



# When is the Bloch–Okounkov $q$ -bracket modular?

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## Abstract

We obtain a condition describing when the quasimodular forms given by the Bloch–Okounkov theorem as  $q$ -brackets of certain functions on partitions are actually modular. This condition involves the kernel of an operator  $\Delta$ . We describe an explicit basis for this kernel, which is very similar to the space of classical harmonic polynomials.

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## 1 Introduction

Given a family of quasimodular forms, the question which of its members are modular often has an interesting answer. For example, consider the family of theta series

$$\theta_P(\tau) = \sum_{\underline{x} \in \mathbb{Z}^r} P(\underline{x}) q^{x_1^2 + \dots + x_r^2} \quad (q = e^{2\pi i \tau})$$

given by all homogeneous polynomials  $P \in \mathbb{Z}[x_1, \dots, x_r]$ . The quasimodular form  $\theta_P$  is modular if and only if  $P$  is harmonic (i.e.  $P \in \ker \sum_{i=1}^r \frac{\partial^2}{\partial x_i^2}$ ) [10]. (As quasimodular forms were not yet defined, Schoeneberg only showed that  $\theta_P$  is modular if  $P$  is harmonic. However, for every polynomial  $P$  it follows that  $\theta_P$  is quasimodular by decomposing  $P$  as in Formula (1).) Also, for every two modular forms  $f, g$ , one can consider the linear combination of products of derivatives of  $f$  and  $g$  given by

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$$\sum_{r=0}^n a_r f^{(r)} g^{(n-r)} \quad (a_r \in \mathbb{C}).$$

This linear combination is a quasimodular form which is modular precisely if it is a multiple of the Rankin–Cohen bracket  $[f, g]_n$  [4,9]. In this paper, we provide a condition to decide which member of the family of quasimodular forms provided by the Bloch–Okounkov theorem is modular. Let  $\mathcal{P}$  denote the set of all partitions of integers and  $|\lambda|$  denote the integer that  $\lambda$  is a partition of. Given a function  $f : \mathcal{P} \rightarrow \mathbb{Q}$ , define the  $q$ -bracket of  $f$  by

$$\langle f \rangle_q := \frac{\sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}} q^{|\lambda|}}.$$

The celebrated Bloch–Okounkov theorem states that for a certain family of functions  $f : \mathcal{P} \rightarrow \mathbb{Q}$  (called shifted symmetric polynomials and defined in Sect. 2) the  $q$ -brackets  $\langle f \rangle_q$  are the  $q$ -expansions of quasimodular forms [2].

Besides being a wonderful result, the Bloch–Okounkov theorem has many applications in enumerative geometry. For example, a special case of the Bloch–Okounkov theorem was discovered by Dijkgraaf and provided with a mathematically rigorous proof by Kaneko and Zagier, implying that the generating series of simple Hurwitz numbers over a torus are quasimodular [5,7]. Also, in the computation of asymptotics of geometrical invariants, such as volumes of moduli spaces of holomorphic differentials and Siegel–Veech constants, the Bloch–Okounkov theorem is applied [3,6].

Zagier gave a surprisingly short and elementary proof of the Bloch–Okounkov theorem [13]. A corollary of his work, which we discuss in Sect. 3, is the following proposition:

**Proposition 1** *There exists actions of the Lie algebra  $\mathfrak{sl}_2$  on both the algebra of shifted symmetric polynomials  $\Lambda^*$  and the algebra of quasimodular forms  $\tilde{M}$  such that the  $q$ -bracket  $\langle \cdot \rangle_q : \Lambda^* \rightarrow \tilde{M}$  is  $\mathfrak{sl}_2$ -equivariant.*

The answer to the question in the title is provided by one of the operators  $\Delta$  which defines this  $\mathfrak{sl}_2$ -action on  $\Lambda^*$ . Namely letting  $\mathcal{H} = \ker \Delta|_{\Lambda^*}$ , we prove the following theorem:

**Theorem 1** *Let  $f \in \Lambda^*$ . Then  $\langle f \rangle_q$  is modular if and only if  $f = h + k$  with  $h \in \mathcal{H}$  and  $k \in \ker \langle \cdot \rangle_q$ .*

The last section of this article is devoted to describing the graded algebra  $\mathcal{H}$ . We call  $\mathcal{H}$  the space of *shifted symmetric harmonic polynomials*, as the description of this space turns out to be very similar to the space of classical harmonic polynomials. Let  $\mathcal{P}_d$  be the space of polynomials of degree  $d$  in  $m \geq 3$  variables  $x_1, \dots, x_m$ , let  $\|x\|^2 = \sum_i x_i^2$ , and recall that the space  $\mathcal{H}_d$  of degree  $d$  harmonic polynomials is given by  $\ker \sum_{i=1}^m \frac{\partial^2}{\partial x_i^2}$ . The main theorem of harmonic polynomials states that every polynomial  $P \in \mathcal{P}_d$  can uniquely be written in the form

$$P = h_0 + \|x\|^2 h_1 + \dots + \|x\|^{2d'} h_{d'} \tag{1}$$

with  $h_i \in \mathcal{H}_{d-2i}$  and  $d' = \lfloor d/2 \rfloor$ . Define  $K$ , the Kelvin transform, and  $D^\alpha$  for  $\alpha$  an  $m$ -tuple of non-negative integers by

$$f(x) \mapsto \|x\|^{2-m} f\left(\frac{x}{\|x\|^2}\right) \quad \text{and} \quad D^\alpha = \prod_i \frac{\partial_i^{\alpha_i}}{\partial x_i^{\alpha_i}}.$$

