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L^q-valued Burkholder–Rosenthal inequalities and sharp estimates for stochastic integrals

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Abstract

We prove sharp maximal inequalities for L^q-valued stochastic integrals with respect to any Hilbert space-valued local martingale. Our proof relies on new Burkholder–Rosenthal type inequalities for martingales taking values in an L^q-space.

Contents

1. *Introduction*

This work is motivated by the semigroup approach to stochastic partial differential equations. In this approach, one first reformulates an SPDE as a stochastic ordinary differential equation in a suitable infinite-dimensional state space X and then establishes existence, uniqueness and regularity properties of a mild solution via a fixed point argument. An important ingredient for this argument is a maximal inequality for the X-valued stochastic convolution associated with the semigroup generated by the operator in the stochastic evolution equation. The semigroup approach for equations driven by Gaussian noise in Hilbert spaces is well established and can be found in [**7**]. This theory has more recently been developed in two directions. First, the theory for equations driven by Gaussian noise has been extended to the context of UMD Banach spaces, see, for example, [**38**, **39**]. In particular, the latter results cover L^q-spaces and Sobolev spaces and, as a consequence, allow to achieve better regularity results than the Hilbert space theory. Second, there has been increased interest in equations driven by discontinuous noise, for example, Poisson- and Lévy-type noise [**3**, **11**, **13**, **30**–**32**, **42**]. The latter results are mostly restricted to the Hilbert space setting. The development of this theory in a non-Hilbertian setting is hindered by the fact that maximal inequalities for vector-valued stochastic convolutions with respect to discontinuous noise are not yet well understood. In general, only some non-sharp maximal estimates based on geometric assumptions on the Banach space are available [**9**, **54**]. In fact, even the theory for 'vanilla' stochastic integrals (corresponding to the trivial semigroup) is incomplete. Sharp maximal inequalities for L^q-valued stochastic integrals with respect to Poisson random measures were obtained only recently [**8**].

The main purpose of the present paper is to contribute to the foundation of the semigroup approach by proving sharp estimates for L^q-valued stochastic integrals with respect to general

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Hilbert-space valued local martingales. In our main result, Theorem 5.32, we identify a suitable norm $|||\cdot|||_{M,p,q}$ so that, for any elementary predictable processes Φ with values in the bounded operators from H into $L^q(S)$,

$$c_{p,q}|||\Phi|||_{M,p,q} \leqslant \left(\mathbb{E}\sup_{0\leqslant s\leqslant t}\left\|\int_0^s \Phi\, dM\right\|_{L^q(S)}^p\right)^{\frac{1}{p}} \leqslant C_{p,q}|||\Phi|||_{M,p,q}, \tag{1.1}$$

with universal constants $c_{p,q}$, $C_{p,q}$ depending only on p and q. Let us emphasize two important points. First, the norm $|||\cdot|||_{M,p,q}$ can be computed in terms of *predictable* quantities, which is important for applications. Second, we call the estimates in (1.1) 'sharp' as these inequalities are two-sided and therefore identify the largest possible class of L^p-stochastically integrable processes. We do not require the constants $c_{p,q}$ and $C_{p,q}$ to be sharp or even to depend optimally on p and q. For applications to stochastic evolution equations, the precise constants in fact do not play a role. In forthcoming work together with Marinelli, we show that the upper bound (1.1) can be transferred to a large class of stochastic convolutions and apply these new estimates to obtain improved well-posedness and regularity results for the associated stochastic evolution equations in L^q-spaces.

Let us roughly sketch our approach to (1.1). As a starting point, we use a classical result due to Meyer [**37**] and Yoeurp [**53**] to decompose the integrator as a sum of three local martingales $M = M^c + M^q + M^a$, where M^c is continuous, M^q is purely discontinuous and quasi-left continuous and M^a is purely discontinuous with accessible jumps. Sharp bounds for stochastic integrals with respect to continuous local martingales were already obtained in a more general setting [**48**].

To estimate the integral with respect to M^a, we prove, more generally, sharp bounds for an arbitrary purely discontinuous L^q-valued local martingale with accessible jumps in Theorem 5.14. To establish this result, we first show that such a process can be represented as an essentially discrete object, namely a sum of jumps occurring at predictable times. Using an approximation argument, the problem can then be further reduced to proving *Burkholder–Rosenthal type inequalities* for L^q-valued discrete-time martingales. In general, if $1 \leqslant p < \infty$ and X is a Banach space, we understand under Burkholder–Rosenthal inequalities estimates for X-valued martingale difference sequences (d_i) of the form

$$c_{p,X}|||(d_i)|||_{p,X} \leqslant \left(\mathbb{E}\left\|\sum_i d_i\right\|_X^p\right)^{\frac{1}{p}} \leqslant C_{p,X}|||(d_i)|||_{p,X}, \tag{1.2}$$

where $|||\cdot|||_{p,X}$ is a suitable norm on (d_i) which can be computed explicitly in terms of the *predictable moments* of the individual differences d_i. In the scalar-valued case, these type of inequalities were proven by Burkholder [**4**], following work of Rosenthal [**47**] in the independent case: for $2 \leqslant p < \infty$

$$\left(\mathbb{E}\left|\sum_{i=1}^n d_i\right|^p\right)^{\frac{1}{p}} \eqsim_p \max\left\{\left(\sum_{i=1}^n \mathbb{E}|d_i|^p\right)^{\frac{1}{p}}, \left(\mathbb{E}\left(\sum_{i=1}^n \mathbb{E}_{i-1}|d_i|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}\right\}. \tag{1.3}$$

Here we write $A \lesssim_\alpha B$ if there is a constant $c_\alpha > 0$ depending only on α such that $A \leqslant c_\alpha B$ and write $A \eqsim_\alpha B$ if both $A \lesssim_\alpha B$ and $B \lesssim_\alpha A$ hold. To state our L^q-valued extension, we fix a filtration $\mathbb{F} = (\mathcal{F}_i)_{i\geqslant 1}$, denote by $(\mathbb{E}_i)_{i\geqslant 1}$ the associated sequence of conditional expectations and set $\mathbb{E}_0 := \mathbb{E}$. Let (S,Σ,ρ) be any measure space. Let us introduce the following norms on the linear space of all finite sequences (f_i) of random variables in $L^\infty(\Omega; L^\infty(S))$. First, for $1 \leqslant p, q < \infty$, we set

$$\|(f_i)\|_{S_q^p} = \left(\mathbb{E}\left\|\left(\sum_i \mathbb{E}_{i-1}|f_i|^2\right)^{\frac{1}{2}}\right\|_{L^q(S)}^p\right)^{\frac{1}{p}}. \tag{1.4}$$

From the work of Junge on conditional sequence spaces [21], one can deduce that this expression is a norm. We let S^p_q denote the completion with respect to this norm. Furthermore, we define

$$\|(f_i)\|_{D^p_{q,q}} = \left(\mathbb{E}\left(\sum_i \mathbb{E}_{i-1}\|f_i\|^q_{L^q(S)}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}},$$
$$\|(f_i)\|_{D^p_{p,q}} = \left(\sum_i \mathbb{E}\|f_i\|^p_{L^q(S)}\right)^{\frac{1}{p}}. \tag{1.5}$$

Clearly these expressions define two norms and we let $D^p_{p,q}$ and $D^p_{q,q}$ denote the completions in these norms. Although these spaces depend on the filtration $\mathbb{F}$, we will suppress this from the notation. We let $\hat{S}^p_q$, $\hat{D}^p_{q,q}$ and $\hat{D}^p_{p,q}$ denote the closed subspaces spanned by all martingale difference sequences in the above spaces.

THEOREM 1.1. *Let $1 < p, q < \infty$ and let S be any measure space. If (d_i) is an $L^q(S)$-valued martingale difference sequence, then*

$$\left(\mathbb{E}\left\|\sum_i d_i\right\|^p_{L^q(S)}\right)^{\frac{1}{p}} \eqsim_{p,q} \|(d_i)\|_{\hat{s}_{p,q}}, \tag{1.6}$$

where $\hat{s}_{p,q}$ is given by

$$\hat{S}^p_q \cap \hat{D}^p_{q,q} \cap \hat{D}^p_{p,q} \quad \text{if} \quad 2 \leqslant q \leqslant p < \infty;$$
$$\hat{S}^p_q \cap (\hat{D}^p_{q,q} + \hat{D}^p_{p,q}) \quad \text{if} \quad 2 \leqslant p \leqslant q < \infty;$$
$$(\hat{S}^p_q \cap \hat{D}^p_{q,q}) + \hat{D}^p_{p,q} \quad \text{if} \quad 1 < p < 2 \leqslant q < \infty;$$
$$(\hat{S}^p_q + \hat{D}^p_{q,q}) \cap \hat{D}^p_{p,q} \quad \text{if} \quad 1 < q < 2 \leqslant p < \infty;$$
$$\hat{S}^p_q + (\hat{D}^p_{q,q} \cap \hat{D}^p_{p,q}) \quad \text{if} \quad 1 < q \leqslant p \leqslant 2;$$
$$\hat{S}^p_q + \hat{D}^p_{q,q} + \hat{D}^p_{p,q} \quad \text{if} \quad 1 < p \leqslant q \leqslant 2.$$

Consequently, if $\mathcal{F} = \sigma(\cup_{i\geqslant 1}\mathcal{F}_i)$, then the map $f \mapsto (\mathbb{E}_i f - \mathbb{E}_{i-1} f)_{i\geqslant 1}$ induces an isomorphism between $L^p_0(\Omega; L^q(S))$, the subspace of mean-zero random variables in $L^p(\Omega; L^q(S))$, and $\hat{s}_{p,q}$.

Let us say a few words about the proof of Theorem 1.1. We derive the upper bound in (1.6) from the known special case that the d_i are independent [8] by applying powerful decoupling techniques due to Kwapień and Woyczyński [25]. In the scalar-valued case, this route was already traveled by Hitczenko [15] to deduce the optimal order of the constant in the classical Burkholder–Rosenthal inequalities (1.3) from the one already known for martingales with independent increments. The lower bound in (1.6) is derived by using a duality argument. For this purpose, we show that for $1 < p, q < \infty$ the spaces $s_{p,q}$ satisfy the duality relation

$$(s_{p,q})^* = s_{p',q'}, \ \frac{1}{p} + \frac{1}{p'} = 1, \ \frac{1}{q} + \frac{1}{q'} = 1.$$

Since it follows from the work of Junge [21] that $(S^p_q)^* = S^{p'}_{q'}$ (see also Appendix A for a short proof valid for Borel probability spaces) and clearly $(D^p_{p,q})^* = D^{p'}_{p',q'}$, the only non-trivial step in proving this duality is to show that $(D^p_{q,q})^* = D^{p'}_{q',q'}$. In Section 4, we prove a more general result: we show that if X is a reflexive separable Banach space, then for the space $H_p^{s_q}(X)$ of all adapted X-valued sequences (f_i) such that

$$\|(f_i)\|_{H_p^{s_q}(X)} = \left(\mathbb{E}\left(\sum_i \mathbb{E}_{i-1}\|f_i\|_X^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} < \infty,$$

the identity $(H_p^{s_q}(X))^* = H_{p'}^{s_{q'}}(X^*)$ holds isomorphically with constants depending only on p and q. Somewhat surprisingly, this result only seems to be known in the literature if $X = \mathbb{R}$ and either $1 < p \leqslant q < \infty$ or $2 \leqslant q \leqslant p < \infty$ (see [**49**]).

Our proof can be modified to extend the result in Theorem 1.1 to martingales taking values in non-commutative L^q-spaces. Since this extension is of interest to a different audience, we choose to defer its development to future work.

Let us now discuss our approach to the integral of Φ with respect to M^q, the purely discontinuous quasi-left continuous part of M. We first show in Lemma 5.18 that this integral can be represented as an integral with respect to $\bar{\mu}^{M^q}$, the compensated version of the random measure μ^{M^q} that counts the jumps of M^q. In Theorem 5.28, we then prove the following sharp estimates for integrals with respect to $\bar{\mu} = \mu - \nu$, where μ is any integer-valued random measure that has a compensator ν that is non-atomic in time. This result covers μ^{M^q} as a special case. To formulate our result, let $(J, \mathcal{J})$ be a measurable space and $\widetilde{\mathcal{P}}$ be the predictable σ-algebra on $\mathbb{R}_+ \times \Omega \times J$. For $1 < p, q < \infty$, we define the spaces $\hat{\mathcal{S}}_q^p$, $\hat{\mathcal{D}}_{q,q}^p$ and $\hat{\mathcal{D}}_{p,q}^p$ as the Banach spaces of all $\widetilde{\mathcal{P}}$-measurable functions $F : \mathbb{R}_+ \times \Omega \times J \to L^q(S)$ for which the corresponding norms

$$\|F\|_{\hat{\mathcal{S}}_q^p} := \left(\mathbb{E}\left\|\left(\int_{\mathbb{R}_+ \times J} |F|^2 \,\mathrm{d}\nu\right)^{\frac{1}{2}}\right\|_{L^q(S)}^p\right)^{\frac{1}{p}},$$

$$\|F\|_{\hat{\mathcal{D}}_{q,q}^p} := \left(\mathbb{E}\left(\int_{\mathbb{R}_+ \times J} \|F\|_{L^q(S)}^q \,\mathrm{d}\nu\right)^{\frac{p}{q}}\right)^{\frac{1}{p}},$$

$$\|F\|_{\hat{\mathcal{D}}_{p,q}^p} := \left(\mathbb{E}\int_{\mathbb{R}_+ \times J} \|F\|_{L^q(S)}^p \,\mathrm{d}\nu\right)^{\frac{1}{p}}$$

are finite.

THEOREM 1.2. *Fix $1 < p, q < \infty$. Let μ be an optional $\widetilde{\mathcal{P}}$-σ-finite random measure on $\mathbb{R}_+ \times J$ and suppose that its compensator ν is non-atomic in time. Then for any $\widetilde{\mathcal{P}}$-measurable $F : \mathbb{R}_+ \times \Omega \times J \to L^q(S)$,*

$$\left(\mathbb{E}\sup_{0\leqslant s\leqslant t}\left\|\int_{[0,s]\times J} F(u,x)\bar{\mu}(\mathrm{d}u,\mathrm{d}x)\right\|_{L^q(S)}^p\right)^{\frac{1}{p}} \eqsim_{p,q} \|F\mathbf{1}_{[0,t]}\|_{\mathcal{I}_{p,q}},$$

where $\mathcal{I}_{p,q}$ is given by

$$\begin{aligned}
&\hat{\mathcal{S}}_q^p \cap \hat{\mathcal{D}}_{q,q}^p \cap \hat{\mathcal{D}}_{p,q}^p && \text{if } 2 \leqslant q \leqslant p < \infty,\\
&\hat{\mathcal{S}}_q^p \cap (\hat{\mathcal{D}}_{q,q}^p + \hat{\mathcal{D}}_{p,q}^p) && \text{if } 2 \leqslant p \leqslant q < \infty,\\
&(\hat{\mathcal{S}}_q^p \cap \hat{\mathcal{D}}_{q,q}^p) + \hat{\mathcal{D}}_{p,q}^p && \text{if } 1 < p < 2 \leqslant q < \infty,\\
&(\hat{\mathcal{S}}_q^p + \hat{\mathcal{D}}_{q,q}^p) \cap \hat{\mathcal{D}}_{p,q}^p && \text{if } 1 < q < 2 \leqslant p < \infty,
\end{aligned}$$

$$\hat{\mathcal{S}}_q^p + (\hat{\mathcal{D}}_{q,q}^p \cap \hat{\mathcal{D}}_{p,q}^p) \quad \textit{if } 1 < q \leqslant p \leqslant 2,$$

$$\hat{\mathcal{S}}_q^p + \hat{\mathcal{D}}_{q,q}^p + \hat{\mathcal{D}}_{p,q}^p \quad \textit{if } 1 < p \leqslant q \leqslant 2.$$

In the scalar-valued case, this result is due to Novikov [**40**]. In the special case that μ is a Poisson random measure, Theorem 5.28 was obtained in [**8**]. A very different proof of the upper bounds in Theorem 5.28, based on tools from stochastic analysis, was discovered independently of our work in [**29**].

The proof of the upper bounds in Theorem 1.2 relies on the Burkholder–Rosenthal inequalities in Theorem 1.1, a Banach space-valued extension of Novikov's inequality in the special case that $\nu(\mathbb{R}_+ \times J) \leqslant 1$ almost surely (see Proposition 5.22), and a time-change argument. For the lower bounds, the non-trivial work is to show that

$$(\hat{\mathcal{S}}_q^p)^* = \hat{\mathcal{S}}_{q'}^{p'}, \quad (\hat{\mathcal{D}}_{q,q}^p)^* = \hat{\mathcal{D}}_{q',q'}^{p'}$$

hold isomorphically with constants depending only on p and q. These duality statements are derived in Appendix B.

Our paper is structured as follows. In Section 3, we prove Theorem 1.1. This proof relies on the duality for the spaces $H_p^{s_q}(X)$, which we prove in Section 4. In Section 5, we prove the sharp bounds (1.1) in several steps. In Section 5.2, we extend the classical martingale decomposition of Meyer [**37**] and Yoeurp [**53**] to the setting of Hilbert-space valued martingales. As has been explained before, this reduces the problem of obtaining (1.1) to proving sharp bounds for integrals with respect to local martingales with accessible jumps, purely discontinuous quasi-left continuous local martingales, and continuous local martingales. We address these problems in Sections 5.3, 5.4, and 5.5, respectively. These three parts can be read independently of each other. Finally, Section 5.6 combines the sharp estimates obtained in these three sections to deduce the main result of this work, Theorem 5.32.

We discuss the necessary preliminaries for our development at relevant locations. In particular, Section 2 discusses some general preliminaries that are used throughout. Section 5.1 discusses terminology regarding stochastic processes, martingales and stopping times. Section 5.4.1 concerns random measures and integration with respect to random measures. Finally, Section 5.5 contains some preliminaries on γ-radonifying operators.

2. *Preliminaries*

Throughout, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space. If X and Y are Banach spaces, then $\mathcal{L}(X, Y)$ denotes the Banach space of bounded linear operators from X into Y. Any X-valued random variable $f : \Omega \to X$ in this paper will be assumed to be strongly $\mathbb{P}$-measurable. Recall that by the Pettis measurability theorem [**17**, Theorem 1.1.20], this is equivalent to assuming that f is a.s. separably valued and that $\langle f, x^* \rangle$ is measurable for any $x^* \in X^*$. In proofs involving countably many random variables, we will therefore typically assume without of loss of generality that the range space X is separable. Similarly, in proofs involving an a.s. right-continuous process $(f_t)_{t \geqslant 0}$, one can typically reduce to the separable case by considering the closed subspace spanned by the images of f_t, $t \in \mathbb{Q}$. We will use these standard reductions throughout without mentioning them explicitly.

In particular, in the case of $X = L^q(S)$, we may always assume that $L^q(S)$ is separable. Indeed, let $X = L^q(S)$ be non-separable. Let $X_0 \subset X$ be a separable subspace (constructed, for example, by the aforementioned argument). Let us show that $X_0 \subset L^q(S_0)$ for some separable $L^q(S_0)$. Fix a dense sequence $(x_n)_{n \geqslant 1} \subset X_0$. Since every x_n has σ-finite support, $S_0 := \cup_n \text{supp}\,(x_n)$ is σ-finite and hence $L^q(S_0)$ is separable. Moreover, $x_n \in L^q(S_0)$ for all $n \geqslant 1$ and so $X_0 = \overline{(x_n)_{n \geqslant 1}} \subset L^q(S_0)$, proving the claim.

In the following, we will frequently use duality arguments for sums and intersections of Banach spaces. Let us recall some basic facts in this direction. If (X, Y) is a compatible couple of Banach spaces, that is, X, Y are continuously embedded in a Hausdorff topological vector space, then their intersection $X \cap Y$ and sum $X + Y$ are Banach spaces under the norms

$$\|z\|_{X \cap Y} = \max\{\|z\|_X, \|z\|_Y\}$$

and

$$\|z\|_{X+Y} = \inf\{\|x\|_X + \|y\|_Y : z = x + y,\ x \in X,\ y \in Y\}.$$

If $X \cap Y$ is dense in both X and Y, then

$$(X \cap Y)^* = X^* + Y^*, \quad (X + Y)^* = X^* \cap Y^* \tag{2.1}$$

hold isometrically. The duality brackets under these identifications are given by

$$\langle x^*, x\rangle = \langle x^*|_{X\cap Y}, x\rangle \quad (x^* \in X^* + Y^*)$$

and

$$\langle x^*, x\rangle = \langle x^*, y\rangle + \langle x^*, z\rangle \quad (x^* \in X^* \cap Y^*,\ x = y + z \in X + Y), \tag{2.2}$$

respectively, see, for example, [**24**, Theorem I.3.1].

The following observation facilitates a duality argument that we will use repeatedly below. We leave the straightforward proof to the reader.

LEMMA 2.1. *Let X and Y be Banach spaces, X be reflexive, U be a dense linear subspace of X and let V be a dense linear subspace of X^*. Consider $j_0 \in \mathcal{L}(U, Y)$ and $k_0 \in \mathcal{L}(V, Y^*)$ so that* ran j_0 *is dense in Y and $\langle x^*, x\rangle = \langle k_0(x^*), j_0(x)\rangle$ for each $x \in U, x^* \in V$. Then*

(i) *there exists $j \in \mathcal{L}(X, Y)$, $k \in \mathcal{L}(X^*, Y^*)$ such that $j|_U = j_0$ and $k|_V = k_0$,*
(ii) ran $j = Y$, ran $k = Y^*$, *in particular k and j are invertible, and*
(iii) *for each $x \in X$ and $x^* \in X^*$*

$$\begin{aligned} \frac{1}{\|k\|}\|x\| &\leqslant \|j(x)\| \leqslant \|j\|\|x\|, \\ \frac{1}{\|j\|}\|x^*\| &\leqslant \|k(x^*)\| \leqslant \|k\|\|x^*\|. \end{aligned} \tag{2.3}$$

3. *L^q-valued Burkholder–Rosenthal inequalities*

In this section, we prove Theorem 1.1. Our starting point is the following L^q-valued version of the classical Rosenthal inequalities [**47**]. For all $1 \leqslant p, q < \infty$ let S_q and $D_{p,q}$ be the spaces of all sequences of $L^q(S)$-valued random variables such the respective norms

$$\begin{aligned} \|(f_i)\|_{S_q} &= \left\| \left(\sum_i \mathbb{E}|f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^q(S)}, \\ \|(f_i)\|_{D_{p,q}} &= \left(\sum_i \mathbb{E}\|f_i\|^p_{L^q(S)} \right)^{\frac{1}{p}} \end{aligned} \tag{3.1}$$

are finite. Note that the following result corresponds to a special case of Theorem 1.1, in which the martingale differences d_i are independent.

THEOREM 3.1 [8]. *Let $1 < p, q < \infty$ and let (S, Σ, σ) be a measure space. If $(\xi_i)_{i \geqslant 1}$ is a sequence of independent, mean-zero random variables taking values in $L^q(S)$, then*

$$\left(\mathbb{E}\left\|\sum_{i=1}^{\infty} \xi_i\right\|_{L^q(S)}^p\right)^{\frac{1}{p}} \eqsim_{p,q} \|(\xi_i)\|_{s_{p,q}}, \tag{3.2}$$

where $s_{p,q}$ is given by

$$\begin{aligned}
S_q \cap D_{q,q} \cap D_{p,q} &\quad \text{if} \quad 2 \leqslant q \leqslant p < \infty; \\
S_q \cap (D_{q,q} + D_{p,q}) &\quad \text{if} \quad 2 \leqslant p \leqslant q < \infty; \\
(S_q \cap D_{q,q}) + D_{p,q} &\quad \text{if} \quad 1 < p < 2 \leqslant q < \infty; \\
(S_q + D_{q,q}) \cap D_{p,q} &\quad \text{if} \quad 1 < q < 2 \leqslant p < \infty; \\
S_q + (D_{q,q} \cap D_{p,q}) &\quad \text{if} \quad 1 < q \leqslant p \leqslant 2; \\
S_q + D_{q,q} + D_{p,q} &\quad \text{if} \quad 1 < p \leqslant q \leqslant 2.
\end{aligned}$$

Moreover, the estimate $\lesssim_{p,q}$ in (3.2) remains valid if $p = 1$, $q = 1$ or both.

To derive the upper bound in Theorem 1.1, we use the following decoupling techniques from [25]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, let $(\mathcal{F}_i)_{i \geqslant 0}$ be a filtration and let X be a Banach space. Two $(\mathcal{F}_i)_{i \geqslant 1}$-adapted sequences $(d_i)_{i \geqslant 1}$ and $(e_i)_{i \geqslant 1}$ of X-valued random variables are called *tangent* if for every $i \geqslant 1$ and $A \in \mathcal{B}(X)$

$$\mathbb{P}(d_i \in A | \mathcal{F}_{i-1}) = \mathbb{P}(e_i \in A | \mathcal{F}_{i-1}). \tag{3.3}$$

An $(\mathcal{F}_i)_{i \geqslant 1}$-adapted sequence $(e_i)_{i \geqslant 1}$ of X-valued random variables is said to satisfy *condition* (*CI*) if, first, there exists a sub-σ-algebra

$$\mathcal{G} \subset \mathcal{F}_\infty = \sigma(\cup_{i \geqslant 0} \mathcal{F}_i)$$

such that for every $i \geqslant 1$ and $A \in \mathcal{B}(X)$,

$$\mathbb{P}(e_i \in A | \mathcal{F}_{i-1}) = \mathbb{P}(e_i \in A | \mathcal{G}) \tag{3.4}$$

and, second, $(e_i)_{i \geqslant 1}$ consists of $\mathcal{G}$-independent random variables, that is, for all $n \geqslant 1$ and $A_1, \ldots, A_n \in \mathcal{B}(X)$,

$$\mathbb{E}(\mathbf{1}_{e_1 \in A_1} \cdot \cdots \cdot \mathbf{1}_{e_n \in A_n} | \mathcal{G}) = \mathbb{E}(\mathbf{1}_{e_1 \in A_1} | \mathcal{G}) \cdot \cdots \cdot \mathbb{E}(\mathbf{1}_{e_n \in A_n} | \mathcal{G}).$$

It is shown in [25] that for every $(\mathcal{F}_i)_{i \geqslant 1}$-adapted sequence $(d_i)_{i \geqslant 1}$ there exists an $(\mathcal{F}_i)_{i \geqslant 1}$-adapted sequence $(e_i)_{i \geqslant 1}$ on a possibly enlarged probability space which is tangent to $(d_i)_{i \geqslant 1}$ and satisfies condition (CI). This sequence is called a *decoupled tangent sequence* for $(d_i)_{i \geqslant 1}$ and is unique in law.

To derive the upper bound in Theorem 1.1 for a given martingale difference sequence $(d_i)_{i \geqslant 1}$, we apply Theorem 3.1 conditionally to its decoupled tangent sequence $(e_i)_{i \geqslant 1}$. For this approach to work, we will need to relate various norms on $(d_i)_{i \geqslant 1}$ and $(e_i)_{i \geqslant 1}$. One of these estimates can be formulated as a Banach space property. Following [5], we say that a Banach space X satisfies *the p-decoupling property* if for some $0 < p < \infty$ there is a constant $C_{p,X}$ such that for any complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, any filtration $(\mathcal{F}_i)_{i \geqslant 0}$, and any $(\mathcal{F}_i)_{i \geqslant 1}$-adapted sequence $(d_i)_{i \geqslant 1}$ in $L^p(\Omega, X)$,

$$\left(\mathbb{E}\left\|\sum_{i=1}^{n} d_i\right\|_X^p\right)^{\frac{1}{p}} \leqslant C_{p,X}\left(\mathbb{E}\left\|\sum_{i=1}^{n} e_i\right\|_X^p\right)^{\frac{1}{p}}, \tag{3.5}$$

for all $n \geqslant 1$, where $(e_i)_{i\geqslant 1}$ is the decoupled tangent sequence of $(d_i)_{i\geqslant 1}$. It is shown in [**5**, Theorem 4.1] that this property is independent of p, so we may simply say that X satisfies the *decoupling property* if it satisfies the p-decoupling property for some (then all) $0 < p < \infty$. Known examples of spaces satisfying the decoupling property are the $L^q(S)$-spaces for any $1 \leqslant q < \infty$ and UMD Banach spaces. If X is a UMD Banach space, then one can also *recouple*, meaning that for all $1 < p < \infty$ there is a constant $c_{p,X}$ such that for any martingale difference sequence $(d_i)_{i\geqslant 1}$ and any associated decoupled tangent sequence $(e_i)_{i\geqslant 1}$,

$$\left(\mathbb{E}\left\|\sum_{i=1}^{n} e_i\right\|_X^p\right)^{\frac{1}{p}} \leqslant c_{p,X}\left(\mathbb{E}\left\|\sum_{i=1}^{n} d_i\right\|_X^p\right)^{\frac{1}{p}}. \tag{3.6}$$

Conversely, if both (3.5) and (3.6) hold for some (then all) $1 < p < \infty$, then X must be a UMD space. This equivalence is independently due to McConnell [**35**] and Hitczenko [**14**].

To further relate a sequence with its decoupled tangent sequence, we use the following technical observation, which is a special case of [**5**, Lemma 2.7].

LEMMA 3.2. *Let X be a Banach space and for every $i \geqslant 1$ let $h_i : X \to X$ be a Borel measurable function. Let $(d_i)_{i\geqslant 1}$ be an $(\mathcal{F}_i)_{i\geqslant 1}$-adapted sequence and $(e_i)_{i\geqslant 1}$ a decoupled tangent sequence. Then $(h_i(e_i))_{i\geqslant 1}$ is a decoupled tangent sequence for $(h_i(d_i))_{i\geqslant 1}$.*

We are now ready to prove the announced result.

Proof of Theorem 1.1. *Step* 1: *upper bounds.* We will only give a proof in the case $1 \leqslant q \leqslant 2 \leqslant p < \infty$. The other cases are proved analogously. Let us write $\mathbb{E}_{\mathcal{G}} = \mathbb{E}(\cdot|\mathcal{G})$ for brevity. By density we may assume that the d_i take values in $L^q(S) \cap L^\infty(S)$. Fix an arbitrary decomposition $d_i = d_{i,1} + d_{i,2}$, where $d_{i,1}, d_{i,2}$ are $L^q(S) \cap L^\infty(S)$-valued martingale difference sequences. Let $e_i = (e_{i,1}, e_{i,2})$ be the decoupled tangent sequence for the martingale difference sequence $(d_{i,1}, d_{i,2})$ which takes values in the space $(L^q(S) \cap L^\infty(S)) \times (L^q(S) \cap L^\infty(S))$. Lemma 3.2 implies that $d_{i,\alpha}$ is the decoupled tangent sequence for $e_{i,\alpha}$, $\alpha = 1, 2$, and $e_{i,1} + e_{i,2}$ is the decoupled tangent sequence for d_i. By the decoupling property for $L^q(S)$,

$$\left(\mathbb{E}\left\|\sum_i d_i\right\|_{L^q(S)}^p\right)^{\frac{1}{p}} \lesssim_{p,q} \left(\mathbb{E}\left\|\sum_i e_{i,1} + e_{i,2}\right\|_{L^q(S)}^p\right)^{\frac{1}{p}}.$$

Since the summands $e_{i,1} + e_{i,2}$ are $\mathcal{G}$-conditionally independent and $\mathcal{G}$-mean zero, we can apply Theorem 3.1 conditionally to find, a.s.,

$$\left(\mathbb{E}_{\mathcal{G}}\left\|\sum_i e_{i,1} + e_{i,2}\right\|_{L^q(S)}^p\right)^{\frac{1}{p}} \lesssim_{p,q} \max\Bigg\{\left\|\left(\sum_i \mathbb{E}_{\mathcal{G}}|e_{i,1}|^2\right)^{\frac{1}{2}}\right\|_{L^q(S)} \\ + \left(\sum_i \mathbb{E}_{\mathcal{G}}\|e_{i,2}\|_{L^q(S)}^q\right)^{\frac{1}{q}}, \left(\sum_i \mathbb{E}_{\mathcal{G}}\|e_{i,1} + e_{i,2}\|_{L^q(S)}^p\right)^{\frac{1}{p}}\Bigg\}.$$

Now, we take L^p-norms on both sides and apply the triangle inequality to obtain

$$\Big(\mathbb{E}\Big\|\sum_i d_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \max\Bigg\{\Bigg(\mathbb{E}\Bigg\|\Big(\sum_i \mathbb{E}_{\mathcal{G}}|e_{i,1}|^2\Big)^{\frac{1}{2}}\Bigg\|_{L^q(S)}^p\Bigg)^{\frac{1}{p}} + \Bigg(\mathbb{E}\Big(\sum_i \mathbb{E}_{\mathcal{G}}\|e_{i,2}\|_{L^q(S)}^q\Big)^{\frac{p}{q}}\Bigg)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|e_{i,1}+e_{i,2}\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\Bigg\}.$$

By the properties (3.4) and (3.3) of a decoupled tangent sequence,

$$\mathbb{E}_{\mathcal{G}}|e_{i,1}|^2 = \mathbb{E}_{i-1}|e_{i,1}|^2 = \mathbb{E}_{i-1}|d_{i,1}|^2,$$

and therefore

$$\Big(\sum_i \mathbb{E}_{\mathcal{G}}|e_{i,1}|^2\Big)^{\frac{1}{2}} = \Big(\sum_i \mathbb{E}_{i-1}|d_{i,1}|^2\Big)^{\frac{1}{2}}.$$

Similarly,

$$\mathbb{E}_{\mathcal{G}}\|e_{i,2}\|_{L^q(S)}^q = \mathbb{E}_{i-1}\|d_{i,2}\|_{L^q(S)}^q.$$

We conclude that

$$\Big(\mathbb{E}\Big\|\sum_i d_i\Big\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \lesssim_{p,q} \max\Bigg\{\Bigg(\mathbb{E}\Bigg\|\Big(\sum_i \mathbb{E}_{i-1}|d_{i,1}|^2\Big)^{\frac{1}{2}}\Bigg\|_{L^q(S)}^p\Bigg)^{\frac{1}{p}} + \Bigg(\mathbb{E}\Big(\sum_i \mathbb{E}_{i-1}\|d_{i,2}\|_{L^q(S)}^q\Big)^{\frac{p}{q}}\Bigg)^{\frac{1}{p}}, \Big(\sum_i \mathbb{E}\|d_i\|_{L^q(S)}^p\Big)^{\frac{1}{p}}\Bigg\}.$$

Taking the infimum over all decompositions as above yields the inequality '$\lesssim_{p,q}$' in (1.6).

