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Computational Section

On the algebraic Brauer classes on open degree four del Pezzo surfaces

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A B S T R A C T

We study the algebraic Brauer classes on open del Pezzo surfaces of degree 4. I.e., on the complements of geometrically irreducible hyperplane sections of del Pezzo surfaces of degree 4. We show that the 2-torsion part is generated by classes of two different types. Moreover, there are two types of 4-torsion classes. For each type, we discuss methods for the evaluation of such a class at a rational point over a p-adic field.

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1. Introduction

In this article, we systematically study the algebraic Brauer classes that may arise on an open del Pezzo surface of degree four. By which we mean the complement U of a geometrically irreducible hyperplane section of a del Pezzo surface of degree four. When U is defined over a field k, the algebraic part of the Brauer group of U is contained in the Galois cohomology group $H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Pic}\, U_{k^{\mathrm{sep}}})$. It equals that group, at least when k is a number field.

Let X be a del Pezzo surface of degree four and $U \subset X$ be an open degree four del Pezzo surface. Then, on X, there are exactly 16 lines, which generate the geometric Picard lattice $\mathrm{Pic}\, X_{k^{\mathrm{sep}}}$, and therefore $\mathrm{Pic}\, U_{k^{\mathrm{sep}}}$. The automorphism group of the geometric Picard lattice is isomorphic to the Weyl group $W(D_5) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ of order 1920. We thus have to study the cohomology groups $H^1(G, \mathfrak{P})$, for $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ any subgroup and $\mathfrak{P}$ the $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$-module $\mathrm{Pic}\, U_{k^{\mathrm{sep}}}$, which is the same for all open del Pezzo surfaces of degree four.

The group $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ has 197 conjugacy classes of subgroups [BCP]. Among them, there are five maximal subgroups, which are of indices 2, 5, 6, 10, and 16, respectively. The first four of these are just the preimages under the natural projection $p\colon (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5 \to S_5$ of the maximal subgroups A_5, S_4, $\mathrm{AGL}_1(\mathbb{F}_5)$, and $S_3 \times S_2$ of S_5. Note that there are, in fact, four conjugacy classes of subgroups of index 10 in $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, but only one of them, that of $p^{-1}(S_3 \times S_2)$, is maximal.

It has been known for a while as an experimental result that $H^1(G, \mathfrak{P})$ may only be $(\mathbb{Z}/2\mathbb{Z})^e$, for $0 \leq e \leq 4$, or $\mathbb{Z}/4\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^e$, for $0 \leq e \leq 2$ [JS,BL]. In Section 3 of this article, we give a formal proof, not relying on any computer work, for the fact that every class is annihilated by 4. Moreover, we classify the nontrivial classes that may arise. It turns out that $H^1(G, \mathfrak{P})_2$ is generated by classes of two different kinds, which we call Brauer classes of types I and II. A 2-torsion class of type I occurs when G is the index 5 subgroup, and every 2-torsion class of type I is a restriction of that. Similarly, a 2-torsion class of type II occurs when G is the maximal index 10 subgroup, and every 2-torsion class of type II is its restriction. Moreover, there are two types of 4-torsion classes possible, which occur when G is contained in specific subgroups of orders 24 and 64, respectively.

The Brauer–Manin obstruction

Let k be a number field and let Σ_k denote the set of all its places. The Grothendieck–Brauer group is a contravariant functor from the category of schemes to the category of abelian groups. In particular, given a scheme U and a k_ν-rational point $x\colon\ \mathrm{Spec}\, k_\nu \to U$, for k_ν the completion of k with respect to any $\nu \in \Sigma_k$, there is the restriction homomorphism $x^*\colon\ \mathrm{Br}(X) \to \mathrm{Br}(k_\nu)$. For a Brauer class $\alpha \in \mathrm{Br}(U)$, one calls

$$\mathrm{ev}_{\alpha,\nu}\colon U(k_\nu) \longrightarrow \mathbb{Q}/\mathbb{Z}\,, \quad x \mapsto \mathrm{inv}(x^*(\alpha))\,,$$

the *local evaluation map,* associated with α. Here, $\mathrm{inv}\colon \mathrm{Br}(k_\nu) \hookrightarrow \mathbb{Q}/\mathbb{Z}$ denotes the invariant map, as usual. The local evaluation map $\mathrm{ev}_{\alpha,\nu}$ is continuous with respect to the ν-adic topology on $U(k_\nu)$ [Ja, Chapter IV, Proposition 2.3.a.ii)].

Let U be a separated scheme that is smooth and of finite type over k and let $\mathscr{U}$ be a *model* of U (cf. Definition 5.1) over the ring $\mathscr{O}_k$ of integers in k. The choice of the model $\mathscr{U}$ determines a set $\mathscr{U}(\mathscr{O}_k)$ of $\mathscr{O}_k$*-integral* points, together with a canonical injection $\mathscr{U}(\mathscr{O}_k) \hookrightarrow U(k)$. Similarly, $\mathscr{U}$ determines a set $\mathscr{U}(\mathscr{O}_{k,\nu}) \hookrightarrow U(k_\nu)$ of $\mathscr{O}_{k,\nu}$*-integral* points, for every non-archimedean place ν, as well as a set $\mathscr{U}(\mathscr{O}_{k,S}) \hookrightarrow U(k)$ of S*-integral* points, for $S \subset \Sigma_k$ any finite set of places that includes all archimedean ones. Here, by $\mathscr{O}_{k,\nu} \subset k_\nu$, we denote the ring of all integral elements in k_ν, and by $\mathscr{O}_S \subset k$ the ring of all elements that are integral outside S (cf. [CX, §1]).

It is well known that, for every $\alpha \in \mathrm{Br}(U)$, there exists a finite set $S_{\mathscr{U},\alpha} \subset \Sigma_k$ [Sk, §5.2] of places such that the restriction

$$\mathrm{ev}_{\alpha,\nu}|_{\mathscr{U}(\mathscr{O}_{k,\nu})}\colon \mathscr{U}(\mathscr{O}_{k,\nu}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is the zero map for each $\nu \in \Sigma_k \setminus S_{\mathscr{U},\alpha}$. Consequently, for the set of all S-integral points, one has the inclusions (cf. [CX, §1])

$$\mathscr{U}(\mathscr{O}_{k,S}) \subset \Big(\prod_{\nu\in S} U(k_\nu) \times \prod_{\nu\in\Sigma_k\setminus S} \mathscr{U}(\mathscr{O}_{k,\nu})\Big)^{\mathrm{Br}(U)} \subset \prod_{\nu\in S} U(k_\nu) \times \prod_{\nu\in\Sigma_k\setminus S} \mathscr{U}(\mathscr{O}_{k,\nu}),$$

for

$$\Big(\prod_{\nu\in S} U(k_\nu) \times \prod_{\nu\in\Sigma_k\setminus S} \mathscr{U}(\mathscr{O}_{k,\nu})\Big)^{\mathrm{Br}(U)} :=$$
$$\Big\{(x_\nu)_\nu \in \prod_{\nu\in S} U(k_\nu) \times \prod_{\nu\in\Sigma_k\setminus S} \mathscr{U}(\mathscr{O}_{k,\nu}) \;\Big|\; \forall\alpha \in \mathrm{Br}(U)\colon \sum_{\nu\in\Sigma_k} \mathrm{ev}_{\alpha,\nu}(x_\nu) = 0\Big\}.$$

Here, $S \subset \Sigma_k$ may be any finite set of places including all archimedean ones.

This phenomenon is called the Brauer–Manin obstruction and it is, at least from our point of view, the most important application of the Brauer group. In the form described here, for S-integral points, it is due to J.-L. Colliot-Thélène and F. Xu. In [CX], many classical counterexamples to the integral Hasse principle or strong approximation (cf. [DW, Definition 2.3] or [PR, §7.1]) off certain primes were explained by the Brauer–Manin obstruction. In another direction, the Brauer-Manin obstruction has been pursued by J.-L. Colliot-Thélène and O. Wittenberg [CW] for families of affine cubic surfaces, such as the representation problem of an integer by the sum of three cubes. Moreover, we advise the reader to consult the work of J. Berg [Be] on affine Châtelet surfaces and that of M. Bright and J. Lyczak [BL] concerning certain $\log K3$ surfaces. Let us note at this point that open del Pezzo surfaces of degree four, too, form a particular type of $\log K3$ surfaces.

For k-rational points in the case of a proper scheme over a number field k, the Brauer–Manin obstruction is due to Yu. I. Manin [Ma, Chapter VI] and takes the form that a nontrivial Brauer class may exclude certain adelic points from being approximated by k-rational points. In particular, the Brauer–Manin obstruction may explain counterexamples to the Hasse principle.

Methods for explicit evaluation

The two final sections of this article are devoted to methods for the evaluation of the Brauer classes on open degree four del Pezzo surfaces. In doing so, we distinguish between the four types.

First of all, 2-torsion classes of type I have been evaluated before [JS]. These are, in fact, *cyclic* classes, which means that there exists a normal subgroup $G' \subset G$ with cyclic quotient G/G' such that the class vanishes under restriction to $H^1(G', \mathfrak{P})$. In this case, Manin's original [Ma, §45] class field theoretic approach to the evaluation applies. A 2-torsion class of type I occurs when one of the degenerate quadrics in $\mathbf{P}^4$ defining U is k-rational and the corresponding two linear systems of conics are interchanged by $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. For these concepts related to the geometry of U, cf. Fig. 1 below and the discussion prior to it.

We show that 4-torsion classes of type I are cyclic, too. They allow a beautiful geometric interpretation, as well. There is a quadrilateral responsible for such a class, which is cut out of U by a k-rational hyperplane. The edges of this quadrilateral are acted upon cyclically by $\mathrm{Gal}(k^{\mathrm{sep}}/k)$.

However, it happens that 2-torsion classes of type II are usually non-cyclic, a phenomenon that has no analogue, e.g., for proper cubic surfaces [EJ10] and shows that the present case is more difficult. Worse, $H^1(G, \mathfrak{P})_2$ does not even need to be generated by cyclic classes. Nonetheless, the explicit evaluation of 2-torsion classes of type II is not hard. Indeed, they turn out to be corestrictions of 2-torsion classes of type I.

The generic algorithm

Finally, the 4-torsion classes of type II are the most mysterious to us. For their evaluation, in general, only a generic algorithm helps. We describe such an algorithm (Algorithm 6.14) as well as our implementation in the final section of this article. It seems to us that nothing of this kind has ever been tried before, although similar ideas occur in the work of T. Preu [Pr].

The basic idea is to describe the Brauer class under consideration by an explicit 2-cocycle c with values in the function field $l(U)^*$, for l the field of definition of the 16 lines. In order to evaluate at a point ξ over a p-adic field k_ν, c has to be restricted, at first. The result is a 2-cocycle $c_{\nu,\xi}$ describing a Brauer class of k_ν.

It is, however, not entirely obvious how to compute from $c_{\nu,\xi}$ the invariant as an element in $\mathbb{Q}/\mathbb{Z}$. The difficulty occurs when the 2-cocycle $c_{\nu,\xi}$ does not come via inflation from a cyclic quotient of $\mathrm{Gal}(\overline{k}_\nu/k_\nu)$. We solve this problem by a computer-algebraic approach, cf. Algorithm 6.12. In the case when the groups involved are sufficiently close to being cyclic, such as dihedral groups, it is known that the same may be achieved essentially without using a computer, cf. [Pr, Example 4.4] or [Be, §4].

Convention. Throughout this article, we assume that k is a field of characteristic $\neq 2$.

2. The Picard group and its automorphism group

Generalities on del Pezzo surfaces of degree four

A del Pezzo surface is a non-singular, proper algebraic surface X over a field k with an ample anti-canonical sheaf $\mathscr{K}^{-1}$. Every non-singular complete intersection of two quadrics in $\mathbf{P}^4$ is del Pezzo, according to the adjunction formula, and clearly of degree four [Ha, §I.7]. The converse is true, as well. For every del Pezzo surface of degree four, its anticanonical image is the complete intersection of two quadrics in $\mathbf{P}^4$ [Do, Theorem 8.6.2].

Thus, associated with a degree four del Pezzo surface X, there is a pencil $(\lambda q_1 + \mu q_2)_{(\lambda:\mu)\in\mathbf{P}^1}$ of quadrics in $\mathbf{P}^4$, the base locus of which is X. This pencil is uniquely determined up to an automorphism of $\mathbf{P}^4$. It contains exactly five degenerate fibres, each of which is exactly of rank 4 [Wi, Proposition 3.26.iv)].

Over an algebraically closed field, a degree four del Pezzo surface contains exactly 16 lines, which generate the Picard group. The same is still true if the base field is only separably closed [Va, Theorem 1.6]. Of more fundamental importance for us, however, are the ten one-dimensional linear systems of conics [VAV, §2.3], which are obtained as follows. Take the five degenerate quadrics in the pencil, which, again, are defined over k as soon as k is only separably closed [Wi, Proposition 3.26.ii)]. As they are of rank 4, each of them contains two one-dimensional linear systems of planes. Intersecting with a second quadric from the pencil, one finds one-dimensional linear systems of conics. In particular, the ten linear systems of conics naturally break into five pairs. Numbering the degenerate quadrics from 1 to 5, we denote the linear systems of conics as follows.

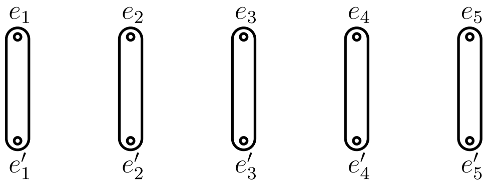

Fig. 1. The combinatorial structure of the ten linear systems of conics.

The picture might suggest that the automorphism group of $\operatorname{Pic} X$ is the wreath product $S_2 \wr S_5 = (\mathbb{Z}/2\mathbb{Z})^5 \rtimes S_5$. There is, however, an additional fine structure that reduces this group by a further index of 2.

Lemma 2.1 *(Automorphisms of the Picard lattice). Let X be a del Pezzo surface of degree four over a separably closed field.*

i) *Then the group W of all automorphisms of* $\operatorname{Pic} X$ *respecting the canonical class K and the intersection pairing is isomorphic to the Weyl group $W(D_5) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$. Here, $(\mathbb{Z}/2\mathbb{Z})^4 \subset (\mathbb{Z}/2\mathbb{Z})^5$ means the subgroup*

$$\{(a_0, \ldots, a_4) \in (\mathbb{Z}/2\mathbb{Z})^5 \mid a_0 + \cdots + a_4 = 0\} \tag{1}$$

being acted upon by S_5 in the natural way.

ii) *The operation of W permutes the ten classes of the linear systems of conics faithfully, according to the natural embedding $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5 \hookrightarrow A_{10}$.*

Proof. The isomorphism $W(D_5) \cong (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ is well-known [Hu, §12] and the main part of assertion i) is [Ma, Theorem 23.9]. Observe that $\operatorname{Pic} X = \operatorname{Pic} X_{\overline{k}}$, as the latter is generated by the classes of the 16 lines, which occur in $\operatorname{Pic} X$ already. Assertion ii) is explained in [KST, pp. 8–10]. □

Remark 2.2. $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ is a subgroup of the wreath product $S_2 \wr S_5 = (\mathbb{Z}/2\mathbb{Z})^5 \rtimes S_5$ in a natural way, and we work in its standard imprimitive permutation representation $\iota\colon S_2 \wr S_5 \hookrightarrow S_{10}$ in degree 10 [DM, §2.6]. Note that Condition (1) yields that

$$(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5 = \iota^{-1}(A_{10})\,. \tag{2}$$

Facts 2.3. *Let X be a del Pezzo surface of degree four over a separably closed field.*

i) *Then, together with the canonical class $K = -H$, the classes of the linear systems $[e_1], \ldots, [e_5]$ generate a subgroup of index 2 in the Picard group. More precisely, after a suitable rearrangement of the type $e_i \leftrightarrow e'_i$, the group* $\operatorname{Pic} X \cong \mathbb{Z}^6$ *is freely generated by $[e_1], \ldots, [e_5]$, and $\frac{1}{2}(K - [e_1] - \cdots - [e_5])$.*

ii) *One has $[e_i] + [e'_i] = -K$, for $i = 1, \ldots, 5$.*

iii) *The classes in* $\operatorname{Pic} X$ *of the 16 lines are all the $\frac{1}{2}(-3K - [e_1^{(\prime)}] - \cdots - [e_5^{(\prime)}])$, in which $[e_j^{(\prime)}] \in \{[e_j], [e'_j]\}$, for $j = 1, \ldots, 5$, and the total number of terms $[e'_j]$ is even.*

Proof. i) is shown in [VAV, Proposition 2.2], while ii) is clear from the construction of the linear systems e_i and e'_i.

iii) Put $D := \frac{1}{2}(-3K - [e_1] - \cdots - [e_5])$. Then direct calculations show that $DK = \frac{1}{2}(-12 + 2 \cdot 5) = -1$ and $D^2 = \frac{1}{4}(36 - 6 \cdot 5 \cdot 2 + 2\binom{5}{2}) = -1$, such that the Riemann–Roch Theorem yields

$$h^0(X, D) + h^0(X, K - D) \geq 1 + \tfrac{1}{2}(D - K)D = 1\,.$$

However, $h^0(X, K - D) = 0$, since $(K - D)(-K) = -5$ although $(-K)$ is ample. Thus, D is represented by an effective curve of degree $D(-K) = 1$, a line. The operation of W sends D to an orbit of length 16, consisting exactly of the divisor classes described. □

2.4 *(The blown-up model).* a) A del Pezzo surface of degree four over a separably closed field is isomorphic to $\mathbf{P}^2$, blown up in five points $x_1, \ldots, x_5$ in general position [Va, Theorem 1.6]. In the blown-up model, the 16 lines are given as follows, cf. [Ma, Theorem 26.2].

i) E_i, for $i = 1, \ldots, 5$, the exceptional curve lying above the blow-up point x_i.
ii) L_{ij}, for $1 \leq i < j \leq 5$, the strict transform of the line through x_i and x_j. The class of L_{ij} in $\operatorname{Pic} X$ is $L - E_i - E_j$, for L the inverse image of the class of a general line in $\mathbf{P}^2$.
iii) C, the strict transform of the conic through all five blow-up points. The class of C in $\operatorname{Pic} X$ is $2L - E_1 - \cdots - E_5$.

b) In addition, the ten linear systems of conics are easily identified to lie in the classes

i) $[e_i] = L - E_i$ and
ii) $[e'_i] = 2L - E_1 - \cdots - E_5 + E_i$, for $i = 1, \ldots, 5$.

In fact, it is well known that lines through a point and conics through four points in general position form pencils of effective curves in $\mathbf{P}^2$. Furthermore, the intersection numbers with the hyperplane section $H = 3L - E_1 - \cdots - E_5$ are directly seen to be equal to 2, so that the curves are indeed conics. Moreover, the arrangement of the ten divisor classes given above is such that $[e_i] + [e'_i] = H = -K$, for $i = 1, \ldots, 5$.

c) One may now identify by a direct calculation the 16 lines in the above form $\frac{1}{2}(-3K - [e_1^{(\prime)}] - \cdots - [e_5^{(\prime)}])$ with those in the blown-up model. It turns out that

i) (No term $[e'_\cdot]$)

$$\tfrac{1}{2}(-3K - [e_1] - \cdots - [e_5]) = [C]\,,$$

ii) (Two terms $[e'_\cdot]$)

$$\tfrac{1}{2}\big(-3K - [e_1] - \cdots - [e_5] + ([e_i] - [e'_i]) + ([e_j] - [e'_j])\big) = [L_{ij}]\,,$$

for $1 \leq i < j \leq 5$, and
iii) (Four terms $[e'_\cdot]$)

$$\tfrac{1}{2}\big(-3K - [e'_1] - \cdots - [e'_5] + ([e'_i] - [e_i])\big) = [E_i]\,,$$

for $i = 1, \ldots, 5$.

Open del Pezzo surfaces of degree four

The open degree four del Pezzo surfaces, which are the subject of this note, are the following.

Definition 2.5. By an *open del Pezzo surface of degree four,* we mean the complement $U = X \setminus H$ of a geometrically irreducible hyperplane section $X \cap H$ of a del Pezzo surface X of degree four.

Convention. In order to have a clear terminology, from this point on, a del Pezzo surface of degree four in the usual sense is called a *proper* degree four del Pezzo surface.

Lemma 2.6. *Let $U = X \setminus H$ be an open del Pezzo surface of degree four over a separably closed field.*

i) *Then the classes of the linear systems $[e_1], \ldots, [e_5]$ freely generate a subgroup of index 2 in the Picard group $\operatorname{Pic} U$. The group $\operatorname{Pic} U$ itself is generated by $[e_1], \ldots, [e_5]$, and $\frac{1}{2}([e_1] + \cdots + [e_5])$.*
ii) *One has $[e_i] = -[e'_i]$, for $i = 1, \ldots, 5$.*
iii) *After a suitable rearrangement of the type $e_i \leftrightarrow e'_i$, the classes in $\operatorname{Pic} U$ of the 16 lines are all the $\frac{1}{2}(\pm[e_1] \pm \cdots \pm [e_5])$, where the total number of plus signs is even.*

Proof. As $X \cap H$ is geometrically irreducible, [Ha, Proposition II.6.5] shows that $\operatorname{Pic} U = \operatorname{Pic} X/\langle H\rangle = \operatorname{Pic} X/\langle K\rangle$. Thus, assertions ii) and iii) are immediate consequences of their counterparts formulated in Facts 2.3. Concerning i), the direct implication of Fact 2.3.i) is a sixth generator of the form $\frac{1}{2}\big((-1)^{i_1}[e_1] + \cdots + (-1)^{i_5}[e_5]\big)$, which, however, yields the same group. □

3. The Brauer group

A purely algebraic fact

The following elementary fact is repeatedly used in this section.