An explicit basis for  $\mathcal{H}_d$  is given by

$$\{K D^\alpha K(1) \mid \alpha \in \mathbb{Z}_{\geq 0}^m, \sum_i \alpha_i = d, \alpha_1 \leq 1\},$$

see for example [1]. We prove the following analogous results for the space of shifted symmetric polynomials:

**Theorem 2** *For every  $f \in \Lambda_n^*$  there exists unique  $h_i \in \mathcal{H}_{n-2i}$  ( $i = 0, 1, \dots, n'$  and  $n' = \lfloor \frac{n}{2} \rfloor$ ) such that*

$$f = h_0 + Q_2 h_1 + \dots + Q_2^{n'} h_{n'},$$

where  $Q_2$  is an element of  $\Lambda_2^*$  given by  $Q_2(\lambda) = |\lambda| - \frac{1}{2}$ . □

**Theorem 3** *The set*

$$\{\text{pr } K \Delta_\lambda K(1) \mid \lambda \in \mathcal{P}(n), \text{ all parts are } \geq 3\}$$

is a vector space basis of  $\mathcal{H}_n$ , where  $\text{pr}$ ,  $K$ , and  $\Delta_\lambda$  are defined by (4), Definition 4, respectively, Definition 6.

The action of  $\mathfrak{sl}_2$  given by Proposition 1 makes  $\Lambda^*$  into an infinite-dimensional  $\mathfrak{sl}_2$ -representation for which the elements of  $\mathcal{H}$  are the lowest weight vectors. Theorem 2 is equivalent to the statement that  $\Lambda^*$  is a direct sum of the (not necessarily irreducible) lowest weight modules

$$V_n = \bigoplus_{m=0}^{\infty} Q_2^m \mathcal{H}_n \quad (n \in \mathbb{Z}).$$

## 2 Shifted symmetric polynomials

Shifted symmetric polynomials were introduced by Okounkov and Olshanski as the following analogue of symmetric polynomials [8]. Let  $\Lambda^*(m)$  be the space of rational polynomials in  $m$  variables  $x_1, \dots, x_m$  which are *shifted symmetric*, i.e. invariant under the action of all  $\sigma \in \mathfrak{S}_m$  given by  $x_i \mapsto x_{\sigma(i)} + i - \sigma(i)$  (or more symmetrically  $x_i - i \mapsto x_{\sigma(i)} - \sigma(i)$ ). Note that  $\Lambda^*(m)$  is filtered by the degree of the polynomials. We have forgetful maps  $\Lambda^*(m) \rightarrow \Lambda^*(m-1)$  given by  $x_m \mapsto 0$ , so that we can define the space of shifted symmetric polynomials  $\Lambda^*$  as  $\varprojlim_m \Lambda^*(m)$  in the category of

filtered algebras. Considering a partition  $\lambda$  as a non-increasing sequence  $(\lambda_1, \lambda_2, \dots)$  of non-negative integers  $\lambda_i$ , we can interpret  $\Lambda^*$  as being a subspace of all functions  $\mathcal{P} \rightarrow \mathbb{Q}$ .

One can find a concrete basis for this abstractly defined space by considering the generating series

$$w_\lambda(T) := \sum_{i=1}^{\infty} T^{\lambda_i - i + \frac{1}{2}} \in T^{1/2} \mathbb{Z}[T][[T^{-1}]] \tag{2}$$

for every  $\lambda \in \mathcal{P}$  (the constant  $\frac{1}{2}$  turns out to be convenient for defining a grading on  $\Lambda^*$ ). As  $w_\lambda(T)$  converges for  $T > 1$  and equals

$$\frac{1}{T^{1/2} - T^{-1/2}} + \sum_{i=1}^{\ell(\lambda)} \left( T^{\lambda_i - i + \frac{1}{2}} - T^{-i + \frac{1}{2}} \right)$$

one can define shifted symmetric polynomials  $Q_i(\lambda)$  for  $i \geq 0$  by

$$\sum_{i=0}^{\infty} Q_i(\lambda) z^{i-1} := w_\lambda(e^z) \quad (0 < |z| < 2\pi). \tag{3}$$

The first few shifted symmetric polynomials  $Q_i$  are given by

$$Q_0(\lambda) = 1, \quad Q_1(\lambda) = 0, \quad Q_2(\lambda) = |\lambda| - \frac{1}{24}.$$

The  $Q_i$  freely generate the algebra of shifted symmetric polynomials, i.e.  $\Lambda^* = \mathbb{Q}[Q_2, Q_3, \dots]$ . It is believed that  $\Lambda^*$  is maximal in the sense that for all  $Q : \mathcal{P} \rightarrow \mathbb{Q}$  with  $Q \notin \Lambda^*$  it holds that  $\langle \Lambda^*[Q] \rangle_q \not\subseteq \tilde{M}$ .

