COVARIANTS OF BINARY SEXTICS AND MODULAR FORMS OF DEGREE 2 WITH CHARACTER

FABIEN CLÉRY, CAREL FABER, AND GERARD VAN DER GEER

Abstract. We use covariants of binary sextics to describe the structure of modules of scalar-valued or vector-valued Siegel modular forms of degree 2 with character, over the ring of scalar-valued Siegel modular forms of even weight. For a modular form defined by a covariant we express the order of vanishing along the locus of products of elliptic curves in terms of the covariant.

1. Introduction

In [4] we describe a map from covariants of binary sextics to Siegel modular forms of degree 2. If V denotes the standard 2-dimensional representation of $\mathrm{GL}(2,\mathbb{C})$ with basis x_1, x_2 we consider the space $\mathrm{Sym}^6(V)$ of binary sextics. A general element $f \in \mathrm{Sym}^6(V)$ will be written as

$$f = \sum_{i=0}^{6} a_i \binom{6}{i} x_1^{6-i} x_2^i \, .$$

The group $\mathrm{GL}(2,\mathbb{C})$ acts on $\mathrm{Sym}^6(V)$. We denote by $\mathcal{C}$ the ring of covariants of binary sextics. A bihomogeneous covariant has a bi-degree (a,b), meaning that it can be seen as a homogeneous expression of degree a in the coefficients a_i of f and as a form of degree b in x_1, x_2; such a covariant will be denoted by $C_{a,b}$. The map from covariants to Siegel modular forms defined in [4] is a map

$$\nu : \mathcal{C} \to M_{\chi_{10}} \, ,$$

where M is the ring of vector-valued modular forms of degree 2 on $\Gamma_2 = \mathrm{Sp}(4,\mathbb{Z})$ and the subscript χ_{10} means that Igusa's cusp form χ_{10} of weight 10 is inverted. It sends the binary sextic f to the meromorphic vector-valued modular form $\chi_{6,8}/\chi_{10}$ of weight $(6,-2)$, where $\chi_{6,8}$ is the unique holomorphic modular form of weight $(6,8)$ (it is a cusp form). Using modular forms with character, we can also write this as $\chi_{6,3}/\chi_5$. This map provides us with a very effective method for constructing Siegel modular forms on Γ_2 with or without character. We used it in [4, 5] to construct modular forms.

Since the image of a covariant under ν may be meromorphic on $\mathcal{A}_2$, with possible poles along the locus $\mathcal{A}_{1,1}$ of abelian surfaces that are products of elliptic curves, it is important to have a method to determine the order of vanishing of modular forms obtained from covariants along this locus. In this paper we give such a method. In our earlier papers [4] and [5] we relied on restriction of the corresponding modular forms to the diagonal in the Siegel upper half space instead.

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To exhibit the effectiveness of our method, we use it here to construct generators for certain modules of vector-valued Siegel modular forms of degree 2.

We denote by $M_{j,k}(\Gamma_2)$ (resp. $S_{j,k}(\Gamma_2)$) the vector space of Siegel modular forms (resp. of cusp forms) of weight (j,k) on Γ_2, that is, the weight corresponds to the irreducible representation $\mathrm{Sym}^j(\mathrm{St}) \otimes \det{}^k(\mathrm{St})$ with St the standard representation of GL(2). The group Γ_2 admits a character ϵ of order 2 and χ_5, the square root of χ_{10}, is a modular form of weight 5 with this character. We refer to the last section for a way to calculate the character. We denote the space of modular forms (resp. of cusp forms) of weight (j,k) with character ϵ by $M_{j,k}(\Gamma_2,\epsilon)$ (resp. by $S_{j,k}(\Gamma_2,\epsilon)$).

Let $R = \oplus_{k\,\mathrm{even}} M_k(\Gamma_2)$ be the ring of scalar-valued Siegel modular forms of degree 2 of even weight. Igusa showed that it is a polynomial ring generated by E_4, E_6, χ_{10} and χ_{12}.

We are interested in the structure of the R-modules

$$\mathcal{M}_j^{\mathrm{ev}}(\Gamma_2,\epsilon) = \oplus_{k\,\mathrm{even}} M_{j,k}(\Gamma_2,\epsilon) \quad \text{and} \quad \mathcal{M}_j^{\mathrm{odd}}(\Gamma_2,\epsilon) = \oplus_{k\,\mathrm{odd}} M_{j,k}(\Gamma_2,\epsilon)\,.$$

The structure of the analogous modules for modular forms without character

$$\mathcal{M}_j^{\mathrm{ev}}(\Gamma_2) = \oplus_{k\,\mathrm{even}} M_{j,k}(\Gamma_2) \quad \text{and} \quad \mathcal{M}_j^{\mathrm{odd}}(\Gamma_2) = \oplus_{k\,\mathrm{odd}} M_{j,k}(\Gamma_2)$$

is known for some values of j by work of Satoh, Ibukiyama, van Dorp, Kiyuna, and Takemori, see [17, 12, 8, 15, 19]. The next table summarizes the results.

j	2	4	6	8	10
even	Satoh [17]	Ibukiyama [12]	Ibukiyama [12]	Kiyuna [15]	Takemori [19]
odd	Ibukiyama [12]	Ibukiyama [12]	van Dorp [8]	Kiyuna [15]	Takemori [19]

The difficult part is the construction of the generators and the authors just mentioned used an array of methods to construct generators. For example, Satoh used generalized Rankin-Cohen brackets, Ibukiyama used theta series for even unimodular lattices and Rankin-Cohen brackets, van Dorp used differential operators, and so on. Here we produce the generators we need by a uniform method via the covariants of binary sextics. We treat the cases $j = 0, 2, 4, 6, 8, 10$ even and odd. In all these cases the module turns out to be a free R-module.

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2. The ring of covariants of binary sextics

We recall some facts about the ring $\mathcal{C}$ of covariants of binary sextics. For a description of $\mathcal{C}$ we refer to [4, 5] and the classical literature mentioned there. The book of Grace and Young [11, p. 156] gives 26 generators for this ring. All these generators can be obtained as (repeated) so-called transvectants of the binary sextic f. The kth transvectant of two forms $g \in \mathrm{Sym}^m(V)$, $h \in \mathrm{Sym}^n(V)$ is defined as

$$(g,h)_k = \frac{(m-k)!(n-k)!}{m!\,n!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\partial^k g}{\partial x_1^{k-j} \partial x_2^j} \frac{\partial^k h}{\partial x_1^j \partial x_2^{k-j}}$$

and the index k is usually omitted if $k = 1$. If g is a covariant of bi-degree (a, m) and h a covariant of bi-degree (b, n), then $(g, h)_k$ is a covariant of bi-degree $(a + b, m + n - 2k)$ (cf. [3]). The following table summarizes the construction of the 26 generators.

1	$C_{1,6} = f$			
2	$C_{2,0} = (f, f)_6$	$C_{2,4} = (f, f)_4$	$C_{2,8} = (f, f)_2$	
3	$C_{3,2} = (f, C_{2,4})_4$	$C_{3,6} = (f, C_{2,4})_2$	$C_{3,8} = (f, C_{2,4})$	$C_{3,12} = (f, C_{2,8})$
4	$C_{4,0} = (C_{2,4}, C_{2,4})_4$	$C_{4,4} = (f, C_{3,2})_2$	$C_{4,6} = (f, C_{3,2})$	$C_{4,10} = (C_{2,8}, C_{2,4})$
5	$C_{5,2} = (C_{2,4}, C_{3,2})_2$	$C_{5,4} = (C_{2,4}, C_{3,2})$	$C_{5,8} = (C_{2,8}, C_{3,2})$	
6	$C_{6,0} = (C_{3,2}, C_{3,2})_2$	$C^{(1)}_{6,6} = (C_{3,6}, C_{3,2})$	$C^{(2)}_{6,6} = (C_{3,8}, C_{3,2})_2$	
7	$C_{7,2} = (f, C^2_{3,2})_4$	$C_{7,4} = (f, C^2_{3,2})_3$		
8	$C_{8,2} = (C_{2,4}, C^2_{3,2})_3$			
9	$C_{9,4} = (C_{3,8}, C^2_{3,2})_4$			
10	$C_{10,0} = (f, C^3_{3,2})_6$	$C_{10,2} = (f, C^3_{3,2})_5$		
12	$C_{12,2} = (C_{3,8}, C^3_{3,2})_6$			
15	$C_{15,0} = (C_{3,8}, C^4_{3,2})_8$			

3. Covariants and modular forms

The group Γ_2 acts on the Siegel upper half space $\mathfrak{H}_2$ and the orbifold quotient $\Gamma_2\backslash\mathfrak{H}_2$ can be identified with the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces. If $\mathcal{M}_2$ denotes the moduli space of complex smooth projective curves of genus 2 we have the Torelli map $\mathcal{M}_2 \hookrightarrow \mathcal{A}_2$. This is an embedding and the complement of the image is the locus $\mathcal{A}_{1,1}$ of products of elliptic curves. This is the image of the 'diagonal'

$$\{\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \in \mathfrak{H}_2 : \tau_{12} = 0\}$$

and also the zero locus of the cusp form χ_{10} that vanishes with order 2 there.

The moduli space $\mathcal{M}_2$ has another description as a stack quotient of the action of $\mathrm{GL}(2, \mathbb{C})$ on the space of binary sextics. We take the opportunity to correct an erroneous representation of this stack quotient in [4].

Let V be a 2-dimensional vector space, say generated by x_1, x_2, and consider $\mathrm{Sym}^6(V)$, the space of binary sextics. The group $\mathrm{GL}(V)$ acts from the right; an element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ sends $f(x_1, x_2)$ to $f(ax_1 + bx_2, cx_1 + dx_2)$. We twist the action by $\det^{-2}(V)$ and consider then

$$\mathcal{X} = \mathrm{Sym}^6(V) \otimes \det{}^{-2}(V)\,.$$

We let $\mathcal{X}^0 \subset \mathcal{X}$ be the open set of binary sextics with non-vanishing discriminant. An element f of $\mathcal{X}^0$ defines a nonsingular curve of genus 2 via the equation $y^2 = f(x)$. The action on the equation $y^2 = f(x)$ is now induced by

$$x \mapsto (ax + b)/(cx + d), \quad y \mapsto (ad - bc)\, y/(cx + d)^3\,.$$

Then $\eta \mathrm{id}_V$ acts on the binary sextics as η^2, so that only $\pm\mathrm{id}_V$ acts trivially. The action of $-\mathrm{id}_V$ on (x, y) is $(x, y) \mapsto (x, -y)$ and induces the hyperelliptic involution. So the stack quotient $[\mathcal{X}^0/\mathrm{GL}(V)]$ equals the stack $\mathcal{M}_2$. Let $\alpha : \mathcal{X}^0 \to \mathcal{M}_2$ be the quotient map.

