they are not independent, they cannot be thought of as additional physical degrees of freedom such as another particle species.

As a first approximation to the inflationary scenario at hand, it turns out that it is good enough to keep only one instead of four constraint equations. We can achieve this by picking the metric perturbation with only one gravitational potential  $\Psi$  in the following form

$$\delta g_{00} = -2\Psi, \quad \delta g_{ij} = -2a^2\delta_{ij}\Psi, \quad (1.20)$$

which refers to a metric with scalar perturbations in the longitudinal gauge without gravitational slip (referring to the difference between the temporal scalar and trace of the spatial perturbation). Moreover, all perturbations are considered to be small during inflation such that we can linearize the equations of motion. Perturbing the homogeneous energy conservation equation (1.8) yields the equation of motion for the inflaton perturbation  $\varphi$ ,

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial^2 V(\bar{\phi})}{\partial \bar{\phi}^2} - a^{-2}\delta^{ij}\partial_i\partial_j\varphi = -2\Psi\frac{\partial V(\bar{\phi})}{\partial \bar{\phi}} + 4\dot{\Psi}\dot{\varphi}, \quad (1.21)$$

where  $V(\bar{\phi})$  is the inflaton potential as a function of the background condensate  $\bar{\phi}(t)$ . The perturbed Friedmann equation (1.5) fixes the gravitational potential

$$\left(\dot{H} + a^{-2}\delta^{ij}\partial_i\partial_j\right)\Psi = 4\pi G\left(-\dot{\bar{\phi}}\dot{\varphi} + \ddot{\bar{\phi}}\varphi\right). \quad (1.22)$$

Before we can solve these equations in some approximation, we still have to ask what the field perturbations  $\varphi$  and  $\Psi$  actually represent during inflation. It is one of the most fascinating aspects of the inflationary model that these perturbations, and hence the seeds of late-time structures like galaxies, are genuinely due to quantum fluctuations. Thus, the field perturbations of the early universe are not classical variables but quantum field operators with non-trivial commutation relations. For the inflaton perturbations the non-vanishing equal-time commutator reads

$$\left[\hat{\varphi}(t, \vec{x}), a^3(t)\dot{\hat{\varphi}}(t, \vec{y})\right] = i\hbar\delta^3(\vec{x} - \vec{y}), \quad (1.23)$$

where we made the operator nature explicit by putting a hat on the field and made use of the three-dimensional Dirac delta distribution. Let us decompose

the quantum operators associated to the field perturbations in Fourier modes,

$$\hat{\varphi}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \varphi_k(t) \hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \varphi_k^*(t) \hat{a}_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}}, \quad (1.24)$$

$$\hat{\Psi}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \Psi_k(t) \hat{a}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} + \Psi_k^*(t) \hat{a}_{\vec{k}}^\dagger e^{-i\vec{k}\cdot\vec{x}}, \quad (1.25)$$

where we introduced the mode function  $\varphi_k(t)$  and its complex conjugate which depend only on the modulus of the co-moving momentum  $\vec{k}$  as well as the creation and annihilation operators  $\hat{a}_{\vec{k}}^\dagger$  and  $\hat{a}_{\vec{k}}$  with full momentum  $\vec{k}$  dependence. Momenta satisfying  $k \ll aH$  are called super-Hubble modes since the corresponding wavelength is bigger than the one set by the inverse Hubble rate. Correspondingly, momenta satisfying  $k \gg aH$  are called sub-Hubble modes. We emphasize that the operators  $\hat{\varphi}$  and  $\hat{\Psi}$  are not independent as they are written in terms of the same creations and annihilation operators. This should of course be the case in view of the constraint equation (1.22), which is not dynamical.

The mode function  $\varphi_k$  can be found by solving the dynamical equation (1.21) together with the constraint (1.22). In a first approximation, one can take the Hubble rate during inflation to be constant and finds

$$\varphi_k(\eta) = -i \frac{H}{\sqrt{2k^3}} (1 - ik\eta) e^{ik\eta}, \quad (1.26)$$

where we introduced the conformal time  $\eta$  by the relation

$$ad\eta = dt, \quad (1.27)$$

such that  $\eta < 0$  during inflation with

$$a(\eta) = -\frac{1}{H\eta}. \quad (1.28)$$

Corrections to the solution (1.26) are suppressed by the slow-roll parameters (1.11) and (1.12). We note that the expression  $-k\eta$  conveniently describes whether the physical momentum  $ka^{-1}$  associated to the co-moving modulus  $k$  is smaller or bigger than the Hubble scale  $H$ .

We note that the mode functions (1.26) behave on sub-Hubble scales as the mode functions of the Minkowski vacuum with a conformal rescaling

$$a^{3/2} \varphi_k \xrightarrow{k \gg aH} \sqrt{\frac{a}{2k}} e^{-i \int ka^{-1} dt}. \quad (1.29)$$

So far, we wrote down the dynamics of the inflaton operator but what about the particle content of the early, inflationary universe? The initial particle content is determined by the initial quantum state. If we subtract the contributions from the condensate  $\bar{\phi}$ , the state should not contain too many particles since fluctuations around the background are otherwise not small. Indeed, the most

popular choice is a vacuum-like state called Bunch-Davies vacuum and we denote it by  $|0\rangle$ . The Bunch-Davies vacuum satisfies

$$\hat{a}_{\vec{k}}|0\rangle = 0, \quad (1.30)$$

for all modes with co-moving wave vector  $\vec{k}$  that are observable today and which were sub-Hubble at the beginning of inflation. For such modes, particle production due to the expansion of the universe remains adiabatically small while they are sub-Hubble and their fluctuations are the analogue of pure vacuum fluctuation in Minkowski space.

However, if we look at the fluctuation for the same co-moving momentum  $\vec{k}$  at later times such that it becomes super-Hubble, we find that it freezes out to a constant,

$$\varphi_k \xrightarrow{k \ll aH} -i \frac{H}{\sqrt{2k^3}}, \quad (1.31)$$

which corresponds to a huge amplification in terms of the physically rescaled field since the scale factor expands exponentially. Consequently, the power spectrum of these fluctuations given in terms of the two-point function in the Bunch-Davies vacuum becomes constant on super-Hubble scales,

$$\mathcal{P}_\varphi(t, k) \equiv \int d^3r e^{i\vec{k}(\vec{x}-\vec{y})} \langle 0 | \hat{\varphi}(t, \vec{x}) \hat{\varphi}(t, \vec{y}) | 0 \rangle = |\varphi_k|^2(t) \xrightarrow{k \ll aH} \frac{H^2}{2k^3}. \quad (1.32)$$

Thus, inhomogeneous small-scale, sub-Hubble quantum fluctuations of the inflaton field are blown up to large-scale, super-Hubble perturbations where they freeze out and become stochastic since the commutator becomes negligible. They will eventually serve as seeds for today's observable CMB anisotropies and large-scale structure (LSS) inhomogeneities where the fact that these perturbations freeze out on super-Hubble scales results in coherent oscillations of the CMB anisotropy power-spectrum as one can see in figure 1.3. These acoustic oscillations set in once perturbations of the photon-baryon fluid re-enter the Hubble scale which happens due to the decelerated expansion in the radiation and matter era.

But how are the quantum fluctuations of the inflaton  $\delta\varphi$  related to temperature fluctuations? So far, all the matter was given in terms of the inflaton field and it did not decay yet into other particles. The key to this question is to consider a linearly gauge invariant variable called curvature perturbation  $\mathcal{R}$  which reads in the gauge we chose in (1.20),

$$\mathcal{R} \equiv \Psi - H\delta u, \quad (1.33)$$

where  $\delta u$  is now the velocity potential associated with the spatial part of the four-velocity of any energy-momentum tensor that is perturbed around an FLRW-metric. In the special case where energy and momentum are solely due to the inflaton, we have

$$\mathcal{R} = \Psi + \frac{H}{\dot{\phi}}\varphi. \quad (1.34)$$

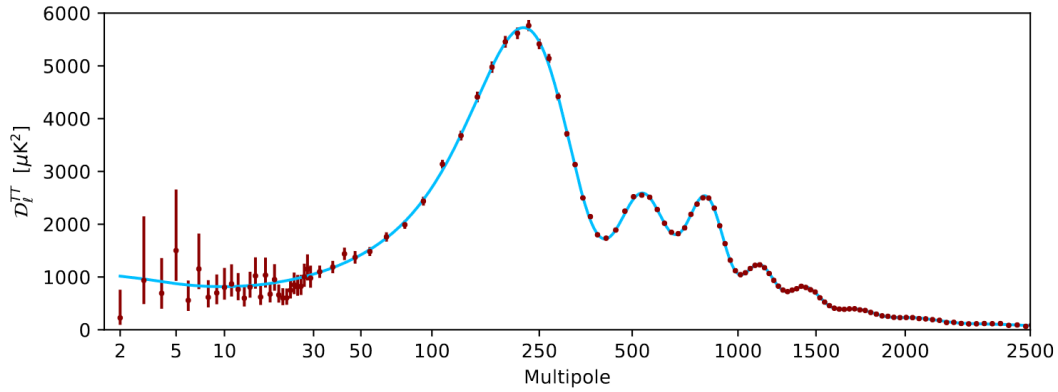


FIGURE 1.3: Coherent oscillations of the power spectrum  $\mathcal{D}_l^{TT}$  of temperature fluctuations in the cosmic microwave background anisotropies as observed by the Planck satellite in terms of multipole moments  $l \propto k/(aH)_{\text{today}}$ . These modes of the photon-baryon fluid originate from inflaton perturbations that were pushed beyond the Hubble scale and froze out. While the expansion slows down in the radiation era, the perturbations re-enter the Hubble scale and start oscillating where the damping is due to diffusion. Photons decouple eventually and provide us today with the image of that moment in time. Source: Planck Collaboration [5].

If we write the gravitational potential  $\Psi$  in terms of the inflaton perturbation  $\varphi$  with the help of the constraint equation (1.22) and plug in the linear solution (1.26), we find that  $\mathcal{R}$  is also conserved on super-Hubble scales similarly to  $\varphi_k$  in (1.31). This is in fact a special case of a more general theorem due to Weinberg [10] which states that linear perturbations are conserved on super-Hubble scales irrespective of the content of the energy-momentum tensor. If the inflaton decays during reheating and the universe becomes radiation dominated, the gauge invariant curvature perturbation (1.33) is numerically the same on super-Hubble scales with the only difference that it is now mostly determined by the velocity potential of a radiative fluid instead of the inflaton. More importantly, some time before recombination, the gauge invariant curvature perturbation is related to the initial photon temperature fluctuation and we can conclude that the CMB power spectrum in figure 1.3 is initially sourced by the power spectrum of quantum fluctuations during inflation (1.32).

Although successful, the aforementioned scenario is a rather simplistic version of inflation that moreover relies on a linear treatment. Possible extensions include non-Gaussian features that stem from non-linear corrections [11] or multi-field models [12] whose non-trivial background evolution in field space results in additional entropy fluctuations that could for example wash out the coherent CMB oscillations. Another way to extend single-field inflationary models consists of including additional light spectator fields which do not contribute to the background evolution but whose vacuum fluctuations interact with those of the inflaton perturbations. In the first part of this thesis, we will study this scenario in detail. In contrast to the standard linear analysis, the loop corrections due to the inflaton perturbation will produce more and more entropy

towards the end of inflation which results in independent fluctuations of the momentum associated to the curvature perturbation that could in principle wash out the coherent CMB oscillations. Studying this extended scenario in general and in particular from several perspectives is desirable not only since experimental precision bounds in cosmology are constantly improving but also since a rigorous quantum field theoretical treatment of inflationary loop interactions has barely been carried out.

### 1.3 Cold dark matter

While the universe is radiation dominated, acoustic oscillations of the photon-baryon fluids hamper the formation of large-scale structures. However, as radiation is eventually diluted and non-relativistic matter becomes the main source of energy, this picture changes and matter perturbations start collapsing on sub-Hubble scales. According to the  $\Lambda$ CDM model, the main source of matter is cold, dark and its origin unknown [13]. It is dark since it has not been detected and (self-)interactions are constrained by astrophysical experiments to be very weak. However, since visible, baryonic matter also interacts gravitationally, it will be dragged along the dark matter distributions and serve as the basis for cosmological experiments. This property was in fact one of the original motivations to introduce dark matter since the rotation velocities at the edges of galaxies could otherwise not be explained. Similar to the power spectrum of cosmic microwave photons, one can measure the power spectrum of cold dark matter by analysing galaxy distributions (cf. figure 1.4).

One of the first candidates for dark matter was the neutrino, comprising a dark matter candidate which is in fact hot, i.e. relativistic. However, dynamics of galaxy clusters favour cold, i.e. non-relativistic, dark matter which led to consider weakly interacting massive particles (WIMPs) as the next promising dark matter candidate [15]. Weakly interacting particles also give the right abundance of dark matter for large masses once they freeze-out from the cosmic plasma. After years of searches no WIMPs have been detected and although they are not excluded, constraints on mass and interaction ranges continue to increase [16] which raises interest in other dark matter models such as axion dark matter. Axions are light scalar particles which are proposed as a solution to the strong CP-problem. For cosmological applications they are effectively described by a real scalar field in a condensate state which is minimally coupled to gravity. This effective description in the context of cosmology is more generally referred to as fuzzy dark matter [17–22] where the mass is as light as  $m \gtrsim 10^{22}$  eV and we have more to say about fuzzy dark matter later on in this introduction. Finally, let us mention primordial black holes as a dark matter candidate. Primordial black holes are assumed to form quickly in the first seconds of the universe where large fluctuations could lead to local collapses. However, the parameter space for primordial black holes is also constrained [23].

In this thesis we are concerned with particle dark matter where we are agnostic about interactions with other particles and focus mainly on its gravitational effects for large-scale structure formation. From the perspective of cosmological perturbation theory present in the CMB and LSS, one is usually ignorant



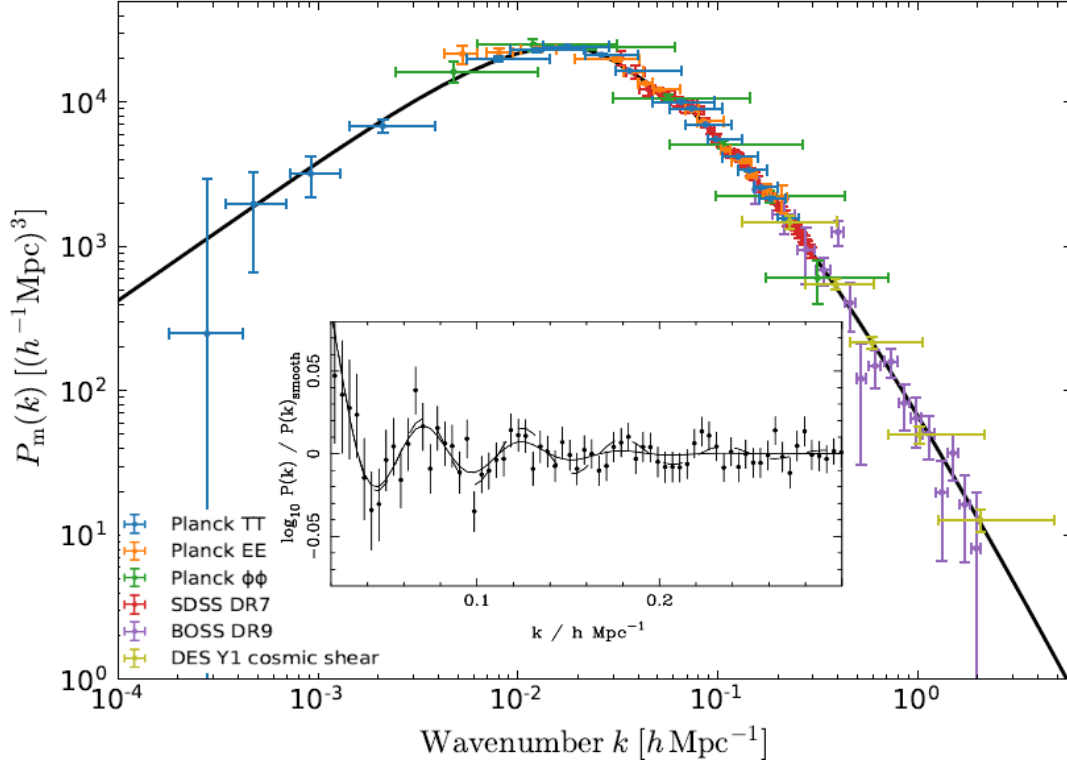


FIGURE 1.4: The power spectrum of cold and baryonic matter constrained by several experiments. The peak corresponds roughly to the scale  $k_{eq} \equiv a_{eq}H_{eq}$  where matter perturbations enter the horizon at the time  $t_{eq}$  at which matter and radiation energy density become equal. The left tail corresponds to super-Hubble matter perturbations that, during radiation, remained frozen in the almost scale-invariant inflationary power spectrum. They become sub-Hubble only later in the matter era where they grow as the scale factor. The right part of the graph entails matter perturbations that were sub-Hubble during radiation and underwent a logarithmical growth despite being suppressed by  $k_{eq}^2/k^2$ . The evolution can be studied in linear perturbation theory up to the scale of non-linearity  $k_{nl} \sim (10 \times \text{Mpc})^{-1}$  which becomes important in matter domination due to the unsuppressed gravitational collapse. Moreover, for scales bigger than  $(h\text{Mpc})^{-1}$  baryon-acoustic oscillations set in which is shown in the inset.

Source: Planck Collaboration [5] and Sloan Digital Sky Survey [14].

about the microscopic formulation of cold dark matter. The standard description starts by assuming a one-particle density in phase-space  $f(t, \vec{x}, \vec{p})$  which gives the probability of having a particle at time  $t$  and position  $\vec{x}$  with momentum  $\vec{p}$  and which may be obtained by smearing a collection of point particles. The dynamics of the phase-space density is determined by the Vlasov equation which in conformal time ( $ad\eta = dt$ ) takes the form

$$\frac{\partial f}{\partial \eta} + \frac{\vec{p}}{ma} \cdot \nabla f - ma \nabla \Psi \cdot \frac{\partial f}{\partial \vec{p}} = 0, \quad (1.35)$$

where the field  $\Psi$  is again the gravitational potential. The Vlasov equation in (1.35) is the non-relativistic limit of the classical Boltzmann equation in general relativity where one assumes that a metric of the type (1.20) with scalar perturbations, in longitudinal (Newtonian) gauge and without gravitational slip

is a good enough approximation. The phase-space density  $f(\eta, \vec{x}, \vec{p})$  contains the homogeneous background matter density  $\bar{\rho}_m(\eta)$  and its perturbation  $\delta(\eta, \vec{x})$ ,

$$\bar{\rho}_m(\eta) \equiv \int d^3p \bar{f}(\eta, \vec{p}), \quad (1.36)$$

$$\delta(\eta, \vec{x}) \equiv \frac{1}{\bar{\rho}_m} \int d^3p f(\eta, \vec{x}, \vec{p}) - 1, \quad (1.37)$$

where  $\bar{f}$  is the spatial average of the phase-space density  $f$ .

It is astonishing that for a large fraction of observable scales the evolution of cold dark matter can accurately be modelled by a linearized perfect fluid. This is achieved by first assuming the non-linear term in the definition of the fluid velocity  $\vec{v}$  to be negligible,

$$(1 + \delta)\vec{v} \equiv \frac{1}{\bar{\rho}_m} \int d^3p \frac{\vec{p}}{ma}. \quad (1.38)$$

Second, one assumes the second moment  $\sigma_{ij}$ ,

$$(1 + \delta)\sigma_{ij} \equiv \frac{1}{\bar{\rho}_m} \int d^3p \frac{p_i}{ma} \frac{p_j}{ma} - (1 + \delta)v_i v_j, \quad (1.39)$$

to be negligible. The second moment (1.39) is related to the fluids pressure and approximating it to be small reflects the fact that dark matter is modelled as a cold non-relativistic fluid. While structure formation on sub-Hubble scales is still slow during radiation domination and grows only logarithmically due to pressure oscillations, it becomes efficient in the matter era where the density contrast grows on large, but sub-Hubble, scales as the scale factor  $\delta \propto a$  which slows down at late-time due to dark energy. We can understand this by considering for simplicity a universe in the matter dominated era where the gravitational potential appearing in the Vlasov equation (1.35) is sourced by the density contrast via the Poisson equation,

$$\Delta\Psi = \frac{3}{2}H^2 a^2 \delta, \quad (1.40)$$

and the scale factor evolves according to

$$3H^2 a^3 = \frac{\hbar}{M_P^2} \bar{\rho}_m = \text{const}. \quad (1.41)$$

Taking time derivatives of the density contrast (1.37) and the velocity (1.38) yields,

$$(\partial_\eta + \vec{v} \cdot \nabla)\delta = -(1 + \delta)\nabla \cdot \vec{v}, \quad (1.42)$$

$$(\partial_\eta + \vec{v} \cdot \nabla)\vec{v} = -\mathcal{H}\vec{v} - \nabla\Psi, \quad (1.43)$$

where the conformal Hubble rate  $\mathcal{H}$  is given by

$$\mathcal{H} \equiv \frac{1}{a} \frac{da}{d\eta}. \quad (1.44)$$

Since vector perturbations decay at the linear level, we focus on scalar perturbations and define the velocity divergence  $\theta \equiv \nabla \cdot \vec{v}$ . We make use of the Poisson equation (1.40) and find at the linear level,

$$\partial_\eta \delta + \theta = 0, \quad (1.45)$$

$$(\partial_\eta + \mathcal{H})\theta + \frac{3}{2}\mathcal{H}^2\delta = 0, \quad (1.46)$$

which is solved by  $\delta \propto a \propto \eta^2$ . The initial conditions at some time in the matter era are provided by a transfer function  $T(k)$  that takes the sub-horizon evolution during radiation into account.

We already remarked the stochastic nature of cosmological perturbations. The power spectrum  $\mathcal{P}_M(k)$  in figure 1.4 is today's value of the following expectation of matter density perturbations in Fourier space

$$\mathcal{P}_M(\eta, k) = \langle \delta(\eta, \vec{k}) \delta(\eta, -\vec{k}) \rangle. \quad (1.47)$$

As opposed to quantum expectation values during inflation (1.32), we think of late-time power spectra such as the matter power spectrum (1.47) (and also the CMB power spectrum) as averages with respect to a stochastic ensemble that reflects the statistical properties of macroscopic perturbations in the sky. Quantum fluctuations from inflation are the favoured origin of these stochastic perturbations where the initial Bunch-Davies vacuum together with a linear evolution result in nearly Gaussian stochastic distributions for perturbations with a given wave vector  $\vec{k}$ .

Ultimately, non-linear effects have to be considered for wavelengths smaller than  $\lambda_{NL} \sim 10\text{Mpc}$  where the single-stream approximation – modelling the cold dark matter phase-space density only in terms of the density contrast and the velocity – breaks down and higher moments become important. These effects can still be captured to some degree within a perturbative expansion [24] and are verified by point-particle simulations (cf. figure 1.5). Eventually, the analytical approach by perturbation theory is invalidated and one will either have to solve the full Vlasov equation in phase-space (1.35) which was so far not successful, reformulate the problem and/or introduce alternative models or UV-completions whose behaviour deviates on smaller scales from that of a (smoothed) cloud of point-particles which we considered up to this point. The latter point is also motivated by a number of small-scale challenges arising around wavelengths smaller than  $\sim 1\text{Mpc}$  [25] one of which is, for example, the missing satellite problem that arises from over-predicting the abundance of dark matter on small scales.

It was realized that fuzzy dark matter, which models dark matter as a field condensate, can address these problems and interest in it increased [26–29]. Moreover, the non-relativistic limit of this condensate representation of cold dark matter has been used to gain access to the non-linear regime by relying on a Schrödinger type equation [30–33]. Although modelling cold dark matter as a field condensate is popular and is motivated by other high-energy physics problems, it is a rather particular way of describing cold dark matter from a field-theoretic point of view since one is referring to a special state that has

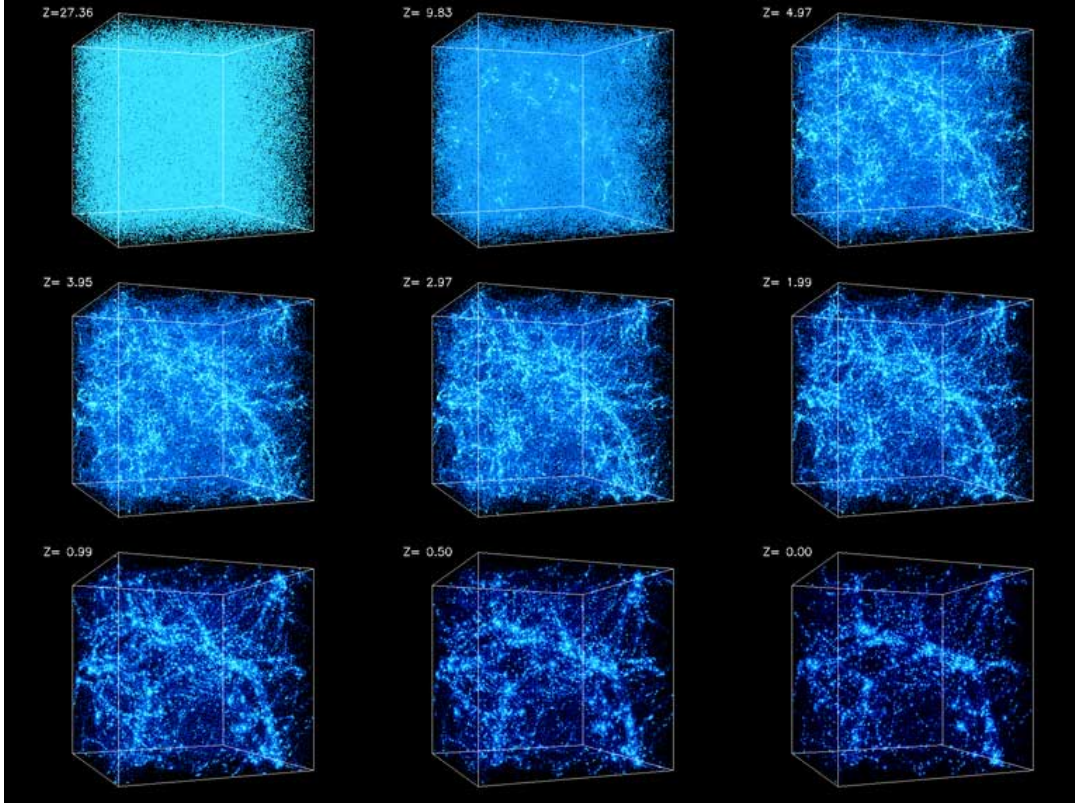


FIGURE 1.5: Simulation of particles in a box as a function of the redshift  $z$  where  $z(t) \equiv a_{\text{today}}/a(t) - 1$ . The simulation shows formation of clusters and large-scale filaments in a cold dark matter model with dark energy. Source/Credit: NCSA (University of Illinois), Andrey Kravtsov (University of Chicago), Anatoly Klypin (New Mexico State University).

much less degrees of freedom than the classical particle phase-space density in (1.35).

The motivation for the work presented in the second part of this thesis is to improve our understanding on more general grounds of how quantum field theory in cosmology is related to the dynamics of classical point-particle distributions and under which conditions small-scale effects may become important. No matter whether we are dealing with inflation, photon-baryon fluids or cold dark matter, the fundamental starting point of a cosmological problem is *a priori* the theory of quantum fields on curved space-time, at least, if we agree on fundamental descriptions in terms of experimentally verified theories. Here, we think of curved space-time as a classical background (typically with a perturbed FLRW metric) which is sourced by the semi-classical Einstein equations,

$$G_{\mu\nu} = \frac{1}{M_P^2} \langle \psi | \hat{T}_{\mu\nu} | \psi \rangle, \quad (1.48)$$

whereas in the beginning in equation (1.2) the Einstein tensor  $G_{\mu\nu}$  measures the (macroscopic) curvature of space-time and may incorporate stochastic properties in a cosmological set-up. Gravitation can be assumed to be classical, possibly stochastic, as quantum gravitational effects are expected to appear only

on very short scales. The energy-momentum tensor  $\hat{T}_{\mu\nu}$  on the other hand is now a quantum field operator whose expectation value is taken in a quantum mechanical sense with respect to a given state  $|\psi\rangle$ . The expectation value then describes a macroscopic form of matter that may incorporate (on top) stochastic properties similar to those of the Einstein tensor in a cosmological set-up. Now, one may average again over these macroscopic stochastic fluctuations as in (1.47) to obtain expectation values in the framework of cosmology. We emphasize that both expectation values, quantum mechanical and cosmological stochastic, coincide only in the early inflationary universe (1.32) and should a priori be disentangled. We explain this in detail in chapter 3 of this thesis.

We note that cosmological models comprise a ladder of scales which include

- the Planck scale  $M_P$  above which Einstein gravity breaks down as an effective theory,
- the inflationary Hubble scale  $H \approx \text{const}$  denoting the almost constant expansion rate of the early universe above which linear inflationary perturbations with inverse wavelengths  $k$  freeze out,
- mass scales  $m$  below which particles with momentum  $p$  get non-relativistic,
- decaying late-time Hubble scales  $\sim t^{-1}$  below which large-scale structures start forming slowly during the radiation era and efficiently during matter domination,
- de Broglie wavelength  $\lambda_{dB}$  at which particle momenta  $p$  become comparable to the inverse system size  $k$  which calls for taking the particle-wave duality into account,

and more, some of which are spelled out in chapter 3. How quantum fields effectively appear within cosmological dynamics depends mainly on the perspective on a given cosmological problem and the initial conditions that come with it. The perspective on the problem refers to the time/energy, distance/momentum scales one is interested in or needs to consider whereas the initial conditions refer to the type of quantum state  $|\psi\rangle$  which is being studied. The perspective on a problem often depends on the state and vice versa since the initial state determines the type of particles present and on which scales and to which amount those particles are distributed. So are vacuum fluctuations of field perturbations the most important ingredient during inflation since actual particle excitation decay during the rapid expansion, whereas, in the standard treatment of CMB photons, the quantum field formulation is coarse-grained or approximated by a fluid description which one justifies with the huge number of photons that were created during reheating. However, since cosmology involves so many different scales it is sometimes an intricate question which picture suits a given problem best and fuzzy dark matter competing with a fluid or point-particle description represents a good example for this.

In the second part of this thesis we provide a field-theoretic framework generalizing fuzzy dark matter. We focus mostly on the foundations of large-scale structure formation, namely gravitationally interacting cold dark matter, and

consecutively develop a general formalism for the minimalistic set-up of a real scalar field coupled to gravity. We consider this derivation instructive for its own sake and, moreover, hope that it provides a different starting point on how non-linear large-scale structure may be tackled by presenting it in a different formulation that can maintain control over all scales.

## 1.A Outline

This thesis is structured into a first part (chapter 2) on entropy production in inflation via spectator fields, and a second part (chapters 3, 4 and 5) which covers field theoretical aspects of large-scale structure formation.

- In chapter 2, we study the production of entropy during inflation from loop corrections of spectator fields. At the end of inflation, this leads to a more general state than the initial squeezed state which is usually employed at the beginning of the radiation era.
- In chapter 3, we derive a classical particle phase-space density for dark matter based on equal-time two-point functions of canonical operators which are associated to a massive real scalar field that couples to semi-classical Einstein gravity with scalar metric perturbations in longitudinal gauge.
- In chapter 4, we extend the framework of chapter 3 to include arbitrary metrics that allow for vector and tensor perturbations, a non-minimal coupling to gravity and scalar self-interactions to one-loop accuracy.
- In chapter 5, we go beyond the semi-classical framework of chapters 3 and 4 by integrating out the gravitational perturbations in a non-relativistic limit which provides an action principle for the dynamical degrees of freedom in a general state.

## 1.B List of publications

The content within this thesis is based on the following publications:

- Chapter 2 is based on  
P. Friedrich, T. Prokopec, *Entropy production in inflation from spectator loops*. Phys. Rev. D, 100, 083505, (2019).
- Chapter 3 is based on  
P. Friedrich, T. Prokopec, *Scalar field dark matter in hybrid approach*. Phys. Rev. D, 96, 083504, (2017).
- Chapter 4 is based on  
P. Friedrich, T. Prokopec, *Kinetic theory and classical limit for real scalar quantum field in curved space-time*. Phys. Rev. D, 98, 025010, (2018).
- Chapter 5 is based on  
P. Friedrich, T. Prokopec, *Field-theoretic approach to large-scale structure formation*. Accepted for publication in Phys. Rev. D, arXiv:1909.10049, (2019).

The thesis concludes with a summary of our results and a discussion of open questions and future work.

## 1.C A very brief introduction to differential geometry

Tensorial structures are the building block of general relativity and cosmology. Let us give a very brief introduction for readers not familiar with those concepts. A tensor  $\mathcal{T}^{\mu_1 \dots \mu_n}_{\nu_1 \dots \nu_n}(x^\mu)$  is a rather generic object that generalizes functions on the space-time manifold (here, the universe). Its indices are usually denoted by Greek letters and also generic in the sense that, for example, an index  $\mu$  can be exchanged by an index  $\beta$  without changing any meaning. The indices run from 0 to 3 where 0 refers to a time-like direction and 1 to 3 refer to space-like directions. The number of indices  $n$  indicates the type of a tensor which is called its rank. A function  $\mathcal{T}(x^\mu)$  has no indices, it yields a number for every space-time point  $P$  whose four coordinates  $x^\nu$  are also written with an index although they are not tensors. Tensors  $\mathcal{T}^\mu(x^\nu)$  carrying one upper index are called four-vectors. Since they depend on every space-time point they contain four times as much information as a function. One can think of a four-vector  $\mathcal{T}^\mu(x^\nu)$  as a tangent to a curve in four-dimensional space-time, called world-line, which is determined by its direction as well as its modulus. A four-vector can thus be thought of as a four dimensional velocity where the time-like component corresponds to the rate of change of some clock and the spatial components correspond to usual three-dimensional velocities. Tensors with one or more lower indices  $\mathcal{T}_{\mu_1 \dots \mu_n}(x^\nu)$  serve mostly the purpose of being contracted with four-vectors to yield a number for every space-time point. Contraction is achieved by multiplying tensors and four-vectors at the same

space-time point together and summing over equal indices. Since tensors with rank two or higher can be contracted with any four-vector, they are not only functions of space-time points but also of all directions associated to a given space-time point. In this sense, the Einstein tensor  $G_{\mu\nu}$ , the metric  $g_{\mu\nu}$  and the energy-momentum tensor  $T_{\mu\nu}$  are providing space-time curvature, distances and energy-momentum in all possible directions of a given space-time point. Apart from the main features of acting like generalized functions, tensors come with another important property which is their general covariant transformation behaviour with respect to coordinate transformations in space-time. General covariance makes sure that all descriptions of a physical process are consistent, independently from which perspective they are made. However, allowing for these different perspectives introduces at the same time redundancies since the physical process itself is ignorant about the perspective it is being described from. The redundancies are called gauge degrees of freedom and appear, for example, also in the theory of electromagnetism, although they are of different origin there. Moreover, these redundancies automatically come with local conservation laws despite introducing forces between particles with local couplings. In the case of Einstein's theory of gravity, we have that energy and momentum are covariantly conserved

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma_{\mu\alpha}^\mu T^{\alpha\nu} + \Gamma_{\mu\alpha}^\nu T^{\mu\alpha} = 0, \quad (1.49)$$

where the number of terms involving the metric connection

$$\Gamma_{\mu\alpha}^\nu = \frac{1}{2} g^{\nu\beta} (\partial_\alpha g_{\mu\beta} + \partial_\mu g_{\alpha\beta} - \partial_\beta g_{\mu\alpha}), \quad (1.50)$$

depends on the rank of the tensor. Although general covariance is a very strong concept to formulate consistent theories, it complicates the analysis of physical degrees of freedom due to the gauge redundancies which we will also encounter in this thesis.



## Chapter 2

# Entropy production in inflation from spectator loops

Perturbations in cosmic microwave background (CMB) photons and large-scale structure (LSS) of the universe are sourced primarily by the curvature perturbation which is widely believed to be produced during inflation. In this chapter we present a two-field inflationary model in which the inflaton couples biquadratically to a spectator field. We show that the spectator induces a rapid growth of the momentum of the curvature perturbation and the associated Gaussian van Neumann entropy during inflation such that the initial conditions at the end of inflation are substantially different from the standard ones. Consequently, one ought to reconsider the kinetic equations describing evolution of the photon, dark matter and baryonic fluids in radiation and matter eras and take account of the fact that the curvature perturbation and its canonical momentum are two *a priori* independent stochastic fields. We also briefly analyze possible imprints on the CMB temperature fluctuations from the more general inflationary scenario which contains light spectator fields coupled to the inflaton.

## 2.1 Overview and motivation

It is a remarkable fact that all of the modern cosmic microwave background data, together with various large-scale structure probes, can be described by a class of simple cosmological models containing six parameters [34, 35]. Two of these parameters – the amplitude ( $A_s$ ) and spectral slope ( $n_s - 1$ ) of the curvature spectrum – are primordial in origin, while four – the Hubble parameter today ( $H_0$ ), the reionization optical depth ( $\tau_r$ ), (relative) baryonic density ( $\Omega_b$ ) and cold dark matter density ( $\Omega_c$ ) – are late time observables. Since the simplest ‘vanilla’ cosmological model assumes a spatially flat universe ( $\Omega_\kappa = 0$ ), the dark energy density  $\Omega_{de} = \Omega_\Lambda$  is not an independent parameter, *i.e.*  $\Omega_{de} = 1 - \Omega_b - \Omega_c$ . For more details we refer to [35, 36].

Cosmological models have been tested for various other features, that include various probes of isotropy and homogeneity, statistical Gaussianity (the amplitude of primordial bispectrum and trispectrum), the amplitude and slope of tensor perturbations, but for all of these only upper bounds exist, albeit there is a statistically weak evidence supporting some of the probes that indicate deviation from statistical isotropy or Gaussianity [37].

Another interesting class of features is encoded in isocurvature modes (see *e.g.* Ref. [12]). Even though there are many potential physical degrees of freedom which can play the role of isocurvature modes, there is no strong evidence in the data that would suggest that any of these contribute dominantly to the CMB photon temperature fluctuations. Indeed, [36] has looked for traces of cold dark matter density isocurvature (CDI), neutrino density isocurvature (NDI) and neutrino velocity isocurvature (NVI) modes in the data, and places upper limits on the relative amount of CDI, NDI and NVI of 2.5%, 7.4%, and 6.8%, respectively, at the scale of  $k = 0.002 \text{ Mpc}^{-1}$ . Signatures that are analogous to isocurvature modes are produced by topological defects and therefore similar upper bounds can be placed on the contribution of various classes of topological defects (which include cosmic strings, monopoles and textures) to the observed spectrum [38].

In this chapter we study an idea with similar effects, namely how spectator fields during inflation decohere the Gaussian density matrix of the curvature perturbation on super-Hubble scales by means of quantum loop interactions.<sup>1</sup> This decoherence is manifested as an increase of entropy during inflation and can produce similar signals as isocurvature modes and topological defects in the effective CMB temperature fluctuations. This is so because isocurvature modes tend to produce peaks which are out-of phase with the adiabatic mode, and therefore tend to wash out the coherent CMB oscillations. Let us be a bit more precise about the last statement and recap the form of the effective photon temperature fluctuation  $\Delta\hat{T}$  in momentum space before recombination in a simple approximation which we review in Appendix 2.B,

$$\Delta\hat{T}(\vec{k}, \eta) \approx \frac{1}{2}\hat{\Psi}(\eta_{\text{cmb}}, \vec{k}) \cos[kr_s(\eta)] + 2\frac{\Psi'(\eta_{\text{cmb}}, \vec{k})}{kc_s(\eta_{\text{cmb}})} \sin[kr_s(\eta)]. \quad (2.1)$$

Here,  $c_s(\eta)$  denotes the speed of sound and the sound horizon  $r_s(\eta)$  is its integral over conformal time. The stochastic variable  $\hat{\Psi}(\eta_{\text{cmb}}, \vec{k})$  is the gauge invariant perturbation of the trace of the spatial metric at conformal time  $\eta = \eta_{\text{cmb}}$  within the radiation era some time before recombination such that it is observable in the CMB. Its derivative in conformal time,  $\hat{\Psi}'(\eta_{\text{cmb}}, \vec{k})$ , is an *a priori* stochastically independent variable. We can conclude that coherent CMB oscillations are possible if the stochastic operators  $\hat{\Psi}(\eta_{\text{cmb}}, \vec{k})$  and  $\hat{\Psi}'(\eta_{\text{cmb}}, \vec{k})$  are linearly related (which induces a phase-shift) or either of them is much smaller than the other. As we pointed out above, Planck data is mostly consistent with coherent CMB oscillation such that the standard case is to discard the initial time-derivative of the gravitational potential and consider only the adiabatic mode whose associated operator is conserved on super-Hubble scales. Still, the constraints to wash out the CMB oscillation reside in the range of percent so its worth studying mechanisms that can contribute to it. This allows us to either target those effects by precision cosmology or to rule them out. We remind ourselves in Appendix 2.C that the linear dynamics of single-field inflation on super-Hubble scales effectively decreases the number of independent stochastic

<sup>1</sup> While this work was nearing completion, we became aware that a similar problem had recently been addressed by [39].

operators to the aforementioned adiabatic mode. Thus, one way of obtaining a non-vanishing and stochastic independent time-derivative of the initial gravitational potential in (2.1) is to work with non-trivial background trajectories in multi-field inflationary model, leading to the aforementioned isocurvature modes whose stochastic independence can be traced back to independent quantum fluctuations whose presence is guaranteed by vacuum expectation values of the additional fields.

We obtain a significant amount of decoherence at the end of inflation by going beyond the tree-level analysis and relying purely on *interactions* of the inflaton perturbation  $\varphi$  with a *spectator field*  $\chi$  that has a zero expectation value. We chose such a simple model because the inflaton coupling to the spectator field is controlled by a separate coupling constant, which is independent on the loop counting parameter of quantum gravity,  $\kappa^2 H^2 \sim H^2/M_{\text{P}}^2 \sim 10^{-12}$  (here  $H$  is the inflationary Hubble parameter and  $M_{\text{P}} \simeq 2.4 \times 10^{18}$  GeV is the reduced Planck mass), which governs the strength of interactions in the inflation sector. Moreover, since the spectator field does not acquire an expectation value, it is invariant under coordinate transformations to linear order. Thus, if we express corrections to the inflaton propagator in terms of the gauge invariant curvature perturbation  $\mathcal{R}$  and take corrections to the inflaton expectation value  $\bar{\phi}$  into account, our results are to linear order gauge invariant and we may compare them to the tree-level analysis at the end of inflation.

The effect of quantum corrections to the power spectrum of the curvature perturbation has been studied in [40, 41] with the conclusion that loop corrections on super-Hubble scales can at most be enhanced as powers of logarithms of the scale factor. However, the power spectrum of the  $\mathcal{R}\mathcal{R}$ -correlator remains approximately frozen due to the coupling constant suppression and the limit on how long inflation lasts. In this chapter, we reconsider these observations with a concrete calculation in the above mentioned model involving spectator fields. The model consists of two canonical scalar fields on locally de Sitter background that interact *via* a cubic interaction which is derived by expanding a biquadratic action around the vev of the inflaton. While the interactions with the spectator indeed produce logarithmic corrections to the comoving curvature perturbations, the corrections to the canonical momentum of the comoving curvature perturbations grow exponentially in time (inverse power in conformal time) and may induce considerable fluctuations. The question whether these field excitations are stochastically independent can be answered by calculating the Gaussian part of the von Neumann entropy  $S_{\text{vN}}$  associated to  $\hat{\mathcal{R}}$  and  $\hat{\pi}_{\mathcal{R}}$ , which is conveniently represented in momentum space by,

$$S_{\text{vN}}[\mathcal{R}, \pi_{\mathcal{R}}] = \frac{1}{2} \sum_{\vec{k}} s_{\text{vN}}(\eta, k), \quad (2.2)$$

$$s_{\text{vN}} = \frac{\Delta_{\mathcal{R}} + 1}{2} \log \frac{\Delta_{\mathcal{R}} + 1}{2} - \frac{\Delta_{\mathcal{R}} - 1}{2} \log \frac{\Delta_{\mathcal{R}} - 1}{2}, \quad (2.3)$$

which depends on the Gaussian invariant  $\Delta_{\mathcal{R}}^2$  (see e.g. [42]),

$$\Delta_{\mathcal{R}}^2(\eta, k) = 4 [\Delta_{\mathcal{R}\mathcal{R}}(\eta, k) \Delta_{\pi_{\mathcal{R}}\pi_{\mathcal{R}}}(\eta, k) - \Delta_{\mathcal{R}\pi_{\mathcal{R}}}^2(\eta, k)], \quad (2.4)$$

where  $\Delta_{\mathcal{R}\mathcal{R}}$ ,  $\Delta_{\pi_{\mathcal{R}}\pi_{\mathcal{R}}}$  and  $\Delta_{\mathcal{R}\pi_{\mathcal{R}}}$  are the equal-time momentum space two-point functions. The Gaussian invariant  $\Delta_{\mathcal{R}}^2$  is identical to *one* for linearly evolved fields prepared in a pure Gaussian initial state (an important example of which is the Bunch-Davies vacuum) and thus yields *zero* Gaussian von Neumann entropy. A large Gaussian invariant on the other hand would indicate a big uncertainty in the phase-space which is spanned by the operators  $\hat{\mathcal{R}}$  and  $\hat{\pi}_{\mathcal{R}}$ .

In order to see how quantum interactions with spectators during inflation influence the CMB, we relate the gauge-invariant gravitational potential  $\hat{\Psi}$  shortly before the end of inflation to the gauge-invariant curvature perturbation  $\hat{\mathcal{R}}$  and evolve it to the radiation era where we assume a simple scenario in which we switch of the interactions after inflation. In Appendix 2.B we review that equation (2.1) then takes the following form,

$$\begin{aligned} \Delta\hat{T}(\eta, \vec{k}) \approx & \frac{1}{2} \left[ \frac{2}{3} \hat{\mathcal{R}}(\eta_e, \vec{k}) - \frac{a^3(\eta_e)}{a^3(\eta_{\text{cmb}})} \frac{H}{2M_p^2 k^2 a(\eta_e)} \hat{\pi}_{\mathcal{R}}(\eta_e, \vec{k}) \right] \cos[kr_s(\eta)] \\ & + \frac{6H}{kc_s(\eta_{\text{cmb}})} \frac{a^4(\eta_e)}{a^4(\eta_{\text{cmb}})} \left[ \frac{H}{2M_p^2 k^2} \hat{\pi}_{\mathcal{R}}(\eta_e, \vec{k}) + a(\eta_e) \hat{\mathcal{R}}(\eta_e, \vec{k}) \right] \sin[kr_s(\eta)], \quad (2.5) \end{aligned}$$

where the parameter  $H$  is the Hubble scale at beginning of inflation and the argument  $\eta_e$  is some time shortly before the end of inflation such that the slow-roll parameter  $\epsilon = 1 - \mathcal{H}^{-2} \mathcal{H}'$ , with  $\mathcal{H}a = a'$ , is still small,  $\epsilon(\eta_e) \ll 1$ . We see that the main contribution  $\propto \hat{\pi}_{\mathcal{R}}$  in (2.5) could wash out the Sakharov oscillations if it was able to balance the heavy suppression by the pre factor  $\propto a^{-4}$ , which is for initially small amplitudes only possible if  $\hat{\pi}_{\mathcal{R}}$  was growing during inflation. As we review in Appendix 2.C, linear single-field inflation yields the following relation on super-Hubble scales in slow roll regime,

$$\hat{\pi}_{\mathcal{R}}^{(\text{lin})}(\eta_e, \vec{k}) = - \frac{2M_p^2 a(\eta_e) \epsilon(\eta_e)}{H} [\hat{\mathcal{R}}(\eta_e, \vec{k}) + \mathcal{O}(k\eta_e)], \quad (2.6)$$

such that stochastic independent off-peak contribution in (2.5) can safely be neglected. However, in models in which the inflaton couples to other matter fields with unsuppressed couplings (a notable example being Higgs inflation), there is no reason to *a priori* expect that the standard tree level results apply and thus spectator fields without vevs might still contribute to stochastic independent modes.

While this work is inspired by the large literature on decoherence and classicalization of cosmological perturbations [43–50], it also differs from it in important aspects. In contrast with the effective approaches based on studying the approximate evolution of the reduced density matrix [51], we use standard perturbative methods of the quantum field theory [42, 52–55]. Furthermore, we identify the late time (CMB) observables that can be used to quantify the amount of decoherence in the curvature perturbation (expressed through the Gaussian part of the von Neumann entropy) that occurs during inflation and subsequent epochs, while most of the existing works base their analysis on standard criteria for classicalization often used in condensed matter systems, such as the diagonalization rate of the reduced density matrix in a suitably chosen pointer basis. While early works [43–50] used the late time observer’s inability to

get a complete access to the state of cosmological perturbations as the principal source of decoherence and classicalization (the so-called ‘decoherence without decoherence’), later works used more realistic settings, in which (dissipative) interactions among quantum fields during (or after) inflation is the principal cause for decoherence. The interactions considered range from self-interactions of the inflaton field [56–59], interactions with gravitational waves [60, 61], interactions with other scalar fields [62–65], as well as interactions with massive fermionic fields [66].

Encouraged by the result of [50] we decided to investigate the effect of one-loop interactions between the spectator and the inflaton where the fields interact biquadratically. When this work was nearing completion, we became aware that a similar problem was addressed in [39] based on the density matrix formalism developed in [67, 68]. While the authors of references [39, 67, 68] start from a cubic interaction and make use of the density matrix formalism, we start from a bi-quadratic interaction which provides a stable theory for a positive coupling. By expanding around the inflaton condensate, we also obtain an effective cubic vertex which turns out to yield the dominate contributions to decoherence. However, we approach the problem differently by providing a one loop evaluation of the inflaton propagator  $\Delta_{\varphi\varphi}(\eta, \eta', k)$  from which we can fully reconstruct the Gaussian part of the density matrix.

This chapter is organized as follows. In section 2.2 we explain the model set up and how to relate the various two-point functions. In the follow-up section 2.3, we present the main steps in the calculation, including renormalization, the solution of the equation of motion for the statistical propagator, symmetry properties and the super-Hubble limit. In section 2.4 we come back to the implications of our results and discuss extensions of the presented analysis. Moreover, we make a comparison with the findings of references [39, 67, 68]. Some important technical details of the calculations are presented in several appendices.

We work in natural units in which  $c = \hbar = 1$  and with the metric tensor with a mostly plus signature,  $(-, +, +, +)$ .

## 2.2 Growing curvature momentum from quantum interactions

Coupling of the comoving curvature perturbation to other fields can be mediated not only via tree level processes, but can be also studied at the quantum (loop) level. Take a simple two-scalar-field inflationary model with a bi-quadratic interaction term,

$$S[\phi, \chi] = S_{\text{EH}} + \int d^D x \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) - \frac{1}{2} g^{\mu\nu} (\partial_\mu \chi) (\partial_\nu \chi) - V(\phi, \chi) \right), \quad (2.7)$$

where  $S_{\text{EH}}$  is the Einstein-Hilbert action,

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^D x \sqrt{-g} R, \quad (2.8)$$

where  $D$  is the number of space-time dimensions,  $R = R[g_{\mu\nu}]$  is the Ricci curvature scalar, the field  $\phi$  is the inflaton with the perturbation,

$$\hat{\phi} = \phi - \bar{\phi}, \quad \bar{\phi}(t) = \langle \phi(x) \rangle \quad (2.9)$$

the field  $\chi$  is a spectator with a vanishing expectation value,  $\langle \hat{\chi} \rangle = 0$  and the potential  $V(\phi, \chi)$  reads,

$$V(\phi, \chi) = \frac{m_\phi^2}{2} \phi^2 + \frac{m_\chi^2}{2} \chi^2 + \frac{g}{4} \chi^2 \phi^2, \quad (2.10)$$

where both fields are assumed to be light,

$$H \gg m_\chi, m_\phi. \quad (2.11)$$

We are interested in studying the dynamics of the metric and field perturbations around a cosmological background, with the metric tensor (in the plasma rest frame) given by,

$$\bar{g}_{\mu\nu} = \text{diag} \left( -\bar{N}^2(t), \underbrace{a^2(t), \dots, a^2(t)}_{D-1 \text{ times}} \right), \quad \bar{g} = \det[\bar{g}_{\mu\nu}] = \bar{N}^2 a^{2(D-1)}, \quad (2.12)$$

where  $\bar{N}(t)$  is the lapse function and  $a(t)$  is the scale factor. While it would be of interest to study both the dynamics of the quantum gravitational and quantum scalar perturbations, for simplicity in this work we limit ourselves to studying the dynamics of the scalar curvature perturbation induced by its bi-quadratic interaction term given in Eq. (2.10). This process is controlled by the coupling constant  $g$  which is generally different from the gravitational coupling constant  $\kappa = 1/\sqrt{16\pi G}$ , where  $G$  denotes the Newton constant, and therefore can be separately studied. To show that, in what follows we recall some of the basics of the quantum perturbative gravity in inflationary space-times.

The theory (2.7) has two dynamical scalar degrees of freedom, which in the comoving gauge ( $\varphi = 0$ ) are the scalar metric perturbation  $\psi = -\text{Tr}[\delta g_{ij}]/(6a^2)$  and the isocurvature field,  $\chi$ , and one transverse, traceless tensor perturbation  $h_{ij} = \delta g_{ij}/a^2$  with  $\delta_{ij} h_{ij} = 0 = \partial_i h_{ij}$ . In addition, there are constraint degrees of freedom: one scalar and one transverse vector degree of freedom, namely the lapse function  $N(x)$  and the shift vector  $N_i(x)$  (with  $\partial_i N_i = 0$ ). Since one can choose a gauge in which the lapse and shift decouple from the dynamical degrees of freedom, one can ignore them [69, 70].

The dynamics of the linear scalar cosmological perturbations is governed by the well-known Mukhanov-Sasaki action [71]. When written for the curvature perturbation  $\mathcal{R}$ , the action reads [9, 71, 72],

$$S_s^{(2)}[\mathcal{R}] = \int d^D x \bar{N} a^{D-1} 2\epsilon M_P^2 \left[ \frac{1}{2} \dot{\mathcal{R}}^2 - \frac{1}{2a^2} (\partial_i \mathcal{R})^2 \right], \quad (2.13)$$

where

$$\epsilon = -\frac{\dot{H}}{H^2}, \quad M_P^2 \equiv \frac{1}{8\pi G}, \quad (2.14)$$

and the quadratic action for the tensor perturbations,  $h_{ij} = \delta g_{ij}/a^2$ , which in the traceless and transverse gauge ( $\delta_{ij}h_{ij} = 0 = \partial_i h_{ij}$ ) reduces to,

$$S_t^{(2)} = \frac{M_{\text{P}}^2}{8} \int d^D x \bar{N} a^{D-1} \left[ \dot{h}_{ij}^2 - \frac{1}{a^2} (\partial_l h_{ij})^2 \right], \quad (2.15)$$

where a *dot* signifies a reparametrization invariant derivative with respect to time,  $\dot{X} \equiv \bar{N}^{-1} \partial_t X$ . Note that both actions (2.13) and (2.15) are manifestly gauge invariant, as they are written for the gauge invariant curvature perturbation  $\mathcal{R}$  and gauge invariant tensor perturbation  $h_{ij}$ . If one fixes a gauge completely, one can easily get the corresponding gauge fixed action from (2.13). For example, in the comoving gauge ( $\varphi = 0$ ), in which  $\mathcal{R} \rightarrow \psi$ , the action for  $\psi$  is identical in form as the action (2.13) for  $\mathcal{R}$ ; in the zero-curvature gauge ( $\psi = 0$ ), the action for  $\varphi$  is obtained by exacting the replacement,  $\mathcal{R} \rightarrow \varphi/(\sqrt{2\epsilon}M_{\text{P}})$  in (2.13),

$$S_s^{(2)}[\varphi] = \int d^D x \bar{N} a^{D-1} \left[ \frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} \left( \frac{\partial_i \varphi}{a} \right)^2 + \frac{1}{4} \left( \frac{(a^{D-1} \dot{\epsilon})}{a^{D-1} \epsilon} - \frac{1}{2} \frac{\dot{\epsilon}^2}{\epsilon^2} \right) \varphi^2 \right], \quad (2.16)$$

such that the linear dynamics of the inflaton perturbation corresponds to that of a harmonic oscillator with a time dependent frequency. Since the spectator field  $\chi$  remains invariant to linear order under gauge transformations, the quadratic action for the spectator field  $\chi$  is by itself gauge invariant,

$$S_s^{(2)}[\chi] = \int d^D x \bar{N} a^{D-1} \left[ \frac{1}{2} \dot{\chi}^2 - \frac{1}{2} \left( \frac{\partial_i \chi}{a} \right)^2 - \frac{1}{2} \left( m_\chi^2 + \frac{g}{2} \bar{\phi}^2 \right) \chi^2 \right]. \quad (2.17)$$

In addition, there are two physical constraint fields - the lapse and (transverse) shift function, but they decouple from the dynamical degrees of freedom  $\mathcal{R}$  and  $h_{ij}$ . While this decoupling is clearly evident (from the Helmholtz decomposition) at the linear order in the perturbations, one has to work harder to show that it also works at higher order in perturbations [69, 70]. In fact, there are gauges in which the constraint fields can play an important role [73]. The leading order actions (2.15–2.17) are supplemented by the higher order actions describing cubic, quartic and higher order interactions [11, 69, 70]. Generically, while all gravitational interactions are suppressed by powers of the gravitational coupling constant  $\kappa = 1/\sqrt{16\pi G}$ , the interactions involving the scalar curvature perturbation are in addition suppressed by powers of the slow roll parameters,  $\epsilon = -\dot{H}/H^2$  and/or its derivatives (no such suppression occurs in the tensor interactions). However, that does not mean that scalar loops are suppressed when compared with the tensor loops, since the scalar curvature propagator is enhanced by a factor  $\sim 1/\epsilon$  when compared with the tensor propagator, thus nullifying the slow-roll vertex suppression. The result is that, quite generically, each gravitational loop contributes as,  $\sim \kappa^2 H^2 \sim H^2/M_{\text{P}}^2$ . In addition, Weinberg's theorem [40, 41] allows for a secular enhancement in the form of powers of the number of e-foldings,  $\mathcal{N} = \ln(a)$ . Since not much is known about such secular enhancements of the gravitational loops (because the problem of

gauge dependence of gravitational loops is *not* well understood [74, 75]), for the sake of simplicity we neglect them in what follows.

From Eq. (2.17) we see that the inflaton condensate  $\bar{\phi} \sim HM_{\text{P}}/m_{\phi}$  generates a mass for the spectator field  $\chi$  of the order,

$$\delta m_{\chi}^2 = \frac{g}{2} \bar{\phi}^2 \sim g H^2 \frac{M_{\text{P}}^2}{m_{\phi}^2}. \quad (2.18)$$

Since light scalar field fluctuations grow during inflation, their effect on the inflaton fluctuation will be larger than from a heavy scalar field. Demanding that the spectator  $\chi$  remains light during inflation,  $\delta m_{\chi}^2 \ll H^2$ , leads to the following condition on the coupling constant,

$$0 < g \lesssim \frac{m_{\phi}^2}{M_{\text{P}}^2} \sim 10^{-12}. \quad (2.19)$$

Let us first consider the tadpole contribution to the expectation value of the inflaton field  $\bar{\phi}$ , which contributes to the inflaton equation of motion as,

$$(\square - m_{\phi}^2) \bar{\phi} = \frac{g}{2} \bar{\phi} i \Delta_{\chi}(x; x). \quad (2.20)$$

This ought to be renormalized by the non-minimal coupling counterterm,  $\int d^D x \left( -\frac{1}{2} \delta \xi R \bar{\phi}^2 \right)$ . According to (2.11), we assume that the coincident scalar propagator is that of the massless scalar in de Sitter space. The finite part of the coincident propagator is given by

$$i \Delta_{\chi}(x; x)_{\text{fin}} \simeq [H^2/(4\pi^2)] \ln(a), \quad (2.21)$$

which exhibits a secular growth and modifies the inflaton mass by  $\delta m_{\phi}^2 = [g H^2/(8\pi^2)] \ln(a) \ll m_{\phi}^2$  by a negligibly small amount. Moreover, this contribution changes the expansion rate and slow roll parameters, but by a small amount. These corrections are important for maintaining gauge invariance of the corrected comoving curvature perturbation at linear order. The reason is that the inflaton vev enters the definition of the curvature perturbation and its corrections are of the similar order as the non-local self-mass corrections. However, local terms will not induce dissipative effects that could affect the entropy of cosmological perturbations [76] and they are negligible for the canonical momentum of the comoving curvature perturbation and correlators thereof, as we will see explicitly later on.

The interaction between the inflaton and spectator fields is governed by cubic and quartic interactions, whose actions are,

$$S_s^{(3)}[\varphi, \chi] = \int d^D x \bar{N} a^{D-1} \left( -\frac{h}{2} \varphi \chi^2 \right), \quad h = g \bar{\phi}, \quad (2.22)$$

$$S_s^{(4)}[\varphi, \chi] = \int d^D x \bar{N} a^{D-1} \left( -\frac{g}{4} \varphi^2 \chi^2 \right). \quad (2.23)$$



Let us first make a rough comparison of the effects induced by these two interactions on the dynamics of the inflaton perturbation.

The one-loop  $\mathcal{O}(g)$  contribution generated by the quartic interaction (2.23) will (upon renormalization) generate a time dependent mass term for the inflaton fluctuations,  $\delta m_\phi^2 = (g/2)i\Delta_\varphi(x; x)$ , where  $i\Delta_\varphi(x; x) = \langle \hat{\varphi}(x)^2 \rangle$  denotes the coincident two-point function for the inflaton perturbation) and thus will not generate any entropy or any other dissipative effects in the scalar sector of the theory.

Next, at order  $g^2$  there are two contributions: the one-loop contribution in figure 2.1a which is generated by the cubic action (2.22) and the two-loop contribution in figure 2.1b generated by the quartic interaction (2.23). Since we are primarily interested in super-Hubble fluctuations, we shall compare the size of these two diagrams for super-Hubble distances,  $\|\vec{x} - \vec{x}'\| \gg 1/H$  and at equal time,  $t = t'$ . It is not hard to see that the ratio of the two-loop to the one-loop contribution scales roughly as,

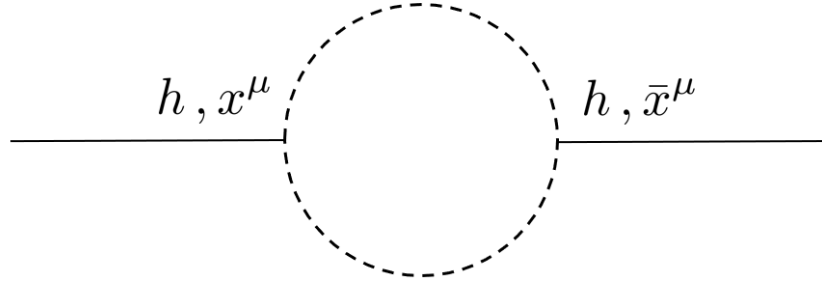
$$\frac{i\Delta_\varphi(t, \vec{x}; t, \vec{x}')}{\bar{\phi}(t)^2} \sim \frac{m_\phi^2 \ln(a)}{M_{\text{P}}^2} \ll 1, \quad (2.24)$$

where we made use of  $i\Delta_\varphi(t, \vec{x}; t, \vec{x}') \sim H^2 \ln(a)$ ,  $\bar{\phi} \sim HM_{\text{P}}/m_\phi$  and  $m_\phi \ll H$  (in the above estimate, factors of order one such as powers of  $\pi$  have been neglected). This means that the principal diagram that contributes (in a dissipative manner) to the dynamics of the inflaton perturbation, and therefore also to the curvature perturbation, is the one-loop diagram in figure 2.1a.

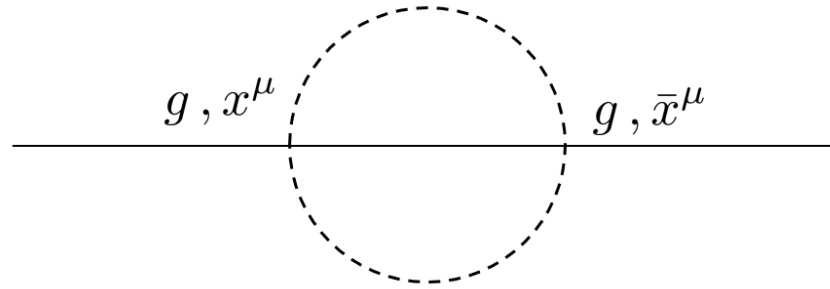
In what follows we shall compare the size of the one loop spectator diagram with that of the quantum gravitational loops. From Eq. (2.32) we see that the ratio of the one-loop to the tree level Hadamard function is of the order  $\delta F_\varphi/F_{\varphi, dS} \sim (h^2/H^2) \ln^3(a)$ , which ought to be compared with the corresponding quantum gravitational contribution,  $\kappa^2 H^2 \ln^{n_g}(a) \sim (H^2/M_{\text{P}}^2) \ln^{n_g}(a)$ , where  $n_g$  as an unspecified positive integer which parametrizes our ignorance of the quantum gravitational loops. Upon dividing the two contributions we get,

$$\frac{(g^2 \bar{\phi}^2/H^2) \ln^3(a)}{\kappa^2 H^2 \ln^{n_g}(a)} \lesssim \frac{m_\phi^2}{H^2} [\ln(a)]^{3-n_g}, \quad (2.25)$$

Knowing the secular terms can be crucial, since each power of  $\ln(a)$  produces an enhancement by a factor  $\sim 10^2$ , and that can be detrimental for determining whether the quantum gravitational or spectator contributions in (2.25) dominate. From the estimate in (2.25) we see that the condition that  $\chi$  remains light in inflation implies that the contribution from the spectator loop can be comparable to the quantum gravitational loops. This means that, before one makes any definite conclusion concerning the strength of decoherence during inflation, one also ought to investigate the effect of the quantum gravitational loops. In fact, there have been several attempts to do precisely that [58, 59, 62, 63, 77]. In addition, a lot of work has been invested into a much easier set of problems, namely into studying how the inflaton coupling with the other quantum fields (scalar, fermionic or vector) induces decoherence in the inflaton sector [48, 50, 63, 66]. While the earlier works considered simple models



(A) The one-loop Feynman diagram for the inflaton two-point function (solid lines) generated by the cubic interaction in (2.22). The spectator field  $\chi$  (dashed lines) runs in the loop. The vertex coupling strength is  $h = g\bar{\phi}$ .



(B) The two-loop diagram generated by the quartic interaction in (2.23) with the spectator  $\chi$  (dashed lines) and inflaton (solid lines) running in the loops.

FIGURE 2.1: Interactions between the inflaton perturbation  $\varphi$  and the spectator field  $\chi$ .

with bilinear couplings [48, 50] (since these couplings are non-dissipative, they are not true interactions), more recent works studied true interactions [63, 66]. These type of studies are much easier, since the hardest problem – the problem of gauge dependence – is absent in these studies.

While these attempts represent important first steps, it is fair to say that no definite answer to that question has been given as yet. The principal reason is that none of the existing works has seriously addressed the issue of gauge (in-)dependence, neither have the authors performed a complete quantum calculation which must include: (a) a complete set of Feynman rules, with all relevant vertices and propagators included (currently there exists no propagator for that encompasses the dynamics of both scalar and tensor perturbations in inflation); (b) a complete calculation of the one-loop diagrams that includes (preferably dimensional) regularization and renormalization, with the notable exception of Refs. [39, 67], where normal ordering was used to renormalize the self-mass; (c) a study of how the inflaton two-point function gets modified by the one-loop quantum fluctuations, which also includes a detailed analysis of how it depends on the choice of gauge. Before we have good understanding of all of these steps and problems, we cannot say anything definite regarding the importance of the quantum gravitational loops for the evolution of cosmological perturbations.

As a final remark, we point out that, because the spectator loop is controlled by a different coupling constant ( $g$ ) from that governing the quantum gravitational loops ( $\kappa$ ), one can unambiguously separate the two. In other words, the quantum gravitational loops cannot cancel or compensate the effects of the spectator loop studied in this work.

In principle we could include slow-roll corrections in our study. However, including them would significantly complicate the spectator propagator, and thus also the whole calculation. Therefore, for simplicity, we shall consider a nearly de Sitter inflation, in which the effects due to slow roll corrections are negligibly small. We point out that the spectator field is very different from the inflaton in that taking the limit  $\epsilon \rightarrow 0$  in the scalar sector of the graviton is a delicate one, because the curvature propagator is in that limit enhanced as  $\propto 1/\epsilon$ , *cf.* the action for the curvature perturbation (2.16). No such enhancement is present in the spectator sector of the theory, implying that there is no subtlety involved in taking the limit  $\epsilon \rightarrow 0$ . Moreover, the tensor-to-scalar ratio  $r \simeq 16\epsilon \leq 0.065$  is known to be small, implying that  $\epsilon < 1/200$ , such that taking the limit  $\epsilon \rightarrow 0$  should give reasonably accurate answers. Next, the spectral slope of the curvature perturbation is also quite small,  $n_s - 1 \simeq -0.035 = -2\epsilon - \epsilon_2 \approx -\epsilon_2$ , and it is controlled by the second slow roll parameter  $\epsilon_2 = \dot{\epsilon}/(\epsilon H) \simeq 0.035$ . This near scale invariance of the scalar perturbation also tells us that approximating the tree level equation for the inflaton perturbation by that of a massless scalar,  $\square\varphi = 0$ , constitutes a reasonably accurate approximation, where  $\square = g^{\mu\nu}\nabla_\mu\nabla_\nu$  is the d'Alembertian operator.

With these remarks in mind, we can now proceed to the calculation of the Hadamard function induced by the one-loop diagram shown in figure 2.1a. The calculation will be done entirely on spatially flat sections of de Sitter space (Poincaré patch), in which the scale factor in conformal time  $d\eta = dt/a$  reads,

$$a(\eta) = -\frac{1}{H\eta}, \quad (\eta < 0). \quad (2.26)$$

The relevant action is simply,

$$S[\varphi, \chi] \approx \int d^4x \sqrt{-\bar{g}_{dS}} \left( -\frac{1}{2} \bar{g}_{dS}^{\mu\nu} (\partial_\mu \varphi) (\partial_\nu \varphi) - \frac{1}{2} \bar{g}_{dS}^{\mu\nu} (\partial_\mu \chi) (\partial_\nu \chi) - \frac{h}{2} \varphi \chi^2 \right), \quad (2.27)$$

with a de Sitter background metric  $\bar{g}_{\mu\nu}^{dS}$ . The free theory is solved in momentum space, with  $k = \|\vec{k}\|$ , for each field by the Bunch-Davies vacuum whose positive (+) and negative (−) frequency mode functions are given by

$$u_{dS}^\pm(\eta, k) = \frac{H}{\sqrt{2k^3}} (1 \pm ik\eta) e^{\mp ik\eta}. \quad (2.28)$$

In Appendix 2.A, we give the definition of the Wightman functions  $\Delta_\varphi^{\mp\pm}$  as well as the spectral (causal) two-point function  $\Delta_\varphi^c$  and the Hadamard (statistical) two-point function  $F_\varphi$  in momentum space. For the Bunch-Davies vacuum they

read

$$i\Delta_{\varphi,dS}^{\mp\pm}(\eta, \eta', k) = \frac{H^2}{2k^3}(1 \pm ik\eta)(1 \mp ik\eta')e^{\mp ik(\eta-\eta')}, \quad (2.29)$$

$$\Delta_{\varphi,dS}^c(\eta, \eta', k) = \frac{H^2}{k^3} \left[ k(\eta-\eta') \cos[k(\eta-\eta')] - (1+k^2\eta\eta') \sin[k(\eta-\eta')] \right], \quad (2.30)$$

$$F_{\varphi,dS}(\eta, \eta', k) = \frac{H^2}{2k^3} \left[ (1+k^2\eta\eta') \cos[k(\eta-\eta')] + k(\eta-\eta') \sin[k(\eta-\eta')] \right]. \quad (2.31)$$

In the following section 2.3, we compute the one-loop correction to the statistical propagator in the super-Hubble limit as,

$$\begin{aligned} \delta F_{\varphi}(\eta, \eta', k) &= [F_{\varphi} - F_{\varphi,dS}](\eta, \eta', k) \\ &= \frac{h^2}{2^6 3^3 k^3 \pi^2} \left\{ 4 \left[ \log^3(-2k\eta) + \log^3(-2k\eta') \right] \right. \\ &\quad + \left[ 12 \log\left(\frac{H}{2k}\right) - 5 \right] \left[ \log^2(-2k\eta) + \log^2(-2k\eta') \right] \\ &\quad + 6 \left[ 4 \log\left(\frac{H}{2k}\right) - 4\gamma_E + 5 \right] \log(-2k\eta) \log(-2k\eta') \\ &\quad - \left[ (106 - 48\gamma_E) \log\left(\frac{H}{2k}\right) - 18 \log\left(\frac{\mu}{k}\right) + 36\gamma_E(\gamma_E - 3) \right. \\ &\quad \left. \left. + \pi^2 + \frac{208}{3} \right] \log(4k^2\eta\eta') + \mathcal{O}(k\eta, k\eta') \right\}, \quad (2.32) \end{aligned}$$

where the parameter  $\mu$  is the renormalization scale and the quantity  $\gamma_E = -\psi(1) \approx 0.577216$  is Euler's constant, where  $\psi(z) = (d/dz) \ln(\Gamma(z))$  is the digamma function (not to be confused with the spatial scalar metric perturbation  $\psi$ ). We now have to express these results in terms of the comoving curvature perturbation which we achieve in a first approximation by using linear relations. The comoving curvature perturbation  $\mathcal{R}$  and its canonical momentum  $\pi_{\mathcal{R}}$  read to linear order in zero curvature gauge  $\psi = 0$ ,

$$\mathcal{R} \equiv \psi + \frac{H}{\dot{\phi}}\varphi \longrightarrow \frac{H}{\dot{\phi}}\varphi = \frac{1}{\sqrt{2\epsilon}} \frac{\varphi}{M_p}, \quad (2.33)$$

$$\pi_{\mathcal{R}} \equiv 2a^2 M_p^2 \epsilon \partial_{\eta} \mathcal{R} \longrightarrow \sqrt{2\epsilon} M_p a^2 \left[ \partial_{\eta} \varphi - (\partial_{\eta} \epsilon) \frac{\varphi}{2\epsilon} \right]. \quad (2.34)$$

This procedure gives results that are gauge invariant to linear order in coordinate gauge transformations if the one-loop corrections discussed above are consistently taken into account.

Using the linear relations (2.34) we can express the statistical two-point functions of the comoving curvature perturbation and its canonical momentum

in terms of the inflaton correlator to linear order as

$$\Delta_{\mathcal{R}\mathcal{R}}(\eta, k) \equiv F_{\mathcal{R}}(\eta, \eta, k) = \frac{1}{2\epsilon M_p^2} F_\varphi(\eta, \eta, k), \quad (2.35)$$

$$\begin{aligned} \Delta_{\mathcal{R}\pi_{\mathcal{R}}}(\eta, k) &\equiv 2a^2 M_p^2 \epsilon \partial_{\eta'} F_{\mathcal{R}}(\eta, \eta', k) \Big|_{\eta=\eta'} \\ &= a^2 \left[ \frac{1}{2} \partial_\eta - \frac{(\partial_\eta \epsilon)}{2\epsilon} \right] F_\varphi(\eta, \eta, k), \end{aligned} \quad (2.36)$$

$$\begin{aligned} \Delta_{\pi_{\mathcal{R}}\pi_{\mathcal{R}}}(\eta, k) &\equiv (2a^2 M_p^2 \epsilon)^2 \partial_\eta \partial_{\eta'} F_{\mathcal{R}}(\eta, \eta', k) \Big|_{\eta=\eta'} \\ &= 2\epsilon M_p^2 a^4 \left[ \partial_\eta \partial_{\eta'} F_\varphi(\eta, \eta', k) \Big|_{\eta=\eta'} \right. \\ &\quad \left. - \frac{(\partial_\eta \epsilon)}{2\epsilon} \partial_\eta F_\varphi(\eta, \eta, k) + \frac{(\partial_\eta \epsilon)^2}{4\epsilon^2} F_\varphi(\eta, \eta, k) \right]. \end{aligned} \quad (2.37)$$

Thus, shortly before the end of inflation at  $\eta = \eta_e$  such that the slow-roll parameter  $\epsilon(\eta_e)$  is still small and to leading order a constant, we have the following leading order corrections to the comoving curvature correlators on super-Hubble scales  $|k\eta_e| \ll 1$ ,

$$\begin{aligned} \Delta_{\mathcal{R}\mathcal{R}}(\eta_e, k) &= \frac{H^2}{4M_p^2 k^3 \epsilon(\eta_e)} \left\{ 1 + \frac{h^2}{108\pi^2 H^2} \left[ \log^3(-2k\eta_e) \right. \right. \\ &\quad \left. \left. + \mathcal{O}(\log^2(-2k\eta_e)) \right] + \mathcal{O}(\epsilon(\eta_e), k\eta_e) \right\}, \end{aligned} \quad (2.38)$$

$$\begin{aligned} \Delta_{\mathcal{R}\pi_{\mathcal{R}}}(\eta_e, k) &= -\frac{Ha(\eta_e)}{2k} \left\{ 1 + \frac{a^2(\eta_e)h^2}{72\pi^2 k^2} \left[ \log^2(-2k\eta_e) \right. \right. \\ &\quad \left. \left. + \mathcal{O}(\log(-2k\eta_e)) \right] + \mathcal{O}(\epsilon(\eta_e), k\eta_e) \right\}, \end{aligned} \quad (2.39)$$

$$\begin{aligned} \Delta_{\pi_{\mathcal{R}}\pi_{\mathcal{R}}}(\eta_e, k) &= kM_p^2 a^2(\eta_e) \epsilon(\eta_e) \left\{ 1 + \frac{h^2 a^4(\eta_e) H^4}{36\pi^2 H^2 k^4} \left[ \log\left(\frac{H}{2k}\right) + \frac{5}{4} - \gamma_E \right] \right. \\ &\quad \left. + \mathcal{O}(\epsilon(\eta_e), k\eta_e) \right\}. \end{aligned} \quad (2.40)$$

We note that our result satisfies Weinberg's theorem, since the  $\Delta_{\mathcal{R}\mathcal{R}}$  correlator in (2.38) receives only logarithmic corrections in time multiplying a constant

$$\propto h^2 H^{-2} = g^2 \bar{\phi}^2 H^{-2} \sim g^2 M_P^2 m^{-2} \lesssim 10^{-12}. \quad (2.41)$$

The one-loop corrections to  $\epsilon$  are also small as argued below (2.20). Although corrections to  $\Delta_{\mathcal{R}\mathcal{R}}$  are negligible, the corrections to  $\Delta_{\mathcal{R}\pi_{\mathcal{R}}}$  and  $\Delta_{\pi_{\mathcal{R}}\pi_{\mathcal{R}}}$ , which are induced by dissipative effects, can become very large since they multiply powers of the scale factor.