Step 2: *lower bounds*. We deduce the lower bounds by duality. It follows from Junge's work on conditional sequence spaces [**21**] that $(S_q^p)^* = S_{q'}^{p'}$ holds isomorphically with constants depending only on p and q. Since moreover $(D_{p,q}^p)^* = D_{p',q'}^{p'}$ and $(D_{q,q}^p)^* = D_{q',q'}^{p'}$ (see Theorem 4.1), it follows from (2.1) that $s_{p,q}^* = s_{p',q'}$ with duality bracket

$$\langle (f_i),(g_i)\rangle = \sum_i \mathbb{E}\langle f_i, g_i\rangle \quad ((f_i)\in s_{p,q},\ (g_i)\in s_{p',q'}).$$

Let $\hat{x}^* \in (\hat{s}_{p,q})^*$. Define the map $P: s_{p,q}\to \hat{s}_{p,q}$ by

$$P((f_i)) = (\Delta_i f_i),$$

where $\Delta_i := \mathbb{E}_i - \mathbb{E}_{i-1}$. By the triangle inequality and Jensen's inequality, one readily sees that P is a bounded projection. As a consequence, we can define $x^* \in s_{p,q}^*$ by $x^* = \hat{x}^*\circ P$. Let $(g_i)\in s_{p',q'}$ be such that

$$x^*((f_i)) = \sum_i \mathbb{E}\langle f_i, g_i\rangle \quad ((f_i)\in s_{p,q}).$$

Then, for any $(f_i)\in \hat{s}_{p,q}$,

$$\hat{x}^*((f_i)) = \sum_i \mathbb{E}\langle f_i, g_i\rangle = \sum_i \mathbb{E}\langle f_i, \Delta_i g_i\rangle = \langle (f_i), P(g_i)\rangle.$$

This shows that $(\hat{s}_{p,q})^* = \hat{s}_{p',q'}$ isomorphically. Let U and V be the dense linear subspaces spanned by all finite martingale difference sequences in $\hat{s}_{p,q}$ and $\hat{s}_{p',q'}$, respectively. Define

$$Y = \overline{\text{span}}\left\{\sum_i d_i \ : \ (d_i) \in U\right\} \subset L^p(\Omega; L^q(S)).$$

By Step 1, we can define two maps $j_0 \in \mathcal{L}(U, Y)$, $k_0 \in \mathcal{L}(V, Y^*)$ by

$$j_0((d_i)) = \sum_i d_i, \quad k_0((\tilde{d}_i)) = \sum_i \tilde{d}_i.$$

By the martingale difference property,

$$\langle j_0((d_i)), k_0((\tilde{d}_i))\rangle = \mathbb{E}\left\langle \sum_i d_i, \sum_i \tilde{d}_i \right\rangle = \sum_i \mathbb{E}\langle d_i, \tilde{d}_i\rangle = \langle (d_i), (\tilde{d}_i)\rangle. \tag{3.7}$$

The lower bounds now follow immediately from Lemma 2.1.

For the final assertion of the theorem, suppose $\mathcal{F} = \sigma(\cup_{i\geqslant 0}\mathcal{F}_i)$. Let $f \in L^p_0(\Omega; L^q(S))$ and define $f_n = \mathbb{E}_n f$. Then $\lim_{n\to\infty} f_n = f$ (see, for example, [**17**, Theorem 3.3.2]). Conversely, let $(f_n)_{n\geqslant 1}$ be a martingale with $\sup_{n\geqslant 1} \|f_n\|_{L^p(\Omega;L^q(S))} < \infty$. By reflexivity of $L^q(S)$, we have that

$$L^p(\Omega; L^q(S)) = (L^{p'}(\Omega; L^{q'}(S)))^*,$$

and hence its unit ball is weak*-compact. Let f be the weak*-limit of (f_n). It is easy to check that $f_n = \mathbb{E}_n f$. In conclusion, any martingale difference sequence $(d_i)_{i\geqslant 1}$ of a bounded martingale in $L^p(\Omega; L^q(S))$ corresponds uniquely to an $f \in L^p(\Omega; L^q(S))$ such that

$$f - \mathbb{E}f = \sum_i d_i, \quad d_i = \mathbb{E}_i f - \mathbb{E}_{i-1} f.$$

The two-sided inequality (1.6) now implies that the map

$$f \mapsto (\mathbb{E}_i f - \mathbb{E}_{i-1} f)_{i\geqslant 1}$$

is a linear isomorphism between $L^p_0(\Omega; L^q(S))$ and $\hat{s}_{p,q}$, with constants depending only on p and q.

Later in the proof of Theorem 5.6 we will need the following version of Theorem 1.1.

PROPOSITION 3.3. *Let $1 < p, q < \infty$. Define $\hat{S}^{p,\mathrm{odd}}_q$, $\hat{D}^{p,\mathrm{odd}}_{q,q}$ and $\hat{D}^{p,\mathrm{odd}}_{p,q}$ as the closed subspaces of $\hat{S}^p_q$, $\hat{D}^p_{q,q}$ and $\hat{D}^p_{p,q}$, respectively, spanned by all L^q-valued martingale difference sequences $(d_i)_{i\geqslant 1}$ such that $d_{2i} = 0$ for each $i \geqslant 1$. Then, any L^q-valued martingale difference sequence $(d_i)_{i\geqslant 1}$ such that $d_{2i} = 0$ for each $i \geqslant 1$ satisfies*

$$\left(\mathbb{E}\left\|\sum_i d_i\right\|^p_{L^q(S)}\right)^{\frac{1}{p}} \eqsim_{p,q} \|(d_i)\|_{\hat{s}^{\mathrm{odd}}_{p,q}},$$

where $\hat{s}^{\mathrm{odd}}_{p,q}$ is given by

$$\begin{aligned}
&\hat{S}^{p,\mathrm{odd}}_q \cap \hat{D}^{p,\mathrm{odd}}_{q,q} \cap \hat{D}^{p,\mathrm{odd}}_{p,q} && \text{if } 2 \leqslant q \leqslant p < \infty;\\
&\hat{S}^{p,\mathrm{odd}}_q \cap (\hat{D}^{p,\mathrm{odd}}_{q,q} + \hat{D}^{p,\mathrm{odd}}_{p,q}) && \text{if } 2 \leqslant p \leqslant q < \infty;\\
&(\hat{S}^{p,\mathrm{odd}}_q \cap \hat{D}^{p,\mathrm{odd}}_{q,q}) + \hat{D}^{p,\mathrm{odd}}_{p,q} && \text{if } 1 < p < 2 \leqslant q < \infty;\\
&(\hat{S}^{p,\mathrm{odd}}_q + \hat{D}^{p,\mathrm{odd}}_{q,q}) \cap \hat{D}^{p,\mathrm{odd}}_{p,q} && \text{if } 1 < q < 2 \leqslant p < \infty;
\end{aligned}$$

$$\hat{S}_q^{p,\mathrm{odd}} + (\hat{D}_{q,q}^{p,\mathrm{odd}} \cap \hat{D}_{p,q}^{p,\mathrm{odd}}) \quad \text{if} \quad 1 < q \leqslant p \leqslant 2;$$

$$\hat{S}_q^{p,\mathrm{odd}} + \hat{D}_{q,q}^{p,\mathrm{odd}} + \hat{D}_{p,q}^{p,\mathrm{odd}} \quad \text{if} \quad 1 < p \leqslant q \leqslant 2.$$

To prove Proposition 3.3, it suffices by Theorem 1.1 to show that $\|(d_i)\|_{\hat{s}_{p,q}^{\mathrm{odd}}} = \|(d_i)\|_{\hat{s}_{p,q}}$ for any $(d_i)_{i\geqslant 1} \in \hat{s}_{p,q}^{\mathrm{odd}}$. The latter follows from the following lemma.

LEMMA 3.4. *Let $1 < p, q < \infty$, X be any finite $+$ and $\cap$ combination of the spaces $\hat{S}_q^p$, $\hat{D}_{q,q}^p$ and $\hat{D}_{p,q}^p$. Let X^{odd} be the same $+$ and $\cap$ combination of the spaces $\hat{S}_q^{p,\mathrm{odd}}$, $\hat{D}_{q,q}^{p,\mathrm{odd}}$ and $\hat{D}_{p,q}^{p,\mathrm{odd}}$. Then $\|(d_i)\|_{X^{\mathrm{odd}}} = \|(d_i)\|_X$ for any $(d_i)_{i\geqslant 1} \in X^{\mathrm{odd}}$. Moreover, there exists a projection $P_X \in \mathcal{L}(X)$ that maps $(d_i)_{i\geqslant 1} \in X$ to $(d_i \mathbf{1}_{i \text{ is odd}})_{i\geqslant 1}$ with $\|P_X\| \leqslant 1$.*

Proof. The proof will be by induction with the induction parameter $P(X)$ being the minimal total number of $+$ and $\cap$ involved in the definition of X.

Induction basis. Let X be equal to $\hat{S}_q^p$, $\hat{D}_{q,q}^p$ or $\hat{D}_{p,q}^p$. Then the assertion of the lemma follows directly from the definition of $\hat{S}_q^p$, $\hat{D}_{q,q}^p$ and $\hat{D}_{p,q}^p$.

Induction step. If X is not equal to neither $\hat{S}_q^p$, $\hat{D}_{q,q}^p$ nor $\hat{D}_{p,q}^p$, then $P(X) > 0$. Hence, there exist X_1 and X_2, which are both $+$ and $\cap$ combinations of the spaces $\hat{S}_q^p$, $\hat{D}_{q,q}^p$ and $\hat{D}_{p,q}^p$, such that either $X = X_1 + X_2$ or $X = X_1 \cap X_2$, and $P(X_1), P(X_2) < P(X)$. Let X_1^{odd} and X_2^{odd} be the odd versions of X_1 and X_2. By the induction hypothesis, we know that for any $j = 1, 2$, $\|P_{X_j}\| \leqslant 1$ and

$$\|(d_i)\|_{X_j^{\mathrm{odd}}} = \|(d_i)\|_{X_j}, \quad \text{for all } (d_i)_{i\geqslant 1} \in X_j^{\mathrm{odd}}.$$

We assume that $X = X_1 + X_2$, the case $X = X_1 \cap X_2$ can be proven analogously. Fix $(d_i)_{i\geqslant 1} \in X^{\mathrm{odd}}$. Then

$$\|(d_i)\|_{X^{\mathrm{odd}}} = \|(d_i)\|_{X_1^{\mathrm{odd}} + X_2^{\mathrm{odd}}} \geqslant \|(d_i)\|_{X_1 + X_2} = \|(d_i)\|_X,$$

since X_1^{odd} and X_2^{odd} are isometrically embedded into X_1 and X_2, respectively. To show the converse inequality, fix $\varepsilon > 0$. Then there exist $(e_i^1)_{i\geqslant 1} \in X_1$ and $(e_i^2)_{i\geqslant 1} \in X_2$ such that $e_i^1 + e_i^2 = d_i$ for any $i \geqslant 1$ and

$$\|(e_i^1)\|_{X_1} + \|(e_i^2)\|_{X_2} \leqslant \|(d_i)\|_X + \varepsilon.$$

Then $P_{X_1}(e_i^1) + P_{X_2}(e_i^2) = (d_i)$ (since $d_{2i} = 0$ for all $i \geqslant 1$), and since $\|P_{X_1}\|, \|P_{X_2}\| \leqslant 1$

$$\begin{aligned}\|(d_i)\|_{X^{\mathrm{odd}}} &\leqslant \|P_{X_1}(e_i^1)\|_{X_1^{\mathrm{odd}}} + \|P_{X_2}(e_i^2)\|_{X_2^{\mathrm{odd}}} \\ &\leqslant \|(e_i^1)\|_{X_1} + \|(e_i^2)\|_{X_2} \leqslant \|(d_i)\|_X + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\|(d_i)\|_{X^{\mathrm{odd}}} \leqslant \|(d_i)\|_X$.

Now, let us show that $\|P_X\| \leqslant 1$. Fix $(d_i)_{i\geqslant 1} \in X$ and $\varepsilon > 0$. Then there exist $(e_i^1)_{i\geqslant 1} \in X_1$ and $(e_i^2)_{i\geqslant 1} \in X_2$ such that $e_i^1 + e_i^2 = d_i$ for any $i \geqslant 1$ and

$$\|(e_i^1)\|_{X_1} + \|(e_i^2)\|_{X_2} \leqslant \|(d_i)\|_X + \varepsilon.$$

Therefore

$$\begin{aligned}\|P_X(d_i)\|_{X^{\mathrm{odd}}} &\leqslant \|P_{X_1}(e_i^1)\|_{X_1^{\mathrm{odd}}} + \|P_{X_2}(e_i^2)\|_{X_2^{\mathrm{odd}}} \\ &\leqslant \|(e_i^1)\|_{X_1} + \|(e_i^2)\|_{X_2} \leqslant \|(d_i)\|_X + \varepsilon\end{aligned}$$

and the claim follows by taking $\varepsilon \to 0$. □

REMARK 3.5. Let us compare our result to the literature. As was mentioned in the introduction, the scalar-valued version of Theorem 1.1 is due to Burkholder [**4**], following work of Rosenthal [**47**]. A version for non-commutative martingales, as well as a version of (1.3) for $1 < p \leqslant 2$, was obtained by Junge and Xu [**22**]. Various upper bounds for the moments of a martingale with values in a uniformly 2-smooth (or equivalently, cf. [**45**], martingale type 2) Banach space were obtained by Pinelis [**43**], with constants of optimal order. For instance, if $2 \leqslant p < \infty$, then (see [**43**, Theorem 4.1])

$$\left(\mathbb{E}\left\|\sum_i d_i\right\|_X^p\right)^{\frac{1}{p}} \lesssim p(\mathbb{E}\sup_i \|d_i\|_X^p)^{\frac{1}{p}} + \sqrt{p}\tau_2(X)\left(\mathbb{E}\left(\sum_i \mathbb{E}_{i-1}\|d_i\|_X^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \tag{3.8}$$

where $\tau_2(X)$ is the 2-smoothness constant of X. As was already remarked in [**43**], due to the presence of the second term on the right-hand side, this type of inequality cannot hold in a Banach space which is not 2-uniformly smooth (or equivalently, has martingale type 2). On the other hand, one can show that the reverse inequality holds (with different constants) if and only if X is 2-uniformly convex (or equivalently, has martingale cotype 2). Thus, a two-sided inequality involving the norm on the right-hand side of (3.8) can only hold in a space with both martingale type and cotype equal to 2. Such a space is necessarily isomorphic to a Hilbert space by a well-known result of Kwapień (see, for example, [**1**, Theorem 7.4.1]).

REMARK 3.6. It is beyond the scope of this article to determine the optimal dependence of the implicit constants on p and q in (1.6). To give an impression of the constants produced by our proof, let us consider the upper bound in the case of real-valued martingales (corresponding to $q = 2$) for $p \geqslant 2$. By performing bookkeeping on the constants in the proof in [**8**] and using that the decoupling inequality (3.5) holds with a constant independent of p if $X = \mathbb{R}$ [**16**], one finds the inequality

$$\left(\mathbb{E}\left|\sum_i d_i\right|^p\right)^{\frac{1}{p}} \lesssim \alpha(p)\left(\mathbb{E}\left(\sum_i \mathbb{E}_{i-1}|d_i|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} + \beta(p)\left(\sum_i \mathbb{E}\|d_i\|^p\right)^{\frac{1}{p}} \tag{3.9}$$

with $\alpha(p) \leqslant \sqrt{p}$ and $\beta(p) \leqslant p$. This inequality is optimal in the sense that it is known that the order of $\alpha(p)$ cannot be reduced and, moreover, the order of $\beta(p)$ cannot be reduced without increasing the order of $\alpha(p)$ (see [**44**, Proposition 2]). On the other hand, it is known that if one considers a single constant in front of both factors on the right-hand side of (3.9), that is,

$$\left(\mathbb{E}\left|\sum_i d_i\right|^p\right)^{\frac{1}{p}} \lesssim \gamma(p)\left[\left(\mathbb{E}\left(\sum_i \mathbb{E}_{i-1}|d_i|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} + \left(\sum_i \mathbb{E}\|d_i\|^p\right)^{\frac{1}{p}}\right],$$

then the order $\gamma(p) = O(p/\log(p))$ is optimal [**15**] (see also [**41**]). It is clear that the latter result cannot be recovered from our proof. We leave the study of the best constants in (1.6) for all values of p and q as an interesting open problem.

4. *The dual of* $H_p^{s_q}(X)$

In the proof of Theorem 1.1, we used the fact that $(D_{q,q}^p)^* = D_{q',q'}^{p'}$ holds isomorphically (with constants depending only on p and q) for all $1 < p, q < \infty$. In this section, we will prove a more general statement.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_k)_{k\geqslant 0}$ and with $\mathcal{F}_0$ being generated by all the negligible sets, X be a Banach space and let $1 < p, q < \infty$. For an adapted sequence $f = (f_k)_{k\geqslant 1}$ of X-valued random variables, we define

$$s_q^n(f) := \left(\sum_{k=1}^{n} \mathbb{E}_{k-1}\|f_k\|^q\right)^{1/q}, \quad s_q(f) := \left(\sum_{k=1}^{\infty} \mathbb{E}_{k-1}\|f_k\|^q\right)^{1/q},$$

where $\mathbb{E}_k = \mathbb{E}(\cdot|\mathcal{F}_k)$, $\mathbb{E}_0 = \mathbb{E}$. We let $H_p^{s_q}(X)$ be the space of all adapted sequences $f = (f_k)_{k\geqslant 1}$ satisfying

$$\|f\|_{H_p^{s_q}(X)} := (\mathbb{E}s_q(f)^p)^{1/p} < \infty.$$

Similarly, we define $H_p^{s_q^n}(X)$. We will prove the following result, which was only known before if $X = \mathbb{R}$ and either $1 < p \leqslant q < \infty$ or $2 \leqslant q \leqslant p < \infty$ (see [**49**, Theorem 15] and the remark following it).

THEOREM 4.1. *Let X be a reflexive separable Banach space, $1 < p, q < \infty$. Then $\big(H_p^{s_q}(X)\big)^* = H_{p'}^{s_{q'}}(X^*)$ isomorphically. The isomorphism is given by*

$$g \mapsto F_g, \quad F_g(f) = \mathbb{E}\left(\sum_{k=1}^{\infty}\langle f_k, g_k\rangle\right) \quad \Big(f \in H_p^{s_q}(X), g \in H_{p'}^{s_{q'}}(X^*)\Big), \tag{4.1}$$

and

$$\min\left\{\frac{q}{p}, \frac{q'}{p'}\right\}\|g\|_{H_{p'}^{s_{q'}}(X^*)} \leqslant \|F_g\|_{(H_p^{s_q}(X))^*} \leqslant \|g\|_{H_{p'}^{s_{q'}}(X^*)}. \tag{4.2}$$

In particular, $H_p^{s_q}(X)$ is a reflexive Banach space.

To prove this result, we will first extend an argument of Csörgő [**6**] to show that $\Big(H_p^{s_q^n}(X)\Big)^*$ and $H_{p'}^{s_{q'}^n}(X^*)$ are isomorphic if $1 < p, q < \infty$, with isomorphism constants depending on p, q and n. In particular, this shows that $H_p^{s_q^n}(X)$ is reflexive. In a second step, we exploit this reflexivity to show that the isomorphism constants do not depend on n. The proof of this result, Theorem 4.5, relies on an argument of Weisz [**49**]. After this step, it is straightforward to deduce Theorem 4.1.

We start by introducing an operator that serves as a replacement for the sign-function in a vector-valued context.

LEMMA 4.2. *Let X be a Banach space with a separable dual. Fix $\varepsilon > 0$. Then there exists a discrete-valued Borel-measurable function $P_\varepsilon : X^* \to X$ such that $\|P_\varepsilon(x^*)\| = 1$ and*

$$(1-\varepsilon)\|x^*\| \leqslant \langle P_\varepsilon(x^*), x^*\rangle \leqslant \|x^*\| \tag{4.3}$$

for each $x^* \in X^*$.

Proof. Let $(x_n^*)_{n\geqslant 1}$ be a dense subset of the unit sphere U of X^*. For each $n \geqslant 1$, define $U_n = U \bigcap B(x_n^*, \frac{\varepsilon}{2})$, where $B(y^*, r)$ denotes the ball in X^* with radius r and center y^*. Define $V_1 = U_1$ and

$$V_n = U_n \setminus \left(\bigcup_{k=1}^{n-1} V_k\right), \quad n \geqslant 2.$$

For each $n \geqslant 1$, one can find an $x_n \in X$ such that $\|x_n\| = 1$ and $\langle x_n, x_n^* \rangle \geqslant 1 - \frac{\varepsilon}{2}$. Now, define

$$P_\varepsilon(x^*) := \sum_{n=1}^{\infty} \mathbf{1}_{V_n}\left(\frac{x^*}{\|x^*\|}\right) x_n, \quad x^* \in X^*.$$

This function is Borel since the V_n are Borel sets. As the V_n form a disjoint cover of the unit sphere, for every $x^* \in X^*$ there exists a unique $n = n(x^*)$ so that $x^*/\|x^*\| \in V_n$. Hence, $\|P_\varepsilon(x^*)\| = 1$ and

$$\langle P_\varepsilon(x^*), x^* \rangle = \|x^*\| \left\langle x_n, \frac{x^*}{\|x^*\|} \right\rangle \geqslant \|x^*\| \langle x_n, x_n^* \rangle - \frac{\varepsilon}{2}\|x^*\| \geqslant (1-\varepsilon)\|x^*\|,$$

so (4.3) follows. □

Let us now show that $\left(H_p^{s_q^n}(X)\right)^* = H_{p'}^{s_{q'}^n}(X^*)$ for any fixed $n \geqslant 1$.

THEOREM 4.3. *Let X be a reflexive separable Banach space, $1 < p, q < \infty$, $n \geqslant 1$. Then $\left(H_p^{s_q^n}(X)\right)^* = H_{p'}^{s_{q'}^n}(X^*)$ isomorphically (with constants depending on p, q and n). The isomorphism is given by*

$$g \mapsto F_g, \quad F_g(f) = \mathbb{E}\left(\sum_{k=1}^{n} \langle f_k, g_k \rangle\right) \quad \left(f \in H_p^{s_q^n}(X), g \in H_{p'}^{s_{q'}^n}(X^*)\right). \tag{4.4}$$

In particular, $H_p^{s_q^n}(X)$ is a reflexive Banach space.

Proof. The main argument is inspired by the proof of [6, Theorem 1]. By the conditional Hölder inequality and the usual version of Hölder's inequality,

$$\begin{aligned} |F_g(f)| &\leqslant \mathbb{E}\left(\sum_{k=1}^{n} \mathbb{E}_{k-1}(\|f_k\|\|g_k\|)\right) \\ &\leqslant \mathbb{E}\left(\sum_{k=1}^{n} (\mathbb{E}_{k-1}\|f_k\|^q)^{1/q} (\mathbb{E}_{k-1}\|g_k\|^{q'})^{1/q'}\right) \\ &\leqslant \|f\|_{H_p^{s_q^n}(X)} \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}. \end{aligned} \tag{4.5}$$

Hence, the functional F_g is bounded and $\|F_g\| \leqslant \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}$.

To prove that $\|F_g\| \gtrsim_{p,q,n} \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}$, we need to construct an appropriate $f \in H_p^{s_q^n}(X)$ with

$$\|f\|_{H_p^{s_q^n}(X)} \lesssim_{p,q,n} 1, \quad \langle F_g, f \rangle \gtrsim_{p,q,n} \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}.$$

Fix $0 < \varepsilon < 1$. We define f by setting

$$f_k := P_\varepsilon(g_k) \frac{\|g_k\|^{q'-1}}{\|g\|_{H_{p'}^{s_{q'}^n}(X^*)}^{p'-1}} (\mathbb{E}_{k-1}\|g_k\|^{q'})^{\frac{p'-q'}{q'}}, \quad 0 \leqslant k \leqslant n$$

where P_ε is as in Lemma 4.2. Using $pp' = p + p'$ and $qq' = q + q'$, we find

$$\|f\|^p_{H_p^{s_q^n}(X)} = \mathbb{E}\bigg(\sum_{k=1}^n \mathbb{E}_{k-1}\|f_k\|^q\bigg)^{p/q} = \frac{1}{\|g\|^{p(p'-1)}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\bigg(\sum_{k=1}^n (\mathbb{E}_{k-1}\|g_k\|^{q'})^{\frac{(p'-1)q}{q'}}\bigg)^{\frac{p}{q}}$$

$$\eqsim_{n,p,q} \frac{1}{\|g\|^{p'}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\bigg(\sum_{k=1}^n (\mathbb{E}_{k-1}\|g_k\|^{q'})\bigg)^{\frac{p'}{q'}} = 1,$$

so $f \in H_p^{s_q^n}(X)$. Moreover,

$$\langle F_g, f\rangle \geqslant (1-\varepsilon)\frac{1}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\sum_{k=1}^n \|g_k\|^{q'}(\mathbb{E}_{k-1}\|g_k\|^{q'})^{\frac{p'-q'}{q'}}$$

$$= (1-\varepsilon)\frac{1}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\sum_{k=1}^n (\mathbb{E}_{k-1}\|g_k\|^{q'})^{\frac{p'}{q'}}$$

$$\eqsim_{p,q,n} (1-\varepsilon)\frac{1}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\bigg(\sum_{k=1}^n \mathbb{E}_{k-1}\|g_k\|^{q'}\bigg)^{\frac{p'}{q'}} = \|g\|_{H_{p'}^{s_{q'}^n}(X^*)},$$

as desired, since ε was arbitrary and can be chosen, say, $\frac{1}{2}$.

Now, we will show that every $F \in \big(H_p^{s_q^n}(X)\big)^*$ is equal to F_g for a suitable $g \in H_{p'}^{s_{q'}^n}(X^*)$. For this purpose, we consider the disjoint direct sum of $(\Omega, \mathcal{F}_k, \mathbb{P})$, $k = 1, \ldots, n$. Formally, we set $\Omega_k = \Omega \times \{k\}$, $\widetilde{\mathcal{F}}_k = \mathcal{F}_k \times \{k\}$ and define a probability measure $\mathbb{P}_k$ on $\widetilde{\mathcal{F}}_k$ by $\mathbb{P}_k(A \times \{k\}) = \mathbb{P}(A)$. Now, the disjoint direct sum $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ is defined by

$$\Omega^n = \bigcup_{k=1}^n \Omega_k, \quad \mathcal{F}^n = \{A \subset \Omega^n \ : \ A \cap \Omega_k \in \widetilde{\mathcal{F}}_k, \text{ for all } 1 \leqslant k \leqslant n\}$$

and

$$\mathbb{P}^n(A) = \sum_{k=1}^n \mathbb{P}_k(A \cap \Omega_k), \quad A \in \mathcal{F}^n.$$

Let $P_k : (\Omega, \mathcal{F}_k) \to (\Omega^n, \mathcal{F}^n)$, $P_k(\omega) = (\omega, k)$, be the measurable bijection between $(\Omega, \mathcal{F}_k)$ and its disjoint copy. We can now define an X^*-valued set function μ by

$$\langle \mu(A), x\rangle := F\Big((x \cdot \mathbf{1}_{P_k^{-1}(A\cap\Omega_k)})_{k=1}^n\Big), \quad A \in \mathcal{F}^n, \ x \in X.$$

We will show that μ is σ-additive, absolutely continuous with respect to $\mathbb{P}^n$ and of finite variation. Let us first show that μ is of finite variation. Let $(A_m)_{m=1}^M \subset \mathcal{F}^n$ be disjoint such that $\cup_m A_m = \Omega^n$. Then

$$\sum_{m=1}^M \|\mu(A_m)\| = \sum_{m=1}^M \sup_{x_m\in X:\|x_m\|=1} F\Big((x_m \cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)})_{k=1}^n\Big)$$

$$= \sup_{(x_m)_{m=1}^M \subset X:\|x_m\|=1} \sum_{m=1}^M F\Big((x_m \cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)})_{k=1}^n\Big)$$

$$
\begin{aligned}
&= \sup_{(x_m)_{m=1}^M \subset X : \|x_m\|=1} F\left(\left(\sum_{m=1}^M x_m \cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\right)_{k=1}^n\right)\\
&\leqslant \|F\| \sup_{(x_m)_{m=1}^M \subset X : \|x_m\|=1} \left\|\left(\sum_{m=1}^M x_m \cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\right)_{k=1}^n\right\|_{H_p^{s_q^n}(X)}\\
&= \|F\| \sup_{(x_m)_{m=1}^M \subset X : \|x_m\|=1} \left(\mathbb{E}\left(\sum_{k=1}^n \mathbb{E}_{k-1}\left\|\sum_{m=1}^M x_m \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\right\|^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}\\
&\leqslant \|F\| \left[\mathbb{E}\left(\sum_{k=1}^n \mathbb{E}_{k-1}\left(\sum_{m=1}^M \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\right)^q\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}\\
&= \|F\| \left(\mathbb{E}\left(\sum_{k=1}^n \mathbb{E}_{k-1}\mathbf{1}_\Omega\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} = \|F\|(n+1)^{\frac{1}{q}}.
\end{aligned}
$$

Now, let us prove the σ-additivity. Obviously μ is additive. Let the family of sets $(A_m)_{m\geqslant 0} \subset \mathcal{F}_n$ be such that $A_m \searrow \varnothing$. Then

$$
\begin{aligned}
\|\mu(A_m)\| &= \sup_{x\in X:\|x\|=1} |F((x\cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)})_{k=1}^n)|\\
&\leqslant \|F\| \sup_{x\in X:\|x\|=1} \|(x\cdot \mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)})_{k=1}^n\|_{H_p^{s_q^n}(X)}\\
&= \|F\| \left(\mathbb{E}\left(\sum_{k=1}^n \mathbb{E}_{k-1}\mathbf{1}_{P_k^{-1}(A_m\cap\Omega_k)}\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \to 0 \text{ as } m\to\infty,
\end{aligned}
$$

by the monotone convergence theorem. This computation also shows that μ is absolutely continuous with respect to $\mathbb{P}^n$.

Since X is reflexive, X^* has the Radon–Nikodym property (see, for example, [**17**, Theorem 1.3.21]). Thus, there exists a $g \in L^1(\Omega^n; X^*)$ so that

$$
\mu(A) = \int_A g \,\mathrm{d}\mathbb{P}^n = \sum_{k=1}^n \int_{A\cap\Omega_k} g\,\mathrm{d}\mathbb{P}_k.
$$

If we now define $g_k := g\circ P_k$, then g_k is $\mathcal{F}_k$-measurable and

$$
\mu(A) = \sum_{k=1}^n \int_{P_k^{-1}(A\cap\Omega_k)} g_k\,\mathrm{d}\mathbb{P}.
$$

It now follows for $f = (f_k)_{k=1}^n \in H_p^{s_q^n}(X)$ with f_k bounded for all $k = 1,\dots,n$ that

$$
F(f) = F_g(f) = \mathbb{E}\sum_{k=1}^n \langle f_k, g_k\rangle. \tag{4.6}
$$

Now, fix general $f \in H_p^{s_q^n}(X)$. Fix $0<\varepsilon<1$ and let

$$
h := (h_k)_{k=1}^n = (\|f_k\| P_\varepsilon(g_k))_{k=1}^n.
$$

Define $h^m := (h_k^m)_{k=1}^n = (h_k \mathbf{1}_{\|h_k\|\leqslant m})_{k=1}^n$ for each $m \geqslant 1$. Then formula (4.6) holds for h^m. But $F(h^m) \to F(h)$ as m goes to infinity, so by the monotone convergence theorem, $F(h) = \mathbb{E}\sum_{k=1}^n \langle h_k, g_k\rangle$. This shows that

$$\mathbb{E}\sum_{k=1}^n |\langle f_k, g_k\rangle| \leqslant \mathbb{E}\sum_{k=1}^n \|f_k\|\|g_k\| \leqslant (1-\varepsilon)^{-1}\mathbb{E}\sum_{k=1}^n \langle h_k, g_k\rangle < \infty. \tag{4.7}$$

Now, consider $f^m := (f_k^m)_{k=1}^n = (f_k \mathbf{1}_{\|f_k\|\leqslant m})_{k=1}^n$. Since (4.6) holds for f^m and $F(f^m) \to F(f)$, we can use (4.7) and the dominated convergence theorem to conclude that f satisfies (4.6).

It remains to prove that $g \in H_{p'}^{s_{q'}^n}(X^*)$. For each $m \geqslant 1$, we consider the approximation $g^m := (g_k \mathbf{1}_{\|g_k\|\leqslant m})_{k=1}^n$. Then

$$\|g^m\|_{H_{p'}^{s_{q'}^n}(X^*)} \lesssim_{p,q,n} \|F_{g^m}\| \leqslant \|F\|.$$

Therefore, by the monotone convergence theorem, $\|g\|_{H_{p'}^{s_{q'}^n}(X^*)} \lesssim_{p,q,n} \|F\|$. □

One can easily show the following simple lemma.

Lemma 4.4. *Let X and Y be reflexive Banach spaces such that X^* is isomorphic to Y and*

$$a\|x^*\|_Y \leqslant \|x^*\|_{X^*} \leqslant b\|x^*\|_Y, \quad x^* \in X^*.$$

Then Y^* is isomorphic to $X^{**} = X$ and

$$a\|x\|_X \leqslant \|x\|_{Y^*} \leqslant b\|x\|_X, \quad x \in X.$$

Let us now show that the isomorphism constants in Theorem 4.3 do not depend on n.