Lemma 3.1. *Let G be a finite group and M a G-module that is finitely generated and torsion-free as a $\mathbb{Z}$-module. Then an element $m \in M$ is G-invariant if and only if there exists some integer $c \neq 0$ such that $cm \in M$ is a norm,*

$$cm = \sum_{g \in G} g(m_0)$$

for some $m_0 \in M$.

Proof. Norms are clearly G-invariant. As M is supposed torsion-free, this implies that the elements described are G-invariant, too. On the other hand, the quotient $\widehat{H}^0(G, M) = M^G/N_G(M)$ of invariant elements modulo norms is known to be an abelian group that is annihilated by the order of G [AW, Corollary 6.1]. The claim immediately follows from this. □

Generalities

The cohomological Grothendieck–Brauer group $\mathrm{Br}(U)$ of a scheme U is, by definition [Gr, Remarque 2.7], the étale cohomology group $H^2_{\text{ét}}(U, \mathbb{G}_m)$. If U is defined over a field k then the Hochschild–Serre spectral sequence [SGA4, Exp. VIII, Proposition 8.4]

$$H^p(\mathrm{Gal}(k^{\mathrm{sep}}/k), H^q_{\text{ét}}(U_{k^{\mathrm{sep}}}, \mathbb{G}_m)) \Longrightarrow H^{p+q}_{\text{ét}}(U, \mathbb{G}_m)$$

yields a three-step filtration

$$0 \subseteq \mathrm{Br}_0(U) \subseteq \mathrm{Br}_1(U) \subseteq \mathrm{Br}(U)\,.$$

In the case that U is an open degree four del Pezzo surface (Definition 2.5), one has that $\Gamma(U_{k^{\mathrm{sep}}}, \mathbb{G}_m) = (k^{\mathrm{sep}})^*$, cf. [JS, Lemma 4.1]. Hence, $\mathrm{Br}_0(U)$ is a quotient group of $H^2(\mathrm{Gal}(k^{\mathrm{sep}}/k), (k^{\mathrm{sep}})^*) = \mathrm{Br}(k)$. It is true that $\mathrm{Br}_0(U) \cong \mathrm{Br}(k)$ if U has a k-rational point, or, in the case that k is a number field, an adelic point [Co, Proposition 1.3.4.1]. (Note that the assumption of projectivity made in [Co] is not used in the proof of this particular statement.)

The subquotient $\mathrm{Br}_1(U)/\mathrm{Br}_0(U) = \mathrm{Br}_1(U)/\mathrm{im}(\mathrm{Br}(k))$ is called the *algebraic* part of the Brauer group. It is, in general, isomorphic to

$$\ker d_2^{1,1}\colon H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Pic}\, U_{k^{\mathrm{sep}}}) \longrightarrow H^3(\mathrm{Gal}(k^{\mathrm{sep}}/k), \Gamma_{\text{ét}}(U_{k^{\mathrm{sep}}}, \mathbb{G}_m))\,,$$

for $d_2^{1,1}$ the differential in the Hochschild–Serre spectral sequence. Because of $\Gamma_{\text{ét}}(U_{k^{\mathrm{sep}}}, \mathbb{G}_m) = (k^{\mathrm{sep}})^*$, the right hand side simplifies to $H^3(\mathrm{Gal}(k^{\mathrm{sep}}/k), (k^{\mathrm{sep}})^*)$. Moreover, if k is a number field then, as a by-product of class field theory [Ta, section 11.4], it is known that the latter group vanishes.

Let us note here that open del Pezzo surfaces of degree four may well have *transcendental* Brauer classes [JS, Examples 6.1 and 7.1], i.e. such not contained in $\mathrm{Br}_1(U)$.

Facts 3.2. i) *The composition*

$$r\colon H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Pic}\, U_{k^{\mathrm{sep}}}) \longrightarrow \mathrm{Br}_1(U)/\mathrm{Br}_0(U) \xrightarrow{\overline{\mathrm{res}}} \mathrm{Br}(k(U))/\mathrm{im}\,\mathrm{Br}(k)$$

of the homomorphism defined by the spectral sequence with that induced by the restriction to the generic point has a more direct description as follows.

The homomorphism r factors via the quotient of

$$H^2(\mathrm{Gal}(k^{\mathrm{sep}}/k), k(U)^{\mathrm{sep}*}) \subset H^2(\mathrm{Gal}(k(U)^{\mathrm{sep}}/k(U)), k(U)^{\mathrm{sep}*}) = \mathrm{Br}(k(U))$$

modulo $\mathrm{im}\,\mathrm{Br}(k)$*, which itself is clearly contained in* $H^2(\mathrm{Gal}(k^{\mathrm{sep}}/k), k(U)^{\mathrm{sep}*}/k^{\mathrm{sep}*})$*. And the resulting homomorphism is the boundary map in cohomology associated with the short exact sequence*

$$0 \longrightarrow k(U)^{\mathrm{sep}*}/k^{\mathrm{sep}*} \longrightarrow \mathrm{Div}\, U_{k^{\mathrm{sep}}} \longrightarrow \mathrm{Pic}\, U_{k^{\mathrm{sep}}} \longrightarrow 0\,.$$

ii) *For example, when* U *is an open del Pezzo surface of degree four (Definition 2.5), the homomorphism* $H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Pic}\, U_{k^{\mathrm{sep}}}) \to \mathrm{Br}_1(U)/\mathrm{Br}_0(U)$ *is characterised by this description of* r.

Proof. i) is shown in [Li, §2]. Concerning ii), one just needs to note that the subsequent homomorphism $\overline{\mathrm{res}}$ is injective. This is an immediate consequence of the injectivity of $\mathrm{res}\colon \mathrm{Br}_1(U) \to \mathrm{Br}(k(U))$, which is true under some minor assumptions [Gr, Corollaire 1.8] that are clearly satisfied in our situation. □

The algebraic part of the Brauer group

The observations made in the previous section show that, as an abelian group, $\mathrm{Pic}\, U_{k^{\mathrm{sep}}}$ is always the same, independently of the concrete open del Pezzo surface U of degree four. Thus, let us fix once and for all that $\mathfrak{P}$ is the free abelian group of rank 5, generated by the symbols $[e_1], \ldots, [e_5]$, together with the element $\frac{1}{2}([e_1] + \cdots + [e_5])$, and let $\mathfrak{P}$ be acted upon by $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ in the natural way, S_5 permuting the indices and $(\mathbb{Z}/2\mathbb{Z})^4$ operating by reversing signs.

Notation 3.3. Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be any subgroup. Then we write $p\colon G \to S_5$ for the natural projection, and put $T := \ker p$ and $S := \mathrm{im}\, p$. One thus has $S \subseteq S_5$, $T \subseteq (\mathbb{Z}/2\mathbb{Z})^4$, and there is a short exact sequence $0 \to T \to G \to S \to 0$.

Notation 3.4 *(The submodule generated by the linear systems of conics).* We write P for the subgroup of $\mathfrak{P}$ generated by the symbols $[e_1], \ldots, [e_5]$. Then P is actually a sub-G-module of $\mathfrak{P}$, and one has the short exact sequence

$$0 \longrightarrow P \longrightarrow \mathfrak{P} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \tag{3}$$

of G-modules, in which $\mathbb{Z}/2\mathbb{Z}$ represents a module with trivial G-operation. The associated long exact sequence in cohomology reads

$$0 \to P^G \to \mathfrak{P}^G \to \mathbb{Z}/2\mathbb{Z} \stackrel{\delta}{\to} H^1(G,P) \stackrel{j}{\to} H^1(G,\mathfrak{P}) \to \mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z})\,. \tag{4}$$

Lemma 3.5. *Let* $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ *be an arbitrary subgroup. Then every element of* $H^1(G,P)$ *is annihilated by* 2.

Proof. *First step.* Inflation.
In the inflation-restriction sequence

$$0 \longrightarrow H^1(S,P^T) \longrightarrow H^1(G,P) \longrightarrow H^1(T,P)\,,$$

let us first consider the term $H^1(S,P^T)$ to the left. One has $P^T = \mathbb{Z}[X]$, for X a subset of $\{1,\ldots,5\}$ that is invariant under $S \subseteq S_5$, hence

$$P^T \cong \bigoplus_x \mathbb{Z}[S/S_x] \cong \bigoplus_x \mathrm{CoInd}_{S_x}^S \mathbb{Z}\,,$$

where the direct sum runs over a system of representatives of the S-orbits of X, and S_x denotes the stabiliser of x. Consequently, according to Shapiro's lemma,

$$H^1(S, P^T) = \bigoplus_x H^1(S, \mathrm{CoInd}_{S_x}^S \mathbb{Z}) = \bigoplus_x H^1(S_x, \mathbb{Z}) = 0\,.$$

Observe here that the operation of S_x on $\mathbb{Z}$ is trivial. Thus, $H^1(G, P)$ injects into $H^1(T, P)$ and it suffices to show that $H^1(T, P)$ is annihilated by 2.

Second step. Restriction.
As a T-module, P splits into the direct sum $P \cong \bigoplus_{i=1}^5 \mathbb{Z} \cdot [e_i]$, so that we only have to show that each $H^1(T, \mathbb{Z} \cdot [e_i])$ is annihilated by 2. But, clearly,

$$H^1(T, \mathbb{Z} \cdot [e_i]) = H^1(T/T_{[e_i]}, \mathbb{Z} \cdot [e_i])$$

for $T_{[e_i]} \subset T$ the stabiliser of $[e_i]$. As $\#(T/T_{[e_i]}) \leq 2$, the proof is complete. □

Theorem 3.6. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be an arbitrary subgroup. Then every element of $H^1(G, \mathfrak{P})$ is annihilated by* 4.

Proof. In the exact sequence (4), the term to the far right is obviously annihilated by 2, while $H^1(G, P)$ is annihilated by 2, according to Lemma 3.5. The assertion follows. □

Remark 3.7. This result has been known to us before as an experimental finding, just calculating $H^1(G, \mathfrak{P})$ in `magma` for all 197 conjugacy classes of subgroups of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ [JS, Remark 4.6.(i)]. Calculations of the same kind have recently been carried out more generally for open del Pezzo surfaces of an arbitrary degree $d \leq 7$ by M. Bright and J. Lyczak [BL].

2-*torsion*

Lemma 3.8. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be an arbitrary subgroup.*
a) *Then there are the natural isomorphisms*

i) $\iota_P\colon H^1(G, P) \stackrel{\cong}{\longleftarrow} (P/2P)^G/(P^G/2P^G)$ *and*
ii) $\iota_{\mathrm{Pic}}\colon H^1(G, \mathfrak{P})_2 \stackrel{\cong}{\longleftarrow} (\mathfrak{P}/2\mathfrak{P})^G/(\mathfrak{P}^G/2\mathfrak{P}^G)$.

Under ι_P and ι_{Pic}, the homomorphism $j\colon H^1(G, P) \to H^1(G, \mathfrak{P})_2$ (cf. (4)) goes over into the homomorphism $(P/2P)^G/(P^G/2P^G) \to (\mathfrak{P}/2\mathfrak{P})^G/(\mathfrak{P}^G/2\mathfrak{P}^G)$ induced by the inclusion $P \stackrel{\subset}{\longrightarrow} \mathfrak{P}$.
b) *For $G' \subseteq G$ another subgroup, the isomorphisms ι_P and ι_{Pic} are compatible with the restriction $\mathrm{res}^G_{G'}$, while the corestriction $\mathrm{cores}^G_{G'}$ goes over into the norm map.*

c) *The boundary homomorphism* $\delta\colon \mathbb{Z}/2\mathbb{Z} \to H^1(G, P)$ *(cf. (4)) satisfies*

$$\iota_P^{-1}(\delta(\overline{1})) = \overline{[e_1] + \cdots + [e_5]}\,.$$

Proof. a) By Lemma 3.5, $H^1(G, P)$ is annihilated by 2. Moreover, the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{\cdot 2} & P & \longrightarrow & P/2P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{P} & \xrightarrow{\cdot 2} & \mathfrak{P} & \longrightarrow & \mathfrak{P}/2\mathfrak{P} & \longrightarrow & 0
\end{array}$$

of short exact sequences induces the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & P^G & \xrightarrow{\cdot 2} & P^G & \longrightarrow & (P/2P)^G & \twoheadrightarrow & H^1(G, P) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathfrak{P}^G & \xrightarrow{\cdot 2} & \mathfrak{P}^G & \longrightarrow & (\mathfrak{P}/2\mathfrak{P})^G & \twoheadrightarrow & H^1(G, \mathfrak{P})_2
\end{array}$$

of long exact sequences in cohomology.

b) $\mathrm{res}^G_{G'}$, and $\mathrm{cores}^G_{G'}$ commute with arbitrary boundary homomorphisms, in particular with ι_P and ι_{Pic}. The assertion therefore follows from the naive description of restriction and corestriction on zeroth cohomology [Se, Chapitre VII, §7].

c) We observe that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{\subset} & \mathfrak{P} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \cdot 2} & & \downarrow{\scriptstyle \overline{1} \mapsto \overline{[e_1]+\cdots+[e_5]}} & & \\
0 & \longrightarrow & P & \xrightarrow{\cdot 2} & P & \longrightarrow & P/2P & \longrightarrow & 0
\end{array}$$

of short exact sequences is commutative. Therefore, $\delta(\overline{1}) = \iota_P(\overline{[e_1] + \cdots + [e_5]})$. □

As a consequence, we observe that all 2-torsion classes in $H^1(G, \mathfrak{P})$ are induced by $H^1(G, P)$, and hence can be obtained by only considering the submodule P of $\mathfrak{P}$ that is generated by $[e_1], \ldots, [e_5]$.

Corollary 3.9. *Let* $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ *be an arbitrary subgroup. Then the natural homomorphism* $j\colon H^1(G, P) \to H^1(G, \mathfrak{P})_2$ *is surjective.*

Proof. According to Lemma 3.8.a), we have to show that $(\mathfrak{P}/2\mathfrak{P})^G$ is generated by the images of $(P/2P)^G$ and $\mathfrak{P}^G$. For this, let $\alpha \in (\mathfrak{P}/2\mathfrak{P})^G$ be an arbitrary element and choose a representative $\widetilde{\alpha} \in \mathfrak{P}$. The assumption on α then means that $\sigma(\widetilde{\alpha}) - \widetilde{\alpha} \in 2\mathfrak{P}$, for each $\sigma \in G$.

First case. $\widetilde{\alpha} \in P$.

Without restriction, $\widetilde{\alpha} = c_1[e_1]+\cdots+c_5[e_5]$, for some $c_1, \ldots, c_5 \in \{0, 1\}$. Then one clearly has $\sigma(\widetilde{\alpha}) - \widetilde{\alpha} = d_1[e_1] + \cdots + d_5[e_5]$, for certain $d_1, \ldots, d_5 \in \{-2, -1, 0, 1\}$. Moreover, as σ operates as a signed permutation, $d_1 + \cdots + d_5$ is even. However, such an element can lie in $2\mathfrak{P}$ only when every coefficient is even, which implies that $\widetilde{\alpha}$ represents a class in $(P/2P)^G$.

Second case. $\widetilde{\alpha} \notin P$.
Without restriction, $\widetilde{\alpha} = c_1[e_1]+\cdots+c_5[e_5]$ for $c_1, \ldots, c_5 \in \{-\frac{1}{2}, \frac{1}{2}\}$. Then one has $\sigma(\widetilde{\alpha}) - \widetilde{\alpha} = d_1[e_1] + \cdots + d_5[e_5]$, for certain $d_1, \ldots, d_5 \in \{-1, 0, 1\}$. Such a linear combination lies in $2\mathfrak{P}$ when either every coefficient is even or every coefficient is odd. Therefore, unless $\sigma(\widetilde{\alpha}) - \widetilde{\alpha} = 0$ for each $\sigma \in G$ and hence $\widetilde{\alpha} \in \mathfrak{P}^G$, there must be some $\sigma \in G$ such that $\sigma(\widetilde{\alpha}) = -\widetilde{\alpha}$. Then, in particular, σ changes the parity of the number of coefficients of $\widetilde{\alpha}$ that are positive. This is, however, a contradiction, as elements of $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ operate as permutations followed by evenly many sign reversals. □

Definition 3.10. Let $O = \{i_1, \ldots, i_n\} \subseteq \{1, \ldots, 5\}$ be an S-orbit. Then the subset $\{[e_{i_1}], \ldots, [e_{i_n}], -[e_{i_1}], \ldots, -[e_{i_n}]\} \subset \mathfrak{P}$ is acted upon by G either transitively or in such a way that it has exactly two G-orbits. In the former case, we say that O is a *non-split* S-orbit. In the latter case, it is called a *split* S-orbit.

Theorem 3.11 *(Explicit description of $H^1(G, \mathfrak{P})_2$). Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be an arbitrary subgroup and let M be the free $\mathbb{Z}/2\mathbb{Z}$-module generated by the set of all $O_j = \{i_{j,1}, \ldots, i_{j,n_j}\}$, for $j = 1, \ldots, k$, the orbits of $S \subseteq S_5$.*

Then $H^1(G, \mathfrak{P})_2$ is the quotient of M modulo $(O_1 + \cdots + O_k)$ and the split S-orbits.

Proof. One clearly has $P/2P \cong \bigoplus_{i=1}^{5} \mathbb{Z}/2\mathbb{Z} \cdot [e_i]$, hence

$$(P/2P)^G \cong \bigoplus_{j=1}^{k} \mathbb{Z}/2\mathbb{Z} \cdot ([e_{i_{j,1}}] + \cdots + [e_{i_{j,n_j}}]) =: \bigoplus_{j=1}^{k} \mathbb{Z}/2\mathbb{Z} \cdot O_j \, .$$

On the other hand, by Lemma 3.1, P^G is generated by all elements obtained as the sum of a G-orbit. In the non-split case, such a sum vanishes. Otherwise, it is mapped to O_j under the projection to $(P/2P)^G$. Thus, Lemma 3.8.a.i) shows that $H^1(G, P)$ is the quotient of M modulo all split S-orbits. Corollary 3.9 together with Lemma 3.8.c) proves the claim. □

In particular, let us observe the consequence below.

Corollary 3.12. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be an arbitrary subgroup.*

a) *Then $H^1(G, \mathfrak{P}) \neq 0$ if and only if $S := p(G) \subseteq S_5$ is intransitive, and at least two of the orbits are non-split.*

In this case, $\iota_{\mathrm{Pic}}(\overline{[e_{i_1}] + \cdots + [e_{i_n}]}) \neq 0$ *whenever* $\{i_1, \ldots, i_n\}$ *is a non-split orbit.*

b) *In particular,* $H^1(G, \mathfrak{P}) \neq 0$ *implies that* S *is intransitive.*

Proof. a) By Theorem 3.6, $H^1(G, \mathfrak{P}) \neq 0$ is possible only when $H^1(G, \mathfrak{P})_2 \neq 0$. The explicit description of $H^1(G, \mathfrak{P})_2$ given in Theorem 3.11 therefore implies the claim.
b) is a direct consequence of a). □

Like all results in this section, the corollary above immediately translates into a result about open del Pezzo surfaces of degree four (cf. Definition 2.5). For example, part b) yields the following.

Theorem 3.13. *Let* $U \subset X = V(q_1, q_2) \subset \mathbf{P}^4$ *be an open del Pezzo surface of degree four over a field* k *and suppose that* $H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Pic}\, U_{k^{\mathrm{sep}}}) \neq 0$. *Then*

a) *the five degenerate quadrics in the pencil defining* X *are acted upon intransitively by the Galois group* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$.
b) *The binary quintic form* $\det(\lambda q_1 + \mu q_2) \in k[\lambda, \mu]$ *is reducible over* k.

Proof. One has $\mathrm{Pic}\, U_{k^{\mathrm{sep}}} \cong \mathfrak{P}$, according to Lemma 2.6.i). Moreover, the natural operation of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ yields a homomorphism $\mathrm{Gal}(k^{\mathrm{sep}}/k) \to (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, whose image we denote by G, and whose kernel by $\mathfrak{G}$. Then $\mathfrak{G}$ is a pro-finite group acting trivially on $\mathrm{Pic}\, U_{k^{\mathrm{sep}}}$, which is torsion-free. Hence, $H^1(\mathfrak{G}, \mathrm{Pic}\, U_{k^{\mathrm{sep}}}) = 0$. Thus, the inflation-restriction exact sequence [NSW, Proposition 1.6.6] shows that

$$H^1(G, \mathfrak{P}) = H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k)/\mathfrak{G}, \mathrm{Pic}\, U_{k^{\mathrm{sep}}}) \cong H^1(\mathrm{Gal}(k^{\mathrm{sep}}/k), \mathrm{Pic}\, U_{k^{\mathrm{sep}}})\,.$$

a) now immediately follows from Corollary 3.12 and b) is just a reformulation of a). □

2-*torsion classes of types I and II*

Definition 3.14. Let $\alpha \in H^1(G, \mathfrak{P})_2$ be a nonzero element.

i) Suppose that $i \in \{1, \ldots, 5\}$ is S-invariant. I.e., that $\{i\}$ is an S-orbit of length one. If $\alpha = \iota_{\mathrm{Pic}}(\overline{[e_i]})$ then we call α a 2-*torsion class of type I.*
ii) Analogously, if $\{i, j\} \subset \{1, \ldots, 5\}$ is an S-orbit of length two then we call $\alpha = \iota_{\mathrm{Pic}}(\overline{[e_i] + [e_j]})$ a 2-*torsion class of type II.*

Lemma 3.15. *Let* $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ *be an arbitrary subgroup. Then* $H^1(G, \mathfrak{P})_2$ *is generated by* 2-*torsion classes of types I and II.*

Proof. Relying on the explicit description of $H^1(G, \mathfrak{P})_2$, given in Theorem 3.11, one sees that an orbit of length 5 defines the zero class. Furthermore, orbits of lengths 3 and 4 may be replaced by the complementary S-invariant sets, which are of sizes 2 and 1, respectively. This completes the proof. □

Example 3.16. There may be an overlap between the two types. For example, let G be the cyclic group of order 4, generated by an element fixing $[e_4]$ and $[e_5]$ and operating as $[e_1] \mapsto [e_2] \mapsto -[e_1]$ and $[e_3] \mapsto -[e_3]$, otherwise (see Fig. 2).