In order to study the physical implications at the end of inflation on super-Hubble scales, we will rescale  $\pi_{\mathcal{R}}$  by its linear relation to the gauge invariant gravitational potential (2.172),

$$\Psi = -\frac{\mathcal{H}}{2M_p^2 k^2 a^2} \pi_{\mathcal{R}}. \quad (2.42)$$

We quantify possibly large corrections of the  $\pi_{\mathcal{R}}\pi_{\mathcal{R}}$ -correlator to the tree-level result  $\bar{\Delta}_{\pi_{\mathcal{R}}\pi_{\mathcal{R}}}$  by the ratio

$$\begin{aligned}\Delta_{\text{infl}} &\equiv \frac{H}{2M_p^2 k^2 a(\eta_e)} \left| \frac{\Delta_{\pi_{\mathcal{R}}\pi_{\mathcal{R}}} - \bar{\Delta}_{\pi_{\mathcal{R}}\pi_{\mathcal{R}}}}{\Delta_{\mathcal{R}\mathcal{R}}} \right|^{1/2} \\ &\approx \frac{\epsilon(\eta_e) h}{6\pi H} \frac{a^2(\eta_e) H^2}{k^2} \left| \log \left( \frac{H}{k} \right) \right|^{1/2} \\ &\lesssim 10^{-12} \frac{\epsilon(\eta_e)}{6\pi} \frac{a^2(\eta_e) H^2}{k^2} \left| \log \left( \frac{H}{k} \right) \right|^{1/2},\end{aligned}\quad (2.43)$$

where we kept only the dominant logarithmic contribution and substituted the estimate for the coupling constant  $h = g\bar{\phi}$  from (2.19). We note that the quantity  $\Delta_{\text{infl}}$  in (2.43) is of order one after the mode  $k$  spends

$$N_{\text{dec}} \approx \frac{1}{2} \log \left[ \frac{6\pi H}{\epsilon(\eta_e) h |\log(H/k)|^{1/2}} \right] \gtrsim 20 \quad (2.44)$$

e-folds on super-Hubble scales. This marks the time at which quantum corrections dominate the tree-level result for the  $\pi_{\mathcal{R}}\pi_{\mathcal{R}}$ -correlator and the decoherence sets in. In fact, the time scale (2.44) is a couple of e-folds longer than the decoherence time associated with the growth of entropy, which is controlled by the time at which the momentum-momentum correlator (2.40) becomes loop dominated,  $N_{\text{entropy}} \simeq \frac{1}{2} \log \left[ \frac{6\pi H}{h |\log(H/k)|^{1/2}} \right] = N_{\text{dec}} - \frac{1}{2} \log(1/\epsilon(\eta_e))$ . Furthermore, the decoherence time-scale  $N_{\text{dec}}$  in (2.44) differs essentially from the breakdown-time of standard perturbation theory which is governed by the perturbativity time associated with the  $\mathcal{R}\mathcal{R}$ -correlator (2.38),<sup>2</sup>

$$N_{\text{pert}} \approx \left[ \frac{108\pi^2 H^2}{h^2} \right]^{\frac{1}{3}} \gtrsim 10^9, \quad (2.45)$$

which is a much larger time scale because the correlators entering loop calculations grow only logarithmically with the scale factor. We can also quantify possibly large corrections of the  $\mathcal{R}\pi_{\mathcal{R}}$ -correlator to the tree-level result by the ratio

$$\begin{aligned}\theta_{\text{infl}} &\equiv \frac{H}{2M_p^2 k^2 a(\eta_e)} \left| \frac{\Delta_{\mathcal{R}\pi_{\mathcal{R}}} - \bar{\Delta}_{\mathcal{R}\pi_{\mathcal{R}}}}{\Delta_{\mathcal{R}\mathcal{R}}} \right| \\ &\approx \epsilon(\eta_e) \frac{a^2(\eta_e) h^2}{72\pi^2 k^2} \log^2(-2k\eta_e) \\ &\lesssim 10^{-24} \epsilon(\eta_e) \frac{a^2(\eta_e) H^2}{72\pi^2 k^2} \left| \log^2 \left( \frac{Ha(\eta_e)}{k} \right) \right|.\end{aligned}\quad (2.46)$$

From (2.43) and (2.46), we see an enhancement of the  $\hat{\pi}_{\mathcal{R}}$  operator by the

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<sup>2</sup>The standard estimate for the perturbativity time is larger,  $N_{\text{pert}} \sim 10^{13}$  e-folds, and it is based on the assumption that there are only two powers of the logarithms in the  $\mathcal{R}\mathcal{R}$ -correlator (2.38). However, the detailed calculation performed in this work shows that there are in fact three powers of the logarithm, thus shortening significantly  $N_{\text{pert}}$ .

factor  $a^2(\eta_e)H^2/k^2$  at the end of inflation. The source of this amplification, however, lies in the vacuum quantum uncertainty of the spectator field  $\chi$  which is coupled to the inflaton via the interaction term  $\bar{\phi}\phi\chi^2$ . Since the quantum fluctuations of the spectator is independent of the inflaton quantum fluctuations they will lead to an independent, amplified late-time stochastic source. We can make the latter statement quantitative by invoking the Gaussian entropy of the corrected two-point functions. Since we used linear relations as a first approximation, the Gaussian invariant associated with the comoving curvature perturbation is identical to the Gaussian invariant associated with the inflaton perturbation,

$$\begin{aligned} \frac{\Delta_{\mathcal{R}}^2(\eta, k)}{4} &= \Delta_{\mathcal{R}\mathcal{R}}(\eta, k)\Delta_{\pi_{\mathcal{R}}\pi_{\mathcal{R}}}(\eta, k) - \Delta_{\mathcal{R}\pi_{\mathcal{R}}}^2(\eta, k) \\ &= a^4 \left[ F_{\varphi}(\eta, \eta, k) \partial_{\eta} \partial_{\eta'} F_{\varphi}(\eta, \eta', k) \Big|_{\eta=\eta'} - \frac{1}{4} (\partial_{\eta} F_{\varphi}(\eta, \eta, k))^2 \right] \\ &= \frac{\Delta_{\varphi}^2(\eta, k)}{4}. \end{aligned} \quad (2.47)$$

The Gaussian invariant  $\Delta_{\varphi}^2$  of the inflaton perturbation  $\varphi$  and hence of the comoving curvature perturbation  $\Delta_{\mathcal{R}}^2$  is given by

$$\begin{aligned} \frac{\Delta_{\varphi}^2(\eta, k)}{4a^4} &= \frac{\Delta_{\mathcal{R}}^2(\eta, k)}{4a^4} \\ &= F_{\varphi}(\eta, \eta', k) \partial_{\eta} \partial_{\eta'} F_{\varphi}(\eta, \eta', k) - \left[ \partial_{\eta'} F_{\varphi}(\eta, \eta', k) \right]^2 \Big|_{\eta'=\eta}, \end{aligned} \quad (2.48)$$

which can be used to calculate the Gaussian part of the von Neumann entropy

$$S_{\text{vN}}[\mathcal{R}] = \frac{\Delta_{\mathcal{R}} + 1}{2} \log \frac{\Delta_{\mathcal{R}} + 1}{2} - \frac{\Delta_{\mathcal{R}} - 1}{2} \log \frac{\Delta_{\mathcal{R}} - 1}{2} = S_{\text{vN}}[\varphi]. \quad (2.49)$$

The last equality follows from the fact that  $\mathcal{R}$  and  $\varphi$  are related by a (time dependent) rescaling, and since the von Neumann entropy is expressed in terms of the Gaussian invariant of the state  $\Delta_{\varphi}^2$ , it cannot depend on a linear field redefinition. This is one way to understand why local mass corrections changing the vev of the inflaton via (2.20) do not contribute to the entropy.

The mode functions of the non-interacting theory in the Bunch-Davies vacuum yield a Gaussian invariant that is identical to one and hence result zero von Neumann entropy. The same reasoning holds for the spectator field  $\chi$  which we also prepare in the Bunch-Davies vacuum. Thus, the Bunch-Davies vacuum for the fields  $\varphi$  and  $\chi$  represents a state with minimal uncertainty which is solely due to the quantum nature of the theory. However, once interactions are taken into account the Gaussian invariant and hence the entropy get perturbatively

corrected

$$\begin{aligned}
\delta\left[\frac{\Delta_\varphi^2}{4a^4}\right] &= \delta\left[F_\varphi(\eta, \eta)\partial_\eta\partial_{\eta'}F_\varphi(\eta, \eta') - [\partial_{\eta'}F_\varphi(\eta, \eta')]^2\right]\Bigg|_{\eta'=\eta} \\
&= \left[F_{\varphi,dS}(\eta, \eta)\partial_\eta\partial_{\eta'}\delta F_\varphi(\eta, \eta') + \delta F_\varphi(\eta, \eta)\partial_\eta\partial_{\eta'}F_{\varphi,dS}(\eta, \eta') \right. \\
&\quad \left. - 2[\partial_{\eta'}F_{\varphi,dS}(\eta, \eta')]\partial_{\eta'}\delta F_\varphi(\eta, \eta')\right]\Bigg|_{\eta'=\eta} \\
&= \frac{H^2}{2k}\left[(1 + k^2\eta^2)\partial_{k\eta}\partial_{k\eta'}\delta F_\varphi(\eta, \eta') \right. \\
&\quad \left. + \delta F_\varphi(\eta, \eta)k^2\eta^2 - 2\eta\partial_{\eta'}\delta F_\varphi(\eta, \eta')\right]\Bigg|_{\eta'=\eta}. \tag{2.50}
\end{aligned}$$

The correction to the Gaussian invariant of the inflaton perturbation is to leading order in the super-Hubble limit given by

$$\delta\left[\frac{\Delta_\varphi^2}{4}\right] = \frac{1}{9\pi^2}\frac{h^2}{H^2}\left(\frac{Ha}{2k}\right)^6\left[4\log\left(\frac{H}{2k}\right) + 5 - 4\gamma_E + \mathcal{O}(k\eta)\right]. \tag{2.51}$$

This expression is greater than zero for  $H > 2k$ , which is amply satisfied for the scales we will be interested in. We conclude that cubic interactions in inflation of the type  $g\bar{\phi}\varphi\chi^2$  lead to a growth of the Gaussian invariant  $\Delta_\varphi^2$  by a factor of  $a^6$  on super-Hubble scales and correspondingly to a growth of the Gaussian entropy. This growth is to leading order due to the quantum loop corrected  $\pi_{\mathcal{R}}\pi_{\mathcal{R}}$ -correlator which grows much faster than the correction to the  $\mathcal{R}\mathcal{R}$ -correlator. This leads to two conclusions. First, the  $\pi_{\mathcal{R}}\pi_{\mathcal{R}}$ -correlator in (2.40) which was calculated with dissipative corrections is linearly gauge invariant. This follows from the fact that entropy production results only from dissipative effects [76] and the statement that the entropy (or the associated Gaussian invariant (2.48)) are to linear order gauge invariant. The second conclusion is that we can view the quantum loop corrected operators  $\hat{\mathcal{R}}$  and  $\hat{\pi}_{\mathcal{R}}$  as stochastically independent at the end of inflation, in contrast to the tree-level result (2.192).

Let us visualize this statement by three snapshots of a phase-space diagram associated to  $\mathcal{R}(\vec{k})$  and  $\pi_{\mathcal{R}}(\vec{k})$  for a given mode  $\vec{k}$ . The first snapshot in figure 2.2a is taken while the mode is deep in the sub-Hubble regime where it is governed by tree-level dynamics due to the smallness of the coupling constant. The state is then approximately in its adiabatic, Gaussian vacuum, indicated by the circle on the phase space diagram, representing the set of points of equal probability amplitude. In an intermediate step in snapshot in figure 2.2b, the mode becomes super-Hubble but the enhancement due to the factor of  $k^{-2}a^2H^2$  in (2.43) is still too small to compensate the small coupling  $hH^{-1}$ . This phase is thus still dominated by the linear analysis and results in the usual squeezed state [78]. The final snapshot in figure 2.2c represents the end of inflation, more precisely, it is representative for all modes that have evolved for  $\gtrsim 20$  e-folds



on super-Hubble scales, *cf.* the estimate (2.44). For these modes, the enhancement of the  $\pi_{\mathcal{R}}\pi_{\mathcal{R}}$ -correlator due to the factor  $k^{-2}a^2H^2$  in (2.43) is now big enough to overcome the suppression of the small coupling  $hH^{-1}$ . The state is still squeezed, but now mostly in the momentum direction.

A tempting question to ask is how the enhanced  $\pi_{\mathcal{R}}$ -operator at the end of inflation affects the effective temperature perturbation. In order to answer this question we still have to map these correlators to a time deep in the radiation era  $\eta_{\text{cmb}} \approx 10^{-1}\eta_{\text{rec}}$ , some time before recombination at  $\eta = \eta_{\text{rec}}$ . As a first attempt, we pick the simplest possible scenario and assume that the comoving curvature perturbation  $\mathcal{R}$  and the gauge invariant gravitational potential  $\Psi$  will not be further affected on super-Hubble scales during the transition to radiation such that we can make use of standard linear relations. We review this process in Appendix 2.B. The effective photon temperature perturbation relevant for the CMB at  $\eta_{\text{cmb}}$  (which is a conformal time early enough from the decoupling time such that the linear collisionless evolution still applies) may then be expressed according to (2.176) in terms the comoving curvature perturbation just before the end of inflation at  $\eta_e$  as follows,

$$\begin{aligned} \Delta\hat{T}(\eta, \vec{k}) &\approx \frac{1}{2} \left[ \frac{2}{3} \hat{\mathcal{R}}(\eta_e, \vec{k}) - \frac{a^3(\eta_e)}{a^3(\eta_{\text{cmb}})} \frac{H}{2M_p^2 k^2 a(\eta_e)} \hat{\pi}_{\mathcal{R}}(\eta_e, \vec{k}) \right] \cos[kr_s(\eta)] \\ &+ \frac{6H}{kc_s(\eta_{\text{cmb}})} \frac{a^4(\eta_e)}{a^4(\eta_{\text{cmb}})} \left[ \frac{H}{2M_p^2 k^2} \hat{\pi}_{\mathcal{R}}(\eta_e, \vec{k}) + a(\eta_e) \hat{\mathcal{R}}(\eta_e, \vec{k}) \right] \sin[kr_s(\eta)]. \end{aligned} \quad (2.52)$$

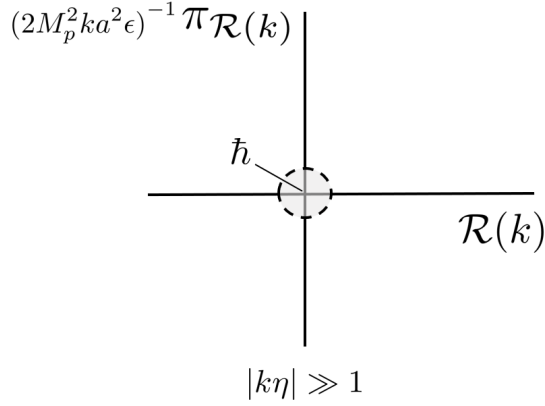
We already know that the tree-level contribution to the sine term in equation (2.52) is insignificant in this scenario. Let us thus define here another quantity that allows us to measure the relative amplitude of orthogonal oscillations in (2.52) if we assume the quantum contributions to the  $\pi_{\mathcal{R}}$  operator to be dominant,

$$\begin{aligned} \frac{\Delta_{\text{sin}}}{\Delta_{\text{cos}}} &\equiv \frac{a^4(\eta_e)}{a^4(\eta_{\text{cmb}})} \frac{18Ha(\eta_e)}{kc_s(\eta_{\text{cmb}})} \Delta_{\text{infl}} \\ &\sim \frac{h}{H} \frac{a^4(\eta_e)}{a^4(\eta_{\text{cmb}})} \frac{3\epsilon(\eta_e)}{\pi} \frac{a^3(\eta_e)H^3}{k^3c_s(\eta_{\text{cmb}})} \left| \log\left(\frac{H}{k}\right) \right|. \end{aligned} \quad (2.53)$$

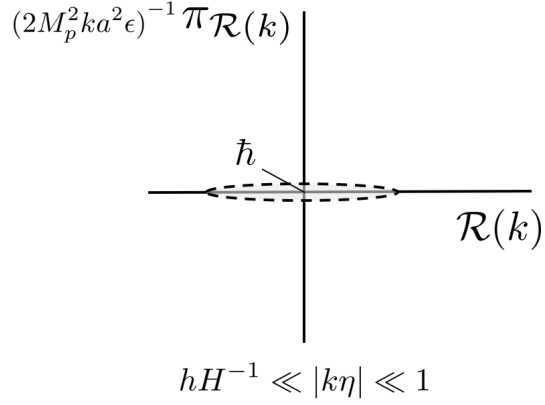
Putting in the estimate for our coupling constant  $h$  from (2.19) we get

$$\frac{\Delta_{\text{sin}}}{\Delta_{\text{cos}}} \lesssim 10^{-12} \frac{a(\eta_e)}{a(\eta_{\text{cmb}})} \frac{3\epsilon(\eta_e)}{\pi} \frac{\mathcal{H}^3(\eta_{\text{cmb}})}{k^3c_s(\eta_{\text{cmb}})} \left| \log\left(\frac{H}{k}\right) \right| \ll 1. \quad (2.54)$$

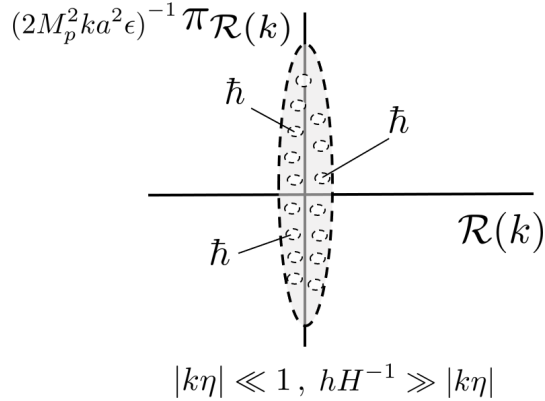
It is thus not sufficient to have quantum loop enhancements of the  $\pi_{\mathcal{R}}\pi_{\mathcal{R}}$ -correlator only during inflation since the linear evolution throughout radiation suppresses it such that at the times of CMB it again becomes small. It is a natural question to ask whether quantum corrections during radiation will hinder this decay in a way that is similar to quantum corrected processes that take place during inflation and we leave this for future studies.



(A) Phase diagram for mode  $k$  early in the sub-Hubble regime. The rescaling for the momentum  $\pi_{\mathcal{R}}$  follows from initial conditions of the linear evolution (2.190) at early times.



(B) Phase diagram for mode  $k$  at intermediate times such that  $k$  is super-Hubble but quantum loop corrections are still negligible. The semi-minor is enlarged to be visible and is substantially smaller than the one in figure 2.2c.



(C) Phase diagram for mode  $k$  which is super-Hubble at late times where quantum loop corrections balance the suppression from the small coupling constant. Note that the axes in this figure are compressed, which was necessary as the surface area of this state is very large when measured in units of  $\hbar$ .

FIGURE 2.2: Phase-diagram for the curvature perturbation  $\mathcal{R}(k)$  and its associated momentum  $\pi_{\mathcal{R}}(k)$  per mode  $\vec{k}$ .

## 2.3 Kadanoff-Baym equation for the statistical propagator

### 2.3.1 Effective action

In this section, we lay out in some detail how we calculate the quantum loop correction to the statistical propagator of the inflaton perturbation that we present in (2.32). We will perform this calculation in the Schwinger-Keldysh formalism for which the first step is to write down the 2-particle-irreducible (2PI) effective action [79]. We will work with an accuracy of a two-loop effective action, where dissipative effects can occur. The two-particle irreducible (2PI) effective action corresponding to the tree-level action (2.27) can be written in the two-loop approximation as,

$$\Gamma[i\Delta_\varphi^{cd}, i\Delta_\chi^{cd}] = \Gamma_0[i\Delta_\varphi^{cd}, i\Delta_\chi^{cd}] + \Gamma_1[i\Delta_\varphi^{cd}, i\Delta_\chi^{cd}] + \Gamma_2[i\Delta_\varphi^{cd}, i\Delta_\chi^{cd}], \quad (2.55)$$

where

$$c, d = \pm, \quad (2.56)$$

and the three constituent functional are given by

$$\begin{aligned} \Gamma_0[i\Delta_\varphi^{cd}, i\Delta_\chi^{cd}] &= \frac{1}{2} \int d^D x d^D x' \sqrt{-\bar{g}_{dS}(x)} \left( \sum_{c,d=\pm} \bar{\square}_x^{dS} \delta^D(x-x') c \delta^{cd} i\Delta_\varphi^{dc}(x', x) \right. \\ &\quad \left. + \sum_{c,d=\pm} \bar{\square}_x^{dS} \delta^D(x-x') c \delta^{cd} i\Delta_\chi^{dc}(x', x) \right), \end{aligned} \quad (2.57)$$

$$\Gamma_1[i\Delta_\varphi^{cd}, i\Delta_\chi^{cd}] = -\frac{i}{2} \text{Tr}[\log(i\Delta_\varphi^{cd}(x; x'))] - \frac{i}{2} \text{Tr}[\log(i\Delta_\chi^{cd}(x; x'))], \quad (2.58)$$

$$\begin{aligned} \Gamma_2[i\Delta_\varphi^{cd}, i\Delta_\chi^{cd}] &= \int d^D x d^D x' \sqrt{-\bar{g}_{dS}(x)} \sqrt{-\bar{g}_{dS}(x')} \\ &\quad \times \sum_{c,d=\pm} cd \frac{i\hbar^2}{4} (i\Delta_\chi^{cd}(x, x'))^2 i\Delta_\varphi^{cd}(x, x'). \end{aligned} \quad (2.59)$$

The elements of the Keldysh propagators  $i\Delta_{\varphi,\chi}^{cd}$  may be identified in terms of the statistical and spectral two-point functions,

$$i\Delta_{\varphi,\chi}^{\mp\pm}(x, x') = F_{\varphi,\chi}(x, x') \pm \frac{1}{2} i\Delta_{\varphi,\chi}^c(x, x'), \quad (2.60)$$

$$i\Delta_{\varphi,\chi}^{\pm\pm}(x, x') = F_{\varphi,\chi}(x, x') \pm \frac{1}{2} \text{sign}[x^0 - (x^0)'] i\Delta_{\varphi,\chi}^c(x, x'). \quad (2.61)$$

Applying the variational principle yields the following equations of motion

$$\begin{aligned} \bar{\square}_x^{dS} i\Delta_\varphi^{ab}(x; x'') &= \frac{a \delta^{ab} i\delta^D(x-x'')}{\sqrt{-\bar{g}_{dS}(x)}} \\ &\quad + \int d^D x' \sqrt{-\bar{g}_{dS}(x')} \sum_{c=\pm} c iM_\varphi^{ac}(x, x') i\Delta_\varphi^{cb}(x', x''), \end{aligned} \quad (2.62)$$

$$\begin{aligned} \bar{\square}_{x''}^{dS} i\Delta_{\varphi}^{ab}(x; x'') &= \frac{a \delta^{ab} i\delta^D(x - x'')}{\sqrt{-\bar{g}_{dS}(x)}} \\ &+ \int d^D x' \sqrt{-\bar{g}_{dS}(x')} \sum_{c=\pm} c i\Delta_{\varphi}^{ac}(x, x') iM_{\varphi}^{cb}(x', x''), \end{aligned} \quad (2.63)$$

where the corresponding self-masses  $iM_{\varphi}^{ab}(x, x')$  read

$$iM_{\varphi}^{ab}(x, x') = -\frac{ih^2}{2} (i\Delta_{\chi}^{cd}(x, x'))^2. \quad (2.64)$$

### 2.3.2 Renormalizing the self-mass

We attempt to solve equation (2.62) by using the expression for the free propagators in the Bunch-Davies vacuum,

$$iM_{\varphi}^{ab}(x, x') = -\frac{ih^2}{2} (i\Delta_{\chi}^{ab}(x, x'))^2 \approx -\frac{ih^2}{2} (i\Delta_{dS}^{ab}(x, x'))^2. \quad (2.65)$$

The self-masses (2.65) are products of distributions that have local contributions  $\propto \delta^D(x, x')$  which would yield indefinite answers when integrated against a test function. The singularities can be isolated by differential, dimensional regularization in position space where they takes the form,  $\propto (D-4)^{-1} \delta^D(x, x')$  (and/or derivatives thereof). We renormalize the self-mass (2.65) by adding suitable local counterterms to the effective action which can be used to subtract these divergent contributions, yielding eventually finite answers in the limit  $D \rightarrow 4$ .

Let us first write down the de Sitter Feynman propagator in position space in  $D$  space-time dimensions which has been computed in terms of the quantity

$$y \equiv y_{++}, \quad (2.66)$$

where in de Sitter invariant length functions

$$y_{ab} = aa' H^2 \Delta x_{ab}^2 = a(\eta) a(\eta') H^2 \Delta x_{ab}^2(\eta - \eta', \vec{x} - \vec{x}') = \frac{\Delta x_{ab}^2(\eta - \eta', \vec{x} - \vec{x}')}{\eta \eta'}, \quad (2.67)$$

can be expressed with the Lorentz invariant length functions

$$\Delta x_{\pm\pm}^2 = -(|\eta - \eta'| \mp i\varepsilon)^2 + \|\vec{x} - \vec{x}'\|^2, \quad (2.68)$$

$$\Delta x_{\pm\mp}^2 = -(\eta - \eta' \pm i\varepsilon)^2 + \|\vec{x} - \vec{x}'\|^2. \quad (2.69)$$

The de Sitter propagator in position space has been given by [80],

$$i\Delta_{dS}^{++} = \frac{H^{D-2}}{(4\pi)^{D/2}} \left[ - \sum_{n=0}^{\infty} \frac{1}{n - \frac{D}{2} + 1} \frac{\Gamma[n + \frac{D}{2}]}{\Gamma[n + 1]} \left(\frac{y}{4}\right)^{n - \frac{D}{2} + 1} \right. \\ \left. - \frac{\Gamma[D-1]}{\Gamma[\frac{D}{2}]} \pi \cot\left[\pi \frac{D}{2}\right] + \sum_{n=1}^{\infty} \frac{1}{n} \frac{\Gamma[n + D - 1]}{\Gamma[n + \frac{D}{2}]} \left(\frac{y}{4}\right)^n \right. \\ \left. + \frac{\Gamma[D-1]}{\Gamma[\frac{D}{2}]} \log[aa'] \right], \quad (2.70)$$

where we use in this section the notation  $a' = a(\eta')$  and it should be clear from the context whether a prime denotes a time derivative or refers to a coordinate. We can expand expression (2.70) around  $D = 4$  and get

$$i\Delta_{dS}^{++} = \frac{H^{D-2}}{(4\pi)^{D/2}} \left[ \Gamma\left[\frac{D-2}{2}\right] \left(\frac{y}{4}\right)^{1 - \frac{D}{2}} - 2 \log\left[\frac{\sqrt{e}y}{4aa'}\right] \right] + \mathcal{O}(D-4). \quad (2.71)$$

Taking the square leads to

$$\left(i\Delta_{dS}^{++}\right)^2 = \frac{H^{2D-4}}{(4\pi)^D} \left[ \Gamma^2\left[\frac{D-2}{2}\right] \left(\frac{y}{4}\right)^{2-D} - \frac{16}{y} \log\left[\frac{\sqrt{e}y}{4aa'}\right] \right. \\ \left. + 4 \log^2\left[\frac{\sqrt{e}y}{4aa'}\right] \right] + \mathcal{O}(D-4), \quad (2.72)$$

and we note that the non-integrable piece of the self-mass is contained in the first term  $\propto y^{2-D}$ . Let us simplify the notation and denote the de Sitter d'Alembert operator as<sup>3</sup>

$$\frac{\square}{H^2} \equiv \frac{\overline{\square}^{dS}}{H^2} = \eta^2 \left[ - \frac{\partial^2}{\partial \eta^2} + \frac{D-2}{\eta} \frac{\partial}{\partial \eta} + \delta^{ij} \frac{\partial^2}{\partial x^i \partial x^j} \right]. \quad (2.74)$$

We will make use of two relations that were established in [81],

$$\left(\frac{y_{\pm\pm}}{4}\right)^{2-D} = \left[ \frac{2}{(D-3)(D-4)} \frac{\square}{H^2} - \frac{D(D-2)}{2(D-3)(D-4)} + \frac{D-6}{2(D-3)} \right] \left(\frac{y_{\pm\pm}}{4}\right)^{3-D} \\ - \left[ \frac{2}{(D-3)(D-4)} \frac{\square}{H^2} - \frac{D(D-2)}{2(D-3)(D-4)} \right] \left(\frac{y_{\pm\pm}}{4}\right)^{1-(D/2)} \\ \pm \frac{2(4\pi)^{D/2}}{(D-3)(D-4)\Gamma[\frac{D}{2}-1]} \frac{i\delta^D(x-x')}{(Ha)^D}, \quad (2.75)$$

---

<sup>3</sup>We would like to remark that due to symmetry reasons we may use in the following derivations also derivatives acting on primed coordinates

$$\frac{\square'}{H^2} = (\eta')^2 \left[ - \frac{\partial^2}{\partial (\eta')^2} + \frac{D-2}{\eta'} \frac{\partial}{\partial \eta'} + \delta^{ij} \frac{\partial^2}{\partial (x^i)' \partial (x^j)'} \right]. \quad (2.73)$$

as well as

$$\frac{\square}{H^2} \left( \frac{y_{\pm\pm}}{4} \right)^{1-(D/2)} = \pm \frac{(4\pi)^{D/2}}{\Gamma[\frac{D}{2}-1]} \frac{i\delta^D(x-x')}{(Ha)^D} + \frac{D(D-2)}{4} \left( \frac{y_{\pm\pm}}{4} \right)^{1-(D/2)}. \quad (2.76)$$

Let us introduce the renormalization parameter  $\mu$  with energy dimension one. We can rewrite (2.75) by adding a  $\mu$ -dependent term that vanishes on  $D = 4$  in such a way that the divergence in the self-mass may be removed with a mass counterterm in the action  $\propto (D-4)^{-1} \mu^{D-4} a^{-D} \delta^D(x-x')$ . Moreover, we use

$$\left( \frac{y_{\pm\pm}}{4} \right)^{3-D} = \left( \frac{y_{\pm\pm}}{4} \right)^{1-(D/2)} \left[ 1 - \frac{D-4}{2} \log[y_{\pm\pm}] + \mathcal{O}[(D-4)^2] \right], \quad (2.77)$$

and expand the non-singular terms in (2.75),

$$\begin{aligned} \left( \frac{y_{\pm\pm}}{4} \right)^{2-D} &= \pm \frac{2(4\pi)^{D/2}}{(D-3)(D-4)\Gamma[\frac{D}{2}-1]} \left( \frac{\mu}{H} \right)^{D-4} \frac{i\delta^D(x-x')}{(Ha)^D} \\ &\quad - \frac{\square}{H^2} \left( \frac{4}{y_{\pm\pm}} \log \left[ \frac{\mu^2 y_{\pm\pm}}{H^2} \right] \right) - \frac{4}{y_{\pm\pm}} \left( 2 \log \left[ \frac{\mu^2 y_{\pm\pm}}{H^2} \right] - 1 \right) + \mathcal{O}(D-4), \end{aligned} \quad (2.78)$$

which leads to

$$\begin{aligned} \left( i\Delta_{dS}^{\pm\pm} \right)^2 &= \pm \frac{2\Gamma[\frac{D}{2}-1]\mu^{D-4}}{(4\pi)^{D/2}(D-3)(D-4)} \frac{i\delta^D(x-x')}{a^D} \\ &\quad - \frac{H^{2D-4}}{(4\pi)^D} \left[ \frac{\square}{H^2} \left( \frac{4}{y_{\pm\pm}} \log \left[ \frac{\mu^2 y_{\pm\pm}}{H^2} \right] \right) - \frac{4}{y_{\pm\pm}} \left( 2 \log \left[ \frac{\mu^2 y_{\pm\pm}}{H^2} \right] - 1 \right) \right. \\ &\quad \left. + \frac{16}{y_{\pm\pm}} \log \left[ \frac{\sqrt{e} y}{4aa'} \right] - 4 \log^2 \left[ \frac{\sqrt{e} y_{\pm\pm}}{4aa'} \right] \right] + \mathcal{O}(D-4). \end{aligned} \quad (2.79)$$

The divergent local contribution in the first line of (2.79) yields a divergent contribution to the self-mass (2.65),

$$(iM_\varphi^{cd}(x, x'))_{\text{div}} = h^2 \frac{\Gamma[\frac{D}{2}-1]\mu^{D-4}}{(4\pi)^{D/2}(D-3)(D-4)} \frac{\delta^D(x-x')}{a^D} c\delta^{cd}, \quad (2.80)$$

which can be removed by adding the following counterterm action,<sup>4</sup>

$$S_{\text{ct}} = \int d^D x a^D \left( -\frac{1}{2} \delta m^2 \sum_{c,d=\pm} c\delta^{cd} i\Delta_\varphi^{cd}(x, x) \right), \quad (2.82)$$

---

<sup>4</sup>The corresponding counterterm action in the one-particle irreducible formalism is local in the fields,

$$S_{\text{ct}}^{\text{1PI}} = \int d^D x a^D \left( -\frac{1}{2} \delta m^2 \sum_{c=\pm} c[\varphi^c(x)]^2 \right). \quad (2.81)$$

where  $\delta m^2$  is proportional to the inflaton condensate squared,

$$\delta m^2 = -g^2 \bar{\phi}^2 \frac{\Gamma\left(\frac{D}{2}-1\right) \mu^{D-4}}{(4\pi)^{D/2}(D-3)(D-4)}, \quad (2.83)$$

and diverges as  $\propto 1/(D-4)$ . Clearly, the counterterm (2.82) is the divergent mass counterterm of the 2PI formalism. It is easy to check that varying the action (2.82) and adding it to the equations of motion (2.62–2.63) removes the divergent parts of the self-masses. The resulting renormalized self-mass  $iM_{\phi,\text{ren}}^{++}$  is,

$$\begin{aligned} iM_{\phi,\text{ren}}^{++}(x, x') = \frac{ih^2}{2} \frac{H^4}{(4\pi)^4} & \left[ \frac{\square}{H^2} \left( \frac{4}{y} \log \left[ \frac{\mu^2 y}{H^2} \right] \right) - \frac{4}{y} \left( 2 \log \left[ \frac{\mu^2 y}{H^2} \right] - 1 \right) \right. \\ & \left. + \frac{16}{y} \log \left[ \frac{\sqrt{e} y}{4aa'} \right] - 4 \log^2 \left[ \frac{\sqrt{e} y}{4aa'} \right] \right]. \quad (2.84) \end{aligned}$$

The other renormalized self-masses,  $iM_{\phi,\text{ren}}^{ab}(x, x')$  ( $a, b = \pm$ ), are obtained simply by replacing  $y(x, x') = y_{++}(x, x')$  in (2.84) by  $y_{ab}(x, x')$ .

### 2.3.3 Self-mass in momentum space

Ultimately, we will be interested in the Wigner transform of the spatially dependent piece of the self-mass. This may be conveniently achieved by extracting d'Alembert's operators and dropping homogeneous (momentum independent) contributions. If the d'Alembertian in de Sitter space-time is acting on non-singular functions (not containing  $y^{-1}$ ), we have,

$$\frac{\square}{H^2} f(y) = (4-y)y f''(y) + 4(2-y)f'(y), \quad (2.85)$$

which gives the identities

$$\frac{1}{y} = \frac{1}{4} \frac{\square}{H^2} \log(y) + \frac{3}{4}, \quad (2.86)$$

$$\frac{\log(y)}{y} = \frac{1}{8} \frac{\square}{H^2} \left[ \log^2(y) - 2 \log(y) \right] + \frac{3}{4} \log(y) - \frac{1}{2}. \quad (2.87)$$

These identities allow us to rewrite the self-mass (2.84) as,

$$\begin{aligned}
iM_{\varphi,\text{ren}}^{++}(x, x') = & \frac{ih^2}{2} \frac{H^4}{(4\pi)^4} \left\{ \frac{\square^2}{H^4} \left[ \frac{1}{2} \log^2 \left( \frac{y}{4} \right) + \log \left[ \frac{4\mu^2}{eH^2} \right] \log \left( \frac{y}{4} \right) \right] \right. \\
& + 2 \frac{\square}{H^2} \left[ \frac{1}{2} \log^2 \left( \frac{y}{4} \right) + \log \left[ \frac{eH^2}{4\mu^2} \right] \log \left( \frac{y}{4} \right) \right] \\
& + 2 \left[ 1 - 2 \log(aa') \right] \frac{\square}{H^2} \log \left( \frac{y}{4} \right) \\
& \left. + 2 \left[ 1 + 4 \log(aa') \right] \log \left( \frac{y}{4} \right) - 4 \log^2 \left( \frac{y}{4} \right) \right\} + \text{hom.}, \quad (2.88)
\end{aligned}$$

where hom. encode spatially homogeneous ( $y$  independent) contributions, which are of no importance for this study. At this stage, we would like to emphasize, that the expression for the self-mass (2.88) could have also been written with the de Sitter d'Alembertian operators acting on the primed space-time coordinates. We now perform the spatial Wigner transform of the self-mass (2.88) according to,

$$iM_{\varphi,\text{ren}}^{++}(\eta, \eta', k) = \int d^3(x - x') iM_{\phi,\text{ren}}^{++}(x, x') e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} . \quad (2.89)$$

Furthermore, spatially homogeneous contributions are proportional to delta functions in  $k$ -space or derivatives thereof,

$$\int_0^\infty dr r \sin(kr) = -\pi \partial_k \delta(k) . \quad (2.90)$$

We will drop again such contributions. In Appendix 2.D we establish the following Wigner transformation,

$$\begin{aligned}
& \int d^3(x - x') e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} \left[ \frac{1}{2} \log^2 \left( \frac{y}{4} \right) + f(\eta, \eta') \log \left( \frac{y}{4} \right) \right] \\
& = -\frac{4\pi^2}{k^3} \left[ 2 + [1 + ik|\Delta\eta|] \left( \log \left[ \frac{aa'H^2|\Delta\eta|}{2k} \right] + i\frac{\pi}{2} - \gamma_E + f(\eta, \eta') \right) \right] e^{-ik|\Delta\eta|} \\
& \quad + \frac{4\pi^2}{k^3} (1 - ik|\Delta\eta|) \left[ \text{ci}[2k|\Delta\eta|] - i \text{si}[2k|\Delta\eta|] \right] e^{+ik|\Delta\eta|}, \quad (2.91)
\end{aligned}$$

where  $\Delta\eta = \eta - \eta'$  and  $f(\eta, \eta')$  is some  $k$ -independent function. We make use of the Wigner transform (2.91), rewrite the scale factor as  $a = -(H\eta)^{-1}$  and



obtain, after some simplifications, the self-mass in momentum space as follows,

$$\begin{aligned}
iM_{\varphi,\text{ren}}^{++}(\eta, \eta', k) = & -\frac{4\pi^2}{k^3} \frac{ih^2}{2} \frac{H^4}{(4\pi)^4} \left\{ \right. \\
& \frac{\square_k^2}{H^4} \left( \left[ 2 + [1 + ik|\Delta\eta|] \left( \log \left[ \frac{2|\Delta\eta|\mu^2}{k\eta\eta'H^2} \right] + i\frac{\pi}{2} - \gamma_E - 1 \right) \right] e^{-ik|\Delta\eta|} \right. \\
& \quad \left. - (1 - ik|\Delta\eta|) \left[ \text{ci}[2k|\Delta\eta|] - i \text{si}[2k|\Delta\eta|] \right] e^{+ik|\Delta\eta|} \right) \\
& + 2 \frac{\square_k}{H^2} \left( \left[ 2 + [1 + ik|\Delta\eta|] \left( \log \left[ \frac{|\Delta\eta|H^2}{8k\eta\eta'\mu^2} \right] + i\frac{\pi}{2} - \gamma_E + 1 \right) \right] e^{-ik|\Delta\eta|} \right. \\
& \quad \left. - (1 - ik|\Delta\eta|) \left[ \text{ci}[2k|\Delta\eta|] - i \text{si}[2k|\Delta\eta|] \right] e^{+ik|\Delta\eta|} \right) \\
& + 2[1 + 2\log(H^2\eta\eta')] \frac{\square_k}{H^2} \left[ [1 + ik|\Delta\eta|] e^{-ik|\Delta\eta|} \right] \\
& - 8 \left( \left[ 2 + [1 + ik|\Delta\eta|] \left( \log \left[ \frac{H^2|\Delta\eta|}{2k} \right] + i\frac{\pi}{2} - \gamma_E - \frac{1}{4} \right) \right] e^{-ik|\Delta\eta|} \right. \\
& \quad \left. - (1 - ik|\Delta\eta|) \left[ \text{ci}[2k|\Delta\eta|] - i \text{si}[2k|\Delta\eta|] \right] e^{+ik|\Delta\eta|} \right) \left. \right\} + \text{hom.}, \quad (2.92)
\end{aligned}$$

where

$$\frac{\square_k}{H^2} = -\eta^2 \left( \partial_\eta^2 - \frac{2}{\eta} \partial_\eta + k^2 \right), \quad (2.93)$$

is the d'Alembertian in momentum space. For a computational convenience we shall split the self-mass (2.92) in the following way,

$$M_{\phi,\text{ren}}^{++}(\eta, \eta', k) = -2[1 + 2\log(H^2\eta\eta')] \frac{\square_k}{H^2} \widehat{M}^{++}(|\Delta\eta|, k) \quad (2.94)$$

$$+ \sum_{n=0}^2 \left( \frac{\square_k}{H^2} \right)^n M_{(n)}^{++}(\eta, \eta', k), \quad (2.95)$$

which is based on the definitions,

$$\begin{aligned}
M_{(n)}^{++}(\eta, \eta', k) \equiv & \alpha_{(n)} \left[ \widetilde{M}_I^{++}(|\Delta\eta|, k) + \widetilde{M}_{II}^{++}(|\Delta\eta|, k) \right] \\
& + \beta_{(n)} \widehat{M}^{++}(|\Delta\eta|, k) + \gamma_{(n)} \widehat{M}^{++}(|\Delta\eta|, k) \log \left[ \frac{\eta\eta'H^4}{4\mu^2} \right], \quad (2.96)
\end{aligned}$$

where

$$\alpha_{(n)} = \{-8, 2, 1\}, \beta_{(n)} = \left\{ -2, -2 + 8 \log \left[ \frac{2\mu}{H} \right], 1 \right\}, \gamma_{(n)} = \{0, 2, 1\}, \quad (2.97)$$

and

$$\widehat{M}^{++} \equiv \frac{4\pi^2 h^2}{k^3} \frac{H^4}{2 (4\pi)^4} [1 + ik|\Delta\eta|] e^{-ik|\Delta\eta|}, \quad (2.98)$$

$$\begin{aligned} \widetilde{M}_I^{++} \equiv & -\frac{4\pi^2 h^2}{k^3} \frac{H^4}{2 (4\pi)^4} \left[ 2 \right. \\ & \left. + [1 + ik|\Delta\eta|] \left( \log \left[ \frac{H^2 |\Delta\eta|}{2k} \right] + i\frac{\pi}{2} - \gamma_E \right) \right] e^{-ik|\Delta\eta|}, \end{aligned} \quad (2.99)$$

$$\widetilde{M}_{II}^{++} \equiv -\frac{4\pi^2 h^2}{k^3} \frac{H^4}{2 (4\pi)^4} (1 - ik|\Delta\eta|) \text{E}_1[2ik|\Delta\eta|] e^{+ik|\Delta\eta|}. \quad (2.100)$$

Here, we made use of the identity for the exponential integral function,

$$\text{E}_1[2ik|\Delta\eta|] = i \text{si}[2k|\Delta\eta|] - \text{ci}[2k|\Delta\eta|], \quad (2.101)$$

which holds when  $k > 0$ . The sine (si) and cosine (ci) integrals are defined in Eqs. (2.154) and (2.155), respectively. The calculation of the other self-masses  $iM_{\phi,\text{ren}}^{\pm\mp}$  and  $iM_{\phi,\text{ren}}^{--}$  proceeds similarly. By writing

$$\log(\Delta x_{--}^2) = \log\left(|\Delta\eta^2 - \|\vec{x} - \vec{x}'\|^2|\right) - i\pi\theta(\Delta\eta^2 - \|\vec{x} - \vec{x}'\|^2), \quad (2.102)$$

we see that

$$M_{\phi,\text{ren}}^{--} = \left[ M_{\phi,\text{ren}}^{++} \right]^*. \quad (2.103)$$

Moreover, due to

$$\begin{aligned} \log(\Delta x_{\mp\pm}^2) = & \log\left(|\Delta\eta^2 - \|\vec{x} - \vec{x}'\|^2|\right) \\ & \pm i \text{sign}(\eta, \eta') \pi\theta(\Delta\eta^2 - \|\vec{x} - \vec{x}'\|^2), \end{aligned} \quad (2.104)$$

we see that,

$$\begin{aligned} M_{\phi,\text{ren}}^{ab}(\eta, \eta', k) = & -4[1 + \log(\eta\eta'H^2)] \frac{\square_k}{H^2} \widehat{M}^{ab}(|\Delta\eta|, k) \\ & + \sum_{n=0}^2 \left( \frac{\square_k}{H^2} \right)^n M_{(n)}^{ab}(\eta, \eta', k), \end{aligned} \quad (2.105)$$

where

$$M_{(n)}^{\mp\pm} = M_{(n)}^{\pm\pm}\theta(\Delta\eta) + M_{(n)}^{\mp\mp}\theta(-\Delta\eta), \quad (2.106)$$

$$\widehat{M}_{(n)}^{\mp\pm} = \widehat{M}_{(n)}^{\pm\pm}\theta(\Delta\eta) + \widehat{M}_{(n)}^{\mp\mp}\theta(-\Delta\eta), \quad (2.107)$$

where  $\text{sign}(\eta, \eta') = \theta(\eta - \eta') - \theta(\eta' - \eta)$  and  $\theta$  is the Heaviside step function. It will be convenient to define

$$M_{(n)}^F \equiv \frac{1}{2} [M_{(n)}^{++} + M_{(n)}^{--}] = \text{Re } M_{(n)}^{++}, \quad (2.108)$$

$$\begin{aligned} M_{(n)}^c(\eta, \eta') &\equiv \frac{\text{sign}(\Delta\eta)}{i} [M_{(n)}^{++} - M_{(n)}^{--}](\eta, \eta') \\ &= 2 \text{sign}(\Delta\eta) \text{Im } M_{(n)}^{++}, \end{aligned} \quad (2.109)$$

$$\widehat{M}^F \equiv \frac{1}{2} [\widehat{M}^{++} + \widehat{M}^{--}] = \text{Re } \widehat{M}^{++}, \quad (2.110)$$

$$\begin{aligned} \widehat{M}^c(\eta, \eta') &\equiv \frac{\text{sign}(\Delta\eta)}{i} [\widehat{M}^{++} - \widehat{M}^{--}](\eta, \eta') \\ &= 2 \text{sign}(\Delta\eta) \text{Im } \widehat{M}^{++}. \end{aligned} \quad (2.111)$$

### 2.3.4 Perturbative solution for the statistical propagator

Let us look at the renormalized version of equations of motion (2.62) for the Keldysh propagators  $i\Delta_\varphi^{ab}$ . By rewriting the two-point functions in terms of real and imaginary parts, we obtain

$$\begin{aligned} \square_x F_\varphi(x, x'') &= \frac{i}{2} \int d\eta' d^3x' (\eta' H)^{-4} [M_{\varphi, \text{ren}}^{++} - M_{\varphi, \text{ren}}^{--} \\ &\quad + M_{\varphi, \text{ren}}^{-+} - M_{\varphi, \text{ren}}^{+-}] (x, x') F_\varphi(x', x'') \\ &\quad - \frac{1}{4} \int d\eta' d^3x' (\eta' H)^{-4} [\text{sign}(\eta' - \eta'') (M_{\varphi, \text{ren}}^{++} + M_{\varphi, \text{ren}}^{--}) \\ &\quad - M_{\varphi, \text{ren}}^{-+} - M_{\varphi, \text{ren}}^{+-}] (x, x') \Delta_\varphi^c(x', x''). \end{aligned} \quad (2.112)$$

We will solve for the statistical propagator perturbatively by approximating  $F_\varphi$  and  $\Delta_\varphi^c$  on the right hand side of (2.112) by the expressions for the Bunch-Davies vacuum (2.30) and (2.31), respectively. Inserting the concrete expressions (2.95) for our model in momentum space, we find

$$\begin{aligned} \square_k F_\varphi(\eta, \eta'', k) &\approx \\ &- \sum_{n=0}^2 \left( \frac{\square_k}{H^2} \right)^n \int_{-\infty}^{\eta} d\eta' (\eta' H)^{-4} M_{(n)}^c(\eta, \eta', k) F_{\varphi, dS}(\eta', \eta'', k) \\ &+ \sum_{n=0}^2 \left( \frac{\square_k}{H^2} \right)^n \int_{-\infty}^{\eta''} d\eta' (\eta' H)^{-4} M_{(n)}^F(\eta, \eta', k) \Delta_{\varphi, dS}^c(\eta', \eta'', k) \\ &+ 2 \int_{-\infty}^{\infty} d\eta' \frac{1 + 2 \log(\eta \eta' H^2)}{(\eta' H)^4} \frac{\square_k}{H^2} [\theta(\eta - \eta') \widehat{M}^c(\eta, \eta', k)] F_{\varphi, dS}(\eta', \eta'', k) \\ &- 2 \int_{-\infty}^{\eta''} d\eta' \frac{1 + 2 \log(\eta \eta' H^2)}{(\eta' H)^4} \frac{\square_k}{H^2} [\widehat{M}^F(\eta, \eta', k)] \Delta_{\varphi, dS}^c(\eta', \eta'', k). \end{aligned} \quad (2.113)$$

Expanding the last two terms and rearranging the integration boundaries gives

$$\begin{aligned}
\Box_k F_\phi(\eta, \eta'', k) &\approx \\
&- 2 \operatorname{Im} \sum_{n=0}^2 \left( \frac{\Box_k}{H^2} \right)^n \int_{-\infty}^{\eta} d\eta' (\eta' H)^{-4} M_{(n)}^{++}(\eta, \eta', k) i \Delta_{\varphi, dS}^{+-}(\eta', \eta'', k) \\
&- \sum_{n=0}^2 \left( \frac{\Box_k}{H^2} \right)^n \int_{\eta''}^{\eta} d\eta' (\eta' H)^{-4} \operatorname{Re} M_{(n)}^{++}(\eta, \eta', k) \Delta_{\varphi, dS}^c(\eta', \eta'', k) \\
&+ 4 \operatorname{Im} \int_{-\infty}^{\eta} d\eta' \frac{1 + 2 \log(\eta \eta' H^2)}{(\eta' H)^4} \left[ \frac{\Box_k}{H^2} \widehat{M}^{++}(\eta, \eta', k) \right] i \Delta_{\varphi, dS}^{+-}(\eta', \eta'', k) \\
&+ 2 \int_{\eta''}^{\eta} d\eta' \frac{1 + 2 \log(\eta \eta' H^2)}{(\eta' H)^4} \left[ \frac{\Box_k}{H^2} \operatorname{Re} \widehat{M}^{++}(\eta, \eta', k) \right] \Delta_{\varphi, dS}^c(\eta', \eta'', k), \quad (2.114)
\end{aligned}$$

We will solve equation (2.114) by using a retarded Green's function  $G_{\text{ret}}(\eta, \eta', k)$  (which yields no contributions of the particular solution to the initial values) for the d'Alembertian operator in momentum space

$$\begin{aligned}
\Box_k G_{\text{ret}}(\eta, \eta', k) &= H^2 \eta^2 \left[ -\partial_\eta^2 + \frac{D-2}{\eta} \partial_\eta - k^2 \right] G_{\text{ret}}(\eta, \eta', k) \\
&= a^{-4}(\eta') \delta(\eta - \eta'), \quad (2.115)
\end{aligned}$$

$$\begin{aligned}
G_{\text{ret}}(\eta, \eta', k) &= \theta(\eta - \eta') \frac{H^2}{k^3} \left[ k(\eta - \eta') \cos[k(\eta - \eta')] \right. \\
&\quad \left. - (1 + k^2 \eta \eta') \sin[k(\eta - \eta')] \right]. \quad (2.116)
\end{aligned}$$

We have

$$\begin{aligned}
F_\varphi(\eta, \eta'', k) &= \frac{\Box_k}{H^2} \mathcal{B}_{(2)}(\eta, \eta'', k) + \mathcal{B}_{(1)}(\eta, \eta'', k) \\
&+ H^2 \int_{-\infty}^{\infty} d\tau \frac{G_{\text{ret}}(\eta, \tau, k)}{(\tau H)^4} \left[ \mathcal{B}_{(0)}(\tau, \eta'', k) + \mathcal{B}_{(0)}^{\log}(\tau, \eta'', k) \right] + F_{\text{hom}}(\eta, \eta'', k), \quad (2.117)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}_{(n)}(\eta, \eta'', k) &\equiv -\frac{2}{H^2} \operatorname{Im} \int_{-\infty}^{\eta} \frac{d\eta'}{(\eta' H)^4} M_{(n)}^{++}(\eta, \eta', k) i \Delta_{\varphi, dS}^{+-}(\eta', \eta'', k) \\
&- \frac{1}{H^2} \int_{\eta''}^{\eta} \frac{d\eta'}{(\eta' H)^4} \operatorname{Re} M_{(n)}^{++}(\eta, \eta', k) \Delta_{\varphi, dS}^c(\eta', \eta'', k), \quad (2.118)
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}_{(0)}^{\log}(\eta, \eta'', k) &\equiv 4 \operatorname{Im} \int_{-\infty}^{\eta} d\eta' \frac{1 + 2 \log(\eta \eta' H^2)}{(\eta' H)^4} \left[ \frac{\Box_\eta(k)}{H^2} \widehat{M}^{++}(\eta, \eta', k) \right] \\
&\quad \times i \Delta_{\varphi, dS}^{+-}(\eta', \eta'', k) \\
&+ 2 \int_{\eta''}^{\eta} d\eta' \frac{1 + 2 \log(\eta \eta' H^2)}{(\eta' H)^4} \left[ \frac{\Box_\eta(k)}{H^2} \operatorname{Re} \widehat{M}^{++}(\eta, \eta', k) \right] \\
&\quad \times \Delta_{\varphi, dS}^c(\eta', \eta'', k), \quad (2.119)
\end{aligned}$$

and

$$\square_k F_{\text{hom}}(\eta, \eta'', k) = 0. \quad (2.120)$$

Let us define

$$\widehat{F}(\eta, \eta'', k) \equiv F_\varphi(\eta, \eta'', k) - F_{\text{hom}}(\eta, \eta'', k). \quad (2.121)$$

The homogeneous solution has to be chosen in such a way that the symmetry properties of the statistical two-point function are satisfied

$$F_{\text{hom}}(\eta, \eta'', k) - F_{\text{hom}}(\eta'', \eta, k) = \widehat{F}(\eta'', \eta, k) - \widehat{F}(\eta, \eta'', k), \quad (2.122)$$

and the full solution reads

$$F_\varphi(\eta, \eta'', k) = \frac{1}{2} [\widehat{F}(\eta, \eta'', k) + \widehat{F}(\eta'', \eta, k)] + \frac{1}{2} [F_{\text{hom}}(\eta, \eta'', k) + F_{\text{hom}}(\eta'', \eta, k)]. \quad (2.123)$$

We immediately get the consistency requirement

$$\square_k \square_k'' [\widehat{F}(\eta, \eta'', k) - \widehat{F}(\eta'', \eta, k)] = 0, \quad (2.124)$$

which can be used as a non-trivial check of the result of the calculation. Let us also fix a common prefactor for the subsequent integrals

$$\lambda \equiv \frac{h^2}{256\pi^2 k^3}, \quad (2.125)$$

which gives the statistical two-point function  $F_\varphi$  correct dimensions in momentum space if all other factors and ratios are dimensionless.

Let us proceed with the calculation of (2.117). The integrals with logarithms  $\mathcal{B}_{(0)}^{\text{log}}$  in (2.119) combine to the following expression

$$\begin{aligned} \mathcal{B}_{(0)}^{\text{log}}(\eta, \eta'', k) = & -8\lambda \left\{ \cos[k(\eta - \eta'')] \left( 2 + \log[H^2 \eta^2] \right) \right. \\ & + \sin[k(\eta - \eta'')] \left( 2k(\eta - \eta'') - k\eta'' \log[H^2 \eta^2] + k\eta \log[H^2 \eta \eta''] \right) \\ & + k\eta \left( \text{ci}[-2k\eta] + \text{ci}[-2k\eta''] \right) \left( k\eta'' \cos[k(\eta + \eta'')] - \sin[k(\eta + \eta'')] \right) \\ & \left. + k\eta \left( \pi + \text{si}[-2k\eta] + \text{si}[-2k\eta''] \right) \left( \cos[k(\eta + \eta'')] + k\eta'' \sin[k(\eta + \eta'')] \right) \right\} \\ & \longrightarrow -8\lambda \left( 2 + \log[H^2 \eta^2] \right), \end{aligned} \quad (2.126)$$

where the arrow denotes the super-Hubble limit. The next step is to tackle

the  $\mathcal{B}_{2,1,0}$  terms in (2.118) for which we note that the integrals containing negative infinity as a boundary may be rewritten as

$$\begin{aligned} \int_{-\infty}^{\eta} \frac{d\tau}{(\tau H)^4} M_{(n)}^{++}(\eta, \tau, k) i\Delta_{\varphi, dS}^{+-}(\tau, \eta'', k) = \\ \frac{1}{2}(1 + ik\eta'') \frac{e^{-ik(\eta'' - \eta)}}{H^2} \int_0^{\infty} dx \left[ \alpha_{(n)} \left[ \widetilde{M}_I^{++}\left(\frac{x}{k}, k\right) + \widetilde{M}_{II}^{++}\left(\frac{x}{k}, k\right) \right] \right. \\ \left. + \beta_{(n)} \widehat{M}^{++}\left(\frac{x}{k}, k\right) \right. \\ \left. + \gamma_{(n)} \widehat{M}^{++}\left(\frac{x}{k}, k\right) \log \left[ \frac{\eta(k\eta - x)H^4}{4k\mu^2} \right] \right] \frac{1 - i(k\eta - x)}{(k\eta - x)^4} e^{-ix}. \end{aligned} \quad (2.127)$$

We then have to solve the following integrals ( $\eta, \eta'' < 0, k > 0$ ),

$$I_{\widetilde{R}}(\eta, \eta'', k) \equiv -\frac{1}{\lambda H^2} \int_{\eta''}^{\eta} d\tau (\tau H)^{-4} \text{Re} [\widetilde{M}_I^{++} + \widetilde{M}_{II}^{++}](\eta, \tau, k) \times \Delta_{\varphi, dS}^c(\tau, \eta'', k), \quad (2.128)$$

$$I_{\widehat{R}}(\eta, \eta'', k) \equiv -\frac{1}{\lambda H^2} \int_{\eta''}^{\eta} d\tau (\tau H)^{-4} \text{Re} \widehat{M}(\eta, \tau, k) \Delta_{\varphi, dS}^c(\tau, \eta'', k), \quad (2.129)$$

$$I_{R_{\log}}(\eta, \eta'', k) \equiv -\frac{1}{\lambda H^2} \int_{\eta''}^{\eta} d\tau (\tau H)^{-4} \text{Re} \widehat{M}(\eta, \tau, k) \log \left[ \frac{\eta \tau H^4}{4\mu^2} \right] \times \Delta_{\varphi, dS}^c(\tau, \eta'', k), \quad (2.130)$$

$$I_{\widetilde{M}}(\eta, k) \equiv -\frac{e^{ik\eta}}{2\lambda H^4} \int_0^{\infty} dx \left[ \widetilde{M}_I^{++}\left(\frac{x}{k}, k\right) + \widetilde{M}_{II}^{++}\left(\frac{x}{k}, k\right) \right] \times \frac{1 - i(k\eta - x)}{(k\eta - x)^4} e^{-ix}, \quad (2.131)$$

$$I_{\widehat{M}}(\eta, k) \equiv -\frac{e^{ik\eta}}{2\lambda H^4} \int_0^{\infty} dx \widehat{M}^{++}\left(\frac{x}{k}, k\right) \frac{1 - i(k\eta - x)}{(k\eta - x)^4} e^{-ix}, \quad (2.132)$$

$$I_{M_{\log}}(\eta, k) \equiv -\frac{e^{ik\eta}}{2\lambda H^4} \int_0^{\infty} dx \log \left[ \frac{\eta(k\eta - x)H^4}{4k\mu^2} \right] \widehat{M}^{++}\left(\frac{x}{k}, k\right) \times \frac{1 - i(k\eta - x)}{(k\eta - x)^4} e^{-ix}. \quad (2.133)$$

We are able to solve all integrals except for the first one in terms of finite sums of exponentials, exponential integrals and generalized hypergeometric functions. However, for the integral  $I_{\widetilde{R}}$  we have to define the function

$$\mathcal{J}(\eta, \eta'', k) \equiv \int_0^1 dx E_1[-2ik(x(\eta - \eta'') + \eta'')] \frac{1 - e^{-2ik(\eta - \eta'')(x-1)}}{x-1}. \quad (2.134)$$

We note that (2.134) approaches a constant in the super-Hubble limit. We solve the  $I_R$  integrals in Appendix 2.E and the  $I_M$  integrals in Appendix 2.F. We then

have

$$\begin{aligned} \mathcal{B}_{(n)}(\eta, \eta'', k) = & \lambda \alpha_{(n)} [2 \operatorname{Im}((1 + ik\eta'')e^{-ik\eta''} I_{\widehat{M}}(\eta, k)) + I_{\widehat{R}}(\eta, \eta'', k)] \\ & + \lambda \beta_{(n)} [2 \operatorname{Im}((1 + ik\eta'')e^{-ik\eta''} I_{\widehat{M}}(\eta, k)) + I_{\widehat{R}}(\eta, \eta'', k)] \\ & + \lambda \gamma_{(n)} [2 \operatorname{Im}((1 + ik\eta'')e^{-ik\eta''} I_{M_{\log}}(\eta, k)) + I_{R_{\log}}(\eta, \eta'', k)], \end{aligned} \quad (2.135)$$

where the coefficients are given in (2.97). If we now act with the de Sitter d'Alembertian on  $\mathcal{B}_{(2)}$  we have

$$\begin{aligned} \frac{\square_k}{H^2} \mathcal{B}_{(2)}(\eta, \eta'', k) = & 2\lambda \left\{ \cos[k(\eta - \eta'')] - k(\eta - \eta'') \sin[k(\eta - \eta'')] \right. \\ & + \left( \cos[k(\eta - \eta'')] + k(\eta - \eta'') \sin[k(\eta - \eta'')] \right) \\ & \quad \times \left( \operatorname{ci}(2k|\eta - \eta''|) + \gamma_E - \log \left[ \frac{2\mu^2 |\eta - \eta''|}{H^2 k \eta \eta''} \right] \right) \\ & + \operatorname{sign}(\eta - \eta'') \left[ \pi k(\eta - \eta'') \cos[k(\eta - \eta'')] - \frac{1}{2} \sin[k(\eta - \eta'')] \right] \\ & \left. - \left( k(\eta - \eta'') \cos[k(\eta - \eta'')] - \sin[k(\eta - \eta'')] \right) \operatorname{si}(2k|\eta - \eta''|) \right\} \\ & \longrightarrow 2\lambda \left( 2\gamma_E - 1 + \log \left[ \frac{H^2 k^2 \eta \eta''}{\mu^2} \right] \right), \end{aligned} \quad (2.136)$$

where we made us of

$$\begin{aligned} E_1[2ik(\eta - \eta'')] &= -\operatorname{ci}(2k|\eta - \eta''|) \\ &+ i \operatorname{sign}(\eta - \eta'') \operatorname{si}(2k|\eta - \eta''|) - i \frac{\pi}{2} \operatorname{sign}(\eta - \eta''). \end{aligned} \quad (2.137)$$

The expression for  $\mathcal{B}_{(1)}$  is unfortunately much lengthier which is why we give here only the super-Hubble limit

$$\begin{aligned} \mathcal{B}_{(1)}(\eta, \eta'', k) \longrightarrow & \frac{2}{3} \lambda \left[ \log^2(-2k\eta) + \log^2(-2k\eta'') \right. \\ & + 2 \log(-2k\eta) \log(-2k\eta'') \\ & + \frac{4}{3} \log \left[ 4k^2 \eta \eta'' \right] \left( 3 \log \left[ \frac{2\mu}{H} \right] + 3\gamma_E - 4 \right) \\ & \left. + \frac{17}{4} - \frac{32}{3} \gamma_E + 4\gamma_E^2 + \frac{\pi^2}{3} + 2(4\gamma_E - 5) \log \left[ \frac{2\mu}{H} \right] \right]. \end{aligned} \quad (2.138)$$

We see that the above expressions are already symmetric and we will not need a homogeneous solution for symmetrizing them. Finally, we turn to the integral

that involves the Green's function

$$\mathcal{G}(\eta, \eta'', k) \equiv H^2 \int_{-\infty}^{\infty} d\tau \frac{G_{\text{ret}}(\eta, \tau, k)}{(\tau H)^4} \left[ \mathcal{B}_{(0)}(\tau, \eta'', k) + \mathcal{B}_{(0)}^{\text{log}}(\tau, \eta'', k) \right] + F_{\text{hom}}(\eta, \eta'', k). \quad (2.139)$$

We realize that the integral boundary at negative infinity will lead to logarithmic divergences which is why we add a homogeneous solution to cancel them

$$\mathcal{G}(\eta, \eta'', k) = H^2 \int_{\eta''}^{\infty} d\tau \frac{G_{\text{ret}}(\eta, \tau, k)}{(\tau H)^4} \left[ \mathcal{B}_{(0)}(\tau, \eta'', k) + \mathcal{B}_{(0)}^{\text{log}}(\tau, \eta'', k) \right] + \tilde{F}_{\text{hom}}(\eta, \eta'', k). \quad (2.140)$$

We computed (2.140) in terms of finite sums of exponentials, exponential integrals and generalized hyper geometric functions as well as an additional integral which contains similar functions as (2.134) but is more complicated. We also find that the consistency condition (2.124) applies which is a highly non-trivial statement with regard to how the various terms contribute. However, since the result fills pages and includes a lot of partial integration, we decided to give only the super-Hubble limit in this chapter. We note that the Green's function has the super-Hubble limit

$$G_{\text{ret}}(\eta, \eta', k) \longrightarrow \theta(\eta - \eta') \frac{H^2}{k^3} \left[ -\frac{1}{3} k^3 (\eta - \eta')^3 \right], \quad (2.141)$$

such that the full integral in the super-Hubble limit reduces to a rather simple expression

$$\begin{aligned} \mathcal{G}(\eta, \eta'', k) \longrightarrow & -\frac{1}{3} \int_{\eta''}^{\eta} d\tau \frac{(\eta - \tau)^3}{\tau^4} \left[ \frac{16}{3} \log^2(-2k\tau) \right. \\ & + \frac{4}{3} \left( 8 \log \left[ \frac{H}{2k} \right] - 9 \right) \log(-2k\tau) \\ & + \frac{4}{3} \left( 8 \log \left[ \frac{H}{2k} \right] + 7 - 8\gamma_E \right) \log(-2k\eta'') \\ & - \frac{70}{3} + 40\gamma_E - 16\gamma_E^2 - \frac{4}{9}\pi^2 + \frac{64}{3}(\gamma_E - 2) \log \left[ \frac{H}{2k} \right] \Big] \\ & + \tilde{F}_{\text{hom}}(\eta, \eta'', k). \end{aligned} \quad (2.142)$$

The last step is to symmetrize the result by means of a homogeneous solutions that should also include the tree-level solution for the Bunch-Davies vacuum,

$$\begin{aligned} \tilde{F}_{\text{hom}}(\eta, \eta'', k) = & F_{\varphi, dS}(\eta, \eta'', k) + \lambda \left[ h_1(\eta'') + i(k\eta'')^{-3} h_2(\eta'') \right] (1 + ik\eta) e^{-ik\eta} \\ & + \lambda \left[ h_1(\eta'') - i(k\eta'')^{-3} h_2(\eta'') \right] (1 - ik\eta) e^{ik\eta}, \end{aligned} \quad (2.143)$$



where  $h_{1,2}$  are real functions that we determine perturbatively as

$$\begin{aligned}
h_1(\eta'') \longrightarrow & -\frac{733}{486} + \frac{1}{81} \left( 18\gamma_E(3 + 2\gamma_E) + 7\pi^2 + 16(6\gamma_E - 11) \log \left[ \frac{H}{2k} \right] \right. \\
& \quad \left. + 216(\gamma_E - 2) \log \left[ \frac{2\mu}{H} \right] \right) \\
& + \frac{2}{81} \log(-2k\eta'') \left( 355 - 12\gamma_E(47 + 18\gamma_E) + 6\pi^2 \right. \\
& \quad \left. - 288(\gamma_E - 2) \log \left[ \frac{H}{2k} \right] \right) \\
& + \frac{4}{27} \log^2(-2k\eta'') \left( 1 + 12\gamma_E - 24 \log \left[ \frac{H}{2k} \right] \right) - \frac{16}{27} \log^3(-2k\eta''), \quad (2.144)
\end{aligned}$$

$$\begin{aligned}
h_2(\eta'') \longrightarrow & -\frac{353}{81} + \frac{2}{27} \left( 18\gamma_E(2\gamma_E - 5) + \pi^2 - 8(11 - 6\gamma_E) \log \left[ \frac{H}{2k} \right] \right) \\
& - \frac{4}{27} \log(-2k\eta'') \left( 1 - 12\gamma_E + 24 \log \left[ \frac{H}{2k} \right] \right) - \frac{8}{9} \log^2(-2k\eta''). \quad (2.145)
\end{aligned}$$

Adding up all contributions for the statistical two-point function

$$\begin{aligned}
F_\varphi(\eta, \eta'', k) = & F_{\varphi, dS}(\eta, \eta'', k) \\
& + \lambda \left( \frac{\square_k}{H^2} \mathcal{B}_{(2)}(\eta, \eta'', k) + \mathcal{B}_{(1)}(\eta, \eta'', k) + \mathcal{G}(\eta, \eta'', k) \right. \\
& \quad + [h_1(\eta'') + i(k\eta'')^{-3}h_2(\eta'')] (1 + ik\eta) e^{-ik\eta} \\
& \quad \left. + [h_1(\eta'') - i(k\eta'')^{-3}h_2(\eta'')] (1 - ik\eta) e^{ik\eta} \right), \quad (2.146)
\end{aligned}$$

yields expression (2.32) in the super-Hubble limit.

## 2.4 Conclusion and outlook

In the literature on cosmological perturbations, their properties are often specified solely in terms of equal time two-point function of the comoving curvature perturbation  $\mathcal{R}$ . This picture is correct if the fields are Gaussian distributed and if the decaying mode on super-Hubble scales makes the (canonical) momentum perturbation  $\pi_{\mathcal{R}}$  small and/or stochastically dependent on  $\mathcal{R}$ , such that no useful or additional information is contained in it. This rationale can be extended at the linear level to include isocurvature modes stemming from additional field perturbation in a multi field inflation scenario. On the other, one can discuss the self-interactions of the inflaton perturbation. There is another possibility that we discuss in this chapter, namely, that the momentum of the comoving curvature perturbation can become significant at the end of inflation via quantum interactions with a spectator field. We study this scenario for a simple two-field model of inflation (2.7) in which the inflaton field couples biquadratically to a light spectator scalar field. Expanding around the inflaton condensate yields a dominant cubic coupling at the level of perturbations in which the inflaton

perturbation couples linearly to the spectator (cf. figure 2.1a). We investigate how the spectator field affects the curvature perturbation by performing an explicit one-loop calculation with renormalized self-masses in the 2PI formalism. Quantum gravitational interactions during inflation have been addressed in [40, 41] with the conclusion that, in the single field inflationary models, corrections to the curvature perturbation grow on super-Hubble scales at most with powers of logarithms of the scale factor. We confirm this observation for our model in (2.32). However, the momentum correlators (2.39–2.40) grow as powers of the scale factor, such that they are not necessarily suppressed at the end of inflation. We calculate the Gaussian, von Neumann entropy of the curvature perturbation (2.49), and show that during inflation and on super-Hubble scales it grows as  $\sim 6 \ln(a)$ . This rapid growth of the entropy indicates a rapid classicalization of the curvature perturbation on super-Hubble scales during inflation, and it is a consequence of the rapid growth ( $\propto a^6$ , see Eq. (2.51)) of the Gaussian invariant of the state (2.48), which in turn can be attributed to the rapid growth of momentum correlators (2.39–2.40). This then implies that the *momentum operator of the curvature perturbation (2.34) should be regarded as stochastically independent from the curvature perturbation.*

When this work was nearing completion, we became aware that the idea of obtaining decoherence from spectators has been addressed in [39], based on the work in [67, 68]. Strictly speaking, the theory with a cubic interaction studied in [39, 67, 68] is unstable and not the same as the bi-quadratic theory we start from in equations (2.7) to (2.10), which is a stable theory for a positive coupling  $g$ . However, since the two-loop diagram in figure 2.1b is suppressed, the principle source of decoherence in our theory is incidentally a diagram that is topologically the same as the diagram used in [39, 67, 68], provided one identifies our coupling  $h = g\bar{\phi}$  with their coupling  $\lambda$ . Since  $\bar{\phi} = \bar{\phi}(t)$ , this identity is never exact and at some level the theories do differ. We also emphasize that our approach (based on the one loop evaluation of the inflaton two point function) differs significantly from the reduced density matrix approach in [39, 67, 68]. Furthermore, our results qualitatively differ in that we find the leading order growth of the inflaton two-point function correlator to be  $\log^3(-k\eta)$ , which differs by one power from the result obtained in [39, 67, 68]. Moreover, our result differs by a sign. Namely, we get that the two point function increases at late times while the above mentioned references find a suppression. Since both calculational frameworks differ significantly and bear a lot of complexity, we leave it as an important task for the future to explain how this difference comes about.

Our study shows that the effects of interactions are typically *large* at the end of inflation, which can be clearly seen from Eq. (2.43) and is illustrated in figures 2.2a–2.2c. On the other hand, if interactions switch off rapidly after inflation, quite generically by the end of radiation era the momentum fluctuations will decay such that their effects will be too small to leave any observable imprint in the CMB or LSS, which is corroborated by the estimate given in Eqs. (2.53) and (2.54). This conclusion holds however, only if the inflaton-spectator interactions are switched off rapidly enough after inflation, such that the post-inflationary evolution of cosmological perturbations on super-Hubble

scales can be well approximated by the corresponding free, linear evolution, according to which the large curvature momentum perturbation from the end of inflation decays swiftly during radiation. One way to hinder the decay of the momentum correlators is to keep the inflaton-spectator interactions active during early parts of the radiation era. This can be achieved, for example, by delaying the post-inflationary decays of the inflaton and spectator fluctuations, and by demanding that both fields are light enough such that, for some time during radiation, they remain approximately massless, *i.e.*  $m_\phi, m_\chi \ll H(t)$ , where  $H(t) \simeq 1/(2t)$  is the Hubble rate in radiation era. We leave a detailed study of decoherence on super-Hubble scales during radiation for future work.

Broadly speaking, investigations of quantum loop corrections to cosmological perturbations in an inflationary setting, a simple example of which is performed in this work, can be used to test consistency of various inflationary models and can be considered as complementary to effective field theory methods, which can be very useful for studying the internal consistency of inflationary models such as Higgs inflation [82, 83]. Furthermore, since quantum loop corrections from light matter fields may leave observable imprints in the CMB and large scale structure, one can use the signatures imprinted in the CMB and large scale structure by the momentum correlators of cosmological perturbations as a means to study inflationary interactions, thus opening a *novel observational window* to inflationary physics.