Theorem 4.5. *Let X be a reflexive separable Banach space, $1 < p, q < \infty$, $n \geqslant 1$. Then*

$$\min\left\{\frac{q}{p}, \frac{q'}{p'}\right\}\|g\|_{H_{p'}^{s_{q'}^n}(X^*)} \leqslant \|F_g\|_{\left(H_p^{s_q^n}(X)\right)^*} \leqslant \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}. \tag{4.8}$$

Proof. We already proved in Theorem 4.3 that $H_p^{s_q^n}(X)$ is reflexive, so by Lemma 4.4 it is enough to show (4.8) for $p \leqslant q$. It was already noted in (4.5) that $\|F_g\| \leqslant \|g\|_{H_{p'}^{s_{q'}^n}(X^*)}$. It is sufficient to show (4.8) for a bounded g. The following construction is in essence the same as in [49, Theorem 15]. Set

$$(v_k)_{k=1}^n = \left(\frac{(s_{q'}^k(g))^{p'-q'}}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\right)_{k=1}^n.$$

Fix $0 < \varepsilon < 1$. Let us define $h \in H_p^{s_q^n}(X)$ by setting

$$h_k = v_k\|g_k\|^{q'-1}P_\varepsilon(g_k),$$

where $P_\varepsilon : X^* \to X$ is as given in Lemma 4.2. Then

$$(s_q^n(h))^q \leqslant \sum_{k=1}^n \frac{(s_{q'}^k(g))^{qp'-qq'}}{\|g\|^{qp'-q}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}_{k-1}\|g_k\|^{q'} \leqslant \frac{(s_{q'}^n(g))^{qp'-(q-1)q'}}{\|g\|^{qp'-q}_{H_{p'}^{s_{q'}^n}(X^*)}}.$$

and therefore

$$\mathbb{E}(s_q^n(h))^p \leqslant \frac{\mathbb{E}(s_{q'}^n(g))^{(qp'-(q-1)q')\frac{p}{q}}}{\|g\|^{pp'-p}_{H_{p'}^{s_{q'}^n}(X^*)}} = 1.$$

As a consequence,

$$\begin{aligned}\|F_g\| &\geqslant |\langle F_g, h\rangle| \\ &\geqslant (1-\varepsilon)\frac{1}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\sum_{k=1}^n (s_{q'}^k(g))^{p'-q'}\mathbb{E}_{k-1}\|g_k\|^{q'} \\ &= (1-\varepsilon)\frac{1}{\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}\sum_{k=1}^n (s_{q'}^k(g))^{p'-q'}((s_{q'}^k(g))^{q'} - (s_{q'}^{k-1}(g))^{q'}).\end{aligned} \tag{4.9}$$

By the mean value theorem,

$$x^\alpha - 1 \leqslant \alpha(x-1)x^{\alpha-1}, \quad x,\alpha \geqslant 1. \tag{4.10}$$

Applying this for $x = \frac{(s_{q'}^k(g))^{q'}}{(s_{q'}^{k-1}(g))^{q'}} \geqslant 1$ and $\alpha = \frac{p'}{q'} \geqslant 1$, we find

$$\frac{q'}{p'}((s_{q'}^k(g))^{p'} - (s_{q'}^{k-1}(g))^{p'}) \leqslant ((s_{q'}^k(g))^{q'} - (s_{q'}^{k-1}(g))^{q'})(s_{q'}^k(g))^{p'-q'}.$$

Combining this with (4.9) and letting $\varepsilon \to 0$,

$$\|F_g\| \geqslant \frac{q'}{p'\|g\|^{p'-1}_{H_{p'}^{s_{q'}^n}(X^*)}}\mathbb{E}(s_{q'}^n(g))^{p'} = \frac{q'}{p'}\|g\|_{H_{p'}^{s_{q'}^n}(X^*)}.$$

□

We can now deduce the main result of this section.

Proof of Theorem 4.1. Let $F \in (H_p^{s_q}(X))^*$. For every $n \geqslant 1$, there exists an $F_n \in \left(H_p^{s_q^n}(X)\right)^*$ such that $\langle F, f\rangle = \langle F_n, (f_k)_{k=1}^n\rangle$ for each $f \in H_p^{s_q}(X)$ satisfying $f_m = 0$ for all $m > n$. Thanks to Theorem 4.3, for each $n \geqslant 1$ there exists a $g^n = (g_k^n)_{k=1}^n \in H_{p'}^{s_{q'}^n}(X)$ such that $F_n = F_{g^n}$. Obviously, $g_k^m = g_k^n$ for each $m, n \geqslant k$, so there exists a unique $g = (g_k)_{k=1}^\infty$ such that $g^n = (g_k)_{k=1}^n$. Moreover, Theorem 4.5 implies

$$\min\left\{\frac{q}{p}, \frac{q'}{p'}\right\}\|g^n\|_{H_{p'}^{s_{q'}^n}(X)} \leqslant \|F_n\|_{\left(H_p^{s_q^n}(X)\right)^*} \leqslant \|F\|_{(H_p^{s_q}(X))^*},$$

so $g \in H_{p'}^{s_{q'}}(X)$ and

$$\min\left\{\frac{q}{p}, \frac{q'}{p'}\right\}\|g\|_{H_{p'}^{s_{q'}}(X)} \leqslant \|F\|_{(H_p^{s_q}(X))^*}.$$

Now, obviously $F = F_g$, as these two functionals coincide on the dense subspace of all finitely non-zero sequences in $H_p^{s_q}(X)$, and (4.1) and (4.2) hold. □

5. *Sharp bounds for L^q-valued stochastic integrals*

We now turn to proving sharp bounds for stochastic integrals. Recall that our aim is to find sharp estimates for L^p-norms of L^q-valued stochastic integrals with respect to general martingales in terms of *predictable* processes. We start by setting notations and recalling some basic facts on local martingales.

5.1. *Preliminaries*

Throughout, H always denotes a Hilbert space. We write $\overline{\mathbb{R}}_+ := [0, +\infty]$. We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ be a filtration that satisfies the usual conditions. We write $\mathcal{P}$ to denote the *predictable σ-algebra* on $\mathbb{R}_+ \times \Omega$, that is, the σ-algebra generated by all càg (that is, continuous from the left-hand side) adapted processes. We use $\mathcal{O}$ to denote the *optional σ-algebra* $\mathbb{R}_+ \times \Omega$, the σ-algebra generated by all càdlàg adapted processes. Let $(J, \mathcal{J})$ be a measurable space. We write $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{J}$ and $\widetilde{\mathcal{O}} := \mathcal{O} \otimes \mathcal{J}$ for the induced σ-algebras on $\widetilde{\Omega} = \mathbb{R}_+ \times \Omega \times J$.

Let $M : \mathbb{R}_+ \times \Omega \to H$ be a local martingale. Then M has a càdlàg version [**46**, Theorem I.9], and hence for each stopping time τ one can a.s. define the jump of M at time τ by $\Delta M_\tau := M_\tau - \lim_{\varepsilon\to 0} M_{(\tau-\varepsilon)\vee 0}$.

Let $N : \mathbb{R}_+ \times \Omega \to H$ be another local martingale. The *covariation* $[M, N] : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and the *quadratic variation*$[M] : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ are defined by

$$[M, N]_t = \mathbb{P} - \lim \sum_{i=1}^{k} \langle M_{t_i} - M_{t_{i-1}}, N_{t_i} - N_{t_{i-1}} \rangle, \quad t \geqslant 0,$$

$$[M]_t = [M, M]_t, \quad t \geqslant 0,$$

where $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k = t$ is a partition and the limit in probability is taken as the mesh of the partition goes to 0. It is well known that the covariation of any two local martingales exists and that $\langle M, N\rangle - [M, N]$ is a local martingale. Note that for any orthogonal basis $(h_n)_{n\geqslant 1}$ of H, for any $t \geqslant 0$ a.s.

$$[M]_t = \sum_{n\geqslant 1} [\langle M, h_n\rangle]_t. \tag{5.1}$$

We will frequently use the *Burkholder–Davis–Gundy inequality*: if M is any local martingale with $M_0 = 0$, $1 \leqslant p < \infty$, and τ is any stopping time, then

$$\left(\mathbb{E} \sup_{0\leqslant t\leqslant \tau} \|M_t\|^p\right)^{1/p} \eqsim_p (\mathbb{E}[M]_\tau^{p/2})^{1/p}.$$

We refer to [**34**] for a self-contained proof.

An non-decreasing càdlàg process $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ is called *pure jump* if for each $t \geqslant 0$ a.s.

$$A_t = A_0 + \sum_{0\leqslant s\leqslant t} \Delta A_s.$$

An H-valued local martingale M is called *purely discontinuous* if $[M]$ is a pure jump process a.s. An equivalent definition of a purely discontinuous local martingale will be given in Proposition 5.5. The reader can find more on purely discontinuous local martingales in [**20**, Chapter I.4; **23**, Chapter 26].

Let τ be a stopping time. We call

$$[\tau] = \{(\omega, t) \in \Omega \times \mathbb{R}_+ \ : \ t = \tau(\omega)\}$$

the graph of τ (although it is strictly speaking, the restriction of the graph of τ to $\Omega \times \mathbb{R}_+$). A stopping time τ is called *predictable* if there exists a non-decreasing sequence $(\tau_n)_{n\geqslant 1}$ of stopping times such that $\tau_n < \tau$ on $\{\tau > 0\}$ for each $n \geqslant 1$ and $\tau_n \nearrow \tau$ a.s. as $n \to \infty$ (see

[20, Definition I.2.7] and [23, Chapter 25]). For a predictable stopping time τ, we define $\mathcal{F}_{\tau-}$ by

$$\mathcal{F}_{\tau-} := \sigma(\mathcal{F}_{\tau_n})_{n\geqslant 1}.$$

Note that $\mathcal{F}_{\tau-}$ does not depend on the choice of the announcing sequence $(\tau_n)_{n\geqslant 1}$ (see, for example, [23, Lemma 25.2(iii)]).

Let X be a Banach space. An X-valued local martingale is called *quasi-left continuous* if $\Delta M_\tau = 0$ a.s. on the set $\{\tau < \infty\}$ for each predictable stopping time τ (see [20, Chapter I.2] for more information). An X-valued local martingale is said to have *accessible jumps* if there exists a sequence of predictable stopping times $(\tau_n)_{n\geqslant 1}$ with disjoint graphs such that a.s.

$$\{t \in \mathbb{R}_+ : \Delta M_t \neq 0\} \subset \cup_{n\geqslant 1}\{\tau_n\}$$

(see [23, p. 499] and [23, Corollary 26.16]).

Note that a local martingale $M : \mathbb{R}_+ \times \Omega \to X$ has accessible jumps if and only if $\langle M, x^*\rangle$ has accessible jumps for each $x^* \in X^*$. To see this, we may assume that X is separable. Let $(x_m)_{m\geqslant 1} \subset X$ be dense in X and let $(x^*_m)_{m\geqslant 1} \subset X^*$ be a sequence satisfying $\|x^*_m\| = 1$ and $\langle x^*_m, x_m\rangle = \|x_m\|$ (the existence of $(x^*_m)_{n\geqslant 1}$ is guaranteed by the Hahn–Banach theorem). One readily checks that $(x^*_m)_{m\geqslant 1}$ is a norming sequence for X and in particular, $x = 0$ if and only if $\langle x, x^*_m\rangle = 0$ for all $m \geqslant 1$. Thus, for any X-valued local martingale M one finds

$$\{t \in \mathbb{R}_+ : \Delta M_t \neq 0\} = \cup_{m\geqslant 1}\{t \in \mathbb{R}_+ : \Delta\langle M_t, x^*_m\rangle \neq 0\}. \tag{5.2}$$

Since each $\langle M, x^*_m\rangle$ has accessible jumps, there exists a sequence of predictable stopping times $(\tau_n)_{n\geqslant 1}$ such that a.s.

$$\{t \in \mathbb{R}_+ : \Delta M_t \neq 0\} \subset \cup_{n\geqslant 1}\{\tau_n\}. \tag{5.3}$$

We can then define a sequence of predictable stopping times $(\tau'_n)_{n\geqslant 1}$ with disjoint graphs that contains all the jump times of M by setting $\tau'_1 = \tau_1$ and

$$\tau'_n := \begin{cases} \tau_n & \text{if } \tau_n \neq \tau'_1, \dots, \tau'_{n-1}, \\ \max\{\tau'_1, \dots, \tau'_{n-1}\} + 1 & \text{otherwise.} \end{cases}$$

Similarly, a local martingale $M : \mathbb{R}_+ \times \Omega \to X$ is quasi-left continuous if and only if $\langle M, x^*\rangle$ is quasi-left continuous for each $x^* \in X^*$. This follows immediately from (5.2).

5.2. *Decomposition of the stochastic integral*

The process $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$ is called *elementary predictable* if it is of the form

$$\Phi(t, \omega) = \sum_{n=1}^N \sum_{m=1}^M \mathbf{1}_{(t_{n-1},t_n]\times A_{mn}}(t, \omega) \sum_{k=1}^K h_k \otimes x_{kmn},$$

where $0 \leqslant t_0 < \ldots < t_n < \infty$, $A_{1n}, \dots, A_{Mn} \in \mathcal{F}_{t_{n-1}}$ for each $n = 1, \dots, N$ and $h_1, \dots, h_K \in H$ are orthogonal. For each elementary predictable Φ and for any H-valued local martingale M, we define the stochastic integral with respect to M as an element of $L^0(\Omega; L^\infty(\mathbb{R}_+; X))$ by

$$\int_0^t \Phi(s)\,\mathrm{d}M(s) = \sum_{n=1}^N \sum_{m=1}^M \mathbf{1}_{A_{mn}} \sum_{k=1}^K \langle M(t_n \wedge t) - M(t_{n-1} \wedge t), h_k\rangle x_{kmn}. \tag{5.4}$$

We will often write $\Phi \cdot M$ for the process $\int_0^\cdot \Phi(s)\,\mathrm{d}M(s)$.

To prove sharp bounds for the stochastic integral, we will decompose it by decomposing the integrator M into three parts. We will need the following statement, which follows from [23, Proposition 25.17].

LEMMA 5.1. *Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be a non-decreasing adapted càdlàg process, $A_0 = 0$ a.s. Then there exist unique non-decreasing adapted càdlàg $A^c, A^q, A^a : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ such that $A_0^c = A_0^q = A_0^a = 0$, A^c is continuous a.s., A^q and A^a are pure jump a.s., A^q is quasi-left continuous, A^a has accessible jumps and $A = A^c + A^q + A^a$.*

The following lemma gives the desired decomposition of the integrator M. It is a generalization of both [**23**, Theorem 26.14] and [**23**, Corollary 26.16] to the Hilbert space-valued case.

LEMMA 5.2 (Decomposition of local martingales, Meyer, Yoeurp). *Let $M : \mathbb{R}_+ \times \Omega \to H$ be a local martingale. Then there exists a unique decomposition $M = M^c + M^q + M^a$, where $M^c : \mathbb{R}_+ \times \Omega \to H$ is a continuous local martingale, $M^q, M^a : \mathbb{R}_+ \times \Omega \to H$ are purely discontinuous local martingales, M^q is quasi-left continuous, M^a has accessible jumps, $M_0^c = M_0^q = 0$ and then $[M^c] = [M]^c$, $[M^q] = [M]^q$ and $[M^a] = [M]^a$, where $[M]^c$, $[M]^q$ and $[M]^a$ are defined as in Lemma 5.1.*

Since the decomposition in Lemma 5.2 is due to [**37**] and Yoeurp [**53**] in the real-valued case, it would be natural to call it the Meyer–Yoeurp decomposition. Unfortunately, this term is already used in the literature for the decomposition of a martingale into its continuous part and purely discontinuous part (see, for example, [**23**]). To avoid confusion, we will therefore refer to the decomposition in Lemma 5.2 as the *canonical* decomposition of M.

It was shown in the recent paper [**50**] that the existence of the canonical decomposition of a general X-valued local martingale is equivalent to X being UMD. Since the proof of this fact is very technical and complicated, we will present an alternative, elementary proof of the Hilbert space case here. For the proof, we will need the following statements. The proof of the first statement is essentially the same as the one given in [**23**, Lemma 26.5] in the real-valued case and therefore omitted.

LEMMA 5.3 (Truncation). *Let H be a Hilbert space, $M : \mathbb{R}_+ \times \Omega \to H$ be a local martingale. Then there exist local martingales M', $M'' : \mathbb{R}_+ \times \Omega \to H$ such that $M = M' + M''$, M' has locally integrable variation and $\|\Delta M_t''\| \leqslant 1$ a.s. for each $t \geqslant 0$.*

The second statement immediately follows from [**23**, Theorem 26.6(viii)].

LEMMA 5.4. *Let $M : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be a local martingale of locally finite variation. Then M is purely discontinuous.*

Proof of Lemma 5.2. The proof consists of two steps. In the first one, we show that we can assume that $\|\Delta M_t\| \leqslant 1$ a.s. for each $t \geqslant 0$. In the second step, we will show the statement in this particular case.

Step 1. First decompose M as in Lemma 5.3. Note that M' is purely discontinuous. Indeed, for each $h \in H$ the martingale $\langle M', h\rangle$ is locally of finite variation. Therefore, due to Lemma 5.4, $[\langle M', h\rangle]$ is a pure jump process, and by (5.1) for any orthogonal basis $(h_n)_{n\geqslant 1}$ of H and for any $t \geqslant 0$

$$[M']_t = \sum_{n\geqslant 1} [\langle M', h_n\rangle]_t = \sum_{n\geqslant 1}\sum_{0\leqslant s\leqslant t} \Delta[\langle M', h_n\rangle]_s = \sum_{0\leqslant s\leqslant t}\sum_{n\geqslant 1} \Delta[\langle M', h_n\rangle]_s$$
$$= \sum_{0\leqslant s\leqslant t} \Delta[M']_s.$$

Hence, $[M']$ is pure jump, and M' is purely discontinuous.

The existence of a decomposition of M' into a purely discontinuous quasi-left continuous part and a purely discontinuous part with accessible jumps follows analogously to [**23**, Corollary 26.16]. The uniqueness holds due to the uniqueness of the decomposition in the real-valued case.

Step 2. By Step 1, we can assume that $\|\Delta M_t\| \leqslant 1$ a.s. for each $t \geqslant 0$. By a localization argument, we may assume that M is an L^2-martingale. Without loss of generality assume also that $M_0 = 0$ and that H is separable. Let $(h_n)_{n\geqslant 1}$ be an orthonormal basis of H. For each $n \geqslant 1$, define a martingale $M^n : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ by $M^n := \langle M, h_n\rangle$. Then by (5.1), for each $t \geqslant 0$ a.s.

$$[M]_t = \sum_{n=1}^{\infty} [M^n]_t. \tag{5.5}$$

For each $n \geqslant 1$ by [**23**, Theorem 26.14] and [**23**, Corollary 26.16], we can define martingales $M^{c,n}, M^{q,n}, M^{a,n} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that

$$M_0^{c,n} = M_0^{q,n} = M_0^{a,n} = 0,$$

$M^{c,n}$ is a continuous martingale, $M^{q,n}$ and $M^{a,n}$ are purely discontinuous, $M^{q,n}$ is quasi-left continuous, $M^{a,n}$ has accessible jumps and

$$[M^n]^c = [M^{c,n}], \ \ [M^n]^q = [M^{q,n}], \ \ [M^n]^a = [M^{a,n}].$$

$[M^{c,n}]_t \leqslant [M^n]_t$ a.s. for all $t \geqslant 0$, so by the Burkholder–Davis–Gundy inequality and (5.5) the series $M_t^c := \sum_{n=1}^{\infty} M_t^{c,n} h_n$ converges in $L^2(\Omega; H)$ for each $t \geqslant 0$. Moreover, since a conditional expectation is a bounded operator on $L^2(\Omega; H)$, $\mathbb{E}(M_t^c|\mathcal{F}_s) = M_s^c$ for each $t \geqslant s \geqslant 0$, so M^c is a martingale. Obviously, $M_0^c = 0$ and M^c is continuous since $\langle M^c, h_n\rangle = M^{c,n}$ is continuous for each $n \geqslant 1$. $[M]^c = [M^c]$ since by (5.1).

$$[M]^c = \sum_{n=1}^{\infty} [M^n]^c = \sum_{n=1}^{\infty} [M^{c,n}] = [M^c].$$

With the same argument we can construct M^q and M^a. The uniqueness follows from the uniqueness of the decomposition

$$\langle M, h_n\rangle = M^{c,n} + M^{q,n} + M^{a,n}, \quad n \geqslant 1. \qquad \square$$

Thanks to Lemma 5.2, one can give an equivalent definition of a purely discontinuous local martingale, which is broadly used in the literature (for example, in [**20**]).

PROPOSITION 5.5. *A local martingale $M : \mathbb{R}_+ \times \Omega \to H$ is purely discontinuous if and only if $\langle M, N\rangle$ is a local martingale for any continuous bounded H-valued martingale N such that $N_0 = 0$.*

Proof. One direction follows from [**23**, Corollary 26.15]. Indeed, by a stopping time argument assume without loss of generality that $\mathbb{E}\|M_t\| < \infty$ for all $t \geqslant 0$. Assume also that N is bounded by 1. Let $(h_n)_{n\geqslant 1}$ be an orthonormal basis of H and define $M^n := \langle M, h_n\rangle$, $N^n := \langle N, h_n\rangle$ for all $n \geqslant 1$. Let

$$L^n := \sum_{i=1}^{n} M^n N^n = \sum_{i=1}^{n} \langle M, h_n\rangle\langle N, h_n\rangle, \quad n \geqslant 1.$$

First note that L^n is a local martingale by [**23**, Corollary 26.15], the fact that $[M^n, N^n] = 0$ and the fact that local martingales form a linear space. Moreover, a.s.

$$|L_t^n| \leqslant \|M_t\|\|N_t\| \leqslant \|M_t\|, \quad t \geqslant 0,$$

so $\mathbb{E}|L_t^n| \leqslant \mathbb{E}\|M_t\| < \infty$ for each $t \geqslant 0$, and thus L^n is a martingale for each $n \geqslant 0$. Note that for any $t \geqslant 0$ a.s., $L_t^n \to \langle M_t, N_t\rangle$. Therefore, by the dominated convergence theorem and the fact that $\langle M_t, N_t\rangle \in L^1(\Omega)$ for all $t \geqslant 0$, one finds that $\langle M_t, N_t\rangle$ is the limit of $(L_t^n)_{n\geqslant 1}$ in $L^1(\Omega)$. Since the conditional expectation is a contraction on $L^1(\Omega)$ and L^n is a martingale for any $n \geqslant 1$, it follows that for all $0 \leqslant s \leqslant t$

$$\mathbb{E}(\langle M_t, N_t\rangle|\mathcal{F}_s) = \mathbb{E}(\lim_{n\to\infty} L_t^n|\mathcal{F}_s) = \lim_{n\to\infty} \mathbb{E}(L_t^n|\mathcal{F}_s) = \lim_{n\to\infty} L_t^s = \langle M_s, N_s\rangle,$$

so $\langle M, N\rangle$ is a martingale.

Let us now prove the reverse implication. Without loss of generality assume that M is a martingale. By Lemma 5.2, there exists a continuous martingale $N : \mathbb{R}_+ \times \Omega \to H$ such that $N_0 = 0$ and $M - N$ is purely discontinuous. Let $(\tau_n)_{n\geqslant 1}$ be a non-decreasing sequence of stopping times such that $\tau_n \nearrow \infty$ as $n \to \infty$ and N^{τ_n} is a bounded continuous martingale for each $n \geqslant 1$. For any $n \geqslant 1$, $\langle M, N^{\tau_n}\rangle$ is a martingale by assumption and by the first part of the proof $\langle M - N, N^{\tau_n}\rangle$ is a martingale as well. Hence, $\|N^{\tau_n}\|^2 = (\langle M, N^{\tau_n}\rangle - \langle M - N, N^{\tau_n}\rangle)^{\tau_n}$ is a non-negative martingale that starts at zero and therefore a zero martingale. By letting $n \to \infty$, find $N = 0$ a.s., so M is purely discontinuous. □

As we will see below (Lemma 5.33), if $M = M^c + M^q + M^a$ is the canonical decomposition of M, then the canonical decomposition of $\Phi \cdot M$ is given by

$$\Phi \cdot M = \Phi \cdot M^c + \Phi \cdot M^q + \Phi \cdot M^a. \tag{5.6}$$

The following four subsections are dedicated to sharp estimates of the respective parts on the right-hand side. In Section 5.6, we combine our work to estimate $\Phi \cdot M$.

5.3. *Purely discontinuous martingales with accessible jumps*

In this section, we prove Burkholder–Rosenthal type inequalities for purely discontinuous martingales with accessible jumps. As an immediate consequence, we find sharp bounds for the accessible jump part in (5.6).

Let us first formulate the main result of this section and outline the steps of the proof. Let $1 < p, q < \infty$ and let $M : \mathbb{R}_+ \times \Omega \to L^q(S)$ be a purely discontinuous martingale with accessible jumps. Let $\mathcal{T} = (\tau_n)_{n\geqslant 1}$ be a sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M. We define three expressions

$$\begin{aligned}
\|M\|_{\widetilde{S}_q^p} &= \left(\mathbb{E}\left\|\left(\sum_{n\geqslant 1} \mathbb{E}_{\mathcal{F}_{\tau_n-}}|\Delta M_{\tau_n}|^2\right)^{\frac{1}{2}}\right\|_{L^q(S)}^p\right)^{\frac{1}{p}},\\
\|M\|_{\widetilde{D}_{q,q}^p} &= \left(\mathbb{E}\left(\sum_{n\geqslant 1} \mathbb{E}_{\mathcal{F}_{\tau_n-}}\|\Delta M_{\tau_n}\|_{L^q(S)}^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}},\\
\|M\|_{\widetilde{D}_{p,q}^p} &= \left(\mathbb{E}\sum_{t\geqslant 1} \|\Delta M_t\|_{L^q(S)}^p\right)^{\frac{1}{p}}.
\end{aligned} \tag{5.7}$$

It is not difficult to show that these expressions do not depend on the choice of the exhausting family $\mathcal{T}$. We let $\widetilde{S}_q^p$, $\widetilde{D}_{q,q}^p$ and $\widetilde{D}_{p,q}^p$ denote the sets of all purely discontinuous martingales with accessible jumps for which the respective expressions in (5.7) are finite. We will prove below that the expressions in (5.7) are norms. Consequently, we can define a normed space $\mathcal{A}_{p,q}$ by

$$\begin{array}{rl} \widetilde{S}_q^p \cap \widetilde{D}_{q,q}^p \cap \widetilde{D}_{p,q}^p & \text{if } 2 \leqslant q \leqslant p < \infty, \\ \widetilde{S}_q^p \cap (\widetilde{D}_{q,q}^p + \widetilde{D}_{p,q}^p) & \text{if } 2 \leqslant p \leqslant q < \infty, \\ (\widetilde{S}_q^p \cap \widetilde{D}_{q,q}^p) + \widetilde{D}_{p,q}^p & \text{if } 1 < p < 2 \leqslant q < \infty, \\ (\widetilde{S}_q^p + \widetilde{D}_{q,q}^p) \cap \widetilde{D}_{p,q}^p & \text{if } 1 < q < 2 \leqslant p < \infty, \\ \widetilde{S}_q^p + (\widetilde{D}_{q,q}^p \cap \widetilde{D}_{p,q}^p) & \text{if } 1 < q \leqslant p \leqslant 2, \\ \widetilde{S}_q^p + \widetilde{D}_{q,q}^p + \widetilde{D}_{p,q}^p & \text{if } 1 < p \leqslant q \leqslant 2. \end{array} \tag{5.8}$$

In addition, consider the space $\mathcal{M}_{p,q}^{\mathrm{acc}}$ of all purely discontinuous martingale with accessible jumps for which the norm

$$\|M\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}} := \|M_\infty\|_{L^p(\Omega;L^q(S))}$$

is finite. Our main result, Theorem 5.14, will show that $\mathcal{A}_{p,q}$ and $\mathcal{M}_{p,q}^{\mathrm{acc}}$ are isomorphic Banach spaces and in particular,

$$\|M\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}} \eqsim_{p,q} \|M\|_{\mathcal{A}_{p,q}} \tag{5.9}$$

for any L^q-valued purely discontinuous martingale M with accessible jumps.

The proof consists of the following steps. As a first step, we prove the statement for a restricted class of purely discontinuous martingales, whose jumps are exhausted by a fixed, *finite* sequence of stopping times (Theorem 5.6). In this case, the statement can be extracted by applying the discrete Burkholder–Rosenthal inequalities (specifically, the version in Proposition 3.3) by identifying M with a suitable discrete martingale difference sequence. In a second step, we derive the general statement by an approximation argument. For this purpose, we construct, for a given martingale M with accessible jumps, a sequence $(M^n)_{n\geqslant 1}$ which approximates M in $\mathcal{M}_{p,q}^{\mathrm{acc}}$ and is such that each M^n is a martingale whose jumps are exhausted by finitely many stopping times (Lemma 5.10). We then use this construction to shown that $\mathcal{M}_{p,q}^{\mathrm{acc}}$ is a Banach space (Proposition 5.11). In addition, we use it to show that $\mathcal{A}_{p,q}$ is a well-defined normed space and that M can be approximated in $\mathcal{A}_{p,q}$ by a sequence of the form $(M^n)_{n\geqslant 1}$ (see Lemma 5.12). Finally, we combine all these ingredients to show that (5.9) holds for any purely discontinuous martingale with accessible jumps; the fact that $\mathcal{A}_{p,q}$ is a Banach space then follows from the fact that $\mathcal{M}_{p,q}^{\mathrm{acc}}$ is a Banach space.

Let us now execute these steps in detail, starting with a version of the main theorem of this subsection (Theorem 5.14 below) for martingales with finitely many jumps. For a fixed family $\mathcal{T} = (\tau_n)_{n\geqslant 1}$ of predictable stopping times with disjoint graphs, we let $\widetilde{S}_q^{p,\mathcal{T}}$, $\widetilde{D}_{q,q}^{p,\mathcal{T}}$ and $\widetilde{D}_{p,q}^{p,\mathcal{T}}$ be the subsets of $\widetilde{S}_q^p$, $\widetilde{D}_{q,q}^p$ and $\widetilde{D}_{p,q}^p$ consisting of martingales M with

$$\{t \in \mathbb{R}_+ : \Delta M_t \neq 0\} \subset \{\tau_1, \tau_2, \dots\} \quad \text{a.s.}$$

THEOREM 5.6. *Let $1 < p, q < \infty$, $N \geqslant 1$, $\mathcal{T} = (\tau_n)_{n=1}^N$ be a finite family of predictable stopping times with disjoint graphs. Then $\widetilde{S}_q^{p,\mathcal{T}}$, $\widetilde{D}_{q,q}^{p,\mathcal{T}}$ and $\widetilde{D}_{p,q}^{p,\mathcal{T}}$ are Banach spaces under the norms in (5.7). As a consequence, $A_{p,q}^{\mathcal{T}}$ given by*

$$
\begin{array}{ll}
\widetilde{S}_q^{p,\mathcal{T}} \cap \widetilde{D}_{q,q}^{p,\mathcal{T}} \cap \widetilde{D}_{p,q}^{p,\mathcal{T}} & \text{if } 2 \leqslant q \leqslant p < \infty, \\
\widetilde{S}_q^{p,\mathcal{T}} \cap (\widetilde{D}_{q,q}^{p,\mathcal{T}} + \widetilde{D}_{p,q}^{p,\mathcal{T}}) & \text{if } 2 \leqslant p \leqslant q < \infty, \\
(\widetilde{S}_q^{p,\mathcal{T}} \cap \widetilde{D}_{q,q}^{p,\mathcal{T}}) + \widetilde{D}_{p,q}^{p,\mathcal{T}} & \text{if } 1 < p < 2 \leqslant q < \infty, \\
(\widetilde{S}_q^{p,\mathcal{T}} + \widetilde{D}_{q,q}^{p,\mathcal{T}}) \cap \widetilde{D}_{p,q}^{p,\mathcal{T}} & \text{if } 1 < q < 2 \leqslant p < \infty, \\
\widetilde{S}_q^{p,\mathcal{T}} + (\widetilde{D}_{q,q}^{p,\mathcal{T}} \cap \widetilde{D}_{p,q}^{p,\mathcal{T}}) & \text{if } 1 < q \leqslant p \leqslant 2, \\
\widetilde{S}_q^{p,\mathcal{T}} + \widetilde{D}_{q,q}^{p,\mathcal{T}} + \widetilde{D}_{p,q}^{p,\mathcal{T}} & \text{if } 1 < p \leqslant q \leqslant 2,
\end{array}
\tag{5.10}
$$

is a well-defined Banach space. Moreover, $(\mathcal{A}_{p,q}^{\mathcal{T}})^* = \mathcal{A}_{p',q'}^{\mathcal{T}}$ *with isomorphism given by*

$$
\begin{aligned}
g \mapsto F_g, \quad F_g(f) &= \mathbb{E} \sum_{t \in \mathcal{T}} \langle \Delta g_t, \Delta f_t \rangle \overset{(*)}{=} \mathbb{E}\langle g_\infty, f_\infty \rangle, \\
\|F_g\|_{(\mathcal{A}_{p,q}^{\mathcal{T}})^*} &\eqsim_{p,q} \|g\|_{\mathcal{A}_{p',q'}^{\mathcal{T}}}.
\end{aligned}
\tag{5.11}
$$

Finally, for any purely discontinuous L^p*-martingale* $M : \mathbb{R}_+ \times \Omega \to L^q(S)$ *with accessible jumps such that* $\{t \in \mathbb{R}_+ : \Delta M_t \neq 0\} \subset \{\tau_1, \ldots, \tau_N\}$ *a.s.,*

$$
\Big(\mathbb{E} \sup_{t \geqslant 0} \|M_t\|_{L^q(S)}^p \Big)^{\frac{1}{p}} \eqsim_{p,q} \|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}}. \tag{5.12}
$$

The idea of the proof is to discretize purely discontinuous martingales with jumps in $\mathcal{T}$ in a suitable way, so that $\widetilde{S}_q^{p,\mathcal{T}}$, $\widetilde{D}_{q,q}^{p,\mathcal{T}}$ and $\widetilde{D}_{p,q}^{p,\mathcal{T}}$ can be identified with discrete martingale spaces S_q^p, $D_{q,q}^p$ and $D_{p,q}^p$ for an appropriate filtration. For this purpose, we need two observations.