The equation $y^2 = f(x)$ defines two differentials xdx/y and dx/y that form a basis of the space of regular differentials on the curve and the action of $\mathrm{GL}(V)$ is by the standard representation. Thus the pullback under α of the Hodge bundle $\mathbb{E}$ from $\mathcal{M}_2$ to $\mathcal{X}^0$ is the equivariant bundle defined by the standard representation $V \times \mathcal{X}^0$. The equivariant bundle $\mathrm{Sym}^6(V) \otimes \det^{-2}(V)$ has the diagonal section $f \mapsto (f, f)$. This diagonal section, the universal binary sextic, thus defines a meromorphic section $\chi_{6,-2}$ of $\mathrm{Sym}^6(\mathbb{E}) \otimes \det(\mathbb{E})^{-2}$. Since the construction extends to the locus of binary sextics with zeroes of multiplicity at most 2, the section extends regularly over $\delta_0 \setminus \delta_1$. (Here, δ_0 corresponds to $\overline{\mathcal{A}}_2 \setminus \mathcal{A}_2$, the divisor at infinity, and δ_1 to the closure of $\mathcal{A}_{1,1}$.) With this construction, the pole order along δ_1 is not yet known, but after multiplication with a power of χ_{10} the section becomes regular.

In fact, it is not hard to see that $\chi_{6,-2}$ has a simple pole along δ_1. Using Taylor series expansions in the normal direction to $\mathfrak{H}_1 \times \mathfrak{H}_1$ with coordinate $t = 2\pi i \tau_{12}$ as in [4, §5] and coordinates c_i on Sym^j corresponding to the monomials $\binom{j}{i} x_1^{j-i} x_2^i$, we see that the coefficient of t^m in c_i in the expansion of a meromorphic section of $\mathrm{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^{\otimes k}$ that is holomorphic outside $\mathcal{A}_{1,1}$, is of the form $g \otimes h$, with g quasimodular of weight $j-i+k+m$ and h quasimodular of weight $i+k+m$. See the Appendix where we prove that we get quasi-modular forms. To get nonzero coefficients, the two weights and hence their sum $j+2k+2m$ must be nonnegative. For $\chi_{6,-2}$, we get $2+2m \geq 0$, hence $m \geq -1$, proving the claim. Multiplying $\chi_{6,-2}$ with χ_{10}, we obtain the holomorphic modular form $\chi_{6,8}$, unique up to a scalar; alternatively, $\chi_{6,-2}$ can be written as $\chi_{6,3}/\chi_5$, see [6] for $\chi_{6,3}$.

We can interpret modular forms as sections of vector bundles made out of $\mathbb{E}$ by Schur functors, like $\mathrm{Sym}^j(\mathbb{E}) \otimes \det(\mathbb{E})^{\otimes k}$. Since the pullback of the Hodge bundle is the equivariant bundle defined by V, the pullback of such a section can be interpreted as a covariant. Recall that the ring of covariants is the ring of invariants for the action of $\mathrm{SL}(V)$ on $V \oplus \mathrm{Sym}^6(V)$, see for example [18, p. 55]. Conversely, a (bihomogeneous) covariant corresponds to a meromorphic modular form, with poles at most along δ_1, hence to an element of $M_{\chi_{10}}$.

We thus get maps

$$M \to \mathcal{C} \xrightarrow{\nu} M_{\chi_{10}}$$

with $\mathcal{C}$ the ring of covariants of binary sextics and $M = \oplus_{j,k} M_{j,k}(\Gamma_2)$ and $M_{\chi_{10}}$ its localization at the multiplicative system generated by χ_{10}. For another perspective on the map ν, see [4, §6].

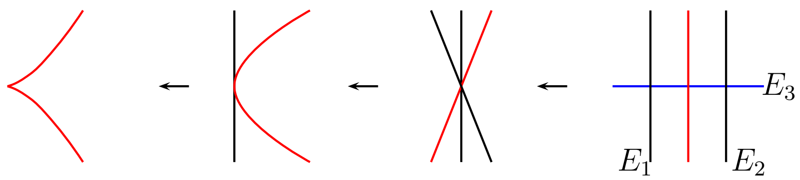
4. The Order of Vanishing

In this section we will describe a way to calculate the order of vanishing along the locus $\mathcal{A}_{1,1}$ of a modular form defined by a covariant. A covariant C has a bi-degree (a, b): if we consider C as a form in the variables $a_0, \ldots, a_6$ and x_1, x_2 then it is of degree a in the a_i and degree b in x_1, x_2. The map $\nu : \mathcal{C} \to M_{\chi_{10}}$ associates to C a meromorphic modular form of weight $(b, a - b/2)$ on Γ_2. It has the property that $\chi_5^a \, \nu(C)$ is a holomorphic modular form on Γ_2, but with character if a is odd.

Recall that $\mathcal{M}_2$ is represented as the stack quotient $[\mathcal{X}^0/\mathrm{GL}(V)]$. The relation with the compactification of $\mathcal{M}_2$ is as follows.

In the (projectivized) space of binary sextics $\mathbb{P}(\mathcal{X})$ the discriminant defines a hypersurface Δ. This hypersurface has a codimension 1 singular locus, one component of which is the locus Δ' of binary sextics with three coinciding roots. So we are in codimension 2 in $\mathbb{P}(\mathcal{X})$ and we take a general plane Π in $\mathbb{P}(\mathcal{X})$ intersecting Δ transversally at a general point of Δ'.

In the plane Π the intersection with Δ gives rise to a curve with a cusp singularity corresponding to the intersection with Δ'; we assume this latter point is the origin of Π. In local coordinates u, v in the plane the discriminant is given by $u^2 = v^3$. One then blows up the plane at the origin three times. This is illustrated in the following picture (cf. the picture in [7, p. 80]).



Then one blows down the exceptional fibres E_1 and E_2. The image of E_3 corresponds in $\overline{\mathcal{M}}_2$ (resp. $\overline{\mathcal{A}}_2$) to the locus δ_1 (resp. $\overline{\mathcal{A}}_{1,1}$) of unions (resp. products) of elliptic curves.

If C is a covariant then it defines a section of an equivariant vector bundle on $\mathcal{X}$ and we can pull this back to the blow-up. It then makes sense to speak of the order of this section along the divisor E_3.

If we consider in the last setting a vertical line that intersects the image of E_3 transversally at a general point, then this corresponds in the original plane with u, v coordinates to a curve $u^2 = c\,v^3$. We can calculate the order of vanishing along E_3 by calculating the order of the covariant on a general family corresponding to $u^2 = c\,v^3$.

The plane Π corresponds to a family of binary sextics of the form

$$g = (x^3 + vx + u)h$$

with h a general cubic polynomial in x. The substitution $u = c^2t^3$, $v = ct^2$ (with c general) gives a family corresponding to $u^2 = c\,v^3$ and the order in t of the covariant after substitution gives the order along E_3.

Theorem 1. *Let C be a covariant of binary sextics of degree a in the a_i and let $\chi_C = \nu(C)$ be the meromorphic modular form obtained by substituting $\chi_{6,-2}$. Then the order of χ_C along $\mathcal{A}_{1,1}$ is given by*

$$\mathrm{ord}_{\mathcal{A}_{1,1}}(\chi_C) = 2\,\mathrm{ord}_{E_3}(C) - a\,.$$

Proof. Since χ_C is obtained by substituting the components of $\chi_{6,-2}$ in C (cf. [4, §6]) and since $\chi_{6,-2}$ has a simple pole along δ_1, the order of χ_C along δ_1 (a.k.a. $\overline{\mathcal{A}}_{1,1}$) is at least $-a$. It can only be larger when C vanishes along E_3, the exceptional divisor of the third blow-up of $\mathcal{X}$. To work this out precisely, note first that the degree (resp. the order) of a product equals the sum of the degrees (resp. the orders) of the factors. Hence, after replacing C by its square if necessary, we may assume that a is even, equal to $2c$. Consider the invariant A of degree 2:

$$A = a_0a_6 - 6a_1a_5 + 15a_2a_4 - 10a_3^2$$

(proportional to $C_{2,0}$). Clearly, it doesn't vanish on E_3, and the associated scalar-valued meromorphic modular form χ_A of weight 2 has a pole of order 2 along δ_1. We can write C as $(C/A^c) \cdot A^c$ and χ_C as $\chi_{C/A^c} \cdot \chi_A^c$, where C/A^c is a meromorphic covariant and χ_{C/A^c} a meromorphic vector-valued modular form, regular along δ_1 but with possible poles along the zero locus of χ_A. The components of C/A^c are meromorphic functions on $\mathbb{P}(\mathcal{X})$ that descend to the components of χ_{C/A^c}. The (minimal) orders of vanishing along E_3 respectively δ_1 are clearly closely related, but since E_3 in the picture above corresponds to the *coarse* moduli space $M_{1,1}$, not to the stack $\mathcal{M}_{1,1}$, the order of χ_{C/A^c} along δ_1 equals twice the order of C/A^c along E_3. □

5. Rings and Modules of Modular Forms

Let $R = \oplus_{k\,\mathrm{even}} M_k(\Gamma_2)$ be the graded ring of scalar-valued Siegel modular forms of even weight on Γ_2. One knows that $R = \mathbb{C}[E_4, E_6, \chi_{10}, \chi_{12}]$ and so its Hilbert-Poincaré series equals $1/(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})$.

We denote by ϵ the unique nontrivial character of order 2 of Γ_2 (see Section 12 for a description of this character). Let $\Gamma_2[2]$ be the principal congruence subgroup of level 2 of Γ_2. The group $\mathrm{Sp}(4, \mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\mathfrak{S}_6$. We fix an explicit isomorphism by identifying the symplectic lattice over $\mathbb{Z}/2\mathbb{Z}$ with the subspace $\{(a_1, \ldots, a_6) \in (\mathbb{Z}/2\mathbb{Z})^6 : \sum a_i = 0\}$ modulo the diagonally embedded $\mathbb{Z}/2\mathbb{Z}$ with form $\sum_i a_ib_i$ as in [1, Section 2]; it is given explicitly on generators of $\mathfrak{S}_6$ in [6, Section 3, (3.2)]. Thus $\mathfrak{S}_6$ acts on the space of modular forms $M_{j,k}(\Gamma_2[2])$ and the space $M_{j,k}(\Gamma_2, \epsilon)$ can be identified with the subspace of $M_{j,k}(\Gamma_2[2])$ on which $\mathfrak{S}_6$ acts via the alternating representation. Since -1_4 belongs to $\Gamma_2[2]$, we have $M_{j,k}(\Gamma_2, \epsilon) = (0)$ for j odd. In the sequel, the integer j will always be even. The following result is in [13]; for the reader's convenience we give an alternative proof.