**Remark 1** The space  $\Lambda^*$  can equally well be defined in terms of the Frobenius coordinates. Given a partition with Frobenius coordinates  $(a_1, \dots, a_r, b_1, \dots, b_r)$ , where  $a_i$  and  $b_i$  are the arm and leg lengths of the cells on the main diagonal, let

$$C_\lambda = \left\{ -b_1 - \frac{1}{2}, \dots, -b_r - \frac{1}{2}, a_r + \frac{1}{2}, \dots, a_1 + \frac{1}{2} \right\}.$$

Then

$$Q_k(\lambda) = \beta_k + \frac{1}{(k-1)!} \sum_{c \in C_\lambda} \text{sgn}(c) c^{k-1},$$

where  $\beta_k$  is the constant given by

$$\sum_{k \geq 0} \beta_k z^{k-1} = \frac{1}{2 \sinh(z/2)} = w_\emptyset(e^z).$$

We extend  $\Lambda^*$  to an algebra where  $Q_1 \neq 0$ . Observe that a non-increasing sequence  $(\lambda_1, \lambda_2, \dots)$  of integers corresponds to a partition precisely if it converges to 0. If, however, it converges to an integer  $n$ , Eqs. (2) and (3) still define  $Q_k(\lambda)$ . In fact, in this case

$$Q_k(\lambda) = (e^{n\partial})Q_k(\lambda - n)$$

by [13, Proposition 1] where  $\partial Q_0 = 0$ ,  $\partial Q_k = Q_{k-1}$  for  $k \geq 1$ , and  $\lambda - n = (\lambda_1 - n, \lambda_2 - n, \dots)$  corresponds to a partition (i.e. converges to 0). In particular,  $Q_1(\lambda) = n$  equals the number the sequence  $\lambda$  converges to. We now define the Bloch–Okounkov ring  $\mathcal{R}$  to be  $\Lambda^*[Q_1]$ , considered as a subspace of all functions from non-increasing eventually constant sequences of integers to  $\mathbb{Q}$ . It is convenient to work with  $\mathcal{R}$  instead of  $\Lambda^*$  to define the differential operators  $\Delta$  and more generally  $\Delta_\lambda$  later. Both on  $\Lambda^*$  and  $\mathcal{R}$ , we define a weight grading by assigning to  $Q_i$  weight  $i$ . Denote the projection map by

$$\text{pr} : \mathcal{R} \rightarrow \Lambda^*. \tag{4}$$

We extend  $\langle \cdot \rangle_q$  to  $\mathcal{R}$ .

The operator  $E = \sum_{m=0}^\infty Q_m \frac{\partial}{\partial Q_m}$  on  $\mathcal{R}$  multiplies an element of  $\mathcal{R}$  by its weight. Moreover, we consider the differential operators

$$\mathfrak{d} = \sum_{m=0}^\infty Q_m \frac{\partial}{\partial Q_{m+1}} \quad \text{and} \quad \mathcal{D} = \sum_{k,\ell \geq 0} \binom{k+\ell}{k} Q_{k+\ell} \frac{\partial^2}{\partial Q_{k+1} \partial Q_{\ell+1}}.$$

Let  $\Delta = \frac{1}{2}(\mathcal{D} - \mathfrak{d}^2)$ , i.e.

$$2\Delta = \sum_{k,\ell \geq 0} \left( \binom{k+\ell}{k} Q_{k+\ell} - Q_k Q_\ell \right) \frac{\partial^2}{\partial Q_{k+1} \partial Q_{\ell+1}} - \sum_{k \geq 0} Q_k \frac{\partial}{\partial Q_{k+2}}.$$

In the following (antisymmetric) table, the entry in the row of operator  $A$  and column of operator  $B$  denotes the commutator  $[A, B]$ , for proofs see [13, Lemma 3].

	$\Delta$	$\mathfrak{d}$	$E$	$Q_1$	$Q_2$
$\Delta$	0	0	$2\Delta$	0	$E - Q_1 \mathfrak{d} - \frac{1}{2}$
$\mathfrak{d}$	0	0	$\mathfrak{d}$	1	$Q_1$
$E$	$-2\Delta$	$-\mathfrak{d}$	0	$Q_1$	$2Q_2$
$Q_1$	0	$-1$	$-Q_1$	0	0
$Q_2$	$-E + Q_1 \mathfrak{d} + \frac{1}{2}$	$-Q_1$	$-2Q_2$	0	0

**Definition 1** A triple  $(X, Y, H)$  of operators is called an  $\mathfrak{sl}_2$ -triple if

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [Y, X] = H.$$

Let  $\hat{Q}_2 := Q_2 - \frac{1}{2}Q_1^2$  and  $\hat{E} := E - Q_1\partial - \frac{1}{2}$ . The following result follows by a direct computation using the above table:

**Proposition 2** *The operators  $(\hat{Q}_2, \Delta, \hat{E})$  form an  $\mathfrak{sl}_2$ -triple. □*

For later reference, we compute  $[\Delta, Q_2^n]$ . This could be done inductively by noting that  $[\Delta, Q_2^n] = Q_2^{n-1}[\Delta, Q_2] + [\Delta, Q_2^{n-1}]Q_2$  and using the commutation relations in the above table. The proof below is a direct computation from the definition of  $\Delta$ .