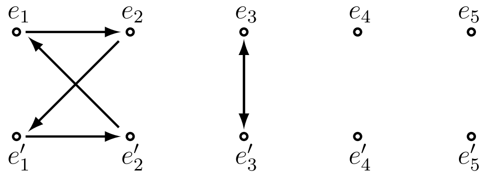

Fig. 2. The generator of the cyclic group G.

Then $H^1(G, \mathfrak{P})_2 \cong \mathbb{Z}/2\mathbb{Z}$ and a generator is provided by the element

$$\iota_{\mathrm{Pic}}(\overline{[e_3]}) = \iota_{\mathrm{Pic}}(\overline{[e_1] + [e_2]}),$$

which may be considered as being of type I, as well as II. Let us note that one has, in fact, $H^1(G, \mathfrak{P}) \cong \mathbb{Z}/2\mathbb{Z}$ in this example.

Corollary 3.17 *(A nontriviality criterion for type I). Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be an arbitrary subgroup and $\{i\} \subset \{1, \ldots, 5\}$ an S-orbit of length one that is non-split. I.e., such that $\sigma([e_i]) = -[e_i]$ for some $\sigma \in G$. Then $\iota_{\mathrm{Pic}}(\overline{[e_i]}) \neq 0$.*

Proof. According to Corollary 3.12, one has to show that there is a second non-split orbit in $\{1, \ldots, 5\}$. Assume, to the contrary, that all the remaining orbits would be split. Then an element $\sigma \in G$ as above operates on $P \subset \mathfrak{P}$ as $\sigma([e_i]) = -[e_i]$ and via cycles

$$[e_j] \mapsto \pm[e_{j_1}] \mapsto \pm[e_{j_2}] \mapsto \cdots \mapsto \pm[e_{j_{l-1}}] \mapsto [e_j]$$

where the $j, j_1, \ldots, j_{l-2}$, and j_{l-1} are all distinct and $\neq i$. Each such cycle, however, comes in a pair together with a second one that has all signs reversed. Hence, σ in total operates as an odd permutation on all the $\pm[e_1], \ldots, \pm[e_5]$, which is a contradiction to formula (2). □

Remark 3.18. Immediately from Definition 3.14, one has the following two observations.

i) The largest subgroup of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ that gives rise to a 2-torsion class of type I is $p^{-1}(S_4)$, i.e. that of index 5. This is clearly a maximal subgroup of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, of order 384.

ii) The largest subgroup of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ that gives rise to a 2-torsion class of type II is $p^{-1}(S_2 \times S_3)$, of index 10. This is a maximal subgroup of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, too, of order 192.

Remark 3.19. An experiment in `magma`, running in a loop over the 197 conjugacy classes of subgroups of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, shows that $H^1(G, \mathfrak{P})_2$ is isomorphic to

i) 0 in 59 cases,
ii) $\mathbb{Z}/2\mathbb{Z}$ in 71 cases,
iii) $(\mathbb{Z}/2\mathbb{Z})^2$ in 47 cases,
iv) $(\mathbb{Z}/2\mathbb{Z})^3$ in 17 cases, and
v) $(\mathbb{Z}/2\mathbb{Z})^4$ in three cases. These are the group $G = (\mathbb{Z}/2\mathbb{Z})^4$ and its subgroups of orders 8 and 4 that are still large enough not to split any of the pairs $\{[e_i], [e'_i]\}$, for $i = 1, \ldots, 5$. The order 4 group may be thought of as being generated by an element reversing the sign of $[e_i]$, for $i = 1, 2$, and that doing the same for $i = 2, 3, 4, 5$ (see Fig. 3).

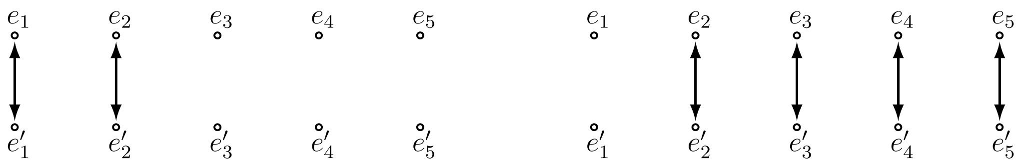

Fig. 3. Generators of the order 4 group.

4-*torsion*

Lemma 3.20. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be an arbitrary subgroup.*
a) *Then there is an isomorphism $\iota_{\mathrm{Pic},4} \colon H^1(G, \mathfrak{P}) \xleftarrow{\cong} (\mathfrak{P}/4\mathfrak{P})^G/(\mathfrak{P}^G/4\mathfrak{P}^G)$.*
b) *If $c \in \mathfrak{P}$ is G-invariant modulo $2\mathfrak{P}$ then $\iota_{\mathrm{Pic},4}(\overline{2c}) = \iota_{\mathrm{Pic}}(\overline{c})$. In particular, if c is G-invariant modulo $4\mathfrak{P}$ then $2\iota_{\mathrm{Pic},4}(\overline{c}) = \iota_{\mathrm{Pic}}(\overline{c})$.*

Proof. By Theorem 3.6, $H^1(G, \mathfrak{P})$ is annihilated by 4. Moreover, the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{P} & \xrightarrow{\cdot 2} & \mathfrak{P} & \longrightarrow & \mathfrak{P}/2\mathfrak{P} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \cdot 2} & & \downarrow{\scriptstyle \overline{\cdot 2}} & & \\ 0 & \longrightarrow & \mathfrak{P} & \xrightarrow{\cdot 4} & \mathfrak{P} & \longrightarrow & \mathfrak{P}/4\mathfrak{P} & \longrightarrow & 0 \end{array}$$

of short exact sequences induces the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{P}^G & \xrightarrow{\cdot 2} & \mathfrak{P}^G & \longrightarrow & (\mathfrak{P}/2\mathfrak{P})^G & \twoheadrightarrow & H^1(G,\mathfrak{P})_2 \\
 & & \| & & \downarrow{\cdot 2} & & \downarrow{\overline{\cdot 2}} & & \downarrow \\
0 & \longrightarrow & \mathfrak{P}^G & \xrightarrow{\cdot 4} & \mathfrak{P}^G & \longrightarrow & (\mathfrak{P}/4\mathfrak{P})^G & \twoheadrightarrow & H^1(G,\mathfrak{P})
\end{array}$$

of long exact sequences in cohomology. This immediately establishes part a) and the first assertion of b). Observe that $2c$ is G-invariant modulo $4\mathfrak{P}$, since c is G-invariant modulo $2\mathfrak{P}$. The second assertion of b) is a direct consequence. □

There are two obvious ways to explicitly construct a 4-torsion class.

3.21 *(4-Torsion classes of type I).* Suppose that $G \subseteq S_3 \times \mathbb{Z}/4\mathbb{Z}$, which occurs as a subgroup of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ in the manner below.
i) The generator $\overline{1} \in \mathbb{Z}/4\mathbb{Z}$ operates as $\tau\colon [e_4] \mapsto [e_5] \mapsto -[e_4] \mapsto -[e_5] \mapsto [e_4]$ and $[e_i] \leftrightarrow -[e_i]$, for $i = 1, 2, 3$.
ii) An element $\sigma \in S_3$ sends $[e_i]$ to $[e_{\sigma(i)}]$, for $i = 1, 2, 3$, and fixes $[e_4]$, as well as $[e_5]$.

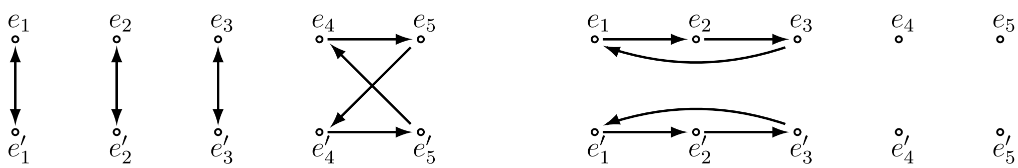

Fig. 4. The element τ and one of the possible elements $\sigma \in S_3$.

Observe that τ clearly commutes with the operation of S_3 and that every element of $S_3 \times \mathbb{Z}/4\mathbb{Z}$ indeed operates as an even permutation on the ten objects $[e_i^{(\prime)}]$ (see Fig. 4).

Proposition 3.22. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be a subgroup of the type described in 3.21.*
a) *Then the element $[e_1] + [e_2] + [e_3] + 2[e_4] \in \mathfrak{P}$ is invariant under G modulo $4\mathfrak{P}$. In particular, one has the class*

$$\alpha := \iota_{\mathrm{Pic},4}(\overline{[e_1] + [e_2] + [e_3] + 2[e_4]}) \in H^1(G, \mathfrak{P})\,.$$

b) *If G surjects onto $\mathbb{Z}/4\mathbb{Z}$ under the natural projection then α is a proper 4-torsion class.*

Proof. a) The element $[e_1] + [e_2] + [e_3] + 2[e_4] \in \mathfrak{P}$ remains fixed under S_3, while it is sent to $-[e_1] - [e_2] - [e_3] + 2[e_5]$ under τ. Hence,

$$\tau([e_1]+[e_2]+[e_3]+2[e_4]) - ([e_1]+[e_2]+[e_3]+2[e_4]) = -2[e_1]-2[e_2]-2[e_3]-2[e_4]+2[e_5]\,,$$

which is an element of $4\mathfrak{P}$.

b) Furthermore, by Lemma 3.20.b), we have

$$2\alpha = \iota_{\mathrm{Pic}}(\overline{[e_1] + [e_2] + [e_3] + 2[e_4]}) = \iota_{\mathrm{Pic}}(\overline{[e_1] + [e_2] + [e_3]}) = \iota_{\mathrm{Pic}}(\overline{[e_4] + [e_5]})\,.$$

In order to show that indeed $2\alpha \neq 0$, let $\tau' \in G$ be an element that is mapped under the canonical projection to a generator of $\mathbb{Z}/4\mathbb{Z}$. Then τ' alone makes sure that the S-orbit $\{4, 5\}$ is non-split. In addition, every S-orbit contained in $\{1, 2, 3\}$ is non-split, as well, so that, in particular, there is at least a second one. Corollary 3.12 shows that $\iota_{\mathrm{Pic}}(\overline{[e_4] + [e_5]}) \neq 0$. □

Definition 3.23. In the situation of Proposition 3.22.b), we call $\alpha \in H^1(G, \mathfrak{P})$ a 4-*torsion class of type I.*

3.24 *(4-torsion classes of type II).* Assume that $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ satisfies the following two conditions.
i) $S = p(G) \subset S_5$ stabilises 1. Moreover, S is contained in the dihedral group D_4 respecting the block [DM, §1.5] system $\{\{2, 4\}, \{3, 5\}\}$. I.e., for $\sigma \in G$, one either has $p(\sigma)(\{2, 4\}) = \{2, 4\}$ or $p(\sigma)(\{2, 4\}) = \{3, 5\}$.
ii) For an element $\sigma \in G$, its image $p(\sigma)$ interchanges the two blocks $\{2, 4\}$ and $\{3, 5\}$ if and only if $\sigma([e_1]) = -[e_1]$.

Examples 3.25.

a) Let G be the cyclic group of order 4 generated by the element depicted below. Then conditions i) and ii) are satisfied.

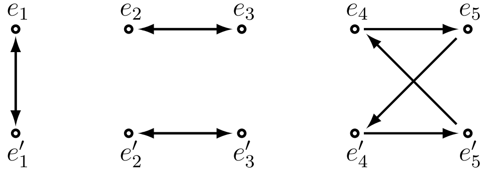

Fig. 5. The generator of the cyclic group G of order 4.

b) Take for G the cyclic group of order 8 generated by the following element. Then G satisfies conditions i) and ii).

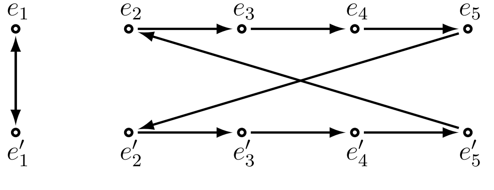

Fig. 6. The generator of the cyclic group G of order 8.

Proposition 3.26. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be a subgroup of the type described in 3.24.*

a) *Then the element $[e_1] + 2[e_2] + 2[e_4] \in \mathfrak{P}$ is invariant under G modulo $4\mathfrak{P}$. In particular, one has the class*

$$\alpha := \iota_{\mathrm{Pic},4}(\overline{[e_1] + 2[e_2] + 2[e_4]}) \in H^1(G, \mathfrak{P})\,.$$

b) *If $\sigma([e_1]) = -[e_1]$ for some $\sigma \in G$ then α is a proper 4-torsion class.*

Proof. a) Let $\sigma \in G$ be an arbitrary element. If σ fixes $[e_1]$ then $p(\sigma)$ leaves the two blocks in place. In this case, $\sigma([e_1] + 2[e_2] + 2[e_4]) = [e_1] \pm 2[e_2] \pm 2[e_4]$, so that

$$\sigma([e_1] + 2[e_2] + 2[e_4]) - ([e_1] + 2[e_2] + 2[e_4])$$

is clearly divisible by 4. Otherwise, $\sigma([e_1]+2[e_2]+2[e_4]) = -[e_1]\pm 2[e_3]\pm 2[e_5]$ and therefore

$$\sigma([e_1] + 2[e_2] + 2[e_4]) - ([e_1] + 2[e_2] + 2[e_4]) = -2[e_1] - 2[e_2] \pm 2[e_3] - 2[e_4] \pm 2[e_5]\,,$$

which is an element of $4\mathfrak{P}$, too.

b) Lemma 3.20.b) shows that $2\alpha = \iota_{\mathrm{Pic}}(\overline{[e_1] + 2[e_2] + 2[e_4]}) = \iota_{\mathrm{Pic}}(\overline{[e_1]})$, which is nonzero according to the criterion given in Corollary 3.17. □

Definition 3.27. In the situation of Proposition 3.26.b), we call $\alpha \in H^1(G, \mathfrak{P})$ a *4-torsion class of type II.*

Occurrence of 4-torsion

Convention. We say that a proper 4-torsion class in $H^1(G, \mathfrak{P})$ is *of type I* or *II,* if it is so after a suitable permutation of the indices $1, \ldots, 5$. In this case, we also say that the subgroup $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ is *of type I* or *II.*

Remark 3.28 *(Occurrence of type I).* a) There are exactly six conjugacy classes of subgroups of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ that lead to a 4-torsion class of type I. These correspond one-to-one to the six conjugacy classes of subgroups of $S_3 \times \mathbb{Z}/4\mathbb{Z}$ that surject onto $\mathbb{Z}/4\mathbb{Z}$ under the natural projection.

i) Among these, there are the groups of the form $T \times \mathbb{Z}/4\mathbb{Z}$, for $T \subseteq S_3$ any of the four conjugacy classes of subgroups.
ii) A fifth conjugacy class is represented by the cyclic subgroup of order four being contained in $S_2 \times \mathbb{Z}/4\mathbb{Z} \subset S_3 \times \mathbb{Z}/4\mathbb{Z}$ as the kernel of

$$S_2 \times \mathbb{Z}/4\mathbb{Z} \xrightarrow{(\mathrm{sgn},\mathrm{pr})} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{+} \mathbb{Z}/2\mathbb{Z}\,.$$

Letting S_2 act on $\{2,3\}$, a generator of this group operates as follows.

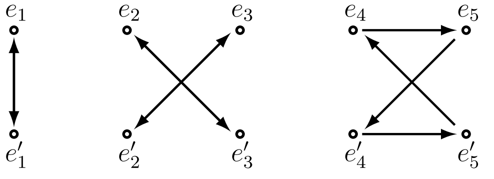

Fig. 7. A generator of a cyclic group of order 4.

iii) Similarly, the sixth conjugacy class is represented by the subgroup of order twelve being the kernel of

$$S_3 \times \mathbb{Z}/4\mathbb{Z} \xrightarrow{(\mathrm{sgn},\mathrm{pr})} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{+} \mathbb{Z}/2\mathbb{Z}\,.$$

As an abstract group, this is the dicyclic group Dic_3. In particular, it is non-isomorphic to $A_3 \times \mathbb{Z}/4\mathbb{Z}$, which is cyclic of order twelve.

b) The largest subgroup that yields a 4-torsion class of type I is isomorphic to $S_3 \times \mathbb{Z}/4\mathbb{Z}$. It is of order 24, thus of index 80 in $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, and contained in the maximal subgroup of index 10, but not in that of index 5.

Remarks 3.29 *(Occurrence of type II).* a) There are exactly eight conjugacy classes of subgroups of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ that give rise to a 4-torsion class of type II.
b) The largest of them is of order 64, thus of index 30 in $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$. In fact, condition 3.24.i) requires an index of at least 15, which is equivalent to saying that G be contained in a 2-Sylow subgroup. Furthermore, condition ii) enforces another index 2. It is contained in the maximal subgroup of index 5, but not in that of index 10.
The groups of type II are of orders 4, 8, 8, 16, 16, 32, 32, and 64, respectively. Each contains an element of order 4 as shown in Fig. 5 or an element of order 8 as shown in Fig. 6.

Theorem 3.30. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be a subgroup and assume that $H^1(G,\mathfrak{P})$ contains a proper 4-torsion element.*
a) *Then G is of type I or of type II.*
b) *The order four group of type II is of type I, as well. There is no further overlap between these two types.*
c) *One has $H^1(G,\mathfrak{P}) \cong \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^e$, for $e = 0$, 1, or 2. If G is of type II then $e \leq 1$.*
In particular, $H^1(G,\mathfrak{P})/H^1(G,\mathfrak{P})_2$ is of order 2, generated by α as in Proposition 3.22 (type I) or Proposition 3.26 (type II).

Proof. a) This is an experimental observation. Running a loop over all 197 conjugacy classes of subgroups of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, one finds 4-torsion in no case other than those described.

b) The element shown in Fig. 5 clearly coincides with that depicted in Fig. 7 after switching $[e_3]$ with $[e'_3]$. This makes the overlap visible.

Assume there would be another overlap. Then the corresponding group G must be a 2-group, which excludes all type I cases, except for the groups $T \times \mathbb{Z}/4\mathbb{Z}$, for T the trivial group or $\mathbb{Z}/2\mathbb{Z}$. In these situations, $S = p(G) \subseteq S_5$ is either a group of order 2, generated by a 2-cycle, or a group of order 4, generated by two disjoint 2-cycles. Neither of these occurs as a subgroup of D_4 that interchanges the two blocks.

c) It is again an experimental observation that $H^1(G, \mathfrak{P})$ contains only one direct summand $\mathbb{Z}/4\mathbb{Z}$, for any of the groups considered. This includes the overlap case, in which $H^1(G, \mathfrak{P})_2 \cong \mathbb{Z}/2\mathbb{Z}$ and therefore $H^1(G, \mathfrak{P}) \cong \mathbb{Z}/4\mathbb{Z}$.

In order to estimate the exponent e, we observe that $H^1(G, \mathfrak{P})_2 \cong (\mathbb{Z}/2\mathbb{Z})^{e+1}$ and that the number of S-orbits, $\{1, \ldots, 5\}$ decomposes into, is at most 4 for type I and at most 3 for type II. Hence, $e + 1 \leq 3$ and $e + 1 \leq 2$, respectively. □

Remarks 3.31. i) For type I, the groups $H^1(G, \mathfrak{P})$ occurring are $\mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ in the case of the naive order 4 group (i.e., T being the trivial group), $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in the case of the order 8 group (i.e., $T \cong \mathbb{Z}/2\mathbb{Z}$), and $\mathbb{Z}/4\mathbb{Z}$, otherwise.
ii) For type II, the group $H^1(G, \mathfrak{P})$ is $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ in two cases and $\mathbb{Z}/4\mathbb{Z}$, otherwise.

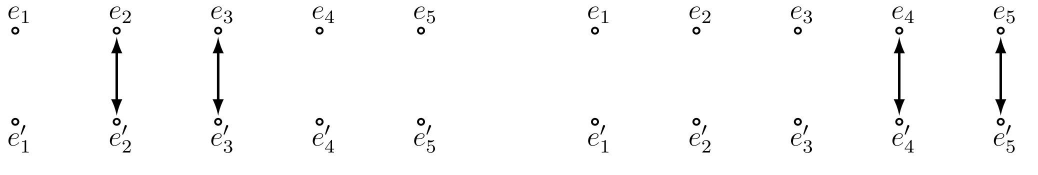

Fig. 8. Further generators that yield $H^1(G, \mathfrak{P}) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

The two groups that lead to $H^1(G, \mathfrak{P}) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ are of orders 8 and 16 and generated by the element of order 4 shown in Fig. 5, together with the element depicted to the left in Fig. 8 and, in the case of the order 16 group, that shown to the right.

4. Behaviour under restriction

Proper cubic surfaces–comparison with a well-known case
For $\mathfrak{P}_{\mathrm{cs}} \cong \mathbb{Z}^6$, the Picard group of a proper cubic surface acted upon by its automorphism group $W(E_6)$, there is the following fact.