Lemma 2. *We have $M_{j,k}(\Gamma_2, \epsilon) = S_{j,k}(\Gamma_2, \epsilon)$ for $(j, k) \neq (0, 0)$.*

Proof. In case $k = 0$ and $j \neq 0$ it is well-known that $M_{j,0}(\Gamma_2, \epsilon) = (0)$, see [9, Satz1]. The Siegel operator Φ_2 maps $M_{j,k}(\Gamma_2[2])$ to $S_{j+k}(\Gamma_1[2])$ which is (0) if k is odd and j is even. Since $M_{j,k}(\Gamma_2, \epsilon) \subseteq M_{j,k}(\Gamma_2[2])$ we find $M_{j,k}(\Gamma_2, \epsilon) = S_{j,k}(\Gamma_2, \epsilon)$ for k odd. For $k \geq 2$ even, the Eisenstein part $E_{j,k}(\Gamma_2[2])$ of $M_{j,k}(\Gamma_2[2])$, that is, the orthogonal complement of $S_{j,k}(\Gamma_2[2])$, was described in [6, Section 13] as an $\mathfrak{S}_6$-representation. From the description there we see that the isotypical component $s[1^6]$ never occurs in $E_{j,k}(\Gamma_2[2])$; the result follows since $S_{j,k}(\Gamma_2, \epsilon) = S_{j,k}(\Gamma_2[2])^{s[1^6]}$. (Note that there is a misprint in the expression in [6, Prop. 13.1]: Sym^k should be read as $\mathrm{Sym}^{(j+k)/2}$.) □

The preceding lemma allows us to study cusp forms only. The dimensions of the spaces $S_{j,k}(\Gamma_2,\epsilon)$ are known by work of Tsushima (private communication) as completed by Bergström (see [2]) and independently by [13, Thm. 6.2 and the tables on p. 203 for $k \geq 5$]. The next table gives the Hilbert-Poincaré series of $\mathcal{M}_j^{\rm odd}(\Gamma_2,\epsilon)$ and $\mathcal{M}_j^{\rm ev}(\Gamma_2,\epsilon)$ as R-modules. We give only the numerators since in all cases we have

$$\sum_{k\equiv_2 0\,(\text{or }1)} \dim S_{j,k}(\Gamma_2,\epsilon)\, t^k = \frac{N_j}{(1-t^4)(1-t^6)(1-t^{10})(1-t^{12})}\,,$$

with N_j a polynomial in t.

j	k mod 2	$N_j(t)$
0	1	t^5
	0	t^{30}
2	1	$t^9+t^{11}+t^{17}$
	0	$t^{16}+t^{22}+t^{24}$
4	1	$t^9+t^{11}+t^{13}+t^{15}+t^{17}$
	0	$t^{14}+t^{16}+t^{18}+t^{20}+t^{22}$
6	1	$t^3+t^5+t^{11}+t^{13}+t^{17}+t^{19}+t^{21}$
	0	$t^8+t^{10}+t^{12}+t^{16}+t^{18}+t^{24}+t^{26}$
8	1	$t^5+t^7+2\,t^9+t^{11}+t^{13}+t^{15}+t^{17}+t^{23}$
	0	$t^4+t^{10}+t^{12}+t^{14}+t^{16}+2\,t^{18}+t^{20}+t^{22}$
10	1	$t^5+t^7+2\,t^9+2t^{11}+2t^{13}+2t^{15}+t^{17}$
	0	$t^8+2\,t^{10}+2\,t^{12}+2t^{14}+2t^{16}+t^{18}+t^{20}$
12	1	$t^3+2\,t^5+t^7+2\,t^9+3\,t^{11}+2\,t^{13}+t^{17}+t^{19}-t^{23}+t^{27}$
	0	$t^2+t^4+t^6+t^8+t^{10}+t^{12}+t^{14}+2\,t^{16}+2\,t^{18}+t^{20}+t^{22}+t^{24}-t^{28}$

For $j \in \{0,2,4,6,8,10\}$ and both for k odd and even the shape of the polynomials N_j is as follows:

$$N_j(t) = a_{k_{j,1}}\, t^{k_{j,1}} + \ldots + a_{k_{j,n}}\, t^{k_{j,n}} \quad \text{with} \quad n, a_{k_{j,i}} \in \mathbb{Z}_{>0} \quad \text{and} \quad \sum_{i=1}^{n} a_{k_{j,i}} = j+1.$$

This suggests that the R-modules $\mathcal{M}_j^{\rm ev}(\Gamma,\epsilon)$ and $\mathcal{M}_j^{\rm odd}(\Gamma,\epsilon)$ are generated by $j+1$ cusp forms with $a_{j,k_{j,i}}$ generators of weight $(j,k_{j,i})$. As the table shows this does not hold for $j=12$.

Therefore the strategy of the proof for the structure of the modules will be to show first that there is no cusp form of weight (j,k) for $k<k_{j,1}$ for $j \in \{0,2,4,6,8,10\}$. In the cases at hand this follows from the above formula and the results in [5]. Then we will construct $j+1$ cusp forms and check that their wedge product is not identically 0. In fact in all cases we find that the wedge product of the $j+1$ forms is a nonzero multiple of a product of powers of χ_5 and χ_{30}. This proves that the submodule they generate has the same Hilbert-Poincaré series as the whole module, hence that we found the whole module. We will give the covariants that define the generators explicitly in a number of cases, but in view of their size we refer for the other cases to [2] where we will make these available.

6. The scalar-valued cases

In this section we deal with the modules of scalar-valued modular forms with character. In this case the weight (j, k) is of the form $(0, k)$ and we simply indicate it by k.

The diagonal element $\gamma_1 = \mathrm{diag}(1, -1, 1, -1) \in \Gamma_2$ defines an involution fixing the coordinates τ_{11} and τ_{22} and replacing τ_{12} by $-\tau_{12}$. Its fixed point set is the locus defined by $\tau_{12} = 0$. This defines the Humbert surface $H_1 = \mathcal{A}_{1,1}$ parametrizing products of elliptic curves in $\mathcal{A}_2$. There is another involution ι_2 given by $\gamma_2 = (a, b; c, d)$ with $b = c = 0$ and $a = d = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which interchanges τ_{11} and τ_{22}, but fixes τ_{12}. The fixed point set of ι_2 is the locus $\tau_{11} = \tau_{22}$ and defines the Humbert surface H_4 in $\mathcal{A}_2$, see [10]. One checks that the action on modular forms is as follows

$$\gamma_1 : f \mapsto (-1)^k f, \quad \gamma_2 : f \mapsto (-1)^{k+1} f \qquad \text{for } f \in M_k(\Gamma_2, \epsilon). \tag{1}$$

Note $\epsilon(\gamma_2) = -1$. It follows that $f \in M_k(\Gamma_2, \epsilon)$ vanishes on H_1 for k odd and on H_4 for k even.

We have two modular forms χ_5 and χ_{30} of weight 5 and 30 whose zero loci in $\mathcal{A}_2$ equal H_1 and H_4. We recall their construction.

The cusp form $\chi_5 \in S_5(\Gamma_2, \epsilon)$ is defined in terms of theta functions. For $(\tau, z) \in \mathfrak{H} \times \mathbb{C}$ and (μ_1, μ_2), (ν_1, ν_2) in $\mathbb{Z}^2$ we have the standard theta series with characteristics

$$\vartheta_{\left[\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right]}(\tau, z) = \sum_{n=(n_1,n_2)\in\mathbb{Z}^2} e^{i\pi(n+\mu/2)(\tau\,(n+\mu/2)^t+2(z+\nu/2))}.$$

By letting μ and ν be vectors consisting of zeroes and ones with $\mu^t\nu \equiv 0 \pmod 2$ and setting $z = 0$ we obtain ten so-called theta constants and their product defines a cusp form of weight 5 on Γ_2 with character ϵ:

$$\chi_5 = -\frac{1}{64} \prod \vartheta_{\left[\begin{smallmatrix} \mu \\ \nu \end{smallmatrix}\right]}.$$

Its Fourier expansion starts with

$$\chi_5(\tau) = (u - 1/u)XY + \ldots$$

where $X = e^{\pi i \tau_1}$, $Y = e^{\pi i \tau_2}$ and $u = e^{\pi i \tau_{12}}$. We note that $\chi_5^2 = \chi_{10}$ and the vanishing locus of χ_{10} in $\mathcal{A}_2$ is $2H_1$.

In order to construct χ_{30} we consider the invariant $C_{15,0}$, given in the table in Section 2. By the procedure of [4] it provides a meromorphic cusp form of weight 15 on Γ_2. One checks using Theorem 1 that the order of this form along $\mathcal{A}_{1,1}$ is -3. So we obtain a holomorphic modular form by multiplying by χ_5^3 and we set

$$\chi_{30} = 2^{-11}3^{11} \cdot 5^{11} \cdot 11 \cdot 13\, \nu(C_{15,0})\chi_5^3\,;$$

it is a cusp form in $S_{30}(\Gamma_2, \epsilon)$ whose Fourier expansion starts with

$$\chi_{30}(\tau) = (u + 1/u)X^3Y^5 - (u + 1/u)X^5Y^3 + \ldots .$$

The following result is due to Igusa, see [14, p. 402-404].

Theorem 3. *We have* $\mathcal{M}_0^{\mathrm{odd}}(\Gamma_2, \epsilon) = R\,\chi_5$ *and* $\mathcal{M}_0^{\mathrm{ev}}(\Gamma_2, \epsilon) = R\,\chi_{30}$.

Proof. Clearly $\mathcal{M}_0^{\rm odd}(\Gamma_2,\epsilon)$ contains $R\,\chi_5$ and $\mathcal{M}_0^{\rm ev}(\Gamma_2,\epsilon)$ contains $R\,\chi_{30}$. The generating function for the dimensions shows that χ_5 (resp. χ_{30}) generates. ☐

Remark 4. We know the cycle classes of the closures of H_1 and H_4 in the compactified moduli space $\tilde{\mathcal{A}}_2$. In the divisor class group with rational coefficients of $\tilde{\mathcal{A}}_2$ we have

$$5\lambda_1 = [\overline{H}_1] + [D], \quad 30\lambda_1 = [\overline{H}_4] + [D]$$

with D the divisor at infinity of $\tilde{\mathcal{A}}_2$, and λ_1 the first Chern class of the determinant of the Hodge bundle, see [10, Thm. 2.6]. From this it follows that the vanishing locus of χ_{30} in $\mathcal{A}_2$ is H_4. Then (1) implies that for k odd (resp. k even) any $f \in M_k(\Gamma_2,\epsilon)$ is divisible by χ_5 (resp. by χ_{30}). This implies the theorem as well.