**Lemma 1** *For all  $n \in \mathbb{N}$ , the following relation holds*

$$[\Delta, Q_2^n] = -\frac{n(n-1)}{2}Q_1^2Q_2^{n-2} - nQ_1Q_2^{n-1}\partial + nQ_2^{n-1}(E + n - \frac{3}{2}).$$

**Proof** Let  $f \in \mathbb{Q}[Q_1, Q_2]$ ,  $g \in \mathcal{R}$ , and  $n \in \mathbb{N}$ . Then

$$\Delta(fg) = \Delta(f)g + \frac{\partial f}{\partial Q_2}(Eg - Q_1\partial g) + f\Delta(g), \tag{5}$$

$$\Delta(Q_2^n) = n(n - \frac{3}{2})Q_2^{n-1} - \frac{n(n-1)}{2}Q_2^{n-2}Q_1^2. \tag{6}$$

By (5) and (6), we find

$$\begin{aligned} \Delta(Q_2^n g) &= (n(n - \frac{3}{2})Q_2^{n-1} - \frac{n(n-1)}{2}Q_1^2Q_2^{n-2})g \\ &\quad + nQ_2^{n-1}(Eg - Q_1\partial g) + Q_2^n\Delta(g). \end{aligned} \tag{□}$$

### 3 An $\mathfrak{sl}_2$ -equivariant mapping

The space of quasimodular forms for  $SL_2(\mathbb{Z})$  is given by  $\tilde{M} = \mathbb{Q}[P, Q, R]$ , where  $P, Q$ , and  $R$  are the Eisenstein series of weight 2, 4, and 6, respectively (in Ramanujan’s notation). We let  $\tilde{M}_k^{(\leq p)}$  be the space of quasimodular forms of weight  $k$  and depth  $\leq p$  (the depth of a quasimodular form written as a polynomial in  $P, Q$ , and  $R$  is the degree of this polynomial in  $P$ ). See [12, Section 5.3] or [13, Section 2] for an introduction into quasimodular forms.

The space of quasimodular forms is closed under differentiation, more precisely the operators  $D = q\frac{d}{dq}$ ,  $\partial = 12\frac{\partial}{\partial P}$ , and the weight operator  $W$  given by  $Wf = kf$  for  $f \in \tilde{M}_k$  preserve  $\tilde{M}$  and form an  $\mathfrak{sl}_2$ -triple. In order to compute the action of  $D$  in terms of the generators  $P, Q$ , and  $R$ , one uses the Ramanujan identities

$$D(P) = \frac{P^2 - Q}{12}, \quad D(Q) = \frac{PQ - R}{3}, \quad D(R) = \frac{PR - Q^2}{2}.$$

In the context of the Bloch–Okounkov theorem, it is more natural to work with  $\hat{D} := D - \frac{P}{24}$ , as for all  $f \in \Lambda^*$  one has  $\langle Q_2 f \rangle_q = \hat{D}\langle f \rangle_q$ . Moreover,  $\hat{D}$  has the property that it increases the depth of a quasimodular form by 1, in contrast to  $D$  for which  $D(1) = 0$  does not have depth 1:

**Lemma 2** *Let  $f \in \tilde{M}$  be of depth  $r$ . Then  $\hat{D}f$  is of depth  $r + 1$ .*

**Proof** Consider a monomial  $P^a Q^b R^c$  with  $a, b, c \in \mathbb{Z}_{\geq 0}$ . By the Ramanujan identities, we find

$$D(P^a Q^b R^c) = \left( \frac{a}{12} + \frac{b}{3} + \frac{c}{2} \right) P^{a+1} Q^b R^c + O(P^a),$$

where  $O(P^a)$  denotes a quasimodular form of depth at most  $a$ . The lemma follows by noting that  $\frac{a}{12} + \frac{b}{3} + \frac{c}{2} - \frac{1}{24}$  is non-zero for  $a, b, c \in \mathbb{Z}$ .  $\square$

Moreover, letting  $\hat{W} = W - \frac{1}{2}$ , the triple  $(\hat{D}, \mathfrak{d}, \hat{W})$  forms an  $\mathfrak{sl}_2$ -triple as well. With respect to these operators, the  $q$ -bracket becomes  $\mathfrak{sl}_2$ -equivariant. The following proposition is a detailed version of Proposition 1:

**Proposition 3** (The  $\mathfrak{sl}_2$ -equivariant Bloch–Okounkov theorem) *The mapping  $\langle \cdot \rangle_q : \mathcal{R} \rightarrow \tilde{M}$  is  $\mathfrak{sl}_2$ -equivariant with respect to the  $\mathfrak{sl}_2$ -triple  $(\hat{Q}_2, \Delta, \hat{E})$  on  $\mathcal{R}$  and the  $\mathfrak{sl}_2$ -triple  $(\hat{D}, \mathfrak{d}, \hat{W})$  on  $\tilde{M}$ , i.e. for all  $f \in \mathcal{R}$ , one has*

$$\hat{D}\langle f \rangle_q = \langle \hat{Q}_2 f \rangle_q, \quad \mathfrak{d}\langle f \rangle_q = \langle \Delta f \rangle_q, \quad \hat{W}\langle f \rangle_q = \langle \hat{E} f \rangle_q.$$

**Proof** This follows directly from [13, Equation (37)] and the fact that for all  $f \in \mathcal{R}$  one has  $\langle Q_1 f \rangle_q = 0$ .  $\square$

#### 4 Describing the space of shifted symmetric harmonic polynomials

In this section, we study the kernel of  $\Delta$ . As  $[\Delta, Q_1] = 0$ , we restrict ourselves without loss of generality to  $\Lambda^*$ . Note, however, that  $\Delta$  does not act on  $\Lambda^*$  as, for example,  $\Delta(Q_3) = -\frac{1}{2}Q_1$ . However,  $\text{pr}\Delta$  does act on  $\Lambda^*$ .