Fact 4.1. *Let $G' \subseteq G \subseteq W(E_6)$ be subgroups such that neither $H^1(G, \mathfrak{P}_{\mathrm{cs}})$ nor $H^1(G', \mathfrak{P}_{\mathrm{cs}})$ vanishes. Then the restriction homomorphism*

$$\mathrm{res}^G_{G'}\colon H^1(G, \mathfrak{P}_{\mathrm{cs}}) \longrightarrow H^1(G', \mathfrak{P}_{\mathrm{cs}})$$

is injective.

Proof. For 2-torsion, this is [EJ10, Corollary 5.9.ii) and iii)], while, for 3-torsion, the same is shown in [EJ12, Corollary 3.19.ii) and iii)]. No higher torsion occurs in this case [SwD]. □

2-*torsion*

Taking Fact 4.1 as a guideline, in the situation of an open degree four del Pezzo surface (Definition 2.5), one may at least say the following.

Lemma 4.2. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be an arbitrary subgroup, $G' \subseteq G$ be another, put $S := p(G)$ and $S' := p(G')$, and let $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, 5\}$ be an S-orbit. Then $\{i_1, \ldots, i_k\}$ is an S'-invariant set, too, and one has*

$$\mathrm{res}^G_{G'}(\iota^G_{\mathrm{Pic}}(\overline{[e_{i_1}] + \cdots + [e_{i_k}]})) = \iota^{G'}_{\mathrm{Pic}}(\overline{[e_{i_1}] + \cdots + [e_{i_k}]})\,.$$

Proof. This is a direct consequence of the compatibility of restriction with the boundary homomorphisms. □

In particular, the two types of 2-torsion classes behave under restriction as follows.

i) The restriction of a 2-torsion class of type I is always a 2-torsion class of type I or 0.
ii) The restriction of a 2-torsion class of type II is a 2-torsion class of type II or a sum of two 2-torsion classes of type I. However, in the latter case, one of the summands or both may degenerate to 0.

These two observations are certainly a justification for the definition of the two types. For 4-torsion classes, there is a stronger result.

4-*torsion*

Theorem 4.3. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be a subgroup and $G' \subseteq G$ be another. Suppose that both $H^1(G, \mathfrak{P})$ and $H^1(G', \mathfrak{P})$ contain proper 4-torsion classes. Then the homomorphism*

$$\overline{\mathrm{res}}^G_{G'}\colon H^1(G, \mathfrak{P})/H^1(G, \mathfrak{P})_2 \longrightarrow H^1(G', \mathfrak{P})/H^1(G', \mathfrak{P})_2\,,$$

induced by restriction, is an isomorphism.

Proof. The groups are of order 2 on either side, so that we only have to show that the homomorphism is nontrivial. For that, let us recall the commutative diagram of exact sequences

$$\begin{array}{ccccc} H^1(G,P) & \longrightarrow & H^1(G,\mathfrak{P}) & \longrightarrow & H^1(G,\mathbb{Z}/2\mathbb{Z}) \\ \downarrow \text{res} & & \downarrow \text{res} & & \downarrow \text{res} \\ H^1(G',P) & \longrightarrow & H^1(G',\mathfrak{P}) & \longrightarrow & H^1(G',\mathbb{Z}/2\mathbb{Z}), \end{array}$$

which, according to Corollary 3.9, induces the commutative diagram

$$\begin{array}{ccccc} H^1(G,\mathfrak{P})/H^1(G,\mathfrak{P})_2 & \overset{i_G}{\hookrightarrow} & H^1(G,\mathbb{Z}/2\mathbb{Z}) & = & \mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z}) \\ \downarrow \overline{\text{res}} & & \downarrow \text{res} & & \\ H^1(G',\mathfrak{P})/H^1(G',\mathfrak{P})_2 & \overset{i_{G'}}{\hookrightarrow} & H^1(G',\mathbb{Z}/2\mathbb{Z}) & = & \mathrm{Hom}(G',\mathbb{Z}/2\mathbb{Z}). \end{array}$$

Thus, the assertion is true, unless G' is contained in the kernel of the nontrivial homomorphism $\chi\colon G \to \mathbb{Z}/2\mathbb{Z}$ in the image of i_G. In order to exclude this possibility, let us explicitly describe χ. We have to distinguish between the two types.

First case. G is of type I.
Then, after a suitable permutation of the indices, the situation is as in 3.21 and $H^1(G,\mathfrak{P})/H^1(G,\mathfrak{P})_2$ is generated by the class of $\alpha = \iota_{\mathrm{Pic},4}(\overline{[e_1]+[e_2]+[e_3]+2[e_4]})$. The homomorphism $\chi\colon G \to \mathbb{Z}/2\mathbb{Z}$ is then the composition of any 1-cocycle representing $\alpha \in H^1(G,\mathfrak{P})$ with the natural projection $\mathfrak{P} \twoheadrightarrow \mathfrak{P}/P \cong \mathbb{Z}/2\mathbb{Z}$. Calculating the boundary map $\delta = \iota_{\mathrm{Pic},4}\colon H^0(G,\mathfrak{P}/4\mathfrak{P}) \to H^1(G,\mathfrak{P})$ in the usual way, one finds that χ is the restriction to G of the homomorphism

$$S_3 \times \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}, \qquad \sigma\tau^i \mapsto (i \bmod 2), \text{ for any } \sigma \in S_3 .$$

Second case. G is of type II.
Here, after permuting indices if necessary, the situation is as described in 3.24 and $H^1(G,\mathfrak{P})/H^1(G,\mathfrak{P})_2$ is generated by the class of $\alpha = \iota_{\mathrm{Pic},4}(\overline{[e_1]+2[e_2]+2[e_4]})$. Calculating the boundary map $\delta = \iota_{\mathrm{Pic},4}\colon H^0(G,\mathfrak{P}/4\mathfrak{P}) \to H^1(G,\mathfrak{P})$ and reducing modulo P, one finds that $\chi\colon G \to \mathbb{Z}/2\mathbb{Z}$ is given by

$$G \to \mathbb{Z}/2\mathbb{Z}, \qquad \begin{cases} \sigma \to \overline{0} & \text{if } \sigma([e_1]) = [e_1], \\ \sigma \to \overline{1} & \text{if } \sigma([e_1]) = -[e_1]. \end{cases}$$

The assumption that $G' \subseteq \ker\chi$ therefore yields that one of the $[e_i]$ is completely fixed under G' or that $\{1,\ldots,5\}$ contains a split orbit of size three. Either conclusion is incompatible with the assumption that $H^1(G',\mathfrak{P})$ contains a proper 4-torsion element of either type. □

Thus, the restriction $\mathrm{res}^G_{G'}(\alpha)$ of a proper 4-torsion class α may be 0 or a proper 2-torsion class, only when $H^1(G',\mathfrak{P})$ does not contain any proper 4-torsion class. Otherwise, Propositions 3.22 and 3.26 imply that $\mathrm{res}^G_{G'}(\alpha)$ is a 4-torsion class of the same type as α.

Theorem 4.4 *(A nontriviality criterion for the restriction of a 4-torsion class). Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be a subgroup such that there is a proper 4-torsion class $\alpha \in H^1(G, \mathfrak{P})$ and let $G' \subseteq G$ be any subgroup.*
i) *Suppose that α is of type I. Then $G \subseteq S_3 \times \mathbb{Z}/4\mathbb{Z}$ and one has the natural surjection $s\colon G \twoheadrightarrow \langle\tau\rangle \cong \mathbb{Z}/4\mathbb{Z}$. The following statements are true.*

- *If $s(G') = \mathbb{Z}/4\mathbb{Z}$ then $\mathrm{res}^G_{G'}(\alpha)$ is a proper 4-torsion class.*
- *If $s(G') = 2\mathbb{Z}/4\mathbb{Z}$ then $\mathrm{res}^G_{G'}(\alpha)$ is a nontrivial 2-torsion class.*
- *If $s(G') = 0$ then $\mathrm{res}^G_{G'}(\alpha)$ is the zero class.*

ii) *If α is of type II, suppose that the indices $1, \dots, 5$ are normalised as in 3.24. Then,*

- *if G' leaves the orbit $\{1\}$ non-split then $\mathrm{res}^G_{G'}(\alpha)$ is a proper 4-torsion class.*
- *If G' splits the orbit $\{1\}$, but either of the sets $\{2,4\}$ and $\{3,5\}$ contains at least one non-split orbit then $\mathrm{res}^G_{G'}(\alpha)$ is a nontrivial 2-torsion class.*
- *Finally, if G' splits the orbit $\{1\}$ and either $\{2,4\}$ or $\{3,5\}$ consists entirely of split orbits then $\mathrm{res}^G_{G'}(\alpha)$ is the zero class.*

Proof. i) The first assertion follows directly from Proposition 3.22.b), together with Theorem 4.3. Otherwise, one has, assuming that the indices $1, \dots, 5$ are normalised as in 3.21, $\mathrm{res}^G_{G'}(\alpha) = \iota^{G'}_{\mathrm{Pic},4}(\overline{[e_1] + [e_2] + [e_3] + 2[e_4]}) = \iota^{G'}_{\mathrm{Pic},4}(2\overline{[e_4]})$. Indeed, in the situation that $s(G') \subsetneqq \mathbb{Z}/4\mathbb{Z}$, the orbits in $\{1,2,3\}$ are all split. Lemma 3.20.b) then shows that $\mathrm{res}^G_{G'}(\alpha) = \iota^{G'}_{\mathrm{Pic}}(\overline{[e_4]})$. Moreover, $\{4\}$ is a non-split orbit and $\{5\}$ is another, when $s(G') = 2\mathbb{Z}/4\mathbb{Z}$, while $\{4\}$ is a split orbit in the case that $s(G') = 0$. The assertion thus follows from Theorem 3.11.
ii) Here, the first claim directly follows from Proposition 3.26.b), together with Theorem 4.3. On the other hand, if $\{1\}$ is a split orbit then Lemma 3.20.b) shows, together with the compatibility of restriction with the boundary homomorphisms, that

$$\mathrm{res}^G_{G'}(\alpha) = \iota^{G'}_{\mathrm{Pic},4}(\overline{[e_1] + 2[e_2] + 2[e_4]}) = \iota^{G'}_{\mathrm{Pic},4}(\overline{2[e_2] + 2[e_4]}) = \iota^{G'}_{\mathrm{Pic}}(\overline{[e_2] + [e_4]})\,.$$

In the terminology of the explicit description of $H^1(G', \mathfrak{P})_2$, given in Theorem 3.11, $\iota^{G'}_{\mathrm{Pic}}(\overline{[e_2] + [e_4]})$ corresponds to $\overline{1} \cdot \{2,4\}$ or $\overline{1} \cdot \{2\} + \overline{1} \cdot \{4\}$, depending on whether $\{2,4\}$ forms one orbit under $S' := p(G')$ or two. The image of this element in the quotient modulo the split orbits is nonzero if and only if $\{2,4\}$ is a non-split orbit or, if it consists of two orbits, if at least one of them is non-split. Assume that this is the case. Then, in order to have $\iota^{G'}_{\mathrm{Pic}}(\overline{[e_2] + [e_4]}) \neq 0$, one needs, in addition, one further non-split orbit, which may only be $\{3,5\}$, $\{3\}$, or $\{5\}$, as required. □

5. Evaluation of the Brauer classes and examples

Convention. In this section and the next, k is always assumed to be a number field.

Definition 5.1. Let U be a separated scheme that is smooth and of finite type over k. Then by a *model* of U, we mean a separated scheme $\mathscr{U}$ that is of finite type over $\mathscr{O}_k$, the ring of integers in k, such that $\mathscr{U} \times_{\operatorname{Spec} \mathscr{O}_k} \operatorname{Spec} k \cong U$.

Caution. In this and the next sections, when considering local evaluation maps on U, we always assume that a model $\mathscr{U}$ is fixed. For the non-archimedean places, we only consider the restrictions $\operatorname{ev}_{\alpha,\nu}|_{\mathscr{U}(\mathscr{O}_{k,\nu})} \colon \mathscr{U}(\mathscr{O}_{k,\nu}) \longrightarrow \mathbb{Q}/\mathbb{Z}$. This is motivated by the results of J.-L. Colliot-Thélène and F. Xu [CX, §1] and the resulting application towards strong approximation. In particular, we do not care whether perhaps $\operatorname{ev}_{\alpha,\nu}$ is constant on the whole of $U(k_\nu)$, for certain non-archimedean places ν.

Example 5.2. Let $q_1, \ldots, q_r, h \in k[X_0, \ldots, X_N]$ be homogeneous forms such that $U := V(q_1, \ldots, q_r) \setminus V(h) \subset \mathbf{P}^N_k$ is a hypersurface complement of a complete intersection. Then, for forms $\widetilde{q}_1, \ldots, \widetilde{q}_r, \widetilde{h} \in \mathscr{O}_k[X_0, \ldots, X_N]$ that are just scalar multiples of $q_1, \ldots, q_r$, and h, respectively, the subscheme $\mathscr{U} := V(\widetilde{q}_1, \ldots, \widetilde{q}_r) \setminus V(\widetilde{h}) \subset \mathbf{P}^N_{\mathscr{O}_k}$ is a model of U.

Constancy of the local evaluation map

Let U be as above, $\alpha \in \operatorname{Br}(U)$, and $\mathscr{U}$ be a model of U. In such a situation, based on the fundamental results [Br07] of M. Bright, one may often indicate an explicit finite set $S_{\mathscr{U},\alpha}$ such that $\operatorname{ev}_{\alpha,\nu}|_{\mathscr{U}(\mathscr{O}_{k,\nu})} \colon \mathscr{U}(\mathscr{O}_{k,\nu}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is the zero map for each place $\nu \in \Sigma_k \setminus S_{\mathscr{U},\alpha}$.

Theorem 5.3 *(Constancy of the local evaluation map). Let X be an irreducible scheme being proper and smooth over k and having the property that $\operatorname{Pic}^0 X_{\overline{k}} = 0$, (i.e., that $\operatorname{Pic} X_{\overline{k}} = \operatorname{NS} X_{\overline{k}}$). Denote by l the field of definition of $\operatorname{NS} X_{\overline{k}}$.*

Moreover, let $\pi\colon B \twoheadrightarrow X$ be an $\mathbf{A}^n$-bundle, for some $n \geq 0$, and $U \subset B$ an open subscheme. Suppose that $\operatorname{Pic} U_{\overline{k}}$ is torsion-free.

Then, for each prime ideal $\mathfrak{p} \subset \mathscr{O}_k$ that is unramified in l, every model $\mathscr{U}$ of U that is smooth above $\mathfrak{p}$ and whose special fibre $\mathscr{U}_\mathfrak{p}$ is irreducible, and any algebraic Brauer class $\alpha \in \operatorname{Br}_1(U)$, the local evaluation map $\operatorname{ev}_{\alpha,\mathfrak{p}}|_{\mathscr{U}(\mathscr{O}_{k,\mathfrak{p}})} \colon \mathscr{U}(\mathscr{O}_{k,\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is constant.

Proof. One has $\operatorname{Pic} B_{\overline{k}} = \operatorname{Pic} X_{\overline{k}}$, according to [Ha, Exercise II.6.3.a)]. Consequently, $\operatorname{Pic} U_{\overline{k}}$ is a quotient of $\operatorname{Pic} X_{\overline{k}}$. In particular, $\operatorname{Pic} U_{\overline{k}}$ is acted upon trivially by $\operatorname{Gal}(\overline{k}/l)$, which shows that

$$\operatorname{Br}_1(U_l)/\operatorname{Br}_0(U_l) = H^1(\operatorname{Gal}(\overline{k}/l), \operatorname{Pic} U_{\overline{k}}) = \operatorname{Hom}(\operatorname{Gal}(\overline{k}/l), \operatorname{Pic} U_{\overline{k}}) = 0\,.$$

In particular, one has $\alpha|_{U_l} \in \operatorname{Br}_0(U_l)$. Let us fix a prime $\mathfrak{q}$ of l that lies above $\mathfrak{p}$. Then, after possibly adding a constant Brauer class, one may assume that $\alpha|_{U_{l_\mathfrak{q}}} = 0$.

Furthermore, according to our assumptions, $\mathscr{U}_{\mathscr{O}_{k,\mathfrak{p}}}$ fulfils M. Bright's [Br07, page 3] Condition (*). (Note that the arguments in [Br07] work over $\mathscr{O}_{k,\mathfrak{p}}$, as well as over $\mathbb{Z}_p$.)Therefore, [Br07, Propositions 6 and 3] show that the local evaluation map $\operatorname{ev}_{\alpha,\mathfrak{p}}|_{\mathscr{U}(\mathscr{O}_{k,\mathfrak{p}})}$ factors via $H^2(\operatorname{Gal}(l_\mathfrak{q}/k_\mathfrak{p}), \mathscr{O}^*_{l_\mathfrak{q}})$. The latter cohomology group, however, vanishes in the unramified case, according to [Se, Chap. V, §2]. □

Remark 5.4 *(Good reduction implies being unramified).* Let X be a surface and $\mathfrak{p} \subset \mathscr{O}_k$ be a prime ideal such that there is a model $\mathscr{X}$ of X that is proper over $\mathscr{O}_k$ and smooth above $\mathfrak{p}$. Then $\mathfrak{p}$ is unramified in the field of definition of $\operatorname{NS} X_{\overline{k}}$. Indeed, this follows from the smooth specialisation theorem for étale cohomology [SGA4, Exp. XVI, Corollaire 2.3]. Cf. [CEJ, Lemma 2.3.5] for a detailed argument.

Remarks 5.5. i) A proper del Pezzo surface of degree four fulfils the general assumptions made on X.

Thus, let X be a proper del Pezzo surface of degree four having a model $\mathscr{X}$ that is irreducible and has an irreducible special fibre $\mathscr{X}_\mathfrak{p}$. Moreover, let $\mathscr{U} \subset \mathscr{X}$ be an open subscheme that excludes the singular points of $\mathscr{X}_\mathfrak{p}$. Then the local evaluation map $\operatorname{ev}_{\alpha,\mathfrak{p}}|_{\mathscr{U}(\mathscr{O}_{k,\mathfrak{p}})}\colon \mathscr{U}(\mathscr{O}_{k,\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is constant, as long as $\mathfrak{p}$ is unramified in the field l of definition of $\operatorname{NS} X_{\overline{k}}$.
ii) A non-singular space quadric X fulfils the general assumptions, too. Having taken out the cusp, the cone above X is an $\mathbf{A}^1$-bundle, to which Theorem 5.3 applies. We make use of this in Corollary 5.15, below.

Corestriction
Let $U' \to U$ be a finite étale morphism of schemes. Then there is a natural corestriction homomorphism $\operatorname{cores}_U^{U'}\colon \operatorname{Br} U' \to \operatorname{Br} U$. In the case of affine schemes, a construction is described in [Sa, Chapter 8] and [Gre, Chapter II, §1]. As that commutes with arbitrary base change [Sa, Theorem 8.1.d)], the corestriction extends directly to the setting of general schemes.

Fact 5.6. *In the case of a finite [separable] field extension, one recovers the usual corestriction in Galois cohomology.*

Proof. This is [Gre, Chapter II, Proposition 1.6]. □

The case relevant to us is that U is an open del Pezzo surface of degree four over k (Definition 2.5) and $U' := U_l$ is a base extension of U. There are the following two results.

Lemma 5.7. *Let U be an open del Pezzo surface of degree four over k and l a finite extension field. Then the diagram*

$$\begin{array}{ccc} \operatorname{Br}_1(U_l)/\operatorname{Br}_0(U_l) & \xrightarrow{\overline{\text{cores}}} & \operatorname{Br}_1(U)/\operatorname{Br}_0(U) \\ \cong\uparrow & & \uparrow\cong \\ H^1(\operatorname{Gal}(\overline{k}/l), \operatorname{Pic} U_{\overline{k}}) & \xrightarrow{\text{cores}} & H^1(\operatorname{Gal}(\overline{k}/k), \operatorname{Pic} U_{\overline{k}}) \end{array}$$

commutes, in which the upper arrow is induced by corestriction, the lower arrow is the corestriction in Galois cohomology, and the upwards arrows are the isomorphisms induced by the Hochschild–Serre spectral sequence (cf. Section 3).