LEMMA 5.7. *Let* $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ *be a locally integrable càdlàg adapted process,* τ *be a predictable stopping time. Let* $G, H : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ *be such that* $G_t = F_\tau \mathbf{1}_{[0,t]}(\tau)$, $H_t = \mathbf{1}_{[0,t]}(\tau)\mathbb{E}_{\mathcal{F}_{\tau-}} F_\tau$ *for each* $t \geqslant 0$*. Then* $G - H$ *is a local martingale.*

Proof. Without loss of generality suppose that F is integrable. First of all note that H is a predictable process, thanks to [**23**, Lemma 25.3(ii)], and G is adapted due to the fact that $G_t = F_{\tau\wedge t}\mathbf{1}_{[0,t]}(\tau)$. Fix $t > s \geqslant 0$. By [**23**, Lemma 25.2(i)], $\mathcal{F}_s \cap \{s < \tau\} \subset \mathcal{F}_{\tau-}$ and $\mathcal{F}_s \cap \{t < \tau\} \subset \mathcal{F}_t \cap \{t < \tau\} \subset \mathcal{F}_{\tau-}$. Hence,

$$
\mathcal{F}_s \cap \{s < \tau \leqslant t\} \subset \mathcal{F}_{\tau-}
$$

and so

$$
\begin{aligned}
\mathbb{E}(G_t - H_t|\mathcal{F}_s) &= \mathbb{E}(F_\tau \mathbf{1}_{\{\tau \leqslant t\}} - \mathbf{1}_{\{\tau \leqslant t\}} \mathbb{E}_{\mathcal{F}_{\tau-}} F_\tau | \mathcal{F}_s) \\
&= \mathbb{E}(F_\tau \mathbf{1}_{\{\tau \leqslant s\}} - \mathbf{1}_{\{\tau \leqslant s\}} \mathbb{E}_{\mathcal{F}_{\tau-}} F_\tau | \mathcal{F}_s) + \mathbb{E}(F_\tau \mathbf{1}_{\{s<\tau \leqslant t\}} - \mathbf{1}_{\{s<\tau \leqslant t\}} \mathbb{E}_{\mathcal{F}_{\tau-}} F_\tau | \mathcal{F}_s) \\
&= G_s - H_s + \mathbb{E}(\mathbb{E}(F_\tau - \mathbb{E}_{\mathcal{F}_{\tau-}} F_\tau | \mathcal{F}_{\tau-}) \mathbf{1}_{\{s<\tau\leqslant t\}} | \mathcal{F}_s \cap \{s < \tau \leqslant t\}) \\
&= G_s - H_s.
\end{aligned}
$$

□

COROLLARY 5.8. *Let* X *be a Banach space,* τ *be a predictable stopping time,* $\xi \in L^1(\Omega; X)$ *be* $\mathcal{F}_\tau$*-measurable such that* $\mathbb{E}_{\mathcal{F}_{\tau-}} \xi = 0$*. Let* $M : \mathbb{R}_+ \times \Omega \to X$ *be such that* $M_t = \xi \mathbf{1}_{[0,t]}(\tau)$*. Then* M *is a martingale.*

Proof. The case $X = \mathbb{R}$ follows from Lemma 5.7 and the fact that $\xi \mathbf{1}_{\tau \leqslant t}$ is $\mathcal{F}_t$-measurable for each $t \geqslant 0$ by the definition of $\mathcal{F}_\tau$. For the general case, we note that $\langle M, x^*\rangle$ is a martingale for each $x^* \in X$ and since M is integrable it follows that M is a martingale. □

Proof of Theorem 5.6. Since $\mathcal{T} = (\tau_n)_{n=1}^N$ is a finite family and the stopping times τ_i have disjoint graphs, we can order them: formally, we can find predictable stopping times $\tau_1', \dots, \tau_N'$ such that

$$\{\tau_1(\omega), \dots, \tau_N(\omega)\} = \{\tau_1'(\omega), \dots, \tau_N'(\omega)\}$$

and $\tau_1'(\omega) < \dots < \tau_N'(\omega)$ for a.e. $\omega \in \Omega$. Indeed, we can set

$$\tau_1' := \min\{\tau_1, \dots, \tau_N\}$$

and

$$\tau_{i+1}' := \min(\{\tau_1, \dots, \tau_N\} \setminus \{\tau_1', \dots, \tau_i'\}), \quad 1 \leqslant i \leqslant N-1.$$

Fix the sequence of σ-algebras $\mathbb{G} = (\mathcal{G}_k)_{k=0}^{2N-1} = (\mathcal{F}_{\tau_1'-}, \mathcal{F}_{\tau_1'}, \dots, \mathcal{F}_{\tau_N'-}, \mathcal{F}_{\tau_N'})$. Using [**23**, Lemma 25.2] and the fact that $(\tau_n')_{n=1}^N$ is a.s. a strictly increasing sequence, one can show that $\mathbb{G}$ is a filtration.

Consider Banach spaces $\hat{S}_q^{p,\mathrm{odd}}$, $\hat{D}_{q,q}^{p,\mathrm{odd}}$ and $\hat{D}_{p,q}^{p,\mathrm{odd}}$ with respect to the filtration $\mathbb{G}$ that were defined in Proposition 3.3. For any purely discontinuous L^q-valued martingale M with accessible jumps in $\mathcal{T}$, we can construct a $\mathbb{G}$-martingale difference sequence $(d_k)_{n=0}^{2N-1}$ by setting $d_{2n} = 0$, $d_{2n-1} = \Delta M_{\tau_n'}$ for $n = 1, \dots, N$. Indeed, by [**23**, Lemma 26.18] (see also [**20**, Lemma 2.27]) for each $n = 1, \dots, N$

$$\mathbb{E}(d_{2n-1}|\mathcal{G}_{2n}) = \mathbb{E}(\Delta M_{\tau_n'}|\mathcal{F}_{\tau_n'-}) = 0.$$

By Lemma 5.7,

$$\|M\|_{\widetilde{S}_q^{p,\mathcal{T}}} = \|(d_n)\|_{S_q^{p,\mathrm{odd}}},\ \|M\|_{\widetilde{D}_{q,q}^{p,\mathcal{T}}} = \|(d_n)\|_{D_{q,q}^{p,\mathrm{odd}}},\ \|M\|_{\widetilde{D}_{p,q}^{p,\mathcal{T}}} = \|(d_n)\|_{D_{p,q}^{p,\mathrm{odd}}}.$$

Moreover, by Corollary 5.8 any element $(d_k)_{k=0}^{2N-1}$ of $\hat{S}_q^{p,\mathrm{odd}}$, $D_{q,q}^{p,\mathrm{odd}}$, or $D_{p,q}^{p,\mathrm{odd}}$ (so in particular, $d_{2n} = 0$ for each $n = 0, \dots, N$) can be converted back to an element M of $\widetilde{S}_q^{p,\mathcal{T}}$, $\widetilde{D}_{q,q}^{p,\mathcal{T}}$ or $\widetilde{D}_{p,q}^{p,\mathcal{T}}$, respectively, by defining

$$M_t = \sum_{n=1}^N d_{2n-1}\mathbf{1}_{[0,t]}(\tau_n'), \quad t \geqslant 0.$$

Using this identification, we find that $\widetilde{S}_q^{p,\mathcal{T}}$, $\widetilde{D}_{q,q}^{p,\mathcal{T}}$ and $\widetilde{D}_{p,q}^{p,\mathcal{T}}$ are Banach spaces. As a consequence, $\mathcal{A}_{p,q}^{\mathcal{T}}$ is a well-defined Banach space that is isometrically isomorphic to $\hat{s}_{p,q}^{\mathrm{odd}}$. The duality statement now follows from the duality for $s_{p,q}^{\mathrm{odd}}$ and $(*)$ in (5.11) follows from (3.7).

Now let us show (5.12). By Doob's maximal inequality,

$$\Big(\mathbb{E}\sup_{t\geqslant 0}\|M_t\|_{L^q(S)}^p\Big)^{\frac{1}{p}} \eqsim_p \Big(\mathbb{E}\|M_\infty\|_{L^q(S)}^p\Big)^{\frac{1}{p}}.$$

Again define a $\mathbb{G}$-martingale difference sequence $(d_n)_{n=0}^{2N-1}$ by setting $d_{2n} = 0$, $d_{2n-1} = \Delta M_{\tau_n'}$, where $n = 1, \dots, N$. Then by Proposition 3.3

$$\|M_\infty\|_{L^p(\Omega;X)} = \Big\|\sum_{n=0}^{2N-1} d_k\Big\|_{L^p(\Omega;X)} \eqsim_{p,q} \|(d_n)_{n=0}^{2N-1}\|_{\hat{s}_{p,q}^{\mathrm{odd}}} = \|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}}. \qquad \square$$

We now proceed to the second part of the proof of Theorem 5.14. We will first show that we can represent a purely discontinuous martingales with accessible jumps as a sum of jumps occurring at predictable times. We need the following simple observation.

LEMMA 5.9. *Let X be a Banach space, $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to X$ be an L^p-martingale, τ be a predictable stopping time. Then $(\Delta M_\tau \mathbf{1}_{[0,t]}(\tau))_{t\geqslant 0}$ is an L^p-martingale as well.*

Proof. By the definition of a predictable stopping time, there exists an increasing sequence of stopping times $(\tau_n)_{n\geqslant 1}$ such that $\tau_n < \tau$ a.s. for each $n \geqslant 1$ on $\{\tau > 0\}$ and $\tau_n \nearrow \tau$ a.s. as $n \to \infty$. Then M^τ, $M^{\tau_1}, \dots, M^{\tau_n}, \dots$ are L^p-martingales. Moreover, $\Delta M_\tau \mathbf{1}_{[0,t]}(\tau)$ is L^p-integrable for each $t \geqslant 0$ by Doob's maximal inequality and the fact that a.s. $\|\Delta M_\tau \mathbf{1}_{[0,t]}(\tau)\| \leqslant 2\sup_{0\leqslant s\leqslant t}\|M_s\|$. In addition, $M_t^\tau - M_t^{\tau_n} \to \Delta M_\tau \mathbf{1}_{[0,t]}(\tau)$ in $L^p(\Omega; X)$ by the dominated convergence theorem and the fact that $\|M_t^\tau - M_t^{\tau_n}\| \leqslant 2\sup_{0\leqslant s\leqslant t}\|M_s\|$ a.s. The continuity of the conditional expectation [17, Corollary 2.6.30] now implies that for any $0 \leqslant s \leqslant t$

$$\begin{aligned}\mathbb{E}(\Delta M_\tau \mathbf{1}_{[0,t]}|\mathcal{F}_s) = \mathbb{E}\Big(\lim_{n\to\infty}(M_t^\tau - M_t^{\tau_n})|\mathcal{F}_s\Big) &= \lim_{n\to\infty}\mathbb{E}(M_t^\tau - M_t^{\tau_n}|\mathcal{F}_s)\\ &= \lim_{n\to\infty}(M_s^\tau - M_s^{\tau_n}) = \Delta M_\tau \mathbf{1}_{[0,s]},\end{aligned}$$

in $L^p(\Omega; X)$. Consequently, $(\Delta M_\tau \mathbf{1}_{[0,t]}(\tau))_{t\geqslant 0}$ is an L^p-martingale. □

We can now prove the assertion.

LEMMA 5.10. *Let $1 < p, q < \infty$, $M : \mathbb{R}_+ \times \Omega \to L^q(S)$ be a purely discontinuous L^p-martingale with accessible jumps. Let $\mathcal{T} = (\tau_n)_{n=0}^\infty$ be any sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M. Then for each $n \geqslant 1$*

$$M_t^n = \sum_{k=1}^n \Delta M_{\tau_k}\mathbf{1}_{[0,t]}(\tau_k), \quad t \geqslant 0, \tag{5.13}$$

defines an L^p-martingale. Moreover, for any $t \geqslant 0$,

$$\|M_t - M_t^n\|_{L^p(\Omega;L^q(S))} \to 0, \quad n \to \infty.$$

If $\sup_{t\geqslant 0}\mathbb{E}\|M_t\|^p < \infty$, then $\|M_\infty - M_\infty^n\|_{L^p(\Omega;L^q(S))} \to 0$ for $n \to \infty$.

Proof. *Step* 1. Let us first suppose that M takes values in a finite-dimensional subspace of $L^q(S)$. Let $|||\cdot|||$ be an equivalent Euclidean norm on this subspace. Then, by Lemma 5.9, (5.13) defines an L^p-martingale. Since M is purely discontinuous, the Burkholder–Davis–Gundy inequality implies

$$\mathbb{E}|||(M - M^n)_t|||^p \eqsim_p \mathbb{E}[M - M^n]_t^{\frac{p}{2}} = \mathbb{E}\Bigg(\sum_{k=n+1}^\infty |||\Delta M_{\tau_k}|||^2 \mathbf{1}_{[0,t]}(\tau_k)\Bigg)^{\frac{p}{2}}. \tag{5.14}$$

Since the sum on the right-hand side is a.s. bounded by $[M]_t$ and monotonically vanishes as $n \to \infty$, the first convergence result follows. If

$$\sup_{t\geqslant 0}\mathbb{E}\|M_t\|^p < \infty,$$

then

$$\mathbb{E}|||(M^n)_t|||^p \eqsim_p \mathbb{E}[M^n]_t^{\frac{p}{2}} = \mathbb{E}\Bigg(\sum_{k=1}^n |||\Delta M_{\tau_k}|||^2 \mathbf{1}_{[0,t]}(\tau_k)\Bigg)^{\frac{p}{2}} \leqslant \mathbb{E}[M]_t{}^{p/2}.$$

Thus, $M^n_\infty = \lim_{t\to\infty} M^n_t$ exists in $L^p(\Omega; L^q(S))$. The second convergence result now follows from the computation (5.14) for $t = \infty$.

Step 2. Let $(x_m)_{m\geqslant 1}$ be a Schauder basis of $L^q(S)$. Then every $x \in L^q(S)$ has a unique decomposition $x = \sum_{m\geqslant 1} a_m x_m$, and according to [**1**, Proposition 1.1.9], there exists $K > 0$ such that for each $N \geqslant 1$

$$\left\| \sum_{m=1}^{N} a_m x_m \right\| \leqslant K\|x\|. \tag{5.15}$$

Let $P_N \in \mathcal{L}(L^q(S))$ be such that $P_N x_m = x_m$ for $m \leqslant N$, and $P_N x_m = 0$ for $m > N$. By Step 1, $P_N M^n$ defines an L^p-martingale for each $N \geqslant 1$ and since $P_N M^n_t \to M^n_t$ in $L^p(\Omega; L^q(S))$ for each $t \geqslant 0$, M^n is an L^p-martingale.

To prove convergence we use a result from [**51**]. Consider a purely discontinuous L^p-martingale $\widetilde{M}$ with accessible jumps. For each $n \geqslant 1$, both martingales $\widetilde{M}$ and $\widetilde{M^n}$ are purely discontinuous and $\widetilde{M^n}$ is weakly differentially subordinated to $\widetilde{M}$, that is, for any $x^* \in L^{q'}(S)$ the process

$$[\langle \widetilde{M}, x^*\rangle] - [\langle \widetilde{M^n}, x^*\rangle]$$

is a.s. non-decreasing and $|\langle \widetilde{M^n_0}, x^*\rangle| \leqslant |\langle \widetilde{M}_0, x^*\rangle|$ a.s. By [**51**],

$$\mathbb{E}\|\widetilde{M^n_t}\|^p \lesssim_{p,q} \mathbb{E}\|\widetilde{M}_t\|^p, \quad t \geqslant 0. \tag{5.16}$$

By (5.16) applied for the martingale $\widetilde{M} = (I - P_N)M$, we obtain

$$\mathbb{E}\|P_N M^n_t - M^n_t\|^p \lesssim_{p,q} \mathbb{E}\|P_N M_t - M_t\|^p$$

for each $N, n \geqslant 1$. Therefore,

$$\begin{aligned}(\mathbb{E}\|M_t - M^n_t\|^p)^{\frac{1}{p}} &\leqslant (\mathbb{E}\|M_t - P_N M_t\|^p)^{\frac{1}{p}} + (\mathbb{E}\|P_N M_t - P_N M^n_t\|^p)^{\frac{1}{p}} + (\mathbb{E}\|P_N M^n_t - M^n_t\|^p)^{\frac{1}{p}}\\ &\lesssim_{p,q} (\mathbb{E}\|M_t - P_N M_t\|^p)^{\frac{1}{p}} + (\mathbb{E}\|P_N M_t - P_N M^n_t\|^p)^{\frac{1}{p}}.\end{aligned} \tag{5.17}$$

For a fixed $\varepsilon > 0$, we can now first pick N so that the first term on the right-hand side of (5.17) is less than $\frac{\varepsilon}{2}$, and subsequently use Step 1 to find an $n = n(N, \varepsilon)$ so that the second term on the right-hand side of (5.17) will be less than $\frac{\varepsilon}{2}$. Hence, for each $\varepsilon > 0$, there exists n_ε such that $(\mathbb{E}\|M_t - M^{n_\varepsilon}_t\|^p)^{\frac{1}{p}} \lesssim_{p,q} \varepsilon$, so $\mathbb{E}\|M^n_t - M_t\|^p \to 0$ as $n \to \infty$.

Finally, if $\sup_{t>0} E\|M_t\|^p < \infty$, then $\sup_{t>0} E\|M^n_t\|^p < \infty$ by (5.16). Hence, $M^n_\infty = \lim_{t\to\infty} M^n_t$ exists in $L^p(\Omega; L^q(S))$ and (5.16) shows that

$$\mathbb{E}\|P_N M^n_\infty - M^n_\infty\|^p \lesssim_{p,q} \mathbb{E}\|P_N M_\infty - M_\infty\|^p$$

for each $N, n \geqslant 1$. Repeating (5.17) for $t = \infty$ now yields the second convergence result. $\square$

Recall that for any $1 < p, q < \infty$, $\mathcal{M}^{\mathrm{acc}}_{p,q}$ is defined to be the linear space of all $L^q(S)$-valued purely discontinuous L^p-martingales with accessible jumps, endowed with the norm $\|M\|_{\mathcal{M}^{\mathrm{acc}}_{p,q}} := \|M_\infty\|_{L^p(\Omega; L^q(S))}$. Using Lemma 5.10, we can deduce the following proposition.

PROPOSITION 5.11. *For any $1 < p, q < \infty$ the space $\mathcal{M}^{\mathrm{acc}}_{p,q}$ is a Banach space.*

Proof. Let $(M^n)_{n\geqslant 1}$ be a Cauchy sequence in $\mathcal{M}^{\mathrm{acc}}_{p,q}$. Since $L^p(\Omega; L^q(S))$ is a Banach space, there exists a limit ξ in $L^p(\Omega; L^q(S))$ of the sequence $(M^n_\infty)_{n\geqslant 1}$. Define an L^p-martingale M by $M_t = \mathbb{E}(\xi|\mathcal{F}_t)$, $t \geqslant 0$. To prove that $M \in \mathcal{M}^{\mathrm{acc}}_{p,q}$, it is enough to show that for each $x^* \in L^{q'}(S)$ the martingale $\langle M, x^*\rangle$ is purely discontinuous with accessible jumps. Fix $x^* \in L^{q'}(S)$. Define $N := \langle M, x^*\rangle$ and $N^n := \langle M^n, x^*\rangle$ for each $n \geqslant 1$. Then $N^n_\infty \to N_\infty$ in

$L^p(\Omega)$. Let $N = N^c + N^q + N^a$ be the canonical decomposition in Lemma 5.2. Since N^n is a purely discontinuous martingale with accessible jumps, it follows from the uniqueness statement in Lemma 5.2 that the canonical decomposition of $N - N_n$ is given by $N^c + N^q + (N^a - N_n)$. By [**23**, Corollaries 26.15, 26.16],

$$[N^c, N^q] = [N^c, N^a - N_n] = [N^q, N^a - N_n] = 0 \quad \text{a.s.}$$

It therefore follows from the Burkholder–Davis–Gundy inequality that

$$\begin{aligned}\mathbb{E}|N_\infty - N^n_\infty|^p &\eqsim_p \mathbb{E}[N - N^n]^{\frac{p}{2}}_\infty \\ &= \mathbb{E}([N]^c_\infty + [N]^q_\infty + [N - N^n]^a_\infty)^{\frac{p}{2}} \geqslant \mathbb{E}([N]^c_\infty + [N]^q_\infty)^{\frac{p}{2}}.\end{aligned}$$

By taking $n \to \infty$, we find $\mathbb{E}\left([N]^c_\infty + [N]^q_\infty\right)^{\frac{p}{2}} = 0$ and so $[N]^c_\infty = [N]^q_\infty = 0$ a.s. We conclude that N is purely discontinuous with accessible jumps. Since this holds for any $x^* \in L^{q'}(S)$, $M \in \mathcal{M}^{\mathrm{acc}}_{p,q}$. □

As a second consequence of Lemma 5.10, we deduce that $\mathcal{A}_{p,q}$ is well defined.

LEMMA 5.12. *Let $1 < p, q < \infty$. Let M be in $\widetilde{S}^p_q$, $\widetilde{D}^p_{q,q}$ or $\widetilde{D}^p_{p,q}$ and let $\mathcal{T} = (\tau_n)_{n\geqslant 1}$ be any sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M. Consider the process M^n defined in (5.13). Then $M^n \to M$ in $\widetilde{S}^p_q$, $\widetilde{D}^p_{q,q}$ or $\widetilde{D}^p_{p,q}$, respectively. As a consequence, $\widetilde{S}^p_q$, $\widetilde{D}^p_{q,q}$, and $\widetilde{D}^p_{p,q}$ are normed linear spaces and $\mathcal{A}_{p,q}$, given in (5.8), is a well-defined normed linear space. If $M \in \mathcal{A}_{p,q}$, then there exists a sequence of predictable stopping times $\mathcal{T}$ with disjoint graphs that exhausts the jumps of M so that $M^n \to M$ in $\mathcal{A}_{p,q}$.*

Proof. We prove the two first statements only for $\widetilde{S}^p_q$. By the dominated convergence theorem, we obtain $M^n \to M$ in $\widetilde{S}^p_q$ and $\|M^n\|_{\widetilde{S}^p_q} \nearrow \|M\|_{\widetilde{S}^p_q}$ as well. Suppose now that $M, N \in \widetilde{S}^p_q$. By [**20**, Lemma I.2.23], there exists a sequence $\mathcal{T} = \{\tau_n\}_{n\geqslant 1}$ of predictable stopping times with disjoint graphs that exhausts the jumps of both M and N. Now clearly,

$$(M + N)^n = M^n + N^n,$$

and so

$$\begin{aligned}\|M + N\|_{\widetilde{S}^p_q} &= \lim_{n\to\infty} \|M^n + N^n\|_{\widetilde{S}^p_q} \\ &\leqslant \lim_{n\to\infty} \left(\|M^n\|_{\widetilde{S}^{p,\mathcal{T}}_q} + \|N^n\|_{\widetilde{S}^{p,\mathcal{T}}_q}\right) = \|M\|_{\widetilde{S}^p_q} + \|N\|_{\widetilde{S}^p_q}.\end{aligned}$$

Let us prove the final statement if $p \leqslant q \leqslant 2$, the other cases are similar. Let $M \in \mathcal{A}_{p,q}$ and let $M_1 \in \widetilde{S}^p_q$, $M_2 \in \widetilde{D}^p_{q,q}$, $M_3 \in \widetilde{D}^p_{p,q}$ be such that $M = M_1 + M_2 + M_3$. Let $\mathcal{T} = \{\tau_n\}_{n\geqslant 1}$ be a sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M_1, M_2 and M_3. Then $M^n = M^n_1 + M^n_2 + M^n_3$ and by the above,

$$\|M - M^n\|_{\mathcal{A}_{p,q}} \leqslant \|M_1 - M^n_1\|_{\widetilde{S}^{p,\mathcal{T}}_q} + \|M_2 - M^n_2\|_{\widetilde{D}^p_{q,q}} + \|M_3 - M^n_3\|_{\widetilde{D}^p_{p,q}} \to 0$$

as $n \to \infty$. □

Intuitively, one would expect the two norms $\|M\|_{\mathcal{A}^{\mathcal{T}}_{p,q}}$ and $\|M\|_{\mathcal{A}_{p,q}}$ to coincide if M happens to have jumps that are exhausted by a finite family $\mathcal{T}$. The following lemma confirms this intuition.

LEMMA 5.13. *Let $1 < p, q < \infty$, $N \geqslant 1$, $\mathcal{T} = (\tau_n)^N_{n=0}$ be a finite family of predictable stopping times with disjoint graphs. Then $\mathcal{A}^{\mathcal{T}}_{p,q} \hookrightarrow \mathcal{A}_{p,q}$ isometrically.*

Proof. We will consider only the case $p \leqslant q \leqslant 2$, the other cases can be shown analogously. Let $M \in \mathcal{A}_{p,q}^{\mathcal{T}}$. Then automatically $M \in \mathcal{A}_{p,q}$ and $\|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}} \geqslant \|M\|_{\mathcal{A}_{p,q}}$. Let us show the reverse inequality. Fix $\varepsilon > 0$, and let $M^1 \in \widetilde{S}_q^p$, $M^2 \in \widetilde{D}_{q,q}^p$ and $M^3 \in \widetilde{D}_{p,q}^p$ be martingales such that $M = M^1 + M^2 + M^3$ and

$$\|M\|_{\mathcal{A}_{p,q}} \geqslant \|M^1\|_{\widetilde{S}_q^p} + \|M^2\|_{\widetilde{D}_{q,q}^p} + \|M^3\|_{\widetilde{D}_{p,q}^p} - \varepsilon.$$

By Lemma 5.9, we can define martingales $\widetilde{M}^1$, $\widetilde{M}^2$ and $\widetilde{M}^3$ by

$$\widetilde{M}_t^i = \sum_{s \in \mathcal{T} \cap [0,t]} \Delta M_s^i, \quad t \geqslant 0,\ i = 1,2,3. \tag{5.18}$$

Note that $|\Delta \widetilde{M}_t^i(\omega)(s)| \leqslant |\Delta M_t^i(\omega)(s)|$ for each $t \geqslant 0$, $\omega \in \Omega$, $s \in S$ and $i = 1,2,3$. Therefore, $\widetilde{M}^1 \in \widetilde{S}_q^p$, $\widetilde{M}^2 \in \widetilde{D}_{q,q}^p$ and $\widetilde{M}^3 \in \widetilde{D}_{p,q}^p$ and $\|\widetilde{M}^1\|_{\widetilde{S}_q^p} \leqslant \|M^1\|_{\widetilde{S}_q^p}$, $\|\widetilde{M}^2\|_{\widetilde{D}_{q,q}^p} \leqslant \|M^2\|_{\widetilde{D}_{q,q}^p}$ and $\|\widetilde{M}^3\|_{\widetilde{D}_{p,q}^p} \leqslant \|M^3\|_{\widetilde{D}_{p,q}^p}$. Moreover, $M = \widetilde{M}^1 + \widetilde{M}^2 + \widetilde{M}^3$. Indeed, since all the martingales here are purely discontinuous with accessible jumps, by (5.18) we find for each $t \geqslant 0$ a.s.

$$\begin{aligned} M_t = \sum_{s \in \mathcal{T} \cap [0,t]} \Delta M_s &= \sum_{s \in \mathcal{T} \cap [0,t]} \left(\Delta M_s^1 + \Delta M_s^2 + \Delta M_s^3\right) \\ &= \widetilde{M}_t^1 + \widetilde{M}_t^2 + \widetilde{M}_t^3. \end{aligned}$$

Therefore,

$$\begin{aligned} \|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}} &\leqslant \|\widetilde{M}^1\|_{\widetilde{S}_q^p} + \|\widetilde{M}^2\|_{\widetilde{D}_{q,q}^p} + \|\widetilde{M}^3\|_{\widetilde{D}_{p,q}^p} \\ &\leqslant \|M^1\|_{\widetilde{S}_q^p} + \|M^2\|_{\widetilde{D}_{q,q}^p} + \|M^3\|_{\widetilde{D}_{p,q}^p} \leqslant \|M\|_{\mathcal{A}_{p,q}} + \varepsilon. \end{aligned}$$

Since ε was arbitrary, we conclude that $\|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}} \leqslant \|M\|_{\mathcal{A}_{p,q}}$, and consequently $\|M\|_{\mathcal{A}_{p,q}^{\mathcal{T}}} = \|M\|_{\mathcal{A}_{p,q}}$. □

We are now ready to deduce the main theorem of this subsection.

THEOREM 5.14. *Let $1 < p,q < \infty$, $M : \mathbb{R}_+ \times \Omega \to L^q(S)$ be a purely discontinuous martingale with accessible jumps. Then,*

$$\left(\mathbb{E} \sup_{t \geqslant 0} \|M_t\|_{L^q(S)}^p\right)^{\frac{1}{p}} \eqsim_{p,q} \|M\|_{\mathcal{A}_{p,q}}, \tag{5.19}$$

where $\mathcal{A}_{p,q}$ is as in (5.8). In particular, $\mathcal{A}_{p,q}$ is a Banach space of L^p-martingales.

Proof. By Doob's maximal inequality $\mathbb{E}\sup_{t \geqslant 0} \|M_t\|^p \eqsim_p \|M\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}}^p$, so we need to show that $\|M\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}} \eqsim_{p,q} \|M\|_{\mathcal{A}_{p,q}}$. Suppose first that $M \in \mathcal{A}_{p,q}$. By Lemma 5.12, there exists a sequence of predictable stopping times $\mathcal{T}$ with disjoint graphs that exhausts the jumps of M so that M^n defined by (5.13) converges to M in $\mathcal{A}_{p,q}$. In particular, $(M^n)_{n \geqslant 1}$ is Cauchy in $\mathcal{A}_{p,q}$. By Lemma 5.13 and Theorem 5.6, it follows that it is also Cauchy in $\mathcal{M}_{p,q}^{\mathrm{acc}}$. By Proposition 5.11, $(M^n)_{n \geqslant 1}$ converges and clearly the limit is M.

Suppose now that $M \in \mathcal{M}_{p,q}^{\mathrm{acc}}$. It suffices to show that $M \in \mathcal{A}_{p,q}$. Indeed, Lemma 5.12 then shows that there is a sequence of predictable stopping times with disjoint graphs that exhausts the jumps of M so that $M^n \to M$ in $\mathcal{A}_{p,q}$. By Lemma 5.10, we also have $M^n \to M$ in $\mathcal{M}_{p,q}^{\mathrm{acc}}$

and so the lower bound in (5.19) follows from Lemma 5.13 and Theorem 5.6. We will show that $M \in \mathcal{A}_{p,q}$ in the two cases $2 \leqslant q \leqslant p$ and $p \leqslant q \leqslant 2$, the other cases can be treated analogously.

Case $2 \leqslant q \leqslant p$. We will show that $\|M\|_{\widetilde{S}_q^p} \lesssim_{p,q} \|M\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}}$. The analogous statements for $\widetilde{D}_{q,q}^p$ and $\widetilde{D}_{p,q}^p$ can be shown in the same way. By Theorem 5.6, $\|M^n\|_{\widetilde{S}_q^p} \lesssim_{p,q} \|M^n\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}}$. Also, by (5.16), we have that $\|M^n\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}} \lesssim_{p,q} \|M\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}}$ for all $n \geqslant 1$. Therefore, $\|M^n\|_{\widetilde{S}_q^p} \lesssim_{p,q} \|M\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}}$ uniformly in n, so by monotone convergence

$$\begin{aligned} \|M\|_{\widetilde{S}_q^p}^p &= \mathbb{E}\left\| \left(\sum_{m \geqslant 1} \mathbb{E}_{\mathcal{F}_{\tau_m -}} |\Delta M_{\tau_m}|^2 \right)^{\frac{1}{2}} \right\|_{L^q(S)}^p \\ &= \lim_{n \to \infty} \mathbb{E}\left\| \left(\sum_{m=0}^{n} \mathbb{E}_{\mathcal{F}_{\tau_m -}} |\Delta M_{\tau_m}|^2 \right)^{\frac{1}{2}} \right\|_{L^q(S)}^p \\ &= \lim_{n \to \infty} \|M^n\|_{\widetilde{S}_q^p}^p \lesssim_{p,q} \|M\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}}^p. \end{aligned} \tag{5.20}$$

Case $p \leqslant q \leqslant 2$. Observe that $\|M^n\|_{\mathcal{A}_{p,q}} \eqsim_{p,q} \|M^n\|_{\mathcal{M}_{p,q}^{\mathrm{acc}}}$ for each $n \geqslant 1$ by Theorem 5.6 and since $(M^n)_{n \geqslant 1}$ is a Cauchy sequence in $\mathcal{M}_{p,q}^{\mathrm{acc}}$ due to Lemma 5.10, it follows that $(M^n)_{n \geqslant 1}$ is a Cauchy sequence in $\mathcal{A}_{p,q}$. Thus, there exists a subsequence $(M^{n_k})_{k \geqslant 0}$ such that

$$\|M^{n_{k+1}} - M^{n_k}\|_{\mathcal{A}_{p,q}} < \frac{1}{2^{k+1}}, \quad k \geqslant 0.$$

Let $N^k = M^{n_k} - M^{n_{k-1}}$, $k \geqslant 1$, $N^0 = M^{n_0}$. Set $n_{-1} = -1$. By Theorem 5.6, for each $k \geqslant 0$ there exist $N^{k,1}$, $N^{k,2}$ and $N^{k,3}$ such that $N^{k,1} \in \widetilde{S}_q^p$, $N^{k,2} \in \widetilde{D}_{q,q}^p$, $N^{k,3} \in \widetilde{D}_{p,q}^p$, $N^k = N^{k,1} + N^{k,2} + N^{k,3}$,

$$\{t : \Delta N_t^{k,i} \neq 0, i = 1,2,3\} \subset \{\tau_{n_{k-1}+1}, \dots, \tau_{n_k}\}, \quad \text{a.s.},$$

and

$$\begin{aligned} \|N^{k,1}\|_{\widetilde{S}_q^p} + \|N^{k,2}\|_{\widetilde{D}_{q,q}^p} + \|N^{k,3}\|_{\widetilde{D}_{p,q}^p} &< \frac{1}{2^k}, \qquad k \geqslant 1, \\ \|N^{0,1}\|_{\widetilde{S}_q^p} + \|N^{0,2}\|_{\widetilde{D}_{q,q}^p} + \|N^{0,3}\|_{\widetilde{D}_{p,q}^p} &\leqslant 2\|M^{n_0}\|_{\mathcal{A}_{p,q}}. \end{aligned} \tag{5.21}$$

Let

$$M^{m,i} := \sum_{k=0}^{m} N^{k,i}, \quad m \geqslant 1, \quad i = 1,2,3.$$

Then by (5.21), $(M^{m,1})_{m \geqslant 1}$, $(M^{m,2})_{m \geqslant 1}$ and $(M^{m,3})_{m \geqslant 1}$ are Cauchy sequences in $\widetilde{S}_q^p$, $\widetilde{D}_{q,q}^p$ and $\widetilde{D}_{p,q}^p$, respectively. By construction, each of $M^{m,i}$, $m \geqslant 1$, $i = 1,2,3$, has finitely many jumps occurring in $\{\tau_0, \dots, \tau_{n_m}\}$, so by Theorem 5.6 the sequences $(M^{m,1})_{m \geqslant 1}$, $(M^{m,2})_{m \geqslant 1}$ and $(M^{m,3})_{m \geqslant 1}$ are Cauchy in $\mathcal{M}_{p,q}^{\mathrm{acc}}$ as well. Due to Proposition 5.11, there exist $\widetilde{M}^1$, $\widetilde{M}^2$ and $\widetilde{M}^3$ such that $M^{m,i} \to \widetilde{M}^i$ in $\mathcal{M}_{p,q}^{\mathrm{acc}}$ as $m \to \infty$ for each $i = 1,2,3$. Since $M^{m,1} + M^{m,2} + M^{m,3} \to M$ in $\mathcal{M}_{p,q}^{\mathrm{acc}}$ as $m \to \infty$ by Lemma 5.10, it follows that $M = \widetilde{M}^1 + \widetilde{M}^2 + \widetilde{M}^3$.