**Definition 2** Let

$$\mathcal{H} = \{f \in \Lambda^* \mid \Delta f \in Q_1 \mathcal{R}\} = \ker \text{pr}\Delta,$$

be the space of *shifted symmetric harmonic* polynomials.

**Proposition 4** *If  $f \in Q_2 \Lambda^*$  is non-zero, then  $f \notin \mathcal{H}$ .*

**Proof** Write  $f = Q_2^n f'$  with  $f' \in \Lambda^*$  and  $f' \notin Q_2 \Lambda^*$ . Then

$$\text{pr}\Delta(f) = Q_2^{n-1} \left( n(n+k - \frac{3}{2})f' + Q_2 \text{pr}\Delta f' \right)$$

by Lemma 1. As  $f'$  is not divisible by  $Q_2$ , it follows that  $\text{pr}\Delta(f) = 0$  precisely if  $f' = 0$ .  $\square$

**Proposition 5** *For all  $n \in \mathbb{Z}$ , one has*

$$\Lambda_n^* = \mathcal{H}_n \oplus Q_2 \Lambda_{n-2}^*.$$

**Proof** For uniqueness, suppose  $f = Q_2g + h$  and  $f = Q_2g' + h'$  with  $g, g' \in \Lambda_{n-2}^*$  and  $h, h' \in \mathcal{H}_n$ . Then,  $Q_2(g - g') = h' - h \in \mathcal{H}$ . By Proposition 4 we find  $g = g'$  and hence  $h = h'$ .

Now, define the linear map  $T : \Lambda_n^* \rightarrow \Lambda_n^*$  by  $f \mapsto \text{pr}\Delta(Q_2f)$ . By Proposition 4 we find that  $T$  is injective, which by finite dimensionality of  $\Lambda_n^*$  implies that  $T$  is surjective. Hence, given  $f \in \Lambda_n^*$  let  $g \in \Lambda_{n-2}^*$  be such that  $T(g) = \text{pr}\Delta(f) \in \Lambda_{n-2}^*$ . Let  $h = f - Q_2g$ . As  $f = Q_2g + h$ , it suffices to show that  $h \in \mathcal{H}$ . That holds true because  $\text{pr}\Delta(h) = \text{pr}\Delta(f) - \text{pr}\Delta(Q_2g) = 0$ .  $\square$

Proposition 5 implies Theorem 2 and the following corollary. Denote by  $p(n)$  the number of partitions of  $n$ .

**Corollary 1** *The dimension of  $\mathcal{H}_n$  equals the number of partitions of  $n$  in parts of size at least 3, i.e.*

$$\dim \mathcal{H}_n = p(n) - p(n - 1) - p(n - 2) + p(n - 3).$$

**Proof** Observe that  $\dim \Lambda_n^*$  equals the number of partitions of  $n$  in parts of size at least 2. Hence,  $\dim \Lambda_n^* = p(n) - p(n - 1)$  and the Corollary follows from Proposition 5.  $\square$

**Proof of Theorem 1** If  $\langle f \rangle_q$  is modular, then  $\langle \Delta f \rangle_q = \mathfrak{d}\langle f \rangle_q = 0$ . Write  $f = \sum_{r=0}^{n'} Q_2^r h_r$  as in Theorem 2 with  $n' = \lfloor \frac{n}{2} \rfloor$ . Then by Lemma 1 it follows that  $\text{pr}\Delta f = \sum_{r=0}^{n'} r(n - r - \frac{3}{2}) Q_2^{r-1} h_r$ . Hence,

$$\sum_{r=1}^{n'} r(n - r - \frac{3}{2}) \hat{D}^{r-1} \langle h_r \rangle_q = 0. \tag{7}$$

As  $\langle h_r \rangle_q$  is modular, either it is equal to 0 or it has depth 0. Suppose the maximum  $m$  of all  $r \geq 1$  such that  $\langle h_r \rangle_q$  is non-zero exists. Then, by Lemma 2 it follows that the left-hand side of (7) has depth  $m - 1$ , in particular is not equal to 0. So,  $h_1, \dots, h_{n'} \in \ker \langle \cdot \rangle_q$ . Note that  $f \in \ker \langle \cdot \rangle_q$  implies that  $Q_2f \in \ker \langle \cdot \rangle_q$ . Therefore,  $k := \sum_{r=1}^{n'} Q_2^r h_r \in \ker \langle \cdot \rangle_q$  and  $f = h + k$  with  $h = h_0$  harmonic.

The converse follows directly as  $\mathfrak{d}\langle h + k \rangle_q = \mathfrak{d}\langle h \rangle_q = \langle \Delta h \rangle_q = 0$ .  $\square$

**Remark 2** A description of the kernel of  $\langle \cdot \rangle_q$  is not known.

Another corollary of Proposition 5 is the notion of *depth* of shifted symmetric polynomials which corresponds to the depth of quasimodular forms:

**Definition 3** The space  $\Lambda_k^{*(\leq p)}$  of shifted symmetric polynomials of depth  $\leq p$  is the space of  $f \in \Lambda_k^*$  such that one can write

$$f = \sum_{r=0}^p Q_2^r h_r,$$

with  $h_r \in \mathcal{H}_{k-2r}$ .