Proof. The diagram

$$\begin{array}{ccc} H^2(\mathrm{Gal}(\overline{k}/l), \overline{k}(U)^*/\overline{k}^*) & \xrightarrow{\text{cores}} & H^2(\mathrm{Gal}(\overline{k}/k), \overline{k}(U)^*/\overline{k}^*) \\ \uparrow & & \uparrow \\ H^1(\mathrm{Gal}(\overline{k}/l), \mathrm{Pic}\, U_{\overline{k}}) & \xrightarrow{\text{cores}} & H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}\, U_{\overline{k}}) \,, \end{array}$$

in which the upwards arrows are the boundary homomorphisms associated with the short exact sequence

$$0 \longrightarrow \overline{k}(U)^*/\overline{k}^* \longrightarrow \mathrm{Div}\, U_{\overline{k}} \longrightarrow \mathrm{Pic}\, U_{\overline{k}} \longrightarrow 0 \,,$$

commutes. Indeed, corestriction commutes with boundary homomorphisms. On the other hand, the natural diagram

$$\begin{array}{ccc} H^2(\mathrm{Gal}(\overline{k}/l), \overline{k}(U)^*/\overline{k}^*) & \xrightarrow{\text{cores}} & H^2(\mathrm{Gal}(\overline{k}/k), \overline{k}(U)^*/\overline{k}^*) \\ \uparrow & & \uparrow \\ H^2(\mathrm{Gal}(\overline{k}/l), \overline{k}(U)^*)/H^2(\mathrm{Gal}(\overline{k}/l), \overline{k}^*) & \xrightarrow{\overline{\text{cores}}} & H^2(\mathrm{Gal}(\overline{k}/k), \overline{k}(U)^*)/H^2(\mathrm{Gal}(\overline{k}/k), \overline{k}^*) \\ \uparrow & & \uparrow \\ \mathrm{Br}_1(U_l)/\,\mathrm{Br}_0(U_l) & \xrightarrow{\overline{\text{cores}}} & \mathrm{Br}_1(U)/\,\mathrm{Br}_0(U) \end{array}$$

commutes as well. For the lower square, this is Fact 5.6, while commutativity of the upper square is a particular case of the fact that corestriction is a morphism of functors. The assertion immediately follows from this, in view of the elementary descriptions of the isomorphism $H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}\, U_{\overline{k}}) \to \mathrm{Br}_1(U)/\,\mathrm{Br}_0(U)$ and its analogue over l, given in Facts 3.2.i) and ii). □

Lemma 5.8. *Let U be any scheme over k and l a finite extension field. Then, for every $\alpha \in \mathrm{Br}\, U_l$, every place ν of k, and every $x \in U(k_\nu)$, one has*

$$\mathrm{ev}_{\mathrm{cores}_U^{U_l}(\alpha),\nu}(x) = \sum_{w|\nu} \mathrm{ev}_{\alpha,w}(x_{l_w}) \,,$$

the sum running over all places w of l that lie above ν.

Proof. There is the Cartesian diagram

$$\begin{array}{ccc} \coprod\limits_{w|\nu} \mathrm{Spec}\, l_w & \xrightarrow{(x_{l_w})_w} & U_l \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k_\nu & \xrightarrow{x} & U \,. \end{array}$$

As the corestriction commutes with base change, $x^* \operatorname{cores}_U^{U_l}(\alpha) = \sum_{w|\nu} \operatorname{cores}_{k_\nu}^{l_w}(x_{l_w}^* \alpha)$. Consequently,

$$\operatorname{ev}_{\operatorname{cores}_U^{U_l}(\alpha),\nu}(x) = \operatorname{inv}_{k_\nu} x^* \operatorname{cores}_U^{U_l}(\alpha) = \sum_{w|\nu} \operatorname{inv}_{k_\nu} \operatorname{cores}_{k_\nu}^{l_w}(x_{l_w}^* \alpha) = \sum_{w|\nu} \operatorname{inv}_{l_w} x_{l_w}^* \alpha$$
$$= \sum_{w|\nu} \operatorname{ev}_{\alpha,w}(x_{l_w}) \,.$$

Here, the next-to-last equality is one of the fundamental properties of the invariant of a Brauer class of a local field [Se, Chapitre XI, §2, Proposition 1.ii)]. □

Cyclic classes
A class $c \in H^1(\operatorname{Gal}(\overline{k}/k), \operatorname{Pic} U_{\overline{k}})$ is called *cyclic* if there is a Galois extension l/k with cyclic Galois group such that the restriction of c to $H^1(\operatorname{Gal}(\overline{k}/l), \operatorname{Pic} U_{\overline{k}})$ vanishes. In the abstract setting of section 3, this means that a class in $H^1(G, \mathfrak{P})$ is cyclic if there is a normal subgroup $H \triangleleft G$ with cyclic quotient G/H such that the restriction to $H^1(H, \mathfrak{P})$ vanishes.

Proposition 5.9 *(Manin's class field theoretic method for evaluation). Let U be an open del Pezzo surface of degree four over k. Assume that U has an adelic point. Moreover, let $c \in H^1(\operatorname{Gal}(\overline{k}/k), \operatorname{Pic} U_{\overline{k}})$ be a cyclic class and let l/k be a cyclic Galois extension that annihilates c.*
a) *Then c is the image under inflation of a class $\widetilde{c} \in H^1(\operatorname{Gal}(l/k), \operatorname{Pic} U_l)$.*
b) *There is a natural isomorphism*

$$\iota\colon H^{-1}(\operatorname{Gal}(l/k), \operatorname{Pic} U_l) \longrightarrow [N_{l/k} \operatorname{Div} U_l \cap \operatorname{Div}_0 U]/N_{l/k} \operatorname{Div}_0 U_l \,,$$

where Div *denotes the Galois module of all divisors,* Div_0 *the submodule of principal divisors, and N the norm.*
c) *Fix a generator $\sigma \in \operatorname{Gal}(l/k)$ and put $c^- := \operatorname{cl}_\sigma \cup \widetilde{c} \in H^{-1}(\operatorname{Gal}(l/k), \operatorname{Pic} U_l)$, for $\operatorname{cl}_\sigma \in H^{-2}(\operatorname{Gal}(l/k), \mathbb{Z})$ the fundamental class. Moreover, let $f \in k(U)$ be a rational function representing a principal divisor in the residue class $\iota(c^-)$.*

Then there is a lift $\alpha \in \operatorname{Br}(U)$ of c such that, for every place ν of k and every $x \in U(k_\nu)$, at which f is defined and nonzero, $\operatorname{ev}_{\alpha,\nu}(x) = (\frac{i}{[l_w:k_\nu]} \bmod \mathbb{Z})$ if and only if

$$(f(x), l_w/k_v) = \sigma^i \,.$$

Here, w is any place of l lying above ν and $(.,l_w/k_v)$ denotes the norm residue symbol [Se, Chapitre XIII, §4].

Proof. a) A direct application of the inflation-restriction sequence yields that c is the image of a class from $H^1(\operatorname{Gal}(l/k), (\operatorname{Pic} U_{\overline{k}})^{\operatorname{Gal}(\overline{k}/l)})$. However, as U has an adelic point, one has $(\operatorname{Pic} U_{\overline{k}})^{\operatorname{Gal}(\overline{k}/l)} = \operatorname{Pic} U_l$, according to [Br02, Proposition 2.21].

b) Yu. I. Manin [Ma, Proposition 31.3] was the first, who came up with such an isomorphism. The non-proper case, which is not substantially different, is considered, for example, in [Ja, Chapter III, Lemma 8.19].

c) This, finally, is Manin's original method for the evaluation of a Brauer class [Ma, §45.2]. Cf. [Ja, Chapter IV, Section 4] for a few more details. □

Remarks 5.10. i) Under the canonical isomorphism

$$H^{-2}(\mathrm{Gal}(l/k), \mathbb{Z}) \xrightarrow{\cong} \mathrm{Gal}(l/k)^{\mathrm{ab}} = \mathrm{Gal}(l/k)\,,$$

the fundamental class cl_σ is that mapped to σ.
ii) A class in $H^{-1}(\mathrm{Gal}(l/k), \mathrm{Pic}\, U_l)$ is represented by an element $\mathscr{L} \in \mathrm{Pic}\, U_l$ of trivial norm. I.e., by a divisor $L \in \mathrm{Div}\, U_l$ whose norm is a principal divisor, $N_{l/k}L = \mathrm{div}\, f \in \mathrm{Div}_0\, U$. One then indeed has that

$$\iota(\overline{\mathscr{L}}) = \overline{\mathrm{div}\, f}\,.$$

This follows immediately from the construction of ι given in the proof of [Ma, Proposition 31.3].
iii) The ambiguity that the rational function f itself is determined by $\mathrm{div}\, f$ only up to a constant factor is absorbed by the possible choices of a lift of $c \in H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}\, U_{\overline{k}}) \cong \mathrm{Br}_1(U)/\operatorname{im}\mathrm{Br}(k)$ to $\mathrm{Br}_1(U)$.
iv) At a point x where f is undefined due to a pole or at which $f(x) = 0$, one can often nevertheless determine $\mathrm{ev}_{\alpha,\nu}(x)$ using the continuity of $\mathrm{ev}_{\alpha,\nu}$. Moreover, there exists a rational function $g \in l(U_l)$ such that $f \cdot N_{l/k}(g)$ is defined and nonzero at x. Indeed, U_l is a regular scheme and therefore the divisor $(-L)$ is locally principal.

2-torsion classes of type I

Lemma 5.11. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be a subgroup. Then*
i) a 2-torsion class $\iota_{\mathrm{Pic}}(\overline{[e_i]}) \in H^1(G, \mathfrak{P})$ of type I is always cyclic. In fact, let $G' \subseteq G$ be the stabiliser of $[e_i]$. Then $G/G' \cong \mathbb{Z}/2\mathbb{Z}$ and $\iota_{\mathrm{Pic}}(\overline{[e_i]})$ is the image under inflation of the class $\widetilde{c} \in H^1(G/G', \mathbb{Z}\cdot[e_i])$ of the 1-cocycle

$$\sigma \mapsto \begin{cases} 0, & \text{for } \sigma \in G/G' \text{ the neutral element}\,, \\ -[e_i], & \text{for } \sigma \in G/G' \text{ the nontrivial element}\,. \end{cases}$$

ii) *The class $\mathrm{cl}_\sigma \cup \widetilde{c} \in H^{-1}(G/G', \mathbb{Z}\cdot[e_i]))$ is represented by $-[e_i]$.*

Proof. i) Observe at first that $\mathbb{Z}\cdot[e_i] \subseteq \mathfrak{P}^{G'}$, according to the definition of G'. The assertion then follows from the explicit description of ι_{Pic}, given in Lemma 3.8.a.ii), together with a calculation in cocycles.
ii) is a consequence of i) and the explicit formula for the cup product, given in [Se, Chapitre XI, Annexe, Lemme 3]. □

5.12 *(Evaluation–planes tangent to a degenerate quadric).* Let $U \subset X = V(q_1, q_2)$ be an open del Pezzo surface of degree four over k (cf. Definition 2.5), on which there is an algebraic 2-torsion Brauer class α of type I. Then $\alpha = \iota_{\mathrm{Pic}}(\overline{[e_i]})$, for a certain linear system e_i of conics. In particular, one of the five degenerate quadrics in the pencil $(\lambda q_1 + \mu q_2)_{(\mu:\nu)\in \mathbf{P}^1}$, that containing the entire system e_i, is k-rational. Let us say that such a quadric *induces* the class α.

Without restriction, assume that $V(q_1)$ induces the Brauer class α. We then take a hyperplane $V(t)$ that is tangent to $V(q_1)$. Over a quadratic extension field $l = k(\sqrt{d})$, the intersection $V(q_1) \cap V(t)$ splits into two planes, so that $\operatorname{div} t$ decomposes into two components, a conic C_i from the class $[e_i]$ and its conjugate from $[e_i']$. Hence, $\operatorname{div} t = N_{l/k} C_i$ and $\operatorname{div} 1/t = N_{l/k}(-C_i)$. The evaluation of the Brauer class is therefore given by the norm residue symbols

$$\big(1/t(x), k_\nu(\sqrt{d})/k_\nu\big) = \big(t(x), k_\nu(\sqrt{d})/k_\nu\big)\,.$$

Note that when the splitting occurs already over k then $\{i\}$ is a split orbit and $\iota_{\mathrm{Pic}}(\overline{[e_i]}) = 0$, according to Theorem 3.11.

Remarks 5.13. i) (Some kind of normal form). One may write the rank-4 quadric in the form $l_1 l_2 - l_3^2 + d l_4^2$, for four linearly independent linear forms $l_1, \ldots, l_4$. The evaluation is then given by the norm residue symbols $(l_1(x), k_\nu(\sqrt{d})/k_\nu)$.
ii) This type of Brauer class has been evaluated before in exactly the same way. Cf., e.g., [JS, Example 8.1], where two linearly independent 2-torsion classes of type I are occurring. Furthermore, the next lemma shows that 2-torsion classes of type I are restrictions of Brauer classes on open rank 4 quadrics, which have been evaluated in the same way by J.-L. Colliot-Thélène and F. Xu [CX, §5.8], already.

Lemma 5.14. *Let $U = X \setminus H \subset X \subset \mathbf{P}^4_k$, for $H \subset \mathbf{P}^4_k$ a hyperplane, be an open del Pezzo surface of degree four over k (cf. Definition 2.5) having a 2-torsion class $\alpha \in \mathrm{Br}_1(U)/\mathrm{Br}_0(U)$ of type I. Suppose that $V(q) =: Q \supset X$ is a k-rational rank 4 quadric inducing the class α, and write $x_0 \in Q$ for its cusp.*

- a) *Then $\mathrm{Pic}(Q \setminus (H \cup \{x_0\}))_{\overline{k}} \cong \mathbb{Z}$.*
- b) *There is a class $\widetilde{\alpha} \in \mathrm{Br}_1(Q \setminus (H \cup \{x_0\}))/\mathrm{Br}_0(Q \setminus (H \cup \{x_0\}))$, the restriction to U of which is α.*

Proof. a) One has $\mathrm{Pic}(Q \setminus \{x_0\})_{\overline{k}} = \mathrm{Pic}\, Q'_{\overline{k}}$, for Q' the non-singular space quadric, the cone of which is $(Q \setminus \{x_0\})_{\overline{k}}$. Hence, $\mathrm{Pic}(Q \setminus \{x_0\})_{\overline{k}} = \mathbb{Z}^2$. Under this isomorphism, the class of the hyperplane section H is mapped to $(1, 1)$, which implies that $\mathrm{Pic}(Q \setminus (H \cup \{x_0\}))_{\overline{k}} = \langle e \rangle$, for e the class of one of the two linear systems of planes. Moreover, the generator e is of infinite order. Indeed, its restriction to U is the class of one of the ten linear systems of conics.

b) Let l be the quadratic extension field that splits the two linear systems of planes on Q. Then the nontrivial element of $\mathrm{Gal}(l/k)$ maps e to $(-e)$, so that

$$\sigma \mapsto \begin{cases} 0, & \text{if } \sigma \in \mathrm{Gal}(\overline{k}/k) \text{ induces the identity on } l\,, \\ -e, & \text{if } \sigma \in \mathrm{Gal}(\overline{k}/k) \text{ induces the conjugation on } l\,, \end{cases}$$

is a 1-cocycle defining a class $\widetilde{\alpha} \in H^1(\mathrm{Gal}(\overline{k}/k), \mathrm{Pic}(Q \setminus (H \cup \{x_0\}))_{\overline{k}})$. In view of Lemma 5.11.i), it is evident that α is the restriction of $\widetilde{\alpha}$ to U. □

Corollary 5.15. *Let q_1 and q_2 be two quadratic forms and h a linear form in five variables over $\mathscr{O}_k$, $\mathscr{X} = V(q_1, q_2) \subset \mathbf{P}^4_{\mathscr{O}_k}$ the associated scheme, and $X \subset \mathbf{P}^4_k$ its generic fibre. Suppose that $\mathscr{U} = \mathscr{X} \setminus V(h)$ is a model of an open del Pezzo surface U of degree four of the kind that there is a 2-torsion class $\alpha \in \mathrm{Br}_1(U)/\mathrm{Br}_0(U)$ of type I. Let $V(q_1) \supset X$ be a degenerate quadric that induces the class α.*

Let $\mathfrak{p} \subset \mathscr{O}_k$ be a prime ideal such that the reduction of q_1 modulo $\mathfrak{p}$ is of rank 4 and suppose that the cusp $x_0 \in V(q_1)_{\mathfrak{p}}$ is not a point on the special fibre $\mathscr{U}_{\mathfrak{p}}$. Then the local evaluation map $\mathrm{ev}_{\alpha,\mathfrak{p}}|_{\mathscr{U}(\mathscr{O}_{k,\mathfrak{p}})} \colon \mathscr{U}(\mathscr{O}_{k,\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is constant.

Proof. Let $\mathscr{C}$ be the Zariski closure in $\mathbf{P}^4_{\mathscr{O}_k}$ of the cusp on $V(q_1)_k$, put $\mathscr{W} := V(q_1) \setminus (V(h) \cup \mathscr{C})$, and let W be the generic fibre of $\mathscr{W}$. Then $\mathscr{W} \supset \mathscr{U}$. Moreover, according to Lemma 5.14, $\alpha \in \mathrm{Br}_1(U)$ is the restriction of an algebraic Brauer class $\widetilde{\alpha} \in \mathrm{Br}_1(W)$. Thus, for every $x \in \mathscr{U}(\mathscr{O}_{k,\mathfrak{p}})$, one has $\mathrm{ev}_\alpha(x) = \mathrm{ev}_{\widetilde{\alpha}}(x)$, so that it suffices to show that $\mathrm{ev}_{\widetilde{\alpha}}|_{\mathscr{W}(\mathscr{O}_{k,\mathfrak{p}})} \colon \mathscr{W}(\mathscr{O}_{k,\mathfrak{p}}) \to \mathbb{Q}/\mathbb{Z}$ is constant.

And this, in fact, is a direct consequence of Theorem 5.3. Indeed, for the generic fibre, one knows that $\big(V(q_1) \setminus \mathscr{C}\big)_k$ is an $\mathbf{A}^1$-bundle over a non-singular space quadric. Furthermore, according to Lemma 5.14.a), $\mathrm{Pic}\, W_{\overline{k}} \cong \mathbb{Z}$, which is torsion-free. Finally, by assumption, $\mathscr{W}$ is smooth above $\mathfrak{p}$ and its special fibre $\mathscr{W}_{\mathfrak{p}}$ is irreducible. □

2-torsion classes of type II

2-torsion classes of type II are, in general, non-cyclic. Indeed, let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be a subgroup and let $\alpha := \iota_{\mathrm{Pic}}(\overline{[e_i] + [e_j]})$ be such a class. The quotient group G' which G induces on the orbit $\{[e_i], [e_j], [e_i'], [e_j']\}$ [DM, §1.6] is a subgroup of the dihedral group D_4, generated by the two elements depicted in Fig. 9.

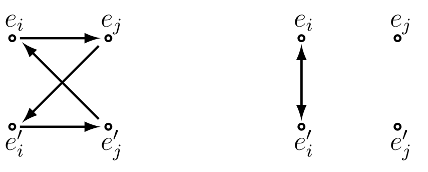

Fig. 9. Two generators for the dihedral group D_4.

When exactly this group occurs then there are two nontrivial subgroups of G' splitting the orbit. Either is of order 2. They are generated by the elements shown in Fig. 10.

One readily sees that the two are conjugate to each other in D_4, which shows in particular that they are non-normal. Thus, the only normal subgroup of D_4 that splits the orbit is the trivial group. However, D_4 is clearly not cyclic, not even abelian.

Fig. 10. Subgroups of D_4 splitting the orbit.

Lemma 5.16. *Let* $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ *and let* $\alpha := \iota_{\mathrm{Pic}}(\overline{[e_i]+[e_j]}) \in H^1(G, \mathfrak{P})$ *be a 2-torsion class of type II. Put* $H \subset G$ *to be the stabiliser of* i *under the operation of* $p(G)$ *on* $\{1, \ldots, 5\}$*. Then*

i) $H \subset G$ *is a subgroup of index* 2.
ii) *One has* $\alpha = \mathrm{cores}_H^G \, \iota_{\mathrm{Pic}}^H(\overline{[e_i]})$.

Proof. i) As $\iota_{\mathrm{Pic}}(\overline{[e_i]+[e_j]})$ is a 2-torsion class of type II, the subset $\{i, j\} \subset \{1, \ldots, 5\}$ is a G-orbit.
ii) This is an immediate consequence of Lemma 3.8.b). Indeed, $N_H^G(\overline{[e_i]}) = \overline{[e_i]+[e_j]}$, for $N_H^G \colon (\mathfrak{P}/2\mathfrak{P})^H \to (\mathfrak{P}/2\mathfrak{P})^G$ the norm map. □

5.17 *(Evaluation–corestriction).* Let $U \subset X = V(q_1, q_2)$ be an open del Pezzo surface of degree four over k (cf. Definition 2.5), on which there is an algebraic 2-torsion Brauer class α of type II. Then $\alpha = \iota_{\mathrm{Pic}}(\overline{[e_i]+[e_j]})$, for e_i and e_j two conjugate linear systems of conics. In particular, two of the five degenerate quadrics in the pencil $(\lambda q_1 + \mu q_2)_{(\lambda:\mu)\in \mathbf{P}^1}$, that containing the entire system e_i and that containing e_j, form a $\mathrm{Gal}(\overline{k}/k)$-orbit. Let us say that such a pair of quadrics *induces* the class α.

Without restriction, assume that the pair $\{V(q_1), V(q_2)\}$ induces α. Then, after scaling by constants, q_1 and q_2 are defined over a quadratic extension field k' and conjugate to each other. Take a hyperplane $V(t)$, for t a linear form defined over k', that is tangent to $V(q_1)$. Over a further quadratic extension $l = k'(\sqrt{d})$, the intersection $V(q_1) \cap V(t)$ then splits into two planes.

According to Lemma 5.16.ii) together with Lemma 5.7, the Brauer class α is the corestriction of the 2-torsion Brauer class on $U_{k'}$ induced by the degenerate quadric $V(q_1)$. Lemma 5.8 shows that the evaluation of the Brauer class at a place ν is hence given by

$$\sum_{w|\nu} (t(x), k'_w(\sqrt{d})/k'_w)\,.$$

Note that, as k'/k is a quadratic extension, the sum is just one summand when ν is inert or ramified in k'. In the split case, it has two summands.