Let us now show that the jumps of $\widetilde{M}^1$, $\widetilde{M}^2$ and $\widetilde{M}^3$ are exhausted by family $\mathcal{T} = (\tau_n)_{n \geqslant 1}$. Indeed, assume that for some $i = 1,2,3$ there exists a predictable stopping time τ such that $\mathbb{P}\{\Delta \widetilde{M}_\tau^i \neq 0, \tau \notin \{\tau_1, \tau_2, \dots\}\} > 0$. Then by separability of $X = L^q(S)$ there exists an $x^* \in X^*$ such that

$$\mathbb{P}\{\langle \Delta \widetilde{M}_\tau^i, x^* \rangle \neq 0, \tau \notin \{\tau_1, \tau_2, \dots\}\} > 0 \tag{5.22}$$

and so, by the Burkholder–Davis–Gundy inequality,

$$\begin{aligned}\mathbb{E}|\langle (M^{m,i}-\widetilde{M}^i)_\infty, x^*\rangle|^p &\eqsim_p \mathbb{E}[\langle M^{m,i}-\widetilde{M}^i, x^*\rangle]_\infty^{\frac{p}{2}}\\ &= \mathbb{E}\Bigg(\sum_{u\geqslant 0}|\langle \Delta(M^{m,i}-\widetilde{M}^i)_u, x^*\rangle|^2\Bigg)^{\frac{p}{2}} \\ &\geqslant \mathbb{E}|\langle \Delta\widetilde{M}^i_\tau, x^*\rangle|^p \mathbf{1}_{\tau\notin\{\tau_1,\tau_2,\dots\}},\end{aligned} \tag{5.23}$$

where the final inequality holds as $\mathbb{P}\{\Delta M^{m,i}_\tau \neq 0, \tau\notin\{\tau_1,\tau_2,\dots\}\} = 0$. But the last expression in (5.23) does not vanish as $m\to\infty$ because of (5.22), which contradicts with the fact that $M^{m,i}\to\widetilde{M}^i$ in $\mathcal{M}^{\mathrm{acc}}_{p,q}$.

By the calculation in (5.20), $\|\widetilde{M}^1\|_{\widetilde{S}^p_q} = \lim_{m\to\infty}\|M^{m,1}\|_{\widetilde{S}^p_q}$ and the right-hand side is bounded as $M^{m,1}$ is Cauchy in $\widetilde{S}^p_q$. By the same reasoning $\widetilde{M}^2\in\widetilde{D}^p_{q,q}$ and $\widetilde{M}^3\in\widetilde{D}^p_{p,q}$, so $M\in\mathcal{A}_{p,q}$. This completes the proof. □

Theorem 5.14 and Lemma 5.33 immediately yield the following sharp estimates for stochastic integrals.

Corollary 5.15. *Let $1<p,q<\infty$, $M:\mathbb{R}_+\times\Omega\to H$ be a purely discontinuous L^p-martingale with accessible jumps, $X=L^q(S)$, $\Phi:\mathbb{R}_+\times\Omega\to\mathcal{L}(H,X)$ be elementary predictable. Then for all $t\geqslant 0$ one has that*

$$\Big(\mathbb{E}\sup_{0\leqslant s\leqslant t}\|(\Phi\cdot M)_s\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} \|(\Phi\mathbf{1}_{[0,t]})\cdot M\|_{\mathcal{A}_{p,q}},$$

where $\mathcal{A}_{p,q}$ is as given in (5.8).

5.4. *Quasi-left continuous purely discontinuous martingales*

We now turn to estimates for the stochastic integral $\Phi\cdot M$ in the case that M is a purely discontinuous quasi-left continuous local martingale. We will first show in Lemma 5.18 that one can (essentially) represent $\Phi\cdot M$ as a stochastic integral $\Phi_H\star\bar{\mu}^M$, where $\bar{\mu}^M$ is the compensated version of the jump measure μ^M of M. Afterwards, in Theorem 5.28, we prove sharp bounds for stochastic integrals of the form $f\star\bar{\mu}$, where μ is any integer-valued random measure with a compensator that is non-atomic in time. By combining these two observations, we immediately find sharp bounds for $\Phi\cdot M$, see Theorem 5.30.

5.4.1. *Facts on random measures.* Let us start by recalling some necessary definitions and facts concerning random measures. Let $(J,\mathcal{J})$ be a measurable space. Then a family $\mu = \{\mu(\omega;\,\mathrm{d}t,\mathrm{d}x),\omega\in\Omega\}$ of non-negative measures on $(\mathbb{R}_+\times J;\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{J})$ is called a *random measure*. A random measure μ is called *integer-valued* if it takes values in $\mathbb{N}\cup\{\infty\}$, that is, for each $A\in\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}\otimes\mathcal{J}$ one has that $\mu(A)\in\mathbb{N}\cup\{\infty\}$ a.s., and if $\mu(\{t\}\times J)\in\{0,1\}$ a.s. for all $t\geqslant 0$. We say that μ is *non-atomic in time* if $\mu(\{t\}\times J)=0$ a.s. for all $t\geqslant 0$.

Recall that $\mathcal{P}$ and $\mathcal{O}$ denote the predictable and optional σ-algebra on $\mathbb{R}_+\times\Omega$ and $\widetilde{\mathcal{P}}=\mathcal{P}\otimes\mathcal{J}$ and $\widetilde{\mathcal{O}}:=\mathcal{O}\otimes\mathcal{J}$ are the induced σ-algebras on $\widetilde{\Omega}=\mathbb{R}_+\times\Omega\times J$. A process $F:\mathbb{R}_+\times\Omega\to\mathbb{R}$ is called *optional* if it is $\mathcal{O}$-measurable. A random measure μ is called *optional* (respectively, *predictable*) if for any $\widetilde{\mathcal{O}}$-measurable (respectively, $\widetilde{\mathcal{P}}$-measurable) non-negative $F:\mathbb{R}_+\times\Omega\times J\to\mathbb{R}_+$ the stochastic integral

$$(F\star\mu)_t(\omega) := \int_{\mathbb{R}_+\times J}\mathbf{1}_{[0,t]}(s)F(s,\omega,x)\mu(\omega;\,\mathrm{d}s,\mathrm{d}x),\quad t\geqslant 0,\ \omega\in\Omega,$$

as a function from $\mathbb{R}_+ \times \Omega$ to $\overline{\mathbb{R}}_+$ is optional (respectively, predictable). Let X be a Banach space.

We can extend stochastic integration to X-valued processes in the following way. Let $F : \mathbb{R}_+ \times \Omega \times J \to X$, μ be a random measure. The integral

$$(F \star \mu)_t := \int_{\mathbb{R}_+ \times J} F(s, \cdot, x) \mathbf{1}_{[0,t]}(s) \mu(\cdot; \mathrm{d}s, \mathrm{d}x), \quad t \geqslant 0,$$

is well defined and optional (respectively, predictable) if μ is optional (respectively, predictable), F is $\widetilde{\mathcal{O}}$-strongly-measurable (respectively, $\widetilde{\mathcal{P}}$-strongly-measurable) and $(\|F\| \star \mu)_\infty$ is a.s. bounded. We refer the reader to [**20**, **29**, **33**, **40**] for further details.

A random measure μ is called $\widetilde{\mathcal{P}}$-σ-finite if there exists an increasing sequence of sets $(A_n)_{n \geqslant 1} \subset \widetilde{\mathcal{P}}$ such that $\int_{\mathbb{R}_+ \times J} \mathbf{1}_{A_n}(s, \omega, x) \mu(\omega; \mathrm{d}s, \mathrm{d}x)$ is finite a.s. and $\cup_n A_n = \mathbb{R}_+ \times \Omega \times J$. According to [**20**, Theorem II.1.8], every $\widetilde{\mathcal{P}}$-σ-finite optional random measure μ has a *compensator*: a unique $\widetilde{\mathcal{P}}$-σ-finite predictable random measure ν such that $\mathbb{E}(W \star \mu)_\infty = \mathbb{E}(W \star \nu)_\infty$ for each $\widetilde{\mathcal{P}}$-measurable real-valued non-negative W. We refer the reader to [**20**, Chapter II.1] for more details on random measures. For any optional $\widetilde{\mathcal{P}}$-σ-finite measure μ, we define the associated compensated random measure by $\bar{\mu} = \mu - \nu$.

For each $\widetilde{\mathcal{P}}$-strongly-measurable $F : \mathbb{R}_+ \times \Omega \times J \to X$ such that $\mathbb{E}(\|F\| \star \mu)_\infty < \infty$ (or, equivalently, $\mathbb{E}(\|F\| \star \nu)_\infty < \infty$, see the definition of a compensator above), we can define a process $F \star \bar{\mu}$ by $F \star \mu - F \star \nu$. The reader should be warned that in the literature $F \star \bar{\mu}$ is often used to denote the integral of F over the whole $\mathbb{R}_+$ (that is, $(F \star \bar{\mu})_\infty$ in our notation).

For further reference, we state the following lemma, which is a straightforward consequence of the definition of a compensator.

LEMMA 5.16. *Let $A \in \widetilde{\mathcal{P}}$, μ_1 be a $\widetilde{\mathcal{P}}$-σ-finite random measure with a compensator ν_1. Then $\mu_2 = \mu_1 \mathbf{1}_A$ is a $\widetilde{\mathcal{P}}$-σ-finite random measure and $\nu_2 = \nu_1 \mathbf{1}_A$ is a compensator for μ_2.*

5.4.2. *Representation of the stochastic integral.* To any purely discontinuous local martingale M with values in a Hilbert space H we can associate an integer-valued random measure μ^M on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(H)$ by setting

$$\mu^M(\omega; B \times A) := \sum_{u \in B} \mathbf{1}_{A \setminus \{0\}}(\Delta M_u(\omega)), \quad \omega \in \Omega,$$

for each $B \in \mathcal{B}(\mathbb{R}_+)$, $A \in \mathcal{B}(H)$. That is, $\mu^M(\omega; B \times A)$ counts the number of jumps within the time set B with size in A on the trajectory belonging to the sample point ω.

Recall that a process $M : \mathbb{R}_+ \times \Omega \to H$ is called quasi-left continuous if $\Delta M_\tau = 0$ a.s. on the set $\{\tau < \infty\}$ for each predictable stopping time τ. If $M : \mathbb{R}_+ \times \Omega \to H$ is a quasi-left continuous local martingale, then μ^M is $\widetilde{\mathcal{P}}$-σ-finite and there exists a compensator ν^M (see, for example, [**20**, Proposition II.1.16] and [**23**, Theorem 25.22]). If M is, in addition, purely discontinuous, then the following characterization holds thanks to [**20**, Corollary II.1.19].

LEMMA 5.17. *Let H be a separable Hilbert space and $M : \mathbb{R}_+ \times \Omega \to H$ be a purely discontinuous local martingale. Let μ^M and ν^M be the associated integer-valued random measure and its compensator. Then M is quasi-left continuous if and only if ν^M is non-atomic in time.*

Let us now show that $\Phi \cdot M$ can (essentially) be represented as a stochastic integral with respect to $\bar{\mu}^M$.

LEMMA 5.18. *Let X be a Banach space, H be a Hilbert space, $1 < p < \infty$, $M : \mathbb{R}_+ \times \Omega \to H$ be a purely discontinuous quasi-left continuous local martingale and $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, X)$*

be elementary predictable. Define $\Phi_H : \mathbb{R}_+ \times \Omega \times H \to X$ *by*

$$\Phi_H(t,\omega,h) := \Phi(t,\omega)h, \quad t \geqslant 0, \omega \in \Omega, h \in H.$$

Then there exists an increasing sequence $(A_n)_{n\geqslant 1} \in \widetilde{\mathcal{P}}$ *such that* $\cup_n A_n = \mathbb{R}_+ \times \Omega \times J$, $\Phi_H \mathbf{1}_{A_n}$ *is integrable with respect to* $\bar{\mu}^M$ *for each* $n \geqslant 1$, *and*

(i) *if* $\Phi \cdot M \in L^p(\Omega; X)$ *then* $(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M \to \Phi \cdot M$ *in* $L^p(\Omega; X)$;
(ii) *if* $\Phi \cdot M \notin L^p(\Omega; X)$ *then* $\|(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M\|_{L^p(\Omega;X)} \to \infty$ *for* $n \to \infty$.

Proof. For each $k, l \geqslant 1$, we define a stopping time $\tau_{k,l}$ by

$$\tau_{k,l} = \inf\{t \in \mathbb{R}_+ : \#\{s \in [0,t] : \|\Delta M_s\| \in [1/k, k]\} = l\}.$$

Since M has càdlàg trajectories, $\tau_{k,l}$ is a.s. well defined and takes its values in $[0,\infty]$. Moreover, $\tau_{k,l} \to \infty$ for each $k \geqslant 1$ a.s. as $l \to \infty$.

Set $B_k = \{h \in H : \|h\| \in [1/k, k]\}$. For each $k, l \geqslant 1$, define

$$A_{k,l} := [0, \tau_{k,l}] \times B_k \subset \widetilde{\mathcal{P}}.$$

Then $\Phi_H \mathbf{1}_{A_{k,l}}$ is integrable with respect to μ^M. Indeed, a.s.

$$\big((\Phi_H \mathbf{1}_{A_{k,l}}) \star \mu^M\big)_\infty \leqslant \sup \|\Phi\| k \big(\mathbf{1}_{A_{k,l}} \star \mu^M\big)_\infty \leqslant \sup \|\Phi\| kl.$$

Since $\tau_{k,l} \to \infty$ for each $k \geqslant 1$ a.s. as $l \to \infty$, we can find a subsequence $(\tau_{k_n,l_n})_{n\geqslant 1}$ such that $k_n \geqslant n$ for each $n \geqslant 1$ and $\inf_{m\geqslant n} \tau_{k_m,l_m} \to \infty$ a.s. as $n \to \infty$. Let $\tau_n = \inf_{m\geqslant n} \tau_{k_m,l_m}$ and define $(A_n)_{n\geqslant 1} \subset \widetilde{\mathcal{P}}$ by

$$A_n = \mathbf{1}_{[0,\tau_n]\times B_n}.$$

Then $\cup_n A_n = \mathbb{R}_+ \times \Omega \times J$ and $\Phi_H \mathbf{1}_{A_n}$ is integrable with respect to $\bar{\mu}^M$ for all $n \geqslant 1$.

Now, prove that $(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M \to \Phi \cdot M$ in $L^p(\Omega; X)$. Since Φ is simple, it takes its values in a finite-dimensional subspace of X, so we can endow X with a Euclidean norm $|||\cdot|||$. First suppose that $(\Phi \cdot M)_\infty \notin L^p(\Omega; X)$. By the Burkholder–Davis–Gundy inequality, this is equivalent to the fact that $[\Phi \cdot M]_\infty^{\frac{1}{2}} \notin L^p(\Omega; X)$. Note that

$$\begin{aligned}\mathbb{E}|||(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M)_\infty|||^p &\eqsim_p \mathbb{E}\big[(\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M\big]_\infty^{\frac{p}{2}} \\ &= \mathbb{E}\left(\sum_{t\in[0,\tau_n]} |||\Delta(\Phi \cdot M)_t|||^2 \mathbf{1}_{\|\Delta M_t\| \in [1/n,n]}\right)^{\frac{p}{2}},\end{aligned}$$

and the last expression monotonically goes to infinity since $\tau_n \to \infty$ a.s. and

$$\mathbb{E}\left(\sum_{t\geqslant 0} \|\Delta(\Phi \cdot M)_t\|^2\right)^{\frac{p}{2}} = \mathbb{E}[\Phi \cdot M]_\infty^{\frac{p}{2}} = \infty.$$

So if $(\Phi \cdot M)_\infty \notin L^p(\Omega; X)$, then $\big\|\big((\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M\big)_\infty\big\|_{L^p(\Omega;X)} \to \infty$ as $n \to \infty$.

Now, assume that $(\Phi \cdot M)_\infty \in L^p(\Omega; X)$. Then

$$\begin{aligned}&\mathbb{E}|||(\Phi \cdot M)_\infty - ((\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M)_\infty|||^p \eqsim_p \mathbb{E}[\Phi \cdot M - (\Phi_H \mathbf{1}_{A_n}) \star \bar{\mu}^M]_\infty^{\frac{p}{2}} \\ &= \mathbb{E}\left(\sum_{t\in[0,\tau_n]} |||\Delta(\Phi \cdot M)_t|||^2 \mathbf{1}_{\|\Delta M_t\| \notin [1/n,n]} + \sum_{t\in(\tau_n,\infty)} |||\Delta(\Phi \cdot M)_t|||^2\right)^{\frac{p}{2}} \to 0, \quad n \to \infty\end{aligned}$$

by the dominated convergence theorem. ∎

By Lemmas 5.17 and 5.18, it now suffices to obtain sharp bounds for the stochastic integral $(F \star \bar{\mu})_\infty$, where μ is any optional integer-valued random measure whose compensator ν is non-atomic in time.

5.4.3. *Integrals with respect to random measures.* Throughout this subsection, *μ denotes an optional integer-valued random measure whose compensator ν is non-atomic in time*, that is, $\nu(\{t\} \times J) = 0$ a.s. for all $t \geqslant 0$. To derive the sharp bounds for stochastic integrals with respect to μ in Theorem 5.28, we will need to collect several observations, stated in Lemma 5.19, Proposition 5.22, Lemma 5.24 and Proposition 5.27. Together with Lemma 2.1 and Corollary B.8, they form the main ingredients for the proof. At the start of the proof of Theorem 5.28, we will discuss how these ingredients are combined.

Let us start by deriving the first observation.

LEMMA 5.19. *Let X be a Banach space, $1 < p < \infty$, μ be a random measure, ν be the corresponding compensator, $F : \mathbb{R}_+ \times \Omega \times J \to X$ and $G : \mathbb{R}_+ \times \Omega \times J \to X^*$ be simple $\widetilde{\mathcal{P}}$-measurable functions. Then for each $A \in \widetilde{\mathcal{P}}$ such that $\mathbb{E}(\mathbf{1}_A \star \mu)_\infty < \infty$ the stochastic integrals $(F\mathbf{1}_A) \star \bar{\mu}$ and $(G\mathbf{1}_A) \star \bar{\mu}$ are well defined and*

$$\mathbb{E}\langle (F\mathbf{1}_A) \star \bar{\mu}, (G\mathbf{1}_A) \star \bar{\mu} \rangle = \mathbb{E}(\langle F, G \rangle \mathbf{1}_A) \star \nu. \tag{5.24}$$

Observe that it suffices to prove this statement if X is finite dimensional and Euclidean. By Lemma 5.16, we can also redefine $F := F\mathbf{1}_A$, $G := G\mathbf{1}_A$. Hence, Lemma 5.19 is immediate from the following statement, which easily follows from [**20**, Theorem II.1.33] (or from [**12**, p.98] and [**40**, (6)] as well).

LEMMA 5.20. *Let H be a Hilbert space, $f : \mathbb{R}_+ \times \Omega \times J \to H$ be strongly-$\widetilde{\mathcal{P}}$-measurable. Then*

$$\mathbb{E}\|f \star \bar{\mu}\|^2 = \mathbb{E}\|f\|^2 \star \nu. \tag{5.25}$$

Equivalently, for each strongly-$\widetilde{\mathcal{P}}$-measurable $f, g : \mathbb{R}_+ \times \Omega \times J \to H$ such that $\mathbb{E}\|f\|^2 \star \nu < \infty$ and $\mathbb{E}\|g\|^2 \star \nu < \infty$

$$\mathbb{E}\langle f \star \bar{\mu}, g \star \bar{\mu} \rangle = \mathbb{E}\langle f, g \rangle \star \nu. \tag{5.26}$$

Proof. The case $H = \mathbb{R}$ can be deduced from [**20**, II.1.34] as ν is assumed to be non-atomic in time. Note that by the Pettis measurability theorem [**17**, Theorem 1.1.20] we may assume that H is separable, thus by applying the real-valued case coordinate-wise, we obtain the general case. □

The second observation is an extension of the following classical result of Novikov [**40**, Theorem 1].

LEMMA 5.21 (Novikov). *Let $f : \mathbb{R}_+ \times \Omega \times J \to \mathbb{R}$ be $\widetilde{\mathcal{P}}$-measurable. Then*

$$\mathbb{E}|f \star \bar{\mu}|^p \lesssim_p \mathbb{E}|f|^p \star \nu \text{ if } 1 \leqslant p \leqslant 2,$$

$$\mathbb{E}|f \star \bar{\mu}|^p \lesssim_p (\mathbb{E}|f|^2 \star \nu)^{\frac{p}{2}} + \mathbb{E}|f|^p \star \nu \text{ if } p \geqslant 2.$$

The following proposition extends Novikov's inequalities in the case that $\nu(\mathbb{R}_+ \times J) \leqslant 1$ a.s. If $X = L^q(S)$, then this result can be seen as a special case of Theorem 5.28. In the proof, we will use the measure $\mathbb{P} \otimes \nu$ on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{J}$ that is defined by setting

$$\mathbb{P} \otimes \nu\left(\bigcup_{i=1}^{n} A_i \times B_i\right) := \sum_{i=1}^{n} \mathbb{E}(\mathbf{1}_{A_i} \nu(B_i)), \tag{5.27}$$

for disjoint $A_i \in \mathcal{F}$ and disjoint $B_i \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{J}$, and extending $\mathbb{P} \otimes \nu$ to $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F} \otimes \mathcal{J}$ via the Carathéodory extension theorem.

PROPOSITION 5.22. *Suppose $\nu(\mathbb{R}_+ \times J) \leqslant 1$ a.s. Let X be a Banach space and $F : \mathbb{R}_+ \times \Omega \times J \to X$ be simple $\widetilde{\mathcal{P}}$-measurable. Then for all $1 < p < \infty$*

$$\mathbb{E}\|F \star \bar{\mu}\|^p \eqsim_p \mathbb{E}\|F\|^p \star \nu. \tag{5.28}$$

In particular,

$$(\mathbb{E}\|(F \star \bar{\mu})_\infty\|^p|\mathcal{F}_0) \eqsim_p (\mathbb{E}(\|F\|^p \star \nu)_\infty|\mathcal{F}_0). \tag{5.29}$$

Proof. We start by proving (5.28). We first prove $\lesssim_p$, and later deduce $\gtrsim_p$ by a duality argument.

Step 1: *upper bounds.* The case $X = \mathbb{R}$ follows from Lemma 5.21 and the fact that $\|\cdot\|_{L^2(\mathbb{R}_+\times\Omega\times J,\mathbb{P}\otimes\nu)} \leqslant \|\cdot\|_{L^p(\mathbb{R}_+\times\Omega\times J,\mathbb{P}\otimes\nu)}$ for each $p \geqslant 2$ since $\mathbb{P} \otimes \nu(\mathbb{R}_+ \times \Omega \times J) \leqslant 1$. Now, let X be a general Banach space. Then

$$\begin{aligned}\mathbb{E}\|F \star \bar{\mu}\|^p &\overset{(i)}{\lesssim_p} \mathbb{E}\|F \star \mu\|^p + \mathbb{E}\|F \star \nu\|^p \overset{(ii)}{\leqslant} \mathbb{E}|\|F\| \star \mu|^p + \mathbb{E}|\|F\| \star \nu|^p \\ &\overset{(iii)}{\lesssim}_p \mathbb{E}|\|F\| \star \bar{\mu}|^p + \mathbb{E}|\|F\| \star \nu|^p \overset{(iv)}{\lesssim}_p \mathbb{E}\|F\|^p \star \nu,\end{aligned}$$

where (i) and (iii) follow from the fact that $\bar{\mu} = \mu - \nu$ and the triangle inequality, (ii) follows from [**17**, Proposition 1.2.2] and (iv) follows from the real-valued case and the fact that a.s.

$$\|\cdot\|_{L^1(\mathbb{R}_+\times J;\nu)} \leqslant \|\cdot\|_{L^p(\mathbb{R}_+\times J;\nu)}.$$

Step 2: *lower bounds.* We can assume that X is finite dimensional since F is simple. Let $Y = L^p(\mathbb{R}_+ \times \Omega \times J, \mathbb{P} \otimes \nu; X)$. Recall that by [**17**, Proposition 1.3.3] $Y^* = L^{p'}(\mathbb{R}_+ \times \Omega \times J, \mathbb{P} \otimes \nu; X^*)$ and $(L^p(\Omega; X))^* = L^{p'}(\Omega; X^*)$. Therefore, due to the upper bounds from Step 1 and Lemma 5.19

$$\begin{aligned}(\mathbb{E}\|F\|^p \star \nu)^{\frac{1}{p}} &= \sup_{G\in Y^*:\|G\|\leqslant 1} \mathbb{E}\langle F, G\rangle \star \nu = \sup_{G\in Y^*:\|G\|\leqslant 1} \mathbb{E}\langle F \star \bar{\mu}, G \star \bar{\mu}\rangle \\ &\lesssim_p \sup_{\xi\in L^{p'}(\Omega;X^*):\|\xi\|\leqslant 1} \mathbb{E}\langle F \star \bar{\mu}, \xi\rangle = (\mathbb{E}\|F \star \bar{\mu}\|^p)^{\frac{1}{p}}.\end{aligned}$$

To derive (5.29), fix any $A \in \mathcal{F}_0$. Then by Lemma 5.16 and (5.28)

$$\mathbb{E}(\|(F \star \bar{\mu})_\infty\|^p \cdot \mathbf{1}_A) = \mathbb{E}\|((F \cdot \mathbf{1}_A) \star \bar{\mu})_\infty\|^p \eqsim_p \mathbb{E}(\|F \cdot \mathbf{1}_A\|^p \star \nu)_\infty = \mathbb{E}((\|F\|^p \star \nu)_\infty \cdot \mathbf{1}_A).$$

Since A is arbitrary, our proof is complete. □

REMARK 5.23. The condition $\nu(\mathbb{R}_+ \times J) \leqslant 1$ a.s. is necessary in general. Indeed, let N be a Poisson process with intensity parameter λ and let μ be the random measure on

$\mathbb{R}_+ \times \{0\}$ defined by $\mu([0,t]\times\{0\}) = N_t$. Then the corresponding compensator ν satisfies $\nu([0,t]\times\{0\}) = \lambda t$. In particular,

$$\mathbb{E}|\mathbf{1}_{[0,1]} \star \bar{\mu}|^4 = \mathbb{E}|N-\lambda|^4 = \sum_{k=0}^{\infty} \frac{(k-\lambda)^4 \lambda^k e^{-\lambda}}{k!} = \lambda(3\lambda+1),$$

which is not comparable with $\mathbb{E}|\mathbf{1}_{[0,1]}|^4 \star \nu = \lambda$ if λ is large.

The condition $\nu(\mathbb{R}_+ \times J) \leqslant 1$ a.s. is, however, not needed for the upper bounds if $1 \leqslant p \leqslant 2$ and X is a Hilbert space. Indeed, for $p = 1$

$$\mathbb{E}\|F \star \bar{\mu}\| \leqslant \mathbb{E}\|F \star \mu\| + \mathbb{E}\|F \star \nu\| \leqslant \mathbb{E}\|F\| \star \mu + \mathbb{E}\|F\| \star \nu = 2\mathbb{E}\|F\| \star \nu,$$

and for case $p = 2$ follows immediately from Lemma 5.20:

$$\mathbb{E}\|F \star \bar{\mu}\|^2 = \mathbb{E}\|F\|^2 \star \nu.$$

Therefore, by the vector-valued Riesz–Thorin theorem [**17**, Theorem 2.2.1] for each $1 \leqslant p \leqslant 2$

$$(\mathbb{E}\|F \star \bar{\mu}\|^p)^{\frac{1}{p}} \leqslant 2(\mathbb{E}\|F\|^p \star \nu)^{\frac{1}{p}}.$$

The third observation needed in the proof of Theorem 5.28 reads as follows.

LEMMA 5.24. *Let $F : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be a non-decreasing continuous predictable function such that $F(t) - F(s) \leqslant C(t-s)$ for all $0 \leqslant s \leqslant t$ a.s. and for some fixed constant $C \geqslant 0$ and $F(0) = 0$ a.s. Then for each fixed $T \geqslant 0$*

$$F(T) = \lim_{m\to\infty} \sum_{n=0}^{[2^m T]-1} \mathbb{E}\Big[F\Big(\frac{n+1}{2^m}\Big) - F\Big(\frac{n}{2^m}\Big)\Big|\mathcal{F}_{\frac{n}{2^m}}\Big],$$

where the last limit holds a.s. and in $L^p(\Omega)$ for all $1 < p < \infty$.

For the proof we will need two lemmas. To state the first one, we fix the following notation. For each $m \geqslant 1$, let $\mathcal{P}_m$ be the σ-field on $\mathbb{R}_+ \times \Omega$ generated by all $\mathcal{P}$-measurable $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that $f|_{(\frac{n}{2^m}, \frac{n+1}{2^m}] \times \Omega}$ is $\mathcal{B}\left((\frac{n}{2^m}, \frac{n+1}{2^m}]\right) \otimes \mathcal{F}_{\frac{n}{2^m}}$-measurable for each $n \geqslant 0$.

LEMMA 5.25. *Let $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ be bounded and $\mathcal{P}$-measurable. Then for each $m \geqslant 1$ a.s. on $\mathbb{R}_+ \times \Omega$*

$$\mathbb{E}(f|\mathcal{P}_m)(s) = \mathbb{E}\big(f(s)\big|\mathcal{F}_{\frac{n}{2^m}}\big), \quad s \in \left(\frac{n}{2^m}, \frac{n+1}{2^m}\right], n \geqslant 0. \tag{5.30}$$

Moreover, $\mathbb{E}(f|\mathcal{P}_m) \to f$ a.s. on $\mathbb{R}_+ \times \Omega$ as $m \to \infty$.

Proof. Let us first show (5.30). Fix $m \geqslant 1$. Fix a simple $\mathcal{P}_m$-measurable process $g : \mathbb{R}_+ \times \Omega \to \mathbb{R}$. Then for each $n \geqslant 0$ and $s \in (\frac{n}{2^m}, \frac{n+1}{2^m}]$ a random variable $g(s)$ is $\mathcal{F}_{\frac{n}{2^m}}$-measurable. Define $\widetilde{f} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ by

$$\widetilde{f}(s) = \sum_{n \geqslant 0} \mathbb{E}\big(f(s)\big|\mathcal{F}_{\frac{n}{2^m}}\big)\mathbf{1}_{s \in (\frac{n}{2^m}, \frac{n+1}{2^m}]}, \quad s \geqslant 0.$$

Then for each $n \geqslant 0$ and $s \in (\frac{n}{2^m}, \frac{n+1}{2^m}]$

$$\begin{aligned}\mathbb{E}\Big[(f(s) - \widetilde{f}(s))g(s)\Big] &= \mathbb{E}\Big[\mathbb{E}\Big((f(s) - \widetilde{f}(s))g(s)\Big|\mathcal{F}_{\frac{n}{2^m}}\Big)\Big] \\ &= \mathbb{E}\Big[\mathbb{E}\Big((f(s) - \widetilde{f}(s))\Big|\mathcal{F}_{\frac{n}{2^m}}\Big)g(s)\Big] = 0.\end{aligned}$$

Therefore,

$$\mathbb{E}\int_{\mathbb{R}_+}\Big(f(s)-\widetilde{f}(s)\Big)g(s)\,\mathrm{d}s=\int_{\mathbb{R}_+}\mathbb{E}\Big[\Big(f(s)-\widetilde{f}(s)\Big)g(s)\Big]\,\mathrm{d}s=0,$$

and hence (5.30) holds. Now, note that $(\mathcal{P}_m)_{m\geqslant 1}$ forms a filtration on $\mathbb{R}_+\times\Omega$, and obviously $\sigma\{\cup_m\mathcal{P}_m\}=\mathcal{P}$. Therefore, the second part of the theorem follows from the martingale convergence theorem (see, for example, [**23**, Theorem 7.23]). ☐

The second lemma is the following statement.

LEMMA 5.26. *Let $F:\mathbb{R}_+\times\Omega\to\mathbb{R}_+$ be a non-decreasing continuous predictable function such that $F(t)-F(s)\leqslant C(t-s)$ for all $0\leqslant s\leqslant t$ a.s. and for some fixed constant $C\geqslant 0$ and $F(0)=0$ a.s. Then there exists a predictable $f:\mathbb{R}_+\times\Omega\to[0,C]$ such that $F(T)=\int_0^T f(s)\,\mathrm{d}s$ for each fixed $T\geqslant 0$.*

Proof. F is a.s. differentiable in t because F is Lipschitz, so there exists $f:\mathbb{R}_+\times\Omega\to[0,C]$ such that for every $\omega\in\Omega$ and $t\geqslant 0$

$$f(t,\omega)=\limsup_{\varepsilon\to 0}\frac{F(t,\omega)-F((t-\varepsilon)\vee 0,\omega)}{\varepsilon}\wedge C,$$

where we consider lim sup instead of lim in order to make f well defined on the whole $\mathbb{R}_+\times\Omega$, even though this lim sup coincides with lim a.s. on $\mathbb{R}_+\times\Omega$, and the constant C is a natural upper bound for f due to the Lipschitz assumption on F. Since F is predictable, $t\mapsto F(t)-F((t-\varepsilon)\vee 0)$ is a predictable process as well for each $\varepsilon\geqslant 0$, so the obtained f is predictable. ☐

We are now ready to prove Lemma 5.24.