For later identifications (for example in the proof of Theorem 11) we need the restriction of $\chi_{6,3}$ to the Humbert surface H_4. This surface can be given by $\tau_{11}=\tau_{22}$, or equivalently by $\tau_{12}=1/2$. Let χ denote the Dirichlet character modulo 4 defined by the Kronecker symbol $\left(\frac{-4}{\cdot}\right)$. The space $S_3^{\rm new}(\Gamma_0(16),\chi)$ is generated by $\eta^6(2\tau)$. The space $S_5^{\rm new}(\Gamma_0(16),\chi)$ has dimension 2 and a basis of eigenforms g', g'' with Fourier expansions

$$q - 8\sqrt{-3}\,q^3 + 18\,q^5 - 16\sqrt{-3}\,q^7 - 111\,q^9 + \ldots$$

and similarly $S_7^{\rm new}(\Gamma_0(16),\chi)$ has dimension 2 and a basis of eigenforms f', f'' with Fourier expansions

$$q - 16\sqrt{-3}\,q^3 - 150\,q^5 - 352\sqrt{-3}\,q^7 - 39\,q^9 + \ldots$$

Lemma 5. *The restriction of $\chi_{6,3}$ to H_4 is given by*

$$\chi_{6,3}\begin{pmatrix} \tau_1 & 1/2 \\ 1/2 & \tau_2 \end{pmatrix} = 2\,i \begin{bmatrix} 16\,\eta^{18}(2\,\tau_1)\otimes\eta^6(2\,\tau_2) \\ 0 \\ F_1(\tau_1)\otimes F_2(\tau_2) \\ 0 \\ F_2(\tau_1)\otimes F_1(\tau_2) \\ 0 \\ 16\,\eta^6(2\,\tau_1)\otimes\eta^{18}(2\,\tau_2) \end{bmatrix}$$

where

$$F_1 = \frac{3+\sqrt{-3}}{6}\,f' + \frac{3-\sqrt{-3}}{6}\,f'' \quad \textit{and} \quad F_2 = \frac{3+\sqrt{-3}}{6}\,g' + \frac{3-\sqrt{-3}}{6}\,g''\,.$$

7. The case $j=2$

We start with the case k odd.

Theorem 6. *The R-module $\mathcal{M}_2^{\rm odd}(\Gamma_2,\epsilon)$ is free with three generators of weight $(2,9)$, $(2,11)$ and $(2,17)$.*

Proof. We recall that the numerator N_2 of the Hilbert-Poincaré series is $t^9+t^{11}+t^{17}$. We construct the three generators by considering the covariants

$$\xi_1 = 4\,C_{2,0}C_{3,2} - 15\,C_{5,2},$$
$$\xi_2 = 32\,C_{2,0}^2C_{3,2} + 135\,C_{2,0}C_{5,2} - 300\,C_{3,2}C_{4,0} - 15750\,C_{7,2},$$
$$\xi_3 = C_{3,2}.$$

These three covariants define meromorphic modular forms vanishing with order -1, -1, -3 along $\mathcal{A}_{1,1}$ (by Theorem 1), so we obtain holomorphic modular forms

$$F_{2,9} = -\frac{3375}{4}\nu(\xi_1)\chi_5, \quad F_{2,11} = -\frac{10125}{8}\nu(\xi_2)\chi_5, \quad F_{2,17} = \frac{1125}{2}\nu(\xi_3)\chi_5^3$$

of weights $(2,9)$, $(2,11)$ and $(2,17)$ and their Fourier expansions start as

$$F_{2,9} = \begin{pmatrix} u-1/u \\ u+1/u \\ u-1/u \end{pmatrix} XY + \ldots \qquad F_{2,11} = \begin{pmatrix} u-1/u \\ u+1/u \\ u-1/u \end{pmatrix} XY + \ldots$$

and

$$F_{2,17} = \begin{pmatrix} u^3+9\,u-9\,u^{-1}-u^{-3} \\ u^3+71\,u+71\,u^{-1}+u^{-3} \\ u^3+9\,u-9\,u^{-1}-u^{-3} \end{pmatrix} X^3Y^3 + \ldots$$

To prove the theorem we have to show that these three generators satisfy

$$F_{2,9} \wedge F_{2,11} \wedge F_{2,17} \neq 0.$$

Note that $\det(\mathrm{Sym}^j(\mathbb{E})) = \det(\mathbb{E})^{j(j+1)/2}$, so this is a form in $S_{40}(\Gamma_2, \epsilon)$. The Fourier expansion of $F_{2,9} \wedge F_{2,11} \wedge F_{2,17}$ starts with

$$86400\,((-u^3 + u + u^{-1} - u^{-3})Y^7X^5 + (u^3 - u - u^{-1} + u^{-3})Y^5X^7 + \ldots)$$

and this shows the result. □

Remark 7. The space $S_{40}(\Gamma_2, \epsilon)$ is 2-dimensional, generated by $\chi_5^2\chi_{30}$ and $E_4E_6\chi_{30}$. We check that $F_{2,9} \wedge F_{2,11} \wedge F_{2,17} = -86400\,\chi_5^2\chi_{30}$.

The case k even is similar.

Theorem 8. *The R-module $\mathcal{M}_2^{\mathrm{ev}}(\Gamma_2, \epsilon)$ is free with generators of weight* $(2,16)$, $(2,22)$ *and* $(2,24)$.

Proof. We use the covariants

$$\xi_1 = 1211\,C_{2,0}^2C_{8,2} - 8910\,C_{2,0}C_{10,2} - 5250\,C_{4,0}C_{8,2} + 277200\,C_{12,2},$$
$$\xi_2 = C_{8,2}, \quad \xi_3 = 7\,C_{2,0}C_{8,2} - 110\,C_{10,2}$$

and set

$$F_{2,16} = \frac{34171875}{2048}\nu(\xi_1)\chi_5 = \begin{pmatrix} 0 \\ 2(u-1/u) \\ u+1/u \end{pmatrix} XY^3 + \begin{pmatrix} -(u+1/u) \\ -2(u-1/u) \\ 0 \end{pmatrix} X^3Y + \ldots$$
$$F_{2,22} = \frac{26578125}{8}\nu(\xi_2)\chi_5^3 = \begin{pmatrix} u+1/u \\ 0 \\ -(u+1/u) \end{pmatrix} X^3Y^3 + \ldots$$
$$F_{2,24} = -\frac{102515625}{16}\nu(\xi_3)\chi_5^3 = \begin{pmatrix} u+1/u \\ 0 \\ -(u+1/u) \end{pmatrix} X^3Y^3 + \ldots$$

By the criterion these are holomorphic modular forms of weight $(2,16)$, $(2,22)$ and $(2,24)$. The Fourier expansion of $F_{2,16} \wedge F_{2,22} \wedge F_{2,24}$ starts with

$$F_{2,16} \wedge F_{2,22} \wedge F_{2,24} = -2880\,(u^3 + u - u^{-1} - u^{-3})\,X^7Y^{11} + \ldots$$

and in fact equals $-2880\,\chi_5\,\chi_{30}^2$. This finishes the proof in view of the Hilbert-Poincaré series. □

8. The case $j=4$.

Theorem 9. *The R-module* $\mathcal{M}_4^{\rm odd}(\Gamma_2,\epsilon)$ *is free with generators of weight* $(4,9)$, $(4,11)$, $(4,13)$, $(4,15)$ *and* $(4,17)$.

Proof. We use the covariants

$$\begin{aligned}
\xi_1 &= 49\, C_{2,0}^2 C_{2,4} + 45\, C_{2,0}C_{4,4} - 375\, C_{2,4}C_{4,0} - 225\, C_{3,2}^2,\\
\xi_2 &= 772\, C_{2,0}^3 C_{2,4} - 1260\, C_{2,0}^2 C_{4,4} - 4875\, C_{2,0}C_{2,4}C_{4,0} - 900\, C_{2,0}C_{3,2}^2,\\
&\quad - 5625\, C_{2,4}C_{6,0} + 13500\, C_{3,2}C_{5,2} + 6750\, C_{4,0}C_{4,4}\\
\xi_3 &= 64\, C_{2,0}^4 C_{2,4} - 1200\, C_{2,0}^2 C_{2,4}C_{4,0} - 3600\, C_{2,0}^2 C_{3,2}^2 + 27000\, C_{2,0}C_{3,2}C_{5,2}\\
&\quad + 5625\, C_{2,4}C_{4,0}^2 - 50625\, C_{5,2}^2,\\
\xi_4 &= C_{2,4}, \qquad \xi_5 = 3\, C_{2,0}C_{2,4} - 5\, C_{4,4}\,.
\end{aligned}$$

The Fourier expansions of

$$F_{4,9} = -\frac{675}{4}\nu(\xi_1)\chi_5, \qquad F_{4,11} = \frac{2025}{8}\nu(\xi_2)\chi_5 \quad \text{and} \quad F_{4,13} = -\frac{30375}{8}\nu(\xi_3)\chi_5$$

all three start as

$$\begin{pmatrix} u-1/u \\ 2(u+1/u) \\ 3(u-1/u) \\ 2(u+1/u) \\ u-1/u \end{pmatrix} XY + \ldots$$

The other two modular forms we need are

$$F_{4,15} = \frac{75}{2}\nu(\xi_4)\chi_5^3 = \begin{pmatrix} u^3-3u+3/u-1/u^3 \\ 2(u^3-u-1/u+1/u^3) \\ 3(u^3+5u-5/u-1/u^3) \\ 2(u^3-u-1/u+1/u^3) \\ u^3-3u+3/u-1/u^3 \end{pmatrix} X^3Y^3 + \ldots$$

$$F_{4,17} = -\frac{675}{2}\nu(\xi_5)\chi_5^3 = \begin{pmatrix} u^3+9u-9/u-1/u^3 \\ 2(u^3-u-1/u+1/u^3) \\ 3(u^3-3u+3/u-1/u^3) \\ 2(u^3-u-1/u+1/u^3) \\ u^3+9u-9/u-1/u^3 \end{pmatrix} X^3Y^3 + \ldots$$

The Fourier expansion of $F_{4,9}\wedge F_{4,11}\wedge F_{4,13}\wedge F_{4,15}\wedge F_{4,17}$ starts with

$$-2866544640\,(u^5-u^3-2u+2/u+1/u^3-1/u^5)X^9Y^{13} + \ldots$$

and by a calculation we get

$$F_{4,9}\wedge F_{4,11}\wedge F_{4,13}\wedge F_{4,15}\wedge F_{4,17} = -2866544640\,\chi_5^3\chi_{30}^2\,.$$

□

Theorem 10. *The R-module* $\mathcal{M}_4^{\rm ev}(\Gamma_2,\epsilon)$ *is free with generators of weight* $(4,14)$, $(4,16)$, $(4,18)$, $(4,20)$ *and* $(4,22)$.