Example 5.18. Let $X \subset \mathbf{P}^4_{\mathbb{Q}}$ be given by the system of equations

$$X_0^2 + 2X_0X_2 - 4X_0X_4 - 2X_1X_2 + 4X_1X_4 - X_1X_3 - X_2^2 = 0\,,$$
$$-2X_0X_4 + X_1X_4 + X_2X_3 - 2X_4^2 = 0\,,$$

$\mathscr{X} \subset \mathbf{P}^4_{\mathbb{Z}}$ the subscheme that is defined by the same system of equations as X, and put $U := X \setminus H$ and $\mathscr{U} := \mathscr{X} \setminus \mathscr{H}$, for $H := V(X_0) \subset \mathbf{P}^4_{\mathbb{Q}}$ and $\mathscr{H} := V(X_0) \subset \mathbf{P}^4_{\mathbb{Z}}$.

A point search using a variant of Elkies' method [El] provides almost 25 000 $\mathbb{Q}$-rational points of height up to 1000, among which 27 are $\mathbb{Z}$-integral, for example $(1:1:0:1:0)$, $(1:1:-2:-3:-2)$, and $(1:331:49:900:252)$.

The 16 lines on X are defined over the Galois hull of $l' := \mathbb{Q}\big(\sqrt{2\sqrt{5}+4}\big)$, which is a field of degree 8 with Galois group D_4. The second of the equations defining X gives rise to a quadric of rank 4, the linear systems of planes on which are $\mathbb{Q}$-rational. The other four degenerate quadrics in the pencil are defined over $k' = \mathbb{Q}(\sqrt{5})$ and form two length two orbits under the operation of $\mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q})$. Thus, having chosen the indices suitably, the absolute Galois group fixes $[e_1]$ and $[e_1']$, and has the two further orbits $\{[e_2],[e_2'],[e_3],[e_3']\}$ and $\{[e_4],[e_4'],[e_5],[e_5']\}$, on which it acts in a totally coupled manner. The operation on either orbit is via the full D_4.

Consequently, one has $\mathrm{Br}_1(U)/\mathrm{Br}_0(U) \cong \mathbb{Z}/2\mathbb{Z}$, generated by the 2-torsion Brauer class $\alpha = \iota_{\mathrm{Pic}}(\overline{[e_2]+[e_3]})$ of type II. Indeed, there is no proper 4-torsion occurring, as the operation of D_4 as given above does not coincide with any of the subgroups of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$, described in Remarks 3.28 and 3.29. Thus, the claim directly follows from Theorem 3.11.

One of the degenerate quadrics in the pencil defining X is

$$V\big((-\sqrt{5}-3)X_0^2 + (-2\sqrt{5}-6)X_0X_2 + (4\sqrt{5}+4)X_0X_4 + (2\sqrt{5}+6)X_1X_2 \\ + (\sqrt{5}+3)X_1X_3 + (-4\sqrt{5}-8)X_1X_4 + (\sqrt{5}+3)X_2^2 + 4X_2X_3 - 8X_4^2\big)\,.$$

The two linear systems of planes on this quadric are defined over l' and conjugate to each other under $\mathrm{Gal}(l'/\mathbb{Q}(\sqrt{5}))$. The tangent hyperplane at the point $(0:0:0:1:0)$ is $V\big((\sqrt{5}+3)X_1 + 4X_2\big)$. The evaluation of α at a place ν of $\mathbb{Q}$ is hence given by

$$\sum_{w|\nu} \big(\tfrac{(\sqrt{5}+3)X_1+4X_2}{X_0}, k'_w(\sqrt{2\sqrt{5}+4})/k'_w\big), \tag{5}$$

where w runs over the places of $k' = \mathbb{Q}(\sqrt{5})$ lying above ν.

The surface X has bad reduction only at the primes 2 and 5, whereas the unique prime above 5 in $\mathbb{Q}(\sqrt{5})$ splits in l'. On the other hand, there is exactly one prime of $\mathbb{Q}(\sqrt{5})$ lying above 2, too, and that actually ramifies in l'. Finally, there are two primes of $\mathbb{Q}(\sqrt{5})$ lying above the infinite place, but one of them splits in l'. Thus, the sum (5) is effectively only one term in either case, $\nu = 2$ or ∞. In particular, the local evaluation map $\mathrm{ev}_{\alpha,\infty}$ just tests the sign of the expression $\frac{(-\sqrt{5}+3)X_1+4X_2}{X_0}$.

An experiment shows that $\mathrm{ev}_{\alpha,2}$ takes both values on $\mathscr{U}(\mathbb{Z}_2)$ and that the same is true for $\mathrm{ev}_{\alpha,\infty}$ on $U(\mathbb{Q})$. Thus, strong approximation is violated. Both combinations $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$ occur, in fact, for $\mathbb{Z}$-integral points.

Remark 5.19 *(Some kind of normal form).* Let U be an open degree 4 del Pezzo surface over k carrying a 2-torsion Brauer class of type II. Then U is as a hyperplane complement of a projective surface that may be defined in the form

$$l_1 l_2 - l_3^2 + d l_4^2 = 0\,,$$
$$l_1^\sigma l_2^\sigma - (l_3^\sigma)^2 + d^\sigma (l_4^\sigma)^2 = 0\,,$$

for a quadratic extension field k', an element $d \in k' \setminus (k')^2$, and linearly independent linear forms $l_1, \dots, l_4$ over k'. Here, σ denotes the conjugation of k'. The evaluation is then given by $\sum\limits_{w|\nu} (l_1(x), k'_w(\sqrt{d})/k'_w)$.

Remark 5.20 *(The cyclic case).* There are clearly 2-torsion classes of type II that are cyclic. Just assume that the quotient group G' induced by G on the orbit $\{[e_i],[e_j],[e'_i],[e'_j]\}$ is cyclic of order 4. It is then generated by the 4-cycle σ depicted to the left in Fig. 9. In this situation, l/k is a cyclic field extension of degree 4 and k' is its intermediate quadratic field.

Moreover, the method for evaluation described in 5.17 may then be directly translated into class field theory. One easily sees that the evaluation of the Brauer class is given by the norm residue symbols

$$(\tfrac{1}{t(x)t^\sigma(x)}, l_w/k_\nu) = (t(x)t^\sigma(x), l_w/k_\nu)\,,$$

for w any place of l that lies above ν. Note that, although l/k is of degree 4, the concrete norm residue symbols above take no values other than σ^2 and the neutral element. Indeed, $t(x)t^\sigma(x)$ is a norm from the intermediate field k'.

4-torsion classes of type I

Lemma 5.21. *Let $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ be a subgroup of the kind that $H^1(G,\mathfrak{P})$ contains a 4-torsion class of type I. Then,*

i) *in the notation of 3.21, G allows a natural surjection $s\colon G \twoheadrightarrow \langle\tau\rangle \cong \mathbb{Z}/4\mathbb{Z}$ and $\iota_{\mathrm{Pic},4}(\overline{[e_1]+[e_2]+[e_3]+2[e_4]}) \in H^1(G,\mathfrak{P})$ is the image under inflation of a class $\widetilde{c} \in H^1(\langle\tau\rangle, \mathfrak{P}^{S_3})$. In particular, a 4-torsion class of type I is always cyclic. The class $\widetilde{c}$ is that of the [unique] 1-cocycle such that $\tau \mapsto \frac{-[e_1]-[e_2]-[e_3]-[e_4]+[e_5]}{2}$.*

ii) *The class $\mathrm{cl}_\tau \cup \widetilde{c} \in H^{-1}(\langle\tau\rangle, \mathfrak{P}^{S_3})$ is represented by $\frac{-[e_1]-[e_2]-[e_3]-[e_4]+[e_5]}{2}$.*

Proof. i) One clearly has $\ker s \subseteq S_3$, hence $\mathfrak{P}^{S_3} \subseteq \mathfrak{P}^{\ker s}$. The assertion itself follows from the description of $\iota_{\mathrm{Pic},4}$, given in Lemma 3.20.a), together with an explicit calculation in cocycles.

ii) is, once again, a consequence of i) and the explicit formula for the cup product, given in [Se, Chapitre XI, Annexe, Lemme 3]. □

5.22 *(Evaluation–k-rational quadrilaterals).* The conjugates of $\frac{-[e_1]-[e_2]-[e_3]-[e_4]+[e_5]}{2}$ under $\langle\tau\rangle$ are $\frac{[e_1]+[e_2]+[e_3]-[e_4]-[e_5]}{2}$, $\frac{-[e_1]-[e_2]-[e_3]+[e_4]-[e_5]}{2}$, and $\frac{[e_1]+[e_2]+[e_3]+[e_4]+[e_5]}{2}$. According to Lemma 2.6.iii), up to sign, these are the classes in $\mathrm{Pic}\, U_{\overline{k}}$ of four of the 16 lines. In the blown-up model (cf. Paragraph 2.4), we can say more precisely that the classes are $-[E_5]$, $-[L_{45}]$, $-[E_4]$, and $-[C]$.

It is not hard to see that these four lines form the edges of a quadrilateral. I.e., in cyclic order, each line intersects its two neighbours, but not the third. Such a quadrilateral is certainly non-planar, as then the lines would pairwise meet each other. On the other hand, the four points of intersection determine a three-dimensional linear subspace of $\mathbf{P}^4_k$. In other words, there is a k-rational hyperplane $V(t)$ cutting the quadrilateral out of the degree four del Pezzo surface X.

Finally, let $\widetilde{l}$ be the field of definition of the 16 lines. Then $\mathrm{Gal}(\widetilde{l}/k) \cong G$ and there is the intermediate field l, corresponding to $\ker s \subseteq G$. Then $\mathrm{Gal}(l/k) \cong \langle\tau\rangle$. I.e., l/k is a cyclic extension of degree 4. The four edges of the quadrilateral are defined over l. Moreover, $\mathrm{Gal}(l/k)$ permutes them cyclically. We therefore have $\operatorname{div} t = N_{l/k}L$, for L one of the edges of the quadrilateral, and $\operatorname{div} 1/t = N_{l/k}(-L)$. The evaluation of the Brauer class is hence given by the norm residue symbols

$$(1/t(x), l_w/k_\nu) = -(t(x), l_w/k_\nu)\,.$$

Remark 5.23. It is not hard to determine the k-rational quadrilaterals algorithmically. The reader might compare Algorithm 6.2 below, which, however, does by far more.

Example 5.24. Let $X \subset \mathbf{P}^4_{\mathbb{Q}}$ be given by the system of equations

$$-X_0^2 + 8X_0X_1 - 4X_0X_2 - 10X_0X_3 + 4X_1^2 - 6X_1X_2 - 8X_1X_3 + 2X_2^2 + 3X_2X_3 - X_3^2 - X_3X_4 = 0\,,$$

$$7X_0^2 + 9X_0X_1 + 6X_0X_3 + 7X_1^2 + X_2X_4 + 3X_3^2 = 0\,,$$

$\mathscr{X} \subset \mathbf{P}^4_{\mathbb{Z}}$ the subscheme that is defined by the same system of equations as X, and put $U := X \setminus H$ and $\mathscr{U} := \mathscr{X} \setminus \mathscr{H}$, for $H := V(X_0) \subset \mathbf{P}^4_{\mathbb{Q}}$ and $\mathscr{H} := V(X_0) \subset \mathbf{P}^4_{\mathbb{Z}}$.

A point search provides almost 1000 $\mathbb{Q}$-rational points of height up to 1000, among which $(1 : 0 : 2 : -1 : -2)$, $(1 : 1 : 2 : -1 : -10)$, $(1 : -20 : -32 : -9 : 88)$, and $(1 : -80 : -62 : 11 : 718)$ are $\mathbb{Z}$-integral.

The 16 lines on X are defined over a number field of degree twelve. The Galois group operating on the 10 linear systems of conics is exactly the dicyclic group Dic_3, described in Remark 3.28.a.iii). Thus, one has $\mathrm{Br}_1(U)/\mathrm{Br}_0(U) \cong \mathbb{Z}/4\mathbb{Z}$, generated by a 4-torsion class α of type I. Note here, $p(\mathrm{Dic}_3) \cong S_3$ acts intransitively with two orbits of sizes 3 and 2. Thus, Theorem 3.11 shows that no further 2-torsion occurs.

There is exactly one $\mathbb{Q}$-rational hyperplane cutting a quadrilateral out of X, namely $V(35X_0 + 50X_1 - 19X_2 - 2X_3 + 5X_4)$. The four edges are defined over $\mathbb{Q}(\zeta_5)$ and acted upon cyclically by $\mathrm{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$. The evaluation of α is hence given by the norm residue symbols

$$-\big(\tfrac{35X_0+50X_1-19X_2-2X_3+5X_4}{X_0}, \mathbb{Q}_\nu(\zeta_5)/\mathbb{Q}_\nu\big).$$

The surface X has bad reduction only at the primes 2, 5, 31, and 251, among which 31 and 251 completely split in $\mathbb{Q}(\zeta_5)$. Moreover, the local evaluation map $\mathrm{ev}_{\alpha,2}|_{\mathscr{U}(\mathbb{Z}_2)} \colon \mathscr{U}(\mathbb{Z}_2) \to \mathbb{Q}/\mathbb{Z}$ is constant, too. Indeed, 2 is an inert prime. Thus, otherwise there would exist a $\mathbb{Z}_2$-integral point on $\mathscr{U}$ such that the linear form above takes an even value. This, however, yields that X_2 and X_4 must have different parity. From this, the second equation defining X shows that X_3 must be odd, which contradicts the first equation.

Finally, an experiment shows that $\mathrm{ev}_{\alpha,5}$ takes all four values on $\mathscr{U}(\mathbb{Z}_5)$ and that $\mathrm{ev}_{\alpha,\infty}$ takes the values 0 and $\frac{1}{2}$ on $U(\mathbb{Q})$. In particular, on a $\mathbb{Z}$-integral point, $\mathrm{ev}_{\alpha,5}$ may not take the values $\frac{1}{4}$ or $\frac{3}{4}$, which alone shows that strong approximation is violated.

Proposition 5.25 *(A normal form). Let U be an open del Pezzo surface of degree four over k that carries a 4-torsion Brauer class of type I. Then U is a hyperplane complement of a projective surface X that may be given by equations of the form*

$$l_1 l_2 = m_0 z\,,$$
$$m_1 m_2 = l_0 z\,,$$

where the linear forms to the left are defined over a cyclic degree four extension l and acted upon by a generator of $\mathrm{Gal}(l/k)$ via the rule $l_1 \mapsto m_1 \mapsto l_2 \mapsto m_2 \mapsto l_1$. Moreover, l_0 and m_0 are linear forms defined over the intermediate quadratic field and conjugate to each other, and z is a linear form defined over k.

Proof. We take z to be a linear form that cuts out of X a quadrilateral inducing the Brauer class. Next, let $L_1, \ldots, L_4$ be the four edges of the quadrilateral, in cyclic order. Then $L_1 \cup L_2$ determines a plane $V(z, l_1)$. We denote the Galois conjugates of the linear form l_1, in this order, by m_1, l_2, and m_2. Then $L_2 \cup L_3$ is contained in the plane $V(z, m_1)$ etc. Consequently, the quadrilateral is given in $V(z) \cong \mathbf{P}^3$ by the system of equations

$$l_1 l_2 = 0\,,$$
$$m_1 m_2 = 0\,.$$

Here, the forms l_1l_2 and m_1m_2 are global sections of the coherent sheaf $\mathscr{I}_X(2)|_{V(z)}$. In order to complete the proof, it has to be shown that both can be lifted to sections of $\mathscr{I}_X(2)$ on the whole of $\mathbf{P}^4$. As there is the exact sequence

$$0 \longrightarrow \mathscr{I}_X(1) \stackrel{\cdot z}{\longrightarrow} \mathscr{I}_X(2) \longrightarrow \mathscr{I}_X(2)|_{V(z)} \longrightarrow 0\,,$$

this indeed follows from the fact that $H^1(\mathbf{P}^4, \mathscr{I}_X(1)) = 0$ [La, Remark 1.8.44]. □

Remark 5.26. The surface above appears when blowing down the line $V(l_0, m_0)$ on the cubic surface

$$C\colon\, l_0l_1l_2 = m_0m_1m_2$$

that is given in the classical Cayley–Salmon form. The nine obvious lines on C are clearly defined over the field l. The combinatorics of the 27 lines on a cubic surface then implies that the other 18 lines may be defined at most over an S_3–extension of l [EJ11, Proposition 4.6]. This shows again that the field of definition of the 16 lines on X has a Galois group that allows an injection into $S_3 \times \mathbb{Z}/4\mathbb{Z}$.

4-torsion classes of type II

5.27. When $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ is the cyclic group of order 4, generated by the element shown in Fig. 5, then this is the overlap case. One has $H^1(G, \mathfrak{P}) \cong \mathbb{Z}/4\mathbb{Z}$, the 4-torsion class being cyclic. The group G operates on the 16 lines such that there are four orbits of length four each. Two of these orbits form the edges of quadrilaterals, while the others consist of mutually skew lines.

5.28. When $G \subseteq (\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ is the cyclic group of order 8, generated by the element shown in Fig. 6, then there is a 4-torsion class of type II in $H^1(G, \mathfrak{P})$, and this, too, is clearly a cyclic class.

Moreover, one easily sees that the orbit of $\frac{-[e_1]-[e_2]-[e_3]-[e_4]-[e_5]}{2}$ consists of eight distinct elements of $\mathfrak{P}$. The remaining eight elements of $\mathfrak{P}$ that are of the type $\frac{\pm[e_1]\pm[e_2]\pm[e_3]\pm[e_4]\pm[e_5]}{2}$ and involve an even number of plus signs form a single orbit, too, and either orbit has sum zero.

Thus, when $U \subset X$ is an open del Pezzo surface of degree four over k such that $\mathrm{Gal}(\overline{k}/k)$ operates on the 16 lines exactly via G then the lines form two orbits of length eight each. As one has $\mathrm{Pic}\, U_{\overline{k}} = \mathrm{Pic}\, X_{\overline{k}}/\langle H\rangle$, the sum over an orbit, considered in $\mathrm{Pic}\, X_{\overline{k}}$, must be a multiple of H, and since the degree is 8, it is $2H$. Moreover, a short calculation in coherent cohomology shows that the restriction homomorphism $\Gamma(\mathbf{P}^4, \mathscr{O}_{\mathbf{P}^2}(2)) \to \Gamma(X, \mathscr{O}_{\mathbf{P}^2}(2)|_X)$ is surjective, cf. [Ha, Exercise III.5.5.a)]. Hence, either of the two configurations of eight lines is cut out of X by a quadric in $\mathbf{P}^4$.

Example 5.29. Let $X \subset \mathbf{P}^4_{\mathbb{Q}}$ be given by the system of equations

$$X_0^2 - X_0X_3 - X_1^2 + X_1X_2 + X_1X_3 + X_2X_3 + X_3^2 - X_3X_4 = 0\,,$$
$$-X_0^2 - X_0X_1 - X_0X_3 + X_1^2 + X_2X_4 = 0\,,$$

$\mathscr{X} \subset \mathbf{P}^4_{\mathbb{Z}}$ the subscheme that is defined by the same system of equations as X, and put $U := X \setminus H$ and $\mathscr{U} := \mathscr{X} \setminus \mathscr{H}$, for $H := V(X_0) \subset \mathbf{P}^4_{\mathbb{Q}}$ and $\mathscr{H} := V(X_0) \subset \mathbf{P}^4_{\mathbb{Z}}$.

A point search provides more than 6000 $\mathbb{Q}$-rational points of height up to 1000, among which $(1 : -1 : 0 : 1 : -1)$, $(1 : 1 : 0 : -1 : -1)$, $(1 : 0 : 1 : 1 : 2)$, $(1 : 2 : 0 : 1 : -1)$, $(1 : 0 : -2 : 1 : -1)$, $(1 : 0 : 0 : -1 : -3)$, $(1 : 0 : 3 : -1 : 0)$, $(1 : 42 : -221 : -47 : 8)$, and $(1 : 42 : 9 : -277 : -222)$ are $\mathbb{Z}$-integral.

The 16 lines on X are defined over $l := \mathbb{Q}(\zeta_{17} + \zeta_{17}^{-1})$. The Galois group $\mathrm{Gal}(l/\mathbb{Q})$ is cyclic of order 8. A generator operates on the 10 linear systems of conics, when they are suitably indexed, exactly as the element g shown in Fig. 6. Thus, $\mathrm{Br}_1(U)/\mathrm{Br}_0(U) \cong \mathbb{Z}/4\mathbb{Z}$, generated by a 4-torsion Brauer class α of type II. Note here that $p(\langle g\rangle) \cong \mathbb{Z}/4\mathbb{Z}$ acts with two orbits of sizes 4 and 1, so that Theorem 3.11 excludes any further 2-torsion.

One finds that the quadrics $V(q_1)$ and $V(q_2)$, for

$$\begin{aligned} q_1 :=\, & 3X_0X_2 - 2X_0X_3 + 12X_0X_4 - 4X_1^2 + 3X_1X_2 - 2X_1X_3 + 5X_1X_4 + X_2^2 + 17X_2X_3 \\ & - 22X_2X_4 + 15X_3^2 - 13X_3X_4 - X_4^2\,, \quad \text{and} \\ q_2 :=\, & 3X_0X_2 - 19X_0X_3 - 5X_0X_4 - 4X_1^2 + 20X_1X_2 + 15X_1X_3 + 5X_1X_4 + X_2^2 \\ & + 17X_2X_3 + 12X_2X_4 + 15X_3^2 - 13X_3X_4 - X_4^2 \end{aligned}$$

cut the two configurations of eight lines out of X. The evaluation of α is hence given by the norm residue symbols

$$-\big(q_1, \mathbb{Q}_\nu(\zeta_{17} + \zeta_{17}^{-1})/\mathbb{Q}_\nu\big),$$

and when one takes the quadratic form q_2, the answer is the same.