**Theorem 4** *If  $f \in \Lambda_k^{*(\leq p)}$ , then  $\langle f \rangle_q \in \tilde{M}_k^{(\leq p)}$ .*

**Proof** Expanding  $f$  as in Definition 3 we find

$$\langle f \rangle_q = \sum_{k=0}^p \langle Q_2^k h_k \rangle_q = \sum_{k=0}^p \hat{D}^k \langle h_k \rangle_q.$$

By Lemma 2, we find that the depth of  $\langle f \rangle_q$  is at most  $p$ . □

Next, we set up notation to determine the basis of  $\mathcal{H}$  given by Theorem 3. Let  $\tilde{\mathcal{R}} = \mathcal{R}[Q_2^{-1/2}]$  and  $\tilde{\Lambda} = \Lambda^*[Q_2^{-1/2}]$  be the formal polynomial algebras graded by assigning to  $Q_k$  weight  $k$  (note that the weights are—possibly negative—integers). Extend  $\Delta$  to  $\tilde{\Lambda}$  and observe that  $\Delta(\tilde{\Lambda}) \subset \tilde{\Lambda}$ . Also extend  $\mathcal{H}$  by setting

$$\tilde{\mathcal{H}} = \{f \in \tilde{\Lambda} \mid \Delta f \in Q_1 \tilde{\mathcal{R}}\} = \ker \text{pr} \Delta|_{\tilde{\Lambda}}.$$

**Definition 4** Define the *partition-Kelvin transform*  $K : \tilde{\Lambda}_n \rightarrow \tilde{\Lambda}_{3-n}$  by

$$K(f) = Q_2^{3/2-n} f.$$

Note that  $K$  is an involution. Moreover,  $f$  is harmonic if and only if  $K(f)$  is harmonic, which follows directly from the computation

$$\Delta K(f) = Q_2^{3/2-n} \Delta f - \left(\frac{3}{2} - n\right) Q_1 Q_2^{\frac{1}{2}-n} \partial f - \frac{1}{2} \left(\frac{3}{2} - n\right) \left(\frac{1}{2} - n\right) Q_1^2 Q_2^{-\frac{1}{2}-n} f.$$

**Example 1** As  $K(1) = Q_2^{3/2}$ , it follows that  $Q_2^{3/2} \in \tilde{\mathcal{H}}$ .

**Definition 5** Given  $\underline{i} \in \mathbb{Z}_{\geq 0}^n$ , let

$$|\underline{i}| = i_1 + i_2 + \dots + i_n, \quad \partial_{\underline{i}} = \frac{\partial^n}{\partial Q_{i_1+1} \partial Q_{i_2+1} \dots \partial Q_{i_n+1}}.$$

Define the  $n$ th order differential operators  $\mathcal{D}_n$  on  $\tilde{\mathcal{R}}$  by

$$\mathcal{D}_n = \sum_{\underline{i} \in \mathbb{Z}_{\geq 0}^n} \binom{|\underline{i}|}{i_1, i_2, \dots, i_n} Q_{|\underline{i}|} \partial_{\underline{i}},$$

where the coefficient is a multinomial coefficient.

This definition generalises the operators  $\partial$  and  $\mathcal{D}$  to higher weights:  $\mathcal{D}_1 = \partial$ ,  $\mathcal{D}_2 = \mathcal{D}$ , and  $\mathcal{D}_n$  reduces the weight by  $n$ .

**Lemma 3** *The operators  $\{\mathcal{D}_n\}_{n \in \mathbb{N}}$  commute pairwise.*

**Proof** Set  $I = |i|$  and  $J = |j|$ . Let  $\underline{a}^{\hat{k}} = (a_1, \dots, a_{k-1}, a_{k+1}, \dots, a_n)$ . Then

$$\begin{aligned} & \left[ \binom{I}{i_1, i_2, \dots, i_n} Q_I \partial_{\underline{i}}, \binom{J}{j_1, j_2, \dots, j_m} Q_J \partial_{\underline{j}} \right] \\ &= \sum_{k=1}^n \delta_{i_k, J-1} J \binom{I}{i_1, i_2, \dots, \hat{i}_k, \dots, i_n, j_1, j_2, \dots, j_m} Q_I \partial_{\underline{i}^k} \partial_{\underline{j}} + \quad (8) \\ & \quad - \sum_{l=1}^m \delta_{j_l, I-1} I \binom{J}{i_1, i_2, \dots, i_n, j_1, j_2, \dots, \hat{j}_l, \dots, j_m} Q_J \partial_{\underline{i}} \partial_{\underline{j}^l}. \end{aligned}$$

Hence,  $[\mathcal{D}_n, \mathcal{D}_m]$  is a linear combination of terms of the form  $Q_{|\underline{a}|+1} \partial_{\underline{a}}$ , where  $\underline{a} \in \mathbb{Z}_{\geq 0}^{n+m-1}$ . We collect all terms for different vectors  $\underline{a}$  which consists of the same parts (i.e. we group all vectors  $\underline{a}$  which correspond to the same partition). Then, the coefficient of such a term equals

$$\begin{aligned} & \sum_{k=1}^n \sum_{\sigma \in S_{m+n-1}} (a_{\sigma(1)} + \dots + a_{\sigma(m)}) \binom{|\underline{a}|+1}{a_1, a_2, \dots, a_{n+m-1}} \\ & \quad - \sum_{l=1}^m \sum_{\sigma \in S_{m+n-1}} (a_{\sigma(1)} + \dots + a_{\sigma(n)}) \binom{|\underline{a}|+1}{a_1, a_2, \dots, a_{n+m-1}} \\ &= (mn - mn) \sum_{\sigma \in S_{m+n-1}} a_{\sigma(1)} \binom{|\underline{a}|+1}{a_1, a_2, \dots, a_{n+m-1}} = 0. \end{aligned}$$