Proof. For weight $(4,14)$ we consider the covariant ξ_1 given as

$$189\, C_{2,0}^3 C_{5,4} + 12390\, C_{2,0}^2 C_{7,4} - 750\, C_{2,0}C_{4,0}C_{5,4} - 63000\,(C_{2,0}C_{9,4} + C_{3,2}C_{8,2} + C_{4,0}C_{7,4})$$

and set $F_{4,14} = -(151875/1024)\nu(\xi_1)\chi_5$. This is holomorphic and its Fourier expansion starts with

$$F_{4,14}(\tau) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2(u-1/u) \\ (u+1/u) \end{pmatrix} XY^3 - \begin{pmatrix} (u+1/u) \\ 2(u-1/u) \\ 0 \\ 0 \\ 0 \end{pmatrix} X^3Y + \dots$$

For weight $(4,16)$ we consider the covariant ξ_2 given as

$$\begin{aligned} &11176\, C_{2,0}^4 C_{5,4} - 82320\, C_{2,0}^3 C_{7,4} + 9576000\, C_{2,0}^2 C_{9,4} - 15750\, C_{2,0} C_{3,2} C_{8,2} \\ &- 220500\, C_{2,0} C_{4,0} C_{7,4} - 176625\, C_{2,0} C_{5,4} C_{6,0} - 414000\, C_{4,0}^2 C_{5,4} + 43213500\, C_{3,2} C_{10,2} \\ &- 47250000\, C_{4,0} C_{9,4} + 20506500\, C_{5,2} C_{8,2} - 9308250\, C_{6,0} C_{7,4} \end{aligned}$$

and set $F_{4,16} = (151875/4096)\nu(\xi_2)\chi_5$; it is holomorphic and its Fourier expansion starts with

$$F_{4,16}(\tau) = \begin{pmatrix} 0 \\ 2(u+1/u) \\ 3(u+1/u) \\ (u-1/u) \\ 0 \end{pmatrix} XY^3 + \dots$$

We get a form $F_{4,18}$ of weight $(4,18)$ by putting $F_{4,18} = (16875/8)\nu(C_{5,4})\chi_5^3$; it is holomorphic and its Fourier expansion starts with

$$F_{4,18}(\tau) = \begin{pmatrix} 3(u+1/u) \\ 2(u-1/u) \\ 0 \\ -2(u-1/u) \\ -3(u+1/u) \end{pmatrix} X^3Y^3 + \dots$$

For weight $(4,20)$ we consider the covariant $\xi_4 = C_{2,0}C_{5,4} + 70C_{7,4}$ and put $F_{4,20} = (151875/32)\nu(\xi_4)\chi_5^3$ with Fourier expansion

$$F_{4,20}(\tau) = \begin{pmatrix} 0 \\ (u-1/u) \\ 0 \\ -(u-1/u) \\ 0 \end{pmatrix} X^3Y^3 + \dots$$

Finally, the covariant $\xi_5 = C_{2,0}^2 C_{5,4} - 10\, C_{2,0}C_{7,4} + 1000\, C_{9,4}$ yields the form $F_{4,22} = (3189375/32)\nu(\xi_5)\chi_5^3$ with Fourier expansion

$$F_{4,22}(\tau) = \begin{pmatrix} (u+1/u) \\ 2(u-1/u) \\ 0 \\ -2(u-1/u) \\ -(u+1/u) \end{pmatrix} X^3Y^3 + \dots$$

The Fourier expansion of $F_{4,14} \wedge F_{4,16} \wedge F_{4,18} \wedge F_{4,20} \wedge F_{4,22}$ starts with

$$-20736\,(u^5 + u^3 - 2u - 2/u + 1/u^3 + 1/u^5)X^{11}Y^{17} + \dots$$

and in fact we checked that it equals $-20736\,\chi_5^2\chi_{30}^3$. □

9. The case $j = 6$

Theorem 11. *The R-module $\mathcal{M}_6^{\mathrm{odd}}(\Gamma_2, \epsilon)$ is free with generators of weight* $(6,3)$, $(6,5)$, $(6,11)$, $(6,13)$, $(6,17)$, $(6,19)$ *and* $(6,21)$.

Proof. We use the covariants

$$
\begin{aligned}
\xi_1 =&C_{1,6}, \qquad \xi_2 = 8\,C_{1,6}C_{2,0} - 75\,C_{3,6},\\
\xi_3 =&125\,C_{1,6}C_{2,0}^2C_{4,0} + 249\,C_{1,6}C_{2,0}C_{6,0} - 840\,C_{1,6}C_{4,0}^2 - 189\,C_{2,0}C_{2,4}C_{5,2}\\
&- 1008\,C_{2,0}C_{3,2}C_{4,4} - 72\,C_{2,0}C_{3,6}C_{4,0} + 630\,C_{3,2}^3 + 132300\,C_{2,4}C_{7,2}\\
&+ 2430\,C_{3,6}C_{6,0} - 1890\,C_{4,4}C_{5,2},\\
\xi_4 =&768\,C_{1,6}C_{2,0}^5 + 768\,C_{2,0}^4C_{3,6} - 487520\,C_{1,6}C_{2,0}^2C_{6,0} - 36075\,C_{2,0}^2C_{2,4}C_{5,2}\\
&+ 33600\,C_{2,0}^2C_{3,2}C_{4,4} - 52500\,C_{2,0}C_{3,2}^3 - 11061300\,C_{1,6}C_{4,0}C_{6,0}\\
&- 314861750\,C_{2,0}C_{2,4}C_{7,2} - 112500\,C_{2,0}C_{3,6}C_{6,0} + 8956675\,C_{2,0}C_{4,4}C_{5,2}\\
&+ 17767100\,C_{2,4}C_{3,2}C_{6,0} + 230625\,C_{2,4}C_{4,0}C_{5,2} - 39779100\,C_{3,2}^2C_{5,2}\\
&+ 17834600\,C_{3,2}C_{4,0}C_{4,4} + 9482503800\,C_{1,6}C_{10,0} - 932772750\,C_{4,4}C_{7,2},\\
\xi_5 =&8\,C_{1,6}C_{2,0}^2 - 125\,C_{2,4}C_{3,2},\\
\xi_6 =&128\,C_{1,6}C_{2,0}^3 + 6600\,C_{2,0}^2C_{3,6} + 6750\,C_{2,4}C_{5,2} - 9000\,C_{3,2}C_{4,4} - 52875\,Cov_{3,6}C_{4,0},\\
\xi_7 =&- 837\,C_{1,6}C_{2,0}^2C_{4,0} + 415\,C_{1,6}C_{2,0}C_{6,0} + 9450\,C_{2,0}C_{2,4}C_{5,2} + 6075\,C_{2,0}C_{3,6}C_{4,0}\\
&+ 3150\,C_{3,2}^3 - 1543500\,C_{2,4}C_{7,2} - 17475\,C_{3,6}C_{6,0} + 14175\,C_{4,4}C_{5,2}\,.
\end{aligned}
$$

We consider the following cusp forms:

$$F_{6,3} = \nu(\xi_1)\chi_5,\; F_{6,5} = -15\,\nu(\xi_2)\chi_5,\; F_{6,11} = \frac{253125}{8}\nu(\xi_3)\chi_5,\; F_{6,13} = \frac{2278125}{16}\nu(\xi_4)\chi_5,$$

and

$$F_{6,17} = -\frac{675}{4}\nu(\xi_5)\chi_5^3, \quad F_{6,19} = -\frac{675}{2}\nu(\xi_6)\chi_5^3, \quad F_{6,21} = -\frac{151875}{4}\nu(\xi_7)\chi_5^3\,.$$

Then

$$W_{110} = F_{6,3} \wedge F_{6,5} \wedge F_{6,11} \wedge F_{6,13} \wedge F_{6,17} \wedge F_{6,19} \wedge F_{6,21}$$

is a cusp form in $S_{0,110}(\Gamma_2, \epsilon)$ and its Fourier expansion starts with

$$2^{30} \cdot 3^5 \cdot 5^8 \cdot 7^3\,(u^7 - u^5 - 3u^3 + 3u + 3/u - 3/u^3 - 1/u^5 + 1/u^7)\,X^{13}Y^{17} + \ldots$$

The order of vanishing of W_{110} along H_1 is 4 while along H_4 it is 3, so W_{110} is a multiple of $\chi_5^4\chi_{30}^3$ and a calculation at the level of covariants yields $W_{110} = 2^{30}\cdot 3^5\cdot 5^8\cdot 7^3\,\chi_5^4\chi_{30}^3$. □

Theorem 12. *The R-module* $\mathcal{M}_6^{\mathrm{ev}}(\Gamma_2, \epsilon)$ *is free with generators of weight* $(6, 8)$*,* $(6, 10)$*,* $(6, 12)$*,* $(6, 16)$*,* $(6, 18)$*,* $(6, 24)$ *and* $(6, 26)$*.*