## 2.A Definitions and conventions

We make use of the d'Alembert operator in de Sitter space-time where

$$\frac{\square}{H^2} \equiv \frac{\overline{\square}_{dS}}{H^2} = \eta^2 \left[ -\partial_\eta^2 + \frac{D-2}{\eta} \partial_\eta + \delta^{ij} \partial_i \partial_j \right], \quad (2.147)$$

where the constant parameter  $H$  is the Hubble rate at the beginning of inflation and  $\eta$  denotes conformal time. We use the following general notation for the Wightman functions, causal and statistical propagators in the cosmological context,

$$i\Delta_\varphi^{\mp\pm}(\eta, \eta', k) = F_\varphi(\eta, \eta', k) \pm \frac{i}{2} \Delta_\varphi^c(\eta, \eta', k), \quad (2.148)$$

$$F_\varphi(\eta, \eta', k) = \int d^3(x - x') e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} F_\phi(x; x'), \quad (2.149)$$

$$i\Delta_\varphi^c(\eta, \eta', k) = \int d^3(x - x') e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} i\Delta_\phi^c(x; x'), \quad (2.150)$$

with

$$F_\varphi(x; x') = \frac{1}{2} \text{Tr} [\hat{\rho}(\eta_0) \{ \hat{\varphi}(x), \hat{\varphi}(x') \} ], \quad (2.151)$$

$$i\Delta_\varphi^c(x; x') = \text{Tr} [\hat{\rho}(\eta_0) [\hat{\varphi}(x'), \hat{\varphi}(x)]] , \quad (2.152)$$

where  $\hat{\rho}_0 \equiv \hat{\rho}(\eta_0)$  is the initial density matrix (defined at  $\eta = \eta_0$ ). Moreover, we define the correlators

$$\Delta_{XY}(x; x') \equiv \frac{1}{2} \text{Tr} \left[ \hat{\rho}(\eta_0) \{ \hat{X}(x), \hat{Y}(x') \} \right]. \quad (2.153)$$

We make frequently use of the following functions,

$$\text{si}(z) = - \int_z^\infty \frac{\sin(t) dt}{t} = \int_0^z \frac{\sin(t) dt}{t} - \frac{\pi}{2} = \text{Si}(z) - \frac{\pi}{2}, \quad (2.154)$$

$$\text{ci}(z) = - \int_z^\infty \frac{\cos(t) dt}{t}, \quad (2.155)$$

$$E_1(ix) = -\gamma_E - \log(ix) - \sum_{k=1}^{\infty} \frac{(-ix)^k}{kk!} = i \text{si}(x) - \text{ci}(x), \quad x > 0, \quad (2.156)$$

$$\text{Ein}(z) = \int_0^z \frac{1 - e^{-t}}{t} dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n!n} = E_1(z) + \log(z) + \gamma_E, \quad (2.157)$$

$$\int_0^z \frac{\text{Ein}(t)}{t} dt = z \times {}_3F_3 \left[ \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}; z \right]. \quad (2.158)$$

## 2.B Photon kinetic equation

The starting point for our recapitulation is the Boltzmann equation for the temperature perturbation of the photon fluids which takes the following form in Fourier space (we follow closely standard literature, see *e.g.* [2, 3]),

$$\begin{aligned} & \partial_\eta \Theta(k, \mu) + ik\mu [\Theta(k, \mu) + \Phi(k)] - \partial_\eta \Psi(k) \\ &= -(\partial_\eta \tau) \left[ \Theta_0(k) - \Theta(k, \mu) + \mu v_b(k) - \frac{1}{2} \mathcal{P}_2(\mu) \Sigma(k) \right], \quad \mu = \frac{\vec{k} \cdot \vec{p}}{kp}. \end{aligned} \quad (2.159)$$

Here,  $\Theta(k, \mu)$  is the time-dependent gauge invariant (integrated) photon temperature perturbation that is obtained from,

$$\Theta(k, \mu) = \sum_{l=0}^{\infty} (-i)^l \theta_l(\eta, k) \mathcal{P}_l(\mu) \propto \int dp p^3 \delta f(\eta, \vec{p}, \vec{k}), \quad (2.160)$$

where  $\delta f$  is the perturbed, gauge-invariant photon distribution function as defined in [3], and  $P_l$  are Legendre polynomials. The Bardeen potentials  $\Phi$  and  $\Psi$  (as defined in [9]) are related to temporal and spatial metric perturbations. The time-dependent variable  $\tau(\eta)$  is the optical depth related to Thomson scattering with  $v_b$  the (longitudinal) baryon velocity perturbation and  $\Sigma$  the anisotropic stress which depends on the polarization and quadrupole moment  $\Theta_2$ , both of which are usually neglected in a first approximation [2, 3]. We note that the gravitational slip is given by

$$\Psi - \Phi = \frac{a^2}{M_p^2} \Sigma, \quad (2.161)$$

and we can identify the two potentials once anisotropic stress is absent or neglected. Moreover, we have to consider the speed of sound  $c_s$  of the photon-baryon fluid which is defined via

$$\delta P = c_s^2 \delta \rho + \delta P_{\text{nad}}, \quad (2.162)$$

where  $\delta P$ ,  $\delta \rho$  and  $\delta P_{\text{nad}}$  are the pressure, density and non-adiabatic pressure perturbations, respectively. The speed of sound in radiation domination is related to the background density of photons  $\rho_\gamma^{(0)}$  and baryons  $\rho_b^{(0)}$  via

$$c_s^2(\eta) = \frac{1}{3(1+R(\eta))}, \quad R(\eta) \equiv \frac{3\rho_b^{(0)}(\eta)}{4\rho_\gamma^{(0)}(\eta)}. \quad (2.163)$$

Since the baryon density is much smaller than the photon density in the radiation dominated phase, we can take as another approximation  $c_s^2 \approx 1/3$  during this time which also determines the baryon velocity to first order through the photon dipole moment as

$$v_b = -3i\Theta_1 + \mathcal{O}(R). \quad (2.164)$$

Putting it all together, one can derive a second-order differential equation for the effective temperature fluctuation  $\Delta T = \Theta_0 + \Phi$  and the gravitational potentials [2, 3],

$$\begin{aligned} \left[ \frac{d^2}{d\eta^2} + \frac{R}{1+R} \mathcal{H} \frac{d}{d\eta} + k^2 c_s^2 \right] \Delta T &= k^2 \left[ c_s^2 - \frac{1}{3} \right] \Phi \\ &+ \left[ \frac{d^2}{d\eta^2} + \frac{R}{1+R} \mathcal{H} \frac{d}{d\eta} \right] [\Psi + \Phi]. \end{aligned} \quad (2.165)$$

We see that if we neglect the damping term by  $c_s^2 \approx 1/3$ , we have a forced harmonic oscillator, whose homogeneous solutions are determined by the monopole density  $\Theta_0(\eta_{\text{cmb}})$  and its time-derivative  $\Theta'_0(\eta_{\text{cmb}})$  as well as the gravitational potential  $\Psi(\eta_{\text{cmb}})$  and its time-derivative  $\Psi'(\eta_{\text{cmb}})$  at some time within the radiation dominated phase  $\eta_{\text{cmb}}$  that is close to recombination  $\eta_{\text{cmb}} \approx 10^{-1} \eta_{\text{rec}}$ ,

$$\begin{aligned} \Delta T(\eta) &\approx [\Theta_0 + \Phi](\eta_{\text{cmb}}) \cos[kr_s(\eta)] + \left[ \frac{\Theta'_0 + \Phi'}{kc_s} \right](\eta_{\text{cmb}}) \sin[kr_s(\eta)] \\ &+ \frac{\sqrt{3}}{k} \int_{\eta_{\text{cmb}}}^{\eta} [\Phi''(\bar{\eta}) + \Psi''(\bar{\eta})] \sin[kr_s(\eta) - kr_s(\bar{\eta})] d\bar{\eta}, \end{aligned} \quad (2.166)$$

where we defined the sound horizon,

$$r_s(\eta) = \int_{\eta_{\text{cmb}}}^{\eta} c_s(\bar{\eta}) d\bar{\eta}, \quad (2.167)$$

and keep the time-dependence of the speed of sound only in the phases. By making use of (2.159) in the super-Hubble limit, in which also the gravitational slip vanishes, we obtain  $\Theta'_0(\eta_{\text{cmb}}) = \Psi'(\eta_{\text{cmb}}) = \Phi'(\eta_{\text{cmb}})$  and the temporal

integration turns out to yield  $2\Theta_0(\eta_{\text{cmb}}) = -\Psi(\eta_{\text{cmb}}) = -\Phi(\eta_{\text{cmb}})$  [2, 3]. We also recall that the gravitational potential  $\Psi$  obeys in the absence of gravitational slip the following differential equation [9],

$$\partial_\eta^2 \Psi + 3(1 + c_s^2)\mathcal{H}\partial_\eta \Psi + \left[2\partial_\eta \mathcal{H} + (1 + 3c_s^2)\mathcal{H}^2 + c_s^2 k^2\right]\Psi = \frac{1}{2M_p^2}\delta P_{\text{nad}}. \quad (2.168)$$

As a first approximation to the inhomogeneous solution in (2.166), we can solve (2.168) for vanishing non-adiabatic pressure,  $\delta P_{\text{nad}} \rightarrow 0$ , with  $c_s^2 \approx 1/3$  during radiation unless it will appear as phase in conjunction with the momentum  $k$ . Thus, we write

$$\partial_\eta^2 \Psi + 4\mathcal{H}\partial_\eta \Psi + c_s^2 k^2 \Psi \approx 0. \quad (2.169)$$

We see that the solution will stay constant on super-Hubble scales or decay otherwise during radiation and we thus neglect the integral in (2.166). We then have

$$\Delta T(k, \eta) \approx \frac{1}{2}\Psi_k(\eta_{\text{cmb}})\cos[kr_s(\eta)] + 2\frac{\Psi'_k(\eta_{\text{cmb}})}{kc_s(\eta_{\text{cmb}})}\sin[kr_s(\eta)]. \quad (2.170)$$

Finally, in order to make contact with the era of inflation, we would like to relate equation (2.170) to the gauge-invariant curvature perturbation  $\mathcal{R}$  in the case of vanishing (linear) non-adiabatic pressure ( $\delta P_{\text{nad}} = 0$ ). First we note, that the gauge-invariant curvature perturbation  $\mathcal{R}$  may be expressed in terms of the gauge-invariant gravitational potential  $\Psi$  *via* [9],

$$\mathcal{R} \equiv \Psi + \frac{\Phi}{\epsilon} + \frac{\partial_\eta \Psi}{\mathcal{H}\epsilon} = \Psi + \frac{\Psi}{\epsilon} + \frac{\partial_\eta \Psi}{\mathcal{H}\epsilon}, \quad \epsilon = 1 - \frac{\partial_\eta \mathcal{H}}{\mathcal{H}^2}, \quad (2.171)$$

where we neglected the gravitational slip in the second equality, which is justified on super-Hubble scales.

The squared adiabatic sound speed may be expressed as

$$c_s^2 \equiv \frac{\partial_\eta \bar{P}}{\partial_\eta \bar{\rho}} = -1 + \frac{2}{3}\epsilon - \frac{\partial_\eta \epsilon}{3\mathcal{H}\epsilon},$$

where  $P$  and  $\rho$  are background pressure and energy density, respectively. Taking a derivative of (2.171), using (2.169), we find

$$\partial_\eta \mathcal{R} = -\frac{c_s^2 k^2}{\epsilon \mathcal{H}} \Psi.$$

Note, that the latter relation holds also in an inflationary context with  $c_s^2$  set equal to 1. We can solve for  $\Psi$  and  $\partial_\eta \Psi$  in terms of  $\mathcal{R}$  and  $\partial_\eta \mathcal{R}$  which yields

$$\begin{aligned} \Psi &= -\frac{\epsilon \mathcal{H}}{c_s^2 k^2} \partial_\eta \mathcal{R} \equiv -\frac{\mathcal{H}}{2M_p^2 k^2 a^2} \pi_{\mathcal{R}}, \\ \partial_\eta \Psi &= \epsilon \mathcal{H} \mathcal{R} + (1 + \epsilon) \frac{\mathcal{H}^2}{2M_p^2 k^2 a^2} \pi_{\mathcal{R}}, \end{aligned} \quad (2.172)$$

where we defined the canonical momentum  $\pi_{\mathcal{R}}$  associated to  $\mathcal{R}$  as in Appendix 2.C. We now would like to evolve the gravitational potential on super-Hubble scales from the end of inflation deep into the radiation era by using linear relations. Therefore, we make use of Weinberg's theorem [10] according to which there are always two solutions for the gravitational potential on super-Hubble scales which take the following form

$$\Psi_{\text{ad}}(\eta) = - \left[ \frac{1}{2M_p^2 k^2} \pi_{\mathcal{R}}(\eta_e) + \frac{a^2(\eta_e)}{\mathcal{H}(\eta_e)} \mathcal{R}(\eta_e) \right] \frac{\mathcal{H}(\eta)}{a^2(\eta)} + \mathcal{R}(\eta_e) \left[ 1 - \frac{\mathcal{H}(\eta)}{a^2(\eta)} \int_{\eta_e}^{\eta} a^2(\bar{\eta}) d\bar{\eta} \right], \quad (2.173)$$

where the time  $\eta_e$  signals some time shortly before the end of inflation such that we still have that  $\epsilon(\eta_e) \ll 1$ . In order to set initial conditions for the CMB spectrum at time  $\eta_{\text{cmb}} \approx 10^{-1} \eta_{\text{rec}}$  close to recombination, we stick to a simplified scenario in which we neglect small contributions due to the transition from inflation to radiation and keep only leading order terms in each variable,

$$\Psi_{\text{ad}}(\eta_{\text{cmb}}) \approx - \frac{1}{2M_p^2 k^2} \frac{H a^2(\eta_e)}{a^3(\eta_{\text{cmb}})} \pi_{\mathcal{R}}(\eta_e) + \frac{2}{3} \mathcal{R}(\eta_e), \quad (2.174)$$

$$\Psi'_{\text{ad}}(\eta_{\text{cmb}}) \approx 3 \frac{H^2 a^3(\eta_e)}{a^4(\eta_{\text{cmb}})} \left[ \frac{1}{2M_p^2 k^2} \pi_{\mathcal{R}}(\eta_e) + \frac{a^2(\eta_e)}{H} \mathcal{R}(\eta_e) \right]. \quad (2.175)$$

Inserting the above super-Hubble initial conditions into the approximate solution for the effective CMB temperature perturbation (2.170) and making the stochastic character of the involved operators manifest, we have

$$\Delta \hat{T}(\eta, \vec{k}) \approx \frac{1}{2} \left[ \frac{2}{3} \hat{\mathcal{R}}(\eta_e, \vec{k}) - \frac{a^3(\eta_e)}{a^3(\eta_{\text{cmb}})} \frac{H}{2M_p^2 k^2 a(\eta_e)} \hat{\pi}_{\mathcal{R}}(\eta_e, \vec{k}) \right] \cos[kr_s(\eta)] + \frac{6H}{kc_s(\eta_{\text{cmb}})} \frac{a^4(\eta_e)}{a^4(\eta_{\text{cmb}})} \left[ \frac{H}{2M_p^2 k^2} \hat{\pi}_{\mathcal{R}}(\eta_e, \vec{k}) + a(\eta_e) \hat{\mathcal{R}}(\eta_e, \vec{k}) \right] \sin[kr_s(\eta)]. \quad (2.176)$$

This relation is used in see Eqs. (2.52–2.54) of section 2.2 to estimate the size of the photon temperature fluctuations induced by the enhanced inflationary momentum perturbation.

## 2.C Linear evolution of curvature perturbation

The gauge-invariant curvature perturbation can be defined in terms of the metric perturbation  $\psi$  and the perturbation of the velocity potential  $\varphi_v$  [7] (in single field inflationary models  $\varphi_v$  reduces to the inflaton field perturbation) as,

$$\mathcal{R} = \psi + \frac{H}{\sqrt{\rho + P}} \varphi_v, \quad (2.177)$$

where  $\psi = -\text{Tr}[\delta g_{ij}]/(6a^2)$ ,  $\rho$  and  $P$  are the background fluid density and pressure (in inflation  $\rho + P \rightarrow (\dot{\phi})^2$ , where  $\phi(t) \equiv \langle \hat{\phi}(x) \rangle$  is the inflaton expectation value). Let us solve for the curvature perturbation  $\mathcal{R}$  in post-inflationary epochs. The quadratic (reparametrization invariant) action for  $\mathcal{R}$  reads (see *e.g.* [7, 70]),

$$S[\mathcal{R}] = (2M_p^2) \int d^3x dt \bar{N}(t) a^3 \epsilon \left( \frac{1}{2c_s^2} \dot{\mathcal{R}}^2 - \frac{1}{2a^2} (\partial_i \mathcal{R})^2 \right), \quad (2.178)$$

where  $\bar{N} = \bar{N}(t)$  is the lapse function of the ADM decomposition (defined on a global equal time hypersurface  $\Sigma_t$ ) and

$$\epsilon(t) = -\frac{\dot{H}}{H^2} \quad (2.179)$$

is the principal slow roll parameter and  $\dot{X}(t) \equiv \bar{N}^{-1} \partial/\partial t$  is the time derivative invariant under time reparametrizations. In inflation  $\epsilon \ll 1$ , in radiation  $\epsilon = 2$  and in matter era  $\epsilon = 3/2$ . From (2.178) one easily finds the canonical momentum of  $\mathcal{R}$ ,

$$\pi_{\mathcal{R}}(t, \vec{x}) \equiv \frac{\delta S}{\delta \partial_t \mathcal{R}(t, \vec{x})} = \frac{2M_p^2 a^3 \epsilon}{\bar{N} c_s^2} \partial_t \mathcal{R} \quad (2.180)$$

and the Hamiltonian,

$$H(t) = \int d^3x \left( \frac{\bar{N} c_s^2}{4M_p^2 a^3 \epsilon} \pi_{\mathcal{R}}^2 + M_p^2 \bar{N} a \epsilon (\partial_i \mathcal{R})^2 \right). \quad (2.181)$$

From (2.181) one easily arrives at the Heisenberg equations,

$$\partial_t \hat{\mathcal{R}} = \frac{\bar{N} c_s^2}{2M_p^2 a^3 \epsilon} \hat{\pi}_{\mathcal{R}}, \quad \partial_t \hat{\pi}_{\mathcal{R}} = 2M_p^2 \bar{N} a \epsilon \partial_i^2 \hat{\mathcal{R}}, \quad (2.182)$$

where  $\hat{\mathcal{R}}$  and  $\hat{\pi}_{\mathcal{R}}$  are the canonical pair obeying,

$$\left[ \hat{\mathcal{R}}(t, \vec{x}), \hat{\pi}_{\mathcal{R}}(t, \vec{x}') \right] = i\hbar \delta^3(\vec{x} - \vec{x}'). \quad (2.183)$$

One can solve (2.182) in space-times of constant  $\epsilon$  as follows. Let us introduce a time,  $a d\eta = \bar{N} dt$  (notice that time  $\eta$  reduces to the usual conformal time in the gauge,  $\bar{N} = a$ ), and (2.182) reduce to,

$$\partial_\eta \left[ a^2 \partial_\eta \hat{\mathcal{R}} \right] - a^2 c_s^2 \nabla^2 \hat{\mathcal{R}} = 0, \quad (2.184)$$



where we made use of  $\dot{\epsilon} = 0$  and  $\dot{c}_s = 0$ . Since we are primarily interested in the spectra, it is convenient to perform the following mode decomposition,

$$\begin{aligned}\hat{\mathcal{R}}(\eta, \vec{x}) &= \int \frac{d^3k}{(2\pi)^3} \left( e^{i\vec{k}\cdot\vec{x}} \mathcal{R}(\eta, k) \hat{a}(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}} \mathcal{R}^*(\eta, k) \hat{a}^+(\vec{k}) \right) \\ &\equiv \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \hat{\mathcal{R}}(\eta, \vec{k}), \\ \hat{\pi}_{\mathcal{R}}(\eta, \vec{x}) &= \int \frac{d^3k}{(2\pi)^3} \left( e^{i\vec{k}\cdot\vec{x}} \pi_{\mathcal{R}}(\eta, k) \hat{a}(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}} \pi_{\mathcal{R}}^*(\eta, k) \hat{a}^+(\vec{k}) \right) \\ &\equiv \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \hat{\pi}_{\mathcal{R}}(\eta, \vec{k}),\end{aligned}\tag{2.185}$$

where

$$\left[ \hat{a}(\vec{k}), \hat{a}^+(\vec{k}') \right] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}'), \quad \mathcal{R}(\eta, k) \pi_{\mathcal{R}}^*(\eta, k) - \mathcal{R}^*(\eta, k) \pi_{\mathcal{R}}(\eta, k) = i.\tag{2.186}$$

The equation of motion for the modes  $\mathcal{R}(\eta, k)$  then becomes,

$$\left[ \frac{d^2}{d\eta^2} + c_s^2 k^2 - (\mathcal{H}^2 + \partial_\eta \mathcal{H}) \right] (a\mathcal{R}) = 0,\tag{2.187}$$

where  $\mathcal{H} = \partial_\eta \ln(a) = aH$  is conformal Hubble rate. For inflation we have  $c_s^2 = 1$  and set to leading order in the slow-roll parameters  $a(\eta) = -H\eta^{-1}$  ( $H \approx \text{const}$ ). Thus, the two fundamental solutions in inflation are to leading order given by

$$\frac{1}{\sqrt{2\epsilon}M_P} \frac{H}{\sqrt{2k^3}} (1 \mp ik\eta) e^{\pm ik\eta},\tag{2.188}$$

such that

$$\begin{aligned}\hat{\mathcal{R}}(\eta, \vec{k}) &= \frac{1}{\sqrt{2\epsilon}M_P} \frac{H}{\sqrt{2k^3}} \left[ (1 + ik\eta) e^{-ik\eta} \hat{a}(-\vec{k}) \right. \\ &\quad \left. + (1 - ik\eta) e^{ik\eta} \hat{a}^+(\vec{k}) \right],\end{aligned}\tag{2.189}$$

$$\hat{\pi}_{\mathcal{R}}(\eta, \vec{k}) = \frac{1}{\sqrt{2\epsilon}M_P} \frac{H}{\sqrt{2k^3}} 2M_P^2 a^2 \epsilon k^2 \eta \left[ e^{-ik\eta} \hat{a}(-\vec{k}) + e^{ik\eta} \hat{a}^+(\vec{k}) \right].\tag{2.190}$$

We now restrict the degrees of freedom of (lets say) a Gaussian state associated to  $\mathcal{R}$  and  $\pi_{\mathcal{R}}$  to the Bunch-Davies vacuum with  $\hat{a}(\vec{k})|0\rangle = 0$  by picking up only the commutator in any two-point function. However, the dynamics of single-field inflation on super-Hubble scales within the standard linear treatment reduces the effective degrees of freedom of the Gaussian state in any case to only one stochastic variable. In other words, if we look on super-Hubble scales  $|k\eta| \ll 1$ , we find that  $\hat{\mathcal{R}}$  and  $\hat{\pi}_{\mathcal{R}}$  are effectively no more independent

operators,

$$\hat{\mathcal{R}}(\eta, \vec{k}) \longrightarrow \frac{1}{\sqrt{2\epsilon}M_P} \frac{H}{\sqrt{2k^3}} \left[ \hat{a}(-\vec{k}) + \hat{a}^+(\vec{k}) + \mathcal{O}(k\eta) \right], \quad (2.191)$$

$$\begin{aligned} \hat{\pi}_{\mathcal{R}}(\eta, \vec{k}) &\longrightarrow \frac{1}{\sqrt{2\epsilon}M_P} \frac{H}{\sqrt{2k^3}} 2M_p^2 a^2 \epsilon k^2 \eta \left[ \hat{a}(-\vec{k}) + \hat{a}^+(\vec{k}) + \mathcal{O}(k\eta) \right] \\ &= -\frac{2\epsilon M_p^2 a^2}{\mathcal{H}} k^2 \left[ \hat{\mathcal{R}}(\eta, \vec{k}) + \mathcal{O}(k\eta) \right]. \end{aligned} \quad (2.192)$$

In conclusion, we would like to emphasize that  $\hat{\mathcal{R}}$  and  $\hat{\pi}_{\mathcal{R}}$  are for every  $\vec{k}$  a priori independent and it is either the choice of state or the dynamics that could effectively cease this independence.

## 2.D Wigner transform of logarithms

In this appendix we show that

$$\begin{aligned} &\int d^3(x - x') e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} \left[ \frac{1}{2} \log^2 \left( \frac{y}{4} \right) + f(\eta, \eta') \log \left( \frac{y}{4} \right) \right] \\ &= -\frac{4\pi^2}{k^3} \left[ 2 + [1 + ik|\Delta\eta|] \left( \log \left[ \frac{aa'H^2|\Delta\eta|}{2k} \right] + i\frac{\pi}{2} - \gamma_E + f(\eta, \eta') \right) \right] e^{-ik|\Delta\eta|} \\ &\quad + \frac{4\pi^2}{k^3} (1 - ik|\Delta\eta|) \left[ \text{ci}[2k|\Delta\eta|] - i \text{si}[2k|\Delta\eta|] \right] e^{+ik|\Delta\eta|}, \end{aligned} \quad (2.193)$$

where  $\Delta\eta = \eta - \eta'$  and  $f(\eta, \eta')$  is some  $k$ -independent function. We need integrals of the following type,

$$\mathcal{I}_n(x) \equiv x^2 \int_0^\infty dz z \sin[xz] \log^n(|1 - z^2|) \quad (2.194)$$

$$= x^2 \left[ \frac{d^n}{db^n} \int_0^\infty dz z \sin[xz] |1 - z^2|^b \right]_{b=0}. \quad (2.195)$$

By using

$$\int_0^\infty dz z \sin[xz] |1 - z^2|^b = \frac{\sqrt{\pi}}{2} \left( \frac{2}{x} \right)^{b+\frac{1}{2}} \Gamma[b+1] [J_{b+\frac{3}{2}}(x) + Y_{-b-\frac{3}{2}}(x)], \quad (2.196)$$

with  $J_n, Y_m$  being the Bessel functions of the first and second kind, we find by analytically extending

$$\mathcal{I}_1(x) = -\pi [\cos(x) + x \sin(x)], \quad (2.197)$$

and

$$\begin{aligned} \mathcal{I}_2(x) = 2\pi \Big[ & -2 \cos(x) + [\cos(x) + x \sin(x)] [\text{ci}(2x) + \gamma_E - \log\left(\frac{2}{x}\right)] \\ & + [\sin(x) - x \cos(x)] \text{si}(2x) \Big], \end{aligned} \quad (2.198)$$

where we used

$$\text{ci}(x) = - \int_x^\infty \frac{\cos(y)}{y} dy, \quad \text{si}(x) = - \int_x^\infty \frac{\sin(y)}{y} dy. \quad (2.199)$$

Remembering that

$$\log(\Delta x_{++}^2) = \log\left(|\Delta\eta^2 - |\vec{x} - \vec{x}'|^2|\right) + i\pi\theta(\Delta\eta^2 - |\vec{x} - \vec{x}'|^2), \quad (2.200)$$

we get

$$\int d^3(x - x') e^{-i\vec{k}\cdot(\vec{x} - \vec{x}')} \log\left(\frac{y}{4}\right) = -\frac{4\pi^2}{k^3} [1 + ik|\Delta\eta|] e^{-ik|\Delta\eta|} + \text{hom.}, \quad (2.201)$$

and

$$\begin{aligned} & \int d^3(x - x') e^{-i\vec{k}\cdot(\vec{x} - \vec{x}')} \log^2\left(\frac{y}{4}\right) \\ &= -\frac{8\pi^2}{k^3} \left[ 2 + [1 + ik|\Delta\eta|] \left( \log\left[\frac{aa'H^2|\Delta\eta|}{2k}\right] + i\frac{\pi}{2} - \gamma_E \right) \right] e^{-ik|\Delta\eta|} \\ &+ \frac{8\pi^2}{k^3} (1 - ik|\Delta\eta|) \left[ \text{ci}[2k|\Delta\eta|] - i \text{si}[2k|\Delta\eta|] \right] e^{+ik|\Delta\eta|} + \text{hom.}, \end{aligned} \quad (2.202)$$

where  $\gamma_E \approx 0.57721$  is Euler's constant. We combine the results (2.201) and (2.202) in order to get (2.193).

## 2.E $I_R$ integrals

In this appendix we calculate the integrals (2.128), (2.129) and (2.130). Let us start with

$$\begin{aligned} I_{\tilde{R}}(\eta, \eta'', k) &= -\lambda^{-1} H^{-2} \text{sign}(\eta, \eta'') \\ &\times \int_{\eta''}^{\eta} d\tau (\tau H)^{-4} \text{Re} [\widetilde{M}_I^{++} + \widetilde{M}_{II}^{++}] (\eta, \tau, k) \Delta_{\phi, BD}^c(\tau, \eta'', k) \\ &= \text{Im} \left\{ \left[ (1 + ik\eta'') e^{-ik\eta''} \right] \right. \\ &\quad \left. \times \left[ e^{ik\eta} \widetilde{\mathcal{R}}_1(\eta, \eta'', k) + e^{-ik\eta} \widetilde{\mathcal{R}}_2(\eta, \eta'', k) \right] \right\}, \end{aligned} \quad (2.203)$$

where

$$\begin{aligned} \tilde{I}_{R_1}(\eta, \eta'', k) \equiv & \frac{1}{k^3} \int_{\eta}^{\eta''} d\tau \left[ 2 + [1 - ik(\eta - \tau)] \left( \log \left[ \frac{iH^2(\eta - \tau)}{2k} \right] \right. \right. \\ & \left. \left. - i\pi \text{sign}(\eta - \eta'') - \gamma_E + E_1[2ik(\eta - \tau)] \right) \right] \frac{1 - ik\tau}{\tau^4}, \quad (2.204) \end{aligned}$$

$$\begin{aligned} \tilde{I}_{R_2}(\eta, \eta'', k) \equiv & \frac{1}{k^3} \int_{\eta}^{\eta''} d\tau \left[ 2 + [1 + ik(\eta - \tau)] \left( \log \left[ -i \frac{H^2(\eta - \tau)}{2k} \right] \right. \right. \\ & \left. \left. + i\pi \text{sign}(\eta - \eta'') - \gamma_E + E_1[-2ik(\eta - \tau)] \right) \right] e^{2ik\tau} \frac{1 - ik\tau}{\tau^4}. \quad (2.205) \end{aligned}$$

We were able to drop the absolute value signs in  $I_{\tilde{R}}$  since the function that effectively appears has no branch cut as one might expect naively due to the logarithm and the exponential integral. The branch cut is exactly cancelled and we are dealing with the entire function  $\text{Ein}(z)$ , the complementary exponential integral,

$$\text{Ein}(z) = \int_0^z \frac{1 - e^{-t}}{t} dt = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n!n} = E_1(z) + \log(z) + \gamma_E, \quad (2.206)$$

converging for all finite values of  $|z|$ . We define

$$\mathcal{J}(\eta, \eta'', k) \equiv \int_0^1 dx E_1[-2ik(x(\eta - \eta'') + \eta'')] \frac{1 - e^{-2ik(\eta - \eta'')(x-1)}}{x-1}, \quad (2.207)$$

and have the following result

$$\begin{aligned}
I_{\tilde{R}}(\eta, \eta'', k) = & \text{Im} \left[ (1 + ik\eta'') e^{-ik\eta''} \left\{ e^{ik\eta} \left( -\frac{4}{3k^3\eta^3} + \frac{4i}{3k^2\eta^2} - \frac{4}{3k\eta} + \frac{2}{3k^3(\eta'')^3} \right. \right. \right. \\
& - \frac{2i}{3k^2(\eta'')^2} + \frac{2}{3k\eta''} + \left( \frac{i}{3} + \frac{1}{3k^3(\eta'')^3} - \frac{ik\eta}{3k^3(\eta'')^3} - \frac{k\eta}{2k^2(\eta'')^2} + \frac{1}{k\eta''} \right) \\
& \times \left( E_1[2ik(\eta - \eta'')] + \log \left[ \frac{iH^2(\eta - \eta'')}{2k} \right] - \gamma_E - i\pi \text{sign}(\eta - \eta'') \right) \\
& - \left( \frac{2}{3k^3\eta^3} - \frac{2i}{3k^2\eta^2} + \frac{1}{k\eta} + \frac{2i}{3} \right) \left( \log \left[ \frac{H^2}{4k^2} \right] - 2\gamma_E \right) + \frac{i}{3} \log \left[ \frac{\eta}{\eta''} \right] \Bigg) \\
& + e^{-ik\eta} \left( E_1[-2ik\eta''] \left( i + \frac{2}{3}(i - k\eta) \left[ E_1[-2ik(\eta - \eta'')] \right. \right. \right. \\
& \left. \left. \left. + \log \left[ \frac{H^2(\eta - \eta'')}{2ik} \right] + i\pi \text{sign}(\eta - \eta'') - \gamma_E \right] \right) \right. \\
& \left. - E_1(-2ik\eta) \left( i + \frac{2}{3}[i - k\eta] \left[ \log \left[ \frac{H^2}{4k^2} \right] + i\pi \text{sign}(\eta - \eta'') - 2\gamma_E \right] \right) \right) \\
& - e^{-ik\eta + 2ik\eta''} \left( \frac{2}{3k^3(\eta'')^3} - \frac{2i}{3k^2(\eta'')^2} + \frac{2}{3k\eta''} \right. \\
& + \left( \frac{1}{3k^3(\eta'')^3} + \frac{ik\eta}{3k^3(\eta'')^3} - \frac{2i}{3k^2(\eta'')^2} + \frac{k\eta}{6k^2(\eta'')^2} + \frac{1}{3k\eta''} + \frac{ik\eta}{3k\eta''} \right) \\
& \times \left( E_1[-2ik(\eta - \eta'')] + \log \left[ \frac{H^2(\eta - \eta'')}{2ik} \right] + i\pi \text{sign}(\eta - \eta'') - \gamma_E \right) \Bigg) \\
& \left. + \frac{2}{3}i(1 + ik\eta)e^{-ik\eta} \mathcal{J}(\eta, \eta'', k) \right\} \Bigg]. \tag{2.208}
\end{aligned}$$

On super-Hubble scales this simplifies to,

$$I_{\tilde{R}}(\eta, \eta'', k) \longrightarrow -\frac{4}{3} \left( \log \left[ \frac{\eta}{\eta''} \right] - \frac{1}{3} + \frac{1}{3} \frac{(\eta'')^3}{\eta^3} \right) \left( \gamma_E - 1 - \log \left[ \frac{H}{2k} \right] \right). \tag{2.209}$$

The next integral we calculate is

$$\begin{aligned}
I_{\widehat{R}}(\eta, \eta'', k) &= -\lambda^{-1} H^{-2} \int_{\eta''}^{\eta} d\tau (\tau H)^{-4} \left[ \text{Re } \widehat{M}(\eta, \tau, k) \right] \Delta_{\phi, BD}^c(\tau, \eta'', k) \\
&= -\text{Im} \left[ (1 + ik\eta'') e^{-ik\eta''} \left\{ -\frac{2}{3} (k\eta - i) e^{-ik\eta} E_1[-2ik\eta] \right. \right. \\
&\quad e^{ik\eta} \left( \frac{i}{3} + \frac{2}{3k^3\eta^3} - \frac{2i}{3k^2\eta^2} + \frac{1}{k\eta} - \frac{1}{3k^3(\eta'')^3} + \frac{ik\eta}{3k^3(\eta'')^3} + \frac{k\eta}{2k^2(\eta'')^2} - \frac{1}{k(\eta'')} \right) \\
&\quad + e^{2ik\eta'' - ik\eta} \left( -\frac{1}{3k^3(\eta'')^3} - \frac{ik\eta}{3k^3(\eta'')^3} + \frac{2i}{3k^2(\eta'')^2} - \frac{k\eta}{6k^2(\eta'')^2} - \frac{1}{3k\eta''} - \frac{ik\eta}{3k\eta''} \right) \\
&\quad \left. \left. + \frac{2}{3} (k\eta - i) e^{-ik\eta} E_1[-2ik\eta''] \right\} \right], \quad (2.210)
\end{aligned}$$

where we again made use of the fact that the absolute value sign does not matter for the real part of  $\widehat{M}$ . On super-Hubble scales we have here

$$I_{\widehat{R}}(\eta, \eta'', k) \longrightarrow -\frac{2}{3} \left( \log \left[ \frac{\eta}{\eta''} \right] - \frac{1}{3} + \frac{1}{3} \frac{(\eta'')^3}{\eta^3} \right). \quad (2.211)$$

We also have to calculate

$$\begin{aligned}
\widehat{\mathcal{R}}_{\log}(\eta, \eta'', k) &= -\lambda^{-1} H^{-2} \int_{\eta''}^{\eta} d\tau (\tau H)^{-4} \left[ \text{Re } \widehat{M}(\eta, \tau, k) \log \left[ \frac{\eta \tau H^4}{4\mu^2} \right] \right] \\
&\quad \times \Delta_{\phi, BD}^c(\tau, \eta'', k). \quad (2.212)
\end{aligned}$$

We have

$$\begin{aligned}
\widehat{\mathcal{R}}_{\log}(\eta, \eta'', k) = & -\text{Im} \left\{ (1 + ik\eta'') e^{-ik\eta''} \left[ \left( -\frac{14i}{9} + \frac{5k\eta}{9} - \frac{2}{3} [i - k\eta] \log \left[ \frac{\eta\tau H^4}{4\mu^2} \right] \right) \right. \right. \\
& \times e^{-ik\eta} E_1[-2ik\tau] \\
& + \left( -\frac{1}{9k^3\tau^3} + \frac{ik\eta}{9k^3\tau^3} + \frac{k\eta}{4k^2\tau^2} - \frac{1}{k\tau} \right. \\
& + \left( -\frac{1}{3k^3\tau^3} + \frac{ik\eta}{3k^3\tau^3} + \frac{k\eta}{2k^2\tau^2} - \frac{1}{k\tau} \right) \log \left[ \frac{\eta\tau H^4}{4\mu^2} \right] \left. \right) e^{ik\eta} \\
& + \left( -\frac{1}{9k^3\tau^3} - \frac{ik\eta}{9k^3\tau^3} + \frac{2i}{9k^2\tau^2} + \frac{k\eta}{36k^2\tau^2} - \frac{7}{9k\tau} - \frac{5ik\eta}{18k\tau} \right. \\
& + \left( -\frac{1}{3k^3\tau^3} - \frac{ik\eta}{3k^3\tau^3} + \frac{2i}{3k^2\tau^2} - \frac{k\eta}{6k^2\tau^2} - \frac{1}{3k\tau} - \frac{ik\eta}{3k\tau} \right) \\
& \times \log \left[ \frac{\eta\tau H^4}{4\mu^2} \right] \left. \right) e^{ik(2\tau-\eta)} \\
& + \left( \frac{\pi}{3} (-14 - 5ik\eta) - \frac{\pi^2}{6} (i - k\eta) - 2\gamma_E (i - k\eta) \log[-2ik\tau] \right. \\
& - \gamma_E^2 (i - k\eta) + 4k\tau (1 + ik\eta) {}_3F_3 \left[ \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}; 2ik\tau \right] \\
& \left. - (i - k\eta) \log[-2ik\tau]^2 \right) \frac{e^{-ik\eta}}{3} \Bigg|_{\tau=\eta}^{\tau=\eta''} \Bigg\}, \tag{2.213}
\end{aligned}$$

where  ${}_3F_3 \left[ \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}; 2ik\tau \right]$  is a generalized hypergeometric function. On super-Hubble scales we get

$$\begin{aligned}
I_{R_{\log}}(\eta, \eta'', k) \longrightarrow & \frac{2}{3} \log(-2k\eta) \log(-2k\eta'') - \log^2(-2k\eta) + \frac{1}{3} \log^2(-2k\eta'') \\
& - \frac{4}{9} \log(-2k\eta) \frac{(\eta'')^3}{\eta^3} + \frac{2}{9} \log(4k^2\eta\eta'') - \frac{4}{3} \log \left[ \frac{\eta}{\eta''} \right] \log \left[ \frac{H^2}{4k\mu} \right] \\
& + \frac{4}{9} \left( 1 - \frac{(\eta'')^3}{\eta^3} \right) \log \left[ \frac{H^2}{4k\mu} \right] + \frac{2}{27} \left( 1 - \frac{(\eta'')^3}{\eta^3} \right). \tag{2.214}
\end{aligned}$$

## 2.F $I_M$ integrals

In this appendix we will calculate the integrals (2.131), (2.132) and (2.133). Let us start by calculating the following integral

$$\begin{aligned}
I_{\widetilde{M}_I}(\eta, k) = & \frac{1}{6k^3} \left[ 2 + (1 + ik\eta) \left( \log \left[ \frac{H^2}{2k^2} \right] + i\frac{\pi}{2} - \gamma_E \right) \right] \partial_\eta^3 \int_0^\infty dx \frac{e^{-2ix}}{x - k\eta} \\
& + \frac{i}{2k^2} \left[ 2 + (2 + ik\eta) \left( \log \left[ \frac{H^2}{2k^2} \right] + i\frac{\pi}{2} - \gamma_E \right) \right] \partial_\eta^2 \int_0^\infty dx \frac{e^{-2ix}}{x - k\eta} \\
& - \frac{1}{k} \left[ \log \left[ \frac{H^2}{2k^2} \right] + i\frac{\pi}{2} - \gamma_E \right] \partial_\eta \int_0^\infty dx \frac{e^{-2ix}}{x - k\eta} \\
& + \frac{1}{6k^3} [1 + ik\eta] \partial_\eta^3 \int_0^\infty dx \frac{\log[x] e^{-2ix}}{x - k\eta} \\
& - \frac{1}{2k^2} [k\eta - 2i] \partial_\eta^2 \int_0^\infty dx \frac{\log[x] e^{-2ix}}{x - k\eta} - \frac{\partial_\eta}{k} \int_0^\infty dx \frac{\log[x] e^{-2ix}}{x - k\eta}. \quad (2.215)
\end{aligned}$$

In order to proceed, we will make use of the following identities,

$$\int_0^\infty dx \frac{e^{-2ix}}{x - k\eta} = e^{-2ik\eta} E_1[-2ik\eta], \quad (2.216)$$

$$\frac{\partial_\eta}{k} \left[ e^{-2ik\eta} E_1[-2ik\eta] \right] = -2ie^{-2ik\eta} E_1[-2ik\eta] - \frac{1}{k\eta}, \quad (2.217)$$

$$\frac{\partial_\eta^2}{k^2} \left[ e^{-2ik\eta} E_1[-2ik\eta] \right] = -4ie^{-2ik\eta} E_1[-2ik\eta] + \frac{1 + 2ik\eta}{k^2\eta^2}, \quad (2.218)$$

$$\frac{\partial_\eta^3}{k^3} \left[ e^{-2ik\eta} E_1[-2ik\eta] \right] = 8ie^{-2ik\eta} E_1[-2ik\eta] - \frac{2 + 2ik\eta - 4k^2\eta^2}{k^3\eta^3}, \quad (2.219)$$

$$\begin{aligned}
\int_0^\infty dx \frac{\log[x] e^{-2ix}}{x - k\eta} = & -e^{-2ik\eta} \left( \gamma_E + i\frac{\pi}{2} + \log(2) \right) E_1[-2ik\eta] \\
& - e^{-2ik\eta} \frac{1}{2} \left[ \gamma_E^2 + \frac{\pi^2}{6} + 4ik\eta {}_3F_3 \left[ \begin{matrix} 1, 1, 1 \\ 2, 2, 2 \end{matrix}; 2ik\eta \right] + 2\gamma_E \log[-2ik\eta] + \log^2[-2ik\eta] \right], \quad (2.220)
\end{aligned}$$

and

$$\frac{d}{dx} \left[ x {}_3F_3 \left[ \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}; x \right] \right] = -\frac{\gamma_E + \log(-x) + E_1(-x)}{x}. \quad (2.221)$$



Plugging these expressions into (2.215), we find

$$\begin{aligned}
I_{\widetilde{M}_I}(\eta, k) = & \left[ \frac{1}{3k^3\eta^3} + \frac{i}{3k^2\eta^2} + \frac{1}{2k\eta} + \frac{2i}{3} \left( 2\gamma_E - 2 - \log \left[ \frac{H^2}{4k^2} \right] \right) \right. \\
& + \frac{2}{3}k\eta \left( \log \left[ \frac{H^2}{4k^2} \right] - 2\gamma_E \right) \left. e^{-2ik\eta} E_1[-2ik\eta] \right. \\
& + \frac{4\gamma_E - 1 - 2\log \left[ \frac{H^2}{4k^2} \right]}{6k^3\eta^3} + i \frac{3 - 4\gamma_E + 2\log \left[ \frac{H^2}{4k^2} \right]}{6k^2\eta^2} \\
& + \frac{6\gamma_E - 5 - 3\log \left[ \frac{H^2}{4k^2} \right]}{6k\eta} + \frac{i}{3} \left( 2\gamma_E - \log \left[ \frac{H^2}{4k^2} \right] \right) \\
& + \frac{1}{3}(i - k\eta)e^{-2ik\eta} \left( \gamma_E^2 + \frac{\pi^2}{6} + 4ik\eta {}_3F_3 \left[ \begin{matrix} 1, 1, 1 \\ 2, 2, 2 \end{matrix}; 2ik\eta \right] \right. \\
& \left. \left. + 2\gamma_E \log[-2ik\eta] + \log^2[-2ik\eta] \right) \right]. \quad (2.222)
\end{aligned}$$

The next integral we calculate is

$$\begin{aligned}
I_{\widetilde{M}_{II}}(\eta, k) &= \int_0^\infty \frac{dx}{(k\eta - x)^4} (1 - ix) E_1[i2x] [1 - i(k\eta - x)] \\
&= -\frac{1}{2k^3\eta^3} + \frac{i}{6k^2\eta^2} + \frac{1}{6k\eta} - \frac{1}{3} \left[ \frac{1}{k^3\eta^3} + \frac{i}{k^2\eta^2} + \frac{3}{2k\eta} - i \right] e^{-2ik\eta} E_1[-2ik\eta], \quad (2.223)
\end{aligned}$$

where we used the indefinite integrals

$$\int \frac{dx}{x^2} E_1[ax + b] = \frac{1}{b} \left[ ae^{-b} E_1[ax] - \frac{1}{x} (ax + b) E_1[ax + b] \right], \quad (2.224)$$

$$\begin{aligned}
\int \frac{dx}{x^3} E_1[ax + b] &= \frac{ae^{-ax-b}}{2bx} - \frac{a^2(1+b)e^{-b} E_1[ax]}{2b^2} \\
&+ \left( \frac{a^2}{2b^2} - \frac{1}{2x^2} \right) E_1[ax + b], \quad (2.225)
\end{aligned}$$

$$\begin{aligned}
\int \frac{dx}{x^4} E_1[ax + b] &= \frac{1}{3} \frac{a^3}{b^3} e^{-ax-b} \left[ -\frac{b}{ax} + \frac{1}{2} (1 - ax) \frac{b^2}{a^2 x^2} \right] \\
&+ \frac{1}{3} \frac{a^3}{b^3} \left[ 1 + b + \frac{b^2}{2} \right] e^{-b} E_1[ax] \\
&- \frac{1}{3} \left[ \frac{1}{x^3} + \frac{a^3}{b^3} \right] E_1[ax + b]. \quad (2.226)
\end{aligned}$$

Adding up the last two major integrals we find the integral (2.131),

$$\begin{aligned}
I_{\widetilde{M}}(\eta, k) &= -\frac{e^{ik\eta}}{2\lambda H^4} \int_0^\infty dx \left[ \widetilde{M}_I^{++}\left(\frac{x}{k}, k\right) + \widetilde{M}_{II}^{++}\left(\frac{x}{k}, k\right) \right] \\
&\quad \times \left[ \frac{1}{(k\eta - x)^4} - \frac{i}{(k\eta - x)^3} \right] e^{-ix} \\
&= e^{ik\eta} \left[ I_{\widetilde{M}_I} + I_{\widetilde{M}_{II}} \right](\eta, k) \\
&= \frac{2}{3} \left[ i \left( 2\gamma_E - \frac{3}{2} - \log \left[ \frac{H^2}{4k^2} \right] \right) + k\eta \left( \log \left[ \frac{H^2}{4k^2} \right] - 2\gamma_E \right) \right] \\
&\quad \times e^{-ik\eta} E_1[-2ik\eta] \\
&\quad + \frac{1}{3} e^{-ik\eta} \left[ \frac{2\gamma_E - 2 - \log \left[ \frac{H^2}{4k^2} \right]}{k^3 \eta^3} + i \frac{2 - 2\gamma_E + \log \left[ \frac{H^2}{4k^2} \right]}{k^2 \eta^2} \right. \\
&\quad \left. + \frac{6\gamma_E - 4 - 3 \log \left[ \frac{H^2}{4k^2} \right]}{2k\eta} + i \left( 2\gamma_E - \log \left[ \frac{H^2}{4k^2} \right] \right) \right] \\
&\quad + \frac{1}{3} (i - k\eta) e^{-ik\eta} \left( \gamma_E^2 + \frac{\pi^2}{6} + 4ik\eta {}_3F_3 \left[ \begin{matrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{matrix}; 2ik\eta \right] \right. \\
&\quad \left. + 2\gamma_E \log[-2ik\eta] + \log^2[-2ik\eta] \right). \tag{2.227}
\end{aligned}$$

The next integral we calculate is (2.132) and we have

$$\begin{aligned}
I_{\widehat{M}}(\eta, k) &= -\frac{e^{ik\eta}}{2\lambda H^4} \int_0^\infty dx \widehat{M}^{++}\left(\frac{x}{k}, k\right) \frac{1 - i(k\eta - x)}{(k\eta - x)^4} e^{-ix} \\
&= e^{ik\eta} \left( \frac{1}{3k^3 \eta^3} - \frac{i}{3k^2 \eta^2} + \frac{1}{2k\eta} - \frac{i}{3} \right) \\
&\quad + \frac{2}{3} (i - k\eta) e^{-ik\eta} E_1[-2ik\eta]. \tag{2.228}
\end{aligned}$$

The last integral we calculate in this appendix is (2.133), where we use similar techniques as above,

$$\begin{aligned}
I_{M_{\log}}(\eta, k) &= -\frac{e^{ik\eta}}{2\lambda H^4} \int_0^\infty dx \log \left[ \frac{\eta(k\eta - x)H^4}{4k\mu^2} \right] \widehat{M}^{++} \left( \frac{x}{k}, k \right) \frac{1 - i(k\eta - x)}{(k\eta - x)^4} e^{-ix} \\
&= e^{ik\eta} \left\{ \left[ \frac{1}{3k^3\eta^3} - \frac{i}{3k^2\eta^2} + \frac{1}{2k\eta} + \frac{i}{3} + \frac{2}{3}(i - k\eta)e^{-2ik\eta}E_1[-2ik\eta] \right] \log \left[ \frac{H^4\eta^2}{4\mu^2} \right] \right. \\
&\quad + \frac{1}{9k^3\eta^3} - i\frac{1}{9k^2\eta^2} + \frac{3}{4k\eta} + i\frac{5}{18} + \frac{e^{-2ik\eta}}{9}(14i - 5k\eta)E_1[-2ik\eta] \\
&\quad + \frac{e^{-2ik\eta}}{3}(i - k\eta) \left( \gamma_E^2 + \frac{\pi^2}{6} + 4ik\eta {}_3F_3 \left[ \begin{matrix} 1, 1, 1 \\ 2, 2, 2 \end{matrix}; 2ik\eta \right] \right. \\
&\quad \left. \left. + 2\gamma_E \log[-2ik\eta] + \log^2[-2ik\eta] \right) \right\}. \quad (2.229)
\end{aligned}$$

We are also interested for super-Hubble limit of the in the integrals we calculated in this appendix. However, let us multiply them with  $(1 + ik\eta'')e^{-ik\eta''}$  before, since these are the expressions that enter the calculation via (2.127). Thus, on super-Hubble scales we have

$$\begin{aligned}
\text{Im} \left[ (1 + ik\eta'')e^{-ik\eta''} I_{\widehat{M}}(\eta, k) \right] &\longrightarrow -\frac{1}{3} \log^2(-2k\eta) \\
&\quad + \left( \frac{2}{3}\gamma_E - 1 - \frac{2}{3} \log \left[ \frac{H^2}{4k^2} \right] \right) \log(-2k\eta) \\
&\quad + \frac{2}{9} \frac{(\eta'')^3}{\eta^3} \left( \gamma_E - 1 - \log \left[ \frac{H}{2k} \right] \right) \\
&\quad + \frac{8}{9} - \frac{26}{9}\gamma_E + \gamma_E^2 + \frac{\pi^2}{36} + \frac{1}{9} (17 - 12\gamma_E) \log \left[ \frac{H}{2k} \right], \quad (2.230)
\end{aligned}$$

$$\text{Im} \left[ (1 + ik\eta'')e^{-ik\eta''} I_{\widehat{M}}(\eta, k) \right] \longrightarrow \frac{2}{3} \log(-2k\eta) + \frac{1}{18} \left( 12\gamma_E - 17 + 2 \frac{(\eta'')^3}{\eta^3} \right), \quad (2.231)$$

$$\begin{aligned}
\text{Im} \left[ (1 + ik\eta'')e^{-ik\eta''} I_{M_{\log}}(\eta, k) \right] &\longrightarrow \log^2(-2k\eta) \\
&\quad + \frac{1}{3} \log(-2k\eta) \left( 4 \log \left[ \frac{H^2}{4k\mu} \right] + 2\gamma_E - 1 + \frac{2}{3} \frac{(\eta'')^3}{\eta^3} \right) \\
&\quad + \frac{1}{27} \frac{(\eta'')^3}{\eta^3} \left( 1 + 6 \log \left[ \frac{H^2}{4k\mu} \right] \right) \\
&\quad - \frac{115}{108} - \frac{14}{9}\gamma_E - \frac{1}{3}\gamma_E^2 + \frac{\pi^2}{36} - \frac{1}{9} (17 - 12\gamma_E) \log \left[ \frac{H^2}{4k\mu} \right]. \quad (2.232)
\end{aligned}$$



## Chapter 3

# Scalar field dark matter in hybrid approach

We develop a hybrid formalism suitable for modeling scalar field dark matter, in which the phase-space distribution associated to the real scalar field is modeled by statistical equal-time two-point functions and gravity is treated by two stochastic gravitational fields in the longitudinal gauge (in this work we neglect vector and tensor gravitational perturbations). Inspired by the commonly used Newtonian Vlasov-Poisson system, we first identify a suitable combination of equal-time two-point functions that defines the phase-space distribution associated to the scalar field and then derive both a kinetic equation that contains relativistic scalar matter corrections as well as linear gravitational scalar field equations whose sources can be expressed in terms of a momentum integral over the phase-space distribution function. Our treatment generalizes the commonly used classical scalar field formalism, in that it allows for modeling of (dynamically generated) vorticity and perturbations in anisotropic stresses of the scalar field. It also allows for a systematic inclusion of relativistic and higher order corrections that may be used to distinguish different dark matter scenarios.

### 3.1 Overview

The standard model of cosmology attributes roughly one third of the universes energy to dark matter, a particle or field whose nature is mostly unknown except for the effect that it interacts with gravity [84]. There has been success in studying large-scale structures of the universe by modeling dark matter as non-relativistic particles that can be described by a pressureless fluid. Linear perturbation theory can be used up to the scale of non-linearity  $k > k_{nl} \sim 0.3 \text{ Mpc}^{-1}$  to predict the distribution of galaxy clusters and the perturbation theory may be used to study higher-order effects [24]. On the other hand, interest has recently [26, 28] been shown in the study of axion or fuzzy dark matter [17–22] which in the end is a real scalar field with a certain mass range minimally coupled to gravity with self-interaction terms playing a minor role. Common to most minimal scalar field dark models is that the mass is much bigger than the Hubble rate. It has been studied in linear perturbation theory in different gauges [85, 86]. The non-relativistic limit of the Klein-Gordon equation of a classical scalar field yields the Schrödinger equation. By defining energy-density

and fluid velocity via the so-called Madelung transformation, one can reproduce non-relativistic, non-linear hydrodynamic equations in FLRW-space-time for real [26] and complex [87] classical scalar field theories. From a quantum field theory point of view, we think of the classical fields entering these models as Bose condensates that are obtained by coherent quantum states whose one-point function defines the classical field. In view of the semi-classical Einstein equations, it is natural to extend the analysis to the statistical limit of the full two-point functions where the additional degrees of freedom can account for all features of a fluid of massive collisionless particles in the classical limit. In fact, it is the expectation value of squares of (non-composite) field operators at equal times that couples to the Einstein tensor in semi-classical gravity. Thus, we should think of these two-point functions as building blocks of the fluid. In the classical limit these equal-time two-point functions reduce to the statistical or Hadamard two-point function. It is a priori not clear why they should reduce only to the product of classical fields, i.e. expectation values of one field operator insertion. Despite that one has to argue on how such condensates are generated in a quadratic potential in late-time cosmology, working only with classical fields cuts down degrees of freedom that might be important for cold dark matter models. We underpin the later point by deriving that statistical two-point functions are in a gradient approximation related to phase-space densities whose position and momentum dependence is initially generic by means of the connected piece of the two-point function, i.e. the part which does not reduce to a product of expectation values. This makes it clear that they contain more features of the scalar field fluid than the one-point functions or classical fields are able to describe. From the perspective of a classical particle that is coupled to gravity, the studies of phase-space dynamics are inevitable when a single-stream fluid *Ansatz* breaks down due to what is called shell-crossing. The kinetic theory underlying dark matter is summarized in the Vlasov equation [88], which represents a phase-space description that does not break down in the non-linear regime since there is no shell-crossing in phase-space. Phase-space densities and the corresponding Vlasov equation have previously been derived by using the Wigner transformation of the non-relativistic Schrödinger-Poisson system [30, 89, 90] and also for a relativistic scalar field [91]. Once again, only one-point functions have been considered and the richness of the connected part of the statistical two-point function is lost.

In this chapter, we want to put forward the discussion about real scalar field dark matter from the perspective of phase-space dynamics which is according to us still incomplete at the moment. We show that instead of using classical fields, the more natural objects are statistical two-point functions which via the additional space-time-dependence can be used to derive a momentum-dependence as it occurs in kinetic theory. Integrating out this momentum dependence still leaves us with a non-homogeneous space-time dependence that is induced by the stochastic gravitational fields. Furthermore, two-point functions naturally arise in quantum field theory, whose broad apparatus might even be used to simplify non-linear calculations once a mapping to observables in cosmology is established as we do in this chapter. Defining phase-space densities from

two-point functions is a known business in Minkowski-space [92], it is a generalization of non-relativistic Wigner transformation [93] to special relativity. The idea is to change the coordinates to a collective and difference coordinate and to Fourier-transform with respect to the difference coordinate to obtain a momentum dependence. However, there are few publications on a generalization of this idea to curved space-times. Two independent works in [94] and in [95] postulate off-shell curved space-time Wigner transformations in a mathematical complex expansion by using geodesics and Riemann normal coordinates, respectively. The transformation is done with respect to a coordinate-independent physical distance between the two points on the space-time manifold. A similar approach has again been discussed by [96]. However, in this chapter we make use of a simpler transformation that allows us to write down exact equations in a one-step transformation. The idea is to think of the two-point function as an object that depends on one point of the space-time manifold and on another point that belongs to the tangent space over that point on the space-time manifold. Consequently, the momentum is a variable of the cotangent space over that point on the space-time manifold. This approach was used in [97] to define particle densities in an unperturbed FLRW-universe where the authors started with off-shell equations and projected them onto on-shell quantities via integration. A similar approach was already proposed in [98] for general space-times and its implication were studied for Fermionic systems, however, no on-shell projection was discussed. As far as we know, none of the previous works makes an attempt to clearly derive a set of equation that reduce in the classical limit to the Newtonian on-shell Vlasov-Poisson system that is used in kinetic theory of dark matter. We consider this as an important gap in the theory of scalar field or axion dark matter and it is the task of our chapter to close it. Once we have a clear pictures on how the dynamics of dark matter is embedded in quantum field theory on curved space-time, we might discover more ways to calculate cosmological quantities in the non-linear regime.

We call the approach in this chapter a hybrid approach for the reason that we start in principle from a quantum field theory for the real scalar field but do not properly integrate out the gravitational constraint. Thus, we approximate the self-interaction terms that would be generated by this procedure in terms of the gravitational potentials treated as external sources which are by means of the semi-classical Einstein equations related to the statistical two-point functions themselves.

The chapter is structured as follows. We start by deriving a dynamical system of on-shell two-point functions that is converted from a pure space-time dependence to a dependence on phase-space variables. We specialize to a scalar linearized longitudinal gauge without vector perturbations and without gravitons. But we keep the gravitational slip (defined as the difference between the two gravitational potentials), which induces higher-order corrections in the fluid dynamics of scalar field dark matter that have not been captured so far in the one-point function approach. We derive Einstein's equations in that gauge and rewrite the energy-momentum tensor as momentum integrals over two-point functions. This allows us in turn to define scalar hydrodynamic variables like

energy density, rest-mass density and pressure. However, hydrodynamic variables containing the four-velocity can only be defined as composite operators leaving space for anisotropy. We then consider a gradient expansion by introducing a variety of perturbation parameters on top of the linearization in the gravitational potentials and show that we indeed recover the generalization of the continuity and Euler equation in the FLRW-space-time. We also use the energy-momentum to identify even and odd phase-space densities which brings us finally to the derivation of the on-shell Vlasov equation by making use of the unintegrated dynamical equations for the statistical two-point functions.

In the hybrid approach that we put forward in this chapter we have two types of two-point functions involved. We have a statistical two-point function (also called Hadamard function) which consists of the expectation value of anti-commutators of the scalar field operators evaluated with respect to some initial density matrix. In our hybrid approach, this initial density matrix is taken to depend on the stochastic gravitational potentials as they appear in the context of cosmological perturbation theory. This dependence makes the Hadamard two-point function itself a stochastic quantity which is also inhomogeneous in space. Thus, we can now take the expectation value of the product of Hadamard two-point functions with respect to the stochastic initial conditions and integrated versions thereof correspond for example to density-density correlators in cosmological perturbation theory. Unless, we do not make a clear distinction, we mean statistical or Hadamard two-point functions whenever we speak generically of two-point functions.



## 3.2 Phase-space distribution from 2-point functions

### 3.2.1 Microscopic theory in the operator formalism

Let us start by writing down the microscopic theory that captures the fundamental dynamics. It is a real scalar quantum field theory that indirectly self-interacts via a minimal and semi-classical coupling to gravity.

We work in units where  $c = 1$ , keeping  $\hbar$ , and write down the action for the system

$$S[\phi, g_{\mu\nu}] = S_g[g_{\mu\nu}] + S_m[\phi, g_{\mu\nu}] , \quad (3.1)$$

where

$$S_m[\phi, g_{\mu\nu}] = -\frac{1}{2} \int d^D x \sqrt{-g} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{m^2}{\hbar^2} \phi^2 \right] , \quad (3.2)$$

is the matter action and

$$S_g[g_{\mu\nu}] = \frac{M_P^2}{2\hbar} \int d^D x \sqrt{-g} R , \quad (3.3)$$

is the classical gravity action with  $R$  being the Ricci scalar. We will work with a metric for linearized scalar perturbations in Newtonian (longitudinal) gauge which is specified by the two gravitational potentials  $\Phi_G$  and  $\Psi_G$ ,

$$\begin{aligned} g_{00}(\eta, x^i) &= -a^2(\eta) [1 + 2\Phi_G(\eta, x^i)] , \\ g_{ij}(\eta, x^i) &= a^2(\eta) \delta_{ij} [1 - 2\Psi_G(\eta, x^i)] . \end{aligned} \quad (3.4)$$

We drop all quadratic terms  $\Phi_G^2$ ,  $\Psi_G^2$ ,  $\Phi_G \Psi_G$  as higher-order corrections from the very beginning. We also drop vector and tensorial perturbations for simplicity although in general we expect them to be generated due to non-linear evolution. Inflation generates gravitational potentials that can be to a good approximation treated as classical stochastic fields that are at large redshifts approximated by a Gaussian distribution. We note that the metric (3.4) is particularly useful to study the Newtonian limit of general relativity. It generalizes the longitudinal metric that has been used in the classical real scalar field theory approach to dark matter in [26] by allowing for a non-zero gravitational slip that we define in  $D$  space-time dimensions as

$$\text{gravitational slip} \equiv \Phi_G - (D - 3)\Psi_G . \quad (3.5)$$

The quantum theory in the operator formalism is specified by the time-evolution or Hamilton operator  $\hat{H}$  which is a functional of the field operator  $\hat{\phi}$  and its canonical momentum field operator  $\hat{\Pi}$ . We work in the Heisenberg picture and the canonical momentum operator evaluates to

$$\hat{\Pi}(x) = a^{(D-2)}(\eta) [1 - \Phi_G(\eta, x^i) - (D - 1)\Psi_G(\eta, x^i)] \hat{\phi}'(\eta, x^i) , \quad (3.6)$$

where

$$x \equiv (\eta, x^i) \quad \text{and} \quad (\cdot)' \equiv \frac{\partial}{\partial \eta}(\cdot). \quad (3.7)$$

The field operators obey equal-time commutation relations

$$\begin{aligned} [\hat{\phi}(\eta, x^i), \hat{\Pi}(\eta, \tilde{x}^i)] &= i\hbar \delta^{D-1}(x^i - \tilde{x}^i), \\ [\hat{\phi}(\eta, x^i), \hat{\phi}(\eta, \tilde{x}^i)] &= 0, \\ [\hat{\Pi}(\eta, x^i), \hat{\Pi}(\eta, \tilde{x}^i)] &= 0. \end{aligned} \quad (3.8)$$

Since we are working in semi-classical gravity the Hamiltonian  $\hat{H}$  additionally depends on the gravitational potentials  $\Phi_G, \Psi_G$  that act as external, stochastic fields,

$$\begin{aligned} \hat{H}(\hat{\phi}, \hat{\Pi}, g_{\mu\nu}) &\equiv \int d^{D-1}x \hat{\Pi} \hat{\phi}' - \int d^{D-1}x \hat{\mathcal{L}}_m[\hat{\phi}, \hat{\phi}', g_{\mu\nu}] \\ &= -\frac{1}{2} \int d^{D-1}x \frac{\hat{\Pi}^2}{\sqrt{-g}g^{00}} \\ &\quad + \frac{1}{2} \int d^{D-1}x \sqrt{-g} \left[ g^{ij} \partial_i \hat{\phi} \partial_j \hat{\phi} + \frac{m^2}{\hbar^2} \hat{\phi}^2 \right]. \end{aligned} \quad (3.9)$$

Using the Heisenberg equations we find the following time evolution of the canonical operators,

$$\hat{\phi}'(x) = a^{-(D-2)}(\eta) [1 + \Phi_G(x) + (D-1)\Psi_G(x)] \hat{\Pi}(x), \quad (3.10)$$

$$\begin{aligned} \hat{\Pi}'(x) &= a^{D-2}(\eta) \delta^{ij} \partial_i \left\{ [1 + \Phi_G(x) - (D-3)\Psi_G(x)] \partial_j \hat{\phi}(x) \right\} \\ &\quad - \frac{m^2}{\hbar^2} a^D(\eta) [1 + \Phi_G(x) - (D-1)\Psi_G(x)] \hat{\phi}(x). \end{aligned} \quad (3.11)$$

We stress that the constraint fields  $\Phi_G, \Psi_G$  do not evolve independently. Thus, we are not fully fixing the gauge by integrating out the constraint fields. We do this in order to make the connection to the Einstein-Vlasov system clearer. This means, at the same time that we are approximating scalar interactions that are induced via gravity by stochastic two-point functions. However, examining the fully gauged fixed theory at the quantum level is discussed in chapter 5.

### 3.2.2 Why a quantum field formalism for a classical problem?

One might object that the quantum field theory framework we presented so far is a completely exaggerated tool to describe effects that arise in the classical treatment of late-time cosmology. However, this description has on the one hand the advantage of being based on a fundamental theory which permits a Lagrangian description and in which in our simple model contains one parameter, the scalar field mass  $m$ . On the other hand, it is related to the typical classical non-relativistic particle description by imposing conditions that approximate a classical stochastic rather than a quantum description as well as a gradient expansion that contains relativistic corrections. We note that quantum

path integrals generalize classical stochastic path integrals where the quantum commutators (3.8) are replaced by Poisson brackets [99]. For us it is important to inherit the stochastic correlations of two-point functions from the quantum field theory framework since late-time cosmology is a classical stochastic theory whose stochastic seeds are given by the gravitational potentials. That quantum effects are potentially present in this approach is a completely negligible add-on rather than a crucial ingredient. Still, this perspective has the advantage that we always keep the bridge to non-equilibrium quantum field techniques as for example the Schwinger-Keldysh formalism [100, 101] that might be useful when studying non-linear evolution. Our formalism is a *hybrid* formalism in the sense that we use a mixture of 2-PI formalism [79] for the scalar field matter and do not fully fix the gravitational gauge in the sense that we do not fully solve (gravitational) constraints of the theory, but instead we leave the gravitational potentials as external stochastic sources (keeping in mind that they are eventually fixed by the linear Einstein equations). A fully gauge-fixed 2PI frame is developed in chapter 5.

The condition of being in the classical stochastic regime of a quantum field theory rather than in the quantum regime can be formulated as follows: the (classical) correlators that contain anti-commutators (and no time ordering) are much larger than the quantum correlators defined in terms of anti-commutators with or without time ordering (example of which include the causal or spectral two-point function  $\langle [\hat{\phi}(x), \hat{\phi}(\tilde{x})] \rangle$ , retarded and advanced propagators, etc.). We therefore assume that our two-point functions obey,

$$2F(x; \tilde{x}) \equiv \langle \{ \hat{\phi}(x), \hat{\phi}(\tilde{x}) \} \rangle \gg | \langle [\hat{\phi}(x), \hat{\phi}(\tilde{x})] \rangle |, \quad (3.12)$$

where  $\{ \hat{\phi}(x), \hat{\phi}(\tilde{x}) \}$  and  $[ \hat{\phi}(x), \hat{\phi}(\tilde{x}) ]$  denote anti-commutator and commutator operation, respectively. Rigorously speaking, the classicality condition (3.12) is never satisfied for all space-time points. By assuming (3.12) we are saying that we restrict ourselves to those space-time points where (3.12) is amply satisfied. Rather than rigorously going through a procedure that would achieve that in practice, here we just sketch how such a procedure can be exacted. In the case of interest for dark matter, the condition (3.12) will be met for sufficiently large spatial separations. One can make use of a suitable window (smearing) function, which projects out of the full two-point function its classical part. When the complementary ('quantum') part of the two-point function is integrated out, one will generate local geometric divergent contributions (that can be renormalized by adding suitable local geometric counterterms). Apart from renormalizing the Newton and cosmological constant to its observable values, the remaining geometric terms, which consist of quadratic and higher-order local gravitational curvature contractions, have a negligible effect on the evolution of late time two-point functions, and we neglect them here. The remaining infrared parts of the two-point functions satisfy the classicality condition (3.12).

To get a better feeling on what classicality really means, it is helpful to assume adiabaticity with respect to gradient expansion (discussed in more detail below), in which case one can perform a Wigner transform with respect to the relative spatial coordinate,  $x^i - \tilde{x}^i$ , resulting in the statistical two-point function,

$F(X^i, p_i, t, t')$ ,  $X^i \equiv (x^i + \tilde{x}^i)/2$ . When  $F(X^i, p_i, t, t')[\partial_t \partial_{t'} F(X^i, p_i, t, t')]|_{t'=t} \gg (\hbar/2)^2$  is satisfied, then one is in the classical regime.<sup>1</sup> More concretely, in the case at study we expect the classicality condition (3.12) to be satisfied for two-point functions that are smeared on distances larger than the co-moving distance corresponding to the end of inflation, which is of the order  $\sim 1\text{m}$ . Since the two-point functions we use to describe dark matter on the scales of large-scale structures, they are in a deeply classical regime and the condition (3.12) is royally satisfied.

The second important condition to get into the regime of non-relativistic particles is related to validity of the gradient expansion. Roughly speaking, the expansion is valid when the following two conditions are met,

$$\|\hbar \partial_{\vec{X}} \cdot \partial_{\vec{p}}\| \ll 1, \quad \|\hbar \partial_\eta \partial_E\| \ll 1, \quad (3.13)$$

where the norm is to be understood in the sense that one derivative acts on one test function (such as a two-point function) and the other on another object (such as a gravitational potential).<sup>2</sup> Assuming that two-point functions vary on scales of momentum (energy) given by the momentum (energy), i.e.  $\hbar \partial_E \sim \hbar/E \sim \hbar/(mc^2) \sim \lambda_C/c$ , and  $\hbar \partial_{\vec{p}} \sim \hbar/|\vec{p}| \sim \hbar/(mv) \sim \lambda_{dB}$ , where  $\lambda_C$  and  $\lambda_{dB}$  denote the Compton and de Broglie wavelength, respectively, the conditions (3.13) can be recast as,

$$L \gg \lambda_{dB}, \quad T \gg \frac{\lambda_C}{c}, \quad (3.14)$$

where  $L$  and  $T$  represent the characteristic length and time scales over which gravitational potentials or two-point functions vary.  $L$  can be as small as the smallest large-scale structures we are interested in (which is of the order  $\sim \text{kpc}$ ) which implies that  $T$  must be much larger than the time light crosses about one mega-parsec which is about a million years (this estimate follows from the observation that  $\lambda_{dB} \sim 10^3 \lambda_C$  as  $v \sim (10^{-3} - 10^{-2})c$ ). To get a feeling on how good that approximation is, note that the inequalities are amply satisfied for a dark matter whose mass is of the electroweak scale  $\sim 10^2 \text{ GeV}$ . However, when one considers ultra-light scalar such as in references [26, 28], the scalar mass is of the order  $m \sim 10^{-22} - 10^{-24} \text{ eV}$ , the Compton and de Broglie wave lengths are  $\lambda_C \sim 10^{-4} - 10^{-2} \text{ kpc}$ ,  $\lambda_{dB} \sim 10^{-2} - 10 \text{ kpc}$ , the quantities in (3.14) can become comparable for smallest scales of interest, and hence one expects significant higher order gradient corrections. One is typically interested in modeling dark matter at an accuracy better than 1% (as it will be tested by upcoming observations), which then defines the order in gradient expansion that one ought to keep. The corrections of the gradient expansion can be subsumed

<sup>1</sup>An alternative (and stricter) criterion for classicality of a state is given by the von Neumann entropy of the Gaussian part of the density matrix being much larger than one [54, 102].

<sup>2</sup>The validity of this expansion depends on the initial density matrix which ought to be classical enough. The initial density matrix can for example be taken to be Gaussian, containing initial one-point functions and connected two-point functions. In particular, without any coarse-graining only the connected part of the two-point function can satisfy the conditions of gradient expansion.

by the following perturbation parameters

$$\varepsilon_h \sim \left\{ \varepsilon_k \sim \hbar \frac{\partial_X}{ma}, \varepsilon_{k/p} \sim \hbar \frac{\partial_X}{p} \sim \hbar \partial_X \partial p, \varepsilon_H \sim \hbar \frac{\mathcal{H}}{ma}, \varepsilon_{\partial\eta} \sim \hbar \frac{\partial_\eta}{ma} \right\}. \quad (3.15)$$

Next, there are relativistic corrections due to the relativistic nature of dark matter. The fact that the energy is not equal to the rest energy we can fully capture in our formalism as long as we keep on-shell energy roughly speaking equal to its quasi-particle value,  $E = \sqrt{m^2 + p_{ph}^2}$ , where  $p_{ph}^2 = g_{ij}p^i p^j$  denotes the physical momentum squared (for simplicity in here we do not include all of the metric corrections). To study these corrections one can systematically include them order by order if,

$$\varepsilon_p \sim \frac{p_{\text{com}}}{ma} \ll 1 \quad (3.16)$$

where  $p_{\text{com}}$  denotes the comoving momentum today. These corrections occur e.g. as relativistic corrections to the energy-momentum tensor whose most important components are energy density and pressure which source the generalized Poisson-like equations for the gravitational potentials.

Furthermore, there are relativistic corrections induced by the relativistic nature of the scalar field Klein-Gordon equations, and these appear as higher order time derivatives in the Vlasov (or collisionless Boltzmann) equation. These corrections are small if the second condition in equations (3.13) and (3.14) is met.

Next, there are relativistic corrections arising from general relativity being different from Newton's gravity. These corrections occur as higher time derivative corrections to gravitational potentials and as the corrections induced by the Universe expansion. The latter are suppressed by the Hubble rate  $H$  and they are small if,

$$T \ll \frac{a}{\mathcal{H}}. \quad (3.17)$$

Furthermore, since general relativity has more degrees of freedom than Newton's gravity, there are general relativistic corrections expressed as a non-vanishing gravitational slip. Finally, we expect that as a result of non-linear interactions between matter and gravity, gravitational vector and tensor perturbations will be (dynamically) generated (even if they are not present at the initial time). In this chapter we neglect these types of perturbations, but they can be included in our formalism by including them in the *Ansatz* for the metric tensor, as it is done in chapter 4.