The surface X has bad reduction only at the primes 2 and 17. The local evaluation map $\mathrm{ev}_{\alpha,\infty}$ is constant, since the field $\mathbb{Q}(\zeta_{17} + \zeta_{17}^{-1})$ is totally real. Moreover, the local evaluation map $\mathrm{ev}_{\alpha,2}|_{\mathscr{U}(\mathbb{Z}_2)} \colon \mathscr{U}(\mathbb{Z}_2) \to \mathbb{Q}/\mathbb{Z}$ is constant, too. Indeed, 2 is an inert prime. Thus, otherwise there would exist a $\mathbb{Z}_2$-integral point on $\mathscr{U}$ such that both quadratic forms q_1 and q_2 evaluate to an even number. This leads to a contradiction as follows. First, the equations defining X imply that X_3 must be odd. Then the first equation shows that $X_1X_2 + X_2 + X_4$ must be odd, too. Using this, the assumptions about q_1 and q_2 reduce to $X_1X_2 + X_1X_4 + X_2$ being odd and $X_2 + X_4 + X_1 + X_1X_4$ being even. These conditions, however, turn out to be fulfilled only when all coordinates are odd, which contradicts the second equation defining X.

Finally, an experiment shows that $\mathrm{ev}_{\alpha,17}$ takes all four values on $\mathscr{U}(\mathbb{Z}_{17})$. In particular, on a $\mathbb{Z}$-integral point, $\mathrm{ev}_{\alpha,17}$ may not take the values $\frac{1}{4}$, $\frac{1}{2}$, or $\frac{3}{4}$, which shows that strong approximation is violated.

Remark 5.30. The six further conjugacy classes of subgroups of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ that yield a 4-torsion Brauer class of type II are non-cyclic. Even worse, it turns out that no normal subgroup, except for the trivial group, annihilates the 4-torsion class. As all other naive approaches, such as that to use a corestriction, do not apply either, it is our conclusion that only a generic algorithm helps.

6. The generic algorithm

The computer-algebraic framework

We implemented a generic algorithm in `magma` relying on some of the data types and functionality already existing. The data types include

i) Finite groups. I.e., permutation groups described by a sequence of generators.
ii) Finitely generated abelian groups, described in the form $\mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_m\mathbb{Z}$, for a sequence of non-negative integers such that $a_m | a_{m-1} | \ldots | a_2 | a_1$.
iii) Finitely generated G-modules. This means that a finitely generated abelian group, as above, is given together with a sequence of matrices describing the operation of the generators of G. The data structure of a finitely generated G-module includes the full information about the underlying finite group G.

`magma` allows to perform fundamental operations in these categories, including the computation of the cokernel $\mathrm{pr}\colon N \to \operatorname{coker}\varphi$ of a homomorphism $\varphi\colon M \to N$ of finitely generated abelian groups.

Also, for a G-module M as above, $H^i(G, M)$ may be computed for $i = 0$, 1, or 2. This means that an isomorphism is asserted to some $H := \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_m\mathbb{Z}$. In addition, for every i-cocycle with values in M, its cohomology class may be expressed as an element of H. And vice versa, there is function returning for every element of H a representing i-cocycle.

Furthermore, let us mention that there is some more functionality that we take to work for granted, such as that to compute Gröbner bases, to compute the unit group of a finite ring, and to compute the Galois group of a polynomial over a number field, as well as the decomposition and inertia groups contained within.

Finally, an open del Pezzo surface of degree 4 over a number field k is, for us, just given by two quadratic forms in five variables over the maximal order $\mathscr{O}_k$. The hyperplane section taken out is supposed to be the vanishing locus of the first variable.

Computing the Picard group as a Galois module

Lemma 6.1. *Let X be a proper del Pezzo surface of degree four over an algebraically closed field and $U \subset X$ an open degree four del Pezzo surface (as in Definition 2.5).*
a) *Then X contains exactly 40 quadrilaterals.*
b) *Let D be the free abelian group over the set of the 16 lines on X and $D_0 \subset D$ the submodule generated by all formal sums $1L_1 + 1L_2 + 1L_3 + 1L_4$, for $L_1, \ldots, L_4$ the edges of a quadrilateral. Then there is a canonical isomorphism $D/D_0 \cong \operatorname{Pic} U$.*

Proof. a) We work in the blown-up model. As the lines E_i are mutually disjoint, every quadrilateral must contain a line of type L_{ij}. Say, a quadrilateral contains L_{12}. The lines intersecting L_{12} are E_1, E_2, and L_{kl}, for $3 \leq k < l \leq 5$. Thus, there are the quadrilaterals

- $[E_1, L_{12}, E_2, C]$ and analogous, which are $\binom{5}{2} = 10$ items, and
- $[E_1, L_{12}, L_{45}, L_{13}]$ and analogous. These are $5 \cdot \binom{4}{2} = 30$ items.

b) By Lemma 2.6, one knows that $\operatorname{Pic} U \cong \mathbb{Z}^5$. On the other hand, the 16 lines generate $\operatorname{Pic} U$. I.e., the canonical homomorphism $D \to \operatorname{Pic} U$ is surjective. Moreover, D_0 is contained in the kernel, so that a homomorphism $D/D_0 \twoheadrightarrow \operatorname{Pic} U$ gets induced. It is therefore sufficient to show that D/D_0 may be generated by five elements. Indeed, every surjection from such a group onto $\mathbb{Z}^5$ is bijective.

For this, again let us work in the blown-up model. First of all, there are the relations $L_{ij} \equiv -E_i - E_j - C \pmod{D_0}$, which show that D/D_0 is generated by $E_1, \ldots, E_5$, and C. Furthermore, one has

$$\begin{aligned} 0 &\equiv (E_1 + L_{12} + E_2 + C) + (E_1 + L_{13} + E_3 + C) + (E_4 + L_{45} + E_5 + C) \\ &\qquad - (E_1 + L_{12} + L_{45} + L_{13}) \pmod{D_0} \\ &\equiv E_1 + E_2 + E_3 + E_4 + E_5 + 3C \pmod{D_0}, \end{aligned}$$

implying that $E_1, \ldots, E_4$, and C form a generating system, as required. □

Algorithm 6.2 *(Computing* $\operatorname{Pic} U_{\overline{k}}$ *as a Galois module).* Given an open degree 4 del Pezzo surface U over a number field k, this algorithm computes the Picard group $\operatorname{Pic} U_{\overline{k}}$ as a Galois module.

i) Represent lines in general position in $\mathbf{P}^4$ by parametrisations of the form $(1 : t : (k_1 + k_2 t) : (k_3 + k_4 t) : (k_5 + k_6 t))$. Write down a system of equations in six variables that encodes the containment of such a line in the surface U. Then calculate a Gröbner basis over k, in order to obtain a univariate polynomial g of degree 16, the zeroes of which correspond one-to-one to the 16 lines on U.
If the Gröbner basis turns out not to be of the expected form then apply an automorphism of $\mathbf{P}^4$, chosen at random, and try again.

ii) Put l to be the splitting field of g and calculate the Galois group G of l. Make sure that the operation of G, i.e. the homomorphism $i\colon G \to \mathrm{Aut}_k\, l$, is stored, too.
iii) Using Gröbner bases again, this time over l, determine the 16 lines on U explicitly. Store them into a list, numbered from 1 to 16.
For each generator $\sigma \in G$, use the automorphism $i(\sigma)$ of l to determine the permutation of the 16 lines that σ induces. Form the associated permutation matrices, thereby transforming $\mathbb{Z}^{16}$ into a G-module Div.
iv) For each pair of lines, determine using linear algebra over l whether they intersect on the proper surface or not. The result is a 16×16-intersection matrix.
v) For each quadruple $\{i_1, \dots, i_4\} \subset \{1, \dots, 16\}$, determine whether it represents a quadrilateral. It suffices to check that exactly four of the six intersection numbers are equal to 1. Store the 40 quadrilaterals found into a 16×40-matrix Q over $\mathbb{Z}$, each row of which contains four ones and zeros otherwise.
Also determine, using linear algebra over l, for each quadrilateral a linear form cutting it out of the surface. Store these linear forms into a list L.
vi) Calculate $\mathrm{Pic}\, U_{\overline{k}}$ as the cokernel of the homomorphism $q\colon \mathbb{Z}^{40} \to \mathrm{Div} = \mathbb{Z}^{16}$ given by Q. Determine the G-module structure on $\mathrm{Pic}\, U_{\overline{k}}$ that is induced from that on Div via the canonical epimorphism $\mathrm{pr}\colon \mathrm{Div} \twoheadrightarrow \mathrm{Pic}\, U_{\overline{k}}$.
vii) Return $\mathrm{Pic}\, U_{\overline{k}}$. As further values, return the homomorphism $\mathrm{pr}\colon \mathrm{Div} \twoheadrightarrow \mathrm{Pic}\, U_{\overline{k}}$ [including the G-module Div], the homomorphism $q\colon \mathbb{Z}^{40} \to \mathrm{Div}$, the number field l, and the list L.

Remark 6.3. Having run Algorithm 6.2, it is just one call of a `magma` intrinsic to compute $H^1(G, \mathrm{Pic}\, U_{\overline{k}})$ in the form $\mathbb{Z}/a_1\mathbb{Z}$ $[\oplus \mathbb{Z}/a_2\mathbb{Z}]$. Just another call returns a 1-cocycle for any [base] element chosen.

Computing a 2-cocycle with values in $l(U)^*$

Algorithm 6.4 *(Computing a 2-cocycle with values in $l(U)^*/l^*$).* Given a 1-cocycle $\varphi\colon G \to \mathrm{Pic}\, U_{\overline{k}}$, as well as the homomorphisms $\mathrm{pr}\colon \mathrm{Div} \twoheadrightarrow \mathrm{Pic}\, U_{\overline{k}}$ and $q\colon \mathbb{Z}^{40} \to \mathrm{Div}$ [together with the list L], this algorithm computes a corresponding 2-cocycle with values in $l(U)^*/l^*$.

i) Lift φ along pr to a cochain $\widetilde{\varphi}\colon G \to \mathrm{Div}$.
ii) Calculate the coboundary $\delta\widetilde{\varphi}\colon G \times G \to \mathrm{Div}$ according to the formula

$$\delta\widetilde{\varphi}(\sigma, \tau) := \varphi(\tau)^\sigma - \varphi(\sigma\tau) + \varphi(\sigma)\,.$$

iii) Lift $\delta\widetilde{\varphi}$ along q to a function $\psi\colon G \times G \to \mathbb{Z}^{40}$. This means to find a particular solution of a 16×40 linear system of equations, for each pair $(\sigma, \tau) \in G \times G$.
iv) Return the function $\psi\colon G \times G \to \mathbb{Z}^{40}$.

Remark 6.5. A vector $\psi(\sigma, \tau) = (e_1, \ldots, e_{40})$ encodes the rational function $L_1^{e_1} \cdots L_{40}^{e_{40}}$ modulo constants. For performance reasons, we do not multiply the product out at this stage, but do so only after the evaluation at a point.

Lemma 6.6 *(Lifting to a 2-cocycle with values in $l(U)^*$). Let U be an integral scheme over a field k having a k-rational point $x \in U(k)$, l/k a finite Galois extension, $G = \mathrm{Gal}(l/k)$ its Galois group, and $c\colon G \times G \to l(U)^*/l^*$ a 2-cocycle.*

Suppose that all the images $c(\sigma, \tau) \in l(U)^*/l^*$, for $\sigma, \tau \in G$, are classes of functions modulo scalars that are defined and nonzero at x. Then the unique lift $\widetilde{c}\colon G \times G \to l(U)^*$ of c so that $\widetilde{c}(\sigma, \tau)(x) = 1$, for all $\sigma, \tau \in G$, is a 2-cocycle.

Proof. By assumption, $\delta c = 0$. Therefore, $\delta\widetilde{c}\colon G \times G \times G \to l(U)^*$ vanishes modulo l^*. In other words, $\delta\widetilde{c}$ takes only constant functions. Thus, in order to exactly determine $\delta\widetilde{c}$, it suffices to calculate the values of the $\delta\widetilde{c}(\sigma, \tau, \upsilon)$ at the point x. But, according to the assumption, we have $\widetilde{c}(\sigma, \tau)(x) = 1$ for all $\sigma, \tau \in G$. Hence, $\delta\widetilde{c}(\sigma, \tau, \upsilon)(x) = 1$, for all $\sigma, \tau, \upsilon \in G$, which completes the proof. □

Remark 6.7. In [Be, Section 3.3], a different approach is presented on how to lift 2-cocycles, which applies to the situation of an affine Châtelet surface.

6.8. Given a 2-cocycle $c\colon \mathrm{Gal}(l/k) \times \mathrm{Gal}(l/k) \to l(U)^*$ representing the Brauer class $\alpha \in \mathrm{Br}(U)$, in order to evaluate α at a point $x \in U(k_\nu)$, two things have to be done as the next step.

a) One has to restrict the 2-cocycle to the local Galois group $\mathrm{Gal}(lk_\nu/k_\nu) \subseteq \mathrm{Gal}(l/k)$.
b) One has to replace the functions $c(\sigma, \tau)$ by their values $c(\sigma, \tau)(x) \in lk_\nu$.

In practice, one usually approximates the coordinates of a k_ν-rational point by elements of k. It is possible to remain in the realm of number fields by doing what follows, instead of a) and b). Fix a prime w of l lying above ν and

a') restrict the 2-cocycle to the decomposition group $D_w \subseteq \mathrm{Gal}(l/k)$.
b') Store the values $c(\sigma, \tau)(x)$ as elements of l. In further calculations, they can be considered as elements of l_w [$\cong lk_\nu$].

The result is a 2-cocycle representing a Brauer class of the local field l_w.

Computing the invariant of a Brauer class of a local field from a given 2-cocycle

Remark 6.9. This problem has been tackled before, in a very thorough manner, by T. Preu [Pr]. We are, however, not aware of any implementation realising Preu's approach. In our implementation, which is sufficient for our purposes, we followed [Pr] only very partially, and took many shortcuts that make use of features built into `magma`. Note that the archimedean case is trivial, as then the local Galois group is of order at most 2.

Lemma 6.10. *Let l/k be a finite Galois extension of non-archimedean local fields.*

a) *Then there is a natural isomorphism*

$$H^2(\mathrm{Gal}(l/k), l^*) \cong \varprojlim_n H^2(\mathrm{Gal}(l/k), l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l}))\,.$$

b) *If $\#\,\mathrm{Gal}(l/k)$ is relatively prime to the residue characteristic of l then*

$$H^2(\mathrm{Gal}(l/k), l^*) \cong H^2(\mathrm{Gal}(l/k), l^*/(1+\mathfrak{m}_{\mathscr{O}_l}))\,.$$

Proof. a) Clearly, $\mathscr{O}_l^* \cong \varprojlim_n \mathscr{O}_l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l})$ and $l^* \cong \varprojlim_n l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l})$. Moreover, for arbitrary natural numbers i and n, the cohomology

$$H^i(\mathrm{Gal}(l/k), \mathscr{O}_l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l}))$$

is a finite abelian group. Indeed, $\mathrm{Gal}(l/k)$ is a finite group and $\mathscr{O}_l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l})$ a finite $\mathrm{Gal}(l/k)$-module. Consequently, [NSW, Corollary 2.7.6] applies and shows that

$$H^i(\mathrm{Gal}(l/k), \mathscr{O}_l^*) \cong \varprojlim_n H^i(\mathrm{Gal}(l/k), \mathscr{O}_l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l}))\,,$$

for every i. Observe here that, in our case, continuous group cohomology agrees with ordinary group cohomology when we equip the finite group $\mathrm{Gal}(l/k)$ with the discrete topology.

Furthermore, for each n, the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{O}_l^* & \longrightarrow & l^* & \xrightarrow{\nu_l} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{O}_l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l}) & \longrightarrow & l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l}) & \xrightarrow{\overline{\nu}_l} & \mathbb{Z} & \longrightarrow & 0
\end{array}$$

of short exact sequences induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & H^2(\mathrm{Gal}(l/k), \mathscr{O}_l^*) & \to & H^2(\mathrm{Gal}(l/k), l^*) & \to & H^2(\mathrm{Gal}(l/k), \mathbb{Z}) & \to & H^3(\mathrm{Gal}(l/k), \mathscr{O}_l^*) \\
 & & \downarrow & & \downarrow & & \| & & \downarrow \\
0 & \to & H^2(\mathrm{Gal}(l/k), \mathscr{O}_l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l})) & \to & H^2(\mathrm{Gal}(l/k), l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l})) & \xrightarrow{\overline{\overline{\nu}}_l} & H^2(\mathrm{Gal}(l/k), \mathbb{Z}) & \to & H^3(\mathrm{Gal}(l/k), \mathscr{O}_l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l}))
\end{array} \tag{6}$$

of exact sequences. Indeed, $H^1(\mathrm{Gal}(l/k), \mathbb{Z}) = 0$.

Putting $C_n := \mathrm{im}\,\overline{\overline{\nu}}_l$, the lower exact sequence may be split into two,

$$0 \to H^2(\mathrm{Gal}(l/k), \mathscr{O}_l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l})) \to H^2(\mathrm{Gal}(l/k), l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l})) \to C_n \to 0$$

and

$$0 \to C_n \to H^2(\mathrm{Gal}(l/k), \mathbb{Z}) \to H^3(\mathrm{Gal}(l/k), \mathscr{O}_l^*/(1+\mathfrak{m}^n_{\mathscr{O}_l}))\,.$$

Either of these sequences remains exact after applying the inverse image functor $\varprojlim_n$. Indeed, this is clear for the second one as $\varprojlim_n$ is left-exact. Moreover, since all the abelian groups $H^2(\mathrm{Gal}(l/k), \mathscr{O}_l^*/(1+\mathfrak{m}_{\mathscr{O}_l}^n))$ are finite, they form an inverse system satisfying the Mittag–Leffler condition. The claim on the first exact sequence therefore results from [NSW, Proposition 2.7.3].

Consequently, the application of $\varprojlim_n$ to (6) yields a commutative diagram of exact sequences, in which all vertical homomorphisms, except possibly for the second from the left, are isomorphisms. A standard diagram argument shows that the latter must be an isomorphism, too, as required.

b) In this situation, for each $n \geq 1$, the quotient $(1+\mathfrak{m}_{\mathscr{O}_l}^n)/(1+\mathfrak{m}_{\mathscr{O}_l}^{n+1})$ is an abelian p-group, for p the residue characteristic of l, and hence cohomologically trivial. From this, the long exact cohomology sequence associated with

$$0 \longrightarrow (1+\mathfrak{m}_{\mathscr{O}_l}^n)/(1+\mathfrak{m}_{\mathscr{O}_l}^{n+1}) \longrightarrow l^*/(1+\mathfrak{m}_{\mathscr{O}_l}^{n+1}) \longrightarrow l^*/(1+\mathfrak{m}_{\mathscr{O}_l}^n) \longrightarrow 0$$

immediately implies that the sequence $H^2(\mathrm{Gal}(l/k), l^*/(1+\mathfrak{m}_{\mathscr{O}_l}^n))$ is stationary for $n \geq 1$. $\square$

Remarks 6.11. i) Part b) may also be shown directly, without relying on a), by applying [Se, Chapitre XII, §3, Lemme 3] for $q := 2$ and 3 to the $\mathrm{Gal}(l/k)$-module $M := 1+\mathfrak{m}_{\mathscr{O}_l}$ and its submodules $M_n := 1+\mathfrak{m}_{\mathscr{O}_l}^n$, for $n \geq 1$.
ii) In the application to open del Pezzo surfaces of degree four carrying an algebraic 4-torsion Brauer class of type II, $\mathrm{Gal}(l/k)$ is always a 2-group. Hence, Lemma 6.10.b) is applicable for the evaluation at all non-archimedean primes, except for those of residue characteristic 2.

For simplicity of the presentation, we restrict ourselves to the case that $\#G$ is a power of 2, although our approach works in general.