Proof. We use the covariants

$$\begin{aligned}
\xi_1 &= 16\, C_{2,0}C_{4,6} + 75\, C_{6,6}^{(1)} - 60\, C_{6,6}^{(2)}, \quad \xi_4 = C_{4,6}, \quad \xi_5 = 4\, C_{2,0}C_{4,6} - 15\, C_{6,6}^{(1)},\\
\xi_2 &= -128\, C_{2,0}^2 C_{4,6} + 75\, C_{2,0}C_{6,6}^{(1)} - 540\, C_{2,0}C_{6,6}^{(2)} - 1500\, C_{3,2}C_{5,4} + 1800\, C_{4,0}C_{4,6},\\
\xi_3 &= 64\, C_{2,0}^3 C_{4,6} - 3975\, C_{2,0}^2 C_{6,6}^{(1)} + 1740\, C_{2,0}^2 C_{6,6}^{(2)} - 189000\, C_{2,4}C_{8,2} + 63000\, C_{3,2}C_{7,4}\\
&\quad + 40500\, C_{4,0}C_{6,6}^{(1)} - 18000\, C_{4,0}C_{6,6}^{(2)} + 4500\, C_{5,2}C_{5,4},\\
\xi_6 &= -17472\, C_{2,0}C_{2,4}C_{8,2} + 31360\, C_{2,0}C_{3,2}C_{7,4} - 513\, C_{2,0}C_{4,0}C_{6,6}^{(1)} + 180\, C_{2,0}C_{4,0}C_{6,6}^{(2)}\\
&\quad - 64\, C_{2,0}C_{4,6}C_{6,0} + 342\, C_{2,0}C_{5,2}C_{5,4} + 39600\, C_{2,4}C_{10,2} - 126000\, C_{3,2}C_{9,4}\\
&\quad - 16800\, C_{4,4}C_{8,2} - 60900\, C_{5,2}C_{7,4} + 600\, C_{6,0}C_{6,6}^{(1)},\\
\xi_7 &= 1024\, C_{2,0}^5 C_{4,6} - 257152000\, C_{2,0}^2 C_{3,2}C_{7,4} + 5375048250\, C_{2,0}C_{2,4}C_{10,2}\\
&\quad - 1808283750\, C_{2,0}C_{3,2}C_{9,4} + 785335250\, C_{2,0}C_{4,4}C_{8,2} + 1144763375\, C_{2,0}C_{5,2}C_{7,4}\\
&\quad + 673186500\, C_{2,4}C_{4,0}C_{8,2} + 656687500\, C_{3,2}^2 C_{8,2} - 938905625\, C_{3,2}C_{4,0}C_{7,4}\\
&\quad + 3150000\, C_{4,0}^2 C_{6,6}^{(2)} + 17435250\, C_{4,0}C_{5,2}C_{5,4} - 378064302000\, C_{2,4}C_{12,2}\\
&\quad - 532125000\, C_{4,4}C_{10,2} - 415800000\, C_{4,6}C_{10,0} + 37292797500\, C_{5,2}C_{9,4}\\
&\quad - 250254270000\, C_{7,2}C_{7,4}\,.
\end{aligned}$$

We consider the following cusp forms:

$$F_{6,8} = \frac{10125}{8}\nu(\xi_1)\chi_5, \quad F_{6,10} = -\frac{30375}{16}\nu(\xi_2)\chi_5, \quad F_{6,12} = \frac{455625}{64}\nu(\xi_3)\chi_5,$$

$$F_{6,16} = -3375\,\nu(\xi_4)\chi_5^3, \quad F_{6,18} = -50625\,\nu(\xi_5)\chi_5^3 \quad F_{6,24} = -\frac{170859375}{32}\nu(\xi_6)\chi_5^3,$$

$$F_{6,26} = -\frac{20503125}{16}\nu(\xi_7)\chi_5^3\,.$$

Then

$$W_{135} = F_{6,8} \wedge F_{6,10} \wedge F_{6,12} \wedge F_{6,16} \wedge F_{6,18} \wedge F_{6,24} \wedge F_{6,26}$$

is a cusp form in $S_{135}(\Gamma_2, \epsilon)$ and its Fourier expansion starts with

$$-2^{32} \cdot 3^8 \cdot 5^8 \cdot 7^2 \cdot 13 \cdot 23\,(u^7 + u^5 - 3u^3 - 3u + 3/u + 3/u^3 - 1/u^5 - 1/u^7)X^{15}Y^{23} + \ldots$$

A calculation shows that the order of vanishing of W_{135} along H_1 is 3, while along H_4 it is 4, so W_{135} is a multiple of $\chi_5^3\chi_{30}^4$ and a calculation at the level of covariants tells us

$$W_{135} = -2^{32} \cdot 3^8 \cdot 5^8 \cdot 7^2 \cdot 13 \cdot 23\,\chi_5^3\chi_{30}^4\,.$$

□

10. The case $j = 8$

Theorem 13. *The R-module $\mathcal{M}_8^{\mathrm{odd}}(\Gamma_2, \epsilon)$ is free with generators of weight* $(8,5)$, $(8,7)$, $(8,9)$, $(8,9)$, $(8,11)$, $(8,13)$, $(8,15)$, $(8,17)$ *and* $(8,23)$.

Proof. We use the covariants

$$
\begin{aligned}
\xi_1 =&160\, C_{1,6}C_{3,2} - 208\, C_{2,0}C_{2,8} + 250\, C_{2,4}^2,\\
\xi_2 =&60\, C_{1,6}C_{2,0}C_{3,2} + 16\, C_{2,0}^2C_{2,8} - 225\, C_{1,6}C_{5,2} - 150\, C_{2,8}C_{4,0},\\
\xi_3^{(1)} =&4032\, C_{2,0}^3C_{2,8} + 55800\, C_{1,6}C_{2,0}C_{5,2} - 25000\, C_{1,6}C_{3,2}C_{4,0} - 46125\, C_{2,0}C_{2,4}C_{4,4},\\
&- 159500\, C_{2,0}C_{3,2}C_{3,6} + 17377500\, C_{1,6}C_{7,2} + 90750\, C_{2,8}C_{6,0} + 675000\, C_{3,6}C_{5,2} - 384375\, C_{4,4}^2\\
\xi_3^{(2)} =&112\, C_{1,6}C_{2,0}^2C_{3,2} - 60\, C_{1,6}C_{2,0}C_{5,2} - 150\, C_{1,6}C_{3,2}C_{4,0} - 135\, C_{2,0}C_{2,4}C_{4,4} - 1440\, C_{2,0}C_{3,2}C_{3,6}\\
&+ 31500\, C_{1,6}C_{7,2} + 450\, C_{2,8}C_{6,0} + 5625\, C_{3,6}C_{5,2} - 1125\, C_{4,4}^2,\\
\xi_4 =&1792\, C_{2,0}^4C_{2,8} + 28750\, C_{1,6}C_{2,0}^2C_{5,2} - 3685500\, C_{1,6}C_{2,0}C_{7,2} - 139200\, C_{1,6}C_{3,2}C_{6,0}\\
&- 229650\, C_{1,6}C_{4,0}C_{5,2} - 93600\, C_{2,0}C_{2,8}C_{6,0} - 183150\, C_{2,0}C_{3,6}C_{5,2} + 166725\, C_{2,4}^2C_{6,0}\\
&- 40500\, C_{2,4}C_{3,2}C_{5,2} - 16875\, C_{2,4}C_{4,0}C_{4,4} - 72450\, C_{2,8}C_{4,0}^2 + 317700\, C_{3,2}^2C_{4,4}\\
&+ 256500\, C_{3,2}C_{3,6}C_{4,0} + 38650500\, C_{3,6}C_{7,2} + 246600\, C_{5,4}^2,\\
\xi_5 =&807424\, C_{2,0}^5C_{2,8} - 6707400000\, C_{1,6}C_{2,0}^2C_{7,2} - 1888920000\, C_{1,6}C_{2,0}C_{3,2}C_{6,0}\\
&- 785694375\, C_{1,6}C_{2,0}C_{4,0}C_{5,2} - 278572500\, C_{1,6}C_{3,2}C_{4,0}^2 - 120600000\, C_{2,0}^2C_{4,4}^2\\
&- 42918750\, C_{2,0}C_{2,8}C_{4,0}^2 + 5193090000\, C_{2,0}C_{3,2}^2C_{4,4} - 271446918750\, C_{1,6}C_{4,0}C_{7,2}\\
&- 5117321250\, C_{1,6}C_{5,2}C_{6,0} + 338190300000\, C_{2,0}C_{3,6}C_{7,2} + 1145700000\, C_{2,0}C_{5,4}^2\\
&+ 62962200000\, C_{2,4}C_{3,2}C_{7,2} - 450720000\, C_{2,4}C_{4,4}C_{6,0} - 1831612500\, C_{2,4}C_{5,2}^2\\
&+ 4053206250\, C_{2,8}C_{4,0}C_{6,0} - 12202200000\, C_{3,2}C_{3,6}C_{6,0} + 20030895000\, C_{3,2}C_{4,4}C_{5,2}\\
&+ 6489787500\, C_{3,6}C_{4,0}C_{5,2} - 8640074520000\, C_{2,8}C_{10,0} - 245226240000\, C_{4,6}C_{8,2}\\
&+ 170775360000\, C_{5,4}C_{7,4},\\
\xi_6 =&8\, C_{2,0}C_{2,8} - 25\, C_{2,4}^2, \qquad \xi_7 = 48\, C_{2,0}^2C_{2,8} - 475\, C_{1,6}C_{5,2} + 625\, C_{3,2}C_{3,6},\\
\xi_8 =&2588867072\, C_{2,0}^5C_{2,8} - 2215180800000\, C_{1,6}C_{2,0}^2C_{7,2} + 13431825000\, C_{1,6}C_{2,0}C_{4,0}C_{5,2}\\
&- 97632787500\, C_{2,0}C_{2,8}C_{4,0}^2 - 125273250000\, C_{2,0}C_{3,2}^2C_{4,4} + 1345443750000\, C_{1,6}C_{4,0}C_{7,2}\\
&+ 7597800000000\, C_{2,0}C_{3,6}C_{7,2} + 95399876250000\, C_{2,4}C_{3,2}C_{7,2} - 968719500000\, C_{2,4}C_{4,4}C_{6,0}\\
&- 248030859375\, C_{2,4}C_{5,2}^2 - 178311712500\, C_{2,8}C_{4,0}C_{6,0} + 1077259500000\, C_{3,2}C_{4,4}C_{5,2}\\
&- 143877610800000\, C_{2,8}C_{10,0} - 5470416000000\, C_{4,6}C_{8,2} - 25300674000000\, C_{5,4}C_{7,4}\,.
\end{aligned}
$$

We consider the following cusp forms:

$$
F_{8,5} = \frac{135}{8}\nu(\xi_1)\chi_5, \quad F_{8,7} = -\frac{405}{4}\nu(\xi_2)\chi_5,
$$

$$
F_{8,9}^{(1)} = \frac{675}{16}\nu(\xi_3^{(1)})\chi_5, \quad F_{8,9}^{(2)} = \frac{10125}{4}\nu(\xi_3^{(2)})\chi_5,
$$

$$
F_{8,11} = \frac{18225}{16}\nu(\xi_4)\chi_5 \quad F_{8,13} = \frac{54675}{16}\nu(\xi_5)\chi_5, \quad F_{8,15} = -\frac{675}{4}\nu(\xi_6)\chi_5^3,
$$

$$
F_{8,17} = \frac{2025}{2}\nu(\xi_7)\chi_5^3, \quad F_{8,23} = -\frac{382725}{32}\nu(\xi_8)\chi_5^3\,.
$$