Of course, there are also higher order gravitational perturbations. However, since on the large scales we are interested in gravitational potentials do not grow much beyond their initial value,

$$\varepsilon_g^2 \sim \Phi_G, \Psi_G \sim 10^{-5} \ll 1, \quad (3.18)$$

we can safely neglect terms of the form  $\Phi_G^2$  and  $\Psi_G^2$ ; higher order vector and tensor perturbations can be also neglected since vectors and tensors remain smaller than gravitational scalars throughout the evolution. We will also encounter the

parameter

$$\varepsilon_{\text{H/k}} \sim \mathcal{H} \partial_X^{-1}, \quad (3.19)$$

that controls whether we are on sub- or super-Hubble scales.

### 3.2.3 Phase-space distributions from Wigner transformation

The concept of the Wigner transformation (see for example [92] or [103]) was introduced to extract phase-space distributions and its Boltzmann equation from particle wave functions, then generalized to field theory in Minkowski space-time and even later to field theory in curved space-time to yield a covariant Boltzmann or Vlasov equation. However, the generalization of the Wigner transformation to arbitrary curved space-times must still be considered as an active research field since there are as far as we know merely four major contributions to this area [94–96, 98] which agree only in the limit where  $\hbar$  goes to zero. Apart from the proposal by [98], all approaches are based on perturbative expressions. On the other hand, all of these papers cover almost entirely off-shell phase-space distribution  $f(X^\mu, p_\nu)$  in the sense that the momentum conjugate to the time difference  $\Delta\eta$  in the two-point function,  $p_0$ , is still an independent variable that needs to be put on shell by integrating it out since the starting point for the Wigner transformation is a non-equal-time two-point function,  $\Delta\eta \neq 0$ . As it was to our knowledge first pointed out by [104] for electrodynamics in Minkowski space-times, going on-shell requires more than one moment in  $p_0$  space, i.e.  $\int \frac{dp_0}{2\pi\hbar} p_0^n f(X^\mu, p_\nu)$ . This approach of taking several moments and relating them has been applied to homogeneous cosmological backgrounds by [97] to define a particle number density. Our goal is to obtain candidates for on-shell phase-space distributions  $f(X^\mu, p_i)$  and their dynamics for the non-homogeneous linearized longitudinal metric (3.4) we provided in the beginning. We seek them by computing the dynamics of two-point functions on-shell or at equal-times in the operator formalism and then performing a  $(D-1)$ -dimensional Wigner transformation of these equal-time two-point functions. Apart from our interest in scalar field dark matter, we want to use the linearized longitudinal metric as a guideline to gain some intuition for on-shell Wigner transformation in curved space-time and generalize it in future work to arbitrary non-perturbative metrics.

Let us start by looking at the four equal-time two-point functions whose combination might provide candidates for phase-space distributions after a  $(D - 1)$ -dimensional Wigner transformation

$$\begin{aligned} F_{00}(\eta, x^i, \tilde{x}^i) &\equiv \langle \hat{\phi}(\eta, x^i) \hat{\phi}(\eta, \tilde{x}^i) \rangle \\ &= \text{Tr} \left[ \hat{\rho}_{\text{ini}} [\hat{\phi}, \hat{\Pi}, \Phi_G, \Psi_G] \hat{\phi}(\eta, x^i) \hat{\phi}(\eta, \tilde{x}^i) \right], \end{aligned} \quad (3.20)$$

$$\begin{aligned} F_{10}(\eta, x^i, \tilde{x}^i) &\equiv \langle \hat{\Pi}(\eta, x^i) \hat{\phi}(\eta, \tilde{x}^i) \rangle \\ &= \text{Tr} \left[ \hat{\rho}_{\text{ini}} [\hat{\phi}, \hat{\Pi}, \Phi_G, \Psi_G] \hat{\Pi}(\eta, x^i) \hat{\phi}(\eta, \tilde{x}^i) \right], \end{aligned} \quad (3.21)$$

$$\begin{aligned} F_{01}(\eta, x^i, \tilde{x}^i) &\equiv \langle \hat{\phi}(\eta, x^i) \hat{\Pi}(\eta, \tilde{x}^i) \rangle \\ &= \text{Tr} \left[ \hat{\rho}_{\text{ini}} [\hat{\phi}, \hat{\Pi}, \Phi_G, \Psi_G] \hat{\phi}(\eta, x^i) \hat{\Pi}(\eta, \tilde{x}^i) \right], \end{aligned} \quad (3.22)$$

$$\begin{aligned} F_{11}(\eta, x^i, \tilde{x}^i) &\equiv \langle \hat{\Pi}(\eta, x^i) \hat{\Pi}(\eta, \tilde{x}^i) \rangle \\ &= \text{Tr} \left[ \hat{\rho}_{\text{ini}} [\hat{\phi}, \hat{\Pi}, \Phi_G, \Psi_G] \hat{\Pi}(\eta, x^i) \hat{\Pi}(\eta, \tilde{x}^i) \right], \end{aligned} \quad (3.23)$$

where the expectation values are taken with respect to some initial density matrix  $\hat{\rho}_{\text{ini}}$  which functionally depends on the operators  $\hat{\phi}$  and  $\hat{\Pi}$  at some initial time  $\eta_{\text{ini}}$ . Note that in the context of cosmology the initial density matrix  $\hat{\rho}_{\text{ini}}$  depends also functionally on the stochastic gravitational potentials  $\Phi_G, \Psi_G$  at this initial time  $\eta_{\text{ini}}$ , and in its general form it allows for implementation of effects of coherent states, squeezing and state mixing. In this way, two-point functions can be stochastic quantities,

$$\begin{aligned} F_{00}(\eta, x^i, \tilde{x}^i) &\neq \langle F_{00} \rangle_{(\Phi_G, \Psi_G)}(\eta, x^i, \tilde{x}^i) \equiv \left\langle \langle \hat{\phi}(\eta, x^i) \hat{\phi}(\eta, \tilde{x}^i) \rangle_{\hat{\rho}} \right\rangle_{(\Phi_G, \Psi_G)} \\ &\equiv \int \mathcal{D}\Phi_G \mathcal{D}\Psi_G \mathcal{P}[\Phi_G, \Psi_G] \text{Tr} \left[ \hat{\rho}_{\text{ini}} [\hat{\phi}, \hat{\Pi}, \Phi_G, \Psi_G] \hat{\phi}(\eta, x^i) \hat{\phi}(\eta, \tilde{x}^i) \right], \end{aligned} \quad (3.24)$$

where  $\mathcal{P}$  is a probability distribution for the gravitational potentials. The reason to introduce this formalism lies in its application to cosmology in the sense that we want to bridge a gap from semi-classical quantum field theory to cosmological perturbation theory.<sup>3</sup> Thus,  $\Phi_G$  and  $\Psi_G$  are stochastic, homogeneously distributed fields that evolve into non-Gaussian fields due to the evolution of large-scale structures. We want to think of this model more as a conceptional test case on how to relate full quantum microscopic theories to models in cosmological perturbation theory as for example the cold dark matter model. Once we find that this is a fruitful *Ansatz*, we will provide generalizations for arbitrary metrics and even wave the semi-classical approach by integrating out the gravitational constraint fields  $\Phi_G, \Psi_G$ , which at the moment act as approximate self-interactions of the scalar field theory since the Einstein equations constrain them to be related to the scalar field's two-point functions. We also want to point out that the reducible and connected pieces of the two-point functions

<sup>3</sup>We remark, that this might be closely related to the stochastic gravity framework proposed in [105] although we did not investigate this further. For us, the stochasticity of two-point functions is more an ad-hoc *Ansatz* that turns out to be very convenient in relation to cosmological perturbation theory.

entering (3.20) to (3.23) do not decouple in general which is due to the gravitational fields since the semi-classical Einstein equation will constrain them to be related to both one-point functions as well as the connected matter field two-point function. On the other hand, this makes clear that setting initially either the one-point functions of the matter field or connected parts of the matter field two-point function to zero removes them from (3.20) to (3.23) for all times. Before we turn to the Wigner transformation itself, we write down the dynamics of the equal-time two-point functions (3.20) to (3.23). Let us therefore define the differential operator

$$\begin{aligned} \mathcal{D}(\eta, x^i) \equiv & a^{D-2}(\eta) \delta^{ij} \left[ [\partial_i \Phi_G](\eta, x^i) - (D-3) [\partial_i \Psi_G](\eta, x^i) \right] \frac{\partial}{\partial x^j} \\ & + a^{D-2}(\eta) [1 + \Phi_G(\eta, x^i) - (D-3) \Psi_G(\eta, x^i)] \Delta^x \\ & - \frac{m^2}{\hbar^2} a^D(\eta) [1 + \Phi_G(\eta, x^i) - (D-1) \Psi_G(\eta, x^i)] , \end{aligned} \quad (3.25)$$

where we used the Laplace operator  $\Delta$  which on conformally flat cosmological spaces equals  $\delta^{ij} \partial_i \partial_j$ . We also define the following function as a shorthand

$$h(\eta, x^i) \equiv a^{-(D-2)}(\eta) [1 + \Phi_G(\eta, x^i) + (D+1) \Psi_G(\eta, x^i)] . \quad (3.26)$$

Then, based on the Hamilton's equation for the canonical operators (3.10) and (3.11), we get the following system of equations

$$F'_{00}(\eta, x^i, \tilde{x}^i) = h(\eta, x^i) F_{10}(\eta, x^i, \tilde{x}^i) + F_{01}(\eta, x^i, \tilde{x}^i) h(\eta, \tilde{x}^i) , \quad (3.27)$$

$$F'_{10}(\eta, x^i, \tilde{x}^i) = \mathcal{D}(\eta, x^i) F_{00}(\eta, x^i, \tilde{x}^i) + F_{11}(\eta, x^i, \tilde{x}^i) h(\eta, \tilde{x}^i) , \quad (3.28)$$

$$F'_{01}(\eta, x^i, \tilde{x}^i) = \mathcal{D}(\eta, \tilde{x}^i) F_{00}(\eta, x^i, \tilde{x}^i) + h(\eta, x^i) F_{11}(\eta, x^i, \tilde{x}^i) , \quad (3.29)$$

$$F'_{11}(\eta, x^i, \tilde{x}^i) = \mathcal{D}(\eta, x^i) F_{01}(\eta, x^i, \tilde{x}^i) + \mathcal{D}(\eta, \tilde{x}^i) F_{10}(\eta, x^i, \tilde{x}^i) . \quad (3.30)$$

This is a system of four first-order differential equations with four independent initial conditions. However, we have to keep in mind that these two-point functions obey certain symmetry properties and we realize by combining  $F_{10}$  and  $F_{01}$  that we can specify three symmetric functions and one anti-symmetric function as initial conditions. We remark, that this system of equations closes in the sense that we need no information about higher n-point functions. This is due to the following reasons: firstly, we neglected manifest self-interactions of the scalar field (e.g.  $\sim \lambda \phi^4$ ) that are not due to gravity, secondly, we approximate the self-interactions that are induced via gravity. These interactions are non-local in space but local in time via an inversion of the generalized Poisson equation. It means that the scalar field couples in this approximation only to its two-point functions since the gravitational potentials  $\Phi_G$ ,  $\Psi_G$  are - via the semi-classical Einstein equations - entirely expressible in terms of the scalar field two-point functions. We call this the hybrid approach. Thirdly, the scalar field does not interact with dynamical part of gravity, the gravitons, since we put them to zero by hand as an assumed negligible effect.



Let us continue to manipulate the system of equations (3.27) to (3.30) by switching to collective and difference coordinates for the spatial parts,

$$X^i \equiv \frac{x^i + \tilde{x}^i}{2}, \quad r^i \equiv x^i - \tilde{x}^i. \quad (3.31)$$

We define the Wigner transform with respect to covariant momenta  $p_i$  and its zeroth moment denoted by a bar as

$$F_*(\eta, X^i, p_i) \equiv \int d^{D-1}r e^{-\frac{i}{\hbar}p_i r^i} F_*(\eta, X^i, r^i), \quad (3.32)$$

$$\bar{F}_*(\eta, X^i) \equiv \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} F_*(\eta, X^i, p_i). \quad (3.33)$$

This definition is our equal-time, thus on-shell, version of the several curved space-time generalizations of the Minkowski space-time Wigner transformation of course for the our specific choice of a longitudinal linearized metric without gravitons. It is a direct generalization of the *Ansatz* in [97] from the homogeneous FLRW-space-time to its non-homogeneous perturbed form. This definition identifies the time coordinate  $\eta$  and spatial collective coordinates  $X^i$  as a single point on the curved space-time manifold, whereas the spatial difference coordinates  $r^i$  and the momenta  $p_i$  belong to the tangent and cotangent space, respectively, that is associated to that point. We neither make use of any geodesic expansion nor do we use Riemannian coordinates. This implies also that our equations are exact apart from the linearization in the gravitational potentials. We would like to mention that the on-shell operator formalism for curved space-times we are using resolves the problem of perturbatively solving the off-shell constraint equation which always accompanies the off-shell Vlasov equation by providing a manifest closure for on-shell correlators [94–96, 98].

It will turn out that upon Wigner transforming the equations (3.27) to (3.30), two other correlators are much more useful than  $F_{01}$  and  $F_{10}$ . Thus, we define the following combination of equal-time two-point functions

$$\begin{aligned} F_+(\eta, x^i, \tilde{x}^i) &\equiv \frac{1}{2} \left[ F_{10}(\eta, x^i, \tilde{x}^i) + F_{01}(\eta, x^i, \tilde{x}^i) \right] \\ &= \frac{1}{4} \left\langle \left\{ \hat{\Pi}(\eta, x^i), \hat{\phi}(\eta, \tilde{x}^i) \right\} + \left\{ \hat{\phi}(\eta, x^i), \hat{\Pi}(\eta, \tilde{x}^i) \right\} \right\rangle, \end{aligned} \quad (3.34)$$

$$\begin{aligned} F_-(\eta, x^i, \tilde{x}^i) &\equiv \frac{i}{2} \left[ F_{10}(\eta, x^i, \tilde{x}^i) - F_{01}(\eta, x^i, \tilde{x}^i) \right] - \frac{\hbar}{2} \delta^{D-1}(x^i - \tilde{x}^i) \\ &= \frac{i}{4} \left\langle \left\{ \hat{\Pi}(\eta, x^i), \hat{\phi}(\eta, \tilde{x}^i) \right\} - \left\{ \hat{\Pi}(\eta, \tilde{x}^i), \hat{\phi}(\eta, x^i) \right\} \right\rangle, \end{aligned} \quad (3.35)$$

where  $\{.,.\}$  denotes the anti-commutator. The calculation of transforming the two-point function dynamics into Wigner space is shown in appendix 3.A. We remark that this calculation is exact up to the linearization in the gravitational potentials. Since we want to identify phase-space distributions we have to treat the problem by utilizing the gradient approximation. Therefore, we consider again the perturbation parameters in (3.15) to (3.18) and drop the  $\varepsilon_h^2 \cdot \varepsilon_g^2$  corrections which is at least naively consistent with our linearization in gravity,

i.e. consistent with not keeping  $\varepsilon_g^4$  terms as we did from the very beginning of this chapter. We do not expect the corrections  $\varepsilon_h^2$  from the gradient expansion to become important unless we have a very light scalar field ( $m \approx 10^{-22}\text{eV}$ ) as pointed out in section 3.2.2 which is considered an extreme case. The reader interested in those corrections is invited to consider the coupled equations in appendix 3.A. However, for typical masses at scales  $\gtrsim \text{eV}$ , the following equations are perfectly accurate and we drop the  $\varepsilon_h^2$  corrections,

$$\frac{F'_{00}}{a^{2-D}} = 2[1 + \Phi_G + (D-1)\Psi_G]F_+ + \hbar \frac{\partial}{\partial X^i} [\Phi_G + (D-1)\Psi_G] \frac{\partial}{\partial p_i} F_- , \quad (3.36)$$

$$\begin{aligned} F'_+ &= \frac{\Delta_X}{4} [a^{D-2} F_{00}] - \frac{p^2}{\hbar^2} [1 + \Phi_G - (D-3)\Psi_G] [a^{D-2} F_{00}] \\ &\quad - \frac{m^2}{\hbar^2} a^2 [1 + \Phi_G - (D-1)\Psi_G] [a^{D-2} F_{00}] \\ &\quad + [1 + \Phi_G + (D-1)\Psi_G] [a^{-(D-2)} F_{11}] , \end{aligned} \quad (3.37)$$

$$\begin{aligned} F'_- &= \frac{1}{2\hbar} \frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] \frac{\partial}{\partial p_k} [p^2 a^{D-2} F_{00}] \\ &\quad - \frac{p}{\hbar} \cdot \partial_X [\Phi_G - (D-3)\Psi_G] [a^{D-2} F_{00}] \\ &\quad - [1 + \Phi_G - (D-3)\Psi_G] \frac{p_k}{\hbar} \frac{\partial}{\partial X^k} [a^{D-2} F_{00}] \\ &\quad + \frac{2m^2}{\hbar} a^2 \frac{\partial}{\partial X^i} [\Phi_G - (D-1)\Psi_G] \frac{\partial}{\partial p_i} [a^{D-2} F_{00}] \\ &\quad + \frac{\hbar}{2} \frac{\partial}{\partial X^i} [\Phi_G + (D-1)\Psi_G] \frac{\partial}{\partial p_i} [a^{-(D-2)} F_{11}] , \end{aligned} \quad (3.38)$$

$$\begin{aligned} \frac{F'_{11}}{a^{D-2}} &= -\frac{2m^2 a^2}{\hbar^2} [1 + \Phi_G - (D-1)\Psi_G] F_+ + \frac{\Delta_X}{2} F_+ \\ &\quad - 2 \frac{p^2}{\hbar^2} [1 + \Phi_G - (D-3)\Psi_G] F_+ \\ &\quad + \frac{1}{\hbar} \frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] \frac{\partial}{\partial p_k} [p^2 F_-] \\ &\quad - 2[1 + \Phi_G - (D-3)\Psi_G] \frac{p}{\hbar} \cdot \partial_X F_- - \frac{2p}{\hbar} \cdot \partial_X [\Phi_G - (D-3)\Psi_G] F_- \\ &\quad + \frac{m^2 a^2}{\hbar} \frac{\partial}{\partial X^i} [\Phi_G - (D-1)\Psi_G] \frac{\partial}{\partial p_i} F_- . \end{aligned} \quad (3.39)$$

We conclude that to this order in the gradient approximation, we still keep all degrees of freedom that were contained on the original first-order system (3.27) to (3.30). Before we try to recover a Vlasov equation from these equations let us pause a bit and make it clear how this equation reflects the difference between products of one-point functions and connected two-point functions. We will also realize that the higher-order time-derivatives implied by equations (3.36) to (3.39) correspond to oscillatory degrees of freedom. For simplicity, we set  $D = 4$  and focus on the homogeneous and the large mass limit ( $p \ll m$ ),

where we denote the homogeneous approximation of  $F_{00}$  by  $F_{00}^{\text{hom}}$ ,

$$\begin{aligned} \frac{1}{2} \left[ a^2 (F_{00}^{\text{hom}})' \right]'' + \mathcal{H} \left[ a^2 (F_{00}^{\text{hom}})' \right]' \\ + 2\mathcal{H} \frac{m^2}{\hbar^2} a^4 F_{00}^{\text{hom}} + 2 \frac{m^2}{\hbar^2} a^2 \left( a^2 F_{00}^{\text{hom}} \right)' \approx 0. \end{aligned} \quad (3.40)$$

Expanding this equation, we arrive at

$$\begin{aligned} \left( a^3 F_{00}^{\text{hom}} \right)''' - 3\mathcal{H} \left( a^3 F_{00}^{\text{hom}} \right)'' + \left( 4 \frac{m^2}{\hbar^2} a^2 - \mathcal{H}^2 - 7\mathcal{H}' \right) \left( a^3 F_{00}^{\text{hom}} \right)' \\ + 3 \left( \mathcal{H}^3 + \mathcal{H}\mathcal{H}' - \mathcal{H}'' \right) \left( a^3 F_{00}^{\text{hom}} \right) \approx 0. \end{aligned} \quad (3.41)$$

In order to make progress, we also have to provide the Einstein equations, which we derive in appendix 3.B in equations (3.124) to (3.125). The spatially homogeneous equations with neglected momenta  $p \ll m$  read in  $D = 4$  dimensions (all terms  $p^2 m^{-2}$  are dropped),

$$-2\mathcal{H}' - \mathcal{H}^2 \approx \frac{\hbar}{2M_P^2} \left\{ a^{-4} F_{11}^{\text{hom}} - \frac{m^2 a^2}{\hbar^2} F_{00}^{\text{hom}} \right\}, \quad (3.42)$$

$$3\mathcal{H}^2 \approx \frac{\hbar}{2M_P^2} \left\{ a^{-4} F_{11}^{\text{hom}} + a^2 \frac{m^2}{\hbar^2} F_{00}^{\text{hom}} \right\}, \quad (3.43)$$

and from (3.39), we have

$$F_{11}^{\text{hom}} \approx \frac{a^2}{2} \left[ a^2 (F_{00}^{\text{hom}})' \right]' + \frac{m^2}{\hbar^2} a^6 F_{00}^{\text{hom}}. \quad (3.44)$$

The equations (3.41) to (3.44) admit three independent solutions. We can guess them quickly by noting once more that  $F_{00}^{\text{hom}}$  is constructed out of one-point functions and a connected piece

$$F_{00}^{\text{hom}} = \langle \hat{\phi} \rangle^{\text{hom}} \langle \hat{\phi} \rangle^{\text{hom}} + \langle \hat{\phi} \hat{\phi} \rangle_{\text{connected}}^{\text{hom}}. \quad (3.45)$$

In the limit  $\mathcal{H} \ll \hbar^{-1} m a$ , the solutions for the one-point functions are through the Klein-Gordon equations approximately given by

$$\langle \hat{\phi} \rangle^{\text{hom, sol 1}} \approx a^{-3/2} \cos \left( \int d\eta m a \right), \quad (3.46)$$

$$\langle \hat{\phi} \rangle^{\text{hom, sol 2}} \approx a^{-3/2} \sin \left( \int d\eta m a \right). \quad (3.47)$$

Had we used only one-point functions  $\langle \hat{\phi} \rangle$  to construct  $F_{00}^{\text{hom}}$ , our analysis would be complete at this stage since we can only impose two initial conditions for  $\langle \hat{\phi} \rangle$  and they would completely determine  $F_{00}^{\text{hom}}$  which in this case has always an oscillatory contribution. However, let us forget about the one-point functions and focus on the connected part of  $F_{00}^{\text{hom}}$ . We see that the following functions

are two independent solutions to (3.41),<sup>4</sup>

$$F_{00}^{\text{hom, sol 1}} \approx a^{-3} \cos^2 \left( \int d\eta ma \right), \quad (3.48)$$

$$F_{00}^{\text{hom, sol 2}} \approx a^{-3} \sin^2 \left( \int d\eta ma \right), \quad (3.49)$$

$$F_{11}^{\text{hom, sol 1}} \approx \frac{m^2}{\hbar^2} a^3 \sin^2 \left( \int d\eta ma \right), \quad (3.50)$$

$$F_{11}^{\text{hom, sol 2}} \approx \frac{m^2}{\hbar^2} a^3 \cos^2 \left( \int d\eta ma \right). \quad (3.51)$$

The corresponding solutions for the Hubble rate are given by the following leading order terms

$$\left[ \mathcal{H}^2 a \right]^{\text{sol 1,2}} \approx \text{const}, \quad (3.52)$$

$$\left[ 2\mathcal{H}' + \mathcal{H}^2 \right]^{\text{sol 1,2}} \approx \pm 3\mathcal{H}^2 \times \cos \left( 2 \int d\eta ma \right), \quad (3.53)$$

$$\left[ \mathcal{H}'' + \mathcal{H}\mathcal{H}' \right]^{\text{sol 1,2}} \approx \mp 3ma\mathcal{H}^2 \sin \left( 2 \int d\eta ma \right). \quad (3.54)$$

We than choose a linear combination of these solutions which is not oscillatory and can only be provided by means of the connected part of  $F_{00}^{\text{hom}}$ . It is to leading order simply given by

$$F_{00}^{\text{hom, non-osc}} = \frac{1}{2} [F_{00}^{\text{hom, sol 1}} + F_{00}^{\text{hom, sol 2}}] \approx a^{-3}, \quad F_{11}^{\text{hom, non-osc}} \approx \frac{m^2}{\hbar^2} a^3. \quad (3.55)$$

The, corresponding solutions for the Hubble rate are given by the following leading order terms

$$\left[ \mathcal{H}^2 a \right]^{\text{non-osc}} \approx \text{const}, \quad (3.56)$$

$$\left[ 2\mathcal{H}' + \mathcal{H}^2 \right]^{\text{non-osc}} \approx 0. \quad (3.57)$$

We of course get this solution by dropping all higher-order time derivatives on  $F_{00}^{\text{hom}}$  in the limit  $\mathcal{H} \ll \hbar^{-1}ma$ . To summarize, we have shown that – by means of the connected part – the two-point function formalism allows to overcome the oscillatory behaviour of time derivatives of the Hubble rate, and equivalently of pressure, without any averaging procedure. It is in this respect richer in comparison to the approach based on classical real scalar fields.

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<sup>4</sup>Being ignorant about any  $p$  dependence for the moment, the initial density matrix for homogeneous two-point functions contains five initial conditions (see e.g. [99]): the one-point functions  $\langle \hat{\phi} \rangle$  and  $\langle \hat{\Pi} \rangle$ , and the connected parts of the two-point functions  $F_{00}$ ,  $F_{11}$  and  $F_+$  or equivalently  $F_{00}$ ,  $F'_{00}$  and  $F''_{00}$ . The fourth condition for the connected part of  $F_-$  is trivially satisfied in the homogeneous case.

### 3.3 Generalized on-shell Vlasov equation

#### 3.3.1 Identification of on-shell phase-space distributions

We now compare the expectation value of the real scalar field energy-momentum tensor (see (3.120) to (3.122) in appendix 3.B) with a general energy-momentum tensor in kinetic theory. This will allow us to identify phase-space distributions based on the scalar field. The phase-space distribution  $f_{\text{cl}}$  of classical collisionless particles in general relativity obeys the Vlasov equation [106],

$$\left[ \frac{\partial}{\partial \eta} + \frac{p_{\text{cl}}^i}{p_{\text{cl}}^0} \frac{\partial}{\partial X^i} + \Gamma_{i\beta}^\alpha \frac{p_{\text{cl}}^\alpha p_{\text{cl}}^\beta}{p_{\text{cl}}^0} \frac{\partial}{\partial p_{\text{cl}}^i} \right] f_{\text{cl}}(\eta, X^j, p_{\text{cl}}^i) = 0. \quad (3.58)$$

The energy-momentum tensor in kinetic theory is then given by

$$T_{\mu\nu}^{\text{kin}}(\eta, X^i) = \int d^{D-1} p^{\text{cl}} \left[ \gamma^{-1/2} \frac{p_\mu^{\text{cl}} p_\nu^{\text{cl}}}{E_{\text{cl}}} \right] (\eta, X^i, p_i^{\text{cl}}) f_{\text{cl}}(\eta, X^i, p_i^{\text{cl}}). \quad (3.59)$$

Here, the quantity  $\gamma$  is the determinant of the spatial metric. The particle energy  $E_{\text{cl}}$  and the temporal momentum  $p_0^{\text{cl}}$  are related to the on-shell condition

$$p_\mu^{\text{cl}} p_{\text{cl}}^\mu = -m_{\text{cl}}^2, \quad (3.60)$$

which gives in longitudinal gauge

$$p_0^{\text{cl}} = g_{00} \sqrt{m_{\text{cl}}^2 + g^{ij} p_i^{\text{cl}} p_j^{\text{cl}}}, \quad g_{0i} = 0, \quad (3.61)$$

$$E_{\text{cl}} = -|g^{00}|^{1/2} p_0^{\text{cl}}. \quad (3.62)$$

Of course we want to identify similar quantities through the covariant Wigner momenta  $p_i$ . By using a tilde from now on, we want to clearly distinguish between the covariant Wigner momentum  $p_i$ , which is an integration variable and derived quantities that are related to it via the metric,

$$E(\eta, X^i, p_i) = \left[ m^2 + \frac{p^2}{a^2(\eta)} \right]^{1/2} \left[ 1 + \frac{p^2}{m^2 a^2(\eta) + p^2} \Psi_G(\eta, X^i) \right], \quad (3.63)$$

$$\tilde{p}^0(\eta, X^i, p_i) = a^{-1}(\eta) [1 - \Phi_G(\eta, X^i)] E(\eta, X^i, p_i), \quad (3.64)$$

$$\tilde{p}_0(\eta, X^i, p_i) = -a^2(\eta) [1 + 2\Phi_G(\eta, X^i)] \tilde{p}^0(\eta, X^i, p_i), \quad (3.65)$$

$$\tilde{p}^k(\eta, X^i, p_i) = a^{-2}(\eta) [1 + 2\Psi_G(\eta, X^i)] \delta^{ki} p_i. \quad (3.66)$$

In particular, we emphasize that there is no independent integration variable  $p_0$  which is encountered in off-shell Wigner transformations, we have only the on-shell quantity  $\tilde{p}_0(p_i)$ . Using the expressions (3.120) and (3.121) in appendix 3.B, we can rewrite the 00- and 0i-components of the expectation value of the

real scalar field energy-momentum tensor are to leading order in  $\hbar$  as,

$$\begin{aligned} \langle \hat{T}_{00} \rangle(\eta, X^i) &= \frac{m^2 a^2}{\hbar^2} [1 + 2\Phi_G] \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} F_{00}(\eta, X^i, p_i) \\ &\quad + [1 + 2\Phi_G + 2\Psi_G] \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} \frac{p^2}{\hbar^2} F_{00}(\eta, X^i, p_i), \end{aligned} \quad (3.67)$$

and

$$\langle \hat{T}_{0i} \rangle(\eta, X^i) = -\frac{1 + \Phi_G + (D-1)\Psi_G}{a^{D-2}} \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} \frac{p_i}{\hbar} F_{-}(\eta, X^i, p_i). \quad (3.68)$$

We realize that according to equation (3.67) a phase-space density candidate in the classical particle limit would be given by

$$f_{\text{cl}}^{\text{even}} \longrightarrow f_{\phi}^{\text{even}} \equiv \frac{E\gamma^{1/2}}{(2\pi\hbar)^{D-1}} \frac{F_{00}}{\hbar^2}. \quad (3.69)$$

However, when looking at the  $T_{0i}$  equation (3.68), we would rather come to the conclusion that the classical phase-space-density should be given by

$$f_{\text{cl}}^{\text{odd}} \longrightarrow f_{\phi}^{\text{odd}} \equiv \frac{1}{(2\pi\hbar)^{D-1}} \frac{F_{-}}{\hbar}. \quad (3.70)$$

In equations (3.101) and (3.102) in appendix 3.A we summarize that according to their fundamental definitions the two-point function  $F_{00}$  is of even parity in  $p_i$  whereas the two-point function  $F_{-}$  is an odd parity function in  $p_i$ . Thus, up to a rescaling and corrections in our perturbation parameters, the quantity  $F_{00}$  seems to play the role of the phase-space-density for even moments and  $F_{-}$  seems to play the role of the phase-space-density for odd moments in  $p_i$ .

We note that our identification of phase-space densities relies here on dropping contributions  $\sim \varepsilon_h^2$  where we also dropped higher time derivatives of  $F_{00}$  and we have to leading order in  $\varepsilon_h^2$  the relation  $F_{11} = \gamma E^2 F_{00}$ . This observation will lead us to a more accurate definition of the even phase-space density in chapter 4.

### 3.3.2 Dynamics of on-shell phase-space distributions

Based on the identifications in the previous paragraph, we rewrite (3.36) to (3.39) in terms of the definitions (3.69) and (3.70). Except for terms involving time derivatives, we drop all terms of order  $\varepsilon_h^2$  and find

$$\begin{aligned} \tilde{p}^0 \frac{\partial}{\partial \eta} f_{\phi}^{\text{odd}} + \tilde{p}^k \frac{\partial}{\partial X^k} f_{\phi}^{\text{even}} - \tilde{p}^i p_i \frac{\partial}{\partial X^k} [\Phi_G + \Psi_G] \frac{\partial}{\partial p_k} f_{\phi}^{\text{even}} \\ - m^2 \frac{\partial}{\partial X^k} \Phi_G \frac{\partial}{\partial p_k} f_{\phi}^{\text{even}} = 0, \end{aligned} \quad (3.71)$$

$$\begin{aligned}
& \tilde{p}^0 \frac{\partial}{\partial \eta} f_\phi^{\text{even}} + \tilde{p}^k \frac{\partial}{\partial X^k} f_\phi^{\text{odd}} - \tilde{p}^i p_i \frac{\partial}{\partial X^k} [\Phi_G + \Psi_G] \frac{\partial}{\partial p_k} f_\phi^{\text{odd}} \\
& - m^2 \frac{\partial}{\partial X^k} \Phi_G \frac{\partial}{\partial p_k} f_\phi^{\text{odd}} + \frac{\hbar^2}{4} a^{-2(D-1)} \left[ a^{D-2} \frac{\partial}{\partial \eta} \right]^3 \left[ a^{-(D-2)} (m^2 a^2 + p^2)^{-1/2} f_\phi^{\text{even}} \right] \\
& - \hbar^2 a^{-2} \frac{\Delta_X}{4} \frac{\partial}{\partial \eta} \left[ (m^2 a^2 + p^2)^{-1/2} f_\phi^{\text{even}} \right] = 0. \quad (3.72)
\end{aligned}$$

We emphasize again that  $\tilde{p}^0$  is on-shell. We set<sup>5</sup>

$$f_\phi \equiv f_\phi^{\text{even}} + f_\phi^{\text{odd}} = (2\pi\hbar)^{-(D-1)} \left[ E \gamma^{1/2} \frac{F_{00}}{\hbar^2} + \frac{F_-}{\hbar} \right], \quad (3.74)$$

and find

$$\begin{aligned}
& \left[ \tilde{p}^0 \frac{\partial}{\partial \eta} + \tilde{p}^k \frac{\partial}{\partial X^k} - \tilde{p}^i p_i \frac{\partial}{\partial X^k} [\Phi_G + \Psi_G] \frac{\partial}{\partial p_k} - m^2 \frac{\partial}{\partial X^k} \Phi_G \frac{\partial}{\partial p_k} \right] f_\phi \\
& + \frac{\hbar^2}{4} a^{-2(D-1)} \left[ a^{D-2} \frac{\partial}{\partial \eta} \right]^3 \left[ a^{-(D-2)} (m^2 a^2 + p^2)^{-1/2} f_\phi^{\text{even}} \right] \\
& - \hbar^2 a^{-2} \frac{\Delta_X}{4} \frac{\partial}{\partial \eta} \left[ (m^2 a^2 + p^2)^{-1/2} f_\phi^{\text{even}} \right] = 0. \quad (3.75)
\end{aligned}$$

Equation (3.75) is the main result of this chapter. It tells us that we can obtain a corrected on-shell Vlasov equation from statistical two-point functions of a scalar field theory. It includes a third-order time derivative such that we keep all degrees of freedom from the initial first-order system (3.27) to (3.30) originating from the relativistic Klein-Gordon equation. However, dropping these third-order time derivatives as small corrections we are left with the degrees of freedom of a one-particle phase-space distribution. We note, that dropping these third-order time derivatives can be justified by either using certain initial conditions in the case that  $F_{00}$  is given only in terms of the connected two-point functions (concretely  $F_+ \sim \mathcal{H}F_{00}$ ). In the case where  $F_{00}$  is only given by a product of oscillatory one-point functions, we can drop those contributions only after an averaging procedure. Apart from the third-order time derivatives the generalized Vlasov equation (3.75) contains corrections in the gradient expansion. Dropping the later, we find the same form as for the classical, collisionless on-shell Vlasov equation given in (3.58). We note that since we used a spin zero field, we expect corrections due to non-trivial spin in other quantum field theoretical settings.

A few more comments are in order. We acknowledge that an off-shell version of this equation has been derived much earlier by [94, 95] for arbitrary metrics. As we pointed out above and explicitly derived here, the on-shell Vlasov cannot be derived from a single on-shell two-point function which came apparent

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<sup>5</sup>Note, that we could also have considered

$$f_\phi^{\text{time-rev}} \equiv f_\phi^{\text{even}} - f_\phi^{\text{odd}}, \quad (3.73)$$

which yields phase-space dynamics for reversed momenta or equally for reversed times. However, the equation for the time-reversed density amounts only to a flip of the sign of the momentum  $p_i$  and thus yields no new information.

from the derivation of real scalar field particle densities by [97] in homogeneous FLRW-space-time and which was pointed out earlier by [104] for a QED Vlasov equation in Minkowski space-time. An equation similar to (3.75) may be derived by employing a quasi-particle approximation that is restricted to positive off-shell energies which are then integrated over [107, 108]. However, this quasi-particle picture does not need to hold and it is a priori not settled why negative off-shell energies should not contribute. Thus, as far as we know, this is the first explicit derivation of the on-shell Vlasov equation for a non-homogeneous metric from a fundamental theory that is not employing a quasi-particle approximation i.e. that does not restrict excitation to be lumped in region of positive off-shell energies. We also acknowledge that equation (3.75) has been derived on phenomenological grounds from a Schrödinger equation based on one-point functions as discussed for example in [30], however without any coarse-graining this approach does not allow for independent moments in momentum space since they will be related by spatial derivatives and thus, this one-point function approach cannot model a generic non-perfect fluid with gravitational interactions without any coarse-graining.

Let us comment a bit more on the physical meaning of this real scalar field phase-space distribution  $f_\phi$  and why we think that it can model a general fluid with gravitational interactions. First, we note that it is no surprise that the definition (3.74) yields a generalization of the Vlasov equation since it is related to the Wigner transformation of the Schrödinger operators  $\hat{\psi}$  in the non-relativistic limit. We see this by writing

$$\hat{\phi}(\eta, x^i) \propto \hat{\psi}(\eta, x^i) \exp \left[ -i \frac{m}{\hbar} \int^\eta d\tilde{\eta} a(\tilde{\eta}) \right] + \text{cc}, \quad (3.76)$$

and after some manipulations we find

$$\begin{aligned} f_\phi(\eta, X^i, p_i) \propto a^{D-1} \int d^{D-1}r e^{-\frac{i}{\hbar} r^i p_i} \langle \hat{\psi}(\eta, X^k + r^k/2) \hat{\psi}^\dagger(\eta, X^k - r^k/2) \rangle \\ + \mathcal{O}(\varepsilon_g^2) + \mathcal{O}(\varepsilon_\hbar^2) + \text{oscillatory terms}. \end{aligned} \quad (3.77)$$

We already remarked in section 3.2.3 that the oscillatory terms can be removed by an appropriate initial density matrix. Thus, we rediscover in the large mass limit the definition of the non-relativistic Wigner quasi-probability distribution based on an initial density matrix [93]. This is another strong hint that our on-shell approach to construct a phase-space distribution  $f_\phi$  has a classical interpretation provided the classically condition and the gradient expansion we discussed in section 3.2.2 apply. This implies a suitable choice for the initial density matrix, in other words the state it represents has to be classical enough. We can also formulate the approximate equivalence between a classical one-particle phase-space model and the classical limit of the real scalar quantum field theory in the following way: since the dynamics for the scalar field phase-space distribution  $f_\phi$  and the classical distribution  $f_{cl}$  agree on length and time-scales that are associated to the classical limit, the difference between the two quantities is encoded in the possibility to formulate arbitrary initial conditions.



For any smooth classical phase-space distribution we can write

$$\begin{aligned} f_{cl}(\eta_{\text{ini}}, X^i, p_i) &= \int d^{D-1}r e^{ir^i p_i} f_{cl}(\eta_{\text{ini}}, X^k, r^k) \\ &= \int d^{D-1}(x-y) e^{i(x-y)^i p_i} \tilde{f}_{cl}(\eta_{\text{ini}}, x^k, y^k), \end{aligned} \quad (3.78)$$

which in particular means that any moment can be written as

$$\begin{aligned} f_{cl}^{(k_1, \dots, k_n)}(\eta_{\text{ini}}, X^i) &= \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} p_{k_1} \dots p_{k_n} f_{cl}(\eta_{\text{ini}}, X^i, p_i) \\ &= \frac{\partial^n}{\partial(x-y)^{k_1} \dots \partial(x-y)^{k_n}} \tilde{f}_{cl}(\eta_{\text{ini}}, x^i, y^i) \Big|_{(x-y)^i=0}. \end{aligned} \quad (3.79)$$

From this we conclude that an arbitrary smooth, classical, one-particle phase-space distribution is initially specified by an arbitrary function in two spatial coordinates. However, we can always provide such a function based on a Gaussian initial density matrix for the quantum theory which is encoded in the connected part of the two-point function,

$$\begin{aligned} \langle \hat{\psi}(\eta_{\text{ini}}, x^k) \hat{\psi}^*(\eta_{\text{ini}}, y^k) \rangle &= \langle \hat{\psi}(\eta_{\text{ini}}, x^k) \rangle \langle \hat{\psi}^*(\eta_{\text{ini}}, y^k) \rangle \\ &\quad + \langle \hat{\psi}(\eta_{\text{ini}}, x^k) \hat{\psi}^*(\eta_{\text{ini}}, y^k) \rangle_{\text{connected}}. \end{aligned} \quad (3.80)$$

Note that the product of two one-point functions is not general enough to cover an arbitrary function of two arguments, so we really need the connected term. We can provide similar arguments for the fully relativistic scalar field theory by splitting the classical distribution into even and odd parts whose arbitrary initial conditions can always be specified by providing the initial connected parts of the two-point functions  $F_{00}$  and  $F_-$  as well as  $F_+$  and  $F_{11}$  to fix oscillatory behaviour.<sup>6</sup>

Finally, let us go to a coarser approximation of equation (3.75) by pursuing the large mass limit in order to clearly see the relation to the cold dark matter particle picture,

$$\begin{aligned} \left[ \frac{\partial}{\partial\eta} + \frac{p_i}{ma(\eta)} \frac{\partial}{\partial X^i} - ma(\eta) \frac{\partial}{\partial X^i} \Phi_G(\eta, X^i) \frac{\partial}{\partial p_i} \right] f_\phi(\eta, X^i, p_i) \\ \times \left[ 1 + \mathcal{O}(\varepsilon_g^2) + \mathcal{O}(\varepsilon_h^2) + \mathcal{O}(\varepsilon_p^2) \right] = 0. \end{aligned} \quad (3.81)$$

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<sup>6</sup>It is worth pointing out that the formalism in which one uses a single particle wave function resulting from the evolution of a certain class of initial states may possess no classical limit in the sense that the phase-space distribution (Wigner) function can exhibit rapid (space and/or time) oscillations in the limit when  $\hbar \rightarrow 0$ , thus invalidating the gradient expansion. Since the Schrödinger equation is the non-relativistic limit of the Klein-Gordon equation this argument could in principle also apply to our scalar field phase-space distribution  $f_\phi$ . However, the question whether a spatial gradient expansion does apply or not is tied to the specification of the initial density matrix: it ought to be such that it yields two-point functions that satisfy the classicality criteria spelled out in section 3.2.2.

The last equation is used for a collisionless gas in the context of cold dark matter as for example in [24] or [109] and is simply the non-relativistic limit of (3.58),

$$\left[ \frac{\partial}{\partial \eta} + \frac{p_i^{cl}}{m_{cl}a(\eta)} \frac{\partial}{\partial X^i} - m_{cl}a(\eta) \frac{\partial}{\partial X^i} \Phi_G(\eta, X^i) \frac{\partial}{\partial p_i^{cl}} \right] f_{cl}(\eta, X^j, p_k^{cl}) = 0. \quad (3.82)$$

We thus conclude that collisionless dark matter obeying a smooth phase-space distribution can always be mimicked by a real scalar field theory based on scales where the mass dominates. Initial non-trivial moments of the phase-space distribution can be provided by non-trivial initial density matrices for the connected parts of the two-point functions. Taking moments of equation (3.81) shows that this generates in principle an infinite hierarchy of moments.

Apart from the evolution of the phase-space distribution we also have to specify the Einstein equations that determine the gravitational potentials. If we stay within the non-relativistic limit and set again  $D = 4$ , we have according to equation (3.120) ,

$$T_{00} = \frac{m}{a} \int d^3p f_\phi \times \left[ 1 + \mathcal{O}(\varepsilon_g^2) + \mathcal{O}(\varepsilon_h^2) + \mathcal{O}(\varepsilon_p^2) \right], \quad (3.83)$$

and thus with the Einstein equations,

$$3\mathcal{H}^2 + 2\Delta\Psi_G - 6\mathcal{H}\Psi'_G \approx \frac{\hbar}{M_P^2} \frac{m}{a} \int d^3p f_\phi. \quad (3.84)$$

We introduce

$$\rho_\phi(\eta, X^i) \equiv \int d^3p f_\phi(\eta, X^i, p_k), \quad (3.85)$$

$$\bar{\rho}_\phi(\eta) \equiv \int d^3p \bar{f}_\phi(\eta, p_k), \quad (3.86)$$

$$\delta_\phi \equiv \frac{\rho_\phi}{\bar{\rho}_\phi} - 1 \quad (3.87)$$

and get the analogue of the Poisson equation where  $\bar{f}_\phi = \bar{f}_\phi$  is the homogeneous part of the phase-space density  $f_\phi$ . This leads on sub-Hubble scales and with the assumption of zero gravitational slip  $\Psi_G = \Phi_G$  to the analogue of the Poisson equation for usual cold dark matter description,

$$\Delta\Phi_G = \frac{3}{2}\mathcal{H}^2\delta_\phi, \quad (3.88)$$

where

$$3\mathcal{H}^2a = \frac{\hbar}{M_P^2} \bar{\rho}_\phi, \quad (3.89)$$

is constant according to equation (3.81), as it should be in matter domination.

## 3.4 Conclusion

In this chapter we developed a formalism for the dynamics of dark matter in which we start with a tree-level relativistic action for a real scalar field and obtain via an operator formalism an effective description that includes leading order interactions mediated by gravity. Our formalism is relativistic, in that it allows for a systematic inclusion of both relativistic matter field effects as well as relativistic gravitational effects. Furthermore, we identify a classical phase-space distribution  $f_\phi$  in (3.74) based on four on-shell, equal-time real scalar field statistical two-point functions ((3.20) to (3.23)). The statistical two-point functions obey a system of first order differential equations ((3.36) to (3.39)) that closes because we first neglected manifest self-interactions of the matter field and the dynamical gravitational fields and second, did not integrate out the gravitational constraint fields. In the language of Feynman diagrams this amounts to approximating loop contributions with external sources whose evolution is determined by the semi-classical Einstein equation. The evolution of the phase-space distribution  $f_\phi$  is determined by a generalized Vlasov equation including relativistic corrections, third-order time derivatives and corrections in a gradient expansion (3.75). Dropping the third-order time-derivatives as small corrections reduces the degrees of freedom to those of classical one-particle phase-space distribution. These corrections are typically small for large masses and large scales if the state is non-squeezed. The statistical two-point functions of the scalar matter field entering the definition of  $f_\phi$  are evaluated with respect to an initial density matrix and thus have generically reducible and connected pieces, in other words they contain a part given by one-point functions or classical fields. Focusing on the connected piece of the statistical matter two-point function makes the major distinction from previous approaches of modeling real scalar field fluids that focused on one-point functions. The reason is that it allows for generic initial conditions in two arguments without coarse-graining, either in position space or in Wigner space which then translates into a hierarchy of non-related moments in momentum space and in particular enables us to model a fluid that generically can include vorticity and anisotropy.

We note that using statistical two-point functions from the beginning allows us to treat gravity on a semi-classical level where we introduced linearized stochastic gravitational potentials that couple to the statistical two-point functions and thereby make the phase-space distribution  $f_\phi$  stochastic. This is what we call hybrid approach and it bridges the gap to cosmological perturbation theory since we now can calculate in a second step two-point function with respect to the gravitational potentials or density perturbations.

The motivations and insights we lay out in this chapter are the starting point for the more general framework that we develop in chapter 4 where we consider arbitrary metrics in the semi-classical approximation, include quartic self-interactions of the scalar field as well as a non-minimal coupling to gravity.

### 3.A Wigner transformation of 2-point function dynamics

In this appendix we drop the ubiquitous  $\eta$  dependence to save some space. We define the following operator

$$\left[ f(X^i, p_i) \right] \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \left[ g(X^i, p_i) \right] \equiv \left[ \frac{\partial f}{\partial X^k} \right] (X^i, p_i) \left[ \frac{\partial g}{\partial p_k} \right] (X^i, p_i). \quad (3.90)$$

By using partial integration, one can show that the following relation holds up to boundary terms

$$\begin{aligned} & \int d^{D-1}(x - \tilde{x}) e^{-ip_i(x^i - \tilde{x}^i)} \int d^{D-1}z A(x^i, z^i) B(z^i, \tilde{x}^i) \\ &= A(X^i, p_i) e^{i\frac{\hbar}{2}(\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_X)} B(X^i, p_i) + \text{boundary terms}. \end{aligned} \quad (3.91)$$

We rewrite equations (3.27) to (3.30) in Wigner space in the following way,

$$\begin{aligned} \frac{F'_{00}}{a^{2-D}} &= [1 + \Phi_G + (D-1)\Psi_G] e^{i\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p} F_{10} \\ &+ F_{01} e^{-i\frac{\hbar}{2}\overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_X} [1 + \Phi_G + (D-1)\Psi_G], \end{aligned} \quad (3.92)$$

$$\begin{aligned} F'_{10} &= \frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] e^{i\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p} \left\{ \left[ \frac{1}{2} \frac{\partial}{\partial X^k} + i \frac{p_k}{\hbar} \right] [a^{D-2} F_{00}] \right\} \\ &+ [1 + \Phi_G - (D-3)\Psi_G] e^{i\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p} \left\{ \left[ \frac{\Delta_X}{4} + i \frac{p}{\hbar} \cdot \partial_X - \frac{p^2}{\hbar^2} \right] [a^{D-2} F_{00}] \right\} \\ &- \frac{m^2}{\hbar^2} a^2 [1 + \Phi_G - (D-1)\Psi_G] e^{i\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p} [a^{D-2} F_{00}] \\ &+ [a^{-(D-2)} F_{11}] e^{-i\frac{\hbar}{2}\overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_X} [1 + \Phi_G + (D-1)\Psi_G], \end{aligned} \quad (3.93)$$

$$\begin{aligned} F'_{01} &= \left\{ \left[ \frac{1}{2} \frac{\partial}{\partial X^k} - i \frac{p_k}{\hbar} \right] [a^{D-2} F_{00}] \right\} e^{-i\frac{\hbar}{2}\overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_X} \frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] \\ &+ \left\{ \left[ \frac{\Delta_X}{4} - i \frac{p}{\hbar} \cdot \partial_X - \frac{p^2}{\hbar^2} \right] [a^{D-2} F_{00}] \right\} e^{-i\frac{\hbar}{2}\overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_X} [1 + \Phi_G - (D-3)\Psi_G] \\ &- \frac{m^2}{\hbar^2} a^2 [a^{D-2} F_{00}] e^{-i\frac{\hbar}{2}\overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_X} [1 + \Phi_G - (D-1)\Psi_G] \\ &+ [1 + \Phi_G + (D-1)\Psi_G] e^{i\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p} [a^{-(D-2)} F_{11}], \end{aligned} \quad (3.94)$$

$$\begin{aligned} \frac{F'_{11}}{a^{D-2}} &= \frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] e^{i\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p} \left\{ \left[ \frac{1}{2} \frac{\partial}{\partial X^k} + i \frac{p_k}{\hbar} \right] F_{01} \right\} \\ &+ [1 + \Phi_G - (D-3)\Psi_G] e^{i\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p} \left\{ \left[ \frac{\Delta_X}{4} + i \frac{p}{\hbar} \cdot \partial_X - \frac{p^2}{\hbar^2} \right] F_{01} \right\} \\ &- \frac{m^2}{\hbar^2} a^2 [1 + \Phi_G - (D-1)\Psi_G] e^{i\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p} F_{01} \\ &+ \left\{ \left[ \frac{1}{2} \frac{\partial}{\partial X^k} - i \frac{p_k}{\hbar} \right] F_{10} \right\} e^{-i\frac{\hbar}{2}\overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_X} \frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] \\ &+ \left\{ \left[ \frac{\Delta_X}{4} - i \frac{p}{\hbar} \cdot \partial_X - \frac{p^2}{\hbar^2} \right] F_{10} \right\} e^{-i\frac{\hbar}{2}\overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_X} [1 + \Phi_G - (D-3)\Psi_G] \\ &- \frac{m^2 a^2}{\hbar^2} F_{10} e^{-i\frac{\hbar}{2}\overleftarrow{\partial}_p \cdot \overrightarrow{\partial}_X} [1 + \Phi_G - (D-1)\Psi_G], \end{aligned} \quad (3.95)$$

where we used

$$p^2 \equiv \delta^{ij} p_i p_j. \quad (3.96)$$

By using the definitions (3.34) and (3.35), we get the following system of equations

$$\begin{aligned} \frac{F'_{00}}{a^{2-D}} &= 2[1 + \Phi_G + (D-1)\Psi_G] \cos \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] F_+ \\ &\quad + 2[\Phi_G + (D-1)\Psi_G] \sin \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] F_-, \end{aligned} \quad (3.97)$$

$$\begin{aligned} F'_+ &= \frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] \cos \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] \frac{1}{2} \frac{\partial}{\partial X^k} [a^{D-2} F_{00}] \\ &\quad + [1 + \Phi_G - (D-3)\Psi_G] \cos \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] \left\{ \left[ \frac{\Delta_X}{4} - \frac{p^2}{\hbar^2} \right] [a^{D-2} F_{00}] \right\} \\ &\quad - [\Phi_G - (D-3)\Psi_G] \sin \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] \left\{ \frac{p}{\hbar} \cdot \partial_X [a^{D-2} F_{00}] \right\} \\ &\quad - \frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] \sin \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] \cdot \left\{ \frac{p_k}{\hbar} [a^{D-2} F_{00}] \right\} \\ &\quad - \frac{m^2 a^2}{\hbar^2} [1 + \Phi_G - (D-1)\Psi_G] \cos \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] [a^{D-2} F_{00}] \\ &\quad + [1 + \Phi_G + (D-1)\Psi_G] \cos \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] [a^{-(D-2)} F_{11}], \end{aligned} \quad (3.98)$$

$$\begin{aligned} F'_- &= -\frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] \sin \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] \frac{1}{2} \frac{\partial}{\partial X^k} [a^{D-2} F_{00}] \\ &\quad - [\Phi_G - (D-3)\Psi_G] \sin \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] \left\{ \left[ \frac{\Delta_X}{4} - \frac{p^2}{\hbar^2} \right] [a^{D-2} F_{00}] \right\} \\ &\quad - [1 + \Phi_G - (D-3)\Psi_G] \cos \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] \left\{ \frac{p}{\hbar} \cdot \partial_X [a^{D-2} F_{00}] \right\} \\ &\quad - \frac{\partial}{\partial X^k} [\Phi_G - (D-3)\Psi_G] \cos \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] \cdot \left[ \frac{p_k}{\hbar} [a^{D-2} F_{00}] \right] \\ &\quad + \frac{m^2}{\hbar^2} a^2 [\Phi_G - (D-1)\Psi_G] \sin \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] [a^{D-2} F_{00}] \\ &\quad + [\Phi_G + (D-1)\Psi_G] \sin \left[ \frac{\hbar}{2} \overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p \right] [a^{-(D-2)} F_{11}]. \end{aligned} \quad (3.99)$$

$$\begin{aligned}
\frac{F'_{11}}{a^{D-2}} = & -2\frac{m^2}{\hbar^2}a^2[1 + \Phi_G - (D-1)\Psi_G] \cos\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] F_+ \\
& + \frac{2m^2a^2}{\hbar^2}[1 + \Phi_G - (D-1)\Psi_G] \sin\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] F_- \\
& + 2[1 + \Phi_G - (D-3)\Psi_G] \cos\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] \left\{\left[\frac{\Delta_X}{4} - \frac{p^2}{\hbar^2}\right] F_+\right\} \\
& - 2[\Phi_G - (D-3)\Psi_G] \sin\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] \left\{\left[\frac{\Delta_X}{4} - \frac{p^2}{\hbar^2}\right] F_-\right\} \\
& - 2[\Phi_G - (D-3)\Psi_G] \sin\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] \left\{\frac{p}{\hbar} \cdot \partial_X F_+\right\} \\
& - 2[1 + \Phi_G - (D-3)\Psi_G] \cos\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] \left\{\frac{p}{\hbar} \cdot \partial_X F_-\right\} \\
& + \left\{\frac{\partial}{\partial X^k}[\Phi_G - (D-3)\Psi_G]\right\} \cos\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] \cdot \left\{\frac{\partial}{\partial X^k} F_+\right\} \\
& - \left\{\frac{\partial}{\partial X^k}[\Phi_G - (D-3)\Psi_G]\right\} \sin\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] \cdot \left\{\frac{\partial}{\partial X^k} F_-\right\} \\
& - 2\left\{\frac{\partial}{\partial X^k}[\Phi_G - (D-3)\Psi_G]\right\} \sin\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] \cdot \left\{\frac{p_k}{\hbar} F_+\right\} \\
& - 2\left\{\frac{\partial}{\partial X^k}[\Phi_G - (D-3)\Psi_G]\right\} \cos\left[\frac{\hbar}{2}\overleftarrow{\partial}_X \cdot \overrightarrow{\partial}_p\right] \cdot \left\{\frac{p_k}{\hbar} F_-\right\}. \quad (3.100)
\end{aligned}$$

For later discussion, we remark the following important equal time properties

$$\begin{aligned}
F_{00}(X^i, p_i) &= \frac{1}{2} \int d^{D-1}r e^{-\frac{i}{\hbar}p_i r^i} \left\langle \left\{ \hat{\phi}\left(X^i + \frac{r^i}{2}\right), \hat{\phi}\left(X^i - \frac{r^i}{2}\right) \right\} \right\rangle \\
&= F_{00}(X^i, -p_i), \quad (3.101)
\end{aligned}$$

$$\begin{aligned}
F_-(X^i, p_i) &= \frac{i}{4} \int d^{D-1}r e^{-\frac{i}{\hbar}p_i r^i} \left[ \left\langle \left\{ \hat{\Pi}\left(X^i + \frac{r^i}{2}\right), \hat{\phi}\left(X^i - \frac{r^i}{2}\right) \right\} \right\rangle \right. \\
&\quad \left. - \left\langle \left\{ \hat{\phi}\left(X^i + \frac{r^i}{2}\right), \hat{\Pi}\left(X^i - \frac{r^i}{2}\right) \right\} \right\rangle \right] \\
&= -F_-(X^i, -p_i), \quad (3.102)
\end{aligned}$$

These equations tell us that  $F_{00}$  is even in  $p_i$  and that  $F_-$  is odd in  $p_i$ . After an appropriate rescaling that also accounts for the right dimensions, these two quantities will play the role of even and odd phase-space densities.

### 3.B Einstein equations

The dynamical equations of the previous section are supplemented by the Einstein equations. Since we neglected gravitons with the choice of our metric, the Einstein equations are constraint equations that will determine the gravitational potentials in terms of two-point functions of the scalar field and thereby induce non-linear interactions. We write down the metric with scalar perturbations in longitudinal gauge

$$g_{00}(\eta, x^i) = -a^2(\eta) [1 + 2\Phi_G(\eta, x^i)] , \quad (3.103)$$

$$g_{ij}(\eta, x^i) = a^2(\eta) \delta_{ij} [1 - 2\Psi_G(\eta, x^i)] , \quad (3.104)$$

$$\sqrt{-g}(\eta, x^i) = a^D(\eta) [1 + \Phi_G(\eta, x^i) - (D-1)\Psi_G(\eta, x^i)] . \quad (3.105)$$

Let us collect the linearized connection coefficients in longitudinal gauge

$$\Gamma_{00}^0 = \mathcal{H} + \Phi'_G , \quad (3.106)$$

$$\Gamma_{0i}^0 = \partial_i \Phi_G , \quad (3.107)$$

$$\Gamma_{00}^i = \delta^{ij} \partial_j \Phi_G , \quad (3.108)$$

$$\Gamma_{ij}^0 = \mathcal{H} \delta_{ij} - [2\mathcal{H}(\Phi_G + \Psi_G) + \Psi'_G] \delta_{ij} , \quad (3.109)$$

$$\Gamma_{j0}^i = \mathcal{H} \delta_j^i - \Psi'_G \delta_j^i , \quad (3.110)$$

$$\Gamma_{jk}^i = -\partial_j \Psi_G \delta_k^i - \partial_k \Psi_G \delta_j^i + \partial_l \Psi_G \delta^{il} \delta_{jk} . \quad (3.111)$$

We have the temporal Ricci tensor components

$$R_{00} = -(D-1)\mathcal{H}' + \Delta\Phi_G + (D-1)\Psi''_G + (D-1)\mathcal{H}[\Phi'_G + \Psi'_G] , \quad (3.112)$$

$$R_{0i} = (D-2)\partial_i \Psi'_G + (D-2)\mathcal{H}\partial_i \Phi_G . \quad (3.113)$$

as well as the purely spatial part

$$\begin{aligned} R_{ij} = & [\mathcal{H}' + (D-2)\mathcal{H}^2] \delta_{ij} + (D-3)\partial_i \partial_j \Psi_G - \partial_i \partial_j \Phi_G + \Delta\Psi_G \delta_{ij} \\ & - [\Psi''_G + 2(D-2)\mathcal{H}^2(\Phi_G + \Psi_G) + 2\mathcal{H}'(\Phi_G + \Psi_G)] \delta_{ij} \\ & - [\mathcal{H}\Phi'_G + (2D-3)\mathcal{H}\Psi'_G] \delta_{ij} . \end{aligned} \quad (3.114)$$

This leaves us with

$$\begin{aligned} a^2 R = & (D-1)[2\mathcal{H}' + (D-2)\mathcal{H}^2] \\ & - 2\Delta\Phi_G + (2D-4)\Delta\Psi_G - 2(D-1)\Psi''_G \\ & - 2(D-1)\Phi_G[2\mathcal{H}' + (D-2)\mathcal{H}^2] \\ & - 2(D-1)\mathcal{H}\Phi'_G - 2(D-1)^2\mathcal{H}\Psi'_G . \end{aligned} \quad (3.115)$$

The 00- and 0*i*-components of the linearized Einstein tensor then read

$$G_{00} = \frac{1}{2}(D-1)(D-2)\mathcal{H}^2 + (D-2)\Delta\Psi_G - (D-1)(D-2)\mathcal{H}\Psi'_G , \quad (3.116)$$

$$G_{0i} = (D-2)\partial_i \Psi'_G + (D-2)\mathcal{H}\partial_i \Phi_G . \quad (3.117)$$



The  $ij$ -components of the linearized Einstein tensor read

$$\begin{aligned}
G_{ij} = & - \left[ (D-2)\mathcal{H}' + \frac{1}{2}(D-2)(D-3)\mathcal{H}^2 \right] \delta_{ij} \\
& + (D-2)\Psi_G'' \delta_{ij} + \Delta[\Phi_G - (D-3)\Psi_G] \delta_{ij} - \partial_i \partial_j [\Phi_G - (D-3)\Psi_G] \\
& + (D-2)[2\mathcal{H}' + (D-3)\mathcal{H}^2](\Phi_G + \Psi_G) \delta_{ij} \\
& + (D-2)\mathcal{H}[\Phi_G' + (D-2)\Psi_G'] \delta_{ij}.
\end{aligned} \tag{3.118}$$

The energy-momentum tensor operator of the scalar field theory is given by

$$\hat{T}_{\mu\nu} = \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} - \frac{g_{\mu\nu}}{2} \left[ g^{\alpha\beta} \partial_\alpha \hat{\phi} \partial_\beta \hat{\phi} + \frac{m^2}{\hbar^2} \hat{\phi}^2 \right]. \tag{3.119}$$

We find

$$\hat{T}_{00} = \frac{1}{2} \hat{\phi}' \hat{\phi}' + \frac{1}{2} [1 + 2\Phi_G + 2\Psi_G] \partial_k \hat{\phi} \partial_k \hat{\phi} + \frac{a^2}{2} [1 + 2\Phi_G] \frac{m^2}{\hbar^2} \hat{\phi}^2. \tag{3.120}$$

$$\hat{T}_{0i} = \hat{\phi}' \partial_i \hat{\phi}, \tag{3.121}$$

$$\begin{aligned}
\hat{T}_{ij} = & \partial_i \hat{\phi} \partial_j \hat{\phi} + [1 - 2(\Phi_G + \Psi_G)] (\hat{\phi}')^2 \frac{\delta_{ij}}{2} \\
& - \frac{1}{2} \partial_k \hat{\phi} \partial_k \hat{\phi} \delta_{ij} - \frac{m^2 a^2}{2} [1 - 2\Psi_G] \hat{\phi}^2 \delta_{ij}.
\end{aligned} \tag{3.122}$$

The composite operator (3.119) needs to be renormalized by introducing on the gravity side higher order geometrical counterterms ( $R^2$ , square of the Weyl tensor, Gauss-Bonnet term) as well as lower order geometrical counterterms containing a bare Newton constant and a bare cosmological constant such that after subtraction of UV-divergences we are left with the observable values of the renormalized Newton constant and the renormalized cosmological constant [110]. Contributions of the renormalized higher order geometrical terms completely irrelevant for the studies of large scale structures. Regularizing the two-point functions contained in the energy-momentum tensor (3.119) related to the conditions we spelled out in (3.12). In order to regularize the two-point functions we will split them into infrared and ultraviolet parts by introducing a cut-off in such a way that the conditions (3.12) can be satisfied up to that cut-off. However, for the scope of this chapter we will not have to worry more about renormalization issues.

The semi-classical Einstein equation relates the gravitational potentials to two-point functions of the scalar field in the coincidence limit

$$G_{\mu\nu} = \frac{\hbar}{M_P^2} T_{\mu\nu} = \frac{\hbar}{M_P^2} \langle \hat{T}_{\mu\nu} \rangle. \tag{3.123}$$

We rewrite all Einstein equations in terms of the two-point functions  $F_{00}$ ,  $F_+$ ,  $F_-$  and  $F_{11}$ ,

$$\begin{aligned}
& -(D-2)\mathcal{H}' - \frac{1}{2}(D-2)(D-3)\mathcal{H}^2 \\
& + (D-2)\Psi_G'' + \frac{D-2}{D-1}\Delta[\Phi_G - (D-3)\Psi_G] \\
& + (D-2)[2\mathcal{H}' + (D-3)\mathcal{H}^2](\Phi_G + \Psi_G) + (D-2)\mathcal{H}[\Phi_G' + (D-2)\Psi_G'] \\
& = \frac{\hbar}{2M_P^2} \left\{ [1 + 2(D-2)\Psi_G] a^{-2(D-2)} \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} F_{11} \right. \\
& \quad - \frac{D-3}{D-1} \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} \left( \frac{\Delta_X}{4} + \frac{p^2}{\hbar^2} \right) F_{00} \\
& \quad \left. - \frac{m^2 a^2}{\hbar^2} [1 - 2\Psi_G] \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} F_{00} \right\}, \tag{3.124}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2}(D-1)(D-2)\mathcal{H}^2 + (D-2)\Delta\Psi_G - (D-1)(D-2)\mathcal{H}\Psi_G' \\
& = \frac{\hbar}{2M_P^2} \left\{ a^{-2(D-2)} [1 + 2\Phi_G + 2(D-1)\Psi_G] \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} F_{11} \right. \\
& \quad + [1 + 2\Phi_G + 2\Psi_G] \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} \left( \frac{\Delta_X}{4} + \frac{p^2}{\hbar^2} \right) F_{00} \\
& \quad \left. + a^2 \frac{m^2}{\hbar^2} [1 + 2\Phi_G] \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} F_{00} \right\}, \tag{3.125}
\end{aligned}$$

$$\begin{aligned}
& (D-2)\partial_X\Psi_G' + (D-2)\mathcal{H}\partial_X\Phi_G \\
& = \frac{\hbar}{2a^{(D-2)}M_P^2} [1 + \Phi_G + (D-1)\Psi_G] \\
& \quad \times \left[ \partial_X \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} F_+ - 2\frac{\vec{p}}{\hbar} \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} F_- \right], \tag{3.126}
\end{aligned}$$

$$\begin{aligned}
& \left[ \frac{\delta_{ij}\Delta_X}{D-1} - \partial_i^X \partial_j^X \right] [\Phi_G - (D-3)\Psi_G] \\
& = \frac{\hbar}{M_P^2} \int \frac{d^{D-1}p}{(2\pi\hbar)^{D-1}} \left[ \frac{\partial_i^X \partial_j^X}{4} + \frac{p_i p_j}{\hbar^2} - \frac{\delta_{ij}}{(D-1)} \frac{\Delta_X}{4} - \frac{\delta_{ij}}{D-1} \frac{p^2}{\hbar^2} \right] F_{00}. \tag{3.127}
\end{aligned}$$

## Chapter 4

# Kinetic theory for real scalar fields in curved space-time

Starting from a real scalar quantum field theory with quartic self-interactions and non-minimal coupling to classical gravity, we define four equal-time, spatially covariant phase-space operators through a Wigner transformation of spatially translated canonical operators within a 3+1 decomposition. A subset of these operators can be interpreted as fluctuating particle densities in phase-space whenever the quantum state of the system allows for a classical limit. We come to this conclusion by expressing hydrodynamic variables through the expectation values of these operators and moreover, by deriving the dynamics of the expectation values within a spatial gradient expansion and a 1-loop approximation which subsequently yields the Vlasov equation with a self-mass correction as a limit. We keep an arbitrary classical metric in the 3+1 decomposition which is assumed to be determined semi-classically. Our formalism allows to systematically study the transition from quantum field theory in curved space-time to a classical particle picture for this minimal model of self-interacting, gravitating matter. As an application we show how to include relativistic and self-interaction corrections to existing dark matter models in a kinetic description by taking into account the gravitational slip, vector perturbations and tensor perturbations.

## 4.1 Introduction

It is interesting to understand how the fundamental theory of quantum fields is related to kinetic theory - a description of physics in terms of momentum distributions that is closer to the physics of classical particles. The relation of quantum field theory and kinetic theory has mainly been studied in flat space-times [92] via the generalization of the non-relativistic Wigner transformation [93] to special relativity. However, there are few publications on a generalization of this idea to general curved space-times. The early works by [94, 95] start from an off-shell formulation of two-point functions of the real scalar field and make use of Riemann normal coordinates to obtain a Wigner transformation. Later, reference [96] proposed an off-shell transformation via exponentiated covariant derivatives lifted on the tangent bundle while [98] also

proposed to keep the Wigner transform as an operator without taking expectation values. In this chapter, we consider again a covariant Wigner transformation by combining the ideas of the last two references, but this time by using a formulation in terms of canonical quantum field operators that exhibits on-shell closure and that was already proposed within the longitudinal scalar gauge for the metric in [111] (cf. chapter 3). Thus, we extend our previous work to general curved space-times and derive dynamics for spatially covariant, phase-space operators of the real scalar field, i.e. quadratic operators with a space-time and a spatial momentum dependence. We derive conditions for classical states under which these phase-space operators describe stochastic distributions of classical particles. The metric is assumed to be derived from the semi-classical Einstein equations, thus fixed through expectation values of the field operators. However, it would not change the formalism within this chapter if the metric is assumed to be fixed by an unknown source without any back reaction. In the hybrid approach in [111] (cf. chapter 3) we assumed a stochastic initial density matrix for these two-point functions to account for stochasticity in cosmological perturbations and we have a similar setting in mind for equations in this chapter, although we are focussing primarily on the evolution.

When studying how classical equations emerge from a quantum field description, there are two major limits which are a priori of different nature. The first limit concerns the classical stochastic field theory limit of quantum field theory which approximates non-commutating field operators with commuting field operators. At the level of two-point functions, we can also rephrase this limit as having a particle number (the state dependent part) which is much bigger than the quantum contribution originating from the non-commutativity of canonical field operators.<sup>1</sup> It is thus the particle number and not the state independent (vacuum) part of the propagator that dominates loop calculations in this limit. The classical stochastic field theory limit of quantum field theory should a priori be considered separately from the classical particle limit of quantum field theory which involves an expansion in temporal and spatial gradients with respect to energies and momenta,  $\Delta E \Delta t \gg \hbar$  and  $\Delta p \Delta x \gg \hbar$ . Such a limit is possible after subtracting quantum UV-modes or virtual particles of the Wigner-transformed two-point function which can also be viewed as a special case of coarse-graining. Here, we follow a procedure that is closely related to normal-ordering and involves subtraction of the state-independent part of the two-point function in a normal neighbourhood. The spatial gradient expansion results from assuming that the remaining, state-dependent two-point functions around a collective point on a spatial hypersurface are correlated only in a small neighbourhood relative to spatial gradients taken with respect to that collective point. Thus, the spatial gradient expansion constitutes a separation of spatial scales and it gives rise to an infinite series in the dynamical equations which needs to be truncated. The situation is different for temporal gradients, at least in a one-loop or Gaussian state approximation, since such a state allows for an on-shell closure of the involved two-point functions and takes the form of four first-order differential equations in time. However, enforcing additionally the

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<sup>1</sup>Note that we are dealing with bosons, a large particle number per momentum for fermions can only be achieved by coarse graining in phase-space.

limit  $\Delta E \Delta t \gg \hbar$  for this closed set of equations reduces them to leading order in spatial gradients to the dynamics of non-interacting classical particles, i.e. dynamics described by a collisionless Boltzmann equation or in the context of curved space-time, the Vlasov equation. The effect of self-interactions is analogous to Minkowski space-time where one-loop corrections provide a space-time dependent mass shift [99]. Out of the scope of this chapter is classical particle scattering as it appears on the right-hand-side of the Boltzmann equation. It can be obtained from the quantum field theory by including two-loop processes which are subsequently approximated with a quasi-particle picture to close on-shell.

A direct application for the transition from quantum field theory in curved space-time to classical particle physics lies in cosmology and astrophysics. This transition should be studied carefully since in the context of cosmology and astrophysics, physics has many different faces and so it happens that for example dark matter [84] is - among many other possible models - a priori believed to be equally well described by a stochastic distribution function of classical non-relativistic, non-interacting, massive particles or a condensate of a stochastic scalar field. The condensate description is easily related to the microscopic scalar field theory whereas the relation between classical particle dark matter to a microscopic theory is less clear and the result of this chapter is not only to show that it is indeed well described by a real quantum scalar field but also to systematically keep track of all approximations that lead to the classical particle picture. Maintaining the classical particle picture is a question of scales as we pointed out in the paragraph above and the natural question to ask is whether these scales can be related to other significant scales in the study of large-scale structures (i.e. the scale of non-linearity  $k < k_{nl} \sim 0.3 \text{ Mpc}^{-1}$  [24] or galactic scales  $\sim 10 \text{ kpc}^{-1}$ ). Apart from the predictability, fundamental dark matter descriptions may also lead to a transfer of calculational techniques from quantum field theory to make progress on analytical or numerical results in the studies of large-scale structures.<sup>2</sup>

Let us give an overview of the chapter. We will start from an interacting real scalar field theory that is non-minimally coupled to gravity via the Ricci scalar with an arbitrary classical metric  $g_{\mu\nu}$  in a 3+1 decomposition and discuss equations of motion and renormalization in the operator formalism. We introduce four composite spatially covariant quantum field operators  $\hat{F}_{\phi\phi}, \hat{F}_{\Pi\phi}, \hat{F}_{\phi\Pi}, \hat{F}_{\Pi\Pi}$  by integrating combinations of covariantly translated canonical operators over the tangent space of spatial hypersurfaces. We rescale them to yield four dimensionally equivalent phase-space density operators  $\hat{f}_1^\pm(x^\mu, p_j), \hat{f}_{2,3}(x^\mu, p_j)$  with a dependence on the on-shell momenta. These operators are in fact scalars on the tangent bundle of spatial hypersurfaces for any time. Moreover, we discuss that

<sup>2</sup>This point has already been pursued by deriving a Vlasov equation from Wigner transforming the non-relativistic Schrödinger-Poisson system for condensates [30, 31, 89, 90, 112]. However, the degrees of freedom for one-point functions suffice only to provide an independent mass density and momentum density on the microscopic level. By taking into account coarse-graining, certain momentum distributions may be modelled by exchanging microscopic degrees of freedom below the cut-off for higher moments in phase-space. The connected part of the phase-operators considered in this chapter does account for these degrees of freedom without any coarse-graining.

the state-independent part of these operators should be subtracted in a normal neighbourhood to yield a finite energy-momentum tensor. As a first step towards the classical particle limit, we rewrite hydrodynamic variables like energy density, pressure and fluid velocity in terms of the phase-space operators  $\langle \hat{f}_{1,2,3} \rangle$ . Afterwards, we derive the dynamics for expectation values  $\langle \hat{f}_{1,2,3} \rangle$  in a spatial gradient expansion  $\Delta x \Delta p \gg \hbar$  and a one-loop approximation. We then combine two out of these four into the most important phase-space density operator  $\hat{f}_1 = \hat{f}_1^+ + \hat{f}_1^-$  whose expectation value can be related to a classical Boltzmann distribution under certain conditions. Namely, to leading order in the classical particle limit  $\Delta x \Delta p \gg \hbar$ ,  $\Delta t \Delta E \gg \hbar$ , the dynamics of the expectation value  $\langle \hat{f}_1 \rangle$  resembles the dynamics of the classical on-shell one-particle phase-space density  $f_{\text{cl}}$  for gravitating particles which is given by the truncated BBGKY hierarchy, which is to leading order the Vlasov equation in curved space-time,

$$\left[ p^\mu \partial_\mu + p_\mu p^\nu \Gamma^\mu_{\nu i} \frac{\partial}{\partial p_i} \right] f_{\text{cl}}(x^\mu, p_j) = 0, \quad (4.1)$$

$$p^0(x^\mu, p_j) \equiv \sqrt{(g^{0j} p_j)^2 - g^{00}(m^2 + g^{ij} p_i p_j)}. \quad (4.2)$$

One could in principle capture higher n-particle distributions by integrating out the gravitational constraint fields. However, here we are mostly interested in giving a field-theoretic description of cold dark matter with massive particles where two- and higher n-particle densities  $f_n(x^0, (x^i, p_j)^1, (x^i, p_j)^2, \dots)$  can be neglected [109].