Hence,  $[\mathcal{D}_n, \mathcal{D}_m] = 0$ . □

It does not hold true that  $[\mathcal{D}_n, Q_1] = 0$  for all  $n \in \mathbb{N}$ . Therefore, we introduce the following operators:

**Definition 6** Let

$$\Delta_n = \sum_{i=0}^n (-1)^i \binom{n}{i} \mathcal{D}_{n-i} \partial^i.$$

For  $\lambda \in \mathcal{P}$  let

$$\Delta_\lambda = \binom{|\lambda|}{\lambda_1, \dots, \lambda_{\ell(\lambda)}} \prod_{i=1}^{\infty} \Delta_{\lambda_i}.$$

(Note that  $\Delta_0 = \mathcal{D}_0 = 1$ , so this is in fact a finite product.)

**Remark 3** By Möbius inversion

$$\mathcal{D}_n = \sum_{i=0}^n \binom{n}{i} \Delta_{n-i} \mathfrak{d}^i.$$

The first three operators are given by

$$\Delta_0 = 1, \quad \Delta_1 = 0, \quad \Delta_2 = \mathcal{D} - \mathfrak{d}^2 = 2\Delta.$$

**Proposition 6** *The operators  $\Delta_\lambda$  satisfy the following properties: for all partitions  $\lambda, \lambda'$*

- (a) *the order of  $\Delta_{|\lambda|}$  is  $|\lambda|$ ;*
- (b)  $[\Delta_\lambda, \Delta_{\lambda'}] = 0$ ;
- (c)  $[\Delta_\lambda, Q_1] = 0$ .

**Proof** Property (a) follows by construction and (b) is a direct consequence of Lemma 3. For property (c), let  $f \in \tilde{\Lambda}$  be given. Then

$$\begin{aligned} \Delta_n(Q_1 f) &= \sum_{i=0}^n (-1)^i \binom{n}{i} \mathcal{D}_{n-i} \mathfrak{d}^i (Q_1 f) \\ &= \sum_{i=0}^n (-1)^i \binom{n}{i} \left( (n-i) \mathcal{D}_{n-i-1} \mathfrak{d}^i f + Q_1 \mathcal{D}_{n-i} \mathfrak{d}^i f + i \mathcal{D}_{n-i} \mathfrak{d}^{i-1} f \right) \\ &= Q_1 \Delta_n(f) + \sum_{i=0}^n (-1)^i \binom{n}{i} \left( (n-i) \mathcal{D}_{n-i-1} \mathfrak{d}^i f + i \mathcal{D}_{n-i} \mathfrak{d}^{i-1} f \right). \end{aligned}$$

Observe that by the identity

$$(n-i) \binom{n}{i} = (i+1) \binom{n}{i+1},$$

the sum in the last line is a telescoping sum, equal to zero. Hence  $\Delta_n(Q_1 f) = Q_1 \Delta_n(f)$  as desired. □

In particular, the above proposition yields  $[\Delta_\lambda, \Delta] = 0$  and  $[\Delta_\lambda, \text{pr}] = 0$ .

Denote by  $(x)_n$  the falling factorial power  $(x)_n = \prod_{i=0}^{n-1} (x-i)$  and for  $\lambda \in \mathcal{P}_n$  define  $Q_\lambda = \prod_{i=1}^\infty Q_{\lambda_i}$ . Let

$$h_\lambda = \text{pr} K \Delta_\lambda K(1).$$

Observe that  $h_\lambda$  is harmonic, as  $\text{pr} \Delta$  commutes with  $\text{pr}$  and  $\Delta_\lambda$ .

**Proposition 7** For all  $\lambda \in \mathcal{P}_n$  there exists an  $f \in \Lambda_{n-2}^*$  such that

$$h_\lambda = \left(\frac{3}{2}\right)_n n! Q_\lambda + Q_2 f.$$

**Proof** Note that the left-hand side is an element of  $\Lambda^*$  of which the monomials divisible by  $Q_2^i$  correspond precisely to terms in  $\Delta_\lambda$  involving precisely  $n - i$  derivatives of  $K(1)$  to  $Q_2$ . Hence, as  $\Delta_\lambda$  has order  $n$  all terms not divisible by  $Q_2$  correspond to terms in  $\Delta_\lambda$  which equal  $\frac{\partial^n}{\partial Q_2^n}$  up to a coefficient. There is only one such term in  $\Delta_\lambda$  with coefficient  $\binom{|\lambda|}{\lambda_1, \dots, \lambda_r} \lambda_1! \dots \lambda_r! Q_\lambda$ .  $\square$

For  $f \in \mathcal{R}$ , we let  $f^\vee$  be the operator where every occurrence of  $Q_i$  in  $f$  is replaced by  $\Delta_i$ . We get the following unusual identity:

**Corollary 2** If  $h \in \mathcal{H}_n$ , then

$$h = \frac{\text{pr} K h^\vee K(1)}{n! \left(\frac{3}{2}\right)_n}. \tag{9}$$

**Proof** By Proposition 7, we know that the statement holds true up to adding  $Q_2 f$  on the right-hand side for some  $f \in \Lambda_{n-2}^*$ . However, as both sides of (9) are harmonic and the shifted symmetric polynomial  $Q_2 f$  is harmonic precisely if  $f = 0$  by Proposition 4, it follows that  $f = 0$  and (9) holds true.  $\square$