The Fourier expansion of

$$
W_{145} = F_{8,5} \wedge F_{8,7} \wedge F_{8,9}^{(1)} \wedge F_{8,9}^{(2)} \wedge F_{8,11} \wedge F_{8,13} \wedge F_{8,15} \wedge F_{8,17} \wedge F_{8,23}
$$

starts with

$$
c\,(u^9 - u^7 - 4u^5 + 4u^3 + 6u - 6/u - 4/u^3 + 4/u^5 + 1/u^7 - 1/u^9)X^{17}Y^{25} + \dots
$$

with $c = -2^{17} \cdot 3^{10} \cdot 5^3 \cdot 7 \cdot 59 \cdot 67 \cdot 103 \cdot 429$. The order of vanishing of W_{145} along H_1 is 5, while along H_4 it is 4, so W_{145} is a multiple of $\chi_5^5\chi_{30}^4$ and a computation at the level of covariants gives

$$W_{145} = -2^{17} \cdot 3^{10} \cdot 5^3 \cdot 7 \cdot 59 \cdot 67 \cdot 103 \cdot 429\, \chi_5^5\chi_{30}^4\,.$$

□

Theorem 14. *The R-module $\mathcal{M}_8^{\mathrm{ev}}(\Gamma_2,\epsilon)$ is free with generators of weight $(8,4)$, $(8,10)$, $(8,12)$, $(8,14)$, $(8,16)$,$(8,18)$, $(8,18)$, $(8,20)$ and $(8,22)$.*

Proof. We use the following covariants

$$\begin{aligned}
\xi_1 &= C_{3,8}, \qquad \xi_5 = C_{5,8},\\
\xi_2 &= 8\,C_{2,0}^3C_{3,8} - 360\,C_{2,0}^2C_{5,8} - 600\,C_{2,0}C_{3,2}C_{4,6} + 28000\,C_{1,6}C_{8,2} - 1875\,C_{3,2}C_{6,6}^{(1)} + 1500\,C_{3,2}C_{6,6}^{(2)} + 3000\,C_{4,0}C_{5,8},\\
\xi_3 &= 64\,C_{2,0}^3C_{5,8} + 960\,C_{2,0}^2C_{3,2}C_{4,6} - 26880\,C_{1,6}C_{2,0}C_{8,2} - 32760\,C_{2,0}C_{2,4}C_{7,4} - 600\,C_{2,0}C_{4,0}C_{5,8} + 405\,C_{3,8}C_{4,0}^2\\
&\quad - 974160\,C_{1,6}C_{10,2} + 705600\,C_{2,4}C_{9,4} + 267120\,C_{3,6}C_{8,2} - 471240\,C_{4,4}C_{7,4} + 3263400\,C_{4,6}C_{7,2} - 44280\,C_{5,2}C_{6,6}^{(1)}\\
&\quad + 41760\,C_{5,8}C_{6,0},\\
\xi_4 &= -450785280\,C_{1,6}C_{2,0}C_{10,2} - 209672400\,C_{1,6}C_{4,0}C_{8,2} - 107933000\,C_{2,0}C_{2,4}C_{9,4} + 322793520\,C_{2,0}C_{3,6}C_{8,2}\\
&\quad - 93936640\,C_{2,0}C_{4,4}C_{7,4} + 708825600\,C_{2,0}C_{4,6}C_{7,2} + 27870759840\,C_{1,6}C_{12,2} - 6460961760\,C_{3,6}C_{10,2}\\
&\quad - 10179070440\,C_{3,8}C_{10,0} - 6501163200\,C_{4,4}C_{9,4} + 2887120425\,C_{7,2}C_{6,6}^{(1)} + 4910108700\,C_{7,2}C_{6,6}^{(2)}\\
&\quad - 19333170\,C_{2,0}C_{5,2}C_{6,6}^{(1)} + 6700200\,C_{2,0}C_{5,2}C_{6,6}^{(2)} + 8466560\,C_{2,0}C_{5,8}C_{6,0} + 104073340\,C_{2,4}C_{3,2}C_{8,2}\\
&\quad + 42245700\,C_{2,4}C_{4,0}C_{7,4} + 26659470\,C_{2,4}C_{5,4}C_{6,0} - 21600\,C_{4,0}^2C_{5,8} + 1024\,C_{2,0}^3C_{3,2}C_{4,6} + 1024\,C_{2,0}^5C_{3,8},\\
\xi_6^{(1)} &= 8\,C_{2,0}C_{5,8} + 25\,C_{2,4}C_{5,4} + 30\,C_{3,2}C_{4,6}, \qquad \xi_6^{(2)} = C_{2,0}^2C_{3,8} - 5\,C_{2,0}C_{5,8} - 25\,C_{3,2}C_{4,6},\\
\xi_7 &= 128\,C_{2,0}^3C_{3,8} + 158200\,C_{1,6}C_{8,2} + 214200\,C_{2,4}C_{7,4} - 88275\,C_{3,2}C_{6,6}^{(1)} + 33900\,C_{3,2}C_{6,6}^{(2)} + 39900\,C_{4,0}C_{5,8},\\
\xi_8 &= 768\,C_{2,0}^4C_{3,8} + 2800000\,C_{1,6}C_{2,0}C_{8,2} - 2782500\,C_{2,0}C_{2,4}C_{7,4} - 11979000\,C_{1,6}C_{10,2} + 66990000\,C_{2,4}C_{9,4}\\
&\quad - 27636000\,C_{3,6}C_{8,2} + 30838500\,C_{4,4}C_{7,4} - 117232500\,C_{4,6}C_{7,2} + 880875\,C_{5,2}C_{6,6}^{(1)} - 1039500\,C_{5,2}C_{6,6}^{(2)}\\
&\quad - 1342500\,C_{5,8}C_{6,0}.
\end{aligned}$$

We consider the following cusp forms:

$$\begin{aligned}
&F_{8,4} = -225\,\nu(\xi_1)\chi_5, \quad F_{8,10} = -\frac{6075}{512}\nu(\xi_2)\chi_5, \quad F_{8,12} = -\frac{6834375}{4}\nu(\xi_3)\chi_5,\\
&F_{8,14} = \frac{102515625}{256}\nu(\xi_4)\chi_5, \quad F_{8,16} = 50625\nu(\xi_5)\chi_5^3 \quad F_{8,18}^{(1)} = \frac{151875}{4}\nu(\xi_6^{(1)})\chi_5^3,\\
&F_{8,18}^{(2)} = -\frac{6075}{16}\nu(\xi_6^{(2)})\chi_5^3, \quad F_{8,20} = \frac{151875}{32}\nu(\xi_7)\chi_5^3, \quad F_{8,22} = -\frac{1366875}{16}\nu(\xi_8)\chi_5^3\,.
\end{aligned}$$

Then

$$W_{170} = F_{8,4} \wedge F_{8,10} \wedge F_{8,12} \wedge F_{8,14} \wedge F_{8,16} \wedge F_{8,18}^{(1)} \wedge F_{8,18}^{(2)} \wedge F_{8,20} \wedge F_{8,22}$$

is a cusp form in $S_{170}(\Gamma_2,\epsilon)$ and its Fourier expansion starts with

$2^{36}\cdot 3^{13}\cdot 5^8\cdot 7^3\cdot 19\,(u^9+u^7-4u^5-4u^3+6u+6/u-4/u^3-4/u^5+1/u^7+1/u^9)X^{19}Y^{29}+\ldots$

One can check that the order of vanishing of W_{170} along H_1 is 4 while along H_4 it is 5, so W_{170} is a multiple of $\chi_5^4\chi_{30}^5$. A calculation with the covariants shows

$$W_{170} = 2^{36} \cdot 3^{13} \cdot 5^8 \cdot 7^3 \cdot 19\, \chi_5^4\chi_{30}^5\,.$$

□

11. The case $j = 10$

Theorem 15. *The R-module $\mathcal{M}_{10}^{\rm odd}(\Gamma_2, \epsilon)$ is free with generators of weight* $(10,5)$*,* $(10,7)$*,* $(10,9)$*,* $(10,9)$*,* $(10,11)$*,* $(10,11)$*,* $(10,13)$*,* $(10,13)$*,* $(8,15)$*,* $(10,15)$ *and* $(10,17)$*.*

Theorem 16. *The R-module $\mathcal{M}_{10}^{\rm ev}(\Gamma_2, \epsilon)$ is free with generators of weight* $(10,8)$*,* $(10,10)$*,* $(10,10)$*,* $(10,12)$*,* $(10,12)$*,* $(10,14)$*,*$(10,14)$*,* $(10,16)$*,* $(10,16)$*,* $(10,18)$ *and* $(10,20)$*.*

The proofs in both cases are similar to the cases above. The covariants used are quite big and we refer for these to [2].

12. The character ϵ of Γ_2

Maaß showed in [16] that the abelianization of Γ_2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. So Γ_2 has one non-trivial character ϵ and it is of order 2. It can be described as the composition

$$\mathrm{Sp}(4,\mathbb{Z}) \xrightarrow{\text{mod } 2} \mathrm{Sp}(4,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cong} \mathfrak{S}_6 \xrightarrow{\text{sign}} \{\pm 1\}\,.$$

The following rules may help in easily determining the value $\epsilon(\gamma)$. If

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then one has

$$\epsilon(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \epsilon(\begin{pmatrix} c & d \\ a & b \end{pmatrix}) = \epsilon(\begin{pmatrix} b & a \\ d & c \end{pmatrix}) = \epsilon(\begin{pmatrix} d & c \\ b & a \end{pmatrix})$$

as one sees by applying $J = (0, 1_g; -1_g, 0)$ on the left and/or on the right.

If γ satisfies

$$\det(a) \equiv \det(b) \equiv \det(c) \equiv \det(d) \equiv 0 \,(\mathrm{mod}\, 2)$$

then we have $\epsilon(\gamma) = -\epsilon(\gamma_0)$ with γ_0 obtained from γ by replacing the first row by minus the third row and the third row by the first row. For this matrix γ_0 at least one of $\det(a_0)$, $\det(b_0)$, $\det(c_0)$, $\det(d_0)$ is not zero modulo 2.

Using this we arrive at the case where γ has the property that $\det(c) \not\equiv 0 (\mathrm{mod}\, 2)$.

Proposition 17. *For $\gamma = (a,b;c,d) \in \Gamma_2$ with $\det(c) \not\equiv 0(\mathrm{mod}\ 2)$ we have $\epsilon(\gamma) = (-1)^\rho$ with ρ given by*

$$a_1c_1+a_2c_1+a_2c_2+a_3c_3+a_4c_3+a_4c_4+c_1c_2+c_2c_3+c_3c_4+c_1d_4+c_2d_3+c_2d_4+c_3d_2+c_4d_1+c_4d_2$$

where the 2×2 matrices are written as $\left(\begin{smallmatrix} x_1 & x_2 \\ x_3 & x_4 \end{smallmatrix}\right)$.