We work in units where  $c = 1$  with a mostly plus signature  $(-, +, +, +)$ .

## 4.2 Canonical operator formalism in curved space-time

As opposed to previous approaches for off-shell Wigner transformations in curved space-times [94–96, 98], we want to obtain on-shell phase-space densities from the very beginning by working with a Hamiltonian formulation as we have done it for the scalar, linearized longitudinal gauge in [111]. For convenience we recap the Hamiltonian formulation at the classical level for a massive, real, self-interacting scalar field  $\phi$  in curved space-time with metric  $g_{\mu\nu}$  in the ADM formalism which is for example discussed in [113] or [114]. We then quantize the matter field and write down the Heisenberg equations for the canonical field operators.

### 4.2.1 ADM decomposition and equations of motion

We begin by writing down the classical action for the theory

$$S_{\text{tot}}[\phi, g_{\mu\nu}] = S_g[g_{\mu\nu}] + S_m[\phi, g_{\mu\nu}] , \quad (4.3)$$

where the matter action  $S_m$  is given by

$$S_m[\phi, g_{\mu\nu}] = - \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} \frac{m^2}{\hbar^2} \phi^2 + \frac{1}{2} \xi R \phi^2 + \frac{1}{4!} \frac{\lambda}{\hbar} \phi^4 \right] , \quad (4.4)$$

and the gravitational action  $S_g$  reads

$$S_g[g_{\mu\nu}] = \frac{M_P^2}{2\hbar} \int d^4x \sqrt{-g} R . \quad (4.5)$$

Here,  $R$  denotes the four-dimensional Ricci scalar, we have a tree level mass parameter  $m^2$  and we allow for a non-minimal coupling as well as a self-interaction given by the tree-level parameters  $\xi$  and  $\lambda$ , respectively. We continue by slicing the space-time into spatial hypersurfaces  $\Sigma_t$  that are determined by constant values of a four-scalar field  $t(x^\mu)$  whose corresponding vector field  $t^\mu$ , obeying  $t^\mu \nabla_\mu t = 1$ , is given by

$$t^\mu = N n^\mu + N^\mu , \quad \partial_t = t^\mu \partial_\mu , \quad (4.6)$$

where  $N$  is the lapse function and  $N^\mu$  is the shift vector such that  $n^\mu$  is the vector normal to the spatial hypersurface [115]. We note that  $N$  is a four-scalar given by

$$g(\partial_t, \partial_t) = -N^2 + N_\mu N^\mu , \quad (4.7)$$

and that

$$\partial_0 \neq \partial_t \text{ in general ,} \quad (4.8)$$

i.e. we can in principle choose to work with a zero coordinate that is different from our choice of time  $t$  used for the slicing into hypersurfaces. The next step

is to define a projection tensor

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \quad (4.9)$$

This allows us to write down the extrinsic curvature associated with our choice of the normal vector field as

$$K_{\mu\nu} = -\nabla_\nu n_\mu - a_\mu n_\nu = -\gamma_\mu^\alpha \nabla_\alpha n_\nu = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}, \quad (4.10)$$

where  $\mathcal{L}_n$  denotes the Lie derivative along  $n^\mu$  and the acceleration is given by

$$a_\mu = n^\alpha \nabla_\alpha n_\mu = \gamma_\mu^\nu \nabla_\nu \log N. \quad (4.11)$$

The Ricci scalar can be rewritten as [116]

$$R = {}^{(3)}R + K^2 + K_{\mu\nu} K^{\mu\nu} - \frac{2}{N} (\partial_t - N^\mu \partial_\mu) K - \frac{2}{N} {}^{(3)}\nabla_\mu {}^{(3)}\nabla^\mu N, \quad (4.12)$$

where  ${}^{(3)}R$  is the three-dimensional Ricci scalar on spatial hypersurfaces given by

$${}^{(3)}R_{\mu\nu\rho}{}^\sigma [\gamma_\sigma^\alpha v_\alpha] = [{}^{(3)}\nabla_\mu {}^{(3)}\nabla_\nu - {}^{(3)}\nabla_\nu {}^{(3)}\nabla_\mu] [\gamma_\rho^\alpha v_\alpha], \quad (4.13)$$

for some dual vector  $v_\alpha$  and the covariant derivative on spatial hypersurfaces reads

$${}^{(3)}\nabla_\mu [\gamma_\nu^\rho v_\rho] = \gamma_\mu^\rho \gamma_\nu^\sigma \nabla_\rho [\gamma_\sigma^\alpha v_\alpha]. \quad (4.14)$$

We have already remarked that the zero coordinate  $x^0$  and the scalar field  $t$  are a priori not related. However, as is often done in the ADM decomposition, we choose the zero coordinate  $x^0$  to coincide with  $t$ ,

$$x^0 = t. \quad (4.15)$$

We then have the following component decomposition of the metric

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^i N_i & N_i \\ N_i & \gamma_{ij} \end{pmatrix}, \quad (4.16)$$

$$g^{\mu\nu} = \begin{pmatrix} -N^{-2} & N^{-2} N^i \\ N^{-2} N^i & \gamma^{ij} - N^{-2} N^i N^j \end{pmatrix}, \quad (4.17)$$

$$\sqrt{-g} = N \gamma^{1/2}, \quad (4.18)$$

with

$$n^\mu = N^{-1}(1, -N^i), \quad n_\mu = (-N, 0), \quad (4.19)$$

and  $\gamma_{ij}$  being the induced metric on the spatial hypersurface. The action for gravity evaluates up to boundary terms<sup>3</sup> to

$$S_g [N, N^k, \gamma_{ij}] = \frac{M_P^2}{2\hbar} \int N dt \gamma^{1/2} d^3x \left[ {}^{(3)}R - K^2 + K_{ij} K^{ij} \right], \quad (4.20)$$

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<sup>3</sup>Since we are only interested in a classical approximation to gravity, these boundary terms can be safely neglected, as they do not influence the dynamics in the semi-classical approximation that we will be using.



where we used

$$\partial_t \log \gamma^{1/2} = -NK + {}^{(3)}\nabla_i N^i. \quad (4.21)$$

We define the canonical momentum as a classical field configuration by means of the scalar field  $t(x^\mu)$ ,

$$\Pi = \frac{\delta S_m}{\delta[\partial_t \phi]} = \sqrt{-g} \frac{n^\mu}{N} \partial_\mu \phi = \frac{\gamma^{1/2}}{N} [\partial_t - N^j \partial_j] \phi, \quad (4.22)$$

and find for the classical matter action

$$S_m = \int N dt \gamma^{1/2} d^3x \left[ \frac{1}{2} \frac{\Pi^2}{\gamma} - \frac{1}{2} \gamma^{ij} \partial_i \phi \partial_j \phi - \frac{1}{2} \frac{m^2}{\hbar^2} \phi^2 - \frac{1}{2} \xi R \phi^2 - \frac{1}{4!} \frac{\lambda}{\hbar} \phi^4 \right], \quad (4.23)$$

which is manifestly invariant under spatial coordinate transformations. Since we will be dealing mostly with  $3+1$  variables in the main parts of the chapter, we would like to mention once that it is not the spatial Ricci scalar  ${}^{(3)}R$  but the four-dimensional Ricci scalar  $R$  that enters the non-minimal coupling to the matter field  $\phi$  and we will sometime refrain from expanding it in a  $3+1$  split in order to save space.

We intend to quantize the matter field  $\phi$  in a curved space-time with a classical metric  $g_{\mu\nu}$  which is an excellent approximation whenever momenta are much smaller than the Planck mass. The quantum theory in the operator formalism is formally specified by the time-evolution or the Hamilton operator  $\hat{H}$  in (4.24), the Heisenberg equations motion (4.26) and (4.27) as well as the equal-time commutation relation (4.25). The Hamilton operator  $\hat{H}$  is a functional of the canonical (bare) field operators  $\hat{\phi}_B$  and  $\hat{\Pi}_B$ . Moreover, it depends on the bare couplings  $m_B^2$ ,  $\xi_B$ ,  $\lambda_B$  as well as the classical, possibly stochastic metric  $g_{\mu\nu}$  in the  $3+1$  split,

$$\begin{aligned} \hat{H} = \int_{\Sigma_t} N \gamma^{1/2} d^3x & \left[ \frac{1}{2} \gamma^{-1} \hat{\Pi}_B^2 + \gamma^{-1/2} N^{-1} \hat{\Pi}_B N^j \partial_j \hat{\phi}_B + \frac{1}{2} \gamma^{ij} \partial_i \hat{\phi}_B \partial_j \hat{\phi}_B + \frac{1}{2} \frac{m_B^2}{\hbar^2} \hat{\phi}_B^2 \right. \\ & + \frac{1}{2} \xi_B R \hat{\phi}_B^2 + \frac{1}{4!} \frac{\lambda_B}{\hbar} \hat{\phi}_B^4 \left. \right] - \hat{\mathbf{1}} \left( \Lambda_B + \kappa_B R \right. \\ & \left. + \alpha_{1B} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + \alpha_{2B} R_{\mu\nu} R^{\mu\nu} + \alpha_{3B} R^2 \right), \quad (4.24) \end{aligned}$$

where we refrained from a  $3+1$ -split of the gravitational counterterms. The counterterms are unavoidable in order to obtain a finite Hamiltonian, once we solve the Heisenberg equations of motion and impose the equal-time commutation relation

$$[\hat{\phi}_B(x^0, x^i), \hat{\Pi}_B(x^0, \tilde{x}^i)] = i\hbar \delta^{(3)}(x^i - \tilde{x}^i), \quad (4.25)$$

where other combinations of canonical fields at equal time commute. The Heisenberg equations of motion read

$$\mathcal{L}_t \hat{\phi}_B = \partial_t \hat{\phi}_B = \frac{N}{\gamma^{1/2}} \hat{\Pi}_B + N^j \partial_j \hat{\phi}_B, \quad (4.26)$$

$$\begin{aligned} \mathcal{L}_t \hat{\Pi}_B = \partial_t \hat{\Pi}_B + \hat{\Pi}_B \partial_\mu t^\mu &= \partial_j \left[ N^j \hat{\Pi}_B \right] + \partial_i \left[ N \gamma^{1/2} \gamma^{ij} \partial_j \hat{\phi}_B \right] \\ &\quad - N \gamma^{1/2} \left[ \frac{m_B^2}{\hbar^2} \hat{\phi}_B + \xi_B R \hat{\phi}_B + \frac{1}{6} \frac{\lambda_B}{\hbar} \hat{\phi}_B^3 \right]. \end{aligned} \quad (4.27)$$

In covariant notation we find<sup>4</sup>

$$\frac{n^\mu}{N} \nabla_\mu \hat{\phi}_B = \frac{\hat{\Pi}_B}{\sqrt{-g}}, \quad (4.29)$$

$$\nabla_\mu \left[ N n^\mu \frac{\hat{\Pi}_B}{\sqrt{-g}} \right] = \nabla_\mu \left[ \gamma^{\mu\nu} \nabla_\nu \hat{\phi}_B \right] - \frac{m_B^2}{\hbar^2} \hat{\phi}_B - \xi_B R \hat{\phi}_B - \frac{1}{6} \frac{\lambda_B}{\hbar} \hat{\phi}_B^3, \quad (4.30)$$

which is equivalent to

$$\square \hat{\phi}_B = \nabla_\mu \nabla^\mu \hat{\phi}_B = \frac{m_B^2}{\hbar^2} \hat{\phi}_B + \xi_B R \hat{\phi}_B + \frac{1}{6} \frac{\lambda_B}{\hbar} \hat{\phi}_B^3. \quad (4.31)$$

In contrast to their classical counterparts, the latter equations exhibit a couple of subtleties of which we have to be aware if we want to formulate phase-space densities that are based on quantum field operators. A very important remark we would like to spell out right away is that even the renormalized version of (4.31) holds strictly speaking only for n-point functions at non-coincident points  $x_1, x_2, \dots, x_n$  in space-time. The equations of motion do not need to hold for monomials of operators in the coincident limits  $x_i \rightarrow x_j$  due to anomalies that emerge from renormalization (see the summary of section 3.3 in [117]) and we will comment more on this anomaly when we discuss the normal ordered energy-momentum tensor entering the semi-classical Einstein equation in the next section.

Renormalizing n-point functions in the coincident limit or similarly normal ordering the composite operators appearing within them is unavoidable due to the constraints that have to be imposed on the canonical operators via the commutation relation (4.25). These constraints distinguish the quantum field theory from a stochastic field theory whose (commuting) field operators may be given any initial values (formulated in terms of all n-point functions) which are evolved via the analogue of (4.28) and (4.30) or equivalently of (4.31) in the stochastic field theory. Apart from the dynamics, which is altered by the non-commutativity, the quantum field theory is thus also different from a stochastic field theory in the sense that their action is constrained to yield a certain value for any n-point function since the commutation relation (4.25) is independent

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<sup>4</sup>Note that

$$\nabla_\mu \left[ \gamma^{\mu\nu} \nabla_\nu \hat{\phi}_B \right] = {}^{(3)}\nabla_\mu {}^{(3)}\nabla^\mu \hat{\phi}_B + a_\mu {}^{(3)}\nabla^\mu \hat{\phi}_B = {}^{(3)}\nabla_i {}^{(3)}\nabla^i \hat{\phi}_B + {}^{(3)}\nabla_i \log N {}^{(3)}\nabla^i \hat{\phi}_B. \quad (4.28)$$

of any state with respect to which we evaluate these operators. We can see this for example by looking at the two Wightman-functions that solve

$$\begin{aligned} \square_x \langle \hat{\phi}_B(x) \hat{\phi}_B(y) \rangle &= \frac{m_B^2}{\hbar^2} \langle \hat{\phi}_B(x) \hat{\phi}_B(y) \rangle + \xi_B R(x) \langle \hat{\phi}_B(x) \hat{\phi}_B(y) \rangle \\ &\quad + \frac{1}{6} \frac{\lambda_B}{\hbar} \langle \hat{\phi}_B^3(x) \hat{\phi}_B(y) \rangle, \end{aligned} \quad (4.32)$$

where the expectation value refers to an arbitrary state and the same equation holds if the differential operator including the Ricci scalar acts on the other coordinate. No matter which state we choose, the equal-time commutation relation (4.25) forces us to pick up a bi-solution (solution in both arguments) of equation (4.32) which is singular in the limit  $x \rightarrow y$ . It also forces us to bestow the Wightman functions with an imaginary part for non-equal times and we note that it is the same singular behaviour that yields Greens functions for the Klein-Gordon operator (see e.g. [118]). Still, the state of the system can very well possess additional non-singular behaviour which is exactly the part which is suitable to be described by the kinetic equations we derive below in some approximation. A clear example for this contribution of singular and non-singular behaviour to the two-point function is a thermal state in Minkowski space-time which contains the vacuum contribution as well as a finite temperature dependent piece (see e.g. [119]).

Moreover, the distributional nature of quantum fields forces us to renormalize parameters of the theory as soon as composite operators such as  $\hat{\phi}^2(x)$  enter physical observables. This becomes apparent if we consider the energy-momentum tensor such that we have to renormalize gravitational couplings as we will review below and it will continue to do so at the level of self-interactions. Looking at the operator equations (4.31), we can already see that the mass  $m_B$ , the non-minimal coupling parameter  $\xi_B$  and coupling  $\lambda_B$  will get renormalized since they have to balance the divergent pieces of the composite operator  $\hat{\phi}^3$  in the coincident limit which itself may be expressible as a formal series in  $\lambda_B$  of composite free-field operators. A better way of writing (4.31) makes use of a normal ordering procedure that has been developed in the context of algebraic quantum field theory in curved space-time (see [120–122] and references therein, in particular [123]). Defining the renormalized field operator  $\hat{\phi}$  and the renormalized couplings  $m, \xi, \lambda$ , equation (4.31) now reads

$$\square \hat{\phi} = \frac{m^2}{\hbar^2} \hat{\phi} + \xi R \hat{\phi} + \frac{1}{6} \frac{\lambda}{\hbar} : \hat{\phi}^3 : , \quad (4.33)$$

where  $: (\cdot) :$  denotes a normal ordering procedure whose essential ideas are explained in the review [120]. The main observation is that a class of well-defined states - called Hadamard states that cover for example Gaussian states and thermal states - have the same singular behaviour concerning the coincidence limit of their two-point function in the free field limit. The singular behaviour is given in terms of the Hadamard parametrix  $H(x, y)$  which is a local (normal neighbourhood) bi-solution to the free Klein-Gordon equation (4.32) ( $\lambda = 0$ ) up to state-dependent terms that remain smooth in the coincident limit,

it reads

$$H(x, y) = \frac{\hbar}{4\pi^2} \left[ \frac{u(x, y)}{\sigma(x, y) + i0^+ \tau(x, y)} + v(x, y) \log \left[ \mu^2 \left( \sigma(x, y) + i0^+ \tau(x, y) \right) \right] \right], \quad (4.34)$$

where the bi-scalar  $\sigma(x, y)$  is the signed squared geodesic distance between two points  $x, y$  in space-time (+ for space-like and  $-$  for time-like separations),  $\tau(x, y)$  is the difference of some global time function between  $y$  and  $x$  and  $\mu$  is an arbitrary energy scale. Moreover, the bi-scalars  $u$  and  $v$  are smooth, real valued and depend on the squared mass as well as local geometric quantities. The bi-scalar  $v$  may be written as a formal series in the signed squared geodesic distance  $\sigma$  whose coefficients can be determined iteratively [118]. The two-point function  $w_2(x, y)$  of any Hadamard state has then locally the form,

$$\begin{aligned} w_2(x, y) &= \frac{\hbar}{4\pi^2} \left[ \frac{u(x, y)}{\sigma(x, y) + i0^+ \tau(x, y)} + \left( \sum_{n=0}^N v_n(x, y) \sigma^n(x, y) \right) \log \left[ \mu^2 \left( \sigma(x, y) + i0^+ \tau(x, y) \right) \right] \right] + R_{N,w} \\ &= H_N(x, y) + R_{N,w}, \end{aligned} \quad (4.35)$$

where  $R_{N,w}$  is a smooth,  $N + 1$ -times differentiable remainder that depends on the state. Normal ordering of a quadratic monomial of off-shell field operators with  $N$  derivatives at the same space-time point is achieved by covariant point-splitting, subtracting the Hadamard parametrix  $H_N(x, y)$ , taking the coincidence limit and fixing a finite number of ambiguities which can be related to the arbitrary energy-scale  $\mu$ . This fixation may be achieved by demanding a certain state or the value of certain renormalized couplings. The deviation from minimal normal ordering (i.e. subtracting exclusively terms that diverge in the coincidence limit) is for some monomials necessary to fulfil reasonable requirements as for example stress-energy conservation [124]. The latter observation may be understood as a consequence of consistently defining algebraic quantum field theory in curved space-time [123]. The procedure of normal ordering quadratic monomials can also be generalized to higher-order monomials and a rigorous definition is given in equation (59) in [123], where it is also discussed that normal ordering obeys the Leibniz rule for off-shell field operators. An anomaly in the free scalar field theory (i.e. failure of the equations of motion to be satisfied for composite operators) is eventually related to the normal ordered operator  $:\hat{\phi}(x) \left[ \square_x - \frac{m^2}{\hbar^2} - \xi R(x) \right] \hat{\phi}(x) : \propto \hat{1} Q(x)$  where  $Q(x)$  is a classical field constructed from purely geometrical quantities which cannot be set to zero via counterterm ambiguities (a detailed calculation via point-splitting is for example available in [125], where, however, the counterterm ambiguities still need to be applied). It is the latter observation that forbids us to enforce the Heisenberg equations for monomials in the coincident limit. It translates

to the fact the energy-momentum tensor acquires a trace even if it was classically zero. Moreover, since the above anomaly can be written as an operator identity, it is independent of the state in which one would like to evaluate this operator and thus cannot be argued away for example by choosing a state that has some notion of classicality. It might be negligible but is strictly speaking always present. The whole program of algebraic quantum field theory in curved space-time is then carried forward to include also interactions by defining time-ordered products in order to relate free and interacting field operators via a formal power series in the coupling constant.

### 4.2.2 Semi-classical Einstein equations

The difficulties of quantum field theory in curved space-time are in particular revealed if we ask how to determine the classical metric  $g_{\mu\nu}$ . The first option is to postulate the metric to have a certain form by means of additional degrees of freedom that couple to the field  $\phi$  only indirectly via gravity neglecting any back reaction. The second option is to include back-reactions via the renormalized semi-classical Einstein equations,

$$G_{\mu\nu}[g_{\mu\nu}] = \frac{\hbar}{M_P^2} \langle : \hat{T}_{\mu\nu}[\hat{\phi}, g_{\mu\nu}] : \rangle, \quad (4.36)$$

where the normal ordering regularizes the infinite contribution of composite operators such that we are dealing with finite quantities but also with renormalized couplings. The quantum expectation values are taken with respect to some yet unspecified state with possibly stochastic initial conditions (for example in order to account for cosmological setups). Ambiguities in the normal ordering prescription can be interpreted as a change of couplings of a renormalized effective action on the gravitational side. These ambiguities may be fixed by demanding that the left-hand side of the Einstein equation remains in its classical form without a cosmological constant. A standard way to carry out this renormalization is to make use of the effective action that is defined in terms of a path integral which implicitly makes use of a preferred state. This state is unambiguous in Minkowski space but fails to be so for general curved space-times. Nonetheless, in the context of slowly-varying space-times one can pick an adiabatic vacuum and calculate the renormalized effective action by methods such as dimensional regularization (this is discussed for example in the standard reference [110] as well as the more recent textbook [126]). Thus, we choose the renormalization parameters for gravity such that they shall neither contain a cosmological constant, nor higher-order geometrical terms other than the four-dimensional Ricci scalar  $R$  so that it agrees with the classical action  $S_g$  and the renormalized Planck mass is given by  $M_P \approx 2.45 \times 10^{18}$  GeV which is another way of phrasing that corrections to the classical Einstein equations without a cosmological constant can safely be neglected at the energy scales that we are preparing experiments at.

Keeping in mind that composite operators diverge in the coincidence limit

and that we have to be careful evaluating the equations of motion, the energy-momentum operator in (4.36) reads formally

$$\begin{aligned} \hat{T}_{\mu\nu} = & \partial_\mu \hat{\phi}_B \partial_\nu \hat{\phi}_B + \xi_B (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu + R_{\mu\nu}) \hat{\phi}_B^2 \\ & - \frac{g_{\mu\nu}}{2} \left[ \partial^\alpha \hat{\phi}_B \partial_\alpha \hat{\phi}_B + \frac{m_B^2}{\hbar^2} \hat{\phi}_B^2 + \xi_B R \hat{\phi}_B^2 + \frac{1}{12} \frac{\lambda_B}{\hbar} \hat{\phi}_B^4 \right]. \end{aligned} \quad (4.37)$$

Let us rewrite the energy-momentum tensor given by (4.37) in terms of the canonical field operators and the 3+1 decomposition

$$\hat{T}_{\mu\nu} = \hat{E} n_\mu n_\nu + \hat{P}_\mu n_\nu + \hat{P}_\nu n_\mu + \hat{S}_{\mu\nu}. \quad (4.38)$$

One can verify that the non-trivial equation of motion of the canonical field operators is encoded in the bare spatial operator  $\hat{S}_{\mu\nu}$ . Without going into any details we now take for granted that we have a normal ordering procedure available as we sketched it above and that this procedure includes perturbative interactions as well. The finite energy, momentum and stress densities with respect to a normal observer,

$$E = \langle : \hat{E} : \rangle, \quad P_j = \langle : \hat{P}_j : \rangle, \quad S_{jk} = \langle : \hat{S}_{jk} : \rangle, \quad (4.39)$$

are then according to (4.41) to (4.43) expressible in terms of the normal ordered equal-time correlators of renormalized fields  $\langle : \hat{\Pi}^2 : \rangle, \langle : ({}^{(3)}\nabla_k)^m \hat{\phi} ({}^{(3)}\nabla_j)^{2-m} \hat{\phi} : \rangle, \langle : \hat{\phi}^4 : \rangle, \dots$  as well as in terms of the renormalized couplings  $m^2$ ,  $\xi$  and  $\lambda$ . The spatial tensor  $S_{jk}$  will always get an anomalous contribution  $\gamma_{kj}Q$  after evaluating the normal ordered equation of motion, despite the viewpoint that this anomalous contribution may be safely neglected for a certain choice of a

state. We have<sup>5</sup>

$$E = \frac{1}{2} \left[ \gamma^{-1} \langle : \hat{\Pi}^2 : \rangle + \langle : {}^{(3)}\nabla^k \hat{\phi} {}^{(3)}\nabla_k \hat{\phi} : \rangle - 2\xi {}^{(3)}\nabla^k {}^{(3)}\nabla_k \langle : \hat{\phi}^2 : \rangle + \frac{m^2}{\hbar^2} \langle : \hat{\phi}^2 : \rangle \right. \\ \left. - 2\xi K \gamma^{-1/2} \left( \langle : \hat{\phi} \hat{\Pi} : \rangle + \langle : \hat{\Pi} \hat{\phi} : \rangle \right) + \xi ({}^{(3)}R + K^2 - K_{ij} K^{ij}) \langle : \hat{\phi}^2 : \rangle \right. \\ \left. + \frac{1}{12} \frac{\lambda}{\hbar} \langle : \hat{\phi}^4 : \rangle \right], \quad (4.41)$$

$$P_j = -\frac{1}{2} \gamma^{-1/2} \left[ \langle : \hat{\Pi} {}^{(3)}\nabla_j \hat{\phi} : \rangle + \langle : {}^{(3)}\nabla_j \hat{\phi} \hat{\Pi} : \rangle \right] \\ + \xi {}^{(3)}\nabla_j \left[ \gamma^{-1/2} \langle : \hat{\Pi} \hat{\phi} : \rangle + \gamma^{-1/2} \langle : \hat{\phi} \hat{\Pi} : \rangle \right] \\ + \xi \left[ {}^{(3)}\nabla^m K_{jm} - {}^{(3)}\nabla_j K + K_j^m {}^{(3)}\nabla_m \right] \langle : \hat{\phi}^2 : \rangle, \quad (4.42)$$

$$S_{jk} = \langle : {}^{(3)}\nabla_j \hat{\phi} {}^{(3)}\nabla_k \hat{\phi} : \rangle - \xi {}^{(3)}\nabla_j {}^{(3)}\nabla_k \langle : \hat{\phi}^2 : \rangle \\ - \xi K_{jk} \gamma^{-1/2} \left[ \langle : \hat{\Pi} \hat{\phi} : \rangle + \langle : \hat{\phi} \hat{\Pi} : \rangle \right] + 2\xi Q \gamma_{jk} \\ + \xi \left[ {}^{(3)}R_{jk} + K K_{jk} - 2K_{jm} K_k^m - \mathcal{L}_n K_{jk} + N^{-1} {}^{(3)}\nabla_j {}^{(3)}\nabla_k N \right] \langle : \hat{\phi}^2 : \rangle \\ - \frac{1}{2} (1 - 4\xi) \gamma_{jk} \left[ -\gamma^{-1} \langle : \hat{\Pi}^2 : \rangle + \langle : {}^{(3)}\nabla^m \hat{\phi} {}^{(3)}\nabla_m \hat{\phi} : \rangle + \frac{m^2}{\hbar^2} \langle : \hat{\phi}^2 : \rangle \right. \\ \left. + \xi R \langle : \hat{\phi}^2 : \rangle + \frac{1}{12} \frac{1 - 8\xi}{1 - 4\xi} \frac{\lambda}{\hbar} \langle : \hat{\phi}^4 : \rangle \right]. \quad (4.43)$$

It should be clear that the trace  $S$  appearing in these equations is not to be confused with the total classical action  $S_{\text{tot}}$ .

As explained below equation (4.36) we now choose the renormalized couplings on the left-hand-side of the semi-classical Einstein equation - and thus the normal ordering ambiguities - such, that we are dealing with classical gravity without a cosmological constant. We then have the expressions found in [127],

$$\frac{1}{2} \left[ {}^{(3)}R + K^2 - K_{ij} K^{ij} \right] = \frac{\hbar}{M_P^2} E, \quad (4.44)$$

$${}^{(3)}\nabla_j K_i^j - {}^{(3)}\nabla_i K = \frac{\hbar}{M_P^2} P_i, \quad (4.45)$$

$$\mathcal{L}_{Nn} K_{ij} + {}^{(3)}\nabla_i {}^{(3)}\nabla_j N \\ - N \left[ {}^{(3)}R_{ij} + K K_{ij} - 2K_{im} K_j^m \right] = \frac{\hbar}{M_P^2} N \left[ \frac{1}{2} (S - E) \gamma_{ij} - S_{ij} \right], \quad (4.46)$$

where we restricted the expressions to spatial indices for tensors in the spatial hypersurface. As we remarked in the beginning, the split of the two-point function into a part which is singular in the coincident limit (state-independent) and non-singular (state-dependent) part can be read as a split into a manifestly microscopic part inherited from the quantum commutation relation and a part

<sup>5</sup>As a cross-check, we verify that also in this 3+1-split the anomalous trace is indeed given by  $Q$  for the configuration  $m^2 = \lambda = 0$  and  $\xi = 1/6$ ,

$$\langle : \hat{T}_\mu^\mu : \rangle = S - E \equiv S_k^k - E, \quad (S - E)|_{m^2=\lambda=0, \xi=1/6} = Q|_{m^2=\lambda=0, \xi=1/6}. \quad (4.40)$$

that in principle allows for a macroscopic distribution of particles (among many other possibilities). Our goal is now to rewrite the quantities  $E$ ,  $P_i$  and  $S_{ij}$  in terms of integrated phase-space densities which allow for a particle distribution interpretation in certain limits.

### 4.3 Wigner operators from canonical fields

We have reviewed the Hamiltonian formulation for the real scalar field operator and its conjugate momentum in curved space-time with a classical metric that is given through the semi-classical Einstein equation. Up to this point we have a description of matter in terms of canonical field operators. We would like to get to a different description by retaining the operator nature and forming a set of transformed objects  $\hat{f}_i(x^\mu, p_i)$  out of the canonical field operators that depend on phase-space variables, where  $x^\mu$  is a collective space-time point and  $p_i$  labels momenta distributed around it. We will construct four such operators  $\hat{f}_1^\pm$ ,  $\hat{f}_2$  and  $\hat{f}_3$ . The first two  $\hat{f}_1^\pm$  naturally combine into a single operator  $\hat{f}_1$  which may be straightforwardly interpreted as a fluctuating particle distribution in phase-space under certain conditions. This means that whenever the state, that the operator  $\hat{f}_1$  eventually acts on can be characterized as classical, we want to interpret the operator  $\hat{f}_1$  as a classical, fluctuating phase-space density in the sense of statistical mechanics. The remaining two phase-space densities  $\hat{f}_2$  and  $\hat{f}_3$  stem from the relativistic description of the Klein-Gordon equation and do represent degrees of freedom which are absent for classical particle descriptions, so they may be interpreted as giving small back reaction on the operator  $\hat{f}_1$  whenever we are in regime where the contributions of  $\hat{f}_1$  dominate. We remark, that we see no advantage in reformulating the system in terms of these phase-space operators if the state cannot be characterized as classical. This requirement can be understood by looking at the dynamics of the operators  $\hat{f}_{1,2,3}$ , which will involve an infinite series of spatial derivatives that needs to be truncated, which is not possible if the state does not allow for a separation of scales, see also the explanations in [111].

Although we are talking about phase-space *operators* so far, we should note that it is not overly important to retain the operator nature and we will soon drop it by taking expectation values. The reason we mentioned the operator nature in the first place, was to make easier contact to an n-particle distribution for the operator  $\hat{f}_1$ , such as for example the irreducible two-particle distribution  $f_1^{(2)} = \langle \hat{f}_1 \hat{f}_1 \rangle - \langle \hat{f}_1 \rangle \langle \hat{f}_1 \rangle$  which will appear naturally once we switch on interactions or take into account the classical stochastic limit of quantized gravitational perturbations (the role of these higher-order correlators which goes under the name BBGKY hierarchy is discussed for example in [109] in the context of dark matter). However, even if we switch on interactions, we are interested in regimes where the higher connected n-point functions are considered to have a small influence on the dynamics (Gaussian state truncation or resummed 1-loop approximation [128]),

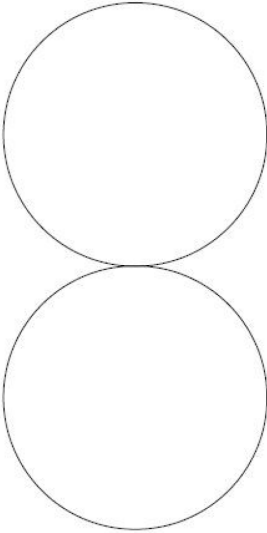
$$\langle : \hat{\phi}(x_1) \dots \hat{\phi}(x_{n+2}) : \rangle_{\text{connected}} \approx 0, \quad n > 2. \quad (4.47)$$



This is the case when the self-coupling  $\lambda$  multiplied by the number of particles running in the loops is small. Moreover, we want to consider a state with vanishing one-point functions

$$\langle \hat{\phi} \rangle = \langle \hat{\Pi} \rangle = 0. \quad (4.48)$$

In principle there is no obstacle in including also one-point functions in the formalism and it is certainly worth studying the influence of condensates. Nonetheless, in order to keep the scope of this chapter focussed we will postpone this discussion. The reason is that densities which are obtained by Wigner transforming products of one-point functions admit a gradient expansion only after a smoothing procedure [30, 31], which makes it necessary to deal separately with their dynamical equations and the way they react back on the connected part of the two-point function (directly via self-interactions or indirectly via gravity). By working with the just mentioned assumptions, we see that the full four-point function entering the energy-momentum tensor becomes

$$\langle : \hat{\phi}^4 : \rangle \approx 3 \langle : \hat{\phi}^2 : \rangle^2 = 3 \times \quad . \quad (4.49)$$


The diagram shows two identical circles that overlap at their top and bottom points. This represents the square of a two-point function, which is a diagram consisting of two vertices connected by two internal lines, forming a figure-eight shape.

After having discussed our assumptions on the quantum state and the self-interaction, let us now get started by gaining some intuition for the phase-space operators  $\hat{f}_i$  that we are after. We know that the energy-momentum tensor for a classical particle distribution in a general relativistic setting is given via second moments for the corresponding classical Boltzmann distribution  $f_{\text{class}}$ ,

$$T_{\mu\nu}^{\text{class}}(x^\mu, p_i) = \int \frac{d^3p}{\gamma^{1/2}} p_\mu p_\nu f_{\text{class}}(x^\mu, p_i), \quad p_0 = \omega(p_i), \quad (4.50)$$

where the zero component of the four-momentum is constrained by an on-shell condition. On the other hand we see that, at least in the absence of self-interactions, the energy-momentum tensor of the scalar field theory (4.38) is given in terms of quadratic monomials of the canonical operators. It is suggestive to look for some kind of Fourier transform of these quadratic monomials with respect to a shift variable  $r^k$  whose conjugate variable  $p_k$  may be interpreted as a spatial momentum such that gradient terms in (4.38) will contain

integrals over these momenta. This spatial Fourier transform is called Wigner transform and it is well known for the special relativistic case with flat metric  $\eta_{\mu\nu}$  in terms of a formulation where the zero component of the momentum variable is off-shell (see e.g. [92] for an introduction),

$$f_{\text{sr}}(x^\mu, p_\mu) \propto \int d^4r e^{-\frac{i}{\hbar} r^\mu p_\mu} \left\langle \hat{\phi}\left(x + \frac{r}{2}\right) \hat{\phi}\left(x - \frac{r}{2}\right) \right\rangle. \quad (4.51)$$

It is important to note that taking moments of these densities in  $k_0$  will yield more than one density as it was worked out in [97] for FLRW space-times and the relation to the energy density in (4.50) has been provided there.

Furthermore, fully general relativistic Wigner transforms have been proposed at the level of two-point functions for scalar fields using local expansions [94, 95] as well as non-perturbative expression based on an operator formulation [96, 98]. However, all of the latter four fully covariant proposals are based descriptions that leave the zero component of the momentum off-shell which does not make a closed set of differential equations for all involved degrees of freedom manifest if we assume a Gaussian state truncation.<sup>6</sup> In [111] we were working with linearized gravitational fields in longitudinal gauge and defined equal-time densities via two-point functions of canonical field operator which did not depend on off-shell momenta. We concluded that in a large mass, non-relativistic limit only a combination of two out of four two-point functions may be regarded as classical Boltzmann distribution whereas the other two two-point functions are to leading order highly oscillatory and otherwise suppressed. Although these oscillatory densities need not to be observable themselves, they still can have the potential to influence the classical particle density.

In addition to the work of [94, 95] [96, 98], it is our goal to make the application of the kinetic representation of quantum field theory for generic curved space-times more feasible by generalizing the description in terms of canonical fields in linearized longitudinal gauge in [111] to arbitrary metrics. This framework would firstly allow us once more to identify the quantity that comes closest to a classical Boltzmann phase-space density (which is non-trivial for a real scalar field in inhomogeneous setups, i.e. it is not simply the integrated version of the various off-shell densities discussed in [94–96, 98]) and secondly, to systematically study the effect of highly oscillatory state contributions on the dynamics of the slowly varying part of the state that comes closest to a classical particle description.<sup>7</sup>

Looking at (4.51), we see that the main ingredient will be a covariant shift of the canonical fields which has been worked out by [96] by treating space and time on equal footing. We apply the idea here to our canonical formulation on spatial hypersurfaces.

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<sup>6</sup>The Gaussian state truncation implies that the effect of interaction via connected higher n-point functions is neglected, if the latter had substantial effect and were taken into account, the system would not close anyway unless a quasi-particle approximation is applied.

<sup>7</sup>The last viewpoint is similar to the analysis in [129] in which relativistic correction on the non-relativistic amplitude of an interacting real scalar field in flat space-time have been worked out to yield.

Before we can present our definition, we introduce the following differential operator on the spatial hypersurfaces  $\Sigma_t$ ,

$${}^{(3)}\nabla_k^H \equiv {}^{(3)}\nabla_k - r^l {}^{(3)}\Gamma_{kl}^n \frac{\partial}{\partial r^n}, \quad (4.52)$$

where  ${}^{(3)}\nabla$  is the time-dependent covariant derivative on the spacelike hypersurfaces  $\Sigma_t$  and  ${}^{(3)}\Gamma_{kl}^n$  are the associated connection coefficients. The differential operator (4.52) is in fact the horizontal lift of the covariant derivative  ${}^{(3)}\nabla$  (induced on  $\Sigma_t$  via the 3+1 decomposition) to the tangent bundle  $T\Sigma_t$  (see for example [130] or [131] for an introduction to induced covariant derivatives on tangent bundles). This covariant derivative satisfies  ${}^{(3)}\nabla_k^H r^j = 0$ .

Let  $\hat{X}$  and  $\hat{Y}$  denote canonical operators  $\hat{\phi}$  and  $\gamma^{-1/2}\hat{\Pi}$ . If we combine a pair of canonical operators  $\{\hat{X}_x, \hat{Y}_y\}$  into a single operator  $\hat{X}_x \hat{Y}_y$  it will yield a state-independent and moreover UV-divergent part that can be defined in a normal neighbourhood around a collective point. In order to capture this region for operators at equal times, let  $\Theta[r^k, l_N(x^\mu)]$  be a cut-off function for the spatial tangent space at  $x^\mu$  that vanishes for values of  $r^k$  which yield a spatial geodesic  $s(x^\mu, r^k)$  with initial tangent vector  $r^k$  emanating from  $x^\mu$  whose associated distance  $||s||_g(x^\mu, r^k)$  is bigger than the radius of a spatial, normal neighbourhood specified by the scalar  $l_N(x^\mu)$  around the point  $x^\mu$  which is much smaller than the scale provided by the curvature but much bigger than a typical momentum scale,  $(\Delta p)^{-1} \ll l_N \ll R^{-1/2}$ . Now we define for each pair of spatially separated canonical operators a function that removes the state-independent part of the operator  $\hat{X}[s(x^\mu, r^k/2)]\hat{Y}[s(x^\mu, -r^k/2)]$  within a normal region around it,

$$H_{\phi\phi}[x^\mu, r^k] \equiv H_\lambda[y, z] \Big|_{y=s(x^\mu, r^k/2), z=s(x^\mu, -r^k/2)}, \quad (4.53)$$

$$H_{\phi\Pi}[x^\mu, r^k] \equiv (n_\nu \nabla^\nu)_z H_\lambda[y, z] \Big|_{y=s(x^\mu, r^k/2), z=s(x^\mu, -r^k/2)}, \quad (4.54)$$

$$H_{\Pi\phi}[x^\mu, r^k] \equiv (n_\nu \nabla^\nu)_y H_\lambda[y, z] \Big|_{y=s(x^\mu, r^k/2), z=s(x^\mu, -r^k/2)}, \quad (4.55)$$

$$H_{\Pi\Pi}[x^\mu, r^k] \equiv (n_\nu \nabla^\nu)_y (n_\rho \nabla^\rho)_z H_\lambda[y, z] \Big|_{y=s(x^\mu, r^k/2), z=s(x^\mu, -r^k/2)}, \quad (4.56)$$

where  $H_\lambda$  has to be computed perturbatively in the self-coupling  $\lambda$  in a normal neighbourhood. The free-field limit  $H_{\lambda=0}$  is given by (4.34). We define the associated, spatially covariant, Wigner operator as

$$\hat{F}_{XY}(x^\mu, p_k) \equiv \gamma^{1/2}(x^\mu) \int_{T\Sigma_t} dr^3 e^{-\frac{i}{\hbar} r^k p_k} : \hat{X}(x^\mu, r^k/2) \hat{Y}(x^\mu, -r^k/2) :, \quad (4.57)$$

where

$$\hat{X}(x^\mu, r^k/2) \equiv \left[ \exp\left(\frac{r^k}{2} {}^{(3)}\nabla_k^H(x^\mu)\right) \hat{X}(x^\mu, r^k) \right], \quad (4.58)$$

$$\hat{Y}(x^\mu, -r^k/2) \equiv \left[ \exp\left(-\frac{r^k}{2} {}^{(3)}\nabla_k^H(x^\mu)\right) \hat{Y}(x^\mu) \right], \quad (4.59)$$

with  $X, Y \in \{\phi, \gamma^{-1/2}\Pi\}$ , the corresponding expectation values

$$F_{\phi\phi} = \langle \hat{F}_{\phi\phi} \rangle, \quad F_{\phi\Pi} = \langle \hat{F}_{\phi\Pi} \rangle, \quad F_{\Pi\phi} = \langle \hat{F}_{\Pi\phi} \rangle, \quad F_{\Pi\Pi} = \langle \hat{F}_{\Pi\Pi} \rangle, \quad (4.60)$$

and the normal ordering prescription

$$\begin{aligned} : \hat{X}(x^\mu, r^k/2) \hat{Y}(x^\mu, -r^k/2) : &\equiv \hat{X}(x^\mu, r^k/2) \hat{Y}(x^\mu, -r^k/2) \\ &- \hat{1}\Theta[r^j, l_N(x^\mu)] H_{XY}[x^\mu, r^k]. \end{aligned} \quad (4.61)$$

In the definition (4.57),  $H_{XY}$  subtracts the state-independent part in a normal neighbourhood around  $x^\mu$  and may be viewed as a off-coincident normal ordering.<sup>8</sup> We can similarly view the subtraction as a coarse-graining with respect to quantum UV-modes and it is interesting to note that boundary terms arising from the normal neighbourhood can give rise to noise terms as they appear in stochastic inflation [132]. Including these type of terms in kinetic equations is however beyond the scope of this thesis. Although a state-independent, off-coincident normal ordering operation is discussed in the context of algebraic quantum field theory for free fields by means of the Hadamard parametrix [120], we are not aware that this definition is extended to interacting fields at the rigorous level that algebraic quantum field theory operates on. However, we think it is important to signal that such a procedure is necessary to obtain operators that describe real particle fluctuation and exclude virtual particles. Moreover, normal ordering is clearly demanded if we take moments in the momenta  $p_k$  of (4.57) which would otherwise result in coincident limit two-point functions that are divergent, instead we have for example

$$\frac{1}{(2\pi\hbar)^3} \int \frac{d^3p}{\gamma^{1/2}} \hat{F}_{\phi\phi}(x^\mu, p_k) = : \hat{\phi}^2 : (x^\mu), \quad (4.62)$$

which is crucial if we want to rewrite the energy-momentum tensor in terms of  $\hat{F}_{\phi\phi}$ ,  $\hat{F}_{\Pi\phi}$ ,  $\hat{F}_{\phi\Pi}$ ,  $\hat{F}_{\Pi\Pi}$ .

On the other hand, the details of the normal ordering procedure should not affect the effective description that we are after whenever infrared and ultraviolet physics decouple which concretely amounts in this chapter to neglect firstly, anomalous contributions of the matter two-point functions, secondly, boundary terms of the state-independent part due to the normal region and thirdly, 2-loop corrections that will again pick up quantum contributions running in the loop.

Let us discuss the other ingredients appearing (4.57). We verify in appendix 4.A that powers of the spatially covariant shift operator  $\frac{r^k}{2} {}^{(3)}\nabla_k^H$  yield the following when acting on scalar densities  $f(x^\mu)$  with weight zero,

$$\left[ r^{k(3)} \nabla_k^H \right]^n f = r^{i_1} \dots r^{i_n} {}^{(3)}\nabla_{i_1} \dots {}^{(3)}\nabla_{i_n} f = \left[ r^k \left( \partial_k - {}^{(3)}\Gamma_{kl}^m r^l \frac{\partial}{\partial r^m} \right) \right]^n f, \quad (4.63)$$

which allows us to consider the definition (4.57) even without the introduction of geometrical objects on the tangent bundles  $T\Sigma_t$ . We also realize that any change of spatial coordinates in (4.57) can be absorbed into the integration

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<sup>8</sup>See also the inclusion of normal ordering for Minkowski space Wigner transformations in [92] which is not restricted to a normal neighbourhood due to vanishing curvature.

variable  $r^k$  that then transforms as a 3-vector and leaves the measure invariant thanks to the spatial determinant factor. The conjugate momentum  $p_k$  then transforms as a covariant 3-vector.

Equation (4.63) also reveals, that the covariant shift operators reduce to spatial translations when working with Riemann normal coordinates since only symmetrized covariant derivatives enter. It shows that the exponentials acting in (4.57) translate the operators  $\hat{X}, \hat{Y}$  from point  $x^\mu$  to the point specified by the spatial geodesic emanating at  $x^\mu$  with tangent vector  $r^k$  in opposite directions. Taking expectation values of the operator  $\hat{f}_{XY}$  will then contain the information in the momentum space representation on how  $\hat{X}$  and  $\hat{Y}$  are correlated around the collective point  $x^\mu$  where the normal ordering takes care that UV correlations that are purely quantum are removed up to boundary terms. However, the cut-off related to the normal neighbourhood around  $x^\mu$  should have small effect as long as the state-dependent field correlations are restricted to regions much smaller than the inverse curvature. The gradients with respect to the center coordinate  $x^\mu$  then quantify how this correlation changes in space-time.

Given the canonical operators of the real scalar field theory, we have four different Wigner operators  $\hat{F}_{\phi\phi}, \hat{F}_{\Pi\phi}, \hat{F}_{\phi\Pi}, \hat{F}_{\Pi\Pi}$  whose dynamics are determined by the dynamics of the operators  $\hat{\phi}$  and  $\hat{\Pi}$ . Unfortunately, the calculation is tedious and some techniques to perform it have to be introduced. Let us therefore make some easier observation before that and save the difficulties for later.

We observe that the operators  $\hat{F}_{\phi\phi}, \hat{F}_{\Pi\phi}, \hat{F}_{\phi\Pi}, \hat{F}_{\Pi\Pi}$  are dimensionally inequivalent and not all of them are real. In order to rescale the Wigner operators in units of energy, we consider the free particle energy via the 3+1 decomposition

$$\omega_p = Np^0 = \left(m^2 + \gamma^{kl}p_k p_l\right)^{1/2}, \quad (4.64)$$

and define the following dimensionally equivalent phase-space density operators and the corresponding expectation values

$$f_1^+ = \langle \hat{f}_1^+ \rangle \equiv \frac{1}{(2\pi\hbar)^3} \frac{1}{2\hbar} \left[ \frac{\omega_p}{\hbar} \langle \hat{F}_{\phi\phi} \rangle + \frac{\hbar}{\omega_p} \langle \hat{F}_{\Pi\Pi} \rangle \right], \quad (4.65)$$

$$f_1^- = \langle \hat{f}_1^- \rangle \equiv \frac{1}{(2\pi\hbar)^3} \frac{i}{2\hbar} \left[ \langle \hat{F}_{\Pi\phi} \rangle - \langle \hat{F}_{\phi\Pi} \rangle \right], \quad (4.66)$$

$$f_2 = \langle \hat{f}_2 \rangle \equiv \frac{1}{(2\pi\hbar)^3} \frac{1}{2\hbar} \left[ \frac{\omega_p}{\hbar} \langle \hat{F}_{\phi\phi} \rangle - \frac{\hbar}{\omega_p} \langle \hat{F}_{\Pi\Pi} \rangle \right], \quad (4.67)$$

$$f_3 = \langle \hat{f}_3 \rangle \equiv \frac{1}{(2\pi\hbar)^3} \frac{1}{2\hbar} \left[ \langle \hat{F}_{\Pi\phi} \rangle + \langle \hat{F}_{\phi\Pi} \rangle \right]. \quad (4.68)$$

We note that the phase-space density operators  $\hat{f}_1^+, \hat{f}_2$  and  $\hat{f}_3$  are even functions of the momentum  $p_k$  whereas  $\hat{f}_1^-$  is an odd function of the momentum. From here on we will work mostly with expectation values of operators which is clarified by omitting the hats.

Making use of delta functions, setting the connected four-point functions to zero, dropping the anomalous contribution and boundary terms, we can express the energy-momentum tensor in terms of the phase-space densities (4.65) to

(4.68) as follows,

$$\begin{aligned}
E = & \int \frac{d^3p}{\gamma^{1/2}} \omega_p f_1^+ - 2\xi\hbar K \int \frac{d^3p}{\gamma^{1/2}} f_3 \\
& + \frac{\hbar^2}{8} [1 - 8\xi] {}^{(3)}\nabla_k {}^{(3)}\nabla^k \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \\
& + \xi \frac{\hbar^2}{2} \left( {}^{(3)}R + K^2 - K_{ij} K^{ij} \right) \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \\
& + \lambda \frac{\hbar^3}{8} \left[ \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \right]^2, \tag{4.69}
\end{aligned}$$

$$\begin{aligned}
P_k = & \int \frac{d^3p}{\gamma^{1/2}} p_k f_1^- - \frac{\hbar}{2} [1 - 4\xi] {}^{(3)}\nabla_k \int \frac{d^3p}{\gamma^{1/2}} f_3 \\
& + \xi \hbar^2 \left[ {}^{(3)}\nabla^m K_{jm} - {}^{(3)}\nabla_j K + K_j^m {}^{(3)}\nabla_m \right] \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p}, \tag{4.70}
\end{aligned}$$

$$\begin{aligned}
S_{km} = & \int \frac{d^3p}{\gamma^{1/2}} \frac{p_k p_m}{\omega_p} (f_1^+ + f_2) - 2\xi\hbar K_{km} \int \frac{d^3p}{\gamma^{1/2}} f_3 \\
& + \frac{\hbar^2}{4} [1 - 4\xi] {}^{(3)}\nabla_k {}^{(3)}\nabla_m \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \\
& - \gamma_{km} [1 - 4\xi] \left[ \int \frac{d^3p}{\gamma^{1/2}} \omega_p f_2 + \frac{\hbar^2}{8} {}^{(3)}\nabla^j {}^{(3)}\nabla_j \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \right] \\
& + 2\xi\hbar \left[ R\gamma_{km} + {}^{(3)}R_{km} + K K_{km} - 2K_{kj} K_m^j \right. \\
& \quad \left. - \mathcal{L}_n K_{km} + N^{-1} {}^{(3)}\nabla_j {}^{(3)}\nabla_k N \right] \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \\
& - \gamma_{km} \lambda \frac{\hbar^3}{2} [1 - 8\xi] \left[ \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \right]^2, \tag{4.71}
\end{aligned}$$

where we used relations of the type

$$: \partial_i \hat{\phi} \partial_j \hat{\phi} : = \frac{1}{4} {}^{(3)}\nabla_i {}^{(3)}\nabla_j : \hat{\phi}^2 : + \int \frac{d^3p}{(2\pi\hbar)^3} \gamma^{-1/2} \frac{p_i p_j}{\hbar^2} \hat{F}_{\phi\phi}. \tag{4.72}$$

We can also write down energy-momentum conservation  $\nabla_\mu \langle : \hat{T}^{\mu\nu} : \rangle = 0$  in terms of this 3+1 decomposition (see for example [127]),

$$\partial_t (\gamma^{1/2} E) + \partial_i [\gamma^{1/2} (N P^i - N^i \hat{E})] = N \gamma^{1/2} (S_{ij} K^{ij} - P^i \partial_i \ln N), \tag{4.73}$$

$$\begin{aligned}
\partial_t (\gamma^{1/2} P_j) + \partial_i [\gamma^{1/2} (N P_j^i - N^i P_j)] = & N \gamma^{1/2} \left( \frac{1}{2} S^{ik} \partial_j \gamma_{ik} + N^{-1} P_i \partial_j N^i \right. \\
& \left. - E \partial_j \ln N \right), \tag{4.74}
\end{aligned}$$

which however does not help very much since the involved quantities are still constrained via the phase-space densities  $f_i$ . We thus have to know their dynamics which we will postpone to a later section as promised.

Let us see what we can read off from the decomposition (4.69) to (4.71). By

writing  $\hat{f}_1 = \hat{f}_1^+ + \hat{f}_1^-$  and using the parity properties of these operators, we arrive at the form

$$E = \int \frac{d^3p}{\gamma^{1/2}} \omega_p f_1 + \hbar^2 \mathcal{O}(f_1) + \hbar^2 \mathcal{O}(f_2) + \mathcal{O}(\xi, \lambda), \quad (4.75)$$

$$P_k = \int \frac{d^3p}{\gamma^{1/2}} p_k f_1 + \hbar \mathcal{O}(f_3) + \mathcal{O}(\xi), \quad (4.76)$$

$$S_{km} = \int \frac{d^3p}{\gamma^{1/2}} \frac{p_k p_m}{\omega_p} f_1 + \mathcal{O}(f_2) + \hbar^2 \mathcal{O}(f_1) + \mathcal{O}(\xi, \lambda). \quad (4.77)$$

We can do the same 3+1 projection with the classical energy-momentum tensor (4.50) whose building blocks can be fluctuating phase-space densities that still need to be averaged over in statistical context. Since we could have written down the equations (4.75) to (4.77) also at the level of renormalized operators, we can tentatively identify the operator  $\hat{f}_1$  as a fluctuating phase-space density at the level of the normal projected energy-momentum tensor, up to certain correction terms. The average  $f_1$  is viewed as the one-particle distribution in phase-space. Let us discuss under which conditions this identification is justified.

The first condition concerns a spatial gradient expansion proportional to the Planck constant  $\hbar$  where spatial gradients with respect to the variable  $x^i$  are compared to either the energy  $\omega_p$  or the momentum  $p_k$  within spatial momentum integrals of a phase-space density  $\langle \hat{f}_i(x^\mu, p_j) \rangle$ . If we picture a non-relativistic setting where  $m \gg |p_k|$  for any  $f_i(x^\mu, p_j)$ , we see that the gradient expansion is applicable if the energy-scales satisfy  $m \gg \Delta p \approx \frac{\hbar}{\Delta r} \gg \frac{\hbar}{\Delta x}$ .<sup>9</sup> The relation between the short distance difference scale  $\Delta r$  and the long distance, center coordinate scale  $\Delta x$  lies at the heart of the Wigner transformation. In the context of general relativity, it corresponds to locally homogeneous two-point functions depending only on the (covariantly generalized) difference coordinate of the involved operators subsequently yielding only a momentum dependence around  $\Delta p$  which is then corrected on larger scales  $\Delta x$  via gravitational inhomogeneities (plus additional effects due to self-interactions). We underpin once more, that it depends on the state in Hilbert space whether or not these corrections are small since the higher-order spatial gradient corrections are strictly speaking always present. Typical correction terms in the dynamics of phase-space operators will include

$$\mathcal{O}(\hbar) f_i(t, x, p) \sim \left\{ \hbar \partial_k \frac{\partial}{\partial p_k}, \frac{\hbar \partial_k}{\sqrt{m^2 + \gamma^{ij} p_i p_j}}, \frac{\hbar}{p^k \partial_k} {}^{(3)}\square, \dots \right\} f_i(t, x, p). \quad (4.78)$$

Another obvious condition that should be satisfied in order to treat the operator  $\hat{f}_1 = \hat{f}_1^+ + \hat{f}_1^-$  similarly to a classical, particle-associated fluctuating phase-space density concerns the expectation values of the two operators  $\hat{f}_{2,3}$  appearing in (4.75) to (4.77). If we wanted an averaged energy-momentum tensor from the field-theoretic description that looks almost identical to the one obtained from an averaged classical particle description, the densities  $f_{2,3}$

<sup>9</sup>We commented more on this expansion in the context of dark matter in [111].

would have to be chosen small initially. Note that the assumption that  $f_1^+$  should be regarded as the dominant density in comparison with  $f_2$  is supported by observation that

$$\int \frac{d^3p}{\gamma^{1/2}} \omega_p f_1 \geq \left| \int \frac{d^3p}{\gamma^{1/2}} \omega_p f_2 \right|, \quad (4.79)$$

which follows from their very definition. The bound can be hit for example for homogeneous condensates  $\langle \hat{\phi} \rangle(t) \propto \sin(mt/\hbar)$ . On the other, only the density  $f_1^-$  can fulfil the job as a classical particle phase-space density since it is the only odd density and thus clearly favoured in comparison to  $f_3$  in (4.76). However, even if we make the identification (4.75) to (4.77) initially, we have to be sure that the dynamics keeps the influence of the fluctuations  $f_{2,3}$  small over time which relates to requirements on the parameters of the theory ( $m, \xi, \lambda$ ). It is clear that we expect for example from a strongly interacting regime  $\lambda \gg 1$  many more effects than a mass renormalization and pressure correction, since the Gaussian state approximation breaks down and higher n-point functions enter the dynamics.

Let us summarize what we have found so far. We have provided a spatially covariant set of three even and one odd quadratic equal-time operators and their expectation values (4.65) to (4.68) that have units of phase-space densities and that depend on a space-time point  $x^\mu$  and spatial three-momentum  $p_i$ . There is no dependence on a off-shell zero-momentum component. By looking at the 3+1 decomposition of the energy-momentum tensor (4.75) to (4.77), we identified a distinguished combination of one even and the odd operator  $\hat{f}_1 = \hat{f}_1^+ + \hat{f}_1^-$  which appears to mimic a fluctuating, classical phase-space density in the sense of statistical mechanics whenever it is acting on a state that is classical enough such that it admits a spatial gradient expansion. The remaining two even operators  $\hat{f}_{2,3}$  represent degrees of freedom that stem from the fundamental relativistic, field-theoretic description and we have argued that the expectation values  $f_{2,3}$  should taken to be small for a purely particle-like interpretation. However, they can in principle have a significant role for the evolution of the system and it is worth studying how such additional components from the field-theoretic description correct the classical particle picture.

## 4.4 Hydrodynamic cold dark matter limit

Our original motivation to identify the phase-space densities  $f_{1,2,3}$  was to study cold dark matter from a field-theoretic description that allows for a systematic inclusion of relativistic effects [111]. This section is devoted to making a closer contact to a cold dark matter description that is formulated in terms of hydrodynamic variables. We want to show as a proof of concept how the hydrodynamical variables, that are used for the classical particle description, can be derived from the theory of scalar quantum field in a certain classical limit. It turns out that this map is already non-trivial even at the level of vanishing self-interaction and minimal coupling to gravity which is why we stick to



the simplified case  $\lambda = \xi = 0$  in this section. We were discussing the energy-momentum tensor in a 3+1 decomposition. The projected quantities

$$\langle : \hat{E} : \rangle = E, \quad \langle : \hat{P}_k : \rangle = P_k, \quad \langle : \hat{S}_{kl} : \rangle = S_{kl}, \quad (4.80)$$

that appear in the energy-momentum tensor (4.69) to (4.71) are related to the observer specified from any other frame by the normal vector  $n^\mu$ . This normal (also referred to as Eulerian) observer measure an energy density  $E$ , a momentum  $P_i$  and a stress tensor  $S_{ij}$ . However, especially in the context of cosmology it is standard to work with a different decomposition that assumes a hydrodynamic representation of energy-momentum which relates to an observer co-moving with the fluid. The fluid is specified from any other frame by the four-velocity  $u^\mu$  that corresponds to an observer moving with a fluid element and the energy-momentum tensor for a hydrodynamic representation is usually written as

$$\langle : \hat{T}_{\mu\nu} : \rangle \equiv T_{\mu\nu} = (e + P)u_\mu u_\nu + P g_{\mu\nu} + \pi_{\mu\nu}, \quad u^\mu \pi_{\mu\nu} = 0, \quad \pi^\mu_\mu = 0. \quad (4.81)$$

In this formulation one assumes that energy and momentum are expressible in terms of a fluid with rest-frame energy density  $e$ , pressure  $P$  and non-isotropic stresses  $\pi_{ij}$ . The energy density  $e$  that is measured by the observer moving with the fluid is in general different from the energy density  $E$  measured by the normal observer. We remark that

$$u^\mu = -n^\nu u_\nu (n^\mu + v^\mu) = W(n^\mu + v^\mu), \quad W = \frac{1}{\sqrt{1 - v^i v^j \gamma_{ij}}}, \quad (4.82)$$

where  $v^\mu$  is the spatial part of the four-velocity with respect to the normal vector and  $W$  is the Lorentz factor [127]. We then have the following relations

$$E = W^2(e + \gamma^{ij}v_i v_j P) + \pi_{ij}v^i v^j, \quad (4.83)$$

$$P^i = (e + P)W^2 v^i + \pi_{kl}v^k \gamma^{li}, \quad (4.84)$$

$$S^{ij} = (e + P)W^2 v^i v^j + P \gamma^{ij} + \gamma^{ik} \gamma^{jl} \pi_{kl}. \quad (4.85)$$

We would now like to invert the relations (4.83) to (4.85), which is in principle a complicated task. However, it can be done in principle exactly and we will do it for the case of the real scalar field stress-energy tensor, where we set the self-coupling  $\lambda$  and the non-minimal coupling to gravity  $\xi$  for simplicity to zero,

$$\langle : \hat{T}_{\mu\nu} : \rangle_{\lambda=\xi=0} = \langle : \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} : \rangle - \frac{g_{\mu\nu}}{2} \left[ \langle : \partial^\alpha \hat{\phi} \partial_\alpha \hat{\phi} : \rangle + \frac{m^2}{\hbar^2} \langle : \hat{\phi}^2 : \rangle \right]. \quad (4.86)$$

It turns out, that it is more convenient at this point to first work without any time-slicing and define the object

$$\chi^\mu_\nu \equiv \langle : \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} : \rangle, \quad (4.87)$$

which is the key ingredient, if we want to consider non-perfect fluids. The energy density is the negative eigenvector of the fluid four-velocity, whereas the

pressure is one third of the sum of the principal stresses, which are eigenvalues belonging to the spatial part of the energy-momentum tensor. The task is thus to find the eigenvalues of energy-momentum (4.86), which amounts to finding the eigenvalues of  $\chi^\mu_\nu$ , which is initially for every point in space an arbitrary matrix that obeys the Cayley-Hamilton equation

$$\begin{aligned} [\chi^4]^\mu_\nu - \text{tr} [\chi] [\chi^3]^\mu_\nu + \frac{1}{2} \left[ (\text{tr} [\chi])^2 - \text{tr} [\chi^2] \right] [\chi^2]^\mu_\nu \\ - \frac{1}{6} \left[ (\text{tr} [\chi])^3 - 3 \text{tr} [\chi^2] \text{tr} [\chi] + 2 \text{tr} [\chi^3] \right] \chi^\mu_\nu + \delta^\mu_\nu \det \chi = 0. \end{aligned} \quad (4.88)$$

The eigenvalues  $\sigma$  of this matrix are then subject to the quartic equation

$$\sigma^4 + \tilde{b}\sigma^3 + \tilde{c}\sigma^2 + \tilde{d}\sigma + \tilde{e} = 0, \quad (4.89)$$

where

$$\tilde{b} = -\text{tr} [\chi], \quad (4.90)$$

$$\tilde{c} = \frac{1}{2} \left[ (\text{tr} [\chi])^2 - \text{tr} [\chi^2] \right], \quad (4.91)$$

$$\tilde{d} = -\frac{1}{6} \left[ (\text{tr} [\chi])^3 - 3 \text{tr} [\chi^2] \text{tr} [\chi] + 2 \text{tr} [\chi^3] \right], \quad (4.92)$$

$$\tilde{e} = \det \chi. \quad (4.93)$$

We express these traces in terms of two-point functions of canonical field operators in appendix 4.B. The solutions of the quartic eigenvalue equation may be written as

$$\sigma_0 = -\frac{\tilde{b}}{4} - |\tilde{S}| - \frac{1}{2} \left[ -4\tilde{S}^2 - 2\tilde{p} + \frac{\tilde{q}}{|\tilde{S}|} \right]^{1/2}, \quad (4.94)$$

$$\sigma_1 = -\frac{\tilde{b}}{4} - |\tilde{S}| + \frac{1}{2} \left[ -4\tilde{S}^2 - 2\tilde{p} + \frac{\tilde{q}}{|\tilde{S}|} \right]^{1/2}, \quad (4.95)$$

$$\sigma_{2,3} = -\frac{\tilde{b}}{4} + |\tilde{S}| \pm \frac{1}{2} \left[ -4\tilde{S}^2 - 2\tilde{p} - \frac{\tilde{q}}{|\tilde{S}|} \right]^{1/2}, \quad (4.96)$$

in terms of the following quantities

$$\tilde{p} \equiv \frac{8\tilde{c} - 3\tilde{b}^2}{8} = \frac{1}{8}(\text{tr} [\chi])^2 - \frac{1}{2}\text{tr} [\chi^2], \quad (4.97)$$

$$\tilde{q} \equiv \frac{\tilde{b}^3 - 4\tilde{b}\tilde{c} + 8\tilde{d}}{8} = -\frac{1}{24}(\text{tr} [\chi])^3 + \frac{1}{4}\text{tr} [\chi^2]\text{tr} [\chi] - \frac{1}{3}\text{tr} [\chi^3], \quad (4.98)$$

$$\tilde{S} \equiv \frac{1}{2} \left[ -\frac{2}{3}\tilde{p} + \frac{1}{3} \left( \tilde{Q} + \frac{\Delta_0}{\tilde{Q}} \right) \right]^{1/2}, \quad (4.99)$$

$$\tilde{Q} \equiv \left[ \frac{\Delta_1}{2} + \frac{1}{2} \left( \Delta_1^2 - 4\Delta_0^3 \right)^{1/2} \right]^{1/3}, \quad (4.100)$$

$$\Delta_0 \equiv \tilde{c}^2 - 3\tilde{b}\tilde{d} + 12\tilde{e}, \quad (4.101)$$

$$\Delta_1 \equiv 2\tilde{c}^3 - 9\tilde{b}\tilde{c}\tilde{d} + 27\tilde{b}^2\tilde{e} + 27\tilde{d}^2 - 72\tilde{c}\tilde{e}. \quad (4.102)$$

We can identify the eigenvalue that will be related to the energy density  $e$  by looking at the limiting case where the full scalar field two-point function reduces into products of classical fields and thus yields a perfect fluid energy-momentum tensor

$$(\chi_{\text{cl}})^\mu{}_\nu = \partial^\mu \langle \hat{\phi} \rangle \partial_\nu \langle \hat{\phi} \rangle = \partial^\mu \phi_{\text{cl}} \partial_\nu \phi_{\text{cl}}. \quad (4.103)$$

In this case, all coefficients of the quartic eigenvalue equation vanish except for  $\tilde{b}$  and we find

$$\sigma_0^{\text{cl}} = \partial^\mu \phi_{\text{cl}} \partial_\mu \phi_{\text{cl}}, \quad \sigma_{1,2,3}^{\text{cl}} = 0. \quad (4.104)$$

Setting

$$\langle \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} \rangle - \frac{g_{\mu\nu}}{2} \left[ \langle \partial^\alpha \hat{\phi} \partial_\alpha \hat{\phi} \rangle + \frac{m^2}{\hbar^2} \langle \hat{\phi}^2 \rangle \right] = (e + P) u_\mu u_\nu + P g_{\mu\nu} + \pi_{\mu\nu}, \quad (4.105)$$

and taking  $-\sigma_0$  as the eigenvalue corresponding to the eigenvector of the four-velocity  $u^\mu$ , we have

$$e = -\sigma_0 + \frac{1}{2} \left[ \langle \partial^\alpha \hat{\phi} \partial_\alpha \hat{\phi} \rangle + \frac{m^2}{\hbar^2} \langle \hat{\phi}^2 \rangle \right], \quad (4.106)$$

$$P = \frac{1}{3} \left[ -\sigma_0 - \frac{1}{2} \langle \partial^\alpha \hat{\phi} \partial_\alpha \hat{\phi} \rangle - \frac{3m^2}{2\hbar^2} \langle \hat{\phi}^2 \rangle \right]. \quad (4.107)$$

We still have to identify the four-velocity itself, which can be done by rewriting the Cayley-Hamilton equation as

$$\prod_{\mu=0}^3 (\chi - \mathbf{1} \sigma_\mu) = 0, \quad (4.108)$$

which tells us that we have four potential eigenvectors for the eigenvalue  $\sigma_0$  and we label them by the letter  $\kappa$ ,

$$(u_\kappa)^\mu \equiv \langle \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} \rangle \langle \partial^\nu \hat{\phi} \partial_\rho \hat{\phi} \rangle \langle \partial^\rho \hat{\phi} \partial_\kappa \hat{\phi} \rangle - (\sigma_1 + \sigma_2 + \sigma_3) \langle \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} \rangle \langle \partial^\nu \hat{\phi} \partial_\kappa \hat{\phi} \rangle \\ + (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_1 \sigma_3) \langle \partial^\mu \hat{\phi} \partial_\kappa \hat{\phi} \rangle - \sigma_1 \sigma_2 \sigma_3 \delta^\mu{}_\kappa. \quad (4.109)$$

However, by considering the homogeneous case for classical fields we see that the only reasonable choice is  $\kappa = 0$ . Taking into account a normalisation factor  $\alpha$ , we are left with

$$u^\mu = \alpha \left[ \langle \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} \rangle \langle \partial^\nu \hat{\phi} \partial_\rho \hat{\phi} \rangle \langle \partial^\rho \hat{\phi} \partial_0 \hat{\phi} \rangle - (\sigma_1 + \sigma_2 + \sigma_3) \langle \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} \rangle \langle \partial^\nu \hat{\phi} \partial_0 \hat{\phi} \rangle \right. \\ \left. + (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_1 \sigma_3) \langle \partial^\mu \hat{\phi} \partial_0 \hat{\phi} \rangle - \sigma_1 \sigma_2 \sigma_3 \delta_0^\mu \right], \quad u^\mu u_\mu = -1. \quad (4.110)$$

Note that the above reproduces the classical field identification ( $\sigma_i = 0$ ),

$$e_{\text{cl}} = -\frac{1}{2} \partial^\mu \phi_{\text{cl}} \partial_\mu \phi_{\text{cl}} + \frac{1}{2} \frac{m^2}{\hbar^2} \phi_{\text{cl}}^2, \quad (4.111)$$

$$P_{\text{cl}} = -\frac{1}{2} \partial^\mu \phi_{\text{cl}} \partial_\mu \phi_{\text{cl}} - \frac{1}{2} \frac{m^2}{\hbar^2} \phi_{\text{cl}}^2, \quad (4.112)$$

$$u_{\text{cl}}^\mu = \alpha (\chi_{\text{cl}}^3)^\mu_0 = \frac{\partial^\mu \phi_{\text{cl}}}{(-\partial^\nu \phi_{\text{cl}} \partial_\nu \phi_{\text{cl}})^{1/2}}, \quad \alpha = \left[ -(\chi_{\text{cl}}^6)_0^0 \right]^{1/2}. \quad (4.113)$$

We would like to check whether our identification of energy density and pressure yields meaningful expressions beyond the special case where the two-point function reduces to classical fields. We consider the limiting case where the mass  $m$  constitutes the biggest energy scale and we can perturbatively expand with respect to this scale. This non-relativistic expansion with parameter  $\varepsilon_p = p^2/m^2$  is another approximation on top of the gradient approximation that we have explained in the previous section and which is denoted by  $\varepsilon_h \propto \hbar^2 m^{-2(3)} \nabla_x^2, \dots$ . We find

$$\tilde{b} = -\langle \partial^\mu \hat{\phi} \partial_\mu \hat{\phi} \rangle = \int \frac{d^3 p}{\gamma^{1/2}} \omega_p (f_1^+ - f_2) - \gamma^{ij} \int \frac{d^3 p}{\gamma^{1/2}} p_i p_j \frac{f_1^+ + f_2}{\omega_p} \\ - \frac{\hbar^2}{4} {}^{(3)}\nabla_i {}^{(3)}\nabla^i \int \frac{d^3 p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \quad (4.114)$$

$$= \int \frac{d^3 p}{\gamma^{1/2}} \omega_p (f_1^+ - f_2) \left[ 1 + \mathcal{O}(\varepsilon_p) + \mathcal{O}(\varepsilon_h) \right]. \quad (4.115)$$

However, this expansion is only meaningful if  $f_1^+ \neq f_2$  which needs not to be satisfied for arbitrary times and initial conditions. Just for illustration one can consider the classical field case in a perturbed FLRW-universe. The solution will be oscillatory

$$\phi_{\text{cl}}(x) \propto \sqrt{\rho_{\text{cl}}(x)} \cos \left[ m \int^{x^0} a d\tilde{x}^0 - v(x) - \theta \right], \quad (4.116)$$

and thus the correlator  $\langle : \hat{\Pi}^2 : \rangle$  is periodically and for short times not determined by the scale  $m$  but by a smaller energy scale

$$\Pi_{\text{cl}} \propto m \sqrt{\rho_{\text{cl}}(x)} \sin [\dots] + \dot{v}_{\text{cl}}(x) \sqrt{\rho_{\text{cl}}(x)} \sin [\dots] - \sqrt{\dot{\rho}_{\text{cl}}(x)} \cos [\dots]. \quad (4.117)$$

However, the case of classical fields is itself not problematic since we already have the exact answer for  $e, P$  and  $u^\mu$ . We only wanted to make the reader aware

that an expansion with respect to the scale  $m$  might be more subtle than one would naively expect. Still, in order to make progress with the non-relativistic limit we will assume that

$$f_2 \propto \mathcal{O}(\varepsilon_{\{p,\hbar\}}) f_1^+, \quad (4.118)$$

which matches one of the conditions for a pure particle limit, that we formulated in the end of the last section. The symbol  $\varepsilon_{\{p,\hbar\}}$  denotes a correction in either  $\varepsilon_p$  or  $\varepsilon_\hbar$ . It is clear from the one-point function analysis in (4.116) that this condition requires a description that goes beyond coherent states (unless an averaging procedure is employed). Once this condition is satisfied, it makes sense to continue the expansion with respect to the scale  $m$  and find

$$\frac{\tilde{c}}{\tilde{b}^2} = \mathcal{O}(\varepsilon_{\{p,\hbar\}}), \quad \frac{\tilde{d}}{\tilde{b}^3} = \mathcal{O}(\varepsilon_{\{p,\hbar\}}^2), \quad \frac{\tilde{e}}{\tilde{b}^4} = \mathcal{O}(\varepsilon_{\{p,\hbar\}}^{5/2}), \quad (4.119)$$

where we used

$$\begin{aligned} \langle : \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} : \rangle \langle : \partial^\nu \hat{\phi} \partial_\mu \hat{\phi} : \rangle &= \left[ \int \frac{d^3 p}{\gamma^{1/2}} \omega_p (f_1^+ - f_2) \right]^2 \\ &- 2\gamma^{ij} \left[ \frac{\hbar^{(3)}}{2} \nabla_i \int \frac{d^3 p}{\gamma^{1/2}} f_3 + \int \frac{d^3 p}{\gamma^{1/2}} p_i f_1^- \right] \left[ \frac{\hbar^{(3)}}{2} \nabla_j \int \frac{d^3 p}{\gamma^{1/2}} f_3 - \int \frac{d^3 p}{\gamma^{1/2}} p_j f_1^- \right] \\ &+ \gamma^{jk} \gamma^{il} \left[ \int \frac{d^3 p}{\gamma^{1/2}} p_i p_j \frac{f_1^+ + f_2}{\omega_p} + \frac{\hbar^2}{4} {}^{(3)}\nabla_i {}^{(3)}\nabla_j \int \frac{d^3 p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \right] \\ &\times \left[ \int \frac{d^3 p}{\gamma^{1/2}} p_k p_l \frac{f_1^+ + f_2}{\omega_p} + \frac{\hbar^2}{4} {}^{(3)}\nabla_k {}^{(3)}\nabla_l \int \frac{d^3 p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} \right], \quad (4.120) \end{aligned}$$

and similar expression for the cubic trace and the determinant.