Proof of Lemma 5.24. Let $f:\mathbb{R}_+\times\Omega\to[0,C]$ be as defined in Lemma 5.26. Then by Lemma 5.25, $\mathbb{E}(f|\mathcal{P}_m)$ exists and converges to f a.s. on $\mathbb{R}_+\times\Omega$. Moreover, f is bounded by C, so $\mathbb{E}(f|\mathcal{P}_m)$ is bounded by C as well. Therefore, for each $m\geqslant 1$ we find using (5.30)

$$\begin{aligned}\sum_{n=0}^{[2^mT]-1}\mathbb{E}\left[F\Big(\frac{n+1}{2^m}\Big)-F\Big(\frac{n}{2^m}\Big)\Big|\mathcal{F}_{\frac{n}{2^m}}\right]&=\sum_{n=0}^{[2^mT]-1}\mathbb{E}\left[\int_{(\frac{n}{2^m},\frac{n+1}{2^m}]}f(s)\,\mathrm{d}s\Big|\mathcal{F}_{\frac{n}{2^m}}\right]\\&\overset{(*)}{=}\sum_{n=0}^{[2^mT]-1}\int_{(\frac{n}{2^m},\frac{n+1}{2^m}]}\mathbb{E}\big(f(s)\big|\mathcal{F}_{\frac{n}{2^m}}\big)\,\mathrm{d}s\\&=\int_{\big(0,\frac{[2^mT]}{2^m}\big]}\mathbb{E}(f|\mathcal{P}_m)(s)\,\mathrm{d}s,\end{aligned}$$

where $(*)$ makes sense since $f|_{(\frac{n}{2^m},\frac{n+1}{2^m}]\times\Omega}$ is $\mathcal{B}((\frac{n}{2^m},\frac{n+1}{2^m}])\otimes\mathcal{F}_{\frac{n+1}{2^m}}$-measurable (because f is predictable and hence adapted) and bounded by C and therefore because of the fact that for a.e. $(s,\omega)\in(\frac{n}{2^m},\frac{n+1}{2^m}]\times\Omega$,

$$\mathbb{E}\big(f(s)\big|\mathcal{F}_{\frac{n}{2^m}}\big)(\omega)=\mathbb{E}\left(f|_{(\frac{n}{2^m},\frac{n+1}{2^m}]\times\Omega}\Big|\mathcal{B}\Big(\Big(\frac{n}{2^m},\frac{n+1}{2^m}\Big]\Big)\otimes\mathcal{F}_{\frac{n}{2^m}}\right)(s,\omega),$$

we have that $\mathbb{E}\left(f(s)|\mathcal{F}_{\frac{n}{2^m}}\right)(\omega)$ is jointly measurable in $(s,\omega)\in(\frac{n}{2^m},\frac{n+1}{2^m}]\times\Omega$, so the integral $\int_{(\frac{n}{2^m},\frac{n+1}{2^m}]}\mathbb{E}\left(f(s)|\mathcal{F}_{\frac{n}{2^m}}\right)\,\mathrm{d}s$ is a.s. well defined and $(*)$ holds by the Fubini theorem. Thus, since

$\frac{[2^m T]}{2^m} \to T$ as $m \to \infty$,

$$\lim_{m\to\infty} \sum_{n=0}^{[2^m T]-1} \mathbb{E}\Big[F\Big(\frac{n+1}{2^m}\Big) - F\Big(\frac{n}{2^m}\Big)\Big|\mathcal{F}_{\frac{n}{2^m}}\Big] = \lim_{m\to\infty} \int_{(0,\frac{[2^m T]}{2^m}]} \mathbb{E}(f|\mathcal{P}_m)(s)\,\mathrm{d}s$$
$$= \int_{(0,T]} f(s)\,\mathrm{d}s = F(T),$$

where the limit holds a.s., and since $F(T) \leqslant CT$ and all the functions above are bounded by CT as well, by the dominated convergence theorem the limit holds in $L^p(\Omega)$ for each $1 < p < \infty$. □

Finally, we will use a time-change argument in the proof of Theorem 5.28. We recall some necessary definitions and results. A non-decreasing, right-continuous family of stopping times $\tau = (\tau_s)_{s\geqslant 0}$ is called a *random time-change*. If $\mathbb{F}$ is right-continuous, then according to [**23**, Lemma 7.3] the same holds true for the *induced filtration* $\mathbb{G} = (\mathcal{G}_s)_{s\geqslant 0} = (\mathcal{F}_{\tau_s})_{s\geqslant 0}$.

For a random time-change $\tau = (\tau_s)_{s\geqslant 0}$ and for a random measure μ, we define $\mu \circ \tau$ in the following way:

$$\mu\circ\tau((s,t]\times B) = \mu((\tau_s,\tau_t]\times A), \quad t \geqslant s \geqslant 0, A \in \mathcal{J},$$

μ is said to be *τ-continuous* if $\mu((\tau_{s-},\tau_s]\times J) = 0$ a.s. for each $s \geqslant 0$, where we let $\tau_{s-} := \lim_{\varepsilon\to 0}\tau_{s-\varepsilon}$, $\tau_{0-} := \tau_0$. Later we will need the following proposition.

Proposition 5.27. *Let $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ be a strictly increasing continuous predictable process such that $A_0 = 0$ and $A_t \to \infty$ as $t \to \infty$ a.s. Then*

$$\tau_s = \{t : A_t = s\}, \quad s \geqslant 0.$$

defines a random time-change $\tau = (\tau_s)_{s\geqslant 0}$. It satisfies

$$(A\circ\tau)(t) = (\tau\circ A)(t) = t$$

a.s. for each $t \geqslant 0$. Let $\mathbb{G} = (\mathcal{G}_s)_{s\geqslant 0} = (\mathcal{F}_{\tau_s})_{s\geqslant 0}$ be the induced filtration. Then $(A_t)_{t\geqslant 0}$ is a random time-change with respect to $\mathbb{G}$. Moreover, for any random measure μ the following hold.

(i) *If μ is $\mathbb{F}$-optional, then $\mu\circ\tau$ is $\mathbb{G}$-optional.*
(ii) *If μ is $\mathbb{F}$-predictable, then $\mu\circ\tau$ is $\mathbb{G}$-predictable.*
(iii) *If μ is an $\mathbb{F}$-optional random measure with a compensator ν, then $\nu\circ\tau$ is a compensator of $\mu\circ\tau$, and for each $\widetilde{\mathcal{P}}$-measurable simple $F : \mathbb{R}_+ \times \Omega \times J \to \mathbb{R}$ such that $\mathbb{E}(F\star\mu)_\infty < \infty$ we have that $\mathbb{E}\left((F\circ\tau)\star(\mu\circ\tau)\right)_\infty < \infty$ and a.s.*

$$\begin{aligned}(F\star\mu)_\infty &= \left((F\circ\tau)\star(\mu\circ\tau)\right)_\infty,\\ (F\star\nu)_\infty &= \left((F\circ\tau)\star(\nu\circ\tau)\right)_\infty,\end{aligned} \tag{5.31}$$

$$(F\star\bar\mu)_{\tau_s} = ((F\circ\tau)\star(\overline{\mu\circ\tau}))_s, \quad s \geqslant 0. \tag{5.32}$$

Proof. First of all note that since A is strictly increasing and continuous a.s., $s \mapsto \tau_s$ is an a.s. continuous function, so any random measure μ is τ-continuous. Therefore, (i) and (ii) follow from [**19**, Theorem 10.27(c,d)]. Let us prove (iii). The fact that $\nu\circ\tau$ is a compensator of $\mu\circ\tau$ holds due to [**19**, Theorem 10.27(e)], while the rest follows from [**19**, Theorem 10.28], and in particular (5.31) follows from the definition of $\mu\circ\tau$ and $\nu\circ\tau$. □

For more information on time-changes for random measures, we refer to [**19**, Chapter X].

Let (S,Σ,ρ) be a measure space. For $1<p,q<\infty$, we define $\hat{\mathcal{S}}_q^p$, $\hat{\mathcal{D}}_{q,q}^p$ and $\hat{\mathcal{D}}_{p,q}^p$ as the Banach spaces of all functions $F:\mathbb{R}_+\times\Omega\times J\to L^q(S)$ that are $\widetilde{\mathcal{P}}$-measurable and for which the corresponding norms are finite:

$$\begin{aligned} \|F\|_{\hat{\mathcal{S}}_q^p} &:= \left(\mathbb{E}\left\|\left(\int_{\mathbb{R}_+\times J}|F|^2\,\mathrm{d}\nu\right)^{\frac12}\right\|_{L^q(S)}^p\right)^{\frac1p},\\ \|F\|_{\hat{\mathcal{D}}_{q,q}^p} &:= \left(\mathbb{E}\left(\int_{\mathbb{R}_+\times J}\|F\|_{L^q(S)}^q\,\mathrm{d}\nu\right)^{\frac pq}\right)^{\frac1p},\\ \|F\|_{\hat{\mathcal{D}}_{p,q}^p} &:= \left(\mathbb{E}\int_{\mathbb{R}_+\times J}\|F\|_{L^q(S)}^p\,\mathrm{d}\nu\right)^{\frac1p}. \end{aligned}\tag{5.33}$$

We show in Appendix B that

$$(\hat{\mathcal{S}}_q^p)^*=\hat{\mathcal{S}}_{q'}^{p'},\quad (\hat{\mathcal{D}}_{q,q}^p)^*=\hat{\mathcal{D}}_{q',q'}^{p'},\quad (\hat{\mathcal{D}}_{p,q}^p)^*=\hat{\mathcal{D}}_{p',q'}^{p'}$$

hold isomorphically with constants depending only on p and q.

THEOREM 5.28. *Fix $1<p,q<\infty$. Let μ be an optional $\widetilde{\mathcal{P}}$-σ-finite random measure on $\mathbb{R}_+\times\mathcal{J}$ and suppose that its compensator ν is non-atomic in time. Then for any simple $\widetilde{\mathcal{P}}$-measurable $F:\mathbb{R}_+\times\Omega\times J\to L^q(S)$ and for any $A\in\widetilde{\mathcal{P}}$ with $\mathbb{E}\mathbf{1}_A\star\mu<\infty$*

$$\left(\mathbb{E}\sup_{0\leqslant s\leqslant t}\|((F\mathbf{1}_A)\star\bar{\mu})_s\|_{L^q(S)}^p\right)^{\frac1p}\eqsim_{p,q}\|F\mathbf{1}_A\mathbf{1}_{[0,t]}\|_{\mathcal{I}_{p,q}},\tag{5.34}$$

where $\mathcal{I}_{p,q}$ is given by

$$\begin{aligned} &\hat{\mathcal{S}}_q^p\cap\hat{\mathcal{D}}_{q,q}^p\cap\hat{\mathcal{D}}_{p,q}^p && \text{if } 2\leqslant q\leqslant p<\infty,\\ &\hat{\mathcal{S}}_q^p\cap(\hat{\mathcal{D}}_{q,q}^p+\hat{\mathcal{D}}_{p,q}^p) && \text{if } 2\leqslant p\leqslant q<\infty,\\ &(\hat{\mathcal{S}}_q^p\cap\hat{\mathcal{D}}_{q,q}^p)+\hat{\mathcal{D}}_{p,q}^p && \text{if } 1<p<2\leqslant q<\infty,\\ &(\hat{\mathcal{S}}_q^p+\hat{\mathcal{D}}_{q,q}^p)\cap\hat{\mathcal{D}}_{p,q}^p && \text{if } 1<q<2\leqslant p<\infty,\\ &\hat{\mathcal{S}}_q^p+(\hat{\mathcal{D}}_{q,q}^p\cap\hat{\mathcal{D}}_{p,q}^p) && \text{if } 1<q\leqslant p\leqslant 2,\\ &\hat{\mathcal{S}}_q^p+\hat{\mathcal{D}}_{q,q}^p+\hat{\mathcal{D}}_{p,q}^p && \text{if } 1<p\leqslant q\leqslant 2. \end{aligned}\tag{5.35}$$

Proof. By Lemma 5.16, we can assume without loss of generality that $F:=F\mathbf{1}_A$, $\mu:=\mu\mathbf{1}_A$ and that there exists a $T\geqslant 0$ such that $F(t)=0$ for each $t\geqslant T$. Since F is simple, it is uniformly bounded on $\mathbb{R}_+\times\Omega\times J$ and, due to the fact that $\mathbb{E}\mathbf{1}_A\star\mu=\mathbb{E}\mu(\mathbb{R}_+\times\Omega)<\infty$, we find $\mathbb{E}\|F\star\mu\|<\infty$. Consequently, $F\star\bar{\mu}$ exists and it is a local martingale. Therefore, Doob's maximal inequality implies

$$(\mathbb{E}\|(F\star\bar{\mu})_t\|^p)^{\frac1p}\eqsim_p\left(\mathbb{E}\sup_{0\leqslant s\leqslant t}\|(F\star\bar{\mu})_s\|_{L^q(S)}^p\right)^{\frac1p}$$

and so it is enough to show that

$$(\mathbb{E}\|(F\star\bar{\mu})_t\|^p)^{\frac1p}\eqsim_{p,q}\|F\mathbf{1}_{[0,t]}\|_{\mathcal{I}_{p,q}}.\tag{5.36}$$

The proof consists of two steps. In the first step, we assume that ν is absolutely continuous with respect to Lebesgue measure. In this case, we can derive the upper bounds in (5.36) from the Burkholder–Rosenthal inequalities, Proposition 5.22 and Lemma 5.24. The lower bounds then follow by a duality argument based on Lemmas 2.1 and 5.19 combined with the duality for the spaces $\mathcal{I}_{p,q}$ derived in the Appendix (see Corollary B.8). In the second step, we deduce the general result via a time-change argument based on Proposition 5.27.

Step 1: $\nu((s,t]\times J)\leqslant (t-s)$ *for each* $t\geqslant s\geqslant 0$ *a.s.* We will consider the cases $2\leqslant q\leqslant p<\infty$ and $1<p\leqslant q\leqslant 2$, the proofs in the other cases are similar.

Case $2\leqslant q\leqslant p<\infty$: Fix $m\geqslant 1$. Let $F_n := F\mathbf{1}_{(\frac{n}{2^m},\frac{n+1}{2^m}]}$ for each $n\geqslant 0$. Then

$$(d_n)_{n\geqslant 0} := ((F_n\star\bar{\mu})_\infty)_{n\geqslant 0}$$

is an $L^q(S)$-valued martingale difference sequence with respect to a filtration $\left(\mathcal{F}_{\frac{n+1}{2^m}}\right)_{n\geqslant 0}$. Theorem 1.1 implies

$$\mathbb{E}\|(F\star\bar{\mu})_\infty\|^p_{L^q(S)} = \mathbb{E}\Big\|\sum_{n\geqslant 0}(F_n\star\bar{\mu})_\infty\Big\|^p_{L^q(S)} = \mathbb{E}\Big\|\sum_{n\geqslant 0}d_n\Big\|^p_{L^q(S)} \eqsim_{p,q} \|(d_n)\|^p_{s_{p,q}}$$
$$\eqsim_p (\|(d_n)\|_{S^p_q} + \|(d_n)\|_{D^p_{q,q}} + \|(d_n)\|_{D^p_{p,q}})^p.$$

To bound $\|(d_n)\|_{S^p_q}$, observe that

$$\begin{aligned}\|(d_n)\|_{S^p_q} &= \left(\mathbb{E}\left\|\left(\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}}|d_n|^2\right)^{\frac12}\right\|^p_{L^q(S)}\right)^{\frac1p} = \left(\mathbb{E}\left\|\left(\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}}|(F_n\star\bar{\mu})_\infty|^2\right)^{\frac12}\right\|^p_{L^q(S)}\right)^{\frac1p}\\ &\overset{(*)}{\eqsim}_p \left(\mathbb{E}\left\|\left(\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}}(|F_n|^2\star\nu)_\infty\right)^{\frac12}\right\|^p_{L^q(S)}\right)^{\frac1p} \qquad (5.37)\\ &= \left(\mathbb{E}\left\|\left(\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}}\Big((|F|^2\star\nu)_{\frac{n+1}{2^m}} - (|F|^2\star\nu)_{\frac{n}{2^m}}\Big)\right)^{\frac12}\right\|^p_{L^q(S)}\right)^{\frac1p},\end{aligned}$$

where $(*)$ holds by Proposition 5.22 and the fact that

$$\nu\left(\left(\frac{n}{2^m},\frac{n+1}{2^m}\right]\times J\right)\leqslant \frac{n+1}{2^m}-\frac{n}{2^m}=\frac{1}{2^m}\leqslant 1.$$

Note that for a.e. $\omega\in\Omega$, all $s\in S$, and each $t\geqslant u\geqslant 0$

$$\begin{aligned}(|F|^2\star\nu)_t(s,\omega)-(|F|^2\star\nu)_u(s,\omega) &\leqslant \sup|F(s)|^2(\nu((u,t]\times J)(\omega))\\ &\leqslant \sup|F(s)|^2(t-u),\end{aligned}$$

so by Lemma 5.24 and the fact that ν is a.s. non-atomic in time

$$\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}}\Big((|F|^2\star\nu)_{\frac{n+1}{2^m}} - (|F|^2\star\nu)_{\frac{n}{2^m}}\Big)\to(|F|^2\star\nu)_T = (|F|^2\star\nu)_\infty$$

a.s. as $m \to \infty$. Therefore, thanks to (5.37),

$$\|(d_n)\|_{S_q^p} \eqsim \left(\mathbb{E}\left\|\left(\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}}\left((|F|^2 \star \nu)_{\frac{n+1}{2^m}} - (|F|^2 \star \nu)_{\frac{n}{2^m}}\right)\right)^{\frac{1}{2}}\right\|_{L^q(S)}^p\right)^{\frac{1}{p}} \tag{5.38}$$
$$\overset{m\to\infty}{\longrightarrow} \left(\mathbb{E}\left\|\left((|F|^2 \star \nu)_\infty\right)^{\frac{1}{2}}\right\|_{L^q(S)}^p\right)^{\frac{1}{p}} = \|F\|_{\mathcal{S}_q^p}.$$

Now, let us estimate $\|(d_n)\|_{D_{q,q}^p}$. Analogously to (5.37)

$$\|(d_n)\|_{D_{q,q}^p} = \left(\mathbb{E}\left(\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}} \|d_n\|_{L^q(S)}^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}$$
$$= \left(\mathbb{E}\left(\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}} \|(F_n \star \bar{\mu})_\infty\|_{L^q(S)}^q\right)^{\frac{p}{q}}\right)^{\frac{1}{p}}$$
$$\eqsim \left(\mathbb{E}\left(\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}} (\|F_n\|_{L^q(S)}^q \star \nu)_\infty\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \tag{5.39}$$
$$= \left(\mathbb{E}\left(\sum_n \mathbb{E}_{\mathcal{F}_{\frac{n}{2^m}}}\left((\|F\|_{L^q(S)}^q \star \nu)_{\frac{n+1}{2^m}} - (\|F\|_{L^q(S)}^q \star \nu)_{\frac{n}{2^m}}\right)\right)^{\frac{p}{q}}\right)^{\frac{1}{p}},$$

and similarly to (5.38) the last expression converges to $\|F\|_{\mathcal{D}_{q,q}^p}$. The same can be shown for $\mathcal{D}_{p,q}^p$.

Case $1 < p \leqslant q \leqslant 2$: Let $\mathcal{I}_{\mathrm{elem}}(\widetilde{\mathcal{P}})$ denote the linear space of all simple $\widetilde{\mathcal{P}}$-measurable $L^q(S)$-valued functions. This linear space is dense in $\hat{\mathcal{S}}_q^p$, $\hat{\mathcal{D}}_{p,q}^p$ and $\hat{\mathcal{D}}_{q,q}^p$. Let $F \in \mathcal{I}_{\mathrm{elem}}(\widetilde{\mathcal{P}})$. Fix a decomposition $F = F_1 + F_2 + F_3$ with $F_\alpha \in \mathcal{I}_{\mathrm{elem}}(\widetilde{\mathcal{P}})$.

Fix $m \geqslant 1$ and set $F_{n,\alpha} = F_\alpha \mathbf{1}_{(\frac{n}{2^m}, \frac{n+1}{2^m}]}$, $d_{n,\alpha} = F_{n,\alpha} \star \bar{\mu}$, $\alpha = 1, 2, 3$, so that

$$(F \star \bar{\mu})_T = (F \star \bar{\mu})_\infty = \sum_n d_{n,1} + d_{n,2} + d_{n,3}.$$

Then by Theorem 1.1, (5.37), (5.38) and (5.39) we conclude that

$$\left(\mathbb{E}\|(F \star \bar{\mu})_\infty\|_{L^q(S)}^p\right)^{\frac{1}{p}} \lesssim_{p,q} \|F_1\|_{\mathcal{S}_q^p} + \|F_2\|_{\mathcal{D}_{p,q}^p} + \|F_3\|_{\mathcal{D}_{q,q}^p}.$$

Since $\mathcal{I}_{\mathrm{elem}}(\widetilde{\mathcal{P}})$ is dense in $\hat{\mathcal{S}}_q^p$, $\hat{\mathcal{D}}_{p,q}^p$ and $\hat{\mathcal{D}}_{q,q}^p$, we conclude by taking the infimum over F_1, F_2, F_3 as above that

$$\left(\mathbb{E}\|(F \star \bar{\mu})_\infty\|_{L^q(S)}^p\right)^{\frac{1}{p}} \lesssim_{p,q} \|F\|_{\mathcal{I}_{p,q}}.$$

The duality argument: Fix $t < \infty$, $1 < p, q < \infty$. Using the upper bounds in (5.36), we can obtain the stochastic integral $(F \star \bar{\mu})_t$ as an L^p-limit of the integrals of the corresponding simple approximations of F in $\mathcal{I}_{p,q}$. Let Y be the closure of the linear subspace $\cup_{F \in \mathcal{I}_{p,q}} (F \star \bar{\mu})_\infty$ in $L^p(\Omega; L^q(S))$ and let $X = \mathcal{I}_{p,q}$. By Corollary B.8, $X^* = \mathcal{I}_{p',q'}$. Let U (respectively, V) be the dense subspace of X (respectively, X^*) consisting of all $\widetilde{\mathcal{P}}$-measurable simple $L^q(S)$-valued (respectively, $L^{q'}(S)$-valued) functions. Define $j_0 : U \to Y$ by $F \mapsto (F \star \bar{\mu})_\infty$. Define $k_0 : V \to Y^*$ in the following way: For every $F^* \in V$, we define $k_0 F^*$ to be such that

$$\langle k_0 F^*, y\rangle := \mathbb{E}\langle (F^* \star \bar{\mu})_\infty, y\rangle, \quad y \in Y.$$

In this case, $k_0 F^*$ is indeed in Y^* since $(F^* \star \bar{\mu})_\infty \in L^{p'}(\Omega; L^{q'}(S))$. By the upper bounds in (5.36), j_0 and k_0 are bounded. Moreover, by the definition of Y, ran j_0 is dense in Y. Finally, by Lemma 5.19 $\langle F^*, F\rangle = \langle k_0 F^*, j_0 F\rangle$ for all $F \in U$ and $F^* \in V$. Now, the lower bounds in (5.34) follow from Lemma 2.1.

Step 2: *general case.* Recall that, due to our assumptions in the beginning of the proof, $\mathbb{E}\mu(\mathbb{R}_+ \times \Omega) = \mathbb{E}\nu(\mathbb{R}_+ \times \Omega) < \infty$. Since ν is non-atomic in time, we can define a continuous strictly increasing predictable process $A : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ by

$$A_t = \nu([0, t] \times J) + t, \quad t \geqslant 0.$$

Let $\tau = (\tau_s)_{s\geqslant 0}$ be the time-change defined in Proposition 5.27. Then according to Proposition 5.27, the random measure $\mu_\tau := \mu \circ \tau$ is $\mathbb{G}$-optional, where $\mathbb{G} := (\mathcal{G}_s)_{s\geqslant 0} = (\mathcal{F}_{\tau_s})_{s\geqslant 0}$. Moreover, $\nu_\tau := \nu \circ \tau$ is $\mathbb{G}$-predictable and a compensator of μ_τ. Let $G := F \circ \tau$. Note that for each $t \geqslant s \geqslant 0$ a.s.

$$\begin{aligned}\nu_\tau((s, t] \times J) &= \nu((\tau_s, \tau_t] \times J) = \nu((0, \tau_t] \times J) - \nu((0, \tau_s] \times J)\\ &\leqslant \nu((0, \tau_t] \times J) - \nu((0, \tau_s] \times J) + (\tau_t - \tau_s)\\ &= (\nu((0, \tau_t] \times J) + \tau_t) - (\nu((0, \tau_s] \times J) + \tau_s) = t - s.\end{aligned} \tag{5.40}$$

Let $\mathcal{I}^\tau_{p,q}$ be defined as $\mathcal{I}_{p,q}$ but for the random measure ν_τ. Due to (5.40), Step 1 yields $\mathbb{E}\|(G \star \bar{\mu}_\tau)_\infty\|^p_{L^q(S)} \eqsim_{p,q} \|G\|^p_{\mathcal{I}^\tau_{p,q}}$. By (5.32),

$$\mathbb{E}\|(G \star \bar{\mu}_\tau)_\infty\|^p_{L^q(S)} = \mathbb{E}\|(F \star \bar{\mu})_\infty\|^p_{L^q(S)}.$$

Moreover, for given F_i and $G_i = F_i \circ \tau$, $i = 1, 2, 3$, it follows from (5.31) that

$$\mathbb{E}\left\|\left(\int_{\mathbb{R}_+\times J} |G_1|^2 \,\mathrm{d}\nu_\tau\right)^{\frac12}\right\|^p_{L^q(S)} = \mathbb{E}\left\|\left(\int_{\mathbb{R}_+\times J} |F_1|^2 \,\mathrm{d}\nu\right)^{\frac12}\right\|^p_{L^q(S)} = \|F_1\|^p_{\mathcal{S}^q_p},$$

$$\mathbb{E}\left(\int_{\mathbb{R}_+\times J} \|G_2\|^q_{L^q(S)} \,\mathrm{d}\nu_\tau\right)^{\frac{p}{q}} = \mathbb{E}\left(\int_{\mathbb{R}_+\times J} \|F_2\|^q_{L^q(S)} \,\mathrm{d}\nu\right)^{\frac{p}{q}} = \|F_2\|^p_{\mathcal{D}^p_{q,q}},$$

$$\mathbb{E}\int_{\mathbb{R}_+\times J} \|G_3\|^p_{L^q(S)} \,\mathrm{d}\nu_\tau = \mathbb{E}\int_{\mathbb{R}_+\times J} \|F_3\|^p_{L^q(S)} \,\mathrm{d}\nu = \|F_3\|^p_{\mathcal{D}^p_{p,q}}.$$

Consequently, $\|G\|_{\mathcal{I}^\tau_{p,q}} = \|F\|_{\mathcal{I}_{p,q}}$. We conclude that

$$\mathbb{E}\|(F \star \bar{\mu})_\infty\|^p_{L^q(S)} \eqsim_{p,q} \|F\|^p_{\mathcal{I}_{p,q}}.$$

□

REMARK 5.29. Let us compare our result to the literature. The upper bounds in Theorem 5.28 were discovered in the scalar-valued case by Novikov in [**40**, Theorem 1]. By exploiting an orthonormal basis, one can easily extend this result to the Hilbert-space valued integrands, see [**33**, Section 3.3] for details. The paper [**33**] contains several other proofs of the Hilbert-space valued version of Novikov's inequality. In the context of Poisson random measures, Theorem 5.28 was obtained in [**8**]. Some one-sided extensions of the latter result in the context of general Banach spaces were obtained in [**9**]. However, these bounds, which are based on the martingale type and cotype of the space, are only matching in the Hilbert-space case and not optimal in general (in particular for L^q-spaces). A very different proof of the upper bounds in Theorem 5.28, which exploits tools from stochastic analysis, was discovered independently of our work by Marinelli in [29].

As a corollary, we obtain the following sharp bounds for stochastic integrals.

THEOREM 5.30. *Fix $1 < p, q < \infty$. Let H be a Hilbert space, (S, Σ, ρ) be a measure space and let $M : \mathbb{R}_+ \times \Omega \to H$ be a purely discontinuous quasi-left continuous local martingale. Let $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, L^q(S))$ be elementary predictable. Then*

$$\Big(\mathbb{E} \sup_{0\leqslant s\leqslant t} \|(\Phi \cdot M)_s\|^p_{L^q(S)}\Big)^{\frac{1}{p}} \eqsim_{p,q} \|\Phi_H \mathbf{1}_{[0,t]}\|_{\mathcal{I}_{p,q}}, \tag{5.41}$$

where $\Phi_H : \mathbb{R}_+ \times \Omega \times H \to L^q(S)$ is defined by

$$\Phi_H(t, \omega, h) := \Phi(t, \omega)h, \quad t \geqslant 0, \omega \in \Omega, h \in H,$$

and $\mathcal{I}_{p,q}$ is given as in (5.35) for $\nu = \nu^M$.

Proof. The result follows from Doob's maximal inequality, Lemma 5.18, Theorem 5.28, and the fact that $\|\Phi_H \mathbf{1}_{A_n}\|_{\mathcal{I}_{p,q}} \nearrow \|\Phi_H\|_{\mathcal{I}_{p,q}}$ as $n \to \infty$ by the monotone convergence theorem. □

5.5. *Integration with respect to continuous martingales*

Finally, let us recall the known sharp bounds for L^q-valued stochastic integrals with respect to continuous local martingales. These bounds are a special case of the main result in [48]. To formulate these, we will need γ-radonifying operators. Let $(\gamma'_n)_{n\geqslant 1}$ be a sequence of independent standard Gaussian random variables on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ (we reserve the notation $(\Omega, \mathcal{F}, \mathbb{P})$ for the probability space on which our processes live) and let H be a separable Hilbert space. A bounded operator $R \in \mathcal{L}(H, X)$ is said to be *γ-radonifying* if for some (and then for each) orthonormal basis $(h_n)_{n\geqslant 1}$ of H the Gaussian series $\sum_{n\geqslant 1} \gamma'_n R h_n$ converges in $L^2(\Omega'; X)$. We then define

$$\|R\|_{\gamma(H,X)} := \Bigg(\mathbb{E}'\Bigg\|\sum_{n\geqslant 1} \gamma'_n R h_n\Bigg\|^2_X\Bigg)^{\frac{1}{2}}.$$

This number does not depend on the sequence $(\gamma'_n)_{n\geqslant 1}$ and the basis $(h_n)_{n\geqslant 1}$, and defines a norm on the space $\gamma(H, X)$ of all γ-radonifying operators from H into X. Endowed with this norm, $\gamma(H, X)$ is a Banach space, which is separable if X is separable. Moreover, if $X = L^q(S)$, $1 < q < \infty$, for some separable measure space (S, Σ, ρ), then thanks to the Trace Duality that is presented, for example, in [18], we have that

$$(\gamma(H, X))^* \simeq \gamma(H^*, X^*). \tag{5.42}$$

We refer to [18] and the references therein for further details on γ-radonifying operators.

For $F : \mathbb{R}_+ \to \mathbb{R}_+$ non-decreasing, we define a measure ρ_F on $\mathcal{B}(\mathbb{R}_+)$ by

$$\rho_F((s, t]) = F(t) - F(s), \quad 0 \leqslant s < t < \infty.$$

If X is a Banach space and $1 \leqslant p \leqslant \infty$, then we write $L^p(\mathbb{R}_+, F; X)$ for the Banach space $L^p(\mathbb{R}_+, \rho_F; X)$.

Let $M : \mathbb{R}_+ \times \Omega \to H$ be a continuous local martingale. Then, thanks to [36, Chapter 14.3], one can define a continuous predictable process $[M] : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and a strongly progressively measurable $q_M : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H)$ (that is, a process q_M such that $q_M h$ is progressively measurable for any $h \in H$) such that $[M]$ is a quadratic variation of M and $\int_0^{\cdot} \langle q_M(s)h, h\rangle \,\mathrm{d}[M]_s$

is a quadratic variation of $[\langle M,h\rangle]$ for each $h\in H$. The following theorem immediately follows from [48, Theorem 4.1] and formula [48, (3.9)].

THEOREM 5.31. *Let H be a Hilbert space, $1<p,q<\infty$. Let $M:\mathbb{R}_+\times\Omega\to H$ be a continuous local martingale, $\Phi:\mathbb{R}_+\times\Omega\to\mathcal{L}(H,L^q(S))$ be elementary predictable. Then*

$$\mathbb{E}\Big(\sup_{0\leqslant s\leqslant t}\|(\Phi\cdot M)_s\|^p\Big)\eqsim_{p,q}\mathbb{E}\|\Phi q_M^{\frac{1}{2}}\mathbf{1}_{[0,t]}\|^p_{\gamma(L^2(\mathbb{R}_+,[M];H),L^q(S))}.$$

5.6. *Integration with respect to general local martingales*

We can now combine the sharp estimates obtained for the three special types of stochastic integrals to obtain sharp estimates for $\Phi\cdot M$, where M is an arbitrary local martingale.

THEOREM 5.32. *Let H be a Hilbert space, $1<p,q<\infty$. Let $M:\mathbb{R}_+\times\Omega\to H$ be a local martingale, $M^c,M^q,M^d:\mathbb{R}_+\times\Omega\to H$ be local martingales such that $M_0^c=M_0^q=0$, M^c is continuous, M^q is purely discontinuous quasi-left continuous, M^a is purely discontinuous with accessible jumps, $M=M^c+M^q+M^a$. Let $\Phi:\mathbb{R}_+\times\Omega\to\mathcal{L}(H,L^q(S))$ be elementary predictable. Then,*

$$\Big(\mathbb{E}\sup_{0\leqslant s\leqslant t}\|(\Phi\cdot M)_s\|^p_{L^q(S)}\Big)^{\frac{1}{p}}\eqsim_{p,q}\Big(\mathbb{E}\|\Phi q_{M^c}^{\frac{1}{2}}\mathbf{1}_{[0,t]}\|^p_{\gamma(L^2(\mathbb{R}_+,[M^c];H),L^q(S))}\Big)^{\frac{1}{p}} + \|\Phi_H\mathbf{1}_{[0,t]}\|_{\mathcal{I}_{p,q}}+\|(\Phi\mathbf{1}_{[0,t]})\cdot M^a\|_{\mathcal{A}_{p,q}}, \tag{5.43}$$

where $\Phi_H:\mathbb{R}_+\times\Omega\times H\to L^q(S)$ is defined by

$$\Phi_H(t,\omega,h):=\Phi(t,\omega)h,\quad t\geqslant 0,\omega\in\Omega,h\in H,$$

$\mathcal{I}_{p,q}$ is given as in (5.35) for $\nu=\nu^{M^q}$, and $\mathcal{A}_{p,q}$ is as defined in (5.8).