The proof is omitted.

13. Appendix on quasi-modularity

We prove here that the Taylor expansion of a Siegel modular form of degree 2 along the diagonal $\mathfrak{H}_1^2$ yields quasi-modular forms. A reference for quasi-modular forms is [20, Section 5]. We write $QM_k(\Gamma_1)$ for the space of quasi-modular forms of weight k on Γ_1. We will write an element τ of $\mathfrak{H}_2$ as $(\tau_1, z; z, \tau_2)$ and develop a modular form $F \in M_{j,k}(\Gamma_2)$ as a Taylor series in z, the normal coordinate of the diagonal.

Proposition 18. *Let $F \in M_{j,k}(\Gamma_2)$ and write $F = (F_0, F_1, \ldots, F_j)^t$. Then the restriction $F_l|_{\mathfrak{H}_1\times\mathfrak{H}_1}$ lies in $M_{j+k-l}(\Gamma_1) \otimes M_{k+l}(\Gamma_1)$ and for $n \geqslant 1$, we have*

$$\frac{\partial^n F_l}{\partial z^n}|_{\mathfrak{H}_1\times\mathfrak{H}_1} \in QM_{j+k-l+n}(\Gamma_1) \otimes QM_{k+l+n}(\Gamma_1).$$

Proof. The boundedness requirements for quasi-modular forms are easily verified. Using the element of Γ_2 that maps $(\tau_1, z; z, \tau_2)$ to $(\tau_2, z; z, \tau_1)$ and which swaps the coordinates of F from bottom to top up to a sign $(-1)^k$, one sees that it suffices to prove

$$\frac{\partial^n F_l}{\partial z^n}\left(\left(\begin{smallmatrix} \gamma\tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)\right) = (c\tau_1 + d)^{k+j-l+n} \sum_{s=0}^{n} f_s\left(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)\right) \left(\frac{c}{c\tau_1 + d}\right)^s$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1$ where the f_s are holomorphic and depend on n, see [20, page 58]. We embed Γ_1 into Γ_2 via

$$\gamma \mapsto \tilde{\gamma} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with action} \quad \tau \mapsto \begin{pmatrix} \gamma\tau_1 & z/(c\tau_1 + d) \\ z/(c\tau_1 + d) & \tau_2 - cz^2/(c\tau_1 + d) \end{pmatrix}.$$

The modularity of F gives $F(\tilde{\gamma}\tau) = (c\tau_1 + d)^k \mathrm{Sym}^j(\left(\begin{smallmatrix} c\tau_1+d & cz \\ 0 & 1 \end{smallmatrix}\right))F(\tau)$ and a direct computation gives for $l = 0, \ldots, j$

$$F_l(\tilde{\gamma}\tau) = (c\tau_1 + d)^{k+j-l} \sum_{m=0}^{j-l} (c\tau_1 + d)^{-m} \binom{l+m}{l} c^m \, z^m F_{l+m}(\tau). \tag{2}$$

Setting $z = 0$ proves that $F_l(\tilde{\gamma}\tau) = (c\tau_1 + d)^{k+j-l} F_l(\tau)$, hence the first statement and the (quasi-)modularity for $n = 0$. We prove the rest by induction on n. We assume that the proposition is true for $a < n$ i.e.

$$\frac{\partial^a F_l}{\partial z^a}\left(\left(\begin{smallmatrix} \gamma\tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)\right) = (c\tau_1 + d)^{k+j-l+a} \sum_{s=0}^{a} f_s\left(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)\right) \left(\frac{c}{c\tau_1 + d}\right)^s.$$

We differentiate n times both sides of the equation (2) with respect to z and evaluate at $z = 0$, and get

$$\frac{\partial^n F_l}{\partial z^n}\left(\left(\begin{smallmatrix} \gamma\tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)\right) \frac{1}{(c\tau_1 + d)^n} + \sum_{\substack{2i+r=n \\ r\neq n}} \frac{\partial^i}{\partial \tau_2^i} \left(\frac{\partial^r F_l}{\partial z^r}\left(\left(\begin{smallmatrix} \gamma\tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)\right)\right) \frac{(-1)^i n!}{r!\, i!} \frac{c^i}{(c\tau_1 + d)^{i+r}}$$
$$= (c\tau_1 + d)^{k+j-l} \left(\sum_{m=0}^{j-l} \left(\frac{c}{c\tau_1 + d}\right)^m \binom{l+m}{l} \frac{n!}{(n-m)!} \frac{\partial^{n-m} F_{l+m}}{\partial z^{n-m}}\left(\left(\begin{smallmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{smallmatrix}\right)\right) \right).$$

By using the induction hypothesis, we arrive at

$$(c\tau_1+d)^{-(k+j-l+n)}\frac{\partial^n F_l}{\partial z^n}\left(\left(\begin{smallmatrix}\gamma\tau_1 & 0\\ 0 & \tau_2\end{smallmatrix}\right)\right)=$$
$$\sum_{m=0}^{j-l}\left(\frac{c}{c\tau_1+d}\right)^m\binom{l+m}{l}\frac{n!}{(n-m)!}\frac{\partial^{n-m}F_{l+m}}{\partial z^{n-m}}\left(\left(\begin{smallmatrix}\tau_1 & 0\\ 0 & \tau_2\end{smallmatrix}\right)\right)$$
$$+\sum_{\substack{2i+r=n\\ r\neq n\\ 0\leqslant s\leqslant r}}\frac{\partial^i f_s}{\partial\tau_2^i}\left(\left(\begin{smallmatrix}\tau_1 & 0\\ 0 & \tau_2\end{smallmatrix}\right)\right)\frac{(-1)^{i+1}n!}{r!\,i!}\frac{c^{i+r+s}}{(c\tau_1+d)^{i+r+s}}$$

and this shows the proposition. □

Using this proposition we can deduce that $\chi_{6,-2}$ has a Taylor expansion along $\mathfrak{H}_1^2$ with quasi-modular coefficients. Indeed, suppose that a is a non-negative integer such that $\chi_{10}^a\chi_{6,-2}$ is holomorphic. We then apply the proposition to χ_{10}^a and $\chi_{10}^a\chi_{6,-2}$ and get Taylor expansions $\sum_{\mu\geq 2a}a_\mu t^\mu$ and $\sum_{\nu\geq\nu_0}c_\nu t^\nu$ with quasi-modular a_μ and c_ν. Writing the Taylor expansion of $\chi_{6,-2}$ as $\sum_\lambda b_\lambda t^\lambda$ with $c_\nu=\sum_{\mu+\lambda=\nu}a_\mu b_\lambda$ we see by induction that the b_λ are tensor products of quasi-modular forms.

References

[1] J. Bergström, C. Faber, and G. van der Geer: *Siegel modular forms of genus* 2 *and level* 2*: conjectures and cohomological computations.* IMRN, Art. ID rnn 100 (2008).

[2] J. Bergström, C. Faber, G. van der Geer: *Siegel modular forms of degree two and three.* 2018, URL http://smf.compositio.nl

[3] J. Chipalkatti: *On the saturation sequence of the rational normal curve.* J. Pure Appl. Algebra **214** (2010), no. 9, 1598–1611.

[4] F. Cléry, C. Faber, G. van der Geer: *Covariants of binary sextics and vector-valued Siegel modular forms of genus two.* Math. Annalen **369** (2017), 1649–1669.

[5] F. Cléry, G. van der Geer: *On vector-valued Siegel modular forms of degree* 2 *and weight* $(j,2)$, (with an Appendix by G. Chenevier). `arXiv:1709.01748`, to appear in Documenta Math..

[6] F. Cléry, G. van der Geer, S. Grushevsky: *Siegel modular forms of genus 2 and level 2.* Int. J. Math. **(26)** (2015), nr. 5.

[7] R. Donagi, R. Smith: *The structure of the Prym map.* Acta Math. **146** (1981), 25–102.

[8] C. H. van Dorp: *Generators for a module of vector-valued Siegel modular forms of degree 2*, `arXiv:1301.2910`.

[9] E. Freitag: *Ein Verschwindungssatz für automorphe Formen zur Siegelschen Modulgruppe.* Math. Zeitschrift **165** (1979), 11–18.

[10] G. van der Geer: *Hilbert Modular Surfaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete **16**. Springer Verlag, Berlin 1988.

[11] J. H. Grace, A. Young: *The algebra of invariants.* 1903, Cambridge: Cambridge University Press.

[12] T. Ibukiyama: *Vector valued Siegel modular forms of symmetric tensor weight of small degrees.* Comment. Math. Univ. St. Pauli **61** (2012), 51–75.

[13] T. Ibukiyama, S. Wakatsuki: *Siegel modular forms of small weight and the Witt operator.* Contemporary Mathematics **493** (2009), 189–209.

[14] J.-I. Igusa: *On Siegel modular forms of genus two (II)*, Amer. J. Math. **86** (1964), 392–412.

[15] T. Kiyuna: *Vector-valued Siegel modular forms of weight* $\det^k \otimes \mathrm{Sym}(8)$, Int. J. Math. **26** (2015), no. 1.
[16] H. Maaß: *Die Multiplikatorsysteme zur Siegelschen Modulgruppe*. Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1964), 125–135.
[17] T. Satoh: *On certain vector valued Siegel modular forms of degree two.* Math. Ann. **274** (1986), 335–352.
[18] T. Springer: Invariant theory. Lecture Notes in Mathematics, Vol. **585**. Springer-Verlag, Berlin-New York, 1977.
[19] S. Takemori: *Structure theorems for vector valued Siegel modular forms of degree* 2 *and weight* $\det^k \otimes \mathrm{Sym}(8)$, Int. J. Math. **27** (2016), no. 12.
[20] D. Zagier: Elliptic modular forms and their applications. In: The 1-2-3 of modular forms., 1–103. Universitext, Springer, Berlin, 2008.

Department of Mathematical Sciences, Loughborough University, UK
E-mail address: cleryfabien@gmail.com

Mathematisch Instituut, Universiteit Utrecht, Postbus 80010, 3508 TA Utrecht, The Netherlands
E-mail address: C.F.Faber@uu.nl

Korteweg-de Vries Instituut, Universiteit van Amsterdam, Postbus 94248, 1090 GE Amsterdam, The Netherlands
E-mail address: G.B.M.vanderGeer@uva.nl