We can now perturb the quartic equation for the eigenvalue  $\sigma_0$  around its zero order solution  $\bar{\sigma}_0 = -\tilde{b}$  and find,

$$\begin{aligned} \sigma_0 &= -\tilde{b} + \frac{\tilde{c}}{\tilde{b}} + \mathcal{O}(\varepsilon_{\{p,\hbar\}}^2) \\ &= \langle : \partial^\mu \hat{\phi} \partial_\mu \hat{\phi} : \rangle - \frac{1}{2} \frac{\langle : \partial^\mu \hat{\phi} \partial_\mu \hat{\phi} : \rangle^2 - \langle : \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} : \rangle \langle : \partial^\nu \hat{\phi} \partial_\mu \hat{\phi} : \rangle}{\langle : \partial^\mu \hat{\phi} \partial_\mu \hat{\phi} : \rangle} + \mathcal{O}(\varepsilon_{\{p,\hbar\}}^2) \\ &= -\gamma^{-1} \langle : \hat{\Pi} \hat{\Pi} : \rangle + \langle : \hat{\Pi} \hat{\Pi} : \rangle^{-1} \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \gamma^{ij} \langle : \partial_j \hat{\phi} \hat{\Pi} : \rangle + \mathcal{O}(\varepsilon_{\{p,\hbar\}}^2) \\ &= - \int \frac{d^3 p}{\gamma^{1/2}} \omega_p (f_1^+ - f_2) + \left[ \int \frac{d^3 p}{\gamma^{1/2}} \omega_p (f_1^+ - f_2) \right]^{-1} \\ &\quad \times \gamma^{ij} \left[ \frac{\hbar^2}{4} {}^{(3)}\nabla_i \int \frac{d^3 p}{\gamma^{1/2}} f_3 {}^{(3)}\nabla_j \int \frac{d^3 p}{\gamma^{1/2}} f_3 \right. \\ &\quad \left. - \int \frac{d^3 p}{\gamma^{1/2}} p_i f_1^- \int \frac{d^3 p}{\gamma^{1/2}} p_j f_1^- \right] + \mathcal{O}(\varepsilon_{\{p,\hbar\}}^2). \quad (4.121) \end{aligned}$$

When we now calculate the energy density up to this order, we find that the leading order contribution containing  $f_2$  drops out

$$e = \int \frac{d^3p}{\gamma^{1/2}} \omega_p f_1^+ - \left[ \int \frac{d^3p}{\gamma^{1/2}} \omega_p (f_1^+ - f_2) \right]^{-1} \\ \times \gamma^{ij} \left[ \frac{\hbar^2}{4} {}^{(3)}\nabla_i \int \frac{d^3p}{\gamma^{1/2}} f_3 {}^{(3)}\nabla_j \int \frac{d^3p}{\gamma^{1/2}} f_3 - \int \frac{d^3p}{\gamma^{1/2}} p_i f_1^- \int \frac{d^3p}{\gamma^{1/2}} p_j f_1^- \right] \\ + \frac{\hbar^2}{8} {}^{(3)}\nabla_j {}^{(3)}\nabla^j \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} + \mathcal{O}(\varepsilon_{\{p,h\}}^2). \quad (4.122)$$

Considering the pressure, we find that the dependence on the density  $f_2$  is still present to leading order,

$$P = \frac{1}{3} \int \frac{d^3p}{\gamma^{1/2}} \gamma^{ij} \frac{p_i p_j}{\omega_p} (f_1^+ + f_2) - \int \frac{d^3p}{\gamma^{1/2}} \omega_p f_2 \\ + \frac{1}{3} \left[ \int \frac{d^3p}{\gamma^{1/2}} \omega_p (f_1^+ - f_2) \right]^{-1} \\ \times \gamma^{ij} \left[ \frac{\hbar^2}{4} {}^{(3)}\nabla_i \int \frac{d^3p}{\gamma^{1/2}} f_3 {}^{(3)}\nabla_j \int \frac{d^3p}{\gamma^{1/2}} f_3 - \int \frac{d^3p}{\gamma^{1/2}} p_i f_1^- \int \frac{d^3p}{\gamma^{1/2}} p_j f_1^- \right] \\ - \frac{\hbar^2}{24} {}^{(3)}\nabla_j {}^{(3)}\nabla^j \int \frac{d^3p}{\gamma^{1/2}} \frac{f_1^+ + f_2}{\omega_p} + \mathcal{O}(\varepsilon_{\{p,h\}}^2). \quad (4.123)$$

Similarly, let us compute the four-velocity to next-to-leading order. This can be done by considering

$$u^\mu = \alpha \left[ \langle : \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} : \rangle \langle : \partial^\nu \hat{\phi} \partial_\rho \hat{\phi} : \rangle \langle : \partial^\rho \hat{\phi} \partial_0 \hat{\phi} : \rangle \right. \\ \left. - (\langle : \partial^\nu \hat{\phi} \partial_\nu \hat{\phi} : \rangle - \sigma_0) \langle : \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} : \rangle \langle : \partial^\nu \hat{\phi} \partial_0 \hat{\phi} : \rangle + \mathcal{O}(\varepsilon_{\{p,h\}}^2) \right], \quad (4.124)$$

with

$$u^\mu u_\mu = -1. \quad (4.125)$$

We compute the non-normalized Lorentz factor first

$$\frac{W}{\alpha} = -\frac{n_\nu u^\nu}{\alpha} = -N \gamma^{-3} \langle : \hat{\Pi} \hat{\Pi} : \rangle^3 - N^k \gamma^{-5/2} \langle \hat{\Pi} \partial_k \hat{\phi} : \rangle \langle : \hat{\Pi} \hat{\Pi} : \rangle^2 \\ + N \gamma^{-2} \langle : \hat{\Pi} \hat{\Pi} : \rangle \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \gamma^{ij} \langle : \partial_j \hat{\phi} \hat{\Pi} : \rangle \\ + N \gamma^{-2} \gamma^{ij} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \langle : \hat{\Pi} \hat{\Pi} : \rangle^2 + \mathcal{O}(\varepsilon_{\{p,h\}}^{3/2}). \quad (4.126)$$

Next, we compute

$$\begin{aligned} \frac{u^k}{\alpha} &= N^k \gamma^{-3} \langle : \hat{\Pi} \hat{\Pi} : \rangle^3 + \frac{N^k N^s}{N} \gamma^{-5/2} \langle : \hat{\Pi} \hat{\Pi} : \rangle^2 \langle : \hat{\Pi} \partial_s \hat{\phi} : \rangle \\ &\quad + \gamma^{kl} N \langle : \partial_l \hat{\phi} \hat{\Pi} : \rangle \gamma^{-2} \langle : \hat{\Pi} \hat{\Pi} : \rangle^2 + \gamma^{kl} N^s \gamma^{-2} \langle : \partial_l \hat{\phi} \hat{\Pi} : \rangle \langle : \hat{\Pi} \hat{\Pi} : \rangle \langle : \hat{\Pi} \partial_s \hat{\phi} : \rangle \\ &\quad - N^k \gamma^{-2} \langle : \hat{\Pi} \hat{\Pi} : \rangle \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \gamma^{ij} \langle : \partial_j \hat{\phi} \hat{\Pi} : \rangle \\ &\quad - N^k \gamma^{-2} \langle : \hat{\Pi} \hat{\Pi} : \rangle^2 \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \gamma^{ij} + \mathcal{O}(\varepsilon_{\{p, \hbar\}}^{3/2}). \end{aligned} \quad (4.127)$$

Finally, we can calculate the spatial part of the four-velocity without the need to explicitly calculate the normalization factor  $\alpha$ ,

$$\begin{aligned} v^k &= \frac{\alpha^{-1} u^k}{\alpha^{-1} W} + \frac{N^k}{N} = -\gamma^{kl} \frac{\langle : \partial_l \hat{\phi} \hat{\Pi} : \rangle}{\langle : \hat{\Pi} \hat{\Pi} : \rangle} + \mathcal{O}(\varepsilon_{\{p, \hbar\}}^{3/2}) \\ &= -\gamma^{kl} \left[ \int \frac{d^3 p}{\gamma^{1/2}} \omega_p (f_1^+ - f_2) \right]^{-1} \left[ \frac{\hbar}{2} {}^{(3)}\nabla_i \int \frac{d^3 p}{\gamma^{1/2}} f_3 - \int \frac{d^3 p}{\gamma^{1/2}} p_i f_1^- \right] + \mathcal{O}(\varepsilon_{\{p, \hbar\}}^{3/2}), \end{aligned} \quad (4.128)$$

and the normalized Lorentz factor is read-off in the standard way

$$W = 1 + \frac{1}{2} v^i v^j \gamma_{ij} + \mathcal{O}(v^4) = 1 + \frac{1}{2} \gamma^{kl} \frac{\langle : \partial_k \hat{\phi} \hat{\Pi} : \rangle \langle : \partial_l \hat{\phi} \hat{\Pi} : \rangle}{\langle : \hat{\Pi} \hat{\Pi} : \rangle^2} + \mathcal{O}(\varepsilon_{\{p, \hbar\}}^2). \quad (4.129)$$

In order to recover expressions in terms of a purely classical particle distribution, we need to assume that the  $\varepsilon_{\hbar}$  corrections are negligible with respect to the  $\varepsilon_p$  corrections and that our initial state is allowing for a hierarchy  $m \gg \Delta p \gg \frac{\hbar}{\Delta x}$ . Once we put forward the identification of even ( $f_1^+$ ) and odd ( $f_1^-$ ) phase-space densities that we discussed in the previous section, we end up with the following expressions,

$$\begin{aligned} e &= \int \frac{d^3 p}{\gamma^{1/2}} \omega_p f_1^+ \\ &\quad + \left[ \int \frac{d^3 p}{\gamma^{1/2}} \omega_p f_1^+ \right]^{-1} \gamma^{ij} \left[ \int \frac{d^3 p}{\gamma^{1/2}} p_i f_1^- \int \frac{d^3 p}{\gamma^{1/2}} p_j f_1^- \right] + \mathcal{O}(\varepsilon_{\hbar}) + \mathcal{O}(\varepsilon_p^2) \\ &= \int \frac{d^3 p}{\gamma^{1/2}} \omega_p f_1 + v^k \int \frac{d^3 p}{\gamma^{1/2}} p_k f_1 + \mathcal{O}(\varepsilon_{\hbar}) + \mathcal{O}(\varepsilon_p^2), \end{aligned} \quad (4.130)$$

$$\begin{aligned} P &= \frac{1}{3} \int \frac{d^3 p}{\gamma^{1/2}} \gamma^{ij} \frac{p_i p_j}{\omega_p} f_1^+ \\ &\quad - \frac{1}{3} \left[ \int \frac{d^3 p}{\gamma^{1/2}} \omega_p f_1^+ \right]^{-1} \gamma^{ij} \left[ \int \frac{d^3 p}{\gamma^{1/2}} p_i f_1^- \int \frac{d^3 p}{\gamma^{1/2}} p_j f_1^- \right] + \mathcal{O}(\varepsilon_{\hbar}) + \mathcal{O}(\varepsilon_p^2) \\ &= \frac{1}{3} \int \frac{d^3 p}{\gamma^{1/2}} \gamma^{ij} \frac{p_i p_j}{m} f_1 - \frac{1}{3} v^k \int \frac{d^3 p}{\gamma^{1/2}} p_k f_1 + \mathcal{O}(\varepsilon_{\hbar}) + \mathcal{O}(\varepsilon_p^2), \end{aligned} \quad (4.131)$$

$$\begin{aligned}
v_k &= \left[ \int \frac{d^3 p}{\gamma^{1/2}} \omega_p f_1^+ \right]^{-1} \left[ \int \frac{d^3 p}{\gamma^{1/2}} p_k f_1^- \right] + \mathcal{O}(\varepsilon_\hbar) + \mathcal{O}(\varepsilon_p^{3/2}) \\
&= \left[ \int \frac{d^3 p}{\gamma^{1/2}} f_1 \right]^{-1} \int \frac{d^3 p}{\gamma^{1/2}} \frac{p_k}{m} f_1 + \mathcal{O}(\varepsilon_\hbar) + \mathcal{O}(\varepsilon_p^{3/2}). \quad (4.132)
\end{aligned}$$

These expressions are identical to the expressions one would obtain for a distribution of classical non-relativistic particles in curved space-time. The above identification shows once more that such classical distributions may be represented by two-point functions of real scalar field operators via the Wigner transformation (4.57) and the subsequent recombination (4.65) to (4.68), always provided we are given a state that behaves classical enough. An example for such a state was given in [133] for an FLRW-universe with a particular vacuum choice. We identify the correlators  $f_{2,3}$  in this thesis with combinations of the squeezing contributions  $\langle \hat{a}_k \hat{a}_{-k} \rangle$  and  $\langle \hat{a}_k^\dagger \hat{a}_{-k}^\dagger \rangle$  in their paper ( $\hat{a}_k$  and  $\hat{a}_k^\dagger$  denote creation and annihilation operators, respectively), which they eventually dropped. The density  $f_1$  in this thesis, that approximates a classical particle phase-space density, is expressible in terms of  $\langle \hat{a}_k^\dagger \hat{a}_k \rangle$  in their paper and gives an intuitive interpretation of  $\hat{f}_1$  as a counting operator. The state-independent (or in this setting vacuum) contributions in [133] were removed by hand, which corresponds to the normal ordering prescription. Let us remark that the starting point in [133] is a phase-space description that makes use of an off-shell momentum variable which makes it in our opinion difficult to take the other degrees of freedom encoded in  $f_{2,3}$  into account (they were dropped in [133] as they are in the review literature [92] for Minkowski space-time).

## 4.5 Dynamics of phase-space densities

In the previous sections, we have interpreted the averaged phase-space densities (4.65) to (4.68) always with respect to the energy-momentum tensor, without self-interactions and without non-minimal couplings to the geometry. The goal of this section is to work out their dynamics in a spatial gradient approximation including the non-minimal coupling to the curvature and even including self-interaction in a one-loop approximation where we assume, for simplicity, that one-point functions  $\langle \hat{\phi} \rangle$ ,  $\langle \hat{\Pi} \rangle$  are absent.<sup>10</sup> Another point we stress again, is that we will not include anomalous contributions in the following kinetic equations. This means that we assume those contributions to be negligible, which remain after the equations of motion have been applied on the terms that normal order the phase-space operators in (4.57), which is well justified for the energy scales we are interested in, since such anomalous contributions are of order  $RM_P^{-2}$  at the level of the energy-momentum tensor. Moreover, we assume contributions, that result from the boundary of the normal neighbourhood, to be negligible which is a requirement on the state that goes hand in hand with the spatial gradient expansion.

<sup>10</sup> One-point functions are straightforwardly included by shifting the canonical operators. This shift is necessary since the gradient approximation cannot simply be applied for Wigner transforms of products of classical fields without a smoothing procedure. We discuss this also in [111] (cf. chapter 3) and list the references where such a procedure is pursued.



The dynamics for the averaged phase-space densities  $f_1^\pm$ ,  $f_2$ ,  $f_3$  given in (4.65) to (4.68) can be derived by first considering the expectation values  $F_{\phi\phi}$ ,  $F_{\phi\Pi}$ ,  $F_{\Pi\phi}$ ,  $F_{\Pi\Pi}$  given via (4.57) and acting with a time-derivative, commuting it with the exponential shift operators, using the equations of motions for the canonical fields, commuting the resulting operators back and rewriting them in such a way that they act on the expectation values of  $F_{\phi\phi}$ ,  $F_{\phi\Pi}$ ,  $F_{\Pi\phi}$ ,  $F_{\Pi\Pi}$  which is the most difficult part of the calculation. Finally, we rewrite everything in terms of the dimensionally rescaled quantities (4.65) to (4.68). The spatial gradient approximation truncates the infinite series of spatial derivatives, that results from commuting various differential operators. We keep on the other hand all time derivatives and thus all degrees of freedom. Since the calculation involves a number of lengthy expressions, it is unavoidable to introduce some notation. A lot of technical details of this procedure are deferred to Appendix 4.C to 4.E.

First, we define

$$\hat{u}^\pm \equiv \exp \left[ \pm \frac{r^k}{2} {}^{(3)}\nabla_k^H \right] \hat{\phi}, \quad (4.133)$$

$$\hat{v}^\pm \equiv \exp \left[ \pm \frac{r^k}{2} {}^{(3)}\nabla_k^H \right] \left[ \gamma^{-1/2} \hat{\Pi} \right], \quad (4.134)$$

$$N^\pm \equiv \exp \left[ \pm \frac{r^k}{2} {}^{(3)}\nabla_k^H \right] N, \quad (4.135)$$

$$(NK)^\pm \equiv \exp \left[ \pm \frac{r^k}{2} {}^{(3)}\nabla_k^H \right] (NK), \quad (4.136)$$

$$[N: \hat{\phi}^2:]^\pm \equiv \exp \left[ \pm \frac{r^k}{2} {}^{(3)}\nabla_k^H \right] [N: \hat{\phi}^2:], \quad (4.137)$$

$$(NR)^\pm \equiv \exp \left[ \pm \frac{r^k}{2} {}^{(3)}\nabla_k^H \right] (NR), \quad (4.138)$$

where  $R$  is the four-dimensional Ricci scalar. Moreover, we will need a couple of differential operators denoted by  $\mathcal{T}_*^\pm$ ,  $\mathcal{M}_*^\pm$ ,  $({}^{(3)}\square)_*^\pm$  and  $({}^{(3)}\nabla N)_*^\pm$ , which are calculated in a gradient approximation in appendix 4.D based on the general identities in 4.C. We find the following expressions, up to anomalous contributions, boundary terms and higher-order correlators which are all assumed to be

small,

$$\begin{aligned} \gamma^{1/2} \partial_t \left( \frac{\langle \hat{F}_{\phi\phi} \rangle}{\gamma^{1/2}} \right) &= \int_{T\Sigma_t} dr^3 \gamma^{1/2} e^{-\frac{i}{\hbar} r^k p_k} \left\{ \frac{1}{2} (N^+ + N^-) \langle : \hat{v}^+ \hat{u}^- + \hat{u}^+ \hat{v}^- : \rangle \right. \\ &\quad + \frac{1}{2} (N^+ - N^-) \langle : \hat{v}^+ \hat{u}^- - \hat{u}^+ \hat{v}^- : \rangle \\ &\quad \left. + (\mathcal{T}_*^+ + \mathcal{T}_*^- + \mathcal{M}_*^+ + \mathcal{M}_*^-) \langle : \hat{u}^+ \hat{u}^- : \rangle \right\}, \quad (4.139) \end{aligned}$$

$$\begin{aligned} \gamma^{1/2} \frac{\partial_t}{2} \left( \frac{\langle \hat{F}_{\Pi\phi} + \hat{F}_{\phi\Pi} \rangle}{\gamma^{1/2}} \right) &= \int_{T\Sigma_t} dr^3_{T\Sigma_t} \gamma^{1/2} e^{-\frac{i}{\hbar} r^k p_k} \left\{ \frac{1}{2} (N^+ + N^-) \langle : \hat{v}^+ \hat{v}^- : \rangle \right. \\ &\quad + \frac{1}{2} \left[ (\mathcal{T}_*^+ + \mathcal{T}_*^- + \mathcal{M}_*^+ + \mathcal{M}_*^-) \right. \\ &\quad \left. + \frac{1}{2} (NK)^+ + \frac{1}{2} (NK)^- \right] \langle : \hat{v}^+ \hat{u}^- + \hat{u}^+ \hat{v}^- : \rangle \\ &\quad + \frac{1}{4} \left( (NK)^+ - (NK)^- \right) \langle : \hat{v}^+ \hat{u}^- - \hat{u}^+ \hat{v}^- : \rangle \\ &\quad + \frac{1}{2} \left[ N^+ ({}^{(3)}\square)_*^+ + N^- ({}^{(3)}\square)_*^- - \frac{m^2}{\hbar^2} (N^+ + N^-) \right. \\ &\quad \left. - \xi [(NR)^+ + (NR)^-] \right. \\ &\quad \left. - \frac{1}{2} \frac{\lambda}{\hbar} ([N \langle : \hat{\phi}^2 : \rangle]^+ + [N \langle : \hat{\phi}^2 : \rangle]^-) \right. \\ &\quad \left. + ({}^{(3)}\nabla N)_*^+ + ({}^{(3)}\nabla N)_*^- \right] \langle : \hat{u}^+ \hat{u}^- : \rangle \Big\}, \quad (4.140) \end{aligned}$$

$$\begin{aligned} \gamma^{1/2} \frac{\partial_t}{2} \left( \frac{\langle \hat{F}_{\Pi\phi} - \hat{F}_{\phi\Pi} \rangle}{\gamma^{1/2}} \right) &= \int_{T\Sigma_t} dr^3 \gamma^{1/2} e^{-\frac{i}{\hbar} r^k p_k} \left\{ -\frac{1}{2} (N^+ - N^-) \langle : \hat{v}^+ \hat{v}^- : \rangle \right. \\ &\quad + \frac{1}{2} \left[ (\mathcal{T}_*^+ + \mathcal{T}_*^- + \mathcal{M}_*^+ + \mathcal{M}_*^-) \right. \\ &\quad \left. + \frac{1}{2} (NK)^+ + \frac{1}{2} (NK)^- \right] \langle : \hat{v}^+ \hat{u}^- - \hat{u}^+ \hat{v}^- : \rangle \\ &\quad + \frac{1}{4} \left( (NK)^+ - (NK)^- \right) \langle : \hat{v}^+ \hat{u}^- + \hat{u}^+ \hat{v}^- : \rangle \\ &\quad + \frac{1}{2} \left[ N^+ ({}^{(3)}\square)_*^+ - N^- ({}^{(3)}\square)_*^- - \frac{m^2}{\hbar^2} (N^+ - N^-) \right. \\ &\quad \left. - \xi [(NR)^+ - (NR)^-] \right. \\ &\quad \left. - \frac{1}{2} \frac{\lambda}{\hbar} ([N \langle : \hat{\phi}^2 : \rangle]^+ - [N \langle : \hat{\phi}^2 : \rangle]^-) \right. \\ &\quad \left. + ({}^{(3)}\nabla N)_*^+ - ({}^{(3)}\nabla N)_*^- \right] \langle : \hat{u}^+ \hat{u}^- : \rangle \Big\}, \quad (4.141) \end{aligned}$$

$$\begin{aligned}
\gamma^{1/2} \partial_t \left( \frac{\langle \hat{F}_{\text{III}} \rangle}{\gamma^{1/2}} \right) = & \int_{T\Sigma_t} dr^3 \gamma^{1/2} e^{-\frac{i}{\hbar} r^k p_k} \left\{ (\mathcal{T}_*^+ + \mathcal{T}_*^- + \mathcal{M}_*^+ + \mathcal{M}_*^- \right. \\
& + (NK)^+ + (NK)^- \rangle \langle : \hat{v}^+ \hat{v}^- : \rangle \\
& - \frac{1}{2} \left[ N^+ ({}^{(3)}\square)_*^+ - N^- ({}^{(3)}\square)_*^- - \frac{m^2}{\hbar^2} (N^+ - N^-) \right. \\
& - \xi [(NR)^+ - (NR)^-] \\
& - \frac{1}{2} \frac{\lambda}{\hbar} ([N \langle : \hat{\phi}^2 : \rangle]^+ - [N \langle : \hat{\phi}^2 : \rangle]^-) \\
& + ({}^{(3)}\nabla N)_*^+ - ({}^{(3)}\nabla N)_*^- \left. \right] \langle : \hat{v}^+ \hat{u}^- - \hat{u}^+ \hat{v}^- : \rangle \\
& + \frac{1}{2} \left[ N^+ ({}^{(3)}\square)_*^+ + N^- ({}^{(3)}\square)_*^- - \frac{m^2}{\hbar^2} (N^+ + N^-) \right. \\
& - \xi [(NR)^+ + (NR)^-] \\
& - \frac{1}{2} \frac{\lambda}{\hbar} ([N \langle : \hat{\phi}^2 : \rangle]^+ + [N \langle : \hat{\phi}^2 : \rangle]^-) \\
& + ({}^{(3)}\nabla N)_*^+ + ({}^{(3)}\nabla N)_*^- \left. \right] \langle : \hat{v}^+ \hat{u}^- + \hat{u}^+ \hat{v}^- : \rangle \left. \right\}. \quad (4.142)
\end{aligned}$$

The dynamical equations for  $\hat{F}_{\phi\phi}$ ,  $\hat{F}_{\Pi\phi}$ ,  $\hat{F}_{\phi\Pi}$ ,  $\hat{F}_{\text{III}}$  take a convenient form in terms of the horizontal lift of the covariant derivative [130] on the cotangent bundle of spatial hypersurfaces

$$D_k \equiv {}^{(3)}\nabla_k + p_l {}^{(3)}\Gamma_{kj}^l \frac{\partial}{\partial p_j}, \quad D_k p_j = 0. \quad (4.143)$$

The latter derivative transforms covariantly under a change of spatial coordinates. For brevity and to illustrate the structure, we write down the dynamics for the Wigner transformed expectation values  $F_{\phi\phi}$ ,  $F_{\Pi\phi}$ ,  $F_{\phi\Pi}$ ,  $F_{\text{III}}$  only to leading order in the spatial gradient expansion and the next-to-leading order expressions may be found in appendix 4.E,

$$\begin{aligned}
\partial_t F_{\phi\phi} = & [N + \mathcal{O}(\hbar^2)] (F_{\Pi\phi} + F_{\phi\Pi}) + \frac{i}{2} \hbar [N_{;k} \frac{\partial}{\partial p_k} + \mathcal{O}(\hbar^2)] (F_{\Pi\phi} - F_{\phi\Pi}) \\
& + \left[ N^k D_k - p_k N_{;m}^k \frac{\partial}{\partial p_m} - NK + \mathcal{O}(\hbar^2) \right] F_{\phi\phi}, \quad (4.144)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial_t}{2} (F_{\Pi\phi} + F_{\phi\Pi}) = & [N + \mathcal{O}(\hbar^2)] F_{\text{III}} \\
& + \frac{1}{2} \left[ N^k D_k - p_k N_{;m}^k \frac{\partial}{\partial p_m} + \mathcal{O}(\hbar^2) \right] (F_{\Pi\phi} + F_{\phi\Pi}) \\
& + \hbar \frac{i}{4} \left[ (NK)_{;j} \frac{\partial}{\partial p_j} + \mathcal{O}(\hbar^2) \right] (F_{\Pi\phi} - F_{\phi\Pi}) \\
& - \frac{1}{\hbar^2} \left[ Nm^2 + N \gamma^{kj} p_k p_j \right. \\
& \left. + \frac{\hbar}{2} \lambda [N + \mathcal{O}(\hbar^2)] \int \frac{d^3 q}{\gamma^{1/2}} F_{\phi\phi}(q) + \mathcal{O}(\hbar^2) \right] F_{\phi\phi}, \quad (4.145)
\end{aligned}$$

$$\begin{aligned}
\frac{i\partial_t}{2}(F_{\Pi\phi} - F_{\phi\Pi}) &= \frac{\hbar}{2} \left[ N_{;k} \frac{\partial}{\partial p_k} + \mathcal{O}(\hbar^2) \right] F_{\Pi\Pi} \\
&+ \frac{i}{2} \left[ N^k D_k - p_k N^k_{;m} \frac{\partial}{\partial p_m} + \mathcal{O}(\hbar^2) \right] (F_{\Pi\phi} - F_{\phi\Pi}) \\
&- \frac{\hbar}{4} \left[ (NK)_{;j} \frac{\partial}{\partial p_j} + \mathcal{O}(\hbar^2) \right] (F_{\Pi\phi} + F_{\phi\Pi}) \\
&- \frac{1}{2\hbar} \left[ 2Np_j D^j - \omega_p^2 N_{;k} \frac{\partial}{\partial p_k} + \mathcal{O}(\hbar^2) \right. \\
&\quad \left. - \frac{\hbar}{2} \lambda \left( \left[ \int \frac{Nd^3q}{\gamma^{1/2}} F_{\phi\phi}(q) \right]_{;k} \frac{\partial}{\partial p_k} + \mathcal{O}(\hbar^2) \right) \right] F_{\phi\phi}, \quad (4.146)
\end{aligned}$$

$$\begin{aligned}
\partial_t F_{\Pi\Pi} &= \left[ N^k D_k + NK - p_k N^k_{;m} \frac{\partial}{\partial p_m} + \mathcal{O}(\hbar^2) \right] F_{\Pi\Pi} \\
&- \frac{i}{2\hbar} \left[ 2Np_j D^j + \mathcal{O}(\hbar^2) \right. \\
&\quad \left. - \frac{\hbar}{2} \lambda \left( \left[ \int \frac{Nd^3q}{\gamma^{1/2}} F_{\phi\phi}(q) \right]_{;k} \frac{\partial}{\partial p_k} + \mathcal{O}(\hbar^2) \right) \right] (F_{\Pi\phi} - F_{\phi\Pi}) \\
&- \frac{1}{\hbar^2} \left[ Nm^2 + N\gamma^{kj} p_k p_j + \mathcal{O}(\hbar^2) \right. \\
&\quad \left. + \frac{\hbar}{2} \lambda \left( \int \frac{Nd^3q}{\gamma^{1/2}} F_{\phi\phi}(q) + \mathcal{O}(\hbar^2) \right) \right] (F_{\Pi\phi} + F_{\phi\Pi}). \quad (4.147)
\end{aligned}$$

The next step is to convert the dynamical equation for the dimensionally unequal expectation values  $F_{\phi\phi}$ ,  $F_{\Pi\phi}$ ,  $F_{\phi\Pi}$  and  $F_{\Pi\Pi}$  into dynamical equations for the dimensionally equal phase-space densities  $f_1^\pm$ ,  $f_2$  and  $f_3$ , that we defined in (4.65) to (4.68). It turns out that several leading order terms cancel in this dimensional rescaling, such that some next-to-leading order terms of the previous equations turn into leading order terms for the equations of the rescaled quantities. We would have to include even higher order terms in the previous calculation for  $F_{\phi\phi}$ ,  $F_{\Pi\phi}$ ,  $F_{\phi\Pi}$  and  $F_{\Pi\Pi}$  in order to get to next-to-leading order terms for rescaled quantities  $f_1^\pm$ ,  $f_2$  and  $f_3$ . However, we see that even certain leading order corrections are of order  $\hbar$  and thus first order terms concerning

the spatial gradient expansion. We find

$$\begin{aligned}
\partial_t f_1^+ = & \left[ N^k D_k - p_k N_{;m}^k \frac{\partial}{\partial p_m} \right] f_1^+ - \left[ NK + \frac{p_m p_k}{\omega_p^2} N_{;m}^{k m} \right] f_2 \\
& - \frac{1}{\omega_p} \left[ N p_j D^j - \omega_p^2 N_{;m} \frac{\partial}{\partial p_m} \right] f_1^- \\
& + \frac{\hbar}{\omega_p} \left[ \frac{1}{2} p_j N_{;k} \frac{\partial}{\partial p_k} D^j + \frac{1}{4} N D_j D^j - \frac{1}{3} N p_i p_j {}^{(3)} R_{qm}^i \frac{\partial^2}{\partial p_q \partial p_m} \right. \\
& \quad \left. - \frac{1}{12} N p_i {}^{(3)} R_{;k}^i \frac{\partial}{\partial p_k} + \frac{1}{6} N {}^{(3)} R - \xi N R \right] f_3 \\
& + \frac{\lambda}{2} \omega_p \left[ N \frac{\hbar^3}{\omega_p^2} \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q) + f_2(q)}{\omega_q} \right]_{;k} \frac{\partial}{\partial p_k} f_1^- \\
& - \frac{\lambda}{2} \frac{\omega_p}{\hbar} \left[ N \frac{\hbar^3}{\omega_p^2} \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q) + f_2(q)}{\omega_q} \right. \\
& \quad \left. - \frac{\hbar^2}{8} \frac{\hbar^3}{\omega_p^2} \left[ N \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q) + f_2(q)}{\omega_q} \right]_{;ks} \frac{\partial^2}{\partial p_k \partial p_s} \right] f_3, \quad (4.148)
\end{aligned}$$

$$\begin{aligned}
\partial_t f_1^- = & \left[ N^k D_k - p_k N_{;m}^k \frac{\partial}{\partial p_m} \right] f_1^- - \frac{\hbar}{2} (NK)_{;j} \frac{\partial}{\partial p_j} f_3 \\
& - \frac{1}{\omega_p} \left[ N p_j D^j - \omega_p^2 N_{;m} \frac{\partial}{\partial p_m} \right] f_1^+ - \frac{1}{\omega_p} \left[ N p_j D^j + N_{;m} p_l \gamma^{lm} \right] f_2 \\
& + \frac{\lambda}{2} \omega_p \left[ N \frac{\hbar^3}{\omega_p^2} \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q) + f_2(q)}{\omega_q} \right]_{;k} \frac{\partial}{\partial p_k} [f_1^+ + f_2], \quad (4.149)
\end{aligned}$$

$$\begin{aligned}
\partial_t f_2 = & 2 \frac{\omega_p}{\hbar} N f_3 + \omega_p N_{;k} \frac{\partial}{\partial p_k} f_1^- \\
& + \left[ N^k D_k - p_k N_{;m}^k \frac{\partial}{\partial p_m} - \frac{p_i p_k}{\omega_p^2} (NK^{ij} - N_{;j}^i) \right] f_2 \\
& - \left[ NK + \frac{p_m p_k}{\omega_p^2} N_{;m}^{k m} \right] f_1^+ + N \frac{p_j}{\omega_p} D^j f_1^- \\
& - \frac{\hbar}{\omega_p} \left[ \frac{1}{2} p_j N_{;k} \frac{\partial}{\partial p_k} D^j + \frac{1}{4} N D_j D^j - \frac{1}{3} N p_i p_j {}^{(3)} R_{qm}^i \frac{\partial^2}{\partial p_q \partial p_m} \right. \\
& \quad \left. - \frac{1}{12} N p_i {}^{(3)} R_{;k}^i \frac{\partial}{\partial p_k} + \frac{1}{6} N {}^{(3)} R - \xi N R \right] f_3 \\
& + \frac{\lambda}{8} \omega_p \left[ N \frac{\hbar^3}{\omega_p^2} \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q) + f_2(q)}{\omega_q} \right]_{;k} f_1^- \\
& + \frac{\lambda}{2} \frac{\omega_p}{\hbar} \left[ N \frac{\hbar^3}{\omega_p^2} \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q) + f_2(q)}{\omega_q} \right. \\
& \quad \left. - \frac{\hbar^2}{8} \frac{\hbar^3}{\omega_p^2} \left[ N \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q) + f_2(q)}{\omega_q} \right]_{;ks} \frac{\partial^2}{\partial p_k \partial p_s} \right] f_3, \quad (4.150)
\end{aligned}$$

$$\begin{aligned}
\partial_t f_3 = & \left[ N^k D_k - p_k N_{;m}^k \frac{\partial}{\partial p_m} \right] f_3 + \frac{\hbar}{2} (NK)_{;j} \frac{\partial}{\partial p_j} f_1^- \\
& - \frac{\hbar}{\omega_p} \left[ 2 \frac{\omega_p^2}{\hbar^2} N - \frac{\omega_p^2}{4} N_{;qm} \frac{\partial^2}{\partial p_q \partial p_m} - \frac{1}{4} \frac{p_i p_j}{\omega_p^2} N_{;ij} \right. \\
& \quad - \frac{1}{2} p_j N_{;m} \frac{\partial}{\partial p_m} D^j + \frac{1}{2} \frac{p_i p_j}{\omega_p^2} N_{;i}^j D^j - \frac{1}{4} N D_j D^j \\
& \quad + \frac{1}{3} N p_i p_j {}^{(3)} R_{qm}^i{}^j \frac{\partial^2}{\partial p_q \partial p_m} + \frac{1}{12} N p_i {}^{(3)} R_m^i \frac{\partial}{\partial p_m} \\
& \quad \left. + \frac{1}{4} N \frac{p_i p_j}{\omega_p^2} {}^{(3)} R^{ij} - \frac{1}{6} N {}^{(3)} R + \xi N R \right] f_2 \\
& - \frac{\hbar}{\omega_p} \left[ \frac{1}{2} p_m N_{;q}^m \frac{\partial}{\partial p_q} + \frac{1}{4} N_{;j}^j - \frac{1}{2} \frac{p_i p_j}{\omega_p^2} N_{;ij} \right. \\
& \quad - \frac{1}{2} p_j N_{;m} \frac{\partial}{\partial p_m} D^j + \frac{1}{2} \frac{p_i p_j}{\omega_p^2} N_{;i}^j D^j - \frac{1}{4} N D_j D^j \\
& \quad + \frac{1}{3} N p_i p_j {}^{(3)} R_{qm}^i{}^j \frac{\partial^2}{\partial p_q \partial p_m} + \frac{1}{12} N p_i {}^{(3)} R_m^i \frac{\partial}{\partial p_m} \\
& \quad \left. + \frac{1}{4} N \frac{p_i p_j}{\omega_p^2} {}^{(3)} R^{ij} - \frac{1}{6} N {}^{(3)} R + \xi N R \right] f_1^+ \\
& - \frac{\lambda \omega_p}{2 \hbar} \left[ N \frac{\hbar^3}{\omega_p^2} \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q) + f_2(q)}{\omega_q} \right. \\
& \quad \left. - \frac{\hbar^2 \hbar^3}{8 \omega_p^2} \left[ N \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q) + f_2(q)}{\omega_q} \right]_{;ks} \frac{\partial^2}{\partial p_k \partial p_s} \right] [f_1^+ + f_2]. \quad (4.151)
\end{aligned}$$

Equations (4.148) to (4.151) are the main result of this chapter.<sup>11</sup> These equations are an effective description of the state-dependent (normal ordered) part of the dynamics of a real scalar field quantum state in curved space-time in the language of phase-space variables  $(x^\mu, p_k)$ , under the assumption that the state admits a gradient and loop expansion. For macroscopic observables of systems, that have some notion of classicality, the quantities  $f_1^\pm(x^\mu, p_k)$  and  $f_{2,3}(x^\mu, p_k)$  should dominate over the state-independent part coming from the quantum commutation relation. They can be given any initial value that is compatible with the spatial gradient approximation and their symmetry properties.

We first note that all equations for the operators  $f_1^\pm$  and  $f_{2,3}$  are spatially covariant and provide in principle candidates for phase-space density operators. However, we also need to realize that  $f_1^+$  and  $f_{2,3}$  are even functions in  $p_k$

<sup>11</sup>Although we excluded states containing one-point functions for simplicity, they will give rise to similar equations subject to a constraint equation due to the lack of degrees of freedom - such equations have been derived for example in [89] where higher time derivatives and thus degrees of freedom were dropped. However, the conditions to obtain a leading order classical particle Vlasov equation (4.153) can only be satisfied on time-averages over the expectation values. This can be understood for example by considering the Minkowski space-time limit where the two solutions of  $\langle \hat{f}_2 \rangle$  that are determined by condensates are given by  $\langle \hat{f}_2 \rangle_{\text{cond}}^{\text{flat}} = \alpha \cos(2\omega_p t) + \beta \sin(2\omega_p t)$ . Fixing the proportionality constants of these two solutions to be zero, as we were able to do it for the general case, would also set  $\langle \hat{f}_1^+ \rangle_{\text{cond}}^{\text{flat}} = (\alpha^2 + \beta^2)^{1/2}$  to zero and yields only a trivial solution of the system. The resolution is thus to keep all the degrees of freedom and perform a time-averaging in this case.

whereas  $f_1^-$  is an odd function in the momentum. Thus, only some combination these two-point functions can account for the degrees of freedom of a classical particle phase-space density. A promising candidate is read off from the first pair of equations (4.148) and (4.149) in the non-interacting limit,

$$\begin{aligned} & \left[ \partial_t - N^{k(3)} D_k + ({}^{(3)}\nabla_j N^k) p_k \frac{\partial}{\partial p_j} + N \frac{p^k}{\omega_p} ({}^{(3)}D_k - \omega_p [\partial_j N] \frac{\partial}{\partial p_j} \right. \\ & \quad \left. + \frac{\lambda}{2} \omega_p \left[ N \frac{\hbar^3}{\omega_p^2} \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1^+(q)}{\omega_q} \right]_{;k} \frac{\partial}{\partial p_k} + \mathcal{O}(\hbar^2) \right] [f_1^+ + f_1^-] = \mathcal{O}(f_{2,3}). \end{aligned} \quad (4.152)$$

By rewriting the above equation for  $f_1 = f_1^+ + f_1^-$ , we find the Vlasov equation with a one loop correction, that can be interpreted as a mass shift, as well as source terms that are due to the additional correlators in the scalar field description and higher-order spatial gradient corrections. Undoing the ADM-decomposition, the equation reads

$$\begin{aligned} & \left[ p^\mu \partial_\mu + p_\mu p^\nu \Gamma_{\nu i}^\mu \frac{\partial}{\partial p_i} \right. \\ & \quad \left. + \frac{\lambda}{2} \omega_p \left[ N \frac{\hbar^3}{\omega_p^2} \int \frac{d^3 q}{\gamma^{1/2}} \frac{f_1(q)}{\omega_q} \right]_{;k} \frac{\partial}{\partial p_k} + \mathcal{O}(\hbar^2) \right] f_1(x^\mu, p_j) = \mathcal{O}(f_{2,3}), \end{aligned} \quad (4.153)$$

where

$$p^0(x^\mu, p_j) \equiv \sqrt{(g^{0j} p_j)^2 - g^{00}(m^2 + g^{ij} p_i p_j)} = \sqrt{(g^{0j} p_j)^2 - g^{00} \omega_p^2}. \quad (4.154)$$

We remark that within the one loop approximation we do not find  $2 \rightarrow 2$  particle scattering processes which come from self-energy diagrams whose first contribution is proportional to  $\lambda^2$ .<sup>12</sup> However, the self-masses  $\propto \lambda$  are included and - depending on the problem - may already give significant corrections to the dynamics of the Vlasov equation.

Combining the other pair of equations (4.150) and (4.151) shows that  $f_2$  and  $f_3$  are to leading order oscillators with frequency of the particle energy  $\omega_p$ . Thus, equations (4.148) to (4.151) generalize the Vlasov equation for relativistic particles in curved space-time by including the additional densities  $f_{2,3}$ . The latter densities can be rewritten as higher-order time derivatives acting on  $f_1^\pm$ . We conclude that if we wanted to recover the limit of a classical particle density, we would have to impose a state such that  $f_2$  is initially of higher order in  $\hbar$  and also remains of higher order in  $\hbar$ , which then translates into a condition for  $f_3$  and finally into  $f_2 \sim \mathcal{O}(\hbar^2)(f_1^+, f_1^-)$  (these are rough estimates and it remains to be studied whether such conditions can be maintained by the dynamics). First-order corrections to (4.153), that are contained in (4.148) to (4.151), may

<sup>12</sup>It is the one-loop approximation that allows the system to close on-shell since two-loop contributions will integrate off-shell energies which are not only supported on the mass-shell. However, one can employ a quasi-particle approximation for the 2-loop contributions which eventually leads also to a  $2 \rightarrow 2$  particle scattering contribution as it appears on the right hand side of the classical particle Boltzmann equation. These properties for the  $\lambda\phi^4$  theory are known in Minkowski space [92, 99], but a general curved space-time discussion is still lacking.

be obtained by expanding the phase-space densities into harmonics and see how the oscillatory terms back-react on the non-oscillatory part of the density  $f_1$  via the self-interaction terms or via non-linear terms that are obtained by making use of the Einstein equations. Also keeping in mind a generalization in terms of higher-loop effects, we think that the advantage of our formalism lies in an end-to-end link between quantum field theory and particle kinetics in curved space-time, that allows one to systemically include field-theoretic corrections, while being able to refer to a (in some sense modified) particle interpretation.

## 4.6 Generalized cold dark matter kinetics in linearized gravity

In the last section we have dealt with a set of fairly general but lengthy equations. The idea of this section is to see how they reduce to more feasible sets of equations once we apply them to the concrete cosmological set up of cold dark matter perturbations between galactic scales and the Hubble horizon. The main result is a generalization of the kinetic description of classical particle cold dark matter as it discussed for example in [24].

Let us fix a linearly perturbed metric in FLRW background in the generalized Newtonian gauge, that includes vector and tensor perturbations and which is also referred to as Poisson gauge [109, 134]. We label equal-time hypersurfaces by the variable  $\eta$  and denote spatial coordinates by  $x$ ,

$$\partial_t \rightarrow \partial_\eta = (\cdot)'. \quad (4.155)$$

Indices for the linear quantities are raised and lowered by the comoving background spatial metric  $\delta_{ij}$  as in [9]. The (3+1)-dimensional metric takes the form

$$g_{\mu\nu} = a^2 \begin{pmatrix} -(1 + 2\Phi_N) & -s_i \\ -s_i & \delta_{ij}(1 - 2\Psi_N) + h_{ij} \end{pmatrix}, \quad (4.156)$$

$$\delta^{kj}\partial_k s_j = 0, \quad \delta^{kj}\partial_k h_{ji} = 0, \quad \delta^{ij}h_{ij} = 0, \quad (4.157)$$

such that the spatial metric, its inverse and its determinant are given to linear order by

$$\gamma_{ij} = a^2 [\delta_{ij}(1 - 2\Psi_N) + h_{ij}], \quad \gamma^{1/2} = a^3(1 - 3\Psi_N), \quad (4.158)$$

and the lapse function and shift vector read

$$N = a(1 + \Phi_N), \quad N^i = -s^i = -\delta^{ij}s_j = a^{-2}\delta^{ij}N_j. \quad (4.159)$$

We define a gravitational perturbation parameter related to the metric perturbations by

$$\varepsilon_g \sim \Phi_N, s_i, \Psi_N, h_{ij} \ll 1. \quad (4.160)$$



We have

$$\begin{aligned} {}^{(3)}\Gamma_{km}^l &= \delta^{ls}\delta_{mk}\partial_s\Psi_N - \delta_m^l\partial_k\Psi_N - \delta_k^l\partial_m\Psi_N \\ &\quad + \frac{1}{2}(\partial_k h_m^l + \partial_m h_k^l - \delta^{sl}\partial_s h_{km}). \end{aligned} \quad (4.161)$$

Let us collect further geometrical quantities, that appear in the Einstein equation in ADM decomposition.

$${}^{(3)}R_{ij} = \delta_{ij}\Delta\Psi_N + \partial_i\partial_j\Psi_N - \frac{1}{2}\Delta h_{ij}, \quad {}^{(3)}R = \frac{4}{a^2}\Delta\Psi_N, \quad (4.162)$$

$$\begin{aligned} K_{ij} &= -a\mathcal{H}[\delta_{ij}(1 - \Phi_N - \mathcal{H}^{-1}\Psi'_N - 2\Psi_N) + h_{ij}] \\ &\quad - \frac{a}{2}[s_{i,j} + s_{j,i} + h'_{ij}], \end{aligned} \quad (4.163)$$

$$K = -3a^{-1}\mathcal{H}(1 - \Phi_N - \mathcal{H}^{-1}\Psi'_N), \quad K^2 = 3K_{ij}K^{ij} + \mathcal{O}(\varepsilon_g^2). \quad (4.164)$$

We split the normal observer momentum vector and stress tensor in scalar, vector and tensor components,

$$P_i = a\partial_i P_L + aP_i^T, \quad \partial^j P_j^T = 0, \quad (4.165)$$

$$S_{ij} = \frac{a^2\delta_{ij}}{3}S + a^2(\partial_i\partial_j S^A - \frac{\Delta}{3}\delta_{ij}S^A + \partial_i S_j + \partial_j S_i + S_{ij}^{TT}), \quad (4.166)$$

$$\partial^j S_j = \partial^j S_{ij}^{TT} = \delta^{ij}S_{ij}^{TT} = 0. \quad (4.167)$$

Indices for the quantities on the right-hand-side of (4.165) and (4.167) will be raised and lowered with the flat three-dimensional metric. Let us write down the Einstein equations in terms of the perturbed metric

$$3\mathcal{H}^2 + 2\Delta\Psi_N = \frac{\hbar}{M_P^2}a^2[E - 3\mathcal{H}P_L], \quad (4.168)$$

$$\frac{1}{2}\Delta s_i = -\frac{\hbar}{M_P^2}a^2P_i^T, \quad (4.169)$$

$$\Phi_N - \Psi_N = -\frac{\hbar}{M_P^2}a^2S_A, \quad (4.170)$$

$$h_{ij}'' + 2\mathcal{H}h'_{ij} - \Delta h_{ij} = \frac{\hbar}{M_P^2}2a^2S_{ij}^{TT}, \quad (4.171)$$

Energy-momentum conservation reads in linearized gravity

$$\begin{aligned} \partial_\eta(a^3[1 - 3\Psi_N]E) + a^3\partial_i\Big([\delta^{ij}(1 + \Phi_N - \Psi_N) - h^{ij}]a^{-1}P_j + \delta^{ij}s_jE\Big) \\ + a^3\Big(\mathcal{H}[1 - \mathcal{H}^{-1}\Psi'_N - 3\Psi_N]S \\ + [s_{i,j} + \frac{1}{2}h'_{ij}]S^{ij} + a^{-1}\delta^{ij}P_j\partial_i\Phi_N\Big) = 0, \end{aligned} \quad (4.172)$$

$$\begin{aligned} & \partial_\eta (a^3 [1 - 3\Psi_N] P_j) + a^3 \partial_i (a [1 + \Phi_N - 3\Psi_N] S_j^i + s^i P_j) \\ &= a^4 \left( a^2 \frac{1}{2} S^{ik} \partial_j h_{ik} - S \partial_j \Psi_N - a^{-1} P_i \partial_j s^i - E \partial_j \Phi_N \right), \end{aligned} \quad (4.173)$$

which does not help much unless we know how  $S_{ij}$  depends on  $E$  and  $P_i$ . Note that it is suggestive to approximate these equations further with the linearized Einstein equations and rewrite  $E, P_i, S_{ij}$  in terms of the gravitational perturbations. However, the resulting non-linear terms are not necessarily small since they involve gradient terms of the type  $\mathcal{H}^{-2} \Delta$ . Some of them may become important around the scale where the density contrast in Fourier space defined via  $E(\eta, k) = \bar{E}(\eta) + \delta E(\eta, k)$  is of order one,  $\bar{E}^{-1} \delta E(k_{NL}) \propto \mathcal{H}^{-2} k_{NL}^2 \Psi_N(k_{NL}) \approx 1$ . This scale is on the order of roughly  $k_{NL}^{-1} \approx 5 \text{ Mpc}$ . We emphasize that linearization in the gravitational perturbations can still be valid on these scales, although the density contrast has to be treated non-linearly. In the context of cosmological large-scale structures one is typically interested in the evolution on sub-Hubble scales ( $k_H^{-1} \lesssim 10^4 \text{ Mpc}$ ). We capture the corrections, that result from separations with respect to this scale, by introducing a perturbation parameter  $\varepsilon_H$ ,

$$\mathcal{O}(\varepsilon_H^{-1}) \delta g_{\mu\nu} \sim \mathcal{H}^{-2} \Delta \delta g_{\mu\nu} \gg \delta g_{\mu\nu}. \quad (4.174)$$

This expansion allows us for example to drop several corrections in  $E, P_i$  and  $S_{ij}$ , that are related to the perturbation of the determinant of the spatial metric  $\delta\gamma^{1/2}$ . On the other hand, the smallest large-scale structures we are interested in are related to galactic scales  $k_g^{-1} \sim 10 \text{ kpc}$ . In order to be consistent with our perturbative schemes, we have to contrast the scale  $k_g^{-1}$  with the de Broglie wavelength  $k_{dB}^{-1} f_i(\vec{k}, \vec{p}) \sim \hbar \parallel \frac{\partial}{\partial p_j} f_i(\vec{k}, \vec{p}) \parallel$  which was related to the spatial gradient expansion that we have used to derive the kinetic equations (4.148) to (4.151). By using typical galaxy velocities of  $v_g \approx 10^{-3} c$  we can express the de Broglie wavelength in terms of the Compton wavelength  $k_C^{-1} \propto \hbar m^{-1}$  as  $k_{dB}^{-1} \sim 10^3 k_C^{-1}$ . For dark matter, the mass of which is at the electroweak scale ( $\sim 10^2 \text{ GeV}$ ), we find that de Broglie wavelength is of order  $k_{dB,EW}^{-1} \sim 10^{-33} \text{ kpc}$  and thus spatial gradient corrections can be safely neglected, whereas for ultralight dark matter with mass  $\sim 10^{-31} \text{ GeV}$  we find  $k_{dB,UL}^{-1} \sim 10^6 \text{ kpc}$  such that gradient corrections can play a role at galactic scales. However, let us focus here on the less exotic case where  $k_g^{-1} \gg k_{dB}^{-1}$ . Moreover, we are given a non-relativistic expansion by means of the galactic velocities

$$\varepsilon_p \sim \frac{p_i p_j \gamma^{ij}}{m^2} \sim 10^{-6}, \quad (4.175)$$

such that the particle energy is dominated by the mass. This relation justifies at least for certain mass ranges the inclusion of a self-coupling term in the kinetic equations, as we will see shortly. We also want to consider small corrections to the classical particle density picture and demand

$$f_1 \gg |f_{2,3}|. \quad (4.176)$$

In order to stick close to the cold dark matter scenario, we also want a first bound on the dark matter interactions such that it does not source the Hubble

rate too much

$$\lambda \frac{\hbar^3}{m^3} \int \frac{d^3 p}{\gamma^{1/2}} f_1^+ \ll 1. \quad (4.177)$$

Moreover, we want to keep the influence of the non-minimal coupling fairly small such that it cannot spoil the smallness of gradients or gravitational perturbations,

$$\hbar^2 |\xi R| \lesssim m^2. \quad (4.178)$$

We express the leading order terms on the right-hand-side of (4.168) to (4.171) in terms of the phase-space densities such that we can plug the constraint equations back in the kinetic equations for  $f_1^\pm$ ,  $f_2$  and  $f_3$  and solve them together with the gravitational wave equation. In accordance with the slightly more general discussion around (4.75) to (4.77), we find that the gravitational perturbations get their leading order contributions between galactic scales and the Hubble scale from the two phase-space densities  $f_1^\pm$  (as is the case for the classical particle cold dark matter scenario if we split the classical density into even and odd parts). The Poisson equation reads

$$3\mathcal{H}^2 + 2\Delta\Psi_N \approx \frac{\hbar}{M_P^2} \frac{m}{a} \int d^3 p f_1^+ = \frac{\hbar}{M_P^2} \frac{m}{a} \int d^3 p f_1, \quad (4.179)$$

and we note that the constraint (4.177) relates the mass and the coupling via

$$\lambda \frac{\hbar^3}{m^3} \int \frac{d^3 p}{\gamma^{1/2}} f_1^+ \ll 1 \quad \longrightarrow \quad \lambda \left( \frac{\hbar^2 \mathcal{H}^2}{a^2 m^2} \right) \frac{M_P^2}{m^2} \sim \lambda \frac{10^{-8} (\text{eV})^4}{m^4} \ll 1. \quad (4.180)$$

It is now clear that for masses around the electroweak scale the interaction energy does not influence the Hubble rate whereas it can become important for ultralight particles already for very small couplings. Moreover, vector perturbations and the gravitational slip are given by

$$\frac{1}{2} \Delta^2 s_i \approx -\frac{\hbar}{M_P^2} a^{-1} \left[ \Delta \int d^3 p p_k f_1^- - \partial_i \partial^k \int d^3 p p_k f_1^- \right], \quad (4.181)$$

$$\Delta^2 (\Phi_N - \Psi_N) \approx \frac{\hbar}{M_P^2} \frac{3}{2} a^{-3} \left[ \frac{\Delta}{3} \delta^{kj} - \partial^k \partial^j \right] \int d^3 p p_k p_j f_1^+. \quad (4.182)$$

Note that we can replace  $f_1^\pm$  with  $f_1 = f_1^+ + f_1^-$  in these equations due to their symmetry properties. The only dynamical gravitational perturbations are the traceless, transverse tensor perturbations which obey

$$\begin{aligned} h''_{ij} + 2\mathcal{H}h'_{ij} - \Delta h_{ij} \approx & \frac{\hbar}{M_P^2} \frac{2}{ma^3} \int d^3 p \left[ p_i p_j - \frac{\delta_{ij}}{3} p_k p_m \delta^{km} \right. \\ & + p_k p_m \frac{\Delta^{-2}}{2} \left( \partial_i \partial_j + \Delta \delta_{ij} \right) \left( \partial^k \partial^m - \frac{\Delta}{3} \delta^{km} \right) \\ & \left. + p_k p_m \delta^{km} \frac{2}{3} \Delta^{-1} \partial_i \partial_j - \Delta^{-1} \left( p_k p_j \partial_i \partial^k + p_k p_i \partial_j \partial^k \right) \right] f_1^+. \end{aligned} \quad (4.183)$$

Thus, the Einstein equations look to leading order in our perturbation parameters the same, whether we use a classical particle phase-space density or the

density derived from the scalar quantum field,  $f_1 = f_1^+ + f_1^-$ . The densities  $f_{2,3}$  enter at higher order. However, the dynamics for this source in the Einstein-equations is generalized by the following set of differential equations including the densities  $f_{2,3}$ . We find for the phase-space density  $f_1^+$ ,

$$\begin{aligned} (f_1^+)' + \left[ s^k \partial_k - (\partial_m s^k) p_k \frac{\partial}{\partial p_m} \right] f_1^+ \approx \\ - \left[ \delta^{jk} \frac{p_j}{ma} \partial_k - ma \partial_k \left[ \Phi_N - \frac{\lambda}{2} \frac{\hbar^3}{m^3 a^3} \int d^3 q f_1^+(q) \right] \frac{\partial}{\partial p_k} \right] f_1^- \\ + 3 \left[ \mathcal{H} - \Psi'_N \right] f_2 - \frac{\hbar}{ma} \left[ \frac{1}{4} \delta^{ij} \partial_i \partial_j + \frac{\lambda}{2} \frac{\hbar^2}{m^2 a^2} \int d^3 q f_1^+(q) \right] f_3, \end{aligned} \quad (4.184)$$

where we drop higher-order terms involving relativistic corrections or gradients that are small compared to the mass scale. Note that the last term in (4.184) may be important for certain combinations of masses and self-couplings, which is still consistent with the constraint (4.180),

$$||\partial_\eta||^{-1} \lambda \frac{\hbar^2}{m^2 a^2} \int d^3 q f_1^+(q) \sim \lambda \frac{\hbar \mathcal{H}}{am} \frac{M_P^2}{m^2} \sim \lambda \frac{10^{-3} (\text{GeV})^3}{m^3}. \quad (4.185)$$

Maybe more importantly, the self-interaction term multiplying  $f_1^-$  in (4.184),

$$\partial_k \left[ \Phi_N - \frac{\lambda}{2} \frac{\hbar^3}{m^3 a^3} \int d^3 q f_1^+(q) \right] \sim \partial_k \left[ \Phi_N - \frac{\lambda \hbar^2 \mathcal{H}^2}{2 m^2 a^2} \frac{M_P^2}{m^2} \frac{\Delta \Psi_N}{\mathcal{H}^2} \right], \quad (4.186)$$

can compete with the potential at the non-linear scale where  $\Delta \Psi_N \sim \mathcal{H}^2$  and still obey the constraint (4.180) for certain combinations of mass and self-coupling,

$$\Phi_N(k_{\text{NL}}) \sim \frac{\lambda \hbar^2 \mathcal{H}^2}{2 m^2 a^2} \frac{M_P^2}{m^2} \frac{k_{\text{NL}}^2}{\mathcal{H}^2} \Psi_N(k_{\text{NL}}) \quad \text{for} \quad \frac{\lambda \hbar^2 \mathcal{H}^2}{2 m^2 a^2} \frac{M_P^2}{m^2} \sim 10^{-5} \ll 1. \quad (4.187)$$

We also note that the gravitational vector perturbations enter at this order as a corrective for the time derivative, which is true for all four densities as we will see in a moment. Tensor perturbations enter in equations like (4.184) in various places, however, such terms are all of higher-order in the spatial gradient expansion. The same is again true for the dynamical equations of the other densities. Also, terms involving the non-minimal coupling  $\xi$  are of higher-order in all equations. For the odd density  $f_1^-$  we find,

$$\begin{aligned} (f_1^-)' + \left[ s^k \partial_k - (\partial_m s^k) p_k \frac{\partial}{\partial p_m} \right] f_1^- \approx \\ - \left[ \delta^{jk} \frac{p_j}{ma} \partial_k - ma \partial_k \left[ \Phi_N - \frac{\lambda}{2} \frac{\hbar^3}{m^3 a^3} \int d^3 q f_1^+(q) \right] \frac{\partial}{\partial p_k} \right] f_1^+ \\ - \frac{\hbar}{2} (\partial_j \Psi'_N) \frac{\partial}{\partial p_j} f_3 - \delta^{jk} \frac{p_j}{ma} \partial_k f_2. \end{aligned} \quad (4.188)$$

The term involving the density  $f_3$  is probably negligible for  $f_1 \gg |f_{2,3}|$ , however

we kept it to see the type of the leading order term for  $f_3$ . The differential equations for  $f_2$  and  $f_3$  read

$$\begin{aligned} (f_2)' + \left[ s^k \partial_k - (\partial_m s^k) p_k \frac{\partial}{\partial p_m} \right] f_2 &\approx 2 \frac{\omega_p}{\hbar} a (1 + \Phi_N) f_3 \\ &+ \left[ \delta^{jk} \frac{p_j}{ma} \partial_k - ma \partial_k \left( \Phi_N - \frac{\lambda}{2} \frac{\hbar^3}{m^3 a^3} \int d^3 q f_1^+(q) \right) \frac{\partial}{\partial p_k} \right] f_1^- \\ &+ 3(\mathcal{H} - \Psi'_N) f_1^+ - \frac{\hbar}{ma} \left[ \frac{1}{4} \delta^{ij} \partial_i \partial_j - \frac{\lambda}{2} \frac{\hbar^2}{m^2 a^2} \int d^3 q f_1^+(q) \right] f_3, \quad (4.189) \end{aligned}$$

$$\begin{aligned} (f_3)' + \left[ s^k \partial_k - (\partial_m s^k) p_k \frac{\partial}{\partial p_m} \right] f_3 &\approx -2 \frac{\omega_p}{\hbar} a (1 + \Phi_N) f_2 + \hbar \frac{3}{2} (\partial_j \Psi'_N) \frac{\partial}{\partial p_j} f_1^- \\ &+ \frac{\hbar}{ma} \left[ \frac{1}{4} \delta^{ij} \partial_i \partial_j - \frac{\lambda}{2} \frac{\hbar^2}{m^2 a^2} \int d^3 q f_1^+(q) \right] (f_1^+ + f_2), \quad (4.190) \end{aligned}$$

where we also included higher-order gradient terms acting on  $f_3$  and the self-coupling, as they might play a role in determining the non-oscillatory behaviour of  $f_2$  and  $f_3$ . Also, the correction to the rest-mass energy may be included for the first term on the right-hand-side of (4.189) and (4.190),

$$\omega_p \approx m \left( 1 + \frac{1}{2} \frac{p_i p_j \delta^{ij}}{m^2 a^2} \right). \quad (4.191)$$

We remark once more, that the equations (4.184) to (4.190) reduce to the classical particle, cold dark matter phase-space dynamics if we can approximate  $f_{2,3} \approx 0$  and set  $\lambda = 0$ . However, we think that the additional densities  $f_{2,3}$  have the potential to significantly alter the evolution of  $f_1^\pm$  for certain combinations of parameters. As a first step, we are currently investigating the effect of the oscillatory densities  $f_{2,3}$  on the density  $f_1$  and thus the Hubble rate  $\mathcal{H}$  in the homogeneous limit.

## 4.7 Conclusion and outlook

Motivated by dark matter models for large-scale structures we introduced a spatially covariant framework based on canonical field operators  $\hat{\phi}$ ,  $\hat{\Pi}$  to study the transition from the quantum theory of a self-interacting real scalar field on curved space-time to the kinetic theory of classical particles by using a spatial gradient expansion. We also included a non-minimal coupling to the Ricci scalar, since it is required at the level of bare parameters and non-renormalized interaction terms. We used a c-number metric that is determined through the semi-classical Einstein equations, although in principle we could have taken any classical metric for our deviation. It was in this sense unrestricted. The metric is a c-number with respect to quantum expectation values but might be taken to be stochastic as a one-point function to account for stochastic features of cosmological settings. Moreover, we considered a Gaussian state or one-loop truncation and neglected the effect of connected higher-order n-point

functions related to the self-coupling, anomalous contributions, that result from the renormalization procedure. These effects can in principle be included and it depends on the scales and couplings of the underlying problem whether they become relevant.

In (4.65) to (4.68), we identified four phase-space operators  $\hat{f}_1^\pm, \hat{f}_{2,3}$  which depend on a space-time point  $x^\mu$  and a three-momentum  $p_k$ . Two of them can be combined and interpreted as a fluctuating phase-space density  $\hat{f}_1 = \hat{f}_1^+ + \hat{f}_1^-$ , the average of which,  $f_1 = \langle \hat{f}_1 \rangle$ , describes a classical statistically-distributed one-particle density, whenever the quantum state of the system is such that the other two phase-space operators are on average small  $f_1 = \langle \hat{f}_1 \rangle \gg |\langle \hat{f}_{2,3} \rangle|$  (expectation values of  $n$  factors of  $\hat{f}_1$  are after subtraction of their disconnected piece similarly interpreted as  $n$ -particle phase-space densities). This picture is consistent when we rewrite the hydrodynamic energy density, pressure and velocity in the non-interacting limit in terms of momentum integrals over  $f_1$ . However, the main result of this chapter are the dynamical equations (4.148) to (4.151) for the phase-space densities  $f_1^\pm, f_{2,3}$  which describe up to one-point functions all degrees of freedom of a Gaussian state. We are not aware that equations (4.148) to (4.151) have been derived elsewhere for general curved space-times. Moreover, these equations support the interpretation that the density  $f_1$  has a limit as a classical one-particle density since the equations (4.148) to (4.151) reduce to the Vlasov equation (4.153) to lowest order in the gradient expansion and upon neglecting the densities  $f_2$  and  $f_3$  and the self-interaction which amounts to a mass correction in the one-loop approximation.

In the derivation of the kinetic description of the real scalar quantum field, we argue that it is necessary to normal order the involved quadratic field operators (4.57) also in the off-coincident limit, since only then one is able to extract quantities, that yield a well-defined renormalized energy-momentum tensor and whose dynamics can be approximated with a finite number of spatial derivatives. As far as we know, this problem has not been addressed in detail in the context of quantum kinetic theory in curved space-time and it should be further investigated whether the boundary terms related to the local subtraction can be given a quantum noise interpretation.

Eventually, we have used the general kinetic equations (4.148) to (4.151) to extend our earlier results on scalar field dark matter with linearized gravity [111] (cf. chapter 3) to include vector and tensor perturbations as well as self-interaction terms. The resulting equations generalize previous cold dark matter descriptions. Note, that we did not include condensates or one-point functions, a popular description of dark matter that goes under the name fuzzy or axion dark matter, which has been around for a long time [17–22, 26, 28]. Equipped with a very small mass, the real scalar field condensate leads to different behaviour on small scales. Recently strong bounds on the mass of fuzzy dark matter have been obtained [135] and more elaborate models combining fuzzy and cold dark matter were suggested [136]. Such a condensate component of the state may be incorporated into our formalism by adding source terms for the Einstein equations (4.69) to (4.71) via the shift  $\hat{f}_{XY} \rightarrow \hat{f}_{(X-\langle X \rangle)(Y-\langle Y \rangle)}$  in (4.57). The dynamics of the condensates can may be derived by taking expectation values of the dynamical equations for the canonical field operators (4.28)

and (4.30) whose non-linear terms have to be expressed in terms of one-point functions and the one-particle phase-space densities (which are related to the connected part of the two-point function). The coupling between one-point functions and the connected two-point functions happens then directly via one-loop self-interactions or indirectly via the gravitational fields and it is promising to study whether and on which scales the particle or the condensate nature dominates (such a dark matter model, that differentiates between different matter phases depending on the scales has been proposed by [137]).<sup>13</sup> Moreover, our formalism can be useful in studying how a Quintessence field [138] that goes beyond a condensate, can play role in large-scale structure dynamics. In this case an additional degree of freedom has to be added to play the role of dark matter itself. Another application we have in mind for our formalism is to study the interplay between gravitational waves and the real scalar field on space and time scales where the other gravitational potentials give negligible effects.

We think our results are important to systematically include special and general relativistic corrections to dark matter models on large scales and study their range of applicability.

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<sup>13</sup>Another closely related framework to study coupling between condensate and two-point function will be developed in chapter 5 where we provide a non-relativistic action for the scalar dark matter field by integrating out the constraint fields, however without performing a spatial gradient expansion with respect to particle momenta.

## 4.A Exponentiating covariant derivatives

Assume that

$$\begin{aligned} \left[ r^k ({}^{(3)}\nabla_k^H) \right]^n \hat{\phi} &= r^{i_1} \dots r^{i_n} ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \\ &= \left[ r^k \left( \partial_k - ({}^{(3)}\Gamma_{kl}^n r^l \frac{\partial}{\partial r^n} \right) \right]^n f(x^\mu), \end{aligned} \quad (4.192)$$

holds for a certain  $n > 2$ . The case  $n = 2$  is satisfied as can be verified by a quick calculation. We now show that the relation holds also for  $n + 1$  provided it holds for  $n$  and we are done

$$\begin{aligned} \left[ r^k ({}^{(3)}\nabla_k^H) \right]^{n+1} f(x^\mu) &= r^{i_1} \dots r^{i_{n+1}} ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{i_{n+1}} f(x^\mu)) \\ &= r^k r^{i_1} \dots r^{i_n} \left[ \partial_k \left( ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \right) \right. \\ &\quad \left. - ({}^{(3)}\Gamma_{ki_1}^{j_1} \left( ({}^{(3)}\nabla_{j_1} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \right) \right) \right. \\ &\quad \left. - \dots - ({}^{(3)}\Gamma_{ki_n}^{j_n} \left( ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{j_n} f(x^\mu)) \right) \right) \right] \\ &= r^k \left[ r^{i_1} \dots r^{i_n} \partial_k \left( ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \right) \right. \\ &\quad \left. - ({}^{(3)}\Gamma_{kl}^{j_1} r^l \frac{\partial}{\partial r^{j_1}} \left( r^{i_1} \dots r^{i_n} ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \right) \right) \right. \\ &\quad \left. + r^{i_1} ({}^{(3)}\Gamma_{kl}^{j_1} r^l \frac{\partial}{\partial r^{j_1}} \left( r^{i_2} \dots r^{i_n} ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \right) \right) \right. \\ &\quad \left. - r^{i_1} \dots r^{i_n} ({}^{(3)}\Gamma_{ki_2}^{j_2} \left( ({}^{(3)}\nabla_{i_1} ({}^{(3)}\nabla_{j_2} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \right) \right) \right. \\ &\quad \left. - \dots - r^{i_1} \dots r^{i_n} ({}^{(3)}\Gamma_{ki_n}^{j_n} \left( ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{j_n} f(x^\mu)) \right) \right) \right] \\ &= r^k \left[ r^{i_1} \dots r^{i_n} \partial_k \left( ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \right) \right. \\ &\quad \left. - ({}^{(3)}\Gamma_{kl}^n r^l \frac{\partial}{\partial r^n} \left( r^{i_1} \dots r^{i_n} ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \right) \right) \right] \\ &= r^k \left( \partial_k - ({}^{(3)}\Gamma_{kl}^n r^l \frac{\partial}{\partial r^n} \right) \left[ r^{i_1} \dots r^{i_n} ({}^{(3)}\nabla_{i_1} \dots ({}^{(3)}\nabla_{i_n} f(x^\mu)) \right] \\ &= \left[ r^k \left( \partial_k - ({}^{(3)}\Gamma_{kl}^n r^l \frac{\partial}{\partial r^n} \right) \right]^{n+1} f(x^\mu). \end{aligned} \quad (4.193)$$

## 4.B Traces in terms of two-point functions of canonical fields

The hydrodynamic representation of the energy-momentum tensor (4.86),

$$\langle : \hat{T}_{\mu\nu} : \rangle_{\lambda=\xi=0} = \langle : \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} : \rangle - \frac{g_{\mu\nu}}{2} \left[ \langle : \partial^\alpha \hat{\phi} \partial_\alpha \hat{\phi} : \rangle + \frac{m^2}{\hbar^2} \langle : \hat{\phi}^2 : \rangle \right], \quad (4.194)$$

is related to its diagonalization which can be written in terms of three independent traces of products of

$$\chi_\nu^\mu \equiv \langle : \partial^\mu \hat{\phi} \partial_\nu \hat{\phi} : \rangle, \quad (4.195)$$



as well as the determinant. In particular, we need

$$\tilde{b} \equiv -\text{tr} [\chi], \quad (4.196)$$

$$\tilde{c} \equiv \frac{1}{2} \left[ (\text{tr} [\chi])^2 - \text{tr} [\chi^2] \right], \quad (4.197)$$

$$\tilde{d} \equiv -\frac{1}{6} \left[ (\text{tr} [\chi])^3 - 3 \text{tr} [\chi^2] \text{tr} [\chi] + 2 \text{tr} [\chi^3] \right], \quad (4.198)$$

$$\tilde{e} \equiv \det \chi. \quad (4.199)$$

The three independent traces read in terms of the two-point functions of canonical field operators as follows,

$$\text{tr} [\chi] = \langle : \partial^\mu \hat{\phi} \partial_\mu \hat{\phi} : \rangle = -\gamma^{-1} \langle : \hat{\Pi} \hat{\Pi} : \rangle + \gamma^{ij} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle, \quad (4.200)$$

$$\begin{aligned} \text{tr} [\chi^2] &= \langle : \partial^\mu \hat{\phi} \partial_\mu \hat{\phi} : \rangle \langle : \partial^\nu \hat{\phi} \partial_\nu \hat{\phi} : \rangle \\ &= \gamma^{-2} \langle : \hat{\Pi} \hat{\Pi} : \rangle^2 - 2\gamma^{-1} \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \gamma^{ij} \langle : \partial_j \hat{\phi} \hat{\Pi} : \rangle \\ &\quad + \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \gamma^{jk} \gamma^{il} \langle : \partial_k \hat{\phi} \partial_l \hat{\phi} : \rangle, \end{aligned} \quad (4.201)$$

$$\begin{aligned} \text{tr} [\chi^3] &= \langle : \partial^\mu \hat{\phi} \partial_\mu \hat{\phi} : \rangle \langle : \partial^\nu \hat{\phi} \partial_\nu \hat{\phi} : \rangle \langle : \partial^\rho \hat{\phi} \partial_\rho \hat{\phi} : \rangle \\ &= -\gamma^{-3} \langle : \hat{\Pi} \hat{\Pi} : \rangle^3 + 3\gamma^{-2} \langle : \hat{\Pi} \hat{\Pi} : \rangle \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \gamma^{ij} \langle : \partial_j \hat{\phi} \hat{\Pi} : \rangle \\ &\quad - 3\gamma^{-1} \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \gamma^{ij} \langle : \partial_j \hat{\phi} \partial_l \hat{\phi} : \rangle \gamma^{kl} \langle : \partial_k \hat{\phi} \hat{\Pi} : \rangle \\ &\quad + \gamma^{mn} \langle : \partial_n \hat{\phi} \partial_j \hat{\phi} : \rangle \gamma^{jk} \langle : \partial_k \hat{\phi} \partial_l \hat{\phi} : \rangle \gamma^{il} \langle : \partial_i \hat{\phi} \partial_m \hat{\phi} : \rangle. \end{aligned} \quad (4.202)$$

We then compute

$$\begin{aligned} \tilde{b} &= \gamma^{-1} \langle : \hat{\Pi} \hat{\Pi} : \rangle - \gamma^{ij} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle, \\ \tilde{c} &= -\gamma^{-1} \langle : \hat{\Pi} \hat{\Pi} : \rangle \gamma^{ij} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle + \gamma^{-1} \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \gamma^{ij} \langle : \partial_j \hat{\phi} \hat{\Pi} : \rangle \\ &\quad - \frac{1}{2} \left[ \gamma^{ij} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \right]^2 + \frac{1}{2} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \gamma^{jk} \gamma^{il} \langle : \partial_k \hat{\phi} \partial_l \hat{\phi} : \rangle, \\ \tilde{d} &= \frac{1}{2} \gamma^{-1} \langle : \hat{\Pi} \hat{\Pi} : \rangle \left[ \left[ \gamma^{ij} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \right]^2 - \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \gamma^{jk} \gamma^{il} \langle : \partial_k \hat{\phi} \partial_l \hat{\phi} : \rangle \right] \\ &\quad + \gamma^{-1} \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \gamma^{ij} \langle : \partial_j \hat{\phi} \partial_l \hat{\phi} : \rangle \gamma^{kl} \langle : \partial_k \hat{\phi} \hat{\Pi} : \rangle \\ &\quad - \frac{1}{3} \gamma^{mn} \langle : \partial_n \hat{\phi} \partial_j \hat{\phi} : \rangle \gamma^{jk} \langle : \partial_k \hat{\phi} \partial_l \hat{\phi} : \rangle \gamma^{il} \langle : \partial_i \hat{\phi} \partial_m \hat{\phi} : \rangle, \\ &\quad - \gamma^{ij} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \left[ \gamma^{-1} \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \gamma^{ij} \langle : \partial_j \hat{\phi} \hat{\Pi} : \rangle \right. \\ &\quad \left. - \frac{1}{2} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \gamma^{jk} \gamma^{il} \langle : \partial_k \hat{\phi} \partial_l \hat{\phi} : \rangle + \frac{1}{6} \left[ \gamma^{ij} \langle : \partial_i \hat{\phi} \partial_j \hat{\phi} : \rangle \right]^2 \right], \\ \tilde{e} &= -\frac{1}{6} \epsilon^{ijk} \left[ \gamma^{-1} \langle : \hat{\Pi} \hat{\Pi} : \rangle + \frac{N^k}{N} \gamma^{-1/2} \langle : \hat{\Pi} \partial_k \hat{\phi} : \rangle \right] \\ &\quad \times \left[ \gamma^{-1} \epsilon^{lmn} \left[ \langle : \partial_i \hat{\phi} \partial_l \hat{\phi} : \rangle \langle : \partial_j \hat{\phi} \partial_m \hat{\phi} : \rangle \langle : \partial_k \hat{\phi} \partial_n \hat{\phi} : \rangle \right] \right. \\ &\quad \left. + \frac{1}{2} \gamma^{-1/2} \tilde{\epsilon}_{lmn} \gamma^{la} \gamma^{mb} \frac{N^n}{N} \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \langle : \partial_a \hat{\phi} \partial_j \hat{\phi} : \rangle \langle : \partial_b \hat{\phi} \partial_k \hat{\phi} : \rangle \right], \end{aligned} \quad (4.203)$$

and the determinant is given by

$$\begin{aligned} \det \chi = & -\frac{1}{6} \tilde{\epsilon}^{ijk} \left[ \gamma^{-1} \langle : \hat{\Pi} \hat{\Pi} : \rangle + \frac{N^k}{N} \gamma^{-1/2} \langle : \hat{\Pi} \partial_k \hat{\phi} : \rangle \right] \\ & \times \left[ \gamma^{-1} \tilde{\epsilon}^{lmn} \langle : \partial_i \hat{\phi} \partial_l \hat{\phi} : \rangle \langle : \partial_j \hat{\phi} \partial_m \hat{\phi} : \rangle \langle : \partial_k \hat{\phi} \partial_n \hat{\phi} : \rangle \right. \\ & \left. + \frac{1}{2} \gamma^{-1/2} \tilde{\epsilon}_{lmn} \gamma^{la} \gamma^{mb} \frac{N^n}{N} \langle : \hat{\Pi} \partial_i \hat{\phi} : \rangle \langle : \partial_a \hat{\phi} \partial_j \hat{\phi} : \rangle \langle : \partial_b \hat{\phi} \partial_k \hat{\phi} : \rangle \right], \quad (4.204) \end{aligned}$$

where  $\tilde{\epsilon}$  denotes is the totally anti-symmetric symbol.