Observe that the upper bound in (5.43) is immediate from the triangle inequality and our estimates for the stochastic integrals $(\Phi\cdot M^c)_t$, $(\Phi\cdot M^q)_t$, and $(\Phi\cdot M^a)_t$. The lower bound, however, requires some care, since it is *a priori* not even clear if the latter three integrals are in $L^p(\Omega;L^q(S))$ if $(\Phi\cdot M)_t\in L^p(\Omega;L^q(S))$. We will derive this using the following two lemmas.

LEMMA 5.33. *Let H be a Hilbert space, X be a finite-dimensional space, $M:\mathbb{R}_+\times\Omega\to H$ be a local martingale, $\Phi:\mathbb{R}_+\times\Omega\to\mathcal{L}(H,X)$ be elementary predictable and $F:\mathbb{R}_+\times\Omega\times H\to X$ be elementary $\widetilde{\mathcal{P}}$-measurable. Then*

(i) *if M is continuous, then $\Phi\cdot M$ is continuous,*
(ii) *if M is purely discontinuous quasi-left continuous, then $F\star\bar{\mu}^M$ is purely discontinuous quasi-left continuous,*
(iii) *if M is purely discontinuous with accessible jumps, then $\Phi\cdot M$ is purely discontinuous with accessible jumps.*

Proof. (i) holds since if M is continuous, then the formula (5.4) defines an a.s. continuous process.

To prove pure discontinuity in (ii), one has to endow X with a Euclidean norm and note that if M is purely discontinuous quasi-left continuous, then by [20, Proposition II.1.28] $[F\star\bar{\mu}^M]_t=\sum_{0\leqslant s\leqslant t}\|F(\Delta M)\|^2$ a.s. for all $t\geqslant 0$ since $F\star\nu^M$ is absolutely continuous, so it does not effect on the quadratic variation. Therefore, $[F\star\bar{\mu}^M]$ is purely discontinuous, and so $F\star\bar{\mu}^M$ is purely discontinuous by [23, Theorem 26.14]. Quasi-left continuity then follows as $\Delta(F\star\bar{\mu}^M)_\tau=F(\Delta M_\tau)=0$ a.s. for any predictable stopping time τ.

Pure discontinuity of $\Phi\cdot M$ in (iii) follows from the same argument as in (ii), and the rest can be proven using the fact that a.s.

$$\{t\in\mathbb{R}_+ : \Delta(\Phi\cdot M)_t\neq 0\}\subset\{t\in\mathbb{R}_+ : \Delta M_t\neq 0\}. \qquad\square$$

LEMMA 5.34. *Let X be a finite-dimensional Banach space, $1<p<\infty$, $M:\mathbb{R}_+\times\Omega\to X$ be a local martingale. Then there exist local martingales $M^c, M^q, M^a:\mathbb{R}_+\times\Omega\to X$ such that $M^c_0=M^q_0=0$, M^c is continuous, M^q is purely discontinuous quasi-left continuous, M^a is purely discontinuous with accessible jumps, $M=M^c+M^a+M^q$ and then for each $t\geqslant 0$*

$$\mathbb{E}\|M_t\|^p \eqsim_{p,X} \mathbb{E}\|M^c_t\|^p+\mathbb{E}\|M^q_t\|^p+\mathbb{E}\|M^a_t\|^p. \tag{5.44}$$

In other words, M is an L^p-martingale if and only if each of M^c, M^q and M^a is an L^p-martingale.

Proof. Since X is finite dimensional, we can endow it with a Euclidean norm $|||\cdot|||$. Then the existence of the decomposition $M=M^c+M^a+M^q$ follows from Lemma 5.2, and for each $t\geqslant 0$ due to the Burkholder–Davis–Gundy inequality and Lemma 5.2

$$\begin{aligned}\mathbb{E}\|M_t\|^p &\eqsim_{p,X} \mathbb{E}|||M_t|||^p \eqsim_p \mathbb{E}[M]_t^{\frac{p}{2}} = \mathbb{E}([M^c]_t+[M^q]_t+[M^a]_t)^{\frac{p}{2}}\\ &\eqsim_p \mathbb{E}[M^c]_t^{\frac{p}{2}}+\mathbb{E}[M^q]_t^{\frac{p}{2}}+\mathbb{E}[M^a]_t^{\frac{p}{2}}\\ &\eqsim_p \mathbb{E}|||M^c_t|||^p+\mathbb{E}|||M^q_t|||^p+\mathbb{E}|||M^a_t|||^p\\ &\eqsim_{p,X} \mathbb{E}\|M^c_t\|^p+\mathbb{E}\|M^q_t\|^p+\mathbb{E}\|M^a_t\|^p. \qquad\square\end{aligned}$$

Returning to the setting of Theorem 5.32, recall that Φ takes values in a finite-dimensional subspace of $L^q(S)$ (since Φ is elementary). Lemmas 5.33 and 5.34 show together with Lemma 5.18 that $(\Phi\cdot M)_t\in L^p(\Omega;L^q(S))$ if and only if $(\Phi\cdot M^c)_t$, $(\Phi\cdot M^q)_t$, and $(\Phi\cdot M^a)_t$ are in $L^p(\Omega;L^q(S))$ and

$$\mathbb{E}\|(\Phi\cdot M)_t\|^p_{L^q} \eqsim_{p,q,\Phi} \mathbb{E}\|(\Phi\cdot M^c)_t\|^p_{L^q}+\mathbb{E}\|(\Phi\cdot M^q)_t\|^p_{L^q}+\mathbb{E}\|(\Phi\cdot M^a)_t\|^p_{L^q}. \tag{5.45}$$

Unfortunately, this still does not yield the lower bounds in Theorem 5.32. Indeed, since the implicit constants in (5.44) depend on the dimension of X, the constants in (5.45) depend on Φ. As it turns out, this is a proof artefact: one can show that the statement of Lemma 5.34 remains valid for a general UMD Banach space [**52**]. Rather than relying on the latter result, we prefer to complete the proof of Theorem 5.32 in an elementary manner. We will directly prove the lower bounds in (5.43) by combining (5.45) with a duality argument. The key for this duality argument is the following technical observation.

LEMMA 5.35. *Let H be a Hilbert space, X be a Banach space, $M^c, M^q:\mathbb{R}_+\times\Omega\to H$ be continuous and purely discontinuous quasi-left continuous martingales, $M^{a,1}:\mathbb{R}_+\times\Omega\to X$, $M^{a,2}:\mathbb{R}_+\times\Omega\to X^*$ be purely discontinuous martingales with accessible jumps, $\Phi_1:\mathbb{R}_+\times\Omega\to\mathcal{L}(H,X)$, $\Phi_2:\mathbb{R}_+\times\Omega\to\mathcal{L}(H,X^*)$ be elementary predictable, $F_1:\mathbb{R}_+\times\Omega\times H\to X$, $F_2:\mathbb{R}_+\times\Omega\times H\to X^*$ be elementary $\widetilde{\mathcal{P}}$-measurable. Assume $(\Phi_1\cdot M^c)_\infty$, $(F_1\star\bar\mu^{M^q})_\infty$, $M^{a,1}_\infty\in L^p(\Omega;X)$ and $(\Phi_2\cdot M^c)_\infty$, $(F_2\star\bar\mu^{M^q})_\infty$, $M^{a,2}_\infty\in L^{p'}(\Omega;X^*)$ for some $1<p<\infty$. Then, for all $t\geqslant 0$,*

$$\begin{aligned}&\mathbb{E}\langle(\Phi_1\cdot M^c+F_1\star\bar\mu^{M^q}+M^{a,1})_t,(\Phi_2\cdot M^c+F_2\star\bar\mu^{M^q}+M^{a,2})_t\rangle\\ &\quad=\mathbb{E}\langle(\Phi_1\cdot M^c)_t,(\Phi_2\cdot M^c)_t\rangle+\mathbb{E}\langle(F_1\star\bar\mu^{M^q})_t,(F_2\star\bar\mu^{M^q})_t\rangle+\mathbb{E}\langle M^{a,1}_t,M^{a,2}_t\rangle.\end{aligned} \tag{5.46}$$

To prove Lemma 5.35, we use the following statement.

LEMMA 5.36. *Let X be a Banach space, $X_0 \subset X$ be a finite-dimensional subspace, $1 < p < \infty$, $M^q : \mathbb{R}_+ \times \Omega \to X_0$ be a purely discontinuous quasi-left continuous L^p-martingale, $M^q_0 = 0$, $M^a : \mathbb{R}_+ \times \Omega \to X^*$ be a purely discontinuous $L^{p'}$-martingale with accessible jumps. Then $\mathbb{E}\langle M^q_t, M^a_t\rangle = 0$ for each $t \geqslant 0$.*

Proof. Let d be the dimension of X_0, $x_1, \dots, x_d$ be a basis of X_0. Then there exist purely discontinuous quasi-left continuous L^p-martingales $M^{q,1}, \dots, M^{q,d} : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ such that $M^q = M^{q,1}x_1 + \cdots + M^{q,d}x_d$. Thus, for any $i = 1, \dots, d$ and any purely discontinuous $L^{p'}$-martingale $N : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ with accessible jumps, $[M^{q,i}, N] = 0$ a.s. by [**23**, Corollary 26.16]. Hence, [**20**, Proposition I.4.50(a)] implies that $M^{q,i}N$ is a local martingale, and due to integrability it is a martingale. Note also that all $M^{q,i}$ start at zero, therefore

$$\mathbb{E}\langle M^q_t, M^a_t\rangle = \sum_{i=1}^d \mathbb{E}M^{q,i}_t\langle x_i, M^a_t\rangle = \sum_{i=1}^d \mathbb{E}M^{q,i}_0\langle x_i, M^a_0\rangle = 0. \qquad \square$$

Proof of Lemma 5.35. Since all the integrands Φ_1, Φ_2, F_1, F_2 are elementary, one can suppose that X and X^* are finite dimensional, so we can endow these spaces with Euclidean norms. Since by Lemma 5.33, $\Phi_1 \cdot M^c$ and $\Phi_2 \cdot M^c$ are continuous, $F_1 \star \bar{\mu}^{M^q}$, $F_1 \star \bar{\mu}^{M^q}$, $M^{a,1}$ and $M^{a,2}$ are purely discontinuous, [**20**, Definition I.4.11] implies that for each $t \geqslant 0$

$$\mathbb{E}\langle(\Phi_1 \cdot M^c)_t, (F_2 \star \bar{\mu}^{M^q})_t\rangle = \mathbb{E}[\Phi_1 \cdot M^c, F_2 \star \bar{\mu}^{M^q}]_t = 0,$$

$$\mathbb{E}\langle(\Phi_2 \cdot M^c)_t, (F_1 \star \bar{\mu}^{M^q})_t\rangle = \mathbb{E}[\Phi_2 \cdot M^c, F_1 \star \bar{\mu}^{M^q}]_t = 0,$$

$$\mathbb{E}\langle(\Phi_1 \cdot M^c)_t, M^{a,2}_t\rangle = \mathbb{E}[\Phi_1 \cdot M^c, M^{a,2}]_t = 0,$$

$$\mathbb{E}\langle(\Phi_2 \cdot M^c)_t, M^{a,1}_t\rangle = \mathbb{E}[\Phi_2 \cdot M^c, M^{a,1}]_t = 0.$$

Moreover, thanks to Lemma 5.33 and Lemma 5.36,

$$\mathbb{E}\langle M^{a,1}_t, (F_2 \star \bar{\mu}^{M^q})_t\rangle = \mathbb{E}\langle M^{a,2}_t, (F_1 \star \bar{\mu}^{M^q})_t\rangle = 0,$$

so (5.46) easily follows. $\square$

We can now combine Theorems 5.31, 5.28 and 5.14 with a duality argument to derive the following.

PROPOSITION 5.37. *Let H be a Hilbert space, $1 < p, q < \infty$. Let M^c, $M^q : \mathbb{R}_+ \times \Omega \to H$ be continuous and purely discontinuous quasi-left continuous local martingales, $M^a : \mathbb{R}_+ \times \Omega \to L^q(S)$ be a purely discontinuous L^p-martingale with accessible jumps, $\Phi : \mathbb{R}_+ \times \Omega \to \mathcal{L}(H, L^q(S))$ be elementary predictable, $F : \mathbb{R}_+ \times \Omega \times H \to L^q(S)$ be elementary $\widetilde{\mathcal{P}}$-measurable. If $\Phi \cdot M^c$ and $F \star \bar{\mu}^{M^q}$ are L^p-martingales, then*

$$\begin{aligned}\Big(\mathbb{E}\Big\|\Big(\Phi \cdot M^c + F \star \bar{\mu}^{M^q} + M^a\Big)_\infty\Big\|^p_{L^q(S)}\Big)^{\frac{1}{p}}&\\ \eqsim_{p,q} \Big(\mathbb{E}\|\Phi q^{\frac{1}{2}}_{M^c}\|^p_{\gamma(L^2(\mathbb{R}_+,[M^c];H),L^q(S))}\Big)^{\frac{1}{p}} + \|F\|_{\mathcal{I}_{p,q}} + \|M^a\|_{\mathcal{A}_{p,q}},&\end{aligned} \tag{5.47}$$

where $\mathcal{I}_{p,q}$ is given as in (5.35) for $\nu = \nu^{M^q}$, $\mathcal{A}_{p,q}$ is given as in (5.8).

Proof. The estimate $\lesssim_{p,q}$ follows from the triangle inequality and Theorems 5.31, 5.28 and 5.14. Let us now prove $\gtrsim_{p,q}$ via duality. Without loss of generality, due to the proof of Theorem 5.14 and due to Lemma 5.13, we can assume that there exists $N \geqslant 1$ and a sequence of predictable stopping times $\mathcal{T} = (\tau_n)_{n=0}^N$ such that M has a.s. at most N jumps and a.s. $\{t \in \mathbb{R}_+ : \Delta M_t \neq 0\} \subset \{\tau_0, \ldots, \tau_N\}$. Define the Banach space

$$X := L^p(\Omega; \gamma(L^2(\mathbb{R}_+, [M^c]; H), L^q(S))) \times \mathcal{I}_{p,q} \times \mathcal{A}_{p,q}^{\mathcal{T}}$$

and let Y be the closure of the linear space

$$\cup_{(\Phi,F,M^a)\in X} (\Phi \cdot M^c + F \star \bar{\mu}^{M^{\mathrm{d}}} + M^a)_\infty$$

in $L^p(\Omega; L^q(S))$. Then by [**17**, Proposition 1.3.3], the Trace duality (5.42), Corollary B.8 and the duality statement in Theorem 5.6

$$X^* = L^{p'}(\Omega; \gamma(L^2(\mathbb{R}_+, [M^c]; H), L^{q'}(S))) \times \mathcal{I}_{p',q'} \times \mathcal{A}_{p',q'}^{\mathcal{T}}.$$

By the upper bounds in (5.47), the maps $j : X \to Y$ and $k : X^* \to Y^*$ defined via $(\Phi, F, M^a) \mapsto (\Phi \cdot M^c + F \star \bar{\mu}^{M^{\mathrm{d}}} + M^a)_\infty$ are both continuous linear mappings. Let $x = (\Phi_1, F_1, M_1^a) \in X$, $x^* = (\Phi_2, F_2, M_2^a) \in X^*$ be such that Φ_1 and Φ_2 are elementary predictable, and F_1 and F_2 are elementary $\widetilde{\mathcal{P}}$-measurable. Then $\langle \tilde{x}^*, \tilde{x}\rangle = \langle k(\tilde{x}^*), j(\tilde{x})\rangle$ by Lemma 5.35 and (5.11) and so Lemma 2.1 yields $\gtrsim_{p,q}$ in (5.47). □

We can now complete the proof of our main result.

Proof of Theorem 5.32. First of all note that $\Phi \cdot M$ is an $L^q(S)$-valued local martingale, so by Doob's maximal inequality

$$\mathbb{E} \sup_{0\leqslant s\leqslant t} \|(\Phi \cdot M)_s\|_{L^q(S)}^p \eqsim_p \mathbb{E}\|(\Phi \cdot M)_t\|_{L^q(S)}^p. \tag{5.48}$$

By (5.45), $(\Phi \cdot M)_t$ is in $L^p(\Omega; L^q(S))$ if and only if $(\Phi \cdot M^c)_t$, $(\Phi \cdot M^q)_t$ and $(\Phi \cdot M^a)_t$ are all in $L^p(\Omega; L^q(S))$. Consequently, (5.43) holds by (5.48), Lemma 5.18 and Proposition 5.37. □

Remark 5.38. Let $M = (M_n)_{n\geqslant 0}$ be a discrete L^q-valued martingale. Then due to the *Strong Doob maximal inequality* (also known as the *Fefferman–Stein inequality*), presented, for example, in [**17**, Theorem 3.2.7] and [**2**, Theorem 2.6],

$$\left(\mathbb{E}\left(\int_S \sup_{n\geqslant 0} |M_n(s)|^q \,\mathrm{d}s\right)^{\frac{p}{q}}\right)^{\frac{1}{p}} \eqsim_{p,q} (\mathbb{E} \sup_{n\geqslant 0} \|M_n\|_{L^q(S)}^p)^{\frac{1}{p}}.$$

As a consequence, for any continuous time martingale $M : \mathbb{R}_+ \times \Omega \to L^q(S)$

$$\left(\mathbb{E}\left\|\sup_{t\geqslant 0} |M_t|\right\|_{L^q(S)}^p\right)^{\frac{1}{p}} \eqsim_{p,q} (\mathbb{E} \sup_{t\geqslant 0} \|M_t\|_{L^q(S)}^p)^{\frac{1}{p}}.$$

Indeed, this follows by the existence of a pointwise càdlàg version of M and by approximating M by a discrete-time martingale. Thus, all the sharp bounds for stochastic integrals proved in this section, in particular Theorems 5.14, 5.28, 5.30, 5.31, and, finally, Theorem 5.32, remain valid if we move the supremum over time inside the L^q-norm.

Appendix A. *Dual of S_q^p*

Recall that for a given σ-algebra $\mathcal{F}$, a given filtration $\mathbb{F} = (\mathcal{F}_i)_{i\geqslant 1}$, and any finitely non-zero sequence $(f_i)_{i\geqslant 1}$ in $L^\infty(\Omega; L^\infty(S))$, we defined

$$\|(f_i)\|_{S_q^p} := \left(\mathbb{E} \left\| \left(\sum_{i\geqslant 1} \mathbb{E}_{i-1}|f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^q(S)}^p \right)^{\frac{1}{p}}, \tag{A.1}$$

where we assume that $\mathbb{E}_0 = \mathbb{E}$, and use S_q^p in order to denote the completion with respect to this norm. The following statement follows from a straightforward modification of the work of Junge on conditional sequence spaces in a non-commutative setting (see [**21**, Section 2]).

THEOREM A.1. *Let $1 < p, q < \infty$. Then $(S_q^p)^* = S_{q'}^{p'}$ isomorphically with constants depending only on p and q. The corresponding duality bracket is given by*

$$\langle (f_i), (g_i) \rangle = \mathbb{E} \sum_{i\geqslant 1} \int_S f_i g_i \,\mathrm{d}\rho, \quad (f_i) \in S_q^p, \quad (g_i) \in S_{q'}^{p'}. \tag{A.2}$$

The argument in [**21**] uses non-commutative analysis. To keep our work accessible to readers that are unfamiliar with these methods, we give a short proof of Theorem A.1 in an important special case based on disintegration. To use this technique, we will assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Borel probability space, that is, Ω is Borel isomorphic to a Borel subset of [0,1]. In particular, Ω may be any complete, separable metric space equipped with its Borel σ-algebra [**23**, Theorem A1.2].

The idea of the proof is to represent the conditional expectations in (A.1) as integrals with respect to random measures. To see that this is possible, consider any sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$ containing all sets of measure zero. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$. We define a probability measure $\mu_\mathcal{G}$ on $\Omega \times \Omega'$ by setting

$$\mu_\mathcal{G}\left(\bigcup_i A_i \times B_i \right) = \sum_i \mathbb{E}[\mathbf{1}_{A_i} \mathbb{E}(\mathbf{1}_{B_i}|\mathcal{G})],$$

for any disjoint $A_1, \ldots, A_n, \ldots \in \mathcal{F}$ and any disjoint $B_1, \ldots, B_n, \ldots \in \mathcal{F}'$ and extending $\mu_\mathcal{G}$ via Carathéodory's extension theorem. For further reference, we will call $\mu_\mathcal{G}$ the *conditional expectation measure* induced by $\mathcal{G}$. Since Ω is Borel, the disintegration measure $\mu_\mathcal{G}(\cdot|\cdot) : \mathcal{F}' \times \Omega \to [0,1]$ of $\mu_\mathcal{G}$ exists a.s. on Ω (see, for example, [**10**, Chapter 10] or [**23**, Chapter 6]). It is straightforward to verify that for any $f \in L^1(\Omega)$

$$\mathbb{E}(f|\mathcal{G})(\omega) = \int_{\Omega'} f(\omega') \,\mathrm{d}\mu_\mathcal{G}(\omega'|\omega) \quad \text{for a.e. } \omega \in \Omega. \tag{A.3}$$

Indeed, (A.3) can be shown directly for step functions, and then for general integrable functions, thanks to the estimates

$$\mathbb{E}|\mathbb{E}(f|\mathcal{G})|, \mathbb{E}\left| \int_{\Omega'} f(\omega') \,\mathrm{d}\mu_\mathcal{G}(\omega'|\cdot) \right| \leqslant \mathbb{E}|f|.$$

Proof of Theorem A.1. Let $S_q^p[N]$ be the closed subspace of S_q^p spanned by all sequences $(f_i)_{i\geqslant 1}$ satisfying $f_i = 0$ for all $i > N$. By a similar reduction as in the proof of Theorem 4.1, it suffices to show that $(S_q^p[N])^* = S_{q'}^{p'}[N]$ isomorphically with constants depending only on p and q and with duality bracket given by (A.2). Let $\mu_0, \ldots, \mu_{N-1}$ be the conditional expectation measures on $\Omega \times \Omega'$ corresponding to $\mathcal{F}_0, \ldots, \mathcal{F}_{N-1}$. By (A.3) we find, for any sequence $(f_i)_{i\geqslant 1}$

in $L^\infty(\Omega; L^\infty(S))$ satisfying $f_i = 0$ for all $i > N$,

$$\begin{aligned}\|(f_i)\|_{S^p_q} &= \left(\mathbb{E}\left\|\left(\sum_i \mathbb{E}_{i-1}|f_i|^2\right)^{\frac12}\right\|^p_{L^q(S)}\right)^{\frac1p}\\ &= \left(\mathbb{E}\left\|\left(\sum_{i=1}^N \int_{\Omega'} f_i^2(\omega')\,\mathrm{d}\mu_{i-1}(\omega'|\cdot)\right)^{\frac12}\right\|^p_{L^q(S)}\right)^{\frac1p}\\ &= \|(f_i)\|_{L^p(\Omega;L^q(S;L^2([N]\times\Omega',\mu)))},\end{aligned}$$

where $L^2([N]\times\Omega',\mu)$ denotes the L^2-space on $\{1,\dots,N\}\times\Omega'$ with respect to the random measure μ defined by $\mu(\{i\}\times A)(\omega) := \mu_{i-1}(A|\omega)$. Since $S^p_q[N]$ and $L^p(\Omega; L^q(S; L^2([N]\times\Omega',\mu)))$ share the same dense set (that is, the set of all sequences in $L^\infty(\Omega; L^\infty(S))$ of length N), we conclude that they are isometrically isomorphic. Therefore, using that

$$(L^p(\Omega; L^q(S; L^2([N]\times\Omega',\mu))))^* = L^{p'}(\Omega; L^{q'}(S; L^2([N]\times\Omega',\mu)))$$

(see Theorem B.5 for a more general statement), we find that $(S^p_q[N])^*$ is isomorphic to $S^{p'}_{q'}[N]$ with duality bracket given by (A.2). □

Remark A.2. In our proof, we did not use the fact that $\mathbb{F}$ is a filtration, so it remains valid for a general family of sub-σ-algebras containing all sets of measure zero. Also, our argument clearly yields the more general duality $(S^p_{q,r})^* = S^{p'}_{q',r'}$, $1 < r < \infty$, for the spaces induced by the norms

$$\|(f_i)\|_{S^p_{q,r}} := \left(\mathbb{E}\left\|\left(\sum_i \mathbb{E}_{i-1}|f_i|^r\right)^{\frac1r}\right\|^p_{L^q(S)}\right)^{\frac1p}.$$

Let us emphasize that our proof is only valid if $(\Omega, \mathcal{F}, \mathbb{P})$ is Borel.

Appendix B. *Duals of $\mathcal{S}^p_q$, $\mathcal{D}^p_{q,q}$, $\mathcal{D}^p_{p,q}$, $\hat{\mathcal{S}}^p_q$, $\hat{\mathcal{D}}^p_{q,q}$, and $\hat{\mathcal{D}}^p_{p,q}$*

In this section, we will find the duals of $\mathcal{S}^p_q$, $\mathcal{D}^p_{q,q}$, $\mathcal{D}^p_{p,q}$, $\hat{\mathcal{S}}^p_q$, $\hat{\mathcal{D}}^p_{q,q}$ and $\hat{\mathcal{D}}^p_{p,q}$ for all $1 < p, q < \infty$. As a consequence, we show the duality for the space $\mathcal{I}_{p,q}$ that was used to prove the lower bounds in Theorem 5.28.

B.1. *$\mathcal{D}^p_{q,q}$ and $\mathcal{D}^p_{p,q}$ spaces*

Let X be a Banach space and consider any σ-finite random measure ν on $\mathbb{R}_+\times J$. In sequel we will assume that $\int_{\mathbb{R}_+\times J} \mathbf{1}_A\,\mathrm{d}\nu$ is an $\overline{\mathbb{R}}_+$-valued random variable for each $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{J}$-measurable $A\subset\mathbb{R}_+\times J$. Note that this condition always holds for any optional $\tilde{\mathcal{P}}$-σ-finite random measure ν.

We define $\mathcal{D}^p_q(X)$ to be the space of all $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}\otimes\mathcal{J}$-strongly measurable functions $f:\mathbb{R}_+\times\Omega\times J\to X$ such that

$$\|f\|_{\mathcal{D}^p_q(X)} := \left(\mathbb{E}\left(\int_{\mathbb{R}_+\times J}\|f\|^q_X\,\mathrm{d}\nu\right)^{\frac pq}\right)^{\frac1p} < \infty.$$

The following result is well known if ν is a deterministic measure (see, for example, [17] for a proof). The argument for random measures is very similar and therefore omitted.

THEOREM B.1. *Let $1 < p, q < \infty$, X be reflexive. Then*

$$(\mathcal{D}_q^p(X))^* = \mathcal{D}_{q'}^{p'}(X^*).$$

Moreover,

$$\|\phi\|_{\mathcal{D}_{q'}^{p'}(X^*)} = \|\phi\|_{(\mathcal{D}_q^p(X))^*}, \quad \phi \in \mathcal{D}_{q'}^{p'}(X^*). \tag{B.1}$$

We now turn to proving a similar duality statement for $\hat{\mathcal{D}}_q^p(X)$, the space of all $\widetilde{\mathcal{P}}$-measurable functions in $\mathcal{D}_q^p(X)$. In the proof we will use the following 'reverse' version of the dual Doob inequality [9, Lemma 2.10].

LEMMA B.2 (Reverse dual Doob inequality). *Fix $0 < p \leqslant 1$. Let $\mathbb{F} = (\mathcal{F}_n)_{n\geqslant 0}$ be a filtration and let $(\mathbb{E}_n)_{n\geqslant 0}$ be the associated sequence of conditional expectations. If $(f_n)_{n\geqslant 0}$ is a sequence of non-negative random variables in $L^1(\mathbb{P})$, then*

$$\left(\mathbb{E}\left|\sum_{n\geqslant 0} f_n\right|^p\right)^{\frac{1}{p}} \leqslant p^{-1}\left(\mathbb{E}\left|\sum_{n\geqslant 0} \mathbb{E}_n f_n\right|^p\right)^{\frac{1}{p}}.$$

Now, let us show that $\left(\hat{\mathcal{D}}_q^p(X)\right)^*$ and $\hat{\mathcal{D}}_{q'}^{p'}(X^*)$ are isomorphic.

THEOREM B.3. *Let X be a reflexive space and let ν be a predictable, $\widetilde{\mathcal{P}}$-σ-finite random measure on $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{J}$ that is non-atomic in time. Then, for $1 < p, q < \infty$,*

$$(L_{\widetilde{\mathcal{P}}}^p(\mathbb{P}; L^q(\nu; X)))^* = L_{\widetilde{\mathcal{P}}}^{p'}(\mathbb{P}; L^{q'}(\nu; X^*))$$

with isomorphism given by

$$g \mapsto F_g, \quad F_g(h) = \mathbb{E}\int_{\mathbb{R}_+\times J} \langle g, h\rangle d\nu \quad \left(g \in L_{\widetilde{\mathcal{P}}}^{p'}(\mathbb{P}; L^{q'}(\nu)), h \in L_{\widetilde{\mathcal{P}}}^p(\mathbb{P}; L^q(\nu))\right).$$

Moreover,

$$\min\left\{\left(\frac{p}{q}\right)^{1/q}\frac{q'}{p'}, \left(\frac{p'}{q'}\right)^{1/q'}\frac{q}{p}\right\}\|g\|_{L_{\widetilde{\mathcal{P}}}^{p'}(\mathbb{P}; L^{q'}(\nu; X^*))} \leqslant \|F_g\| \leqslant \|g\|_{L_{\widetilde{\mathcal{P}}}^{p'}(\mathbb{P}; L^{q'}(\nu; X^*))}. \tag{B.2}$$

Proof. *Step* 1: *reduction*. It suffices to prove the result for $p \leqslant q$. Indeed, once this is known we can deduce the case $q \leqslant p$ as follows. Observe that $L_{\widetilde{\mathcal{P}}}^{p'}(\mathbb{P}; L^{q'}(\nu; X^*))$ is a closed subspace of $\mathcal{D}_{q'}^{p'}(X^*) = L^{p'}(\mathbb{P}; L^{q'}(\nu; X^*))$. By Theorem B.1, $\mathcal{D}_{q'}^{p'}(X^*)$ is reflexive and therefore $L_{\widetilde{\mathcal{P}}}^{p'}(\mathbb{P}; L^{q'}(\nu; X^*))$ is reflexive as well. Therefore, as $p' \leqslant q'$,

$$(L_{\widetilde{\mathcal{P}}}^p(\mathbb{P}; L^q(\nu; X)))^* = L_{\widetilde{\mathcal{P}}}^{p'}(\mathbb{P}; L^{q'}(\nu; X^*))^{**} = L_{\widetilde{\mathcal{P}}}^{p'}(\mathbb{P}; L^{q'}(\nu; X^*)).$$

Hence, if $F \in (L_{\widetilde{\mathcal{P}}}^p(\mathbb{P}; L^q(\nu; X)))^*$, then there exists an $f \in L_{\widetilde{\mathcal{P}}}^{p'}(\mathbb{P}; L^{q'}(\nu; X^*))$ so that for any $g \in L_{\widetilde{\mathcal{P}}}^p(\mathbb{P}; L^q(\nu; X))$

$$F(g) = F_g(f) = \mathbb{E}\int_{\mathbb{R}_+\times J} \langle f, g\rangle d\nu.$$

Moreover, the bounds (B.2) follow from Lemma 4.4. Thus, for the remainder of the proof, we can assume that $p \leqslant q$.

Step 2: *norm estimates.* Let us now show that (B.2) holds. Since the upper bound is immediate from Hölder's inequality, we only need to show that for any $g \in L^{p'}_{\widetilde{\mathcal{P}}}(\mathbb{P}; L^{q'}(\nu; X^*))$,

$$\|F_g\| \geqslant \left(\frac{p}{q}\right)^{1/q} \frac{q'}{p'} \|g\|_{L^{p'}(\mathbb{P};L^{q'}(\nu;X^*))}. \tag{B.3}$$

It suffices to show this on a dense subset of $L^{p'}_{\widetilde{\mathcal{P}}}(\mathbb{P}; L^{q'}(\nu; X^*))$. Indeed, suppose that $g_n \to g$ in $L^{p'}_{\widetilde{\mathcal{P}}}(\mathbb{P}; L^{q'}(\nu; X^*))$ and that (B.3) holds for g_n, for all $n \geqslant 1$. Then,

$$\left(\frac{p}{q}\right)^{1/q} \frac{q'}{p'} \|g_n\|_{L^{p'}(\mathbb{P};L^{q'}(\nu;X^*))} \leqslant \|F_{g_n}\| \leqslant \|F_g\| + \|g - g_n\|_{L^{p'}(\mathbb{P};L^{q'}(\nu;X^*))},$$

and by taking limits on both sides we see that g also satisfies (B.3).