**Proof of Theorem 3** Let  $\mathcal{B}_n = \{h_\lambda \mid \lambda \in \mathcal{P}_n \text{ all parts are } \geq 3\}$ . First of all, observe that by Corollary 1 the number of elements in  $\mathcal{B}_n$  is precisely the dimension of  $\mathcal{H}_n$ . Moreover, the weight of an element in  $\mathcal{B}_n$  equals  $|\lambda| = n$ . By Proposition 7 it follows that the elements of  $\mathcal{B}_n$  are linearly independent harmonic shifted symmetric polynomials.  $\square$

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## Appendix: Tables of shifted symmetric harmonic polynomials up to weight 10

We list all harmonic polynomials  $h_\lambda$  of even weight at most 10. The corresponding  $q$ -brackets  $\langle h_\lambda \rangle_q$  are computed by the algorithm prescribed by Zagier [13] using SageMath [11].

$\lambda$	$h_\lambda$	$\langle h_\lambda \rangle_q$
(0)	1	1
(4)	$\frac{27}{4} (Q_2^2 + 2Q_4)$	$\frac{9}{320} Q$
(6)	$\frac{225}{4} (63Q_6 + 9Q_2Q_4 + Q_3^2)$	$-\frac{55}{384} R$
(3,3)	$\frac{225}{4} (63Q_3^2 - 108Q_2Q_4 + 2Q_2^3)$	$\frac{115}{384} R$
(8)	$\frac{19845}{16} (3960Q_8 + 360Q_2Q_6 + 20Q_2^2Q_4 + Q_2^4)$	$\frac{19173}{4096} Q^2$
(5,3)	$\frac{19845}{2} (495Q_3Q_5 + 45Q_2Q_3^2 - 1350Q_2Q_6 - 50Q_2^2Q_4 + 2Q_2^4)$	$-\frac{2415}{128} Q^2$
(4,4)	$\frac{297675}{8} (132Q_4^2 + 24Q_2Q_3^2 - 440Q_2Q_6 - 28Q_2^2Q_4 + Q_2^4)$	$-\frac{38241}{2048} Q^2$
(10)	$\frac{382725}{8} (450450Q_{10} + 30030Q_2Q_8 + 1155Q_2^2Q_6 + 35Q_2^3Q_4 + Q_2^5)$	$-\frac{2053485}{4096} QR$
(7,3)	$\frac{1913625}{8} (90090Q_3Q_7 + 6006Q_2Q_3Q_5 - 336336Q_2Q_8 + 231Q_2Q_3^2 +$ $-12936Q_2^2Q_6 - 112Q_2^3Q_4 + 10Q_2^5)$	$\frac{11975985}{4096} QR$
(6,4)	$\frac{13395375}{8} (12870Q_4Q_6 + 1716Q_2Q_3Q_5 + 858Q_2Q_4^2 - 96096Q_2Q_8 +$ $+132Q_2^2Q_3^2 - 6501Q_2^2Q_6 - 89Q_2^3Q_4 + 5Q_2^4)$	$\frac{21255885}{4096} QR$
(5,5)	$\frac{8037225}{4} (10725Q_5^2 + 1430Q_2Q_3Q_5 + 1430Q_2Q_4^2 - 10010Q_2Q_8 +$ $+165Q_2^2Q_3^2 - 7700Q_2^2Q_6 - 120Q_2^3Q_4 + 6Q_2^5)$	$\frac{7759395}{1024} QR$
(4,3,3)	$\frac{13395375}{8} (12870Q_3^2Q_4 - 34320Q_2Q_3Q_5 + 10296Q_2Q_4^2 + 363Q_2^2Q_3^2 +$ $+55440Q_2^2Q_6 - 376Q_2^3Q_4 + 10Q_2^5)$	$-\frac{16583805}{4096} QR$

In case  $|\lambda|$  is odd, the harmonic polynomials  $h_\lambda$  up to weight 9 are given in the following table. The  $q$ -bracket of odd degree (harmonic) polynomials is zero, hence trivially modular.

$\lambda$	$h_\lambda$
(3)	$-\frac{9}{4} Q_3$
(5)	$-\frac{135}{4} (5Q_5 + Q_2Q_3)$
(7)	$-\frac{14175}{16} (126Q_7 + 14Q_2Q_5 + Q_2^2Q_3)$
(4, 3)	$-\frac{99225}{16} (18Q_3Q_4 - 40Q_2Q_5 + Q_2^2Q_3)$
(9)	$-\frac{297675}{8} (7722Q_9 + 594Q_2Q_7 + 27Q_2^2Q_5 + Q_2^3Q_3)$
(6, 3)	$-\frac{893025}{4} (1287Q_3Q_6 + 99Q_2Q_3Q_4 - 4158Q_2Q_7 - 162Q_2^2Q_5 + 5Q_2^3Q_3)$
(5, 4)	$-\frac{8037225}{8} (286Q_4Q_5 + 66Q_2Q_3Q_4 - 1540Q_2Q_7 - 117Q_2^2Q_5 + 3Q_2^3Q_3)$
(3, 3, 3)	$-\frac{893025}{4} (1287Q_3^3 - 3564Q_2Q_3Q_4 + 3240Q_2^2Q_5 + 10Q_2^3Q_3)$

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