## 4.C Geometry of tangent bundles: definitions and identities

The expressions involved in (4.139) to (4.142), which appear before integrating over the spatial tangent space (associated to the hypersurface  $\Sigma_t$  at a common space-point  $x^\mu$ ), can be expressed in terms of the geometry of tangent bundles which is covered for example in [130, 131]. We use the general notation

$$X_R = X^k(x^\mu, r^k) \frac{\partial}{\partial r^k}, \quad (4.205)$$

$$X_E = X^k(x^\mu, r^k) e_k = X^k(x^\mu, r^k) \left[ \partial_k - r^{m(3)} \Gamma_{km}^l(x^\mu) \frac{\partial}{\partial r^l} \right], \quad (4.206)$$

where the vector  $X_R \in TT\Sigma_t^V$  lies in the vertical part of the tangent space  $TT\Sigma_t$  of the tangent bundle  $T\Sigma_t$  and the vector  $X_E$  in the remaining horizontal tangent space. The associated derivative operators will act on functions  $f^\pm \in T\Sigma_t$  on the tangent bundle. These functions are obtained by translating the function  $f \in \Sigma_t$  on the spatial hypersurface along a spatial geodesic with initial tangent vector  $r^k$ ,

$$f^\pm(x^\mu, r^k) \equiv \exp \left[ \pm \frac{1}{2} r^{k(3)} \nabla_k^H \right] f(x^\mu), \quad f \in C^\infty(\Sigma_t), \quad (4.207)$$

where in local coordinates

$$r_E = r^k e_k = r^k \left( \partial_k - r^{m(3)} \Gamma_{km}^l \frac{\partial}{\partial r^l} \right). \quad (4.208)$$

We made use of the horizontal lift  ${}^{(3)}\nabla_k^H$  of the covariant derivative  ${}^{(3)}\nabla$  (induced on  $\Sigma_t$  via the 3+1 decomposition) to the tangent bundle  $T\Sigma_t$  that we introduced in (4.52),

$${}^{(3)}\nabla_k^H = {}^{(3)}\nabla_k - r^{l(3)} \Gamma_{kl}^n \frac{\partial}{\partial r^n}. \quad (4.209)$$

Wigner transforming derivative operators will also give rise to the horizontal lift of the covariant derivative on the cotangent bundle of spatial hypersurfaces which we will denote as

$$D_k \equiv {}^{(3)}\nabla_k + p_l {}^{(3)}\Gamma_{kj}^l \frac{\partial}{\partial p_j}, \quad D_k p_j = 0. \quad (4.210)$$

If we want to calculate the differential operators appearing in (4.139) to (4.142), we see that we have to rewrite the differential operators acting on individual fields  $\hat{u}^\pm$  that are translated geodesically in opposite directions in such a way that will act on the product, schematically

$$\langle (\mathcal{D}\hat{u}^+)\hat{u}^- \rangle \rightarrow \mathcal{D}_*^+(\langle \hat{u}^+\hat{u}^- \rangle). \quad (4.211)$$

In order to achieve this, we need to find annihilation operators  $\mathcal{P}^\pm$  such that

$$[X_{E,R}f^\pm](x^\mu, r^k) = [\mathcal{P}^\pm[X_{E,R}]f^\pm](x^\mu, r^k), \quad (4.212)$$

where

$$\mathcal{P}^\pm[X_{E,R}]f^\mp = 0. \quad (4.213)$$

Since derivative with respect to the tangent space coordinate  $\partial_r$  annihilate functions on the spatial hypersurface  $f(x^\mu)$ , we can commute the exponential shift operator with any vertical derivative operator  $X_R$  to obtain the representation

$$X_R f^\pm = - \sum_{n=1}^{\infty} \frac{1}{(\pm 2)^n} \frac{1}{n!} \underbrace{[r_E, [\dots, [r_E, X_R] \dots]]}_{n \text{ times}} f^\pm. \quad (4.214)$$

We find

$$[r_E, X_R] = {}^{(3)}\nabla_{r_E}^H X_R - X_E, \quad (4.215)$$

$$[r_E, X_E] = {}^{(3)}\nabla_{r_E}^H X_E + R_R[X], \quad (4.216)$$

where the horizontal lift of the covariant derivative acting on  $X_R$  and  $X_E$  reads in components

$${}^{(3)}\nabla_{r_E}^H X_R = r^k \left[ {}^{(3)}\nabla_k X^m - r^n {}^{(3)}\Gamma_{kn}^l \frac{\partial}{\partial r^l} X^m \right] \frac{\partial}{\partial r^m}, \quad (4.217)$$

$${}^{(3)}\nabla_{r_E}^H X_E = r^k \left[ {}^{(3)}\nabla_k X^m - r^n {}^{(3)}\Gamma_{kn}^l \frac{\partial}{\partial r^l} X^m \right] e_m, \quad (4.218)$$

and the vector field  $R_R[X]$  is defined as

$$R_R[X] = {}^{(3)}R_{ikj}^l r^i r^j X^k \frac{\partial}{\partial r^l}. \quad (4.219)$$

We will make use of the following commutators,

$$[r_E, (\cdot)]^2[X_R] = \left({}^{(3)}\nabla_{r_E}^H\right)^2 X_R - 2\left({}^{(3)}\nabla_{r_E}^H X_E - R_R[X]\right), \quad (4.220)$$

$$\begin{aligned} [r_E, (\cdot)]^3[X_R] &= \left({}^{(3)}\nabla_{r_E}^H\right)^3 X_R - 3\left({}^{(3)}\nabla_{r_E}^H\right)^2 X_E \\ &\quad - 2R_R\left[{}^{(3)}\nabla_{r_E}^H X\right] - \left({}^{(3)}\nabla_{r_E}^H (R_R[X]) + R_E[X]\right), \end{aligned} \quad (4.221)$$

$$\begin{aligned} [r_E, (\cdot)]^4[X_R] &= \left({}^{(3)}\nabla_{r_E}^H\right)^4 X_R - 4\left({}^{(3)}\nabla_{r_E}^H\right)^3 X_E - 3R_R\left[\left({}^{(3)}\nabla_{r_E}^H\right)^2 X\right] \\ &\quad - 2\left({}^{(3)}\nabla_{r_E}^H (R_R[{}^{(3)}\nabla_{r_E}^H X]) - 2R_E[{}^{(3)}\nabla_{r_E}^H X]\right) \\ &\quad - \left({}^{(3)}\nabla_{r_E}^H\right)^2 (R_R[X]) + 2\left({}^{(3)}\nabla_{r_E}^H (R_E[X]) + R_R[R[X]]\right). \end{aligned} \quad (4.222)$$

Further commutators go beyond the order of derivatives that we want to keep track of in the spatial gradient expansion. We now set

$$H_n^\pm[X] \equiv \frac{1}{(\pm 2)^n n!} \underbrace{[r_E, [\dots, [r_E, X_R] \dots]]}_{n \text{ times}}, \quad (4.223)$$

and find

$$\left(X_R \pm \frac{1}{2}\left({}^{(3)}\nabla_{r_E}^H X_R - X_E\right) + \sum_{n=2}^{\infty} H_n^\pm[X]\right) f^\pm = 0, \quad (4.224)$$

$$\left(\frac{1}{2}X_E \mp X_R - \frac{1}{2}\left({}^{(3)}\nabla_{r_E}^H X_R \mp \sum_{n=2}^{\infty} H_n^\pm[X]\right)\right) f^\pm = 0. \quad (4.225)$$

The last two identities can be used to define the annihilation operators we were looking for. We define

$$\mathcal{P}^\pm[X] \equiv \left(\frac{1}{2}X_E \pm X_R - \frac{1}{2}\left({}^{(3)}\nabla_{r_E}^H X_R \pm \sum_{n=2}^{\infty} H_n^\mp[X]\right)\right), \quad (4.226)$$

$$\mathcal{P}^\pm[X] f^\mp = 0, \quad (4.227)$$

as well as

$$\left({}^{(3)}\nabla R\right)_R[X] \equiv r^s \left({}^{(3)}\nabla_s \left({}^{(3)}R^l_{ikj}\right) X^k r^i r^j \frac{\partial}{\partial r^l}\right), \quad (4.228)$$

and compute up to  $\mathcal{O}({}^{(3)}R^2)$  corrections,

$$\begin{aligned} \mathcal{P}^\pm[X] &= \frac{1}{2}X_E \pm X_R - \frac{1}{2}\left({}^{(3)}\nabla_{r_E}^H X_R\right) \\ &\quad \pm \frac{1}{8}\left\{\left({}^{(3)}\nabla_{r_E}^H\right)^2 X_R - 2\left({}^{(3)}\nabla_{r_E}^H X_E - R_R[X]\right)\right\} \\ &\quad - \frac{1}{48}\left\{\left({}^{(3)}\nabla_{r_E}^H\right)^3 X_R - 3\left({}^{(3)}\nabla_{r_E}^H\right)^2 X_E\right. \\ &\quad \left.- 3R_R\left[{}^{(3)}\nabla_{r_E}^H X\right] - \left({}^{(3)}\nabla R\right)_R[X] + R_E[X]\right\} \\ &\quad \pm \frac{1}{192}\left\{\left({}^{(3)}\nabla R\right)_E[X] - 2\left({}^{(3)}\nabla_{r_E}^H\right)^3 X_E\right\} + \mathcal{O}({}^{(3)}R^2). \end{aligned} \quad (4.229)$$

We also have

$$X_E = \mathcal{P}^+[X] + \mathcal{P}^-[X] + {}^{(3)}\nabla_{r_E}^H X_R + 2 \sum_{n=1}^{\infty} H_{2n+1}^+[X], \quad (4.230)$$

$$X_R = \frac{1}{2} \left( \mathcal{P}^+[X] - \mathcal{P}^-[X] - 2 \sum_{n=1}^{\infty} H_{2n}^+[X] \right). \quad (4.231)$$

Since we are only interested in terms of order  $({}^{(3)}\nabla)^3$  for the gradient expansion, we can re-express the higher order derivative coefficients by iteration

$$\begin{aligned} X_E &= \mathcal{P}^+[X] + \mathcal{P}^-[X] + \frac{1}{2} (\mathcal{P}^+ [{}^{(3)}\nabla_{r_E}^H X] - \mathcal{P}^- [{}^{(3)}\nabla_{r_E}^H X]) \\ &\quad + \frac{1}{8} (\mathcal{P}^+ [({}^{(3)}\nabla_{r_E}^H)^2 X] + \mathcal{P}^- [({}^{(3)}\nabla_{r_E}^H)^2 X]) \\ &\quad + \frac{1}{48} (\mathcal{P}^+ [({}^{(3)}\nabla_{r_E}^H)^3 X] - \mathcal{P}^- [({}^{(3)}\nabla_{r_E}^H)^3 X]) \\ &\quad + \frac{1}{24} (\mathcal{P}^+ [R[X]] + \mathcal{P}^- [R[X]]) \\ &\quad + \frac{1}{48} (\mathcal{P}^+ [R[{}^{(3)}\nabla_{r_E}^H X]] - \mathcal{P}^- [R[{}^{(3)}\nabla_{r_E}^H X]]) + \mathcal{O}({}^{(3)}R^2). \end{aligned} \quad (4.232)$$

$$\begin{aligned} X_R &= \frac{1}{2} (\mathcal{P}^+[X] - \mathcal{P}^-[X]) + \frac{1}{4} (\mathcal{P}^+ [{}^{(3)}\nabla_{r_E}^H X] + \mathcal{P}^- [{}^{(3)}\nabla_{r_E}^H X]) \\ &\quad + \frac{1}{16} (\mathcal{P}^+ [({}^{(3)}\nabla_{r_E}^H)^2 X] - \mathcal{P}^- [({}^{(3)}\nabla_{r_E}^H)^2 X]) \\ &\quad + \frac{1}{16} (\mathcal{P}^+ [R[X]] - \mathcal{P}^- [R[X]]) \\ &\quad + \frac{1}{32} (\mathcal{P}^+ [({}^{(3)}\nabla R)[X]] + \mathcal{P}^- [({}^{(3)}\nabla R)[X]]) + \mathcal{O}({}^{(3)}R^2). \end{aligned} \quad (4.233)$$

Since we will eventually act on functions shifted in the  $\pm r^k$  direction, we introduce the notation

$$\begin{aligned} X_E^\pm &\equiv \mathcal{P}^\pm[X] \pm \frac{1}{2} \mathcal{P}^\pm [{}^{(3)}\nabla_{r_E}^H X] + \frac{1}{8} \mathcal{P}^\pm [({}^{(3)}\nabla_{r_E}^H)^2 X] \pm \frac{1}{48} \mathcal{P}^\pm [({}^{(3)}\nabla_{r_E}^H)^3 X] \\ &\quad + \frac{1}{24} \mathcal{P}^\pm [R[X]] \pm \frac{1}{48} \mathcal{P}^\pm [R[{}^{(3)}\nabla_{r_E}^H X]] + \mathcal{O}({}^{(3)}R^2), \end{aligned} \quad (4.234)$$

$$\begin{aligned} X_R^\pm &\equiv \pm \frac{1}{2} \mathcal{P}^\pm[X] + \frac{1}{4} \mathcal{P}^\pm [{}^{(3)}\nabla_{r_E}^H X] \pm \frac{1}{16} \mathcal{P}^\pm [({}^{(3)}\nabla_{r_E}^H)^2 X] \pm \frac{1}{16} \mathcal{P}^\pm [R[X]] \\ &\quad + \frac{1}{32} \mathcal{P}^\pm [({}^{(3)}\nabla R)[X]] + \mathcal{O}({}^{(3)}R^2), \end{aligned} \quad (4.235)$$

where the pattern on how to include higher-order terms in this definition should be clear. The operators (4.234) and (4.235) can now be pulled out from individual factors to act on the whole product as in the scheme

$$\langle (\mathcal{D}_{E,R}^+ \hat{u}^+) \hat{u}^- \rangle \rightarrow \mathcal{D}_{E,R}^+ (\langle \hat{u}^+ \hat{u}^- \rangle). \quad (4.236)$$

After the differential operators act on the product, we want replace the projection operators again in terms of concrete expressions and we find,

$$\begin{aligned} X_E^\pm &= \frac{1}{2}X_E \pm X_R \pm \frac{1}{32}({}^{(3)}\nabla_{r_E}^H)^3 X_E \mp \frac{1}{12}R_R[X] \\ &\mp \frac{1}{96}R_E[{}^{(3)}\nabla_{r_E}^H X] \mp \frac{1}{192}({}^{(3)}\nabla R)_E[X] + \mathcal{O}({}^{(3)}R^2), \end{aligned} \quad (4.237)$$

$$\begin{aligned} X_R^\pm &= \pm \frac{1}{4}X_E + \frac{1}{2}X_R \mp \frac{1}{96}({}^{(3)}\nabla_{r_E}^H)^3 X_R \pm \frac{1}{48}R_E[X] + \frac{1}{384}({}^{(3)}\nabla R)_E[X] \\ &\pm \frac{1}{96}({}^{(3)}\nabla R)_R[X] - \frac{1}{48}R_E[{}^{(3)}\nabla_{r_E}^H X] \\ &\mp \frac{1}{32}R_R[{}^{(3)}\nabla_{r_E}^H X] + \mathcal{O}({}^{(3)}R^2). \end{aligned} \quad (4.238)$$

We will also have to commute the spatial curved Laplace operator with the spatially covariant shift operators and rewrite it in terms of the annihilation operators. In order to commute the spatial Laplace operator, we apply a trick. We realize that once more that all operations we perform in the integrals that define the Wigner transformed operators take place in the spatial tangent bundle  $T\Sigma_t$  specified by the time-slicing. This bundle can be equipped with some natural metric  $\tilde{\gamma}$  that satisfies

$$\tilde{\gamma}(e_i, e_j) = \gamma_{ij}, \quad (4.239)$$

$$\tilde{\gamma}(e_i, \frac{\partial}{\partial r^j}) = 0. \quad (4.240)$$

The important point is that this metric will satisfy

$$[{}^{(3)}\nabla_{e_k}^H \tilde{\gamma}](e_i, e_j) = 0, \quad (4.241)$$

which follows from  ${}^{(3)}\nabla_{e_k}^H e_i = {}^{(3)}\Gamma_{ki}^j e_j$  and so we can use the usual techniques for computations with a metric compatible covariant derivative. The spatial Laplace operator acting on fields that depend only on the sub manifold  $\Sigma_t$  can be rewritten in terms of the horizontal lift of the covariant derivative to the tangent bundle  $T\Sigma_t$

$$\gamma^{ij}({}^{(3)}\nabla_i({}^{(3)}\nabla_j f(x^\mu)) = \tilde{\gamma}^{ij}({}^{(3)}\nabla_{e_i}^H({}^{(3)}\nabla_{e_j}^H f(x^\mu)). \quad (4.242)$$

It is now a matter of computing commutation relations for the horizontal lift of the covariant derivative to the tangent bundle to find the first correction terms that result from commuting the spatial Laplacian acting on  $\hat{\phi}(x^\mu)$  or  $\hat{\Pi}(x^\mu)$  with the shift exponentials and we give the concrete expressions in the following appendix. Note that since the basis  $e_k$  is not a coordinate basis ( $[e_i, e_j] \neq 0$ ), the commutator of horizontal covariant derivatives acting on functions is also non-zero

$$[{}^{(3)}\nabla_{e_i}^H, {}^{(3)}\nabla_{e_j}^H]f(x^\mu, r^k) = -r^{n({}^{(3)}R^l_{nij})} \frac{\partial}{\partial r^l} f(x^\mu, r^k). \quad (4.243)$$

We find for the adapted basis vectors

$$\left[ {}^{(3)}\nabla_{e_i}^H, {}^{(3)}\nabla_{e_j}^H \right] (e_k) = {}^{(3)}R_{kij}^l e_l, \quad (4.244)$$

$$\left[ {}^{(3)}\nabla_{e_i}^H, {}^{(3)}\nabla_{e_j}^H \right] \left( \frac{\partial}{\partial r^k} \right) = {}^{(3)}R_{kij}^l \frac{\partial}{\partial r^l}, \quad (4.245)$$

and opposite signs for the dual adapted basis. Note that due to the property

$${}^{(3)}\nabla_{\frac{\partial}{\partial r^k}}^H \frac{\partial}{\partial r^j} = {}^{(3)}\nabla_{\frac{\partial}{\partial r^k}}^H e_j = 0, \quad (4.246)$$

the partial derivative  $\frac{\partial}{\partial r^k}$  acts itself covariantly and its commutator with the basis vectors  $e_k$  can be computed to be zero also when acting on functions due to non-vanishing commutator of the partial derivatives  $[e_i, \frac{\partial}{\partial r^k}] \neq 0$ ,

$$\left[ {}^{(3)}\nabla_{e_i}^H, {}^{(3)}\nabla_{\frac{\partial}{\partial r^j}}^H \right] f(x^\mu, r^k) = 0. \quad (4.247)$$

## 4.D Commutating derivative operators

The general expressions from the previous appendix will now be used to compute the concrete expressions that appear in (4.139) to (4.142). We will need the commutator

$$\mathcal{T}^\pm \equiv \left[ \partial_t, \exp \left[ \pm \frac{1}{2} r^k {}^{(3)}\nabla_k^H \right] \right], \quad (4.248)$$

which can be computed by using the three tensor  ${}^{(3)}\dot{\Gamma}_{ik}^l$  on the spatial hypersurface,

$$\begin{aligned} {}^{(3)}\dot{\Gamma}_{ik}^l &= {}^{(3)}\nabla^l [NK_{ik}] - 2 {}^{(3)}\nabla_{(i} [NK_{k)}^l] \\ &\quad + \left[ {}^{(3)}\nabla_{(i} {}^{(3)}\nabla_{k)} N^l + {}^{(3)}\nabla_{(i} {}^{(3)}\nabla^l N_{k)} - {}^{(3)}\nabla^l {}^{(3)}\nabla_{(i} N_{k)} \right], \end{aligned} \quad (4.249)$$

with

$$T_{(ik)} = \frac{T_{ik} + T_{ki}}{2}. \quad (4.250)$$

We can define a vertical vector field

$$\dot{\Gamma}_R \equiv r^i r^k {}^{(3)}\dot{\Gamma}_{ik}^l \frac{\partial}{\partial r^l} = [r_E, \partial_t], \quad (4.251)$$

in terms of which we have

$$\begin{aligned} \mathcal{T}^\pm &= \mp \frac{1}{2} \dot{\Gamma}_R + \frac{1}{8} (\dot{\Gamma}_E - {}^{(3)}\nabla_{r_E}^H \dot{\Gamma}_R) \mp \frac{1}{48} \left[ ({}^{(3)}\nabla_{r_E}^H)^2 \dot{\Gamma}_R - 2 {}^{(3)}\nabla_{r_E}^H \dot{\Gamma}_E - R_R [\dot{\Gamma}] \right] \\ &\quad + \frac{1}{384} (-3 ({}^{(3)}\nabla_{r_E}^H)^2 \dot{\Gamma}_E + R_E [\dot{\Gamma}]) + \mathcal{O}({}^{(3)}R^2). \end{aligned} \quad (4.252)$$

Acting on functions translated in the  $\pm r$  direction leads us to define the operator

$$\begin{aligned} \mathcal{T}_*^\pm \equiv & \mp \frac{1}{2} \dot{\Gamma}_R^\pm + \frac{1}{8} \left( \dot{\Gamma}_E^\pm - {}^{(3)}\nabla_{rE}^H \dot{\Gamma}_R^\pm \right) \mp \frac{1}{48} \left( ({}^{(3)}\nabla_{rE}^H)^2 \dot{\Gamma}_R^\pm - 2 {}^{(3)}\nabla_{rE}^H \dot{\Gamma}_E^\pm - R_R^\pm [\dot{\Gamma}] \right) \\ & + \frac{1}{384} \left( -3 ({}^{(3)}\nabla_{rE}^H)^2 \dot{\Gamma}_E^\pm + R_E^\pm [\dot{\Gamma}] \right) + \mathcal{O}({}^{(3)}R^2). \end{aligned} \quad (4.253)$$

Since we will be dealing only with the sum of this operator with opposite signs, we have

$$\mathcal{T}_*^+ + \mathcal{T}_*^- = -\frac{1}{8} \dot{\Gamma}_E - \frac{1}{24} {}^{(3)}\nabla_{rE}^H \dot{\Gamma}_R + \mathcal{O}({}^{(3)}R^2). \quad (4.254)$$

We will also need the following commutator

$$\widetilde{\mathcal{M}}^\pm \equiv \left[ M_E, \exp \left[ \pm \frac{1}{2} r^k {}^{(3)}\nabla_k^H \right] \right], \quad (4.255)$$

with

$$M_E \equiv N^k e_k. \quad (4.256)$$

We find

$$\begin{aligned} \widetilde{\mathcal{M}}^\pm = & \mp \frac{1}{2} [{}^{(3)}\nabla_{rE}^H M_E + R_R[M]] - \frac{1}{8} ({}^{(3)}\nabla_{rE}^H)^2 M_E - \frac{1}{8} R_R [{}^{(3)}\nabla_{rE}^H M] \\ & - \frac{1}{8} [{}^{(3)}\nabla_{rE}^H (R_R[M]) - R_E[M]] \\ & \mp \frac{1}{48} [({}^{(3)}\nabla_{rE}^H)^3 M_E - R_E [{}^{(3)}\nabla_{rE}^H M] \\ & - 2 {}^{(3)}\nabla_{rE}^H (R_E[M])] + \mathcal{O}({}^{(3)}R^2), \end{aligned} \quad (4.257)$$

and define

$$\begin{aligned} \widetilde{\mathcal{M}}_*^\pm \equiv & \mp \frac{1}{2} [{}^{(3)}\nabla_{rE}^H M_E^\pm + R_R^\pm [M]] - \frac{1}{8} ({}^{(3)}\nabla_{rE}^H)^2 M_E^\pm - \frac{1}{8} R_R^\pm [{}^{(3)}\nabla_{rE}^H M] \\ & \mp \frac{1}{48} [({}^{(3)}\nabla_{rE}^H)^3 M_E^\pm - R_E^\pm [{}^{(3)}\nabla_{rE}^H M] - 2 {}^{(3)}\nabla_{rE}^H (R_E[M])^\pm] \\ & - \frac{1}{8} [{}^{(3)}\nabla_{rE}^H (R_R[M])^\pm - R_E^\pm [M]] + \mathcal{O}({}^{(3)}R^2). \end{aligned} \quad (4.258)$$

It is convenient to define the following operator

$$\mathcal{M}_*^\pm \equiv M_E^\pm - \widetilde{\mathcal{M}}_*^\pm, \quad (4.259)$$

and we have

$$\begin{aligned} \mathcal{M}_*^+ + \mathcal{M}_*^- = & M_E + {}^{(3)}\nabla_{rE}^H M_R + \frac{1}{8} ({}^{(3)}\nabla_{rE}^H)^2 M_E + \frac{1}{24} ({}^{(3)}\nabla_{rE}^H)^3 M_R \\ & + \frac{1}{8} R_E[M] + \frac{1}{24} {}^{(3)}\nabla_{rE}^H (R_R[M]) + \mathcal{O}({}^{(3)}R^2). \end{aligned} \quad (4.260)$$

It is the sum  $\mathcal{T}_*^+ + \mathcal{T}_*^- + \mathcal{M}_*^+ + \mathcal{M}_*^-$  that will enter the dynamical equations (4.139) to (4.142), and this sum happens to reduce to a rather short expressions to second order in the spatial gradient after performing the Wigner transformation (and dropping boundary terms). Making use of the horizontal lift of the



spatial covariant derivative to the cotangent bundle (4.209), we find for example in (4.139) up to boundary terms,

$$\begin{aligned}
& \gamma^{1/2} \int_{\Sigma_t} d^3 r e^{-\frac{i}{\hbar} r^k p_k} [\mathcal{M}_*^+ + \mathcal{M}_*^- + \mathcal{T}_*^+ + \mathcal{T}_*^-] [:\hat{u}^+ \hat{u}^-:] \\
&= \left[ N^k D_k - N_{;k}^k - p_k N_{;m}^k \frac{\partial}{\partial p_m} - \hbar^2 \frac{1}{12} N_{;kqm}^k \frac{\partial^2}{\partial p_q \partial p_m} \right. \\
&\quad \left. + \frac{\hbar^2}{4} \left( \frac{1}{2} {}^{(3)}\nabla^l [N K_{ik}] - {}^{(3)}\nabla_i [N K_k^l] \right) \frac{\partial^2}{\partial p_k \partial p_i} D_l \right. \\
&\quad \left. - \frac{\hbar^2}{12} \left( \frac{1}{2} {}^{(3)}\nabla_l {}^{(3)}\nabla^l [N K_{ik}] - {}^{(3)}\nabla_l {}^{(3)}\nabla_i [N K_k^l] - {}^{(3)}\nabla_i {}^{(3)}\nabla_k [N K] \right) \frac{\partial^2}{\partial p_k \partial p_i} \right. \\
&\quad \left. - \frac{\hbar^2}{12} p_l \left( \frac{1}{2} {}^{(3)}\nabla_s {}^{(3)}\nabla^l [N K_{ik}] - {}^{(3)}\nabla_s {}^{(3)}\nabla_i [N K_k^l] \right) \frac{\partial^3}{\partial p_k \partial p_i \partial p_s} + \mathcal{O}(\hbar^4) \right] \hat{F}_{\phi\phi}.
\end{aligned} \tag{4.261}$$

Similarly, we will have to commute the operator  $\gamma^{ij} \partial_i N \partial_j$  and so we define

$$(\nabla^{(3)} N)_*^\pm \equiv (\nabla^{(3)} N)_E^\pm - \widetilde{(\nabla^{(3)} N)}_*^\pm, \tag{4.262}$$

again with

$$\widetilde{(\nabla^{(3)} N)}^\pm \equiv \left[ (\nabla^{(3)} N)_E, \exp \left[ \pm \frac{1}{2} r^k {}^{(3)}\nabla_k^H \right] \right], \tag{4.263}$$

where

$$(\nabla^{(3)} N)_E \equiv \gamma^{ij} \partial_i N e_j. \tag{4.264}$$

We will be dealing with the sum and differences of those operators. We have

$$(\nabla^{(3)} N)_*^+ + (\nabla^{(3)} N)_*^- = (\nabla^{(3)} N)_E + {}^{(3)}\nabla_{rE}^H (\nabla^{(3)} N)_R + \mathcal{O}({}^{(3)}R^2) \tag{4.265}$$

$$\begin{aligned}
(\nabla^{(3)} N)_*^+ - (\nabla^{(3)} N)_*^- &= 2(\nabla^{(3)} N)_R + \frac{1}{2} {}^{(3)}\nabla_{rE}^H (\nabla^{(3)} N)_E \\
&\quad + \frac{1}{4} ({}^{(3)}\nabla_{rE}^H)^2 (\nabla^{(3)} N)_R \\
&\quad + \frac{1}{12} R_R [(\nabla^{(3)} N)] + \mathcal{O}({}^{(3)}R^2).
\end{aligned} \tag{4.266}$$

We find for the expressions appearing in (4.140) and (4.141) to second order in the spatial gradient expansion and up to boundary terms,

$$\begin{aligned}
& \gamma^{1/2} \int_{\Sigma_t} d^3 r e^{-\frac{i}{\hbar} r^k p_k} [(\nabla^{(3)} N)_*^+ + (\nabla^{(3)} N)_*^-] [:\hat{u}^+ \hat{u}^-:] \\
&= \gamma^{1/2} \int_{\Sigma_t} d^3 r e^{-\frac{i}{\hbar} r^k p_k} \left( N_{;k}^k {}^{(3)}\nabla_{e_k}^H + r^l N_{;l}^k \frac{\partial}{\partial r^k} + \mathcal{O}({}^{(3)}R^2) \right) [:\hat{u}^+ \hat{u}^-:] \\
&= \left( N_{;k}^k {}^{(3)}D^k - N_{;k}^k - p_k N_{;l}^k \frac{\partial}{\partial p_l} + \mathcal{O}(\hbar^4) \right) \hat{F}_{\phi\phi},
\end{aligned} \tag{4.267}$$

$$\begin{aligned}
& \gamma^{1/2} \int_{\Sigma_t} d^3r e^{-\frac{i}{\hbar} r^k p_k} [(\nabla^{(3)} N)_*^+ - (\nabla^{(3)} N)_*^-] [:\hat{u}^+ \hat{u}^-:] \\
&= \gamma^{1/2} \int_{\Sigma_t} d^3r e^{-\frac{i}{\hbar} r^k p_k} \left( 2N_{;k} \frac{\partial}{\partial r^k} + \frac{1}{2} r^l N_{;l} {}^{(3)}\nabla_{e_k}^H + \frac{1}{4} r^l r^m N_{;lm} \frac{\partial}{\partial r^k} \right. \\
&\quad \left. - \frac{1}{12} r^m r^{k(3)} R_{kms}^l N_{;s} \frac{\partial}{\partial r^l} \right) + \mathcal{O}({}^{(3)}R^2) [:\hat{u}^+ \hat{u}^-:] \\
&= \frac{i}{\hbar} \left( 2\gamma^{lk} p_l N_{;k} + \frac{1}{2} N_{;kl} \frac{\partial}{\partial p_l} D^k - \frac{\hbar^2}{4} p_k \gamma^{ks} N_{;slm} \frac{\partial^2}{\partial p_l \partial p_m} - \frac{\hbar^2}{2} N_{;km} \frac{\partial}{\partial p_m} \right. \\
&\quad \left. - \frac{\hbar^2}{6} {}^{(3)}R_l^m N_{;m} \frac{\partial}{\partial p_l} + \frac{\hbar^2}{12} p_l {}^{(3)}R_{kms}^l N_{;s} \frac{\partial^2}{\partial p_k \partial p_m} + \mathcal{O}(\hbar^4) \right) \hat{F}_{\phi\phi}. \quad (4.268)
\end{aligned}$$

Finally, we commute the spatial Laplace operator appearing in (4.140) to (4.142) and let it act on the product of the geodesically shifted canonical operators

$$(N^+({}^{(3)}\square)_*^+ \pm N^-({}^{(3)}\square)_*^-) [f^+ g^-] \equiv [N({}^{(3)}\square f)]^+ g^- \pm f^+ [N({}^{(3)}\square g)]^-, \quad (4.269)$$

where  $f^+$  and  $g^-$  are place holders for the operators  $\hat{u}^\pm$  and  $\hat{v}^\pm$ . The first term that appears in the series when commuting the spatial Laplacian with the shift exponentials is the horizontally lifted operator itself,

$$\begin{aligned}
& N^+ \gamma^{ij} [{}^{(3)}\nabla_{e_i}^H {}^{(3)}\nabla_{e_j}^H f^+] \hat{u}^- \pm N^- \gamma^{ij} f^+ [{}^{(3)}\nabla_{e_i}^H {}^{(3)}\nabla_{e_j}^H g^-] \\
&= [N^+ \gamma^{ij} {}^{(3)}\nabla_{e_i}^H e_j^+ \pm N^- \gamma^{ij} {}^{(3)}\nabla_{e_i}^H e_j^-] [f^+ g^-] + \gamma^{ij} (N^+ \pm N^-) f^+ [[e_i^+, e_j^-] g^-] \\
&\quad - \gamma^{ij} [N^+ e_i^+ e_j^- \pm N^- e_i^- e_j^+] [f^+ g^-]. \quad (4.270)
\end{aligned}$$

The second part is not manifestly spatially covariant, but our calculation should now show that it actually is. We will need the following identity

$$\begin{aligned}
\gamma^{ij} [e_i^+, e_j^-]^- &= -\gamma^{ij} {}^{(3)}\Gamma_{ij}^l \left( -\frac{1}{4} e_l + \frac{1}{2} \frac{\partial}{\partial r^l} \right) \\
&\quad - \gamma^{ij} \frac{1}{12} [e_i, r^m r^{k(3)} R_{kmj}^l \frac{\partial}{\partial r^l}]^-, \quad (4.271)
\end{aligned}$$

The term (4.270) takes then the form

$$\begin{aligned}
& N^+ \gamma^{ij} [{}^{(3)}\nabla_{e_i}^H {}^{(3)}\nabla_{e_j}^H f^+] g^- \pm N^- \gamma^{ij} f^+ [{}^{(3)}\nabla_{e_i}^H {}^{(3)}\nabla_{e_j}^H g^-] \\
&= \gamma^{ij} \left\{ (N^+ \mp N^-) \frac{\partial}{\partial r^i} {}^{(3)}\nabla_{e_j}^H + (N^+ \pm N^-) \left[ \frac{1}{4} {}^{(3)}\nabla_{e_i}^H {}^{(3)}\nabla_{e_j}^H + \frac{\partial}{\partial r^i} \frac{\partial}{\partial r^j} \right] \right. \\
&\quad + \frac{1}{12} (N^+ \mp N^-) r^k r^{m(3)} R_{kmj;i}^l \frac{\partial}{\partial r^l} \\
&\quad + \frac{1}{12} (N^+ \mp N^-) r^k r^{m(3)} R_{kmj}^l \frac{\partial}{\partial r^l} {}^{(3)}\nabla_{e_i}^H \\
&\quad \left. + \frac{1}{6} (N^+ \pm N^-) r^k r^{m(3)} R_{kmj}^l \frac{\partial}{\partial r^l} \frac{\partial}{\partial r^i} \right\} [f^+ g^-] \\
&\quad + (N^+ \pm N^-) \left[ \frac{1}{12} r^{m(3)} R_m^l \frac{\partial}{\partial r^l} \right] [f^+ g^-]. \quad (4.272)
\end{aligned}$$

Let us now include the commutator of the spatial Laplacian with the shift exponentials. We have

$$\begin{aligned}
& \left[ {}^{(3)}\square, \exp \left[ \pm \frac{1}{2} r^i {}^{(3)}\nabla_{e_i}^H \right] \right] f(x^\mu) \\
&= \pm \frac{1}{2} r^k \gamma^{ij} \left[ r^{m(3)} R_{mkj;i}^l \frac{\partial}{\partial r^l} + {}^{(3)}R_{ikj}^l {}^{(3)}\nabla_{e_l}^H \right. \\
&\quad \left. + 2r^{m(3)} R_{mkj}^l \frac{\partial}{\partial r^l} {}^{(3)}\nabla_{e_i}^H + \mathcal{O}({}^{(3)}R^2) \right] \exp \left[ \pm \frac{1}{2} r^i {}^{(3)}\nabla_{e_i}^H \right] f(x^\mu) \\
&+ \frac{1}{8} r^k r^s \gamma^{ij} \left[ {}^{(3)}R_{ikj;s}^l {}^{(3)}\nabla_{e_l}^H \right. \\
&\quad \left. + 2r^{m(3)} R_{mkj;s}^l \frac{\partial}{\partial r^l} {}^{(3)}\nabla_{e_i}^H + \mathcal{O}({}^{(3)}R^2) \right] \exp \left[ \pm \frac{1}{2} r^i {}^{(3)}\nabla_{e_i}^H \right] f(x^\mu). \quad (4.273)
\end{aligned}$$

We are now in the position to compute (4.269) to second order in the spatial gradient expansion,

$$\begin{aligned}
& \left( N^+ ({}^{(3)}\square)_*^+ \pm N^- ({}^{(3)}\square)_*^- \right) [f^+ g^-] \\
&= \gamma^{ij} \left\{ (N^+ \mp N^-) \frac{\partial}{\partial r^i} {}^{(3)}\nabla_{e_j}^H + (N^+ \pm N^-) \left[ \frac{1}{4} {}^{(3)}\nabla_{e_i}^H {}^{(3)}\nabla_{e_j}^H + \frac{\partial}{\partial r^i} \frac{\partial}{\partial r^j} \right] \right. \\
&\quad - \frac{7}{24} (N^+ \mp N^-) r^k r^{m(3)} R_{kmj;i}^l \frac{\partial}{\partial r^l} - \frac{1}{6} (N^+ \mp N^-) r^k r^{m(3)} R_{kmj}^l \frac{\partial}{\partial r^l} {}^{(3)}\nabla_{e_i}^H \\
&\quad - \frac{1}{3} (N^+ \pm N^-) r^k r^{m(3)} R_{kmj}^l \frac{\partial}{\partial r^l} \frac{\partial}{\partial r^i} - \frac{5}{12} (N^+ \pm N^-) r^{m(3)} R_{imj}^l \frac{\partial}{\partial r^l} \\
&\quad - \frac{1}{8} (N^+ \mp N^-) r^s r^k r^{m(3)} R_{mkj;s}^l \frac{\partial}{\partial r^l} \frac{\partial}{\partial r^i} + \mathcal{O}({}^{(3)}R^2) \left. \right\} [f^+ g^-] \\
&- \left\{ \frac{1}{4} (N^+ \mp N^-) r^k \gamma^{ij} {}^{(3)}R_{ikj}^l {}^{(3)}\nabla_{e_l}^H \right. \\
&\quad \left. + \frac{1}{4} (N^+ \mp N^-) r^k \gamma^{ij} r^{m(3)} R_{mkj}^l {}^{(3)}\nabla_{e_l}^H \frac{\partial}{\partial r^i} + \mathcal{O}({}^{(3)}R^2) \right\} [f^+ g^-]. \quad (4.274)
\end{aligned}$$

Performing the  $r$ -integral of the covariant Wigner transform yields the following expressions appearing in (4.140) and (4.141) up to higher order corrections in the spatial gradient expansion and up to boundary terms,

$$\begin{aligned}
& \gamma^{1/2} \int_{\Sigma_t} d^3r e^{-\frac{i}{\hbar} r^k p_k} \left( N^+ ({}^{(3)}\square)_*^+ + N^- ({}^{(3)}\square)_*^- \right) [:\hat{u}^+ \hat{u}^-:] \\
&= \left[ -p_i ({}^{(3)}\nabla_k N) \frac{\partial}{\partial p_k} D^i - ({}^{(3)}\nabla_i N) D^i + 2N \left( \frac{1}{4} D_i D^i - \gamma^{ij} \frac{p_i p_j}{\hbar^2} \right) \right. \\
&\quad + \gamma^{ij} \frac{p_j p_i}{4} ({}^{(3)}\nabla_k {}^{(3)}\nabla_l N) \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_l} + \frac{p_i}{2} ({}^{(3)}\nabla^i {}^{(3)}\nabla_k N) \frac{\partial}{\partial p_k} \\
&\quad + \frac{1}{2} ({}^{(3)}\nabla^i {}^{(3)}\nabla_i N) - \frac{2}{3} N p_l p_i \gamma^{ij} {}^{(3)}R_{kmj}^l \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_m} \\
&\quad \left. - \frac{1}{6} p_i N {}^{(3)}R_k^i \frac{\partial}{\partial p_k} + \frac{1}{6} N {}^{(3)}R + \mathcal{O}(\hbar^4) \right] \hat{F}_{\phi\phi}, \quad (4.275)
\end{aligned}$$

$$\begin{aligned}
& \gamma^{1/2} \int_{\Sigma_t} d^3r e^{-\frac{i}{\hbar} r^k p_k} \left( N^+ ({}^{(3)}\square)_*^+ - N^- ({}^{(3)}\square)_*^- \right) [ : \hat{u}^+ \hat{u}^- : ] \\
&= \left[ \frac{i}{\hbar} \left( 2N p_j - \frac{\hbar^2}{2} ({}^{(3)}\nabla_k ({}^{(3)}\nabla^j N) \frac{\partial}{\partial p_k} - \frac{\hbar^2}{4} p_j ({}^{(3)}\nabla_k ({}^{(3)}\nabla_l N) \frac{\partial^2}{\partial p_k \partial p_l} \right) D^j \right. \\
&\quad + \frac{i}{\hbar} ({}^{(3)}\nabla_k N) \left( \frac{\hbar^2}{4} D^j D_j - \gamma^{jl} p_l p_j \right) \frac{\partial}{\partial p_k} - 2 \frac{i}{\hbar} p_k ({}^{(3)}\nabla^k N) \\
&\quad + \frac{i\hbar}{24} p_r p_s \gamma^{rs} ({}^{(3)}\nabla_k ({}^{(3)}\nabla_l ({}^{(3)}\nabla_m N) \frac{\partial^3}{\partial p_k \partial p_l \partial p_m} \\
&\quad + \frac{i\hbar}{8} p_j ({}^{(3)}\nabla_k ({}^{(3)}\nabla_l ({}^{(3)}\nabla^j N) \frac{\partial^2}{\partial p_k \partial p_l} \\
&\quad + \frac{i\hbar}{4} ({}^{(3)}\nabla_k ({}^{(3)}\nabla_j ({}^{(3)}\nabla^j N) \frac{\partial}{\partial p_k} + \frac{i\hbar}{12} R^l_k ({}^{(3)}\nabla_l N) \frac{\partial}{\partial p_k} \\
&\quad + i\hbar N \left( \frac{1}{12} p_j \gamma^{jl} ({}^{(3)}R^s_{kml;s} \frac{\partial^2}{\partial p_k \partial p_m} - \frac{1}{3} ({}^{(3)}R^j_{k;j} \frac{\partial}{\partial p_k} \right. \\
&\quad \left. - \frac{1}{4} p_i p_l \gamma^{ij} ({}^{(3)}R^l_{mkj;s} \frac{\partial^3}{\partial p_s \partial p_k \partial p_m} + \frac{1}{4} p_j ({}^{(3)}R^j_{m;s} \frac{\partial^2}{\partial p_s \partial p_m} \right) \\
&\quad + i\hbar N \left( \frac{5}{6} p_l ({}^{(3)}R^l_{kmj} \frac{\partial^2}{\partial p_k \partial p_m} D^j - \frac{1}{3} ({}^{(3)}R^j_k \frac{\partial}{\partial p_k} D_j \right) \\
&\quad + i\hbar \left( -\frac{1}{3} \gamma^{ij} p_i p_l ({}^{(3)}\nabla^s N ({}^{(3)}R^l_{kmj} \frac{\partial^3}{\partial p_s \partial p_k \partial p_m} \right. \\
&\quad \left. + \frac{5}{12} ({}^{(3)}\nabla_j N ({}^{(3)}R^j_k \frac{\partial}{\partial p_k} + \frac{1}{12} ({}^{(3)}\nabla^j N ({}^{(3)}R \frac{\partial}{\partial p_j} \right) \\
&\quad + i\hbar \left( -\frac{5}{8} \gamma^{ij} p_l ({}^{(3)}\nabla_i N ({}^{(3)}R^l_{kmj} \frac{\partial^2}{\partial p_k \partial p_m} \right. \\
&\quad \left. + \frac{5}{12} p_j ({}^{(3)}\nabla_s N ({}^{(3)}R^j_k \frac{\partial^2}{\partial p_k \partial p_s} \right) + \mathcal{O}(\hbar^4) \Big] \hat{F}_{\phi\phi} . \tag{4.276}
\end{aligned}$$

## 4.E Dynamics of $F_{\phi\phi}$ , $F_{\phi\Pi}$ , $F_{\Pi\phi}$ and $F_{\Pi\Pi}$

Performing the  $r$ -integral in (4.139) to (4.142) leads to the following equations in phase-space to next-to-leading order in the spatial gradient expansion where we also dropped boundary terms, anomalous contributions and simplified to self-interaction terms by assuming a Gaussian state truncation with vanishing one-point functions,

$$\begin{aligned} \partial_t F_{\phi\phi} = & \left[ N - \frac{\hbar^2}{8} N_{;kl} \frac{\partial^2}{\partial p_k \partial p_l} \right] (F_{\Pi\phi} + F_{\phi\Pi}) \\ & - \frac{\hbar}{2i} \left[ N_{;k} \frac{\partial}{\partial p_k} - \frac{\hbar^2}{24} N_{;klm} \frac{\partial^3}{\partial p_k \partial p_l \partial p_m} \right] (F_{\Pi\phi} - F_{\phi\Pi}) \\ & + \left[ N^k D_k - p_k N_{;m}^k \frac{\partial}{\partial p_m} - NK - \frac{\hbar^2}{12} N_{;kqm}^k \frac{\partial^2}{\partial p_q \partial p_m} \right. \\ & + \frac{\hbar^2}{4} \left( \frac{1}{2} [NK_{ik}]_{;l} - [NK_{kl}]_{;i} \right) \frac{\partial^2}{\partial p_k \partial p_i} D^l \\ & - \frac{\hbar^2}{12} \left( \frac{1}{2} [NK_{ik}]_{;l}^l - [NK_k^l]_{;il} - [NK]_{;ki} \right) \frac{\partial^2}{\partial p_k \partial p_i} \\ & \left. - \frac{\hbar^2}{12} p_l \left( \frac{1}{2} [NK_{ik}]_{;s}^l - [NK_k^l]_{;is} \right) \frac{\partial^3}{\partial p_k \partial p_i \partial p_s} \right] F_{\phi\phi}. \quad (4.277) \end{aligned}$$

$$\begin{aligned} \frac{\partial_t}{2} (F_{\Pi\phi} + F_{\phi\Pi}) = & \left[ N - \frac{\hbar^2}{8} N_{;kl} \frac{\partial^2}{\partial p_k \partial p_l} \right] F_{\Pi\Pi} \\ & + \frac{1}{2} \left[ N^k D_k - p_k N_{;m}^k \frac{\partial}{\partial p_m} - \frac{\hbar^2}{12} N_{;kqm}^k \frac{\partial^2}{\partial p_q \partial p_m} \right. \\ & + \frac{\hbar^2}{4} \left( \frac{1}{2} [NK_{ik}]_{;l} - [NK_{kl}]_{;i} \right) \frac{\partial^2}{\partial p_k \partial p_i} D^l \\ & - \frac{\hbar^2}{12} \left( \frac{1}{2} [NK_{ik}]_{;l}^l - [NK_k^l]_{;il} + \frac{1}{2} [NK]_{;ki} \right) \frac{\partial^2}{\partial p_k \partial p_i} \\ & \left. - \frac{\hbar^2}{12} p_l \left( \frac{1}{2} [NK_{ik}]_{;s}^l - [NK_k^l]_{;is} \right) \frac{\partial^3}{\partial p_k \partial p_i \partial p_s} \right] (F_{\Pi\phi} + F_{\phi\Pi}) \\ & - \frac{\hbar}{4i} \left[ (NK)_{;j} \frac{\partial}{\partial p_j} - \frac{\hbar^2}{24} (NK)_{;jlm} \frac{\partial^3}{\partial p_j \partial p_l \partial p_m} \right] (F_{\Pi\phi} - F_{\phi\Pi}) \\ & - \left[ \left( \frac{m^2}{\hbar^2} + \gamma^{kj} \frac{p_k p_j}{\hbar^2} \right) \left( N - \frac{\hbar^2}{8} N_{;kl} \frac{\partial^2}{\partial p_k \partial p_l} \right) + \xi NR \right. \\ & + \frac{1}{2} \frac{\lambda}{\hbar} \left( N \int \frac{d^3 q}{\gamma^{1/2}} F_{\phi\phi}(q) - \frac{\hbar^2}{8} \left[ N \int \frac{d^3 q}{\gamma^{1/2}} F_{\phi\phi}(q) \right]_{;kl} \frac{\partial^2}{\partial p_k \partial p_l} \right) \\ & - \frac{1}{2} p_j N_{;k} \frac{\partial}{\partial p_k} D^j - \frac{1}{4} N D_j D^j + \frac{1}{3} N p_l p_i \gamma^{ij(3)} R_{kmj}^l \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_m} \\ & \left. + \frac{1}{12} N p_i^{(3)} R_k^i \frac{\partial}{\partial p_k} - \frac{1}{6} N^{(3)} R \right] F_{\phi\phi}. \quad (4.278) \end{aligned}$$

$$\begin{aligned}
\frac{i}{2}\partial_t(F_{\Pi\phi} - F_{\phi\Pi}) = & \frac{\hbar}{2}\left[N_{;k}\frac{\partial}{\partial p_k} - \frac{\hbar^2}{24}N_{;klm}\frac{\partial^3}{\partial p_k\partial p_l\partial p_m}\right]F_{\Pi\Pi} \\
& + \frac{i}{2}\left[N^k D_k - p_k N_{;m}^k \frac{\partial}{\partial p_m} - \frac{\hbar^2}{12}N_{;kqm}^k \frac{\partial^2}{\partial p_q\partial p_m}\right. \\
& + \frac{\hbar^2}{4}\left(\frac{1}{2}[NK_{ik}]_{;l} - [NK_{kl}]_{;i}\right)\frac{\partial^2}{\partial p_k\partial p_i}D^l \\
& - \frac{\hbar^2}{12}\left(\frac{1}{2}[NK_{ik}]_{;l}^l - [NK_k^l]_{;il} + \frac{1}{2}[NK]_{;ki}\right)\frac{\partial^2}{\partial p_k\partial p_i} \\
& \left. - \frac{\hbar^2}{12}p_l\left(\frac{1}{2}[NK_{ik}]_{;s}^l - [NK_k^l]_{;is}\right)\frac{\partial^3}{\partial p_k\partial p_i\partial p_s}\right](F_{\Pi\phi} - F_{\phi\Pi}) \\
& - \frac{\hbar}{4}\left[(NK)_{;j}\frac{\partial}{\partial p_j} - \frac{\hbar^2}{24}(NK)_{;jlm}\frac{\partial^3}{\partial p_j\partial p_l\partial p_m}\right](F_{\Pi\phi} + F_{\phi\Pi}) \\
& - \frac{1}{2}\left[\frac{1}{\hbar}\left(2Np_j - \frac{\hbar^2}{4}p_j N_{;lk}\frac{\partial^2}{\partial p_k\partial p_l}\right)D^j\right. \\
& - \hbar\xi(NR)_{;k}\frac{\partial}{\partial p_k} + \frac{1}{\hbar}({}^{(3)}\nabla_k N)\left(\frac{\hbar^2}{4}D^j D_j\right)\frac{\partial}{\partial p_k} \\
& - \frac{1}{\hbar}\left(m^2 + p_r p_s \gamma^{rs}\right)\left(N_{;k}\frac{\partial}{\partial p_k} - \frac{\hbar^2}{24}N_{;mlk}\frac{\partial^3}{\partial p_k\partial p_l\partial p_m}\right) \\
& - \frac{1}{2}\frac{\lambda}{\hbar}\left(\hbar\left[N\int\frac{d^3q}{\gamma^{1/2}}F_{\phi\phi}(q)\right]_{;k}\frac{\partial}{\partial p_k}\right. \\
& \left. - \frac{\hbar^3}{24}\left[N\int\frac{d^3q}{\gamma^{1/2}}F_{\phi\phi}(q)\right]_{;mlk}\frac{\partial^3}{\partial p_k\partial p_l\partial p_m}\right) \\
& + \hbar N\left(\frac{1}{12}p_j\gamma^{jl(3)}R_{kml;s}^s\frac{\partial^2}{\partial p_k\partial p_m} - \frac{1}{3}({}^{(3)}R_{k;j}^j\frac{\partial}{\partial p_k}\right. \\
& - \frac{1}{4}p_i p_l \gamma^{ij(3)}R_{mkj;s}^l\frac{\partial^3}{\partial p_s\partial p_k\partial p_m} + \frac{1}{4}p_j({}^{(3)}R_{m;s}^j\frac{\partial^2}{\partial p_s\partial p_m}) \\
& + \hbar N\left(\frac{5}{6}p_l({}^{(3)}R_{kmj}^l\frac{\partial^2}{\partial p_k\partial p_m}D^j - \frac{1}{3}({}^{(3)}R_k^j\frac{\partial}{\partial p_k}D_j)\right. \\
& - \hbar\left(\frac{1}{3}\gamma^{ij}p_i p_l N_{;s(3)}R_{kmj}^l\frac{\partial^3}{\partial p_s\partial p_k\partial p_m} + \frac{1}{12}N_{;j(3)}R\frac{\partial}{\partial p_j}\right) \\
& + \frac{\hbar}{12}({}^{(3)}R_k^l N_{;l}\frac{\partial}{\partial p_k} + \hbar\left(+\frac{5}{12}p_j N_{;s(3)}R_k^j\frac{\partial^2}{\partial p_k\partial p_s}\right) \\
& - \frac{\hbar}{8}p_k N_{;lm}^k\frac{\partial^2}{\partial p_l\partial p_m} - \frac{\hbar}{4}N_{;km}^k\frac{\partial}{\partial p_m} - \frac{\hbar}{4}({}^{(3)}R_l^m N_{;m}\frac{\partial}{\partial p_l} \\
& \left. + \frac{13\hbar}{48}p_l({}^{(3)}R_{kms}^l N_{;s}\frac{\partial^2}{\partial p_k\partial p_m})\right]F_{\phi\phi}. \tag{4.279}
\end{aligned}$$

$$\begin{aligned}
\partial_t F_{\Pi\Pi} = & \left[ N^k D_k + NK - p_k N^k_{;m} \frac{\partial}{\partial p_m} - \frac{\hbar^2}{12} N^k_{;kqm} \frac{\partial^2}{\partial p_q \partial p_m} \right. \\
& + \frac{\hbar^2}{4} \left( \frac{1}{2} [NK_{ik}]_{;l} - [NK_{kl}]_{;i} \right) \frac{\partial^2}{\partial p_k \partial p_i} D^l \\
& - \frac{\hbar^2}{12} \left( \frac{1}{2} [NK_{ik}]_{;l}^l - [NK_k^l]_{;il} + 2[NK]_{;ki} \right) \frac{\partial^2}{\partial p_k \partial p_i} \\
& \left. - \frac{\hbar^2}{12} p_l \left( \frac{1}{2} [NK_{ik}]_{;s}^l - [NK_k^l]_{;is} \right) \frac{\partial^3}{\partial p_k \partial p_i \partial p_s} \right] F_{\Pi\Pi} \\
& - \frac{i}{2} \left[ \frac{1}{\hbar} \left( 2N p_j - \frac{\hbar^2}{4} p_j N_{;lk} \frac{\partial^2}{\partial p_k \partial p_l} \right) D^j + \frac{1}{\hbar} ({}^{(3)}\nabla_k N) \left( \frac{\hbar^2}{4} D^j D_j \right) \frac{\partial}{\partial p_k} \right. \\
& - \frac{1}{\hbar} \left( m^2 + p_r p_s \gamma^{rs} \right) \left( N_{;k} \frac{\partial}{\partial p_k} - \frac{\hbar^2}{24} N_{;mlk} \frac{\partial^3}{\partial p_k \partial p_l \partial p_m} \right) \\
& - \hbar \xi (NR)_{;k} \frac{\partial}{\partial p_k} \\
& - \frac{1}{2} \frac{\lambda}{\hbar} \left( \hbar [N \int \frac{d^3 q}{\gamma^{1/2}} F_{\phi\phi}(q)]_{;k} \frac{\partial}{\partial p_k} \right. \\
& \left. - \frac{\hbar^3}{24} [N \int \frac{d^3 q}{\gamma^{1/2}} F_{\phi\phi}(q)]_{;mlk} \frac{\partial^3}{\partial p_k \partial p_l \partial p_m} \right) \\
& + \hbar N \left( \frac{1}{12} p_j \gamma^{jl(3)} R^s_{kml;s} \frac{\partial^2}{\partial p_k \partial p_m} - \frac{1}{3} {}^{(3)}R^j_{k;j} \frac{\partial}{\partial p_k} \right. \\
& - \frac{1}{4} p_i p_l \gamma^{ij(3)} R^l_{mkj;s} \frac{\partial^3}{\partial p_s \partial p_k \partial p_m} + \frac{1}{4} p_j {}^{(3)}R^j_{m;s} \frac{\partial^2}{\partial p_s \partial p_m} \Big) \\
& + \hbar N \left( \frac{5}{6} p_l {}^{(3)}R^l_{kmj} \frac{\partial^2}{\partial p_k \partial p_m} D^j - \frac{1}{3} {}^{(3)}R^j_k \frac{\partial}{\partial p_k} D_j \right) \\
& - \hbar \left( \frac{1}{3} \gamma^{ij} p_i p_l N_{;s} {}^{(3)}R^l_{kmj} \frac{\partial^3}{\partial p_s \partial p_k \partial p_m} - \frac{1}{12} N_{;j} {}^{(3)}R^j \frac{\partial}{\partial p_j} \right) \\
& + \frac{\hbar}{12} {}^{(3)}R^l_k N_{;l} \frac{\partial}{\partial p_k} + \hbar \left( + \frac{5}{12} p_j N_{;s} {}^{(3)}R^j_k \frac{\partial^2}{\partial p_k \partial p_s} \right) \\
& - \frac{\hbar}{8} p_k N_{;lm} \frac{\partial^2}{\partial p_l \partial p_m} - \frac{\hbar}{4} N_{;km} \frac{\partial}{\partial p_m} \\
& - \frac{\hbar}{4} {}^{(3)}R^m_l N_{;m} \frac{\partial}{\partial p_l} + \hbar \frac{13}{48} p_l {}^{(3)}R^l_{kms} N_{;s} \frac{\partial^2}{\partial p_k \partial p_m} \Big] (\hat{F}_{\Pi\phi} - \hat{F}_{\phi\Pi}) \\
& - \left[ \left( \frac{m^2}{\hbar^2} + \gamma^{kj} \frac{p_k p_j}{\hbar^2} \right) \left( N - \frac{\hbar^2}{8} N_{;kl} \frac{\partial^2}{\partial p_k \partial p_l} \right) + \xi NR \right. \\
& + \frac{1}{2} \frac{\lambda}{\hbar} \left( N \int \frac{d^3 q}{\gamma^{1/2}} F_{\phi\phi}(q) - \frac{\hbar^2}{8} [N \int \frac{d^3 q}{\gamma^{1/2}} F_{\phi\phi}(q)]_{;kl} \frac{\partial^2}{\partial p_k \partial p_l} \right) \\
& - \frac{1}{2} p_j N_{;k} \frac{\partial}{\partial p_k} D^j - \frac{1}{4} N D_j D^j + \frac{1}{3} N p_l p_i \gamma^{ij(3)} R^l_{kmj} \frac{\partial}{\partial p_k} \frac{\partial}{\partial p_m} \\
& \left. + \frac{1}{12} N p_i {}^{(3)}R^i_k \frac{\partial}{\partial p_k} - \frac{1}{6} N {}^{(3)}R \right] (F_{\Pi\phi} + F_{\phi\Pi}) . \tag{4.280}
\end{aligned}$$





## Chapter 5

# Field-theoretic approach to large-scale structure formation

We develop a field-theoretic description of large-scale structure formation by taking the non-relativistic limit of a canonically transformed, real scalar field which is minimally coupled to scalar gravitational perturbations in longitudinal gauge. We integrate out the gravitational constraint fields and arrive at a non-local action which is only specified in terms of the dynamical degrees of freedom. In order to make this framework closer to the classical particle description, we construct the corresponding 2PI effective action truncated at two loop order for a non-squeezed state without field expectation values. We contrast the dynamical description of the coincident time phase-space density to the standard Vlasov description of cold dark matter particles and identify momentum and time scales at which linear perturbation theory will deviate from the standard evolution.

### 5.1 Introduction

It lies in the nature of physics that surprising effects happen on the transition between one physical scale to another. In order to study whether such transitioning effects are important one ought to start from the most fundamental description that is available and descend in a controlled way to the scale that is relevant for the problem. Cosmological theories are in particular sensible to such transitions since they attempt to describe various scales and its associated effects range from quantum field physics during inflation up to the evolution of large-scale structures and cold dark matter at later times which is what we are interested in. Even if one assumes only a real scalar particle with gravitational interactions in a non-relativistic limit, there is still room to choose the state which should describe this cold dark matter, be it a classical stochastic state with or without squeezing, or a condensate. In [111, 139], we showed that a non-squeezed, classical stochastic state leads to point-like cold dark matter characteristics on large scales and is thus the field-theoretic generalization of the standard Vlasov description [24, 88]. The condensate description corresponding to a coherent state, on the other hand, is referred to as fuzzy dark matter [17–22]. It also resembles point-like cold dark matter dynamics on large scales but there are, however, significant small scale effects [26–29]. Are such

small scale effects and exclusive features of a condensate state, do they occur for other states, how do they differ?

In order to account for these questions we are after a field-theoretic description of cold dark matter that originates from the QFT tree-level action of a real scalar field with minimal coupling to gravity where we focus on scalar gravitational perturbations in longitudinal gauge in an FLRW universe. We would like to emphasize that using an action of genuine quantum nature does not imply that quantum effects are considered important, field-theoretic effects, however, may be and we will give examples of such effects in this chapter. One of the key ingredients in this work is the generalization of the canonical field transformation developed in [129] where the non-relativistic limit of a self-interacting real scalar field in Minkowski space-time is addressed. We perturb the general relativistic theory (5.1) and rewrite it in terms of the diagonal field representation (5.8). We then take the non-relativistic limit assuming that the mass  $m$  of the scalar is the largest scale apart from the Planck scale  $M_P$ . The resulting action (5.49) contains the classical, non-relativistic particle description as a special case on large-scales. We show this by constructing the corresponding 2PI effective action truncated at two loop order for a virialized state, namely a state that is neither squeezed nor that it has a non-vanishing condensate. Virialized states can contain a large number of particles, if they descend from a mixed density matrix.

The work presented in this chapter is in line with our previous works [111, 139] (cf. chapters 3 and 4). However, the main differences are first, that we perturbatively integrate out the gravitational constraint fields which leads to an additional exchange interaction and second, that we set up a general framework where we *a priori* do not assume that spatial gradients  $\nabla_x$  are small compared to the particle momenta  $p$  which is important if one would like to study small scale effects.

Let us also mention that the development of the framework in this chapter is also motivated by the problem of solving cold dark matter dynamics beyond the linearized, single-stream perfect fluid approximation. Similar to the statistical field theory based on classical point-like particles [140, 141] and as an extended approach to the condensate based Schrödinger model [30–33], we reformulate the problem of cold dark matter dynamics by resorting to a more fundamental description which may be more suitable to get a different analytic and numerical access.

We work in units where  $c = 1$  with a mostly plus signature  $(-, +, +, +)$ .

## 5.2 Gravity through external fields

Let us start by writing down the action for a massive, real scalar field in its canonical form with couplings to gravity in ADM-variables [115]

$$S_\phi = \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \left[ \Pi_\phi \dot{\phi} - \frac{N}{2} \gamma^{1/2} \left( \gamma^{-1} \Pi_\phi^2 + \gamma^{ij} \partial_i \phi \partial_j \phi + \frac{m^2}{\hbar^2} \phi^2 \right) - N_i \Pi_\phi \partial^i \phi \right], \quad (5.1)$$

where  $N$  and  $N^i$  are lapse and shift functions,  $\gamma_{ij}$  is the spatial metric,  $\gamma$  its determinant and  $\pi_\phi$  is the canonical momentum associated with  $\phi$ . We now neglect vector and tensor perturbations in the metric and consider scalar perturbations in the longitudinal gauge with the gravitational potentials  $\Phi_G$  and  $\Psi_G$ , in which we also linearize with a small perturbation parameter  $\varepsilon_g$ ,

$$N = \bar{N}(1 + \Phi_G), \quad N^i = 0, \quad \gamma_{ij} = a^2 \delta_{ij}(1 - 2\Psi_G), \quad (5.2)$$

$$\mathcal{O}(\Phi_G, \Psi_G) = \varepsilon_g^2 \ll 1. \quad (5.3)$$

This leads us to

$$S_\phi \approx \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \left[ \Pi_\phi \dot{\phi} - \frac{1}{2} \bar{N}(1 + \Phi_G) \left( a^{-3} [1 + 3\Psi_G] \Pi_\phi^2 + a [1 - \Psi_G] \delta^{ij} \partial_i \phi \partial_j \phi + a^3 [1 - 3\Psi_G] \frac{m^2}{\hbar^2} \phi^2 \right) \right]. \quad (5.4)$$

We switch to conformal time  $ad\eta = \bar{N}dt$  whose derivative is denoted by a prime ( $a\mathcal{H} = a'$ ) and perform a first canonical transformation (leaving the path-integral measure unchanged) by defining

$$\phi_c \equiv a\phi, \quad \Pi_\phi^c \equiv a^{-1} \Pi_\phi + \mathcal{H}a\phi. \quad (5.5)$$

We integrate by parts and find upon dropping temporal boundary terms

$$S_\phi \approx S_{\phi_c} \equiv \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left[ \Pi_\phi^c \phi_c' - \frac{1}{2} \left\{ [1 + \Phi_G + 3\Psi_G] (\Pi_\phi^c)^2 - 2[\Phi_G + 3\Psi_G] \mathcal{H} \phi_c \Pi_\phi^c + [1 + \Phi_G - \Psi_G] \delta^{ij} \partial_i \phi_c \partial_j \phi_c + [1 + \Phi_G - 3\Psi_G] \frac{m_{\text{eff}}^2}{\hbar^2} \phi_c^2 + [(\mathcal{H}' + 2\mathcal{H}^2) \Phi_G - 3\mathcal{H}' \Psi_G] \phi_c^2 \right\} \right], \quad (5.6)$$

where we identify the effective mass

$$m_{\text{eff}}^2 \equiv m^2 a^2 - \hbar^2 \mathcal{H}' - \hbar^2 \mathcal{H}^2. \quad (5.7)$$

We now propose a straightforward generalization of the non-local field redefinition worked out for Minkowski space-time by [129],

$$\psi \equiv \frac{1}{\sqrt{2\hbar}} \mathcal{E}^* \hat{\Omega}^{1/2} \left( \phi_c + i\hbar \hat{\Omega}^{-1} \Pi_\phi^c \right), \quad \hat{\Omega} \equiv \sqrt{m_{\text{eff}}^2 - \hbar^2 \Delta}, \quad (5.8)$$

where the spatial Laplacians is given by

$$\Delta \equiv \delta^{ij} \partial_i \partial_j, \quad (5.9)$$

and the time-dependent phase  $\mathcal{E}$  is defined as

$$\mathcal{E}(\eta) \equiv \exp \left( -i \int^\eta \frac{m_{\text{eff}}(\tilde{\eta})}{\hbar} d\tilde{\eta} \right). \quad (5.10)$$

The transformation (5.8) is akin to going to creation and annihilation operator variables in which one may diagonalize the Hamiltonian in the free theory. Moreover, it removes *Zitterbewegung* generated by the mass term. The operator  $\hat{\Omega}$  has the interpretation of a particle energy. The reverse transformation of (5.8) reads

$$\phi_c = \sqrt{\frac{\hbar}{2\hat{\Omega}}} (\mathcal{E}\psi + \mathcal{E}^*\psi^*), \quad \Pi_\phi^c = -i\sqrt{\frac{\hat{\Omega}}{2\hbar}} (\mathcal{E}\psi - \mathcal{E}^*\psi^*). \quad (5.11)$$

We note, that the corresponding measure in the path-integral is in the Hamilton formulation related to the real and imaginary parts of  $\psi$ ,

$$\mathcal{D}\phi_c \mathcal{D}\Pi_\phi^c \propto \mathcal{D}\text{Re}\psi \mathcal{D}\text{Im}\psi. \quad (5.12)$$

Thus, we have a canonical transformation between the pair  $\Phi_c$  and  $\Pi_\phi^c$  and the pair  $\text{Re}\psi$  and  $\text{Im}\psi$ . Moreover, one obtains the expected, equal-time commutation relation for the corresponding quantum operators in the non-relativistic theory,

$$[\hat{\psi}(\eta, x^i), \hat{\psi}^\dagger(\eta, y^i)] = \hbar \delta^3(x^i, y^i). \quad (5.13)$$

Plugging in the transformation (5.8) into the action (5.6), we find

$$\begin{aligned} S_{\phi_c} &= S_\psi[\Phi_G, \Psi_G] \\ &\equiv \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left\{ i\psi^*\psi' - \frac{m_{\text{eff}}}{\hbar} \psi \left( \frac{\hat{\Omega}}{m_{\text{eff}}} - 1 \right) \psi^* - \frac{1}{2} \frac{m'_{\text{eff}}}{m_{\text{eff}}} \text{Im} \left[ \mathcal{E}^2 \psi \frac{m_{\text{eff}}^2}{\hat{\Omega}^2} \psi \right] \right. \\ &\quad - \frac{m_{\text{eff}}}{\hbar} [\Phi_G + 3\Psi_G] \left[ \sqrt{\frac{\hat{\Omega}}{m_{\text{eff}}}} \text{Im}(\mathcal{E}\psi) \right]^2 \\ &\quad - \frac{m_{\text{eff}}}{\hbar} [\Phi_G - 3\Psi_G] \left[ \sqrt{\frac{m_{\text{eff}}}{\hat{\Omega}}} \text{Re}(\mathcal{E}\psi) \right]^2 \\ &\quad - \frac{\hbar}{m_{\text{eff}}} [(\mathcal{H}' + 2\mathcal{H}^2)\Phi_G - 3\mathcal{H}'\Psi_G] \left[ \sqrt{\frac{m_{\text{eff}}}{\hat{\Omega}}} \text{Re}(\mathcal{E}\psi) \right]^2 \\ &\quad - \frac{\hbar}{m_{\text{eff}}} (\Phi_G - \Psi_G) \delta^{ij} \sqrt{\frac{m_{\text{eff}}}{\hat{\Omega}}} \partial_i \text{Re}(\mathcal{E}\psi) \sqrt{\frac{m_{\text{eff}}}{\hat{\Omega}}} \partial_j \text{Re}(\mathcal{E}\psi) \\ &\quad \left. + 2[\Phi_G + 3\Psi_G] \mathcal{H} \left[ \sqrt{\frac{m_{\text{eff}}}{\hat{\Omega}}} \text{Re}(\mathcal{E}\psi) \right] \left[ \sqrt{\frac{\hat{\Omega}}{m_{\text{eff}}}} \text{Im}(\mathcal{E}\psi) \right] \right\}. \quad (5.14) \end{aligned}$$

The transformation (5.8) was designed to obtain a non-relativistic description in  $\hbar^2 \|\Delta\| \ll m_{\text{eff}}^2$  such that one can perturbatively correct it in a controlled way. Spatial derivatives  $\nabla = \nabla_{\vec{x}}$  acting on matter fields  $\psi(\vec{x})$  will be mapped on particle momenta  $\vec{p}$  and long-distance gradients  $\nabla_{\vec{X}} \sim \hbar \vec{k}$  once two-point functions of fields such as  $\langle \psi^\dagger(\eta, \vec{x}) \psi(\eta, \vec{y}) \rangle$  are mapped to a particle phase-space density  $f(\eta, \vec{p}, \vec{X})$ . Thus, assuming  $\hbar^2 \|\Delta\| \ll m_{\text{eff}}^2$  corresponds to assuming physical momenta  $p$  and inverse distance scales  $L^{-1} \sim k$  of the underlying physical problem to be much smaller than the scale set by the mass  $m_{\text{eff}}$ . Let us subsume these scale relations in the following expansion parameter

$$\mathcal{O}\left(\frac{\hbar \|\nabla\|}{m}\right) = \varepsilon_{\text{nr}} \ll 1. \quad (5.15)$$

We will only keep leading order contributions in  $\varepsilon_{\text{nr}}$  and also drop multiplicative higher-order terms of the type  $\varepsilon_g^2 \cdot \varepsilon_{\text{nr}}^2$  that involve the gravitational perturbation parameter. Moreover, we want to consider the case where the mass  $m$  is much bigger than the Hubble rate or its logarithmic derivative

$$\mathcal{O}\left(\frac{\hbar \mathcal{H}}{ma}, \frac{\hbar \mathcal{H}'}{\mathcal{H}ma}\right) = \varepsilon_{\text{H/m}} \ll 1, \quad (5.16)$$

In what follows, we shall keep only leading order contributions of order  $\varepsilon_{\text{H/m}}$  and drop multiplicative higher-order terms of order  $\varepsilon_g^2 \cdot \varepsilon_{\text{H/m}}^2$  involving the gravitational potential. However, we keep terms of order  $\varepsilon_g^2 \cdot \varepsilon_{\text{H/m}}$  since they come with phase-factors whose time derivative can reduce the order by one power. We then have

$$\begin{aligned} S_\psi[\Phi_G, \Psi_G] \approx & \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left\{ i\psi^* \psi' + \psi^* \left( \frac{\hbar \Delta}{2ma} - \frac{ma}{\hbar} \Phi_G \right) \psi \right. \\ & \left. + 3 \frac{ma}{\hbar} \Psi_G \text{Re}(\mathcal{E}^2 \psi^2) - \mathcal{H} \left( \frac{1}{2} - \Phi_G - 3\Psi_G \right) \text{Im}(\mathcal{E}^2 \psi^2) \right\}. \end{aligned} \quad (5.17)$$

What we have achieved so far is a different viewpoint on the non-relativistic limits we discussed in [111, 139] (cf. chapters 3 and 4) by assuming small gradients and a small expansion rate of scale factor with respect to the mass. If we promote the field  $\psi$  to an operator, we find that we treated the equal-time correlators

$$\langle \hat{\Pi}_\phi(x) \hat{\Pi}_\phi(y) \rangle, \langle \hat{\Pi}_\phi(x) \hat{\phi}(y) \rangle, \langle \hat{\phi}(x) \hat{\Pi}_\phi(y) \rangle, \langle \hat{\phi}(x) \hat{\phi}(y) \rangle, \quad (5.18)$$

for the equal-time correlators

$$\langle \hat{\psi}(x) \hat{\psi}^\dagger(y) \rangle, \langle \hat{\psi}^\dagger(x) \hat{\psi}(y) \rangle, \langle \hat{\psi}(x) \hat{\psi}(y) \rangle, \langle \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(y) \rangle. \quad (5.19)$$

In [111, 139] we concluded that only a particular combination of suitably transformed correlators constitutes a phase-space density of classical particles, the other ones being highly oscillatory and suppressed if they are initially small. The situation is similar in the new variables and amounts to neglecting  $\langle \hat{\psi}(x) \hat{\psi}(y) \rangle$

and  $\langle \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(y) \rangle$  in comparison to  $\langle \hat{\psi}(x) \hat{\psi}^\dagger(y) \rangle$  and  $\langle \hat{\psi}^\dagger(x) \hat{\psi}(y) \rangle$ . It is usually the case that if one drops these squeezing contribution, one can show that if they are not present initially, the evolutions will generate them only under special circumstances. Apart from the limits we have taken so far, we can consider this requirement on the quantum state as another requirement to obtain a description of classical particles from a real scalar quantum field. We refer to such a state as a *virialized* state since the kinetic energy in field space expressed through the  $\Pi_\phi \Pi_\phi$ -correlator is of the same order as the potential energy expressed through particle energy squared times the  $\phi\phi$ -correlator. A virialized state corresponds to a spherical blob in the phase-space diagram of the real scalar field. This state is more general than a thermal state since no relationship is assumed between phase-space occupancy of different field momenta. Thus, assuming the oscillatory correlators to be small initially, we can omit them from the dynamical description,

$$S_\psi[\Phi_G, \Psi_G] \stackrel{\text{virialized state}}{\approx} \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left[ i\psi^* \psi' + \psi^* \left( \frac{\hbar \Delta}{2ma} - \frac{ma}{\hbar} \Phi_G \right) \psi \right], \quad (5.20)$$

and the operator equation corresponding to this action reads (for classical gravitational fields),

$$i\partial_\eta \hat{\psi}(\eta, x^i) = - \left[ \frac{\hbar \Delta_x}{2ma} - \frac{ma}{\hbar} \Phi_G(\eta, x^i) \right] \hat{\psi}(\eta, x^i). \quad (5.21)$$

Choosing a coherent quantum state such that the connected piece of the two-point functions are negligible and classical fields are a good enough approximation leaves us with the dark matter description coined *fuzzy dark matter*. However, as we advocated in [111, 139], we do not have to restrict ourself to one-point functions since choosing a more-general state allows *a priori* for vorticity and anisotropy without additional course graining. For such a more general state with non-vanishing connected two-point functions, we can define a Wigner transformation (which corresponds to the spatially covariant one in [139] to zeroth order in gravitational perturbations),

$$f(\eta, X^i, p_i) \equiv \frac{1}{(2\pi\hbar)^3 \hbar} \int d^3r e^{-\frac{i}{\hbar} r^k p_k} \langle : \hat{\psi}(\eta, X^i + r^i/2) \hat{\psi}^\dagger(\eta, X^i - r^i/2) : \rangle, \quad (5.22)$$

where we made use of a local normal ordering prescription ":: $\cdot$ " that essentially subtracts the state-independent quantum contribution of the two-point function such that a gradient expansion in  $\hbar p_i \partial_{X^i}$  is possible (in other words, we have a hierarchy of scales  $ma \gg p \gg \hbar \partial_X$  together with  $ma \gg \mathcal{H}$ , for more details see [139]). The dynamical equation for the phase-space density  $f$  approaches the Vlasov equation for cold dark matter to leading order in the spatial gradient expansion

$$\left[ \frac{\partial}{\partial \eta} + \frac{p_k}{ma} \frac{\partial}{\partial X^k} - ma [1 + \mathcal{O}(\hbar^2)] \frac{\partial}{\partial X^k} \Phi_G(\eta, X^i) \frac{\partial}{\partial p_k} \right] f(\eta, X^i, p_i) = 0. \quad (5.23)$$

### 5.3 Integrating out gravitational fields

Instead of treating the gravitational perturbations as part of a classical (possibly stochastic) background metric, we treat them now as quantum fluctuations and integrate them out. This approach enables one to be more accurate in comparison to the one-loop semi-classical expansion and leaves only the true degrees of freedom in the description of the theory. The starting point for the gravitational part is the Einstein-Hilbert action in the ADM formulation

$$S_g = \int_{t_1}^{t_2} dt \int_{\Sigma_t} d^3x \left[ \Pi^{ij} \dot{\gamma}_{ij} - N \mathcal{H}_0^{(g)} - N^i \mathcal{H}_i^{(g)} \right] + \int_{t_1}^{t_2} dt \int_{\partial \Sigma_t} d^2x \mathcal{H}_B, \quad (5.24)$$

where the spatial boundary term  $\mathcal{H}_B$  specified in [142] is of no relevance for us and the Hamilton and momentum constraints of the gravitational sector are given by

$$\mathcal{H}_0^{(g)} = -\frac{M_P^2}{2\hbar} \gamma^{1/2} R^{(n-1)} + \frac{2\hbar}{M_P^2 \gamma^{1/2}} \left[ \Pi_{ij} \Pi^{ij} - \frac{\Pi^2}{2} \right], \quad (5.25)$$

$$\mathcal{H}_i^{(g)} = -2\gamma^{1/2(3)} \nabla^j \frac{\Pi_{ij}}{\gamma^{1/2}}, \quad (5.26)$$

which should not be confused with the conformal Hubble rate  $\mathcal{H}$ . In the gravitational Hamiltonian densities (5.26), we made use of the reduced Planck mass  $M_P$  and the canonical momentum  $\Pi^{ij}$  conjugate to the spatial metric  $\gamma_{ij}$ . We also denoted the trace of the canonical momentum as  $\Pi = \gamma_{ij} \Pi^{ij}$  and introduced the covariant derivative  ${}^{(3)}\nabla$  on spatial sections. As a first step to a non-relativistic limit of gravitating matter in an expanding universe, we will approximate the gravitational action (5.24) as in the semi-classical case with scalar perturbations in the longitudinal gauge. In addition to the decomposition of lapse, shift and spatial metric in (5.2), we also need to compose the canonical momentum of the spatial metric which we do as follows,

$$\Pi^{ij} = \delta^{ij} a^{-2} \Pi_a \left( 1 + \frac{1}{2} \Pi_\Psi \right). \quad (5.27)$$

A few comments on this split into a homogeneous background  $\bar{N}, a, \Pi_a$  and the path-integral perturbations  $\Phi_G, \Psi_G, \Pi_\Psi$  are in order. The obvious difference to the semi-classical analysis lies in the fact that we are treating inhomogeneous perturbations not any more as part of the classical (external) background which allows one to go beyond semi-classical one-loop approximation and include *in principle* quantum effects. This, however, does *not* mean that these perturbations *necessarily* correspond to quantum-sized effects. Whether such effects are important depends on the initial conditions: so are vacuum fluctuations the essential ingredient for inflationary models, whereas they are in most scenarios not at all for non-relativistic set-ups with a highly populated state ("many particles"). Let us also mention some boundary conditions of the perturbations  $\Phi_G, \Psi_G, \Pi_\Psi$ . We will assume that a well chosen background will keep any zero-mode fluctuations negligible such that the perturbations  $\Phi_G, \Psi_G, \Pi_\Psi$  decay at spatial infinity at least as  $1/r$ . For the same reason we will ignore the

boundary term in (5.24). Having said this, we will already make a choice for the background field  $\Pi_a$  such that it evolves according to the background equations of motion

$$\Pi_a = -\frac{M_P^2}{\hbar} a^2 \mathcal{H}. \quad (5.28)$$

After these remarks we expand the gravitational action (5.24) in conformal time for longitudinal scalar perturbations up to quadratic order, drop the zero order contribution  $\bar{S}_g$  from the gravitational part and add the matter action (5.17),

$$\begin{aligned} S[\Phi_G, \Psi_G, \Pi_\Psi, \psi] &\equiv S_\psi[\Phi_G, \Psi_G, \psi] + S_g[\Phi_G, \Psi_G, \Pi_\Psi] - \bar{S}_g \\ &\approx \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left\{ i\psi^* \psi' + \psi^* \left( \frac{\hbar \Delta}{2ma} - \frac{ma}{\hbar} \Phi_G \right) \psi + 3 \frac{ma}{\hbar} \Psi_G \text{Re}(\mathcal{E}^2 \psi^2) \right. \\ &\quad \left. - \mathcal{H} \left( \frac{1}{2} - \Phi_G - 3\Psi_G \right) \text{Im}(\mathcal{E}^2 \psi^2) \right\} \\ &\quad + \frac{M_P^2}{2\hbar} \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left\{ -6a^2(\mathcal{H}^2 + 2\mathcal{H}') \Psi_G - 2a^2 \Psi_G \Delta \Psi_G \right. \\ &\quad + 2a^2 \Phi_G [2\Delta \Psi_G + 6\mathcal{H}' + \mathcal{H}^2(1 + \Pi_\Psi)] \\ &\quad + 6a^2 \mathcal{H} \Pi_\Psi (\mathcal{H} \Psi_G + \Psi_G') + \frac{3}{2} a^2 \mathcal{H}^2 \Pi_\Psi^2 \\ &\quad \left. - 12a^2(2\mathcal{H}^2 - \mathcal{H}') \Psi_G^2 \right\}. \end{aligned} \quad (5.29)$$

We make the important remark that we did not expand the matter field  $\psi$  around a background value. The main reason why we do this lies in the observation that the perturbative expansion in (5.29) is valid if we supply the matter fields with appropriate boundary which are more general than a spatially homogeneous expectation value. We will shortly come back to this issue.