Let us first assume that

$$\nu((s,t] \times J) \leqslant (t-s) \quad \text{a.s.,} \quad \text{for all } 0 \leqslant s \leqslant t. \tag{B.4}$$

By the previous discussion, we may assume that $\|g\|_{L^{p'}(\mathbb{P};L^{q'}(\nu;X^*))} = 1$ and that g is of the form

$$g = \sum_{n=0}^{N_{m_*}} \sum_{\ell=0}^{L} \mathbf{1}_{(n/2^{m_*},(n+1)/2^{m_*}]} \mathbf{1}_{B_\ell} g_{n\ell},$$

where $N_{m_*} < \infty$, $g_{n\ell}$ is simple and $\mathcal{F}_{n/2^{m_*}}$-measurable for all n and ℓ, and the B_ℓ are disjoint sets in $\mathcal{J}$ of finite $\mathbb{P} \otimes \nu$-measure. For $m \geqslant m_*$, define

$$g^{(m)} = \sum_{n=0}^{N_m} \sum_{\ell=0}^{L} \mathbf{1}_{(n/2^m,(n+1)/2^m]} \mathbf{1}_{B_\ell} g^{(m)}_{n\ell}$$

so that $g^{(m)} = g$. Then clearly, $g^{(m)}_{n\ell}$ is $\mathcal{F}_{n/2^m}$-measurable for all n and ℓ. Let us now fix an $m \geqslant m_*$. We define, for any $0 \leqslant k \leqslant N_m$,

$$\bar{s}^k_{q'}(g) := \left(\sum_{n=0}^{k} \sum_{\ell=0}^{L} \|g^{(m)}_{n\ell}\|^{q'} \mathbb{E}_{n/2^m} \nu((n/2^m, (n+1)/2^m] \times B_\ell)\right)^{1/q'}$$

and set

$$\alpha = (\mathbb{E} \bar{s}^{N_m}_{q'}(g^{(m)})^{p'})^{1/p'}.$$

Let P_ε be as in Lemma 4.2. We define a $\widetilde{\mathcal{P}}$-measurable function h by

$$h = \sum_{n=0}^{N_m} \sum_{\ell=0}^{L} \mathbf{1}_{(n/2^m,(n+1)/2^m]} \mathbf{1}_{B_\ell} h_{n\ell},$$

where, for $0 \leqslant n \leqslant N_m$ and $0 \leqslant \ell \leqslant L$, $h_{n\ell}$ is the $\mathcal{F}_{n/2^m}$-measurable function defined by

$$h_{n\ell} = \frac{1}{\alpha^{p'-1}} (\bar{s}^n_{q'}(g^{(m)}))^{p'-q'} \|g^{(m)}_{n\ell}\|^{q'-1} P_\varepsilon g^{(m)}_{n\ell}.$$

Since $p/q \leqslant 1$, Lemma B.2 implies

$$\|h\|_{L^p(\mathbb{P};L^q(\nu))} = \left(\mathbb{E}\left(\sum_{n=0}^{N_m} \sum_{\ell=0}^{L} \|h_{n\ell}\|^q \nu((n/2^m, (n+1)/2^m] \times B_\ell)\right)^{p/q}\right)^{1/p}$$

$$\leqslant \left(\frac{q}{p}\right)^{1/q}\left(\mathbb{E}\left(\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|h_{n\ell}\|^q\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell)\right)^{p/q}\right)^{1/p}$$

$$= \left(\frac{q}{p}\right)^{1/q}(\mathbb{E}\bar{s}_q^{N_m}(h)^p)^{1/p}.$$

Now, observe that

$$\begin{aligned}
\bar{s}_q^{N_m}(h)^q &= \sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|h_{n\ell}\|^q\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell) \\
&\leqslant \frac{1}{\alpha^{(p'-1)q}}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|g_{n\ell}^{(m)}\|^{(q'-1)q}\bar{s}_{q'}^n(g^{(m)})^{(p'-q')q}\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell) \\
&\leqslant \frac{1}{\alpha^{(p'-1)q}}\bar{s}_{q'}^{N_m}(g^{(m)})^{(p'-q')q}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|g_{n\ell}^{(m)}\|^{q'}\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell) \\
&= \frac{1}{\alpha^{(p'-1)q}}\bar{s}_{q'}^{N_m}(g^{(m)})^{p'q-q'q+q'}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|h\|_{L^p(\mathbb{P};L^q(\nu))}^p &\leqslant \left(\frac{q}{p}\right)^{p/q}\frac{1}{\alpha^{(p'-1)p}}\bar{s}_{q'}^{N_m}(g^{(m)})^{(p'q-q'q+q')p/q} \\
&= \left(\frac{q}{p}\right)^{p/q}\frac{1}{\alpha^{p'}}\mathbb{E}\bar{s}_{q'}^{N_m}(g^{(m)})^{p'} = \left(\frac{q}{p}\right)^{p/q}.
\end{aligned}$$

Moreover, by Lemma 4.2,

$$\begin{aligned}
F_g(h) &= \mathbb{E}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\langle g_{n\ell}^{(m)},h_{n\ell}\rangle\nu((n/2^m,(n+1)/2^m]\times B_\ell) \\
&= \mathbb{E}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\langle g_{n\ell}^{(m)},h_{n\ell}\rangle\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell) \\
&\geqslant \frac{(1-\varepsilon)}{\alpha^{p'-1}}\mathbb{E}\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|g_{n\ell}^{(m)}\|^{q'}\bar{s}_{q'}^n(g^{(m)})^{p'-q'}\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell) \\
&= \frac{(1-\varepsilon)}{\alpha^{p'-1}}\mathbb{E}\sum_{n=0}^{N_m}\bar{s}_{q'}^n(g^{(m)})^{p'-q'}(\bar{s}_{q'}^n(g^{(m)})^{q'}-\bar{s}_{q'}^{n-1}(g^{(m)})^{q'}).
\end{aligned}$$

Now, apply (4.10) for $\alpha = p'/q' \geqslant 1$ and $x = \bar{s}_{q'}^n(g^{(m)})^{q'}/\bar{s}_{q'}^{n-1}(g^{(m)})^{q'} \geqslant 1$ to obtain

$$\begin{aligned}
F_g(h) &\geqslant (1-\varepsilon)\frac{1}{\alpha^{p'-1}}\mathbb{E}\sum_{n=0}^{N_m}\frac{q'}{p'}\left(\bar{s}_{q'}^n(g^{(m)})^{p'}-\bar{s}_{q'}^{n-1}(g^{(m)})^{p'}\right) \\
&= (1-\varepsilon)\frac{q'}{p'}\frac{1}{\alpha^{p'-1}}\mathbb{E}\bar{s}_{q'}^{N_m}(g^{(m)})^{p'}
\end{aligned}$$

$$= (1-\varepsilon)\frac{q'}{p'}\Big(\mathbb{E}\bar{s}_{q'}^{N_m}(g^{(m)})^{p'}\Big)^{1/p'}$$

$$= (1-\varepsilon)\frac{q'}{p'}\left(\mathbb{E}\left(\sum_{n=0}^{N_m}\sum_{\ell=0}^{L}\|g_{n\ell}^{(m)}\|^{q'}\mathbb{E}_{n/2^m}\nu((n/2^m,(n+1)/2^m]\times B_\ell)\right)^{p'/q'}\right)^{1/p'}$$

$$= (1-\varepsilon)\frac{q'}{p'}\left(\mathbb{E}\left(\sum_{n=0}^{N_m}\mathbb{E}_{n/2^m}\Big((\|g\|^{q'}\star\nu)_{(n+1)/2^m}-(\|g\|^{q'}\star\nu)_{n/2^m}\Big)\right)^{p'/q'}\right)^{1/p'}.$$

In conclusion, for any $m\geqslant m_*$, we find

$$\|F_g\|\geqslant\left(\frac{p}{q}\right)^{1/q}\frac{q'}{p'}\left(\mathbb{E}\left(\sum_{n=0}^{N_m}\mathbb{E}_{n/2^m}((\|g\|^{q'}\star\nu)_{(n+1)/2^m}-(\|g\|^{q'}\star\nu)_{n/2^m})\right)^{p'/q'}\right)^{1/p'}.$$

Taking $m\to\infty$, we find using Lemma 5.24 and the fact that ν is non-atomic in time that

$$\|F_g\|\geqslant\left(\frac{p}{q}\right)^{1/q}\frac{q'}{p'}\|g\|_{L^{p'}_{\widetilde{\mathcal{P}}}(\mathbb{P};L^{q'}(\nu;X^*))}.$$

Let us now remove the additional restriction (B.4) on ν. In this case, we define a strictly increasing, predictable, continuous process

$$A_t := \nu([0,t]\times J)+t,\quad t\geqslant 0$$

and a random time change $\tau=(\tau_s)_{s\geqslant 0}$ by

$$\tau_s=\{t\ :\ A_t=s\}.$$

By Proposition 5.27, $A\circ\tau(t)=t$ a.s. for any $t\geqslant 0$, and hence by continuity of A and τ, a.s. $A\circ\tau(t)=t$ for all $t\geqslant 0$. As was noted in (5.40), we have $\nu_\tau((s,t]\times J)\leqslant t-s$ a.s. for all $s\leqslant t$. By Proposition 5.27, we can now write

$$\begin{aligned}\|F_g\| &= \sup_{\|h\|_{L^p_{\widetilde{\mathcal{P}}}(\mathbb{P};L^q(\nu;X))}\leqslant 1}\mathbb{E}\int_{\mathbb{R}_+\times J}\langle g,h\rangle d\nu\\ &\geqslant \sup_{\|\tilde{h}\circ A\|_{L^p_{\widetilde{\mathcal{P}}}(\mathbb{P};L^q(\nu;X))}\leqslant 1}\mathbb{E}\int_{\mathbb{R}_+\times J}\langle g,\tilde{h}\circ A\rangle d\nu\\ &= \sup_{\|\tilde{h}\|_{L^p_{\widetilde{\mathcal{P}}}(\mathbb{P};L^q(\nu_\tau;X))}\leqslant 1}\mathbb{E}\int_{\mathbb{R}_+\times J}\langle g\circ\tau,\tilde{h}\rangle d\nu_\tau.\end{aligned}$$

Applying the previous part of the proof for $\nu=\nu_\tau$, we find

$$\|F_g\|\geqslant\left(\frac{p}{q}\right)^{1/q}\frac{q'}{p'}\|g\circ\tau\|_{L^{p'}_{\widetilde{\mathcal{P}}}(\mathbb{P};L^{q'}(\nu_\tau;X^*))}=\|g\|_{L^{p'}_{\widetilde{\mathcal{P}}}(\mathbb{P};L^{q'}(\nu;X^*))}.$$

This completes our proof of (B.2).

Step 3: *representation of linear functionals.* It now remains to show that every $F\in(L^p_{\widetilde{\mathcal{P}}}(\mathbb{P};L^q(\nu;X)))^*$ is of the form F_g for a suitable $\widetilde{\mathcal{P}}$-measurable function g. We will first assume that $\mathbb{E}\nu(\mathbb{R}_+\times J)<\infty$. On $\widetilde{\mathcal{P}}$ we can define an X^*-valued measure θ by setting

$$\langle\theta(A),x\rangle := F(\mathbf{1}_A\cdot x)\quad (A\in\widetilde{\mathcal{P}},x\in X).$$

Then θ is σ-additive, absolutely continuous with respect to the measure $\mathbb{P}\otimes\nu$ defined in (5.27). Moreover, for any disjoint partition $A_1,\dots,A_n\in\widetilde{\mathcal{P}}$ of $\mathbb{R}_+\times\Omega\times J$,

$$\begin{aligned}\sum_{i=1}^n\|\theta(A_i)\| &= \sup_{(x_i)_{i=1}^n\subset B_X}\sum_{i=1}^n F(\mathbf{1}_{A_i}x_i)\\ &= \sup_{(x_i)_{i=1}^n\subset B_X} F\left(\sum_{i=1}^n\mathbf{1}_{A_i}x_i\right)\\ &\leqslant \|F\|_{(\mathcal{D}_q^p(X))^*}\sup_{(x_i)_{i=1}^n\subset B_X}\left(\mathbb{E}\left(\int_{\mathbb{R}_+\times J}\left\|\sum_{i=1}^n\mathbf{1}_{A_i}x_i\right\|_X^q d\nu\right)^{p/q}\right)^{1/p}\\ &= \|F\|_{(\mathcal{D}_q^p(X))^*}\sup_{(x_i)_{i=1}^n\subset B_X}\left(\mathbb{E}\left(\int_{\mathbb{R}_+\times J}\sum_{i=1}^n\mathbf{1}_{A_i}\|x_i\|_X^q\, d\nu\right)^{p/q}\right)^{1/p}\\ &\leqslant \|F\|_{(\mathcal{D}_q^p(X))^*}(\mathbb{E}\nu(\mathbb{R}_+\times J)^{p/q})^{1/p},\end{aligned}$$

so θ is of finite variation. By the Radon–Nikodym property of X^*, there exists a $\widetilde{\mathcal{P}}$-measurable X^*-valued function g such that

$$F(h)=F_g(h)=\mathbb{E}\int_{\mathbb{R}_+\times J}\langle g,h\rangle\,\mathrm{d}\nu$$

for each $h\in L^p_{\widetilde{\mathcal{P}}}(\mathbb{P};L^q(\nu;X))$.

Now, let $\mathbb{E}\nu(\mathbb{R}_+\times J)=\infty$. Since $\mathbb{P}\otimes\nu$ is σ-finite, there exists a sequence $(A_n)_{n\geqslant 1}\subset\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}\otimes\mathcal{J}$ such that $A_n\nearrow\mathbb{R}_+\times\Omega\times J$ as $n\to\infty$ and $\mathbb{P}\otimes\nu(A_n)<\infty$ for each $n\geqslant 1$. By the previous part of the proof, for each $n\geqslant 1$, there exists $f_n\in\mathcal{D}_{q'}^{p'}(X^*)$ with support in A_n such that

$$F(g\cdot\mathbf{1}_{A_n})=F_{f_n}(g\cdot\mathbf{1}_{A_n})=\mathbb{E}\int_{\mathbb{R}_+\times J}\langle f_n,g\rangle\mathbf{1}_{A_n}\,\mathrm{d}\nu$$

and

$$\left(\frac{p}{q}\right)^{1/q}\frac{q'}{p'}\|f_n\|_{\mathcal{D}_{q'}^{p'}(X^*)}\leqslant\|F_{f_n}\|_{(\mathcal{D}_q^p(X))^*}\leqslant\|F\|_{(\mathcal{D}_q^p(X))^*}.$$

Obviously, $f_{n+1}\mathbf{1}_{A_n}=f_n$ for each $n\geqslant 1$, hence there exists $f:\Omega\times\mathbb{R}_+\times J\to X^*$ such that $f\mathbf{1}_{A_n}=f_n$ for each $n\geqslant 1$. But then Fatou's lemma implies

$$\left(\frac{p}{q}\right)^{1/q}\frac{q'}{p'}\|f\|_{\mathcal{D}_{q'}^{p'}(X^*)}\leqslant\left(\frac{p}{q}\right)^{1/q}\frac{q'}{p'}\liminf_{n\to\infty}\|f_n\|_{\mathcal{D}_{q'}^{p'}(X^*)}\leqslant\|F\|_{(\mathcal{D}_q^p(X))^*},$$

so $f\in\mathcal{D}_{q'}^{p'}(X^*)$. On the other hand, by Hölder's inequality

$$\|F\|_{(\mathcal{D}_q^p(X))^*}\leqslant\|f\|_{\mathcal{D}_{q'}^{p'}(X^*)}.$$

Since the bounded linear functionals F and F_f agree on a dense subset of $\mathcal{D}_q^p(X)$, it follows that $F=F_f$ and (B.1) holds. □

REMARK B.4. The reader may wonder whether the duality

$$(L^p_{\widetilde{\mathcal{P}}}(\mathbb{P};L^q(\nu;X)))^*=L^{p'}_{\widetilde{\mathcal{P}}}(\mathbb{P};L^{q'}(\nu;X^*))$$

remains valid if ν is any random measure and $\widetilde{\mathcal{P}}$ is replaced by an arbitrary sub-σ-algebra of $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}\otimes\mathcal{J}$. It turns out that, surprisingly, such a general result does not hold true. Indeed, it was pointed out by Pisier (in a personal communication) that there exist two probability spaces $(\Omega_1,\mathcal{F}_1,\mathbb{P}_1)$, $(\Omega_2,\mathcal{F}_2,\mathbb{P}_2)$ and a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}_1\otimes\mathcal{F}_2$ so that the duality

$$(L^p_{\mathcal{G}}(\mathbb{P}_1;L^q(\mathbb{P}_2)))^* = L^{p'}_{\mathcal{G}}(\mathbb{P}_1;L^{q'}(\mathbb{P}_2))$$

does not even hold isomorphically. This counterexample, in particular, shows that the duality results claimed in [**27**] are not valid without imposing additional assumptions. We refer the reader to [**26**, **28**] for details.

B.2. $\mathcal{S}^p_q$ and $\hat{\mathcal{S}}^p_q$ spaces

Let ν be any σ-finite random measure on $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{J}$. Recall that $\mathcal{S}^p_q$ is the space of all $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{F}\otimes\mathcal{J}$-strongly measurable functions $f:\mathbb{R}_+\otimes\Omega\otimes J\to L^q(S)$ satisfying

$$\|f\|_{\mathcal{S}^p_q} = \left(\mathbb{E}\left\|\left(\int_{\mathbb{R}_+\times J}|f|^2\,\mathrm{d}\nu\right)^{\frac12}\right\|^p_{L^q(S)}\right)^{\frac1p} < \infty. \tag{B.5}$$

The proof of the following result is analogous to the proof of Theorem B.1. We leave the details to the reader.

THEOREM B.5. *Let $1<p,q<\infty$. Then $(\mathcal{S}^p_q)^* = \mathcal{S}^{p'}_{q'}$ and*

$$\|f\|_{\mathcal{S}^{p'}_{q'}} \eqsim_{p,q} \|f\|_{(\mathcal{S}^p_q)^*},\quad f\in\mathcal{S}^{p'}_{q'}.$$

Let us now prove the desired duality for $\hat{\mathcal{S}}^p_q$, the subspace of all $\widetilde{\mathcal{P}}$-strongly measurable functions in $\mathcal{S}^p_q$.

THEOREM B.6. *Let $1<p,q<\infty$. Suppose that ν is a predictable, $\widetilde{\mathcal{P}}$-σ-finite random measure on $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{J}$ that is non-atomic in time. Then $(\hat{\mathcal{S}}^p_q)^* = \hat{\mathcal{S}}^{p'}_{q'}$ and*

$$\|f\|_{\hat{\mathcal{S}}^{p'}_{q'}} \eqsim_{p,q} \|f\|_{(\hat{\mathcal{S}}^p_q)^*},\quad f\in\hat{\mathcal{S}}^{p'}_{q'}. \tag{B.6}$$

For the proof of Theorem B.6, we will use the following assertion. Given a filtration $\mathbb{F}=(\mathcal{F}_n)_{n\geqslant0}$ and $1<p,q<\infty$, we define Q^p_q to be the Banach space of all adapted $L^q(S)$-valued sequences $(f_n)_{n\geqslant0}$ satisfying

$$\|(f_n)_{n\geqslant0}\|_{Q^p_q} := \left(\mathbb{E}\left\|\left(\sum_{n=0}^\infty|f_n|^2\right)^{\frac12}\right\|^p_{L^q(S)}\right)^{\frac1p} < \infty. \tag{B.7}$$

PROPOSITION B.7. *Let $1<p,q<\infty$. Then $(Q^p_q)^* = Q^{p'}_{q'}$ isomorphically, with the duality bracket given by*

$$\langle(f_n)_{n\geqslant0},(g_n)_{n\geqslant0}\rangle := \mathbb{E}\sum_{n=0}^\infty\langle f_n,g_n\rangle\quad ((g_n)_{n\geqslant0}\in Q^{p'}_{q'},\ (f_n)_{n\geqslant0}\in Q^p_q).$$

Moreover,

$$\|(g_n)_{n\geqslant 0}\|_{Q^{p'}_{q'}} \eqsim_{p,q} \|(g_n)_{n\geqslant 0}\|_{(Q^p_q)^*}.$$

Proof. Consider the filtration $\mathbb{G} = (\mathcal{G}_n)_{n\geqslant 0} = (\mathcal{F}_{n+1})_{n\geqslant 0}$. Let S^p_q be the conditional sequence space defined in (1.4) for the filtration $\mathbb{G}$. First note that Q^p_q is a closed subspace and

$$\|(f_n)_{n\geqslant 0}\|_{Q^p_q} = \|(f_n)_{n\geqslant 0}\|_{S^p_q}, \quad \text{for all } (f_n)_{n\geqslant 0} \in Q^p_q.$$

Let F be in $(Q^p_q)^*$. Then by the Hahn–Banach theorem and the duality $(S^p_q)^* = S^{p'}_{q'}$ (see Theorem A.1), there exists $\tilde{g} = (\tilde{g}_n)_{n\geqslant 0} \in S^{p'}_{q'}$ such that $\|\tilde{g}\|_{S^{p'}_{q'}} \eqsim_{p,q} \|F\|_{(Q^p_q)^*}$ and

$$F(f) = F_{\tilde{g}}(f) := \mathbb{E}\sum_{n=1}^{\infty}\langle f_n, \tilde{g}_n\rangle, \quad f = (f_n)_{n\geqslant 0} \in Q^p_q.$$

Now, let $(g_n)_{n\geqslant 0}$ be the $\mathbb{F}$-adapted $L^q(S)$-valued sequence defined by $g_n = \mathbb{E}_n\tilde{g}_n$ for $n \geqslant 0$ (recall that $\mathbb{E}_n(\cdot) := \mathbb{E}(\cdot|\mathcal{F}_n)$). Then $F_g = F_{\tilde{g}}$ on Q^p_q. Moreover, the conditional Jensen inequality yields $\|(g_n)_{n\geqslant 0}\|^{p'}_{Q^{p'}_{q'}} \leqslant \|(\tilde{g}_n)_{n\geqslant 0}\|^{p'}_{S^{p'}_{q'}}$. Finally, $\|F\|_{(Q^p_q)^*} \leqslant \|(g_n)_{n\geqslant 0}\|_{Q^{q'}_{p'}}$ follows immediately from Hölder's inequality. □

Proof of Theorem B.6. The proof contains two parts. In the first part, consisting of several steps, we will show that $\|f\|_{\hat{\mathcal{S}}^{p'}_{q'}} \eqsim_{p,q} \|f\|_{(\hat{\mathcal{S}}^p_q)^*}$. In the second part, we show that $(\hat{\mathcal{S}}^p_q)^* = \hat{\mathcal{S}}^{p'}_{q'}$.

Step 1: *J is finite, ν is a Lebesgue measure.* Let $J = \{j_1, \dots, j_K\}$, $\nu(\omega)$ be the product of Lebesgue measure and the counting measure on $\mathbb{R}_+ \times J$ for all $\omega \in \Omega$ (that is, $\nu((s,t] \times j_k) = t - s$ for each $k = 1, \dots, K$ and $t \geqslant s \geqslant 0$). Fix $f \in \hat{\mathcal{S}}^{p'}_{q'}$. Without loss of generality we can assume that f is simple and that there exist $N, M \geqslant 1$ and a sequence of random variables $(f_{k,m})_{k=1,m=0}^{k=K,m=M}$ such that $f_{k,m}$ is $\mathcal{F}_{\frac{m}{N}}$-measurable and $f(t, j_k) = f_{k,m}$ for each $k = 1, \dots, K$, $m = 0, \dots, M$, and $t \in (\frac{m}{N}, \frac{m+1}{N}]$. Let

$$\mathbb{G} = (\mathcal{G}_{k,m})_{k=1,m=0}^{k=K,m=M} := (\mathcal{F}_{\frac{m}{N}})_{k=1,m=0}^{k=K,m=M}.$$

Then $\mathbb{G}$ forms a filtration with respect to the reverse lexicographic order on the pairs (k,m), $1 \leqslant k \leqslant K$ and $0 \leqslant m \leqslant M$, that is, $\mathcal{G}_{k_1,m_1} \subseteq \mathcal{G}_{k_2,m_2}$ if $m_1 < m_2$ or if $m_1 = m_2$ and $k_1 \leqslant k_2$. Let $Q^{p'}_{q'}$ be as defined in (B.7) for $\mathbb{G}$. Then

$$\|f\|_{\hat{\mathcal{S}}^{p'}_{q'}} = \frac{1}{\sqrt{N}}\Big\|(f_{k,m})_{k=1,m=0}^{k=K,m=M}\Big\|_{Q^{p'}_{q'}}. \tag{B.8}$$

By Proposition B.7, there exists a $\mathbb{G}$-adapted $(g_{k,m})_{k=1,m=0}^{k=K,m=M} \in Q^p_q$ such that

$$\Big\|(g_{k,m})_{k=1,m=0}^{k=K,m=M}\Big\|_{Q^p_q} = 1$$

and

$$\Big\langle (f_{k,m})_{k=1,m=0}^{k=K,m=M}, (g_{k,m})_{k=1,m=0}^{k=K,m=M}\Big\rangle \eqsim_{p,q} \Big\|(f_{k,m})_{k=1,m=0}^{k=K,m=M}\Big\|_{Q^{p'}_{q'}}.$$

Let $g : \mathbb{R}_+ \times \Omega \times J \to L^q(S)$ be defined by setting $g(t, j_k) = \sqrt{N}g_{k,m}$ for each $k = 1, \dots, K$, $m = 1, \dots, M$, and $t \in (\frac{m}{N}, \frac{m+1}{N}]$. Then $g \in \hat{\mathcal{S}}^p_q$, and analogously to (B.8)

$$\|g\|_{\hat{\mathcal{S}}^p_q} = \Big\|(g_{k,m})_{k=1,m=0}^{k=K,m=M}\Big\|_{Q^p_q} = 1.$$

Moreover,

$$\langle f,g\rangle = \mathbb{E}\int_{\mathbb{R}_+\times J}\langle f(t,j),g(t,j)\rangle\,\mathrm{d}t\,\mathrm{d}j = \frac{1}{\sqrt{N}}\mathbb{E}\sum_{k=1,m=0}^{k=K,m=M}\langle f_{k,m},g_{k,m}\rangle$$

$$\eqsim_{p,q}\frac{1}{\sqrt{N}}\Big\|(f_{k,m})_{k=1,m=0}^{k=K,m=M}\Big\|_{Q^{p'}_{q'}} = \|f\|_{\hat{\mathcal{S}}^{p'}_{q'}},$$

which finishes the proof.

Step 2: *J is finite, $\nu((s,t]\times J)\leqslant t-s$ a.s. for each $t\geqslant s\geqslant 0$.* Let ν_0 be the product of Lebesgue measure and the counting measure on $\mathbb{R}_+\times J$ (see Step 1). Then clearly $\mathbb{P}\otimes\nu$ is absolutely continuous with respect to $\mathbb{P}\otimes\nu_0$ and by the Radon–Nikodym theorem there exists a $\widetilde{\mathcal{P}}$-measurable density $\phi:\mathbb{R}_+\times\Omega\times J\to\mathbb{R}_+$ such that $\mathrm{d}(\mathbb{P}\otimes\nu)=\phi\,\mathrm{d}(\mathbb{P}\otimes\nu_0)$. Fix $f\in\hat{\mathcal{S}}^{p'}_{q'}$. Let $\hat{\mathcal{S}}^{p',\nu_0}_{q'}$ be as defined in (B.5) for the random measure ν_0. Then $f_0:=f\cdot\sqrt{\phi}\in\hat{\mathcal{S}}^{p',\nu_0}_{q'}$, and $\|f\|_{\hat{\mathcal{S}}^{p'}_{q'}}=\|f_0\|_{\hat{\mathcal{S}}^{p',\nu_0}_{q'}}$. By Step 1, there exists a $g_0\in\hat{\mathcal{S}}^{p,\nu_0}_q$ such that $\|g_0\|_{\hat{\mathcal{S}}^{p,\nu_0}_q}=1$ and $\langle f_0,g_0\rangle\eqsim_{p,q}\|f_0\|_{\hat{\mathcal{S}}^{p',\nu_0}_{q'}}$. Let $g=g_0\mathbf{1}_{\phi\neq 0}\frac{1}{\sqrt{\phi}}$. Then

$$\langle f,g\rangle = \mathbb{E}\int_{\mathbb{R}_+\times J}\langle f,g\rangle\,\mathrm{d}\nu = \mathbb{E}\int_{\mathbb{R}_+\times J}\langle f,g\rangle\phi\,\mathrm{d}\nu_0 = \mathbb{E}\int_{\mathbb{R}_+\times J}\langle f\sqrt{\phi},g\sqrt{\phi}\rangle\,\mathrm{d}\nu_0$$

$$= \mathbb{E}\int_{\mathbb{R}_+\times J}\langle f_0,g_0\rangle\,\mathrm{d}\nu_0 = \langle f_0,g_0\rangle\eqsim_{p,q}\|f_0\|_{\hat{\mathcal{S}}^{p',\nu_0}_{q'}} = \|f\|_{\hat{\mathcal{S}}^{p'}_{q'}}$$

and

$$\|g\|_{\hat{\mathcal{S}}^p_q} = \left(\mathbb{E}\left\|\left(\int_{\mathbb{R}_+\times J}|g|^2\,\mathrm{d}\nu\right)^{\frac12}\right\|^p\right)^{\frac1p} = \left(\mathbb{E}\left\|\left(\int_{\mathbb{R}_+\times J}|g_0|^2\mathbf{1}_{\phi\neq 0}\frac{1}{\phi}\,\mathrm{d}\nu\right)^{\frac12}\right\|^p\right)^{\frac1p}$$

$$= \left(\mathbb{E}\left\|\left(\int_{\mathbb{R}_+\times J}|g_0|^2\mathbf{1}_{\phi\neq 0}\,\mathrm{d}\nu_0\right)^{\frac12}\right\|^p\right)^{\frac1p} \leqslant \left(\mathbb{E}\left\|\left(\int_{\mathbb{R}_+\times J}|g_0|^2\,\mathrm{d}\nu_0\right)^{\frac12}\right\|^p\right)^{\frac1p}$$

$$= \|g_0\|_{\hat{\mathcal{S}}^{p,\nu_0}_q} = 1.$$

Therefore, $\|f\|_{\hat{\mathcal{S}}^{p'}_{q'}}\eqsim_{p,q}\|f\|_{(\hat{\mathcal{S}}^p_q)^*}$.

Step 3: *J is finite, ν is general.* Without loss of generality we can assume that $\mathbb{E}\nu(\mathbb{R}_+\times J)<\infty$. Then by a time-change argument as was used in the proof of Theorem B.3, we can assume that $\nu((s,t]\times J)\leqslant t-s$ a.s. for each $t\geqslant s\geqslant 0$, and apply Step 2.

Step 4: *J is general, ν is general.* Without loss of generality assume that $\mathbb{E}\nu(\mathbb{R}_+\times J)<\infty$. Let f be simple $\widetilde{\mathcal{P}}$-measurable, that is, there exist a $K\geqslant 1$ and a partition $J=J_1\cup\cdots\cup J_K$ of J into disjoint sets such that

$$f(t,\omega,j)=\sum_{k=1}^K f_k(t,\omega)\mathbf{1}_{j\in J_k},\quad t\geqslant 0,\omega\in\Omega,j\in J,$$

where $f_1,\dots,f_k:\mathbb{R}_+\times\Omega\to L^q(S)$ are predictable. Fix $j_k\in J_k$, $k=1,\dots,K$, and define $\widetilde{J}=\{j_1,\dots,j_K\}$. Let $\widetilde{\nu}$ be a new random measure on $\mathbb{R}_+\times\Omega\times\widetilde{J}$ defined by

$$\widetilde{\nu}(A\times\{j_k\})=\nu(A\times J_k),\quad A\in\mathcal{P},\ k=1,\dots,K.$$

Let $\hat{\mathcal{S}}_{q'}^{p',\widetilde{\nu}}$ be as constructed in (B.5) for the measure $\widetilde{\nu}$. Let $\widetilde{f} \in \hat{\mathcal{S}}_{q'}^{p',\widetilde{\nu}}$ be such that $\widetilde{f}(j_k) = f_k$ for each $k = 1, \ldots, K$. Then $\|\widetilde{f}\|_{\hat{\mathcal{S}}_{q'}^{p',\widetilde{\nu}}} = \|f\|_{\hat{\mathcal{S}}_{q'}^{p'}}$. By Step 3, there exists a $\widetilde{g} \in \hat{\mathcal{S}}_q^{p,\widetilde{\nu}}$ such that $\|\widetilde{g}\|_{\hat{\mathcal{S}}_q^{p,\widetilde{\nu}}} = 1$ and $\langle \widetilde{f}, \widetilde{g}\rangle \eqsim_{p,q} \|\widetilde{f}\|_{\hat{\mathcal{S}}_{q'}^{p',\widetilde{\nu}}}$.

Define $g \in \hat{\mathcal{S}}_q^p$ by setting $g(j) = \widetilde{g}(j_k)$ for each $k = 1, \ldots, K$ and $j \in J_k$. Then $\|g\|_{\hat{\mathcal{S}}_q^p} = \|\widetilde{g}\|_{\hat{\mathcal{S}}_q^{p,\widetilde{\nu}}} = 1$. Moreover,

$$\begin{aligned}\langle f, g\rangle &= \mathbb{E}\int_{\mathbb{R}_+\times J} \langle f(t,j), g(t,j)\rangle \,\mathrm{d}\nu(t,j) = \mathbb{E}\sum_{k=1}^{K}\int_{\mathbb{R}_+\times J_k} \langle f(t,j), g(t,j)\rangle \,\mathrm{d}\nu(t,j)\\ &= \mathbb{E}\int_{\mathbb{R}_+\times \widetilde{J}} \langle \widetilde{f}(t,j), \widetilde{g}(t,j)\rangle \,\mathrm{d}\widetilde{\nu}(t,j) \eqsim_{p,q} \|\widetilde{f}\|_{\hat{\mathcal{S}}_{q'}^{p',\widetilde{\nu}}} = \|f\|_{\hat{\mathcal{S}}_{q'}^{p'}}.\end{aligned}$$

Hence, $\|f\|_{\hat{\mathcal{S}}_{q'}^{p'}} \eqsim_{p,q} \|f\|_{(\hat{\mathcal{S}}_q^p)^*}$.

Step 5: $(\hat{\mathcal{S}}_q^p)^* = \hat{\mathcal{S}}_{q'}^{p'}$. In Step 4, we proved that $\hat{\mathcal{S}}_{q'}^{p'} \hookrightarrow (\hat{\mathcal{S}}_q^p)^*$ isomorphically, so it remains to show that $(\hat{\mathcal{S}}_q^p)^* = \hat{\mathcal{S}}_{q'}^{p'}$. This identity follows from the same Radon–Nikodym argument that was presented in Step 3 of the proof of Theorem B.3. □

Corollary B.8. *Let $1 < p, q < \infty$. Then $\mathcal{I}_{p,q}^* = \mathcal{I}_{p',q'}$, where $\mathcal{I}_{p,q}$ is as defined in (5.35), and*

$$\|f\|_{\mathcal{I}_{p',q'}} \eqsim_{p,q} \|f\|_{\mathcal{I}_{p,q}^*}, \quad f \in \mathcal{I}_{p',q'}. \tag{B.9}$$

Proof. The result follows by combining Theorem B.3 (for $X = L^q(S)$), Theorem B.6 and (2.2). □

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