Algorithm 6.12 *(For the case that $\#G$ is a power of 2, the residue characteristic is $\neq 2$, and the Brauer class is at most 4-torsion).*

Given a 2-cocycle $\varphi\colon G \times G \to l^*$, for l/k a finite Galois extension of non-archimedean local fields and $G \cong \mathrm{Gal}(l/k)$, satisfying the conditions listed, this algorithm computes the invariant in $\mathbb{Q}/\mathbb{Z}$ of the class in $H^2(G, l^*)$ represented by φ.

i) Put $q := \#(\mathscr{O}_l/\mathfrak{m}_{\mathscr{O}_l})$ and $f := \log q / \log \#(\mathscr{O}_k/\mathfrak{m})$. If $f = 1$ or 2 then execute the modification described below in 6.13.c). Calculate the inertia group $E \subseteq G$ and put $e := \#E$.
Run in a loop through G and identify a Frobenius element $\mathrm{Frob} \in G$ by testing the condition that $\nu_l(\mathrm{Frob}(x) - x^{\#(\mathscr{O}_k/\mathfrak{m}_{\mathscr{O}_k})}) > 0$, for $x \in \mathscr{O}_l$ an element such that $\overline{x} \in \mathscr{O}_l/\mathfrak{m}_{\mathscr{O}_l}$ generates that field as an extension of $\mathscr{O}_k/\mathfrak{m}_{\mathscr{O}_k}$.
Run, once again, in a loop through G and create a map $\mathrm{ex}\colon G \to \{0, 1, \ldots, f-1\}$ of the kind that $\mathrm{Frob}^{\mathrm{ex}(g)} \equiv g \pmod{E}$, for every $g \in G$.

ii) Fix a uniformiser π_l of l and a multiplicative generator $u \in \mathscr{O}_l/\mathfrak{m}_{\mathscr{O}_l}$ of the residue field. Then create the G-module $l^*/(1+\mathfrak{m}_{\mathscr{O}_l})$ as follows. Put, at first $M := \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z}$. In addition, associate the matrices

$$A_i := \begin{pmatrix} 1 & 0 \\ \log_u \frac{g_i(\pi_l)}{\pi_l} & \#(\mathscr{O}_k/\mathfrak{m})^{\mathrm{ex}(g_i)} \end{pmatrix},$$

with the generators $g_i \in G$, in order to transform M into a G-module. Here, $\log_u \colon (\mathscr{O}_l/\mathfrak{m}_{\mathscr{O}_l})^* \to \mathbb{Z}/(q-1)\mathbb{Z}$ denotes the discrete logarithm of base u.

iii) Define the standard 2-cocycle $\mathrm{st}\colon G \times G \to M$ of invariant $(1/f \bmod 1)$ by

$$(\sigma,\tau) \mapsto \begin{cases} (0,0) & \text{if } \mathrm{ex}(\sigma)+\mathrm{ex}(\tau) < f\,, \\ (e, \log_u \frac{\pi}{\pi_l^e}) & \text{if } \mathrm{ex}(\sigma)+\mathrm{ex}(\tau) \geq f\,, \end{cases}$$

for π a uniformiser of k.

iv) Calculate the image of the input cocycle by putting

$$\widetilde{\varphi}(\sigma,\tau) := \big(\nu_l(\varphi(\sigma,\tau)), \log_u \tfrac{\varphi(\sigma,\tau)}{\pi_l^{\nu_l(\varphi(\sigma,\tau))}}\big),$$

for every $(\sigma,\tau) \in G \times G$.

v) Calculate the abelian group $H^2(G,M)$ and, as its elements, the classes c_{st} and $c_{\widetilde{\varphi}}$ of the 2-cocycles st and $\widetilde{\varphi}$. The cohomology class c_{st} is a proper f-torsion element and $c_{\widetilde{\varphi}}$ is one of its multiples. Find a natural number m such that $c_{\widetilde{\varphi}} = mc_{\mathrm{st}}$, and output $(m/f \bmod 1) \in \mathbb{Q}/\mathbb{Z}$ as the desired invariant.

6.13. a) The idea behind Algorithm 6.12 is to compare the given 2-cocycle, which is for the extension field l, with the standard 2-cocycle for the maximal unramified subextension $l \cap k^{\mathrm{nr}}$ of degree f. Thus, it may work only for invariants being an integral multiple of $(1/f \bmod 1)$. This is why in step i) we had to suppose that $f \neq 1, 2$.
b) If $f = 1$ or 2 then one might replace l by $lk^{\mathrm{nr},4}$, for $k^{\mathrm{nr},4}$ the unramified degree-4 extension of k, extend the group G accordingly, and then run Algorithm 6.12 as described.
c) However, the standard 2-cocycle has values only in $k \subseteq l$ and the 2-cocycle to be evaluated has values only in l. Moreover, π_l is still a uniformiser for $lk^{\mathrm{nr},4}$. Thus, we can get by without explicitly introducing the larger field $lk^{\mathrm{nr},4}$. Instead,

i') put $G' := G \times \mathbb{Z}/4\mathbb{Z}$ if $f = 1$ and, if $f = 2$,

$$G' := \ker(G \times \mathbb{Z}/4\mathbb{Z} \xrightarrow{(\mathrm{ex} \bmod 2)\circ \mathrm{pr}_1 - (\mathrm{pr}_2 \bmod 2)} \mathbb{Z}/2\mathbb{Z})\,.$$

Moreover, let $\varphi' \colon G' \times G' \to l^*$ be the composition of $\mathrm{pr}_1 \times \mathrm{pr}_1$ with φ. Define $\mathrm{ex}' \colon G' \twoheadrightarrow \{0,\dots,3\}$ to be the lift of the homomorphism $G \to \mathbb{Z}/4\mathbb{Z}$ induced by the projection to the second factor. Put $q' := q^{4/f}$ and $f' := 4$.

ii') Put $\log'_u := \frac{q'-1}{q-1} \cdot \log_u$, at least on $(\mathscr{O}_l/\mathfrak{m}_{\mathscr{O}_l})^*$. Then put $M' := \mathbb{Z} \oplus \mathbb{Z}/(q'-1)\mathbb{Z}$ and transform M' into a G'-module by means of matrices A'_i that are defined as in iv), but using $\log'_u$ and ex' instead of $\log_u$ and ex.
Finally, execute steps iii), iv), and v), accordingly.

To summarise, the generic algorithm runs as follows.

Algorithm 6.14 *(The generic algorithm).* Given an open degree 4 del Pezzo surface U over a number field k and a sequence of $\mathscr{O}_{k,\nu}$-integral points, for ν a prime of k, this algorithm computes the evaluations of the points given, for any algebraic Brauer class on U.

i) Run Algorithm 6.2 to compute the Galois module $\operatorname{Pic} U_{\overline{k}}$. Extract the underlying group G, which is the Galois group operating on the 16 lines on U. Moreover, store the field l, over which the lines are defined.
ii) Call the `magma` intrinsic to compute $H^1(G, \operatorname{Pic} U_{\overline{k}})$. Choose the element to be evaluated.
iii) Run Algorithm 6.4 to compute a 2-cocycle c with values in $l(U)^*/l^*$ representing the cohomology class chosen. Lift c to a 2-cocycle $\widetilde{c}$ with values in $l(U)^*$ as described in Lemma 6.6, using the first of the given points.
iv) Compute the decomposition group $D_w \subseteq G$, for w any prime of l lying above ν. Then localise the 2-cocycle $\widetilde{c}$ at the prime ν as described in 6.8.a'). The result is a 2-cocycle c_ν for the operation of the decomposition group $D_w \subseteq G$.
v) Run in a loop over the given sequence of $\mathscr{O}_{k,\nu}$-integral points. For each of them do the following.
   - Evaluate the 2-cocycle c_ν at the current point ξ, as described in 6.8.b'). This yields a 2-cocycle $c_{\nu,\xi}$ representing a cohomology class in $H^2(D_w, l_w^*)$.
   - Run Algorithm 6.12 to determine the invariant of the 2-cocycle $c_{\nu,\xi}$. Store the value into a list.
vi) Output the list of invariants found.

Remark 6.15. We assume, in practice, that the $\mathscr{O}_{k,\nu}$-integral points are given as k-rational points lying on U. One might have the idea to work with k-rational approximations instead that are not exactly located on U. In such a case, one would have to take care, in addition, about the quality of the approximation.

Example 6.16 *(Requiring the generic algorithm).* Let $X \subset \mathbf{P}^4_{\mathbb{Q}}$ be given by the system of equations

$$X_0^2 + 2X_0X_1 - 3X_0X_3 + X_1^2 - 3X_1X_3 - X_2^2 - X_2X_4 + 2X_3^2 - X_3X_4 = 0\,,$$
$$-2X_0^2 - X_0X_1 + 2X_0X_3 - 2X_1^2 + 2X_1X_3 + X_2X_4 - X_3^2 = 0\,,$$

$\mathscr{X} \subset \mathbf{P}^4_{\mathbb{Z}}$ the subscheme that is defined by the same system of equations as X, and put $U := X \setminus H$ and $\mathscr{U} := \mathscr{X} \setminus \mathscr{H}$, for $H := V(X_0) \subset \mathbf{P}^4_{\mathbb{Q}}$ and $\mathscr{H} := V(X_0) \subset \mathbf{P}^4_{\mathbb{Z}}$.

A point search provides more than 600 $\mathbb{Q}$-rational points of height up to 1000, among which $(1 : 1 : -1 : 2 : -1)$, $(1 : 15 : -5 : 4 : -71)$, $(1 : 20 : 15 : -6 : 74)$, $(1 : -9 : -2 : 1 : -86)$, $(1 : 41 : 15 : -6 : 263)$, $(1 : 223 : -229 : 308 : -247)$, $(1 : 299 : -213 : 312 : -419)$, and $(1 : -96 : -53 : 22 : -434)$ are $\mathbb{Z}$-integral.

The 16 lines on X are defined over a number field l of degree 64. The Galois group $\mathrm{Gal}(l/\mathbb{Q})$ operates on the 10 linear systems of conics as the largest subgroup G of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_5$ fulfilling conditions 3.24.i) and ii). Thus, $\mathrm{Br}_1(U)/\mathrm{Br}_0(U) \cong \mathbb{Z}/4\mathbb{Z}$, generated by a 4-torsion Brauer class α of type II. Note here that $p(G) \cong D_4$ acts intransitively with two orbits of sizes 4 and 1, so that Theorem 3.11 excludes any further 2-torsion.

First of all, it turns out that the local evaluation map $\mathrm{ev}_{\alpha,\infty}\colon U(\mathbb{R}) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ at the infinite place is constantly zero. Indeed, the base extension of α is an element of $\mathrm{Br}_0(U_{\mathbb{R}})$, although $\mathrm{Br}_1(U_{\mathbb{R}})/\mathrm{Br}_0(U_{\mathbb{R}}) \cong \mathbb{Z}/2\mathbb{Z}$. In order to explain this, let us note that exactly three of the five degenerate quadrics in the pencil defining X are real, as is shown by a simple calculation. In particular, the decomposition group D_{w_∞} is indeed of order 2. Among the degenerate quadrics, only one has its linear systems of planes defined over $\mathbb{R}$. Numbering the five degenerate quadrics from 1 to 5 as in 3.24, D_{w_∞} must split the orbit $\{1\}$. This is implied by Theorem 4.4.ii), since an order two group never gives rise to a 4-torsion class. More precisely, the operation of the nonzero element of D_{w_∞} is necessarily of the form indicated in Fig. 11, which implies the claim, again according to Theorem 4.4.ii).

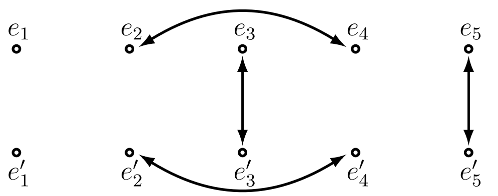

Fig. 11. The operation of the decomposition group at infinity.

The surface X has bad reduction at the primes 3, 5, 7, and 19. The local evaluation map $\mathrm{ev}_{\alpha,19}|_{\mathscr{U}(\mathbb{Z}_{19})}\colon \mathscr{U}(\mathbb{Z}_{19}) \to \mathbb{Q}/\mathbb{Z}$ is constant. Indeed, direct calculations show the following. Three of the five degenerate quadrics in the pencil defining X are defined over $\mathbb{Q}_{19}$, the other two only over the ramified extension $\mathbb{Q}_{19}(\sqrt{-19})$. The linear systems of planes on the latter two are defined over $\mathbb{Q}_{19}(\sqrt{-19})$, already. Finally, two of the former three have their linear systems of planes defined only over the unramified extension field $\mathbb{Q}_{19}(\sqrt{-1})$, while on the final one, they are defined over $\mathbb{Q}_{19}$. Thus, the decomposition group is isomorphic to the Klein four group, and, again following the conventions of 3.24, two of its generators necessarily operate as shown below, which implies the claim according to Theorem 4.4.ii) (see Fig. 12).

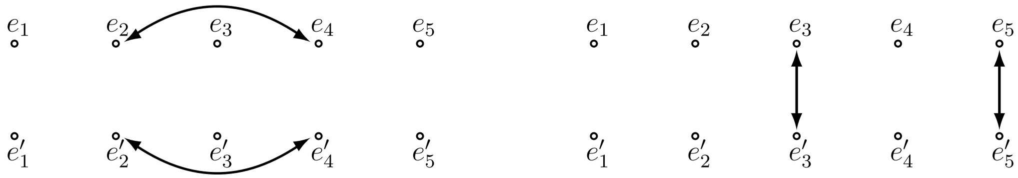

Fig. 12. Generators of the decomposition group at $p = 19$.

The Brauer class 2α is a 2-torsion class of type I. Indeed, using the conventions as above, Theorem 3.11 shows that $2\alpha = \iota_{\mathrm{Pic}}(\overline{[e_1]})$. Moreover, the evaluation of 2α at a prime p is given by the norm residue symbol $\left(\frac{X_2+X_3}{X_0}, \mathbb{Q}_p(\sqrt{5})\right)$. This is in fact obtained by a direct application of the method, described in 5.12. Note here that $V(q_1)$, for q_1 the term to the left in the first equation of X, is the only quadric in the pencil defining X that is defined over $\mathbb{Q}$ and has rank 4.

As a quadratic form of rank 4, q_1 is of discriminant exactly 5. Thus, Corollary 5.15 shows that the local evaluation map $\mathrm{ev}_{2\alpha,p}|_{\mathscr{U}(\mathbb{Z}_p)}$ is constant, for $p = 3$ and 7. Note here that $(1 : (-1) : 0 : 0 : 0)$ is the cusp of $V(q_1)$. Its reduction modulo 7 does not lie on $V(q_2)_7$. On the other hand, its reduction modulo 3 lies on $V(q_2)_3$, but as a singular point, so that no $\mathbb{Q}_3$-rational point on $V(q_2)$ can reduce to it. Consequently, $\mathrm{ev}_{\alpha,p}|_{\mathscr{U}(\mathbb{Z}_p)}$ may take at most two values for either of these primes.

We ran Algorithm 6.14 for each of the four primes $p = 3$, 5, 7, and 19. As our sample of $\mathbb{Z}_p$-integral points, we used the set of all of $\mathbb{Q}$-rational points, found by the point search, that are p-adically integral. The outcome is as follows.

i) Over the prime $p = 3$, the decomposition group is cyclic of order 8 and the inertia group is its subgroup of order 2. In particular, the modification of Algorithm 6.12 does not get called. Among the $\mathbb{Q}$-rational points found, 510 are 3-adically integral. The lift $\alpha \in \mathrm{Br}_1(U)$ of one of the two proper 4-torsion classes in $H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}\, U_{\overline{\mathbb{Q}}})$, chosen running the algorithm, evaluates 208 times to 0 and 202 times to $\frac{1}{2}$.

ii) Over the prime $p = 5$, the decomposition group is cyclic of order 4 and coincides with the inertia group. I.e., the decomposition field is totally ramified. The modification of Algorithm 6.12 runs and enlarges the decomposition group to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Among the $\mathbb{Q}$-rational points found, 452 are 5-adically integral. The Brauer class $\alpha \in \mathrm{Br}_1(U)$ evaluates 163 times to 0, 99 times to $\frac{1}{4}$, 79 times to $\frac{1}{2}$, and 111 times to $\frac{3}{4}$.

iii) Over the prime $p = 7$, the decomposition group is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the inertia group is a subgroup of order 2 with a cyclic quotient. The modification of Algorithm 6.12 is not called. Among the $\mathbb{Q}$-rational points found, 576 are 7-adically integral. The Brauer class α evaluates 341 times to 0 and 235 times to $\frac{1}{2}$.

iv) Over the prime $p = 19$, the algorithm detects that the decomposition group is isomorphic to the Klein four group and that the inertia group is a subgroup of order 2. The modification runs and enlarges the decomposition group to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Among the $\mathbb{Q}$-rational points found, 582 are 19-adically integral and the Brauer class α is found to evaluate to 0 at each of them, in accordance with our theoretical understanding.

On the surface U, strong approximation is violated. For example, the Brauer class 2α already yields that $\mathbb{Z}_5$-integral points of the kind that $\mathrm{ev}_{\alpha,5}(\xi) = \frac{1}{4}$ or $\frac{3}{4}$ cannot be approximated by $\mathbb{Z}$-integral points. The 4-torsion class α enforces additional constraints, for instance that adelic points $\mathbb{x}$ such that $\mathrm{ev}_{\alpha,5}(\mathbb{x}_5) = \frac{1}{2}$, $\mathrm{ev}_{\alpha,3}(\mathbb{x}_3) = 0$, and $\mathrm{ev}_{\alpha,7}(\mathbb{x}_7) = 0$ may not be approximated, either.

Finally, among the $\mathbb{Z}$-integral points within our range, only $(1 : -9 : -2 : 1 : -86)$ has evaluation $\frac{1}{2}$ at 5. In particular, we see only three of the four admissible combinations of values.

Remarks 6.17. i) The calculations described in Example 6.16 took around 2 hours and 12 minutes, running `magma`, version 2.23.4, on one core of an AMD Phenom II X4 955 processor. In comparison with the more specific and more advanced class field theoretic methods, which do not apply here, this is, of course, disappointingly slow.

First of all, the point search using Elkies' method took more than three minutes. Then steps i) to iii) of Algorithm 6.14 play the role of an initialisation procedure. Their running time is dominated by the size of the field l of definition of the 16 lines, which is a number field of degree 64 for the surface above. They take around two minutes and 20 seconds, the lion's share of which accounts for the Gröbner base calculation over l in step iii) of Algorithm 6.2.

For each of the four primes, step iv) of Algorithm 6.14 is another initialisation step. It takes about 90 seconds, each time, which is mainly the time to compute the decomposition group. Then points are evaluated at a frequency of about one per almost 10 seconds, for $p = 5$, and at about one per 1.7 seconds, for the other primes. A considerable part of this time is spent on the determination of the $\#D_w^2$ values of the 2-cocycle $c_{\nu,\xi}$, which means to multiply 40 elements of l, each time.

ii) There is some `magma` code available from the first author's web page that shows our implementation of Algorithm 6.14. It includes the concrete calculations for Example 6.16, as well as the output that is generated.

The modification necessary for the case that the residue characteristic divides the group order

In our application, this is simply the case when the residue characteristic is 2. Our approach to this situation is even more generic and in a certain sense experimental, since Lemma 6.10.a) does not give any indication on how large the integer n has to be chosen in practice. For n fixed, we rely on the short exact sequence

$$0 \longrightarrow {\mathscr{O}_l}^*/(1 + \mathfrak{m}_{\mathscr{O}_l}^n) \longrightarrow l^*/(1 + \mathfrak{m}_{\mathscr{O}_l}^n) \xrightarrow{\overline{\nu}_l} \mathbb{Z} \longrightarrow 0$$

and on the canonical isomorphism ${\mathscr{O}_l}^*/(1 + \mathfrak{m}_{\mathscr{O}_l}^n) \cong (\mathscr{O}_l/\mathfrak{m}_{\mathscr{O}_l}^n)^*$.

Algorithm 6.18 *(For the case that $\#G$ is a power of 2, and the Brauer class is of at most 4-torsion).* Given a positive integer n and a 2-cocycle $\varphi\colon G \times G \to l^*$, for l/k a finite Galois extension of local fields and $G \cong \mathrm{Gal}(l/k)$, satisfying the conditions listed, this algorithm computes the invariant in $\mathbb{Q}/\mathbb{Z}$ of the class in $H^2(G, l^*)$ represented by φ, as soon as n is sufficiently large.

i) Run step i) of Algorithm 6.12. If $f = 1$ or $f = 2$ then run a totally naive modification, which just replaces l by $lk^{\mathrm{nr},4}$. Then f is a multiple of 4.

ii) Fix a uniformiser π_l of l and calculate the unit group of the finite ring $\mathscr{O}_l/\mathfrak{m}^n_{\mathscr{O}_l}$, together with a partial map $\mathrm{pr}\colon \mathscr{O}_l \dashrightarrow \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_m\mathbb{Z}$.
Then create the G-module $l^*/(1 + \mathfrak{m}^n_{\mathscr{O}_l})$ as follows. Put $M := \mathbb{Z} \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_m\mathbb{Z}$. For each generator $g_i \in G$, construct an $(m + 1) \times (m + 1)$-matrix A_i in the following manner.
Put the first row to be $(1, 0, \ldots, 0)$. Complete the first column by the coefficients of $\mathrm{pr}(\frac{g_i(\pi_l)}{\pi_l})$. For $j = 1, \ldots, n$, complete the $(j + 1)$-st column by $\mathrm{pr}(g_i(\widetilde{e}_j))$, for $\widetilde{e}_j$ a lift of the j-th generator e_j of the unit group $\mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_m\mathbb{Z}$ to $\mathscr{O}_l$.
Finally, transform M into a G-module using the matrices A_i.

iii) Define the standard 2-cocycle $\mathrm{st}\colon G \times G \to M$ by

$$(\sigma, \tau) \mapsto \begin{cases} (0, 0, \ldots, 0) & \text{if } \mathrm{ex}(\sigma) + \mathrm{ex}(\tau) < f\,, \\ (e, \mathrm{pr}(\frac{\pi}{\pi_l^e})) & \text{if } \mathrm{ex}(\sigma) + \mathrm{ex}(\tau) \geq f\,, \end{cases}$$

for π a uniformiser of k.

iv) Calculate the image of the input cocycle by putting

$$\widetilde{\varphi}(\sigma, \tau) := \big(\nu_l(\varphi(\sigma, \tau)), \mathrm{pr}(\tfrac{\varphi(\sigma,\tau)}{\pi_l^{\nu_l(\varphi(\sigma,\tau))}})\big),$$

for every $(\sigma, \tau) \in G \times G$.

v) Calculate the abelian group $H^2(G, M)$ and, as its elements, the classes c_{st} and $c_{\widetilde{\varphi}}$ of the 2-cocycles st and $\widetilde{\varphi}$.
If $2c_{\mathrm{st}} = 0$ then output a message saying that a larger value of n has to be chosen and terminate immediately. Otherwise, find a natural number m such that $c_{\widetilde{\varphi}} = mc_{\mathrm{st}}$, and output $(m/f \bmod 1) \in \mathbb{Q}/\mathbb{Z}$ as the desired invariant.

Remark 6.19. We successfully ran Algorithm 6.18 for several examples. In these, the minimal value of n, for which the calculations went through, varied between 3 and 9.

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