If we now vary with respect to  $\Phi_G$ , we get the following constraint

$$\begin{aligned} a^2 \Delta \Psi_G + 3a^2 \mathcal{H}' \Psi_G + \frac{3}{2} a^2 \mathcal{H}^2 (1 + \Pi_\Psi) \\ - \frac{\hbar}{2M_P^2} \frac{ma}{\hbar} \psi^* \psi + \frac{\hbar}{2M_P^2} \mathcal{H} \text{Im}(\mathcal{E}^2 \psi^2) = 0, \end{aligned} \quad (5.30)$$

which means at the level of path integrals, that we generate a delta function by integrating over  $\Phi_G$ . We have

$$\Pi_\Psi = \frac{E_0(\psi)}{3a\mathcal{H}^2} - \frac{2}{3} \frac{\Delta \Psi_G}{\mathcal{H}^2} - \frac{2\mathcal{H}'}{\mathcal{H}^2} \Psi_G, \quad (5.31)$$

where we defined

$$E_0(\psi) \equiv \frac{\hbar}{M_P^2} \frac{m}{\hbar} \psi^* \psi - 3a\mathcal{H}^2 - \frac{\mathcal{H}}{a} \frac{\hbar}{M_P^2} \text{Im}(\mathcal{E}^2 \psi^2). \quad (5.32)$$



Let us also define

$$E_1(\psi) \equiv a\mathcal{H}^2 + 2a\mathcal{H}' - \frac{m}{\hbar} \frac{\hbar}{M_P^2} \text{Re}(\mathcal{E}^2\psi^2) - \frac{\mathcal{H}}{a} \frac{\hbar}{M_P^2} \text{Im}(\mathcal{E}^2\psi^2). \quad (5.33)$$

We are now in the position to integrate out the gravitational fields  $\Phi_G$  and  $\Psi_G$  by plugging the constraint equation (5.31) back into the action (5.29),

$$\begin{aligned} S[\Phi_G, \Psi_G, \Pi_\Psi, \psi] &\longrightarrow S[\Psi_G, \psi] \\ &= \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left[ i\psi^*\psi' + \psi^* \frac{\hbar\Delta}{2ma} \psi - \frac{1}{2} \mathcal{H} \text{Im}(\mathcal{E}^2\psi^2) \right] \\ &\quad + \frac{M_P^2}{2\hbar} \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left\{ \frac{2}{3\mathcal{H}^2} a^2 \Psi_G \Delta^2 \Psi_G - 2a^2 (1 - \mathcal{H}^{-2} \mathcal{H}') \Psi_G \Delta \Psi_G \right. \\ &\quad \left. - 2a^2 \Psi_G \Delta \Psi_G + \frac{E_0^2(\psi)}{6\mathcal{H}^2} - 6a^2 \mathcal{H} (4\mathcal{H}^3 - 2\mathcal{H}\mathcal{H}' - \mathcal{H}'') \Psi_G^2 \right. \\ &\quad \left. - a\mathcal{H}^{-2} \left[ \frac{2}{3} \Delta E_0(\psi) + 6\mathcal{H}^2 E_1(\psi) + 2\mathcal{H} E_0'(\psi) \right] \Psi_G \right\}. \end{aligned} \quad (5.34)$$

Since both,  $E_0$  and  $E_1$  multiply terms linear in the gravitational perturbations, their homogeneous limit will be related to the Einstein equations as we will see shortly. Varying the Hubble action (5.34) with respect to the gravitational potential  $\Psi_G$  yields the following constraint equation,

$$\begin{aligned} \frac{2}{3} a^2 \Delta^2 \Psi_G - 2a^2 (\mathcal{H}^2 - \mathcal{H}') \Delta \Psi_G - 6a^2 \mathcal{H} (4\mathcal{H}^3 - 2\mathcal{H}\mathcal{H}' - \mathcal{H}'') \Psi_G \\ - a \left[ \frac{1}{3} \Delta E_0(\psi) + 3\mathcal{H}^2 E_1(\psi) + \mathcal{H} E_0'(\psi) \right] = 0. \end{aligned} \quad (5.35)$$

If we want to integrate out the gravitational potential via the constraint equation (5.35), we have to invert the Laplace operator and assume that the quantities  $E_0(\psi)$ ,  $E_1(\psi)$  vanish at least as  $1/r$  at spatial infinity since we made the same assumptions for the gravitational perturbations. In other words, we have to impose

$$E_0^\infty(\psi) \equiv \lim_{|\vec{x}| \rightarrow \infty} E_0[\psi(\vec{x})] \stackrel{!}{=} 0, \quad (5.36)$$

and

$$E_1^\infty(\psi) \equiv \lim_{|\vec{x}| \rightarrow \infty} E_1[\psi(\vec{x})] \stackrel{!}{=} 0. \quad (5.37)$$

We were implicitly always dealing with path integrals in this derivation and remark that the conditions (5.36) and (5.37) are in fact operator equations which involve more than the zero mode of the field  $\psi$ . Subtracting the gravitational

background fields, we have<sup>1</sup>

$$\begin{aligned}\hat{\rho}_\infty &\equiv E_0^\infty(\hat{\psi}) + 3a\mathcal{H}^2 \\ &= \frac{\hbar}{M_P^2} \int d^3p \left[ \frac{m}{\hbar} : \hat{\psi}^\dagger(\vec{p})\hat{\psi}(-\vec{p}) : - \frac{\mathcal{H}}{a} \text{Im}(\mathcal{E}^2 : \hat{\psi}(\vec{p})\hat{\psi}(-\vec{p}) :) \right],\end{aligned}\quad (5.38)$$

$$\begin{aligned}\hat{P}_\infty &\equiv E_1^\infty(\hat{\psi}) - a\mathcal{H}^2 - 2a\mathcal{H}' \\ &= -\frac{\hbar}{M_P^2} \int d^3p \left[ \frac{m}{\hbar} \text{Re}(\mathcal{E}^2 : \hat{\psi}(\vec{p})\hat{\psi}(-\vec{p}) :) + \frac{\mathcal{H}}{a} \text{Im}(\mathcal{E}^2 : \hat{\psi}(\vec{p})\hat{\psi}(-\vec{p}) :) \right].\end{aligned}\quad (5.39)$$

Taking expectation value and inserting the conditions (5.36) and (5.37), we recover the semi-classical Einstein equations at spatial infinity,

$$3a\mathcal{H}^2 = \langle \hat{\rho}_\infty \rangle, \quad (5.40)$$

$$-a\mathcal{H}^2 - 2a\mathcal{H}' = \langle \hat{P}_\infty \rangle. \quad (5.41)$$

We realize that the operators  $\hat{\rho}_\infty$  and  $\hat{P}_\infty$  should not fluctuate around their expectation values. Rigorously speaking, only if even by small amounts, they of course do. However, in a more rigorous treatment, we would also have to include zero-mode fluctuations in the gravitational sector which we assumed to negligible from the very beginning. This then resolves the apparent inconsistency.

We can conclude that the boundary conditions (5.36) and (5.37) can be met if we adjust the background metric (which is a priori free to choose) to satisfy equations (5.40) and (5.41) which are determined by the two-point functions of the matter field  $\psi$  at spatial infinity. With these adjustments, we are in the position to integrate out the gravitational potential  $\Psi_G$  in the action (5.34) by completing the squares,

$$\begin{aligned}S[\Psi_G, \psi] \longrightarrow S[\psi] &= \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left[ i\psi^* \psi' + \psi^* \frac{\hbar \Delta}{2ma} \psi - \frac{1}{2} \mathcal{H} \text{Im}(\mathcal{E}^2 \psi^2) \right] \\ &+ \frac{M_P^2}{4\hbar} \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left\{ \frac{E_0^2(\psi)}{3\mathcal{H}^2} - \frac{3}{4\mathcal{H}^2} \left[ \frac{2}{3} \Delta E_0(\psi) + 6\mathcal{H}^2 E_1(\psi) + 2\mathcal{H} E_0'(\psi) \right] \right. \\ &\quad \left. \times \Delta_{\mathcal{H}}^{-2} \left[ \frac{2}{3} \Delta E_0(\psi) + 6\mathcal{H}^2 E_1(\psi) + 2\mathcal{H} E_0'(\psi) \right] \right\},\end{aligned}\quad (5.42)$$

where we introduced the operator

$$\Delta_{\mathcal{H}}^2 \equiv \Delta^2 - 3(\mathcal{H}^2 - \mathcal{H}')\Delta - 9(4\mathcal{H}^4 - 2\mathcal{H}^2\mathcal{H}' - \mathcal{H}\mathcal{H}''). \quad (5.43)$$

While equation (5.42) represents the sought-for action, for the purpose of this chapter, and to make progress, we focus on the sub-Hubble limit of action (5.42)

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<sup>1</sup>Note, that we decided to give here a simpler treatment than for example in [139] (cf. chapter 4), where we gave some remarks on the renormalization of coincident limit operator products in a similar set-up.

and introduce another perturbation parameter

$$\mathcal{O}\left(\frac{\mathcal{H}^2}{|\Delta|}, \frac{\mathcal{H}'}{|\Delta|}\right) = \varepsilon_{\text{H/k}} \ll 1. \quad (5.44)$$

We have

$$\Delta_{\mathcal{H}}^{-2} = \Delta^{-2} [1 + 3(\mathcal{H}^2 - \mathcal{H}')\Delta^{-1} + \mathcal{O}(\varepsilon_{\text{H/k}}^2)]. \quad (5.45)$$

We assume that the back reaction between super- and sub-Hubble modes is negligible and work to leading order in  $\varepsilon_{\text{H/k}}$ . Upon integration by parts we find

$$S[\psi] \approx S_\psi \equiv \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left[ i\psi^* \psi' + \psi^* \frac{\hbar \Delta}{2ma} \psi - \frac{1}{2} \mathcal{H} \text{Im}(\mathcal{E}^2 \psi^2) - \frac{M_P^2}{4\hbar} (E_0(\psi) + 6E_1(\psi)) \Delta^{-1} E_0(\psi) \right]. \quad (5.46)$$

Before we plug in the concrete expressions for  $E_0$  and  $E_1$ , let us for convenience rescale the fields as

$$\psi \rightarrow \hbar^{1/2} \psi, \quad (5.47)$$

such that the two-point function has the dimensions of a number density. We then define

$$\rho_0 \equiv 3a\mathcal{H}^2 \frac{M_P^2}{\hbar m}, \quad (5.48)$$

and find

$$\begin{aligned} S_\psi = \hbar \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \Big\{ & i\psi^* \psi' + \psi^* \frac{\hbar \Delta}{2ma} \psi - \frac{1}{2} \mathcal{H} \text{Im}(\mathcal{E}^2 \psi^2) \\ & - \frac{m^2}{4M_P^2} [\psi^* \psi - \rho_0 - \frac{\hbar \mathcal{H}}{ma} \text{Im}(\mathcal{E}^2 \psi^2)] \Delta^{-1} [\psi^* \psi - \rho_0 - \frac{\hbar \mathcal{H}}{ma} \text{Im}(\mathcal{E}^2 \psi^2)] \\ & - \frac{m^2}{2M_P^2} \left[ \frac{1}{\mathcal{H}} \frac{d\rho_0}{d\eta} - 3\text{Re}(\mathcal{E}^2 \psi^2) - 3\frac{\hbar \mathcal{H}}{ma} \text{Im}(\mathcal{E}^2 \psi^2) \right] \\ & \times \Delta^{-1} [\psi^* \psi - \rho_0 - \frac{\hbar \mathcal{H}}{ma} \text{Im}(\mathcal{E}^2 \psi^2)] \Big\}. \end{aligned} \quad (5.49)$$

The action (5.49) is one of the principal results of this work and it serves as the starting point for a more general discussion of scalar field cold dark matter since it makes less assumptions about the underlying state, we only assumed that its momenta are mainly distributed in a non-relativistic but also sub-Hubble window after the background contributions at spatial infinity have been subtracted. Let us identify some future lines of research. By starting from (5.49) one can approach the theory in the 2PI formulation which captures the dynamics and interplay of the various contributions to the state, namely: the condensate  $\langle \psi \rangle$  ("fuzzy cold dark matter"), the two-point function  $\langle \hat{\psi} \hat{\psi}^\dagger \rangle$  corresponding to a virialized state ("particle cold dark matter" plus field-theoretic corrections) and squeezed two-point functions  $\langle \hat{\psi} \hat{\psi} \rangle$ ,  $\langle \hat{\psi}^\dagger \hat{\psi}^\dagger \rangle$ . Assuming mostly fuzzy cold dark matter, one can study its back reaction on particle dark matter and vice versa. Moreover, a field-theoretic description of cold dark matter can also lead to new

insights on how dark matter behaves on different scales and, due to this reformulation, hopefully even to new techniques on how to tackle non-linear evolution on large scales.

## 5.4 2PI formulation for a virialized state

In order to make the relation between the field-theoretic and the particle picture more concrete, we will study for simplicity an non-squeezed state having no condensate which we call a virialized state. We postpone the more general case for the future. Since interaction terms couple the various state contributions, they cannot be consistently set to zero but they remain, however, small if we assume a large mass in comparison to the Hubble rate as one can see in (5.49),

$$|\langle \hat{\psi} \hat{\psi}^\dagger \rangle| \gg |\langle \hat{\psi} \hat{\psi} \rangle|, \quad |\langle \hat{\psi} \hat{\psi}^\dagger \rangle| \gg |\langle \hat{\psi}^\dagger \hat{\psi}^\dagger \rangle|, \quad \langle \psi \rangle \approx 0. \quad (5.50)$$

From the point of view of Lagrangians, non-vanishing condensates are natural when the scalar field couples linearly to external sources, the two-point function framework without condensate is more natural when the scalar field couples quadratically to external sources (such as in the theory of scalar electrodynamics). First of all, we note that the equations of motion for the scale factors (5.40) and (5.41) reduce to

$$\rho_0 \equiv 3a\mathcal{H}^2 \frac{M_P^2}{\hbar m} \approx \int d^3p \langle : \hat{\psi}^\dagger(\vec{p}) \hat{\psi}(-\vec{p}) : \rangle \approx \text{const}. \quad (5.51)$$

Thus, the scale factor has to evolve as in a matter dominated universe

$$a(\eta) = a_I \frac{\eta^2}{\eta_I^2}, \quad (5.52)$$

and we choose  $a_I = 1$ . Moreover, it will be convenient to define

$$\beta \equiv \frac{\hbar \eta_I^2}{2m} = \frac{6M_P^2}{m^2 \rho_0}. \quad (5.53)$$

Using these relations, the approximation (5.50) and writing out the inverse Laplace operator, we find that the action (5.49) reads,

$$S_\psi \approx \hbar \int_{\eta_1}^{\eta_2} d\eta \int_{\Sigma_\eta} d^3x \left\{ i\psi^* \psi' + \frac{\beta}{\eta^2} \psi^* \Delta \psi + \frac{3}{8\pi\beta\rho_0} \int d^3y \frac{[\psi^*(\vec{x})\psi(\vec{x}) - \rho_0][\psi^*(\vec{y})\psi(\vec{y}) - \rho_0]}{\|\vec{x} - \vec{y}\|} \right\}, \quad (5.54)$$

where we for simplicity suppressed the  $\eta$ -dependence. In the Schwinger-Keldysh formulation, we then have the following effective action truncated at two loops

with  $M_P^{-2} \propto \beta^{-1}$  being the loop counting parameter of gravity,

$$\begin{aligned} \Gamma[iG_{ij}^{cd}] = & \hbar \int d^4x \int d^4y \sum_{c,d=\pm} c \mathcal{D}_{cd}^{ij}(x, y) iG_{ji}^{dc}(x, y) - i \frac{\hbar}{2} \text{Tr} [\log (iG_{ij}^{cd})] \\ & - i \hbar \sum_{c=\pm} \frac{1}{8} \int d^4x_1 \dots d^4x_4 [iG_{12}^{cc}(x_1, x_2) iG_{12}^{cc}(x_3, x_4) + iG_{21}^{cc}(x_1, x_2) iG_{21}^{cc}(x_3, x_4) \\ & + 2iG_{12}^{cc}(x_1, x_2) iG_{21}^{cc}(x_3, x_4) + 2iG_{11}^{cc}(x_1, x_3) iG_{22}^{cc}(x_2, x_4)] \\ & \times [V_H^c(x_1, \dots, x_4) + V_E^c(x_1, \dots, x_4)], \end{aligned} \quad (5.55)$$

where we defined the (formally divergent) derivative operator

$$\begin{aligned} \mathcal{D}_{cd}^{ij} \equiv & \frac{\delta_{cd}}{2} \begin{bmatrix} 0 & -i\partial_\eta + \frac{\beta \Delta_x}{\eta^2} \\ i\partial_\eta + \frac{\beta \Delta_x}{\eta^2} & 0 \end{bmatrix} \delta(\eta - \eta') \delta^3(\vec{x} - \vec{y}) \\ & - \frac{3}{2\beta} \delta_{cd} [\Delta_x^{-1}(1)] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \delta(\eta - \eta') \delta^3(\vec{x} - \vec{y}), \end{aligned} \quad (5.56)$$

which acts on the four propagators

$$iG_{ij}^{++}(x, y) \equiv \begin{bmatrix} \langle T[\hat{\psi}(x) \hat{\psi}(y)] \rangle & \langle T[\hat{\psi}(x) \hat{\psi}^\dagger(y)] \rangle \\ \langle T[\hat{\psi}^\dagger(x) \hat{\psi}(y)] \rangle & \langle T[\hat{\psi}^\dagger(x) \hat{\psi}^\dagger(y)] \rangle \end{bmatrix}, \quad (5.57)$$

$$iG_{ij}^{--}(x, y) \equiv \begin{bmatrix} \langle \bar{T}[\hat{\psi}(x) \hat{\psi}(y)] \rangle & \langle \bar{T}[\hat{\psi}(x) \hat{\psi}^\dagger(y)] \rangle \\ \langle \bar{T}[\hat{\psi}^\dagger(x) \hat{\psi}(y)] \rangle & \langle \bar{T}[\hat{\psi}^\dagger(x) \hat{\psi}^\dagger(y)] \rangle \end{bmatrix}, \quad (5.58)$$

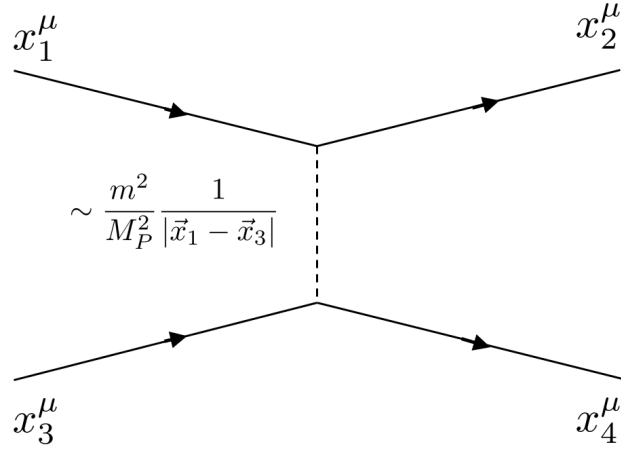
$$iG_{ij}^{-+}(x, y) \equiv \begin{bmatrix} \langle \hat{\psi}(x) \hat{\psi}(y) \rangle & \langle \hat{\psi}(x) \hat{\psi}^\dagger(y) \rangle \\ \langle \hat{\psi}^\dagger(x) \hat{\psi}(y) \rangle & \langle \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(y) \rangle \end{bmatrix}, \quad (5.59)$$

$$iG_{ij}^{+-}(x, y) \equiv \begin{bmatrix} \langle \hat{\psi}(y) \hat{\psi}(x) \rangle & \langle \hat{\psi}^\dagger(y) \hat{\psi}(x) \rangle \\ \langle \hat{\psi}(y) \hat{\psi}^\dagger(x) \rangle & \langle \hat{\psi}^\dagger(y) \hat{\psi}^\dagger(x) \rangle \end{bmatrix}, \quad (5.60)$$

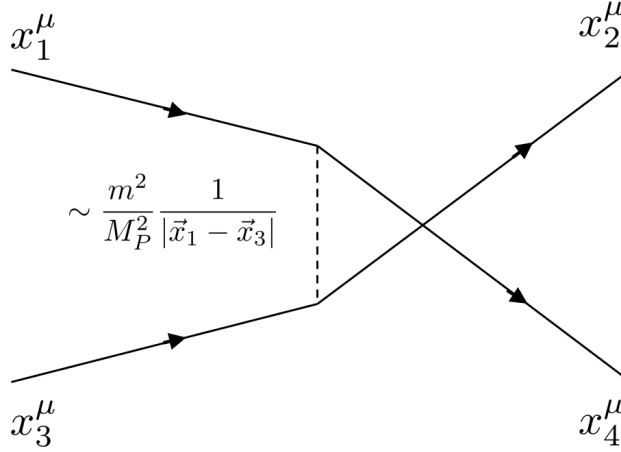
where  $T$  and  $\bar{T}$  denote time ordering and anti-time ordering, respectively. We will soon drop the squeezed state propagators to be consistent with (5.51). The divergent part of the derivative operator (5.56) should be thought of part of the interaction term since it removes homogeneous contributions of the spatially non-local coupling. The two vertices  $V_H$  and  $V_E$  we use in (5.55) are both symmetric under exchange of the first and last pair of coordinates and correspond to Hartree and exchange interaction (cf. figures 5.1a and 5.1b), respectively,

$$V_H^c(x_1, \dots, x_4) \equiv i \frac{3c}{4\pi\beta\rho_0} \frac{\delta^4(x_1 - x_2) \delta(\eta_1 - \eta_3) \delta^4(x_3 - x_4)}{\|\vec{x}_1 - \vec{x}_3\|}, \quad (5.61)$$

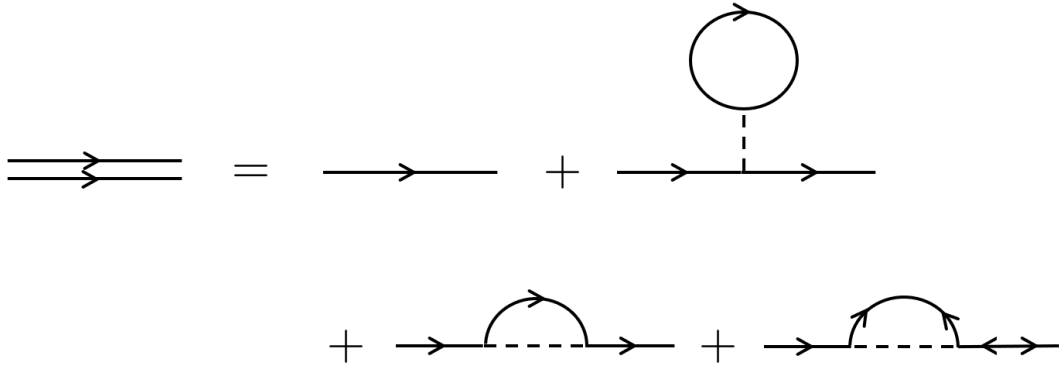
$$V_E^c(x_1, \dots, x_4) \equiv i \frac{3c}{4\pi\beta\rho_0} \frac{\delta^4(x_1 - x_4) \delta(\eta_1 - \eta_3) \delta^4(x_3 - x_2)}{\|\vec{x}_1 - \vec{x}_3\|}. \quad (5.62)$$



(A) The Hartree vertex is local in time but non-local in space. The separation between  $\vec{x}_1$  and  $\vec{x}_3$  (as well as between  $\vec{x}_2$  and  $\vec{x}_4$ ) is denoted by a dashed line.



(B) The exchange vertex is obtained from the Hartree vertex by exchanging the spatially separated coordinates  $\vec{x}_2$  and  $\vec{x}_4$ .



(C) The 2PI equation for the full two-point function  $G_{12}$  from the two-loop effective action (5.55). Dashed lines in the (spatial) loop denote spatial non-locality. Lines with two arrows denote the two-point functions  $G_{11}$  and  $G_{22}$  which are initially absent for non-squeezed states. For brevity we omitted three diagrams with identical topology but reversed flow in the loop.

FIGURE 5.1: Vertices for the action (5.54) and the 2PI equation for the two-point function  $G_{12}$  from the effective action (5.55).

Setting the variation of the 2PI effective action (5.55) with respect to  $G_{ij}^{cd}$  to zero and multiplying the resulting equation again by  $G_{ij}^{cd}$ , we obtain

$$\begin{aligned}
& \left[ \begin{array}{cc} 0 & -i\partial_\eta + \beta\Delta_x \eta^{-2} - 3\beta^{-1}[\Delta_x^{-1}(1)] \\ i\partial_\eta + \beta\Delta_x \eta^{-2} - 3\beta^{-1}[\Delta_x^{-1}(1)] & 0 \end{array} \right]^{ij} iG_{jk}^{cd}(\eta, \vec{x}, \eta', \vec{y}) \\
& - i\frac{c}{2\hbar} \int d^4 z_1 d^4 z_2 d^4 z_3 \left\{ V_{H+E}^c(x_1, z_3, z_1, z_2) (iG_{12}^{cc}(z_1, z_2) + iG_{21}^{cc}(z_1, z_2)) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{ij} \right. \\
& \quad \left. - \begin{bmatrix} V_{H+E}^c(x_1, z_1, z_3, z_2) iG_{22}^{cc}(z_1, z_2) & 0 \\ 0 & V_{H+E}^c(z_1, x_1, z_2, z_3) iG_{11}^{cc}(z_1, z_2) \end{bmatrix}^{ij} \right\} iG_{jk}^{cd}(z_3, \eta', \vec{y}) \\
& = ic\delta^{cd}\delta_k^i \delta(\eta - \eta') \delta^3(\vec{x} - \vec{y}). \quad (5.63)
\end{aligned}$$

In the equations for  $iG_{12}^{cd}$  and  $iG_{21}^{cd}$  it is consistent within our approximation scheme (5.50) to drop the squeezing contributions  $iG_{ii}^{cd}$ . We have the following equation for  $iG_{21}^{\mp\pm}(\eta, \vec{x}, \eta', \vec{y})$  if the time derivative acts on the first argument,

$$\begin{aligned}
& [i\partial_\eta - (\eta)^{-2}\beta\Delta_x] iG_{21}^{\mp\pm}(\eta, \vec{x}, \eta', \vec{y}) \\
& + \frac{3}{8\pi\beta\rho_0} \int d^3 z \frac{iG_{21}^{\mp\mp}(\eta, \vec{z}, \eta, \vec{z}) + iG_{12}^{\mp\mp}(\eta, \vec{z}, \eta, \vec{z}) - 2\rho_0 iG_{21}^{\mp\pm}(\eta, \vec{x}, \eta', \vec{y})}{\|\vec{x} - \vec{z}\|} \\
& + \frac{3}{8\pi\beta\rho_0} \int d^3 z \frac{iG_{21}^{\mp\mp}(\eta, \vec{z}, \eta, \vec{x}) + iG_{12}^{\mp\mp}(\eta, \vec{z}, \eta, \vec{x})}{\|\vec{x} - \vec{z}\|} iG_{21}^{\mp\pm}(\eta, \vec{z}, \eta', \vec{y}) \approx 0. \quad (5.64)
\end{aligned}$$

If the time derivative acts on the second argument of  $iG_{21}^{\mp\pm}(\eta, \vec{x}, \eta', \vec{y})$ , we have

$$\begin{aligned}
& [i\partial_{\eta'} + (\eta')^{-2}\beta\Delta_y] iG_{21}^{\mp\pm}(\eta, \vec{x}, \eta', \vec{y}) \\
& - \frac{3}{8\pi\beta\rho_0} \int d^3 z \frac{iG_{21}^{\pm\pm}(\eta', \vec{z}, \eta', \vec{z}) + iG_{12}^{\pm\pm}(\eta', \vec{z}, \eta', \vec{z}) - 2\rho_0 iG_{21}^{\mp\pm}(\eta, \vec{x}, \eta', \vec{y})}{\|\vec{y} - \vec{z}\|} \\
& - \frac{3}{8\pi\beta\rho_0} \int d^3 z \frac{iG_{21}^{\pm\pm}(\eta, \vec{y}, \eta, \vec{z}) + iG_{12}^{\pm\pm}(\eta, \vec{y}, \eta, \vec{z})}{\|\vec{y} - \vec{z}\|} iG_{21}^{\mp\pm}(\eta, \vec{x}, \eta', \vec{z}) \approx 0. \quad (5.65)
\end{aligned}$$

In order to arrive at a particle density in phase-space, we will make use of the following statistical (Hadamard) two-point function,

$$\begin{aligned}
F(\eta, \vec{x}, \eta', \vec{y}) \equiv F_{21}(\eta, \vec{x}, \eta', \vec{y}) &= \frac{1}{2} [iG_{21}^{+-}(\eta, \vec{x}, \eta', \vec{y}) + iG_{21}^{+ -}(\eta, \vec{x}, \eta', \vec{y})] \\
&= \frac{1}{2} \langle \{ \hat{\psi}^\dagger(\eta, \vec{x}), \hat{\psi}(\eta', \vec{y}) \} \rangle. \quad (5.66)
\end{aligned}$$

The spectral density

$$\begin{aligned}
i\rho_{21}^s(\eta, \vec{x}, \eta', \vec{y}) &= [iG_{21}^{+-}(\eta, \vec{x}, \eta', \vec{y}) - iG_{21}^{+ -}(\eta', \vec{x}, \eta, \vec{y})] \\
&= i \langle [\hat{\psi}^\dagger(\eta, \vec{x}), \hat{\psi}(\eta', \vec{y})] \rangle, \quad (5.67)
\end{aligned}$$

will drop out once we evaluate the coincident time limit. We use collective (average) and difference coordinates to define

$$F(\eta, \vec{X}, \vec{r}) \equiv F(\eta, \eta' = \eta, \vec{x} = \vec{X} + \vec{r}/2, \vec{y} = \vec{X} - \vec{r}/2). \quad (5.68)$$

Adding up the equations for  $G_{21}^{\pm\mp}$  in (5.64) and (5.65) we find in the coincident time limit,

$$\begin{aligned} & \left[ i\partial_\eta + 2\beta\eta^{-2} \frac{\partial}{\partial \vec{X}} \cdot \frac{\partial}{\partial \vec{r}} \right] F(\eta, \vec{X}, \vec{r}) \\ & + \frac{3}{8\pi\beta\rho_0} \int \frac{d^3z}{z} \left\{ 2[F(\eta, \vec{z} + \vec{X} + \vec{r}/2, 0) - F(\eta, \vec{z} + \vec{X} - \vec{r}/2, 0)] F(\eta, \vec{X}, \vec{r}) \right. \\ & + [F(\eta, \vec{X} + (\vec{r} + \vec{z})/2, \vec{z}) + F(\eta, \vec{X} + (\vec{r} - \vec{z})/2, -\vec{z})] F(\eta, \vec{X} + \vec{z}/2, \vec{r} + \vec{z}) \\ & \left. - [F(\eta, \vec{X} - (\vec{r} - \vec{z})/2, \vec{z}) + F(\eta, \vec{X} - (\vec{r} + \vec{z})/2, -\vec{z})] F(\eta, \vec{X} + \vec{z}/2, \vec{r} - \vec{z}) \right\} = 0. \end{aligned} \quad (5.69)$$

We see that the homogeneous and isotropic equation is solved by a function  $F_{\text{hom}}(r)$  which is constant in time and constant in the collective coordinate  $\vec{X}$ ,

$$F_{\text{hom}}(\eta, \vec{X}, \vec{r}) = F_{\text{hom}}(r) \quad \text{with} \quad F_{\text{hom}}(0) = \rho_0, \quad (5.70)$$

which matches the initial conditions at spatial infinity (5.51). Let us switch to momentum space and introduce the inhomogeneous Wigner transformation,

$$F(\eta, \vec{k}, \vec{p}) = \frac{1}{(2\pi\hbar)^6} \int d^3X e^{-\frac{i}{\hbar}\vec{k}\cdot\vec{X}} \int d^3r e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{r}} F(\eta, \vec{X}, \vec{r}). \quad (5.71)$$

We emphasize that we are counting both momenta, small scale momentum  $\vec{p}$  and large scale momentum  $\vec{k}$ , in units of energy. We then have

$$\begin{aligned} & [i\partial_\eta - 2(\hbar\tau)^{-2}\beta\vec{k} \cdot \vec{p}] F(\eta, \vec{k}, \vec{p}) + \frac{3\hbar^2}{2\beta\rho_0} \int d^3w \int d^3u F(\eta, \vec{w}, \vec{u}) \\ & \times \left[ \frac{F(\eta, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2)}{\|\vec{p} + \vec{u} + (\vec{k} - \vec{w})/2\|^2} - \frac{F(\eta, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2)}{\|\vec{p} + \vec{u} - (\vec{k} - \vec{w})/2\|^2} \right. \\ & + \frac{F(\eta, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2)}{\|\vec{p} - \vec{u} + (\vec{k} - \vec{w})/2\|^2} - \frac{F(\eta, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2)}{\|\vec{p} - \vec{u} - (\vec{k} - \vec{w})/2\|^2} \\ & \left. + 2w^{-2} \left( F(\eta, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2) - F(\eta, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2) \right) \right] = 0. \end{aligned} \quad (5.72)$$

Our next goal is to expand around a homogeneous Maxwellian distribution and see which differences we get (at least in the linear theory) in comparison to classical particle cold dark matter. It will turn out to be convenient if we rescale all momenta and times

$$\vec{p} \rightarrow \vec{p}\alpha^{1/2}, \quad \vec{k} \rightarrow \vec{k}\alpha^{1/2}, \quad \alpha \equiv mk_B T, \quad \eta \longrightarrow \eta_I \tau = \frac{\tau}{2\mathcal{H}_I}, \quad (5.73)$$



such that the quantities on the right-hand-side of (5.73) are dimensionless. The dimensionless time  $\tau$  is nothing but the square-root of the scale factor  $a$ . The parameter  $\alpha$  is the geometric mean between the particles mass  $m$  and temperature parameter  $k_B T$  with  $k_B$  being the Boltzmann constant. Thus, the parameter  $\alpha$  corresponds to the averaged particle moment  $\langle p^2 \rangle$  where the expectation value denotes here the integral against a particle distribution in momentum space which we choose to be a Maxwellian distribution. Moreover, it will be handy to define the parameter

$$\xi \equiv \frac{\alpha\beta}{\hbar^2\eta_I} = \frac{mk_B T}{\hbar^2\eta_I} \frac{\hbar\eta_I^2}{2m} = \frac{k_B T}{\hbar\mathcal{H}_I}. \quad (5.74)$$

We will see that the parameter  $\xi$  will decide on which time-scales the exchange interaction term can become important if we are working on scales  $k \ll p$ . Moreover, we rescale the coincident Hadamard function as

$$F \longrightarrow \alpha^{-3}\rho_0 F, \quad (5.75)$$

so that the  $p$ -integral over its inhomogeneous part yields the density contrast. We also assume further, that is only a function of the moduli  $k$  and  $p$  as well as its scalar product

$$F(\tau, \vec{k}, \vec{p}) = F(\tau, k, p, \mu), \quad \mu = \frac{\vec{p} \cdot \vec{k}}{pk}, \quad (5.76)$$

and expand it as<sup>2</sup>

$$F(\tau, k, p, \mu) = (2\pi)^{-3/2} \delta^3(\vec{k}) e^{-p^2/2} + \delta F(\tau, k, p, \mu). \quad (5.77)$$

---

<sup>2</sup>We note that the perturbations  $\delta F$  should in principle be multiplied by stochastic variables  $\hat{a}_{\vec{k}}$  such that the perturbations of the two-point functions  $F(\eta, \vec{k}, \vec{p})$  are stochastic variables in a cosmological context.

We have

$$\begin{aligned}
& [i\partial_\tau - 2\tau^{-2}\xi kp\mu] \delta F(\tau, k, p, \mu) + \frac{6}{\xi k^2} (2\pi)^{-3/2} \exp\left[-\frac{p^2}{2} - \frac{k^2}{8}\right] \sinh\left[\frac{pk\mu}{2}\right] \\
& \quad \times \int d^3u \left[1 + \frac{k^2}{2\|\vec{p} + \vec{u}\|^2} + \frac{k^2}{2\|\vec{p} - \vec{u}\|^2}\right] \delta F(\tau, k, u, \mu_{k,u}) \\
& + \frac{6\delta F(\tau, k, p, \mu)}{\xi} \left[ \frac{2^{1/2} \text{DawsonF}(2^{-1/2}\|\vec{p} - \vec{k}/2\|)}{\|\vec{p} - \vec{k}/2\|} - (\vec{k} \rightarrow -\vec{k}) \right] \\
& + \frac{3}{2\xi} \int d^3w \int d^3u \delta F(\tau, w, u, \mu_{w,u}) \tag{5.78}
\end{aligned}$$

$$\begin{aligned}
& \times \left[ \frac{\delta F(\tau, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2)}{\|\vec{p} + \vec{u} + (\vec{k} - \vec{w})/2\|^2} - \frac{\delta F(\tau, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2)}{\|\vec{p} + \vec{u} - (\vec{k} - \vec{w})/2\|^2} \right. \\
& \quad + \frac{\delta F(\tau, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2)}{\|\vec{p} - \vec{u} + (\vec{k} - \vec{w})/2\|^2} - \frac{\delta F(\tau, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2)}{\|\vec{p} - \vec{u} - (\vec{k} - \vec{w})/2\|^2} \\
& \quad \left. + 2w^{-2} \left( \delta F(\tau, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2) - \delta F(\tau, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2) \right) \right] = 0, \tag{5.79}
\end{aligned}$$

where we made use of the Dawson integral

$$\text{DawsonF}(z) = e^{-z^2} \int_0^z e^{y^2} dy = z e^{-z^2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, z^2\right), \tag{5.80}$$

and  ${}_1F_1$  is the confluent hypergeometric function of the first kind. Let us define

$$F(\tau, k) \equiv \int d^3u \delta F(\tau, k, u, \mu_{k,u}), \tag{5.81}$$

and contrast equation (5.79) with the perturbed Vlasov description in the truncated equation (4.153). We realize that the terms

$$\begin{aligned}
\mathcal{V}[\delta F] & \equiv [i\partial_\tau - 2\tau^{-2}\xi kp\mu] \delta F(\tau, k, p, \mu) \\
& + \frac{6}{\xi k^2} (2\pi)^{-3/2} \exp\left[-\frac{p^2}{2} - \frac{k^2}{8}\right] \sinh\left[\frac{pk\mu}{2}\right] \delta F(\tau, k) \\
& + \frac{3}{\xi} \int d^3w \delta F(\tau, w) w^{-2} \left( \delta F(\tau, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2) - \delta F(\tau, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2) \right), \tag{5.82}
\end{aligned}$$

should correspond to the full non-linear Vlasov equation if we work in the limit where particle momenta are much bigger than large-scale momenta ( $p \sim 1 \gg k$ ) which is amply satisfied for a cold dark matter scenario with galactic scales around  $\sim \text{Mpc} \gg \alpha^{-1/2}$ . The "sinh" term may be expanded in this case and yields the same result as a partial derivative in momentum space acting on a Maxwellian background density. The second difference is the non-linear term in (5.82) which, however, may be converted into a partial derivative for  $k/p \ll 1$  as it appears in the Vlasov equation. In addition to the Vlasov-like terms in (5.79), we note the appearance of exchange interaction corrections which are

of order  $\sim k^2/p^2$ ,

$$\begin{aligned} \mathcal{E}[\delta F] \equiv & \frac{3}{\xi k^2} (2\pi)^{-3/2} \exp \left[ -\frac{p^2}{2} - \frac{k^2}{8} \right] \sinh \left[ \frac{pk\mu}{2} \right] \\ & \times \int d^3u \left[ \frac{k^2}{\|\vec{p} + \vec{u}\|^2} + \frac{k^2}{\|\vec{p} - \vec{u}\|^2} \right] \delta F(\tau, k, u, \mu_{k,u}) \\ & + \frac{3}{2\xi} \int d^3w \int d^3u \delta F(\tau, w, u, \mu_{w,u}) \\ & \times \left[ \frac{\delta F(\tau, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2)}{\|\vec{p} + \vec{u} + (\vec{k} - \vec{w})/2\|^2} - \frac{\delta F(\tau, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2)}{\|\vec{p} + \vec{u} - (\vec{k} - \vec{w})/2\|^2} \right. \\ & \left. + \frac{\delta F(\tau, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2)}{\|\vec{p} - \vec{u} + (\vec{k} - \vec{w})/2\|^2} - \frac{\delta F(\tau, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2)}{\|\vec{p} - \vec{u} - (\vec{k} - \vec{w})/2\|^2} \right]. \end{aligned} \quad (5.83)$$

Since, we are for the moment interested in scales larger or at most comparable to galactic scales, we will assume from now on the limit  $k \ll 1 \sim p$  and postpone the study of this type of corrections for future research, where we expect small scale effects similar to ones for fuzzy dark matter as described for example in [28].

There is another term originating from linearly expanding the exchange interaction term (5.62),

$$\begin{aligned} \mathcal{F}[\delta F] \equiv & \frac{6}{\xi} 2^{1/2} \delta F(\tau, k, p, \mu) \left\{ \frac{\text{DawsonF} \left[ 2^{-1/2} \|\vec{p} - \vec{k}/2\| \right]}{\|\vec{p} - \vec{k}/2\|} \right. \\ & \left. - \frac{\text{DawsonF} \left[ 2^{-1/2} \|\vec{p} + \vec{k}/2\| \right]}{\|\vec{p} + \vec{k}/2\|} \right\}. \end{aligned} \quad (5.84)$$

As we will discuss shortly, it gives rise to late-time corrections and is *not*  $k^2/p^2$  suppressed in contrast to all other terms originating from the exchange interaction.

We would now like to proceed studying (5.79), however, without taking moments in  $p$  to avoid arguing about the smallness of higher moments. Therefore, it is convenient to convert (5.79) into an integral equation for the density contrast by defining

$$\chi(\tau, k, p, \mu) \equiv 2\tau^{-1} \xi k p \mu + 6\tau \xi^{-1} \left[ \frac{2^{1/2} \text{DawsonF}(2^{-1/2} \|\vec{p} - \vec{k}/2\|)}{\|\vec{p} - \vec{k}/2\|} - (\vec{k} \rightarrow -\vec{k}) \right], \quad (5.85)$$

with the series expansion in  $k \ll 1 \sim p$ ,

$$\chi(\tau, k, p, \mu) = 2\tau^{-1} \xi k p \mu + \frac{6\mu k \tau}{\xi p^2} \left[ 2^{1/2} (1 + p^2) \text{DawsonF}(2^{-1/2} p) - p \right] + \mathcal{O}(k^3). \quad (5.86)$$

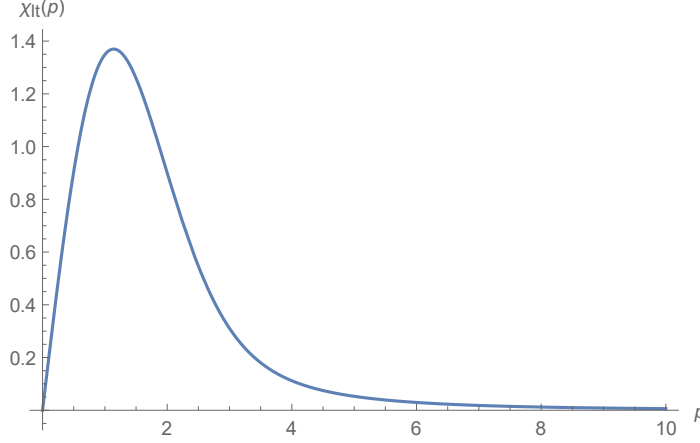


FIGURE 5.2: The function  $\chi_{\text{lt}}$  defined in (5.87) dominating the late-time behaviour of the phase factor (5.85) in the  $k \ll 1$  expansion.

We note that the  $p$ -dependent factor in the expansion of the late-time term (5.86),

$$\chi_{\text{lt}}(p) \equiv \frac{3}{p^2} \left[ 2^{1/2}(1+p^2)\text{DawsonF}(2^{-1/2}p) - p \right], \quad (5.87)$$

is of order 1 for  $p \sim 1$  (cf. figure 5.2) and thus, phase corrections due to the exchange interaction term become only important at late times if we work in the limit  $k \ll 1$ . The transition time from which on the late time phase factor dominates is given by

$$\eta_{\text{trans}} \equiv \xi \eta_I = \frac{k_B T}{\hbar \mathcal{H}_I} \eta_I, \quad (5.88)$$

which is a very large time even on cosmological scales unless the particle temperature is extremely small. We now make use of the phase definition (5.85) and integrate equation (5.79) in time. As just discussed below (5.79), we neglect the  $p^2/k^2$  corrections due to the exchange interaction terms and are left with

$$\begin{aligned} i\delta F(\tau, k) \approx & i \int d^3p \exp[i\chi(\tau, k, p, \mu) - i\chi(\tau_I, k, p, \mu)] \delta F_I(k, p, \mu) \\ & - \int d^3p \exp[i\chi(\tau, k, p, \mu)] \int_1^\tau d\bar{\tau} \exp[-i\chi(\bar{\tau}, k, p, \mu)] \\ & \times \left\{ \frac{6}{\xi k^2} (2\pi)^{-3/2} \exp\left[-\frac{p^2}{2} - \frac{k^2}{8}\right] \sinh\left[\frac{pk\mu}{2}\right] \delta F(\bar{\tau}, k) \right. \\ & \left. + \int d^3w \frac{3\delta F(\bar{\tau}, w)}{\xi w^2} \left( \delta F(\bar{\tau}, \vec{k} - \vec{w}, \vec{p} - \vec{w}/2) - \delta F(\bar{\tau}, \vec{k} - \vec{w}, \vec{p} + \vec{w}/2) \right) \right\}. \end{aligned} \quad (5.89)$$

**Case  $\xi^2 \gg a(\eta)$ , Hartree interaction phase dominates** First, we study the integral equation (5.89) for dimensionless times  $\tau$  which are much smaller than the parameter  $\xi$  (despite this, they can still correspond to galactic time

scales  $\eta_{\text{final}} \sim 10^{5-10} \eta_I$ ),

$$\xi = \frac{k_B T}{\hbar \mathcal{H}_I} \gg \tau = \sqrt{a(\eta)}. \quad (5.90)$$

We can then write equation (5.89) as

$$\begin{aligned} \delta F(\tau, k) \approx & \int d^3 p \exp[2i(\tau^{-1} - 1)\xi k p \mu] \delta F_I(k, p, \mu) \\ & + \frac{6}{k^2 \xi} \int_1^\tau d\bar{\tau} \exp\left[-\frac{2k^2 \xi^2 (\tau - \bar{\tau})^2}{\tau^2 \bar{\tau}^2}\right] \sin\left[\frac{k^2 \xi (\tau - \bar{\tau})}{\tau \bar{\tau}}\right] \delta F(\bar{\tau}, k) \\ & + \frac{6}{\xi} \int d^3 p \int_1^\tau d\bar{\tau} \exp[2i(\tau^{-1} - \bar{\tau}^{-1})\xi k p \mu] \\ & \times \int d^3 w \frac{\delta F(\bar{\tau}, w)}{w^2} \sin\left[\frac{k w \mu_{kw} \xi (\tau - \bar{\tau})}{\tau \bar{\tau}}\right] \delta F(\bar{\tau}, \vec{k} - \vec{w}, \vec{p}). \end{aligned} \quad (5.91)$$

We see two scales in expression (5.91). The first important scale in expression (5.91) appears in the exponential for the linear term. The question, whether this exponential is important may be answered by referring to the scale

$$k_\xi(\eta) \equiv \frac{\alpha^{1/2} \tau}{\xi} = k_H(\eta) a \left( \frac{m}{k_B T} \right)^{1/2}, \quad (5.92)$$

where we introduced the Hubble scale

$$k_H(\eta) \equiv \hbar a(\eta) H(\eta) = \hbar \mathcal{H}(\eta). \quad (5.93)$$

Relative to sub-Hubble scales, the scale  $k_\xi$  is in reach for light and warm particles. We note that the scale  $k_\xi$  results from the free-streaming term in (5.79) and is also present in the Vlasov equation. The second scale in (5.91) appears in the oscillatory terms,

$$k_{\text{osc}}(\eta) \equiv \sqrt{m a(\eta) k_H(\eta)} = \alpha^{1/2} \left( \frac{\tau}{\xi} \right)^{1/2} = k_\xi(\eta) \left( \frac{\xi}{\tau} \right)^{1/2}. \quad (5.94)$$

The scale  $k_{\text{osc}}$  is the geometric mean between the scale of relativistic effects and the sub-Hubble scale

$$k_{\text{rel}} \gtrsim k_{\text{osc}} \gtrsim k_H, \quad (5.95)$$

and we suspect that structure formation is inhibited at these scales due to oscillatory solutions. However, since we are working in the limit  $\xi \gg \tau$  in this paragraph, we have

$$k_{\text{osc}} \gg k_\xi, \quad (5.96)$$

such that the exponential suppression in the linear term in (5.91) begins before oscillatory contributions become important. It is of course tempting to study the full  $k$ -dependence in the linearized version of equation (5.91). However, we are not aware of a solution in terms of the exponential and sinusoidal kernel

$$K[k, \tau, \bar{\tau}] \equiv \exp\left[-\frac{2k^2 \xi^2 (\tau - \bar{\tau})^2}{\tau^2 \bar{\tau}^2}\right] \sin\left[\frac{k^2 \xi (\tau - \bar{\tau})}{\tau \bar{\tau}}\right], \quad (5.97)$$

and leave it for future research. For cold dark matter it is now a reasonable scenario to assume<sup>3</sup>

$$k \ll k_\xi(\eta_I) \ll k_\xi(\eta), \quad (5.98)$$

in which case

$$\int d^3p \exp[2i(\tau^{-1} - 1)\xi k p \mu] \delta F_I(k, p, \mu) \longrightarrow \delta F_I(k) \quad \text{for } \tau \gg 1, \quad k \ll k_\xi. \quad (5.99)$$

Moreover, the exponential suppression in (5.91) is negligible in this scenario

$$\exp\left[-\frac{2k^2\xi^2(\tau - \bar{\tau})^2}{\tau^2\bar{\tau}^2}\right] \longrightarrow 1 \quad \text{for } k \ll k_\xi. \quad (5.100)$$

Since we are working out the case  $\xi \gg \tau$  in this paragraph, the sine can also be expanded around zero. We then have

$$\delta F_{\text{lin}}(\tau, k) \approx \delta F_I(k) + 6 \int_1^\tau d\bar{\tau} \frac{(\tau - \bar{\tau})}{\tau \bar{\tau}} \delta F_{\text{lin}}(\bar{\tau}, k), \quad (5.101)$$

which is solved at late times by the standard linear cold dark matter evolution

$$\delta F_{\text{lin}}(\eta, k) \longrightarrow \frac{3}{5} a(\eta) \delta F_I(k), \quad \text{for } k \ll k_{\text{osc}}(\eta). \quad (5.102)$$

Although the form (5.91) differs slightly from the Vlasov description (4.153), the study of non-linear evolution is still highly non-trivial and we leave the discussion of approximations and perturbative expansions for the future. Let us now discuss the other limit that brings the exchange interaction term into play.

**Case  $\xi^2 \ll a(\eta)$ , exchange interaction phase dominates** For this case, we approximate the phase-factor by (5.86) and drop the free-streaming contributions  $\sim \tau^{-1}$ . We will be able to say something about the linear evolution. Unfortunately, we are not in the position to perform the full momentum integral for the linear term as we could in the case  $\xi^2 \gg a(\eta)$ , which is why we have to restrict ourselves to

$$k \ll k_{\text{osc}}. \quad (5.103)$$

The linearized integral equation (5.89) then reads

$$\delta F_{\text{lin}}(\tau, k) \approx \delta F_I(\tau, k) - \frac{3}{\xi^2} \int_1^\tau d\bar{\tau} (\tau - \bar{\tau}) \delta F_{\text{lin}}(\bar{\tau}, k), \quad (5.104)$$

and is quickly solved in terms of the scale factor by

$$\delta F_{\text{lin}}(\eta, k) = \delta F_I(k) \cos\left[\frac{\sqrt{3}}{\xi}(\sqrt{a(\eta)} - 1)\right] \xrightarrow{a \gg a_I} \delta F_I(k) \cos\left[\frac{\sqrt{3}a(\eta)}{\xi}\right]. \quad (5.105)$$

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<sup>3</sup>For the cold dark matter paradigm we have the limit  $m/(k_B T) \gtrsim 10^{12}$  where WIMPs are far away from this limit with  $m/(k_B T) \gtrsim 10^{24}$  [143].

We conclude that there is no linear growth for a small enough parameter  $\xi$  such that at late times  $\xi \ll \tau$  (on scales  $k \ll k_{\text{osc}}$ ). Thus, the effect of the exchange interaction term is to hinder the growth of linear perturbations for large distances at late time where late times are defined to be greater than the transition  $\eta_{\text{trans}}$  given in (5.102) which depends on the temperature of cold dark matter. If we demand as a rough estimate that the observed power spectrum for linear modes does not oscillate around a constant value, field-theoretic corrections yield a lower bound on the temperature of cold dark matter,

$$k_B T \gtrsim \frac{a_{\text{today}}}{a_I} \frac{H_0}{\hbar} \approx 10^{-38} \text{GeV} , \quad (5.106)$$

where  $a_I$  is the scale factor at the beginning of the matter dominated epoch and  $H_0$  the Hubble rate today.

## 5.5 Conclusion and outlook

We present a new formalism for deriving the non-relativistic limits of a covariant QFT tree-level action in which a real scalar field couples minimally to gravity. The key ingredients are to introduce an approximate diagonal field representation (5.8) for cosmological space-times and integrate out the gravitational constraint fields in a perturbative expansion. We focus on scalar perturbations in the longitudinal gauge but the formalism can be straightforwardly adapted to include also tensor and vector gravitational perturbations and even modified gravitational theories to study their non-relativistic limits in a controlled way. We derive a general non-relativistic, non-local action (5.49) for gravitational interacting matter on sub-Hubble scales that makes no reference to a particular state in the sense that it can contain a condensate, as well as squeezed contributions (all correlators in (5.19)).

Let us summarize the assumptions and approximations which are needed to arrive at the final action (5.49). First of all, we neglect vector and tensor perturbations in the metric and linearize around a homogeneous, spatially flat FLRW-metric with scalar perturbations in the longitudinal gauge with gravitational potentials  $\Phi_G$  and  $\Psi_G$ ,

$$\mathcal{O}(\Phi_G, \Psi_G) = \varepsilon_g^2 \ll 1. \quad (5.107)$$

Secondly, by expanding around these potentials, we assume that gravitational boundary terms and zero-mode fluctuations around the classical and a priori free-to-choose FRLW-metric to be negligible. However, for a consistent perturbative expansion of the action we ultimately pick the classical FRLW-metric in such a way that the boundary conditions (5.40) and (5.41), which are nothing but the homogeneous semi-classical Einstein equations, are satisfied. Thirdly, we are working in a non-relativistic limit with

$$\mathcal{O}\left(\frac{\hbar\|\nabla\|}{m}\right) = \varepsilon_{\text{nr}} \ll 1. \quad (5.108)$$

Spatial derivatives  $\nabla = \nabla_{\vec{x}}$  acting on matter fields  $\psi(\vec{x})$  will be mapped on particle momenta  $\vec{p}$  and long-distance gradients  $\nabla_{\vec{X}} \sim \hbar\vec{k}$  once two-point functions of fields such as  $\langle\psi^\dagger(\eta, \vec{x})\psi(\eta, \vec{y})\rangle$  are mapped to a particle phase-space density  $f(\eta, \vec{p}, \vec{X})$ . Thus, assuming  $\hbar\|\nabla\| \ll m$  corresponds to assuming physical momenta  $p$  and inverse distance scales  $L^{-1} \sim k$  of the underlying physical problem to be much smaller than the scale set by the mass  $m$ . Moreover, we consider the case where the mass  $m$  is much bigger than the Hubble rate or its logarithmic derivative

$$\mathcal{O}\left(\frac{\hbar\mathcal{H}}{ma}, \frac{\hbar\mathcal{H}'}{\mathcal{H}ma}\right) = \varepsilon_{\text{H/m}} \ll 1. \quad (5.109)$$



Finally, we focus on the sub-Hubble limit relevant for structure formation and introduce the perturbation parameter

$$\mathcal{O}\left(\frac{\mathcal{H}^2}{\|\Delta\|}, \frac{\mathcal{H}'}{\|\Delta\|}\right) = \varepsilon_{\text{H}/k} \ll 1. \quad (5.110)$$

We study the derived action (5.49) for a non-squeezed state without condensate contributions and derive the corresponding 2PI two-loop effective action. Because this two-loop action contains only quartic interactions it is non-dissipative, which allows us to get closure for the dynamics of the coincident two-point functions. The resulting equations have a form of classical kinetic equations. By performing an inhomogeneous Wigner transformation, we derive the dynamics of the dark matter phase space density (5.79) and compare it to the standard Vlasov equation describing particle cold dark matter. For large galactic scales and masses, we recover a description close to particle cold dark matter which is confirmed by the linear evolution (5.102). This is, however, the case only if the particles temperature is much bigger than the Hubble scale, since otherwise the exchange interaction (absent in the Vlasov description) becomes important at late times. We also identify a scale  $k_{\text{osc}}$  in (5.94) between the relativistic and the sub-Hubble scale at which we suspect density perturbations to deviate significantly from the standard CDM evolution. These results were derived in the limit where particle momenta  $p$  are much bigger than the large scale momentum  $k$  (or in other words where the distances of the system under study are much bigger than de Broglie wavelength). However, the general formula (5.79) can be used to study also the case  $k \sim p$  where we expect new effects due to the exchange interaction term (5.62) to kick in.

Another route of investigation is to start from the general non-relativistic action (5.49) we derive in section 5.3 and to study the interplay between different state contribution, i.e. the influence of particle dark matter on fuzzy dark matter and vice versa.



## Chapter 6

# Summary and outlook

Large-scale structure formation is arguably one of the most important problems of modern cosmology. The standard treatment is based on a point-particle picture and the corresponding Vlasov equation. Here, we take a field-theoretic approach to this problem and identify how field correlators in curved-space time are related to classical particle distributions in phase-space. We obtain the Vlasov equation in a well-defined limit together with field-theoretic corrections which can be studied in a controlled way. We also take a fresh look at the initial conditions for the cosmic microwave background and large-scale structure formation.

In chapter 2 we present a two-field inflationary model in which the inflaton couples biquadratically to a light spectator field. The spectator induces a rapid production of entropy which significantly alters the initial conditions for the cosmological evolution at the end of inflation by providing large, stochastically independent momentum fluctuations of the gauge invariant curvature perturbation. Still, if interactions are switched off during the radiation era such that the evolution of cosmological perturbations on super-Hubble scales can be well approximated by the corresponding linear evolution, the momentum fluctuations decay and their effect is too small to leave any observable imprint on the cosmic microwave background or large-scale structures. A natural proposal for future research is thus to study a scenario in which the post-inflationary decay is hindered, for example by keeping the inflaton-spectator interactions active during the early parts of the radiation era.

Chapters 3 to 5 are devoted to developing a field-theoretic approach to large-scale structure formation which is based on an action principle for a real scalar field as the dark matter candidate coupled to gravity. In chapter 3 we lay the foundation for such a description by deriving the Vlasov equation for particle cold dark matter from a Lagrangian description in which a massive real scalar field couples minimally to gravity with Newtonian perturbations. Part of this derivation is to identify a suitable combination of equal-time two-point functions of the real scalar field that can represent an on-shell particle density in phase-space. We arrive at a system of partial differential equations that couple the classical particle phase-space density to squeezed state contributions and, moreover, contain relativistic corrections.

These findings are generalized and made spatially covariant in chapter 4 by including arbitrary metrics, non-minimal coupling to gravity as well as quartic

self-interactions to one-loop accuracy. We apply these results in the context of cosmological perturbation theory and discuss how the standard particle cold dark matter dynamics gets field-theoretic corrections. Furthermore, we derive in a spatially covariant manner how hydrodynamic variables are related to equal-time two-point functions of the real scalar field. Both, the Newtonian treatment in chapter 3 and the covariant generalization in chapter 4 are based on a semi-classical approach to gravity as well as a hierarchy between particle momenta and spatial gradients.

The goal of our studies in chapter 5 is to improve further the field-theoretic description of dark matter on large scales by integrating out the gravitational constraint fields and, moreover, assuming a priori no hierarchy between particle momenta and spatial gradients. We focus on a non-relativistic limit where the mass is much larger than the Hubble rate, particle momenta and spatial gradients and perform a canonical field transformation that takes account of that. The resulting action can be written in terms of a complex scalar field with a non-local interaction term. It extends the description of fuzzy dark matter by providing the dynamical description of a more general dark matter state that may contain virialized, squeezed and condensate components. We study a virialized state without squeezing and condensation in the two-particle-irreducible formalism. The resulting phase-space dynamics generalizes the Vlasov equation by an exchange interaction term which originates from integrating out the gravitational potentials and is absent in the semi-classical approach. Moreover, the phase-space dynamics do not contain partial derivatives with respect to particle momenta which we consider as a new starting point for analytical or numerical insights into non-linear structure formation. As a first step to analytical results, we consider linear solutions where we study inverse distance scales which are much smaller than particle momenta and indeed recover the evolution for the density contrast for standard cold dark matter for large galactic scales. We realize that exchange interactions become important at late times when the temperature of particles is much bigger than the Hubble scale. As another important result we identify scales between the relativistic and the sub-Hubble scale at which we suspect the linear evolution to deviate from the standard results. It is desirable to extend the linear analysis within chapter 5 to inverse distance scales that are comparable to particle momenta which is a harder task but in principle well feasible within the framework we provide. Moreover, it is worth investigating with this framework how virialized, squeezed and condensate components of the dark matter state interact with one another.

For future research, we consider it a great opportunity to exploit further our field-theoretic approach to large-scale structure formation on a more quantitative level, such that numerical and analytical predictions can be contrasted with experimental data. On a broader perspective, it is desirable to extend the classical particle limit, that we derived for spinless fields in curved space-time, to include fermions and photons which might lead to novel field-theoretic effects in the context of cosmic microwaves. It is also intriguing to extend our derivation to include modified theories of gravity.

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# Nederlandse samenvatting

Grootschalige structuurvorming is een van de belangrijkste problemen van de moderne kosmologie. Doorgaans wordt dit gemodelleerd door een puntdeeltje met de bijbehorende Vlasov vergelijking. Wij benaderen dit probleem vanuit een veldtheoretische invalshoek en identificeren hoe veldcorrecties in de gekromde ruimtetijd gerelateerd zijn aan de klassieke deeltjesverdeling in de faseruimte. In een welbepaalde limiet kunnen we de Vlasov vergelijking afleiden met de veldtheoretische correcties, welke we op een gecontroleerde manier kunnen bestuderen. We bekijken ook opnieuw de initiële voorwaarden voor de kosmische achtergrondstraling en grootschalige structuurvorming.

In hoofdstuk 2 presenteren we een twee-velden inflatiemodel, waarin de inflaton bikwadratisch koppelt aan een licht toeschouwersveld. Het toeschouwersveld induceert een snelle productie van entropie, welke de beginvoorwaarden voor de kosmologische evolutie aan het einde van de inflatie significant verandert door grote, stochastisch onafhankelijke momentumfluctuaties. Desondanks, als de interacties tijdens het stralingstijdperk zodanig worden uitgeschakeld dat de evolutie van kosmologische perturbaties op super-Hubble schalen goed benaderd kan worden door de overeenkomstige lineaire evolutie, nemen de momentum-schommelingen af en is het effect ervan te klein om waar te nemen in de kosmische achtergrondstraling of grootschalige structuren.

De hoofdstukken 3 tot 5 zijn gewijd aan de ontwikkeling van een veldtheoretische benadering van grootschalige structuurvorming die gebaseerd is op een actieprincipe voor een reëel scalair veld voor de donkere materie welke gekoppeld is aan de zwaartekracht. In hoofdstuk 3 leggen we de basis voor een dergelijke beschrijving door de Vlasov-vergelijking af te leiden voor koude-donkere-materie deeltjes uit een Lagrangiaanse beschrijving waarin een massief reëel scalair veld minimaal aan de zwaartekracht gekoppeld is met een Newtoniaanse potentiaal. Een deel van deze afleiding heeft als doel een geschikte combinatie te identificeren voor de gelijke-tijd twee-punts correlatie functie van het reële scalaire veld dat de on-shell deeltjes dichtheid in de fase ruimte representeert. We komen uit op een systeem van partiele differentiaalvergelijkingen die de dichtheid van deeltjes in de klassieke faseruimte koppelt aan geperste toestandsbijdragen, en die bovendien relativistische correcties bevatten. Deze bevindingen zijn veralgemeend en ruimtelijk covariant gemaakt in hoofdstuk 4 door het opnemen van een arbitraire metrische tensor en niet-minimale koppeling aan de zwaartekracht tot op één-loop nauwkeurigheid. We passen deze resultaten toe in de context van de kosmologische storingstheorie en bespreken hoe de standaard deeltjesdynamica van koude donkere materie veldtheoretische correcties krijgt. Verder leiden we op een ruimtelijk covariante manier af

hoe hydrodynamische variabelen gerelateerd zijn aan twee-punt functies van het reële scalaire veld. Zowel de Newtoniaanse behandeling in hoofdstuk 3 als ook de covariante veralgemening in hoofdstuk 4 zijn gebaseerd op een semi-klassieke benadering van de zwaartekracht en een hiërarchie tussen deeltjesmomenta en ruimtelijke gradiënten. Het doel van onze studies in hoofdstuk 5 is de veldtheoretische beschrijving van donkere materie op grote schaal verder te verbeteren door het uit integreren van de zwaartekrachtbeperkende velden en, bovendien, a priori, geen hiërarchie aan te nemen tussen deeltjesmomenten en ruimtelijke gradiënten. We richten ons op een niet-relativistische grens waar de massa veel groter is dan de Hubble-snelheid, deeltjesmomenten en ruimtelijke gradiënten en voeren een canonieke veldtransformatie uit die daarmee rekening houdt. De resulterende actie kan worden geschreven in termen van een complex scalair veld met een niet-lokale interactieterm. Het breidt de beschrijving van vage donkere materie (fuzzy dark matter) uit door de dynamische beschrijving van een meer algemene donkere materie toestand te geven die klassieke, geperste en condensaat componenten kan bevatten. We bestuderen een klassieke toestand in de twee-deeltjes-irreduceerbare formalisme. De resulterende faseruimte dynamiek veralgemeent de Vlasov-vergelijking door een term voor uitwisselingsinteractie die voortkomt uit de integratie van het gravitatiepotentieel en afwezig is in de semi-klassieke benadering. Bovendien bevat de faseruimtedynamiek geen gedeeltelijke derivaten van deeltjesmomenten die we beschouwen als een nieuw vertrekpunt voor analytisch of numeriek inzicht in niet-lineaire structuurvorming. Als eerste stap naar analytische resultaten overwegen we lineaire oplossingen waarbij we inverse afstandschaal bestuderen die veel kleiner zijn dan deeltjesmomenta en inderdaad de evolutie voor het dichtheidscontrast voor standaard koude donkere materie voor grote galactische schalen herstellen. We realiseren ons dat uitwisselingsinteracties belangrijk worden op late tijdstippen wanneer de temperatuur van deeltjes veel hoger is dan de temperatuur van grote galactische schalen.



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Pavel Friedrich was born on March 1st 1989 in Berlin, Germany, to Peter and Olga Friedrich. He grew up in Berlin-Marzahn where he attended the Sonnenhut primary school until 1999. He moved with his family to Petershagen, Brandenburg, one year later.

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Pavel greatly enjoys listening to music and plays guitar in the progressive metal band The Apex